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I. *The Rare Lichens of Glencroe.* By Rev. Hugh Macmillan, LL.D., F.R.S.E.

During the months of August and September last, after my return from Switzerland, I spent my holidays, along with my family, in a solitary farm-house in Glencroe, at the head of Loch Long. This farm-house is called Larich Park, and is situated about three quarters of a mile from the entrance of the glen, and about three miles from Arrochar. Glencroe is six miles long, leading from Loch Long to the elevated plateau of "Rest and be Thankful," the scene of one of Wordsworth's sonnets, and famous from the use made by Lord Russell of it as a parliamentary phrase. The coach road from Arrochar to Inverary passes through it. The scenery of the glen is remarkably wild and desolate. Rugged mountains from two to three thousand feet high,—their sides strewn with boulders, and seamed with torrents,—overhang the road, and approach at their base from either side so closely, as to leave but a very narrow passage for the river, which has worn for itself a strangely tortuous and deeply indented channel through the rocks. There are only three or four human habitations in the whole glen, and but two small portions of cultivated land,—the rest being given up to pasturage for sheep. That part of the glen, immediately surrounding the farm of Larich Park, combines in a remarkable manner the grand and the beautiful. Ben Arthur, or the Cobbler, rises on one side to a height of 2863 feet; its precipitous sides covered with rich vegetation, and marked by white waterfalls, and its summit consisting of a huge aiguille of rock, which appears and disappears at intervals through the driving mist with a singularly weird and wild effect. On the other side of the glen towers up a ridge of equally steep and rocky peaks, terminating in a great shattered rocky crown called Ben Unack, only 200 feet or so lower than the Cobbler. While at the entrance of the glen, on both sides of the river, which forms here a promontory or delta of alluvial soil projecting into Loch Long, repose in quiet beauty between the mighty masses of rocks and hills, the luxuriant woods of Ardgarten, composed of fine old pines, beeches, and oaks. This part of the glen has a southern exposure; and owing to its narrowness and the loftiness of the mountains around it, enjoys a mild, moist climate, as may be seen by the luxuriant development of grass, bracken, and rushes upon the hills, instead of heather. The prevailing formation is micaceous schist, considerably distorted and of varying hardness, and a quartzose flagstone.

Residing on the spot continuously for so long a period, I had abundant opportunities of exploring the cryptogamic flora of the glen, which I presume had never previously been investigated. The Cobbler and the surrounding mountains yield nothing of any particular interest, producing the common lichens and mosses which are found everywhere at similar heights; in this respect contrasting remarkably with the rich variety of rare Alpine species which are found on the Breadalbane mountains, not twenty miles distant in a straight line, and whose climate and soil are very much the same. The base of the hills at Glencroe, however, more than made up for this deficiency. On the western side of the river, immediately opposite Larich Park, huge fragments of micaceous rock, which had evidently fallen ages ago from the cliffs overhead, strew the green slope of the hill. These fragments are somewhat numerous, and conspicuous in one spot. They are covered with the usual lichenose vegetation of a sub-alpine glen, such as Stereocaulons, Sphærophorons, Cetrarias, and Parmelias, all spreading luxuriantly, and attaining unusual proportions, indicating that they find in this place the peculiar conditions of shade and shelter, and the mild moistness of the air in which lichens delight. Among the common species I picked out several rare lichens which interested me greatly. One of these is the *Parmelia lævigata* of Acharius. This species is one of the largest and handsomest of the British lichens. It forms very broad, conspicuous patches on the rocks; one individual often extending upwards of a foot in length and breadth, and covering continuously the whole exposed side of a boulder. It adheres very loosely to its growing place by means of its black shaggy underside. Its upper surface is smooth and greyish-white, cut up into numerous lobes and segments; the ultimate ones dividing into two or more very narrow branches whose extremities are turned up, and covered with grey powdery warts or soredia. The lichen consists of a series of these segments, often imbricated. The curious abrupt divisions of the larger and broader lobes, into two or more acute linear ones, crowned with mealy tubercles, is a characteristic which cannot be mistaken, and serves effectually to distinguish this from all other lichens. It is, indeed, one of the best defined species we possess. The apothecia, which are very hollow, and of a dark red colour, with an entire crenate or sorediate border, are extremely rare. There were none on the individuals which I gathered, and I believe they have never been found in this country, being described from foreign specimens. The lichen itself is exceedingly rare. The late Mr Borrer is reported to have found it in Ross-shire and at Ballachulish; but his discovery has not been repeated in these localities by any other botanist. The latter region I searched pretty minutely for two days while on a yachting cruise in the summer of 1873; but I failed to find specimens either of this lichen or of the three or four other extremely rare lichens which Mr Borrer, and no one else, gathered there. It is therefore an important circumstance that a new locality for the *P. lævigata* has been found in Scotland, which is more accessible, more easily pointed out, and where it occurs in considerable abundance. In England it has been found on Dartmoor and in St Leonard's Forest, and on rocks in the island of Anglesea, at Dolgelly in Carnarvonshire, and in one or two other places in South Wales, in Jersey and Guernsey, and in Ireland in Askew Wood, County Kerry. In England, Wales, and Scotland its distribution is thus confined to the west coast. We can trace it along a narrow line that never goes into the interior, or curves to the east, from Devonshire to Ballachulish. It is eminently a southern lichen, and loves moist, mild, sheltered localities within reach of the sea, probably depending, like the *Ramalina scopulorum* and its own congener *Parmelia aquila*, for its growth and well-being upon some constituent in the sea-air. It has been found sparingly distributed throughout the continent of Europe in localities similar to those which it affects in this country; and specimens have been sent from the valley of the Amazons in South America, and from gneiss and basalt boulders in Middle Island, New Zealand, where the thallus is slightly yellowish. It admits of being suggested that Glencroe and Ballachulish were colonised by spores from Wales and the south-west of England, blown northwards by the south-western breezes which penetrate to the heads of our western sea-lochs lying directly in their course. Or we may suppose that the lichen exists in other localities on the western sea-board of Scotland, and awaits discovery there; this region being confessedly very imperfectly explored by cryptogamists, and may be expected to yield, as in the case of the discovery of the *Myurium Hebridense* by the side of Loch Coruisk in Skye by myself, and in South Uist and other localities by others, to a thorough search, results as interesting as they are novel. If the latter supposition be adopted, the lichen might be regarded as indigenous to the region; and we might as reasonably imagine it to have migrated from Scotland to the English and Welsh stations, as to have been derived in the northern localities from the southern. I am disposed to class it with a peculiar group of large highly-developed lichens, comprising the tropical Stictas, &c., found in the south-west of Ireland, and in a few isolated spots in Scotland and England, which may probably have had an American origin, and may be connected with the late geological changes that have taken place in the relations between the Old and New Worlds.

Associated with typical specimens of the *Parmelia lævigata* I found several others which appeared to me to differ very considerably from them, and to present several abnormal peculiarities. The more I considered them, the more puzzled I became. At first I took them to be mere varieties due to certain conditions of soil and exposure. But a more attentive examination led me to the conclusion that they belonged to a distinct, but to me unknown species. I laid the specimens before Dr Stirton of Glasgow, upon whose critical judgment I could place the utmost dependence. After subjecting them to microscopical examination of their physical peculiarities, and to those chemical tests initiated by Nylander, which have lately proved so useful in identifying species so closely allied as to be otherwise confounded, he came to the conclusion that they were entirely new to science. Dr Stirton, therefore, did me the honour of naming the lichen after me as *Parmelia Millaniana.* Subsequently, however, I sent a small specimen to the Rev. Mr Leighton; when he assured me that it was identical with his *P. endochlora*, found by Dr Taylor in Askew Wood, County Kerry, Ireland, associated, as in this country, with *P. lævigata.* In these circumstances, although the discrepancies between Mr Leighton's published description, and the characteristics of my plant were sufficient to warrant Dr Stirton and myself in regarding it at first as new, I have now no alternative but to yield, which I do cordially, to Mr Leighton's prior right of nomenclature, and adopt his name. The lichen in question, as I have said, is at first sight very like *P. lævigata.* But by scraping the upper surface, the medulla is found to be of a decidedly yellow colour; whereas the medulla of the *P. lævigata* is pure white. The action of the reagent produces in the new lichen a deeper yellow, amounting almost to an orange tinge; whereas the chemical reaction of the *P. lævigata* is bright unchangeable red. The hairs on the under-surface are also different, being much more dendritic and branched. The whole facies of the lichen is such as to distinguish it at once, when attention has been directed to it. It is less lax in its habit than *P. lævigata;* clinging more closely to the rock, and its ultimate segments are not divided in the same way as in the allied species, but terminate in rounded lobes covered with mealy powder, or else are matted together into a series of little imbricated fragments. Tuckerman, in his "North American Lichens," has recorded a species under the name of *P. aurulenta*, which in some respects resembles the plant under consideration, but differs in others. *P. endochlora* is almost as abundant in Glencroe as *P. lævigata.* It is always found associated with that species, on the boulders which strew the hillsides between Ardgarten and Larich Park farm-house. It occurs on both sides of the river, but is less abundant, and less common on the slopes of the Cobbler. It spreads sometimes in wide masses over the shady sides of loose moorland walls, or isolated stones half sunk in the ground, and partially covered with grass or moss. Creeping twigs of heather and willow are also frequently invested with it. Seeing that it is so common and abundant in this locality, I am astonished that it has not hitherto been observed in Scotland. I am prepared to hear that it has been discovered in other localities along the western sea-board of Scotland, now that attention has been directed to it.

While fishing one day in the river Croe, a little above Larich Park farm-house, I noticed on the upper side of a huge rock in the middle of the stream that rises high above the water, a number of specimens of the very rare *Parmelia diatrypa* of Acharius, adhering closely to the stone. In some the centre had disappeared; while others were perfect in shape and filled up from centre to circumference. This is one of the neatest and most beautiful of our lichens, having a smooth, waxy, shining appearance, and its segments being exceedingly regular, raying out in a close laciniated form. I afterwards found it somewhat abundant on mossy rocks and on the lower part of the trunks of pine trees in the Ardgarten woods beside the path, a little beyond the bridge that spans the river. The specimens on the pine trees differed a little from those on the rocks, being inclined to be more sorediate at the apices of the segments, and greener and more shining in colour. The smaller size, narrower segments, polished waxy appearance, and, above all, the minute perforations of the segments, serve effectually to distinguish this species from *P. physodes*, with certain states of which, especially when growing on rocks, it might at first be confounded. It is exceedingly rare in fructification in this country. In New Zealand, where it seems to be rather common and widely diffused, it fruits freely; and its bright red apothecia, with their greyish entire margin, are a great ornament to the plant, and complete the beauty of its appearance. In Scotland it was previously known only to occur at Ballachulish, where Messrs Turner and Hooker gathered it. In England, Dr Greville found it on moist mossy rocks at the foot of Snowdon; and Dr Taylor and Mr Carrol in several places in the south-west of Ireland. It is exceedingly local in this country, but has a wide distribution over the world, having been found in Europe, Asia, America, Australia, the Sandwich Islands, and New Zealand. With us the *P. physodes* is exceedingly abundant everywhere, and the *P. diatrypa* exceedingly rare and local; but in New Zealand it is exactly the reverse, the *P. diatrypa* being the common, and the *P. physodes* the rare species. I gathered the *P. diatrypa* abundantly on pine trees in the woods of the Kleine Rugen at Interlaken, Switzerland; and on boulders in the Nâerodal at the head of the Sogne Fjord, Norway. The old specific name which Acharius gave it has been changed for the more descriptive new name of *P. pertusa* of Schrank, under which, in recent works on lichenology it is now known. On the same trees of Ardgarten woods, I may mention, I observed magnificent specimens of *P. physodes* in the most luxuriant fructification, every segment loaded with apothecia, varying in size from a pin-point to a fourpenny piece.

On one solitary pine tree, in the same part of the Ardgarten woods, I found a few specimens of a lichen which I identified as the *Parmelia Mougeotii* of Schærer. Leighton includes it as a mere variety under *P. conspersa*. It is found most frequently on boulders; and I observed subsequently a few specimens on a quartz rock near the farm-house. It may be distinguished by its greenish yellow stellated thallus, rarely attaining more than an inch in diameter; the somewhat convex laciniæ being very narrow and multifid, and bearing powdery warts of a lighter colour at their extremities. It is closely fixed to the bark on which it grows. Most of the specimens observed are very minute, but quite characteristic notwithstanding. The fructification is altogether unknown. The lichen has been found in several places in England, Wales, and Ireland; while in Scotland Professor Dickie gathered it near Aberdeen, Dr Hall in the fir woods of Dunkeld, and the Rev. Mr Crombie in the pine woods of Glendee, Braemar. Although considered very rare at present, it is probable that it may yet be found in many subalpine localities where pine woods prevail; its minute size being likely to cause it to be overlooked.

On the same pine trees I also gathered, what I concluded to be one of the numerous forms of that protean lichen, the *Parmelia tiliacea* of Acharius. It grows on stones and trees somewhat abundantly in the district, but it assumes such different appearances, and departs so widely from the normal type, that it is almost impossible to identify all the varieties. The saxicolous specimens as a rule are much more luxuriant, membranous, and wide-spreading, and less divided than the corticolous ones. When growing in favourable localities, and particularly on the ledges of rocks in the beds of streams, where a considerable quantity of soil has lodged, and which are exposed to occasional inundations, the lichen grows to a large size, and presents a very tough and corrugated appearance; portions of the thallus being much broader and thicker than the others, and marked by numerous rough wrinkles. In the typical plant the thallus is orbicular and membranous, inclined to be pruinose in the centre and divided into somewhat rounded lobes, marked by crenated sinuses. The upper surface varies from a greenish or glaucous grey to a silvery white colour. The under surface is brownish black and a little fibrous, not nearly so shaggy as in *P. lævigata.* The flat orange-brown apothecia with an inflexed border are unknown to me. Indeed, none of the rare lichens of Glencroe were observed in fruit. The vegetative system, which in all of them was remarkably luxuriant, seems to have been developed at the expense of the reproductive. What are the conditions upon which the production of apothecia in lichens depends is at present involved in mystery. The *Parmelia physodes*, a species which very rarely fructifies, produces the most abundant and luxuriant apothecia in the same wood, and on the same trees where other species nearly allied are invariably and inveterately barren. We know the laws which determine the appearance of sexual organs on flowering plants, but are ignorant of those which determine the development of the sexual organs of the cryptogamia. This is a field in which most interesting and important discoveries may yet be made. The *Parmelia tiliacea* is peculiarly an English species, being found not unfrequently on trees in the south-west of England, and on rocks in Anglesea and Carnarvonshire, associated with *P. lævigata.* Indeed, where the one lichen grows the other is almost sure to be found in the neighbourhood, for they love the same localities and the same conditions of growth. Previous to my finding the *P. tiliacea* in Glencroe, it had been gathered only in one spot in Scotland, on the battlements of Brodick Castle in the Isle of Arran. In the south-west of France it occurs in abundance on olive trees, and produces rich fructification.

Various forms of the *P. tiliacea* approximate to the typical *P. scortea* of Acharius. I found on a loose dyke running down into Loch Long, at the end of the Ardgarten woods and the entrance of Glencroe, very remarkable specimens which come under this description.* They occur nowhere else than on the above-mentioned wall, growing above high-water mark, covering every exposed stone, and giving to the whole dyke, along with its gay garniture of yellow Parmelias, a peculiarly picturesque appearance. The thallus is orbicular, spreading widely from the centre to the circumference; the central parts decaying and disappearing, and the circumferential parts maintaining the life and growth of the lichen. In texture it is membranous, almost cartilaginous, tough, and wrinkled. It is lobed and sinuated in a very regular manner, and its colour is of a greyish-white, slightly darker at the edges, and covered with innumerable black granulations, which are spermogones, propagating the plant in the absence of its bright chestnut apothecia with an inflexed crenated border, which are extremely rare, having never been observed in this country. In England the typical lichen occurs usually on trees and poles, and grows in sheltered places. I am not aware of its having been found in any other place in Scotland save that which I have described; and the circumstances of its habitat there, growing on stones exposed to the stormy breezes of the sea-shore, alternately moistened by the spray of the waves, and dried and hardened by the hot sunshine, must have modified considerably its appearance, and are sufficient to account for the difference which the Scotch specimens present to the English ones. In regard to such

* Mr Leighton refers them to *P. tiliacea* var. *rugosa* of Taylor.

a plant as this, we feel the truth of a somewhat paradoxical statement made by a great botanist, that "in other tribes of plants the more we study them the more species we discover, whereas among the lichens, the more we study them the fewer species we find."

Since the preceding observations were read to the Society, I have revisited Glencroe; and having lived in the same farm-house for a somewhat longer period, I had ample opportunities of going over the ground again more carefully. I found *Parmelia sinuosa* of Acharius somewhat abundantly on small weather-worn boulders on the slopes of the Cobbler, particularly at the foot of the long incline which rises straight from the Ardgarten woods. I saw it in even finer condition on mossy rocks on the banks of the river opposite the farm-house and a little higher up the glen. I also found it sparingly distributed on the trunks of birch trees on the shores of Lochlomond, about two miles above Tarbet. It occurs on stones and walls throughout Scotland, but is always local and scarce. It is more frequently found on moorland stones in Dumfriesshire than anywhere else; and I have gathered fine specimens on granite boulders near New Abbey. It is easily known by its pale primrose colour, and its dilated cloven lobes and circular sinuses, and the dense black fibres of its underside. It is a very lovely species even in the fragmentary form in which it usually occurs; but when it is well grown and complete, it is one of the most beautiful British lichens.

At the foot of boulders half sunk on the hill sides, I found a few specimens of *Parmelia speciosa* var. *hypoleuca*. This handsome species crept over the mossy turf, among tufts of cup-lichens. It assumed a somewhat broken, fragmentary form, and some of its fragments presented a close resemblance to the segments of the thallus of the cup-lichen among which they grew. Its texture is somewhat thick and cartilaginous. Its colour on the upper surface of the thallus is of a pale greenish-white. The under side of the typical species is pure white, interspersed with a few grey fibres, but the variety *hypoleuca* is distinguished by its black ciliæ and rhizinæ. Its form is stellated, radiating from the centre in numerous linear multifid segments, broadening, and powdery at the extremities, and somewhat imbricated. The apothecia have not been found in this country, where it has been gathered only by Messrs Turner and Hooker, at Ballachulish. It occurs somewhat commonly in North America and in Switzerland; in which latter country I have gathered magnificent specimens in pine woods at a great elevation.

I was fortunate also in finding upon a single ash tree in Ardgarten woods, several diminutive specimens of the rare *Sticta crocata.* This tropical lichen, easily distinguished by the lemon-coloured powdery spots which mark the reticulations of the upper surface, I observed some years ago in the woods of Moness at Aberfeldy. It occurs in the woods at Inverary, to which Glencroe leads; and I am informed that it is very abundant and luxuriant in form on trees in Islay.

Besides the very rare species described in the preceding paragraphs, an abundance of common, though still local lichens, occur in Glencroe; such as *Endocarpon lœté-virens, Bœomyces rufus, Lecidea sanguinaria, Squamaria lanuginosa, Placodium plumbeum, Parmelia caperata, conspersa, perforata, perlata,* and *aquila, Sticta pulmonaria, scrobiculata, limbata, fuliginosa,* and *sylvatica, Collema Burgessii, Peltidea scutata, Nephroma resupinata,* and *Gyrophora pellita.*

I have thus described the small but rare and interesting group of Parmelias which I discovered in Glencroe; and the fact that so many large and beautiful lichens should have been found for the first time in Scotland, in a region so near to our large cities, so much frequented by tourists, and so well known to the general public, shows conclusively that much requires still to be done in ascertaining the geographical distribution of our lichens, and encourages the hope that still more important discoveries will reward the search of the indefatigable cryptogamist in districts less known and frequented.

Reprinted from THE JOURNAL OF THE QUEKETT MICROSCOPICAL CLUB, Vol. IV., Ser. II., Page 114, No. 27, July, 1890.

On "Amphileptus flagellatus," sp. n. A New Infusorian.

BY C. ROUSSELET, F.R.M.S.

(*Read May 16th, 1890.*)

PLATE IX.

Body ovoid, rounded posteriorly, obliquely truncated anteriorly, and often somewhat compressed in this region; the anterior margin prolonged into a long tapering, flexible, trunk-like appendage or flagellum, $\frac{1}{4}$ to $\frac{3}{4}$ the size of the body; in swimming, this appendage is carried well forward, bent in a spiral curve from left to right, whilst the animal is continually revolving on its longer axis from right to left, and only rarely reverses this movement for a short time. The oral aperture is situated in a slight depression at the base of the filament and on the truncated part of the body, but there is no excavated canal as in *Bursaria*. The cuticular surface of the entire body is finely ciliated; the cilia on the outer margin of the trunk-like appendage are slightly larger than those of the body. Contractile vesicles numerous, small and spherical, situated in the body plasma, but none present in the filament; endoplasm white, uniformly granular, and containing two spherical endoplasts.

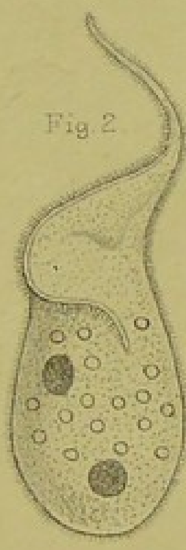
For several years past I have found this *Infusorian* at our yearly excursion to Keston, and as it appears to be a new species, not mentioned by Mr. Saville Kent, I have thought that a short description and record in our journal would be acceptable.

At first sight of this fairly large animalcule one would be tempted to put it down as an abnormal *Trachelius ovum*. On closer examination, however, I found that it belonged to a different genus, namely, *Amphileptus*, and that it agreed with none of the twelve species described by Mr. Saville Kent. Its large size, and its prominent and long trunk-like filament, at once differentiate it from the known species.

Mr. Saville Kent limits the genus *Trachelius* to those forms which, like *T. Ovum*, possess an apparent internally ramified alimentary tract. The present species does not present this appearance, the sarcode being evenly, but coarsely granular, and containing numerous spherical, highly refractile vesicles, which are considered to be contractile vesicles; I have not, however, seen them contract. The body is highly elastic, and the slightest pressure in the live box causes the animal to change its shape considerably, and to withdraw the flagellum. This whip-like appendage varies much in size in different individuals, from $\frac{1}{4}$ to $\frac{3}{4}$ and more the size of the body, and is carried in a graceful spiral curve in front of the body when swimming. The body is thickly furred all over with fine cilia, which are easily seen with a good dark field illumination.

Size $\frac{1}{65}$ to $\frac{1}{55}$ in. including flagellum, width $\frac{1}{100}$ to $\frac{1}{120}$ in.

Fig. 2.



(Journ. R. Micr. Soc., 1897, pp. 467–511.)

VIII.—*A Contribution to the Freshwater Algæ of the South of England.*

By W. West, F.L.S., and G. S. West, A.R.C.S.

Communicated by A. W. Bennett, F.L.S., F.R.M.S.

With Appendix by A. W. Bennett, F.L.S., F.R.M.S.

(*Read Oct. 20th*, 1897.)

Plates VI. and VII.

Although much has already been done towards our knowledge of the distribution of these interesting plants in this part of the country, and this paper is intended to supply a further addition to this knowledge, yet many large districts still remain from which careful collecting is required. The chief workers in this field during recent years have been Mr. W. Joshua, Mr. A. W. Bennett, and Mr. E. D. Marquand; and the districts included in their researches are portions of Surrey, Hampshire, Wiltshire, Devonshire, and Cornwall. The following is a list of the more important papers dealing with Freshwater Algæ from the South of England since the publication of Hassall's 'British Freshwater Algæ' in 1845, and Ralfs' 'British Desmids' in 1848:

1. Joshua, W. Notes on British Desmidieæ. Journ. Bot., xx. (1882) p. 300.
2. Marquand, E. D. The Desmids and Diatoms of West Cornwall. Trans. Penzance Nat. Hist. & Antiq. Soc., i. (1882–3) pt. 3.
3. Joshua, W. Notes on British Desmidieæ. Journ. Bot., xxi. (1883) p. 290.
4. Marquand, E. D. Freshwater Algæ of the Land's End District. Trans. Penzance Nat. Hist. & Antiq. Soc., 1885.
5. Bennett, A. W. Freshwater Algæ of North Cornwall. Journ. Roy. Micr. Soc., Feb. 1887.
6. " " Freshwater Algæ and Schizophyceæ of Hampshire and Devonshire. Journ. Roy. Micr. Soc., Feb. 1890.
7. Roy, J. Freshwater Algæ of Enbridge Lake and Vicinity, Hampshire. Journ. Bot., xxviii. (Nov. 1890).
8. Bennett, A. W. Freshwater Algæ and Schizophyceæ of South-west Surrey. Journ. Roy. Micr. Soc., Feb. 1892.

Also in part the two following papers:—

9. West, W. and G. S. New British Freshwater Algæ. Journ. Roy. Micr. Soc., Dec. 1893.
10. " " On some New and Interesting Freshwater Algæ. Journ. Roy. Micr. Soc., April 1896.

EXPLANATION OF PLATES

(see pp. 468, 469).

The results here set forth have been obtained by the gradual examination for many years of a large and extensive series of algæ-gatherings ranging over many of the counties of the south and south-west of England. A more detailed account of the collections (which were for the most part made by the authors) will be found below.

All the species recorded for Frensham, the New Forest, and Dartmoor, are additional to those found at these respective localities by Mr. Bennett.

In these gatherings no less than 52 species of Desmids have been observed with zygospores, many of them for the first time in that state.

As might be expected, the more alpine and "wet-rock" species (such as *Cosmarium anceps, C. Holmiense, C. nasutum, Staurastrum Meriani, S. Kjellmani*, &c.) are, as far as we have observed, absent from all the eastern counties of the South of England.

As it is impossible to accurately identify, from sterile specimens, the majority of plants belonging to the families Œdogoniaceæ and Zygnemaceæ, no species of these orders are recorded in this paper unless they were seen in abundant fruit, with the exception of *Œdogonium undulatum*, which even in its barren state is unmistakable.

The classification set forth in this paper is the one which we think to be the most natural, after taking into consideration all the

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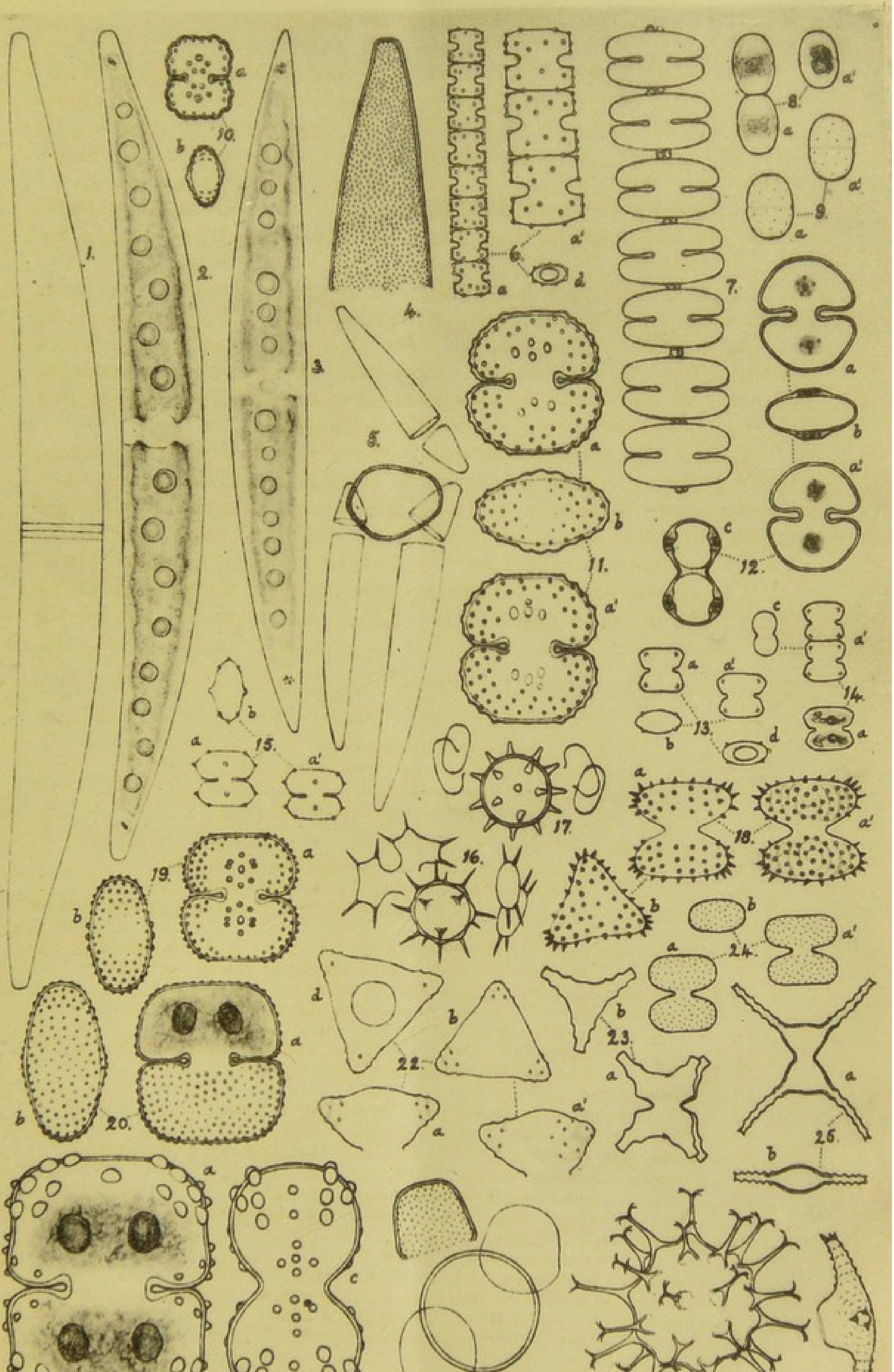
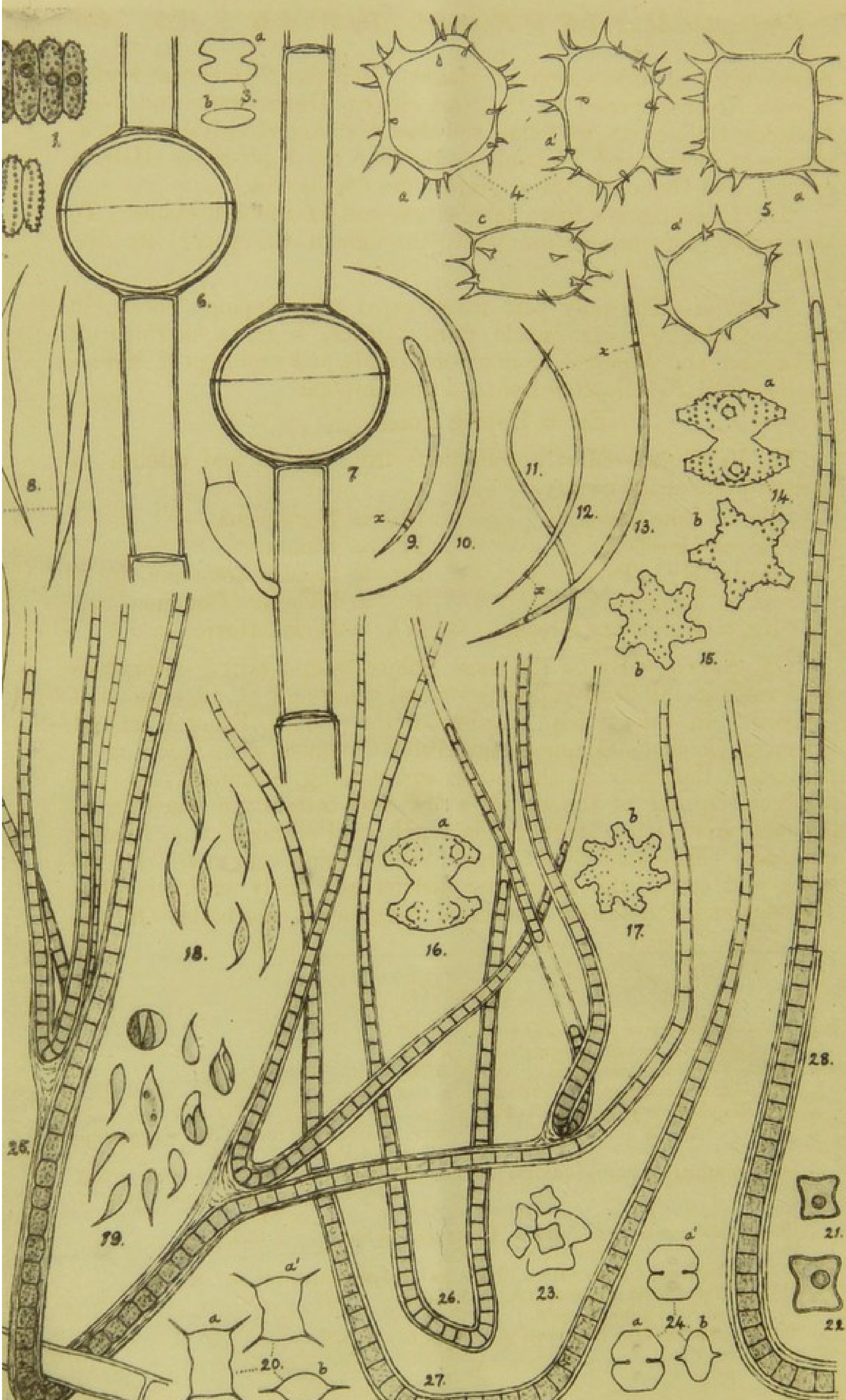
PLATE VI.

(Explicatio iconum.)

a, *a'* = front view (a basi visa).
b = vertical view (a vertice visa).
c = side view (a latere visa).
d = basal view of semi-cell (a basi visa).

Fig. 1, 2.—*Closterium Siliqua* sp. n. × 520.
„ 3. „ *lanceolatum* Kütz. var. *parvum* var. n. × 520.
„ 4. „ *Pritchardianum* Arch. Apex of semi-cell, × 520.
„ 5. „ „ Zygospore formed by the conjugation of three cells. × 120.
„ 6.—*Sphærozosma Wallichii* Jacobs var. *anglicum* var. n. *a* et *d*, × 520; *a'*, × 830.
„ 7. „ *vertebratum* (Bréb.) Ralfs var. *latius* var. n. × 520.
„ 8, 9.—*Penium subtile* sp. n. 8, × 520; 9, × 660.
„ 10.—*Cosmarium Blyttii* Wille var. *Novæ Sylvæ* var. n. × 520.
„ 11. „ *fastidiosum* sp. n. × 520.
„ 12. „ *ocellatum* B. Eichl. and Gutw. var. *incrassatum* var n. × 520.
„ 13, 14. „ *sphagnicolum* sp. n. × 520.
„ 15.—*Xanthidium concinnum* Arch. × 740.
„ 16.—*Arthrodesmus octocornis* Ehrenb. Zygospore × 520.
„ 17.—*Cosmarium bioculatum* Bréb. Zygospore × 520.
„ 18.—*Staurastrum rostellum* Roy and Biss. var. *erostellum* var. n. × 660.
„ 19.—*Cosmarium subpunctulatum* Nordst. × 520.
„ 20. „ *subbroomei* Schmidle, forma. × 520.
„ 21. „ *Ungerianum* (Näg.) De Bary var. *subtriplicatum* var. n. × 520.
„ 22.—*Staurastrum trachytithophorum* sp. n. × 520.
„ 23. „ *nodosum* sp. n. × 660.
„ 24.—*Cosmarium bioculatum* Bréb. var. *hians* var. n. × 520.
„ 25.—*Staurastrum tetracerum* Ralfs var. *validum* var. n. × 520.
„ 26.—*Cosmarium Cucurbita* Bréb. Zygospore × 520.
„ 27.—*Staurastrum gracile* Ralfs. Zygospore × 520.

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recent work at this group of plants. For the filamentous Myxophyceæ (Cyanophyceæ), i.e. the Nostochaceæ Heterocystæ and Nostochaceæ Homocystæ, we follow the classification and also the nomenclature of Bornet and Flahault's 'Revision des Nostocacées Hétérocystées' and Gomont's 'Monographie des Oscillariées.'

The following is a more detailed account of the collections. To save space, contractions are used (throughout the paper) for the localities.

I. *Essex.*—The only part investigated was Epping Forest, in which gatherings were made from ditches, bogs, squeezings of *Sphagnum* and *Hypnum* from several ponds, and washings of *Myriophyllum* and *Potamogeton* from a deep fish-pond.

Ep. = Epping Forest.

II. *Middlesex.*—Mostly collections from ponds and ditches in various parts of the county.

Bg. = Bow Green.	No. = Northwood.
Ha. = Harefield.	Pi. = Pinner.
Hy. = Hyde Park.	Ru. = Ruislip Reservoir.
Ki. = Kingsbury.	U. = Uxbridge Common.
Kg. = Kingsbury Green.	Wh. = Welsh Harp.

III. *Surrey.*—By far the most numerous and varied gatherings were made in this county, during several years and at all seasons of the year. In the northern portion of the county, Esher West-end Common, Chobham Common, and Bisley Common, were much more productive than any of the other surrounding commons. In the south-east corner of the county a few very interesting species were obtained from some large ponds north of Felbridge, one locality quoted as "Mill-pond E. of Chapel Wood" being very rich. In the south-west corner, the Frensham district was fairly productive, but Thursley and Puttenham Commons were much the best. On the

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PLATE VII.

Fig. 1, 2.—*Scenedesmus granulatus* sp. n. × 520.
„ 3.—*Cosmarium truncatellum* Perty. × 520.
„ 4, 5.—*Tetraëdron horridum* sp. n. × 520.
„ 6, 7.—*Œdogonium macrospermum* sp. n. × 520.
„ 8.—*Rhaphidium polymorphum* Fresen. var. *tumidum* var. n. × 520.
„ 9–13. „ „ var. *mirabile* var. n. × 520. *x*, moving corpuscle.
„ 14.—*Staurastrum margaritaceum* (Ehrenb.) Menegh. var. *robustum* var. n. × 520.
„ 15–17. „ „ var. *subcontortum* var. n. × 520.
„ 18.—*Dactylococcus bicaudatus* A. Br. var. *exilis* var. n. × 520.
„ 19. „ *dispar* sp. n. × 520.
„ 20.—*Arthrodesmus Incus* (Bréb.) Hass. var. *subquadratus* var. n. × 520.
„ 21–23.—*Tetraëdron minimum* (A. Br.) Hansg. × 520.
„ 24.—*Cosmarium adoxum* sp. n. × 740.
„ 25–28.—*Ammatoidea Normanii* gen. et sp. n. × 520.

former is a fine bog in which *Rhynchospora fusca* is plentiful, along with abundance of submerged and floating *Sphagnum contortum*, *Hypnum scorpioides*, and better still, *Utricularia minor* coated with a brownish-green slime of almost pure Desmids. Squeezings of *Sphagnum* and *Utricularia minor* from General's Pond, Puttenham Common, also yielded some uncommon species.

B. = Barnes Common.
Bh. = Blindley Heath (S.E.).
Bi. = Bisley Common.
Bo. = Bolder Mere.
Br. = Brockham Green (to Betchworth).
C. = Chobham Common.
Co. = Near Cobham Common.
Cr. = Crowhurst (S.E.).
Di. = Ditton Marsh.
Dj. = Devil's Jumps, Frensham.
Do. = Dorking.
E. = Esher Common.
Ea. = Earlswood Common.
Ew. = Esher West-end Common.
F. = Felbridge.
Fg. = Frensham Great Pond.
Fh. = Frogit Heath (S.E.).
Fl. = Frensham Little Pond.
Hk. = Hackbridge.
M. = Mitcham Common.
Mi. = Mitcham Grove.
Mp. = Mill-pond E. of Chapel Wood (S.E.).
Pt. = Putney Heath (nr. Pt. = Roehampton Lane).
Pu. = Puttenham Common.
Ra. = Ranmore Com. (pond).
Rp. = Richmond Park.
Th. = Thursley Common.
Wa. = Wandsworth Common.
We. = Bog by R. Wey, Frensham.
Wi. = Wimbledon Common.
Wk. = Canal at Woking.
Wo. = Whitemoor Common, Worplesdon.
Wt. = Witley Common.

IV. *Kent.*—Collections were made from a few small ponds on Hayes Common, and also from two large ponds, as well as from a small bog, on Keston Common. These ponds were crowded with *Potamogeton*, *Scirpus*, &c., and yielded many good species.

Bm. = Bromley.
Cb. = Cobham.
Hc. = Hayes Common.
Ks. = Keston Common.

V. *Oxfordshire.*—Collections were kindly made by Mr. S. Wood from a few small ponds in the south-east of the county.

Go. = About 2½ miles N.E. of Goring.

VI. *Hampshire.*—The gatherings in this county were limited to a few ponds and some very rich bogs in the New Forest. Mr. W. West, jun., collected some rich gatherings of Desmids from *Utricularia minor* in Ashurst Bog, near Lyndhurst Road.

N. = New Forest.

VII. *Devonshire.*—The best gatherings made in this county were from some fine bogs in the neighbourhood of Hey Tor, Dartmoor. The Lea on Slapton Sands also yielded some good material.

D. = Dartmoor (Hey Tor).
Dl. = Dawlish.
Gp. = Grimspound, Dartmoor.
S. = Slapton Sands.
Tq. = Torquay.

VIII. *Cornwall.*—These gatherings were obligingly made at our request by Mr. R. V. Tellam, of Bodmin. Some of them were very rich, and all were more or less productive of good things. A small collection was also made by Mr. Ll. J. Cocks from Tremethick Moor.

G. = Gunwen Moor.	P. = Penhargurd Wood.
H. = Halgavor Moor.	R. = Roughter Moor.
Hl. = Helmentor Moor.	T. = Tintagel.
K. = Kynance Valley.	Tm. = Tremethick Moor.
L. = Lanlivery Moor.	W. = Withiel.

Those species (about 60) not previously recorded for the British Isles are prefixed by an asterisk.

We have noted a very large number of Diatoms, but, as much of the material has not as yet been examined with regard to this class of plants, we leave them for future publication.

Class FLORIDEÆ.

Fam. Batrachospermeæ.

Genus *Batrachospermum* Roth.

1. *B. moniliforme* Roth. II. In canal, Ha. IV. Ks. VI. N. VII. D.
   Var. *confusum* (Hass.) Cooke. IV. Cæsar's Well, Ks.
2. *B. vagum* (Roth) Ag. III. In pond, Dj. In bogpool, Th.

Class CHLOROPHYCEÆ.

Ord. *Confervoideæ Heterogamæ.*

Fam. Coleochætaceæ.

Genus *Coleochæte* Bréb.

1. *C. scutata* Bréb. I. Ep. II. Wh. III. Ew., Fg., Wo. V. Go. VI. N.
2. *C. soluta* Prings. I. Ep. III. Bi., Fg.
3. *C. orbicularis* Prings. III. Cr.

Fam. Œdogoniaceæ.

Genus *Bulbochæte* Ag.

1. *B. polyandra* Cleve. VIII. W.
2. *B. setigera* (Roth) Ag. III. Mp., Rp. IV. Ks.

*3. *B. nana* Wittr. V. Go.

4. *B. mirabilis* Wittr. III. Th.
5. *B. rectangularis* Wittr. III. Ew., Mp.

Genus *Œdogonium* Link.

1. *Œ. Petri* Wittr. III. Ew. VI. N.
2. *Œ. cryptoporum* Wittr. var. *vulgare* Wittr. II. Wh.

*3. *Œ. obsoletum* Wittr. V. Go.

*4. *Œ. zig-zag* Cleve.

Vegetative cells slightly thicker and spermogonia bicellular.

Crass. cell. veget.	19–23 μ;	altit. 2½–4-plo major;
„ oogon.	53–57 μ;	„ 50–55 μ;
„ cell. spermogon.	19 μ;	„ 6–7·5 μ.

II. Ha.

*5. *Œ. nobile* Wittr. II. Ha.

6. *Œ. paludosum* (Hass.) Wittr. III. C.

*7. *Œ. oblongum* Wittr. II. Wh.

8. *Œ. platygynum* Wittr. VIII. L.

*9. *Œ. obtruncatum* Wittr.

Œ. dioicum, nannandrium; oogoniis 4-continuis, globoso-ellipsoideis, oosporis oogonia complentibus; nannandribus oblongo-pyriformibus, curvatis, unicellularibus, in oogoniis sedentibus.

Crass. cell. veget.	17–20 μ;	altit. 4–5-plo major;
„ oogon.	48–53 μ;	„ 48–53 μ;
„ oospor.	47–52 μ;	„ 47–51 μ;
„ cell. suffult.	26 μ;	
„ nannandr.	9·5–14 μ.	

II. Wh.

This plant comes nearest to the partially described *Œ. obtruncatum*, agreeing with it in all those characters mentioned by Wittrock. We therefore describe the additional characters observed in the specimens.

10. *Œ. undulatum* (Bréb.) A. Br. VIII. W.

*11. *Œ. cyathigerum* Wittr. II. Ha.

12. *Œ. Braunii* (Kütz.) Prings. II. Ha.

13. Œ. MACROSPERMUM sp. n. (Pl. VII. figs. 6, 7.)

Œ. dioicum, nannandrium, oogoniis singulis, magnis, subdepresso-globosis, circumscissilibus cum rima angustissima submediana; oosporis subdepresso-globosis, oogonia complentibus, membrana glabra; cellulis suffultoriis eadem forma ac cellulis vegetativis ceteris: nannandribus in cellulis suffultoriis sedentibus, paullo curvatis, bicellularibus (?)

Crass. cell. veget.	13–13·5 μ;	altit. 4–5-plo major;
„ oogon.	44–46 μ;	„ 39–40 μ;
„ nannandr.	11·5 μ;	„ 38 μ.

II. Ha.

The nearest species to this is *Œ. propinquum* Wittr., from which it is distinguished by its slightly greater thickness and proportionately longer cells, its much larger oogonia and oospores, which are depressed, and the circumscissile oogonia.

14. *Œ. macrandrum* Wittr. I. Ep.

15. *Œ. cardiacum* (Hass.) Wittr., var. *carbonicum* Wittr. III. M.

*16. *Œ. lautumniarum* Wittr. The spermogonia were about 8-celled, and the supporting cells of the oogonia a little inflated. II. Wh.

Ord. *Siphoneæ.*

Fam. **Vaucheriaceæ.**

Genus *Vaucheria* DC.

1. *V. sericea* Lyngb. II. Ki.
2. *V. aversa* Hass. II. Ru.
3. *V. sessilis* (Vauch.) DC. II. Ki. III. M. VI. N.
4. *V. geminata* (Vauch.) DC. III. C. IV. Ks.
   Var. *racemosa* Walz. III. B., F.
5. *V. hamata* (Vauch.) Lyngb. II. Ki., Ru.
6. *V. terrestris* Lyngb. IV. Bm.

Fam. **Hydrogastraceæ.**

Genus *Botrydium* Wallr.

1. *B. granulatum* (L.) Grev. III. Ew., July 1893; also Oct. 1893, abundant. M. (L. A. Boodle; 1892).

Ord. *Confervoideæ Isogamæ.*

Fam. **Ulvaceæ.**

Genus *Monostroma* Thur.

1. *M. bullosa* (Roth) Wittr. III. M. (L. A. Boodle; 1891).

Genus *Prasiola* Ag.

1. *P. crispa* (Lightf.) Ag. II. Bg., U. III. Kingston. IV. Hc.

Fam. **Ulotrichaceæ.**

Sub-fam. Ulotricheæ.

Genus *Hormidium* Kütz.

1. *H. murale* (Lyngb.) Kütz. (*Ulothrix radicans* Kütz.). I. Ep. III. Wi.
2. *H. parietinum* (Vauch.) Kütz. (*Ulothrix parietina* Kütz.). I. Ep. II. Pi. III. Esher. Frensham, Kingston, Wi. IV. Bm.

Genus *Hormiscia* Fries; Aresch.

*1. *H. subtilis* (Kütz.) De Toni. IV. Hc.
   Var. *variabilis* (Kütz.) Kirchn. (*Ulothrix variabilis* Kütz.). I. Ep. II. Hy. III. Dj., E., Ew., Mp., Wi. IV. Cb., Ks. V. Go.
   Var. *tenerrima* (Kütz.) Kirchn. (*Ulothrix tenerrima* Kütz.). I. Ep. II. Ru. III. E., Rp., Wi.

2. *H. æqualis* (Kütz.) Rabenh. var. *catæniformis* (Kütz.) Rabenh. I. Ep. II. Wh. III. B. IV. Ks.
3. *H. moniliformis* (Kütz.) Rabenh. I. Ep. II. Ha. V. Go.
4. *H. zonata* (Web. et Mohr.) Aresch. II. Canal, Ha.

Genus *Hormospora* Bréb.

1. *H. mutabilis* Bréb. VI. N. VII. D. VIII. W.
2. *H. plena* Bréb. VIII. L.

Sub-fam. Chætophoreæ.

Genus *Herposteiron* Näg.

1. *H. repens* (A. Br.) Wittr. (*Aphanochæte repens* A. Br.). I. Ep. II. Ha., Hy., No., Wh. III. Bh., C., Th. V. Go.

Var. gracilis var. n.

Var. cellulis minoribus, diametro 2–3-plo longioribus, setis tenuioribus et distincte articulatis.

Long. cell. 7–15 μ; lat. cell. 4–7 μ. II. U., on *Œdogonium* sp.

Genus *Nordstedtia* Borzì.

1. *N. globosa* (Nordst.) Borzì. III. E., Fg., M., Th. V. Go. VI. N. VIII. T., W.

Var. *depressa* nob. (*Aphanochæte globosa* var. *depressa* West). III. Mp. VI. N.

Genus *Chætophora* Schrank.

1. *C. pisiformis* (Roth) Ag. II. Ru. III. Ew., Mp.
2. *C. elegans* (Roth) Ag. I. Ep. III. Fg., Canal at Woking. VII. S.
3. *C. Cornu Damæ* (Roth) Ag. (*C. endivæfolia* auct.). III. M. VII. S.
4. *C. tuberculosa* (Roth) Ag. IV. Ks.

Genus *Draparnaudia* Bory.

1. *D. glomerata* (Vauch.) Ag. III. M. IV. Ks. VI. N.

Obtained with spores from Mitcham Common, Surrey, and Keston Common, Kent. Spores thick-walled; walls rusty brown and rough; chlorophyll parietal.

Var. *distans* (Kütz.) —. III. Ew.

2. *D. plumosa* (Vauch.) Ag. I. Ep. II. Ru. III. Ew., Pu. VII. D. VIII. G. H.

Genus *Stigeoclonium* Kütz.

1. *S. tenue* (Ag.) Rabenh. III. Do., E. IV. Hc.
2. *S. protensum* (Dillw.) Kütz. III. Th. VII. D.
3. *S. fastigiatum* Kütz. III. M.

Sub-fam. CONFERVEÆ.

Genus *Conferva* L.; em. Lagerh.

1. *C. bombycina* Ag. Generally distributed and abundant.
   Forma *minor* Wille. General and abundant.
2. *C. Raciborskii* Gutw. VI. N.

Genus *Microspora* Thur.; em. Lagerh.

1. *M. floccosa* (Vauch.) Thur. I. Ep. II. Ha., Pi.
2. *M. Wittrockii* (Wille) Lagerh. III. C.
3. *M. abbreviata* (Rabenh.) Lagerh. II. Pi.
4. *M. fontinalis* (Berk.) De Toni. VII. Street.

Fam. **Chroolepidaceæ.**

Genus *Microthamnion* Näg.

1. *M. strictissimum* Rabenh. 1863. (*M. vexator* Cooke 1882). I. Ep. III. B., Br., Di., Ew., M., Pt., Wi. VI. N.

There is no doubt that Cooke gave the name *M. vexator* to the plant previously described by Rabenhorst as *M. strictissimum.* Dr. Nordstedt (in K. Sv. Vet.-Akad. Handl., Bd. xxii. No. 8, p. 15) suggests that they are identical. Cooke (Brit. Freshw. Alg., p. 188) states that his species is "very much more slender than *M. strictissimum,*" and gives as the diameter ·003 mm. Nordstedt's measurements of *M. vexator* are 3–4 μ (diam.) Rabenhorst's measurements of *M. strictissimum* are 1/700–1/600 of a Paris line (= 3–4 μ); therefore Cooke, in giving the above statement, must have made a mistake in converting Rabenhorst's measurements to mm.

*2. *M. Kützingianum* Näg. I. Ep. III. E., Pu., Rp. IV. Cb. V. Go.

Fam. **Cladophoraceæ.**

Genus *Cladophora* Kütz.

1. *C. flavescens* Ag. II. Wh. III. Cr., Ra.
2. *C. crispata* (Roth) Kütz. II. No. III. Bh.
3. *C. glomerata* (L.) Kütz. II. Ha., Hy. VI. N.

Ord. *Conjugatæ.*

Fam. **Zygnemaceæ.**

Sub-fam. MESOCARPEÆ.

Genus *Mougeotia* Ag.

1. *M. scalaris* Hass. III. Ew.
2. *M. recurva* (Hass.) De Toni. VI. N.
3. *M. parvula* Hass. I. Ep. III. Ew., Wi. VIII. T.
4. *M. genuflexa* (Dillw.) Ag. II. No., U. III. Ew., Wi. VI. N.

*5. *M. calcarea* Wittr. VI. N.
6. *M. capucina.* III. Ew. VI. N.
7. *M. quadrangulata* Hass. III. Dj.
8. *M. viridis* (Kütz.) Wittr. I. Ep. III. Ew., Pu., Th., We.
9. *M. gracillima* (Hass.) Wittr. I. Ep. III. C., Ew., Wi.

Genus *Gonatonema* Wittr.

1. G. Boodlei sp. n.

G. cellulis vegetativis diametro 12–15-plo longioribus; aplanosporis maturis e fronte et e latere visis ellipticis, apicibus late acuteve rotundatis, membrana distincte punctata, lutescente.

Crass. fil. veget. 5–5·5 μ; long. aplanospor. 17–23 μ (plerumque 21 μ); lat. aplanospor. 13–17 μ (plerumque 15 μ). III. M.

This interesting genus has been but once previously found in this country, when Hassall (1845) described from Notting Hill a plant which he named *Mesocarpus notabilis*, and which has since been placed by Wittrock (1878) under *Gonatonema* as *G. notabile* (Hass.) Wittr. *G. Boodlei* differs from *G. ventricosum* Wittr. in being slightly smaller, and in having elliptical aplanospores when seen from either the front or side; these aplanospores are never oblique, the apices are never truncate, and the spore-membrane is punctate. It was found in abundance mixed with *Spirogyra* sp. in a gathering made by Mr. L. A. Boodle in 1894 in a ditch on Mitcham Common, Surrey.

Sub-fam. Zygnemeæ.

Genus *Debarya* Wittr.

1. *D. glyptosperma* (De Bary) Wittr. VI. N. (very fine, April 1897).
2. *D. lævis* (Kütz.) nob. [*Mougeotia lævis* (Kütz.) Arch.]. III. M.

Crass. fil. veget. 25 μ; spor. 42–50 × 28–32 μ.

There seems to be no doubt whatever that this plant is a true *Debarya*, as the whole of the contents of the conjugating cells unites to form a true zygospore. The adult zygospores are dark coloured, with thick walls, and are ornamented with large scrobiculations.

Genus *Zygnema* Ag.; em. De Bary.

1. *Z. anomalum* (Hass.) Cooke. IV. Ks.
2 *Z. leiospermum* De Bary. III. Ew. VIII. T.
3. *Z. atrocœruleum* sp. n.

Z. cellulis vegetativis diametro 2½–5-plo longioribus; zygosporis globosis vel subglobosis, membrana lævis, homogenea, et atrocœrulea, in una cellularum conjugatarum inclusis; cellulis fructiferis valde inflatis circa zygosporas.

Crass. fil. veget. 14·5–17 μ (plerumque 15–16 μ); diam. zygosp. 23–26 μ (usque ad 29 μ long.). V. N.

This species comes nearest to *Z. melanospermum* Lagerh. and *Z. cyanospermum* Cleve. It differs from the former in its narrower filaments, in its subglobose zygospores, and in having the fructiferous cells much inflated round the zygospores; from the latter, in its narrower filaments and much smaller zygospores, which are not situated in the conjugating tube between the filaments.

4. *Z. ericetorum* (Kütz.) Hansg. I. Ep. III. Pu., Th., Wi.
5. *Z. pectinatum* (Vauch.) Ag. VI. N.

Genus *Spirogyra* Link.

1. *S. tenuissima* (Hass.) Kütz. II. Ha., U., Wh. III. M.
2. *S. inflata* (Vauch.) Rabenh. III. Ew., M.
*3. *S. Spréeiana* Rabenh. III. Ew.
4. *S. Weberi* Kütz. II. Ru. III. Ra.
*5. *S. calospora* Cleve.

The cells are 4 to 7 times longer than broad; sporiferous cells very slightly inflated to accommodate the zygospore, which is a little broader than the vegetative cells. The zygospores were usually oblong-elliptic, and only 1½ times longer than broad although occasionally twice as long. The median spore-membrane is covered so densely with large somewhat angular scrobiculations of variable size, that its surface view appears a mere reticulation. Petit's figure (Spirogyræ des envir. Paris, pl. ii. f. 13) of an enlarged portion of the zygospore of this species is very incorrect.

Crass. cell. veget. 31–36·5 μ; crass. cell. fruct. 41–52 μ; long. zygosp. 75–87 μ; lat. zygosp. 40–50 μ. II. Ru.

6. *S. gracilis* (Hass.) Kütz. var. *flavescens* (Hass.) Rabenh. II. Ru.
7. *S. catæniformis* (Hass.) Kütz. III. M. VI. N.
8. *S. affinis* (Hass.) Petit. III. E., M.
9. *S. varians* (Hass.) Kütz. II. Ru. III. B. VII. Street.
10. *S. porticalis* (Müll.) Cleve. IV. Hc.
11. *S. decimina* (Müll.) Kütz. III. Fh.

Var. CYLINDROSPERMA var. n.

Var. zygosporis exacte cylindricis, diametro 2–3½-plo longioribus, apicibus late rotundatis, cellulas fructiferas pæne complentibus.

Crass. cell. veget. 40–44 μ; long. zygosp. 92–140 μ; lat. zygosp. 39–43 μ. III. Fl.

12. *S. nitida* (Dillw.) Link. II. Ha. III. Fl., Wi.

13. *S. setiformis* (Roth) Kütz. III. Fl.
14. *S. bellis* (Hass.) Cleve. II. Ki., U., Wh.
15. *S. crassa* Kütz. III. Cr. IV. Hc.

Fam. **Desmidiaceæ.**

Genus *Gonatozygon* De Bary.

1. *G. Ralfsii* De Bary. III. Ew., M. VI. N. VII. D.
2. *G. Brebissonii* De Bary. III. Ew., Fg., Rp., Th. IV. Ks. VI. N. VIII. H.
3. *G. minutum* West. III. Ew. VI. N. VIII. H.
4. *G. læve* Hilse. I. Ep.
5. *G. Kinahani* (Arch.) Rabenh. III. Rp., Wi.

Genus *Spirotænia* Bréb.

1. *S. condensata* Bréb. III. Th. VII. D. VIII. H., Hl., L., W.
2. *S. obscura* Ralfs. III. Ew., Wi.

Very abundant from Wimbledon Common, Surrey, Oct. 1892, and from Esher West-end Common, Oct. 1893.

Genus *Mesotænium* Näg.

1. *M. De Greyi* W. B. Turn. III. Pu. VIII. T.
2. *M. violascens* De Bary. III. Dj.
*3. *M. Endlicherianum* Näg. var. *grande* Nordst. I. Ep.

Genus *Cylindrocystis* Menegh.

1. *C. diplospora* Lund. I. Ep. III. C., Th. VIII. T.
2. *C. crassa* De Bary. I. Ep. III. B., Bi., C., E., Th., Wi. VIII. H., Hl., T., W.

Pure gatherings from Chobham, Surrey.

3. *C. Brebissonii* Menegh. III. Bo., C., Rp., Th. (with zygospores), Wi., Wo. IV. Ks. VII. Gp. VIII. R., T., W.

Genus *Penium* Bréb.

1. *P. margaritaceum* (Ehrenb.) Bréb. III. E. VI. N. VII. D. VIII. G., H.
   Var. *punctatum* Ralfs. VII. D.
2. *P. cylindrus* (Ehrenb.) Bréb. III. Pu., Th. (with zygospores).
3. *P. cuticulare* West & G. S. West. III. Th. (with zygospores).
4. *P. exiguum* West forma *Lewisii* West & G. S. West (*Penium Lewisii* W. B. Turner). III. Bi., E. VIII. W.
5. *P. spirostriolatum* Barker. III. Pu. VIII. H., T.
6. *P. digitus* (Ehrenb.) Bréb. (inclus. *P. lamellosum* Bréb.).

I. Ep. III. Bi., C., E., Ew., Pu., Th. IV. Ks. VII. D. VIII. G., H., Hl., T., W.

Var. *constrictum* West. VI. N.

7. *P. interruptum* Bréb. III. Pu.

8. *P. Nägelii* Bréb. III. Th. VI. N.

Long. 115 μ; lat. 25 μ.

9. *P. libellula* (Focke) Nordst. (*P. closterioides* Ralfs). III. C., Th., Wo. VII. D. VIII. H.

Forma *interrupta* West. III. C.

10. *P. Navicula* Bréb. I. Ep. III. E., Ew., Fh., Th. IV. Ks. VI. N. VII. D. VIII. H., L.

11. *P. rufescens* Cleve (*P. rufopellitum* Roy). III. Bi.

12. *P. curtum* Bréb. III. Pu., Th. VII. Gp.

Forma *major* Wille. III. Mp.

Long. 59 μ; lat. 30–32 μ.

13. *P. suboctangulare* West. VI. N. (with zygospores).

14. P. SUBTILE sp. n. (Pl. VI. figs. 8, 9.)

P. minutissimum, pæne $1\frac{1}{2}$-plo longius quam latum, ellipticum, apicibus subtruncatis, medio cum sutura mediana delicatissima (levissime constrictum); a vertice visum exacte circulare; membrana achroa, delicatissime et indistincte punctulata, punctulis sparsis circiter 12 in semicellula unaquaque.

Long. 14–15 μ; lat. 10–11 μ. III. Th.

This minute species was in abundance and was perfectly constant in its characters. The widest part of the cell is always *in the middle*, and the apices are always flattened.

It can be compared with *Disphinctium sparsipunctatum* Schmidle (in Œsterr. botan. Zeitschrift, 1895, No. 7, p. 14 (sep.) t. xv. f. 1–6), from the unconstricted form of which it differs in the form of its cells, its much more delicate punctulations, and its circular vertical view.

15. *P. truncatum* Bréb. VII. D. VIII. T., W.

16. *P. polymorphum* Perty. III. C. VIII. P.

17. *P. inconspicuum* West. III. Pu. IV. Ks.

18. *P. cruciferum* (De Bary) Wittr. I. Ep. III. Wi.

19. *P. cucurbitinum* Bissett. I. Ep. III. Th.

20. *P. minutum* (Ralfs.) Cleve. III. C., Pu., Th. VI. N. VII. D. VIII. R., W.

umbellatum Th

Genus *Roya* West & G. S. West.

1. *R. obtusa* (Bréb.) West & G. S. West var. *montana* West & G. S. West. I. Ep. III. Ew.

Genus *Closterium* Nitzsch.

1. *C. abruptum* West. III. Pu., Wi. VI. N. VII. D.

2. *C. didymotocum* Corda. III. C., Th. VII. D. VIII. H., P.

Some slender forms of this species were observed from the New Forest (long. 405–418 μ; lat. 28–34 μ), with rather more attenuate and slightly recurved apices; some of these were distinctly but minutely asperulate, being covered with somewhat irregular and depressed granules.

3. *C. turgidum* Ehrenb. I. Ep. III. Th. VI. N. VIII. L.
4. *C. prælongum* Bréb. III. Mp.
   Forma *brevior* West. I. Ep. II. Ru. (with zygospores). III. Fh.
5. *C. Pritchardianum* Arch. VIII. Tm. (with zygospores).
   Long. 440–590 μ; lat. 35–46 μ; lat. apic. 7–8 μ; diam. zygosp. 83–108 μ.

This species was observed with zygospores in quantity. The cells were of the same proportions as those described by Archer, but the apices were generally a little narrower. The cell-membrane was of a uniform golden-brown colour, and the fine striolations, which consisted of a series of fine puncta, were arranged subspirally as in forma *crassa* Gutw. (in Rosprawy Akad. Umiej. Kratow. Wydzial. mat.-przyr., t. xxiii. (1896) p. 38, t. v. f. 13). Towards the apices, however, these puncta became irregular (cf. Pl. I. fig. 4). One zygospore was noticed which had been produced by the conjugation of three individual cells (Pl. VI. fig. 5).

6. C. Siliqua sp. n. (Pl. VI. figs. 1, 2.)
   C. submediocre, cellulis diametro circiter 10-plo longioribus, leviter curvatis, dorso modice curvato, in medio subrecto, et prope apices levissime concavo, ventre levissime concavo, parte mediana cellularum cum lateribus subparallelis, apices versus gradatim attenuatis, apicibus subangustatis, levissime recurvatis subrotundatisque; membrana achroa glabra; pyrenoidibus in serie singula subirregulariter dispositis, in semicellula unaquaque 7 vel 8; locellis apicalibus terminalibus et corpuscula oblonga singula includentibus.
   Long. 217–250 μ; lat. 21–24 μ; lat. apic. 4 μ. III. Ew.

This species is distinguished from *C. Pritchardianum* Arch. by its much smaller size, its much more tapering and narrower extremities, as well as by its smooth and colourless membrane. From *C. littorale* Gay it differs in being a little longer, in the absence of the slight ventral inflation, and in the blunter and slightly recurved apices. It may also be compared with *C. subangulatum* Gutw., from which it differs in the subparallel median portion of the cells, in the absence of the ventral inflation, in the more convex dorsal margin, as well as in the recurved apices. From all the above, the living examples are distinguished by the terminal locellus possessing but one oblong movable corpuscle.

*7. *C. littorale* Gay. II. Kg. VIII. Tm.

8. *C. acerosum* (Schrank) Ehrenb. General; zygospores from Kingsbury, Middlesex.

9. *C. lanceolatum* Kütz. I. Ep. (with zygospores). II. Ki., Wh. III. B., Do, Fh. IV. Hc.

Var. PARVUM var. n. (Pl. VI. fig. 3).

Var. duplo minor. Long. 183 μ; lat. 21 μ.

III. Do., in ditch amongst *Spirogyra* sp.

10. *C. lunula* (Müll.) Nitzsch. I. Ep. III. B., E., Ew., Pu., Th. IV. Ks. VIII. G., H., L.

11. *C. Ehrenbergii* Menegh. I. Ep. II. Ha. III. B., C., Ew., Fh., Pt., Wi. (with zygospores). IV. Cb.

A pure gathering was obtained from a ditch in Roehampton Lane, Putney Heath, Surrey.

12. *C. Malinvernianum* De Not. III. Ew., Fh.

Forms were observed with rather more attenuate apices, which were obliquely subtruncate. Long. 284–359 μ; lat. 41–48 μ; lat. apic. 7 μ. VIII. Tm.

13. *C. moniliferum* (Bory) Ehrenb. General and abundant.

14. *C. Leibleinii* Kütz. I. Ep. II. Ha., No., Wh. III. Cr., Ew., M., Mp., Ra., Rp., Wi., Wt. V. Go. (with zygospores). VI. N. VII. S.

15. *C. Dianæ* Ehrenb. I. Ep. III. C., D., Ew., Mp., Wi. VIII. H., L., T., Tm., W.

16. *C. pseudodianæ* Roy. III. Th.

17. *C. parvulum* Näg. General; zygospores from Esher West-end and Mitcham Commons, Surrey.

18. *C. calosporum* Wittr. forma *major* West & G. S. West. V. Go. (with zygospores).

19. *C. Venus* Kütz. I. Ep. III. Ew. (very abundant in April 1895), Mp., Pu. VI. N. VIII. G., L.

20. *C. trochiscosporum* West & G. S. West. V. Go. (abundant with zygospores).

21. *C. Jenneri* Ralfs. III. Bi., Mp., Pu., Th. VI. N. VIII. T., Tm.

22. *C. Cynthia* De Not. III. Pu. VI. N.

23. *C. Archerianum* Cleve. VI. N. VIII. G.

24. *C. costatum* Corda. I. Ep. II. Ru. III. E., Ew., Pu., Th. IV. Ks. V. Go. VI. N. VIII. G., H., Hl., T., Tm.

25. *C. regulare* Bréb. III. Pu., not uncommon. Small forms; long. 226–232 μ; lat. 24–26 μ.

26. *C. striolatum* Ehrenb. General; pure gathering from a pool on Wimbledon Common, Surrey.

Var. *orthonotum* Roy. III. Dj., E., Fh., Pt., Wi. IV. Ks. VI. N. VII. D. VIII. H., T.

27. *C. intermedium* Ralfs. III. Bi., E., Th., We., Wi. VII. D., Gp. VIII. H.
   Var. *hibernicum* West. III. Mp.
28. *C. directum* Arch. III. E., Mp.
29. *C. angustatum* Kütz. III. Th. VI. N. VIII. G., R., W.
30. *C. juncidum* Ralfs. I. Ep. III. Bi., Ew., Pu. VIII. G., H., T. (with zygospores).
31. *C. lineatum* Ehrenb. I. Ep. III. Ew. IV. Ks. VI. N. (with zygospores). VIII. H.
32. *C. Ralfsii* Bréb. IV. Ks.
   Var. *hybridum* Rabenh. III. Pu., Th.
33. *C. attenuatum* Ehrenb. I. Ep. III. Ew., M., Pu., Th. VI. N. VIII. Tm.
34. *C. rostratum* Ehrenb. General and abundant. Zygospores from Epping Forest, Essex; Esher West-end, Mitcham, Puttenham and Wimbledon Commons, Richmond Park and Frensham, Surrey.
   Var. *brevirostratum* West. III. Ew., Wi. (with zygospores).
35. *C. setaceum* Ehrenb. III. Pu., Th. (with zygospores). VIII. G.
36. *C. Kützingii* Bréb. I. Ep. III. Ew., Fl., M., Mp., Ra., Rp. V. Go. (with zygospores). VI. N. VIII. L.
37. *C. Cornu* Ehrenb. I. Ep. II. Ru. III. M. IV. Ks. V. Go. VIII. G., H., L., T.
38. *C. gracile* Bréb. I. Ep. III. E., Ew., Pu., Th. V. Go. (with zygospores.) VI. N. (with zygospores). VII. D. VIII. G.
39. *C. pronum* Bréb. I. Ep. VI. N. VIII. K.

A form of this was noticed with the membrane colourless and without any trace of striation. There were ten pyrenoids in each semi-cell. Long. 375 μ; lat. 9 μ. III. Rp.

40. *C. acutum* (Lyngb.) Bréb. I. Ep. II. Ru. III .Fh., M., Wa., Wi. V. Go. (with zygospores). VIII. W.
41. *C. linea* Perty. I. Ep. III. Bi., Dj., E., Ew., M., Mp., Rp., Th. IV. Ks. VIII. W.
42. *C. Ceratium* Perty. III. E., Wi.

Genus *Docidium* Bréb.

1. *D. baculum* Bréb. VI. N.

Genus *Pleurotænium* Näg.

1. *P. coronatum* (Bréb.) Rabenh. III. Th. VI. N.
   Var. *nodulosum* (Bréb.) West. III. E., Th. VI. N. VIII. W.

2. *P. Ehrenbergii* (Bréb.) De Bary. I. Ep. III. C., Ew., Mp., Pu., Th. (with zygospores), Wa., Wi., Wo. IV. Hc., Ks. V. Go. VII. D. VIII. G., H., L., T., Tm., W.
   Diam. zygosp. 70 μ.
3. *P. Trabecula* (Ehrenb.) Näg. (*P. maximum* (Reinsch) Lund. var. *occidentale* West). I. Ep. III. Fg., Mp., Ra., Rp., Th., Wi. VI. N. VII. D.
4. *P. truncatum* (Bréb.) Näg. III. B., C., Di., Ew., M., Mp., Wi.

Genus *Tetmemorus* Ralfs.

1. *T. Brébissonii* (Menegh.) Ralfs. III. C., E., Th. VII. D., Gp. VIII. H.
2. *T. granulatus* (Bréb.) Ralfs. I. Ep. III. Bi., C., E., Th., Wa. IV. Ks. VII. D., Gp. VIII. G., H., L., T. (with zygospores), W.
3. *T. lævis* (Kütz.) Ralfs. I. Ep. III. Bi., C., E., Pu., Th., Wi. IV. Ks. VII. D., Gp. VIII. G., H., Hl., T.

Genus *Euastrum* Ehrenb.

1. *E. verrucosum* Ehrenb. III. Mp., Pu. IV. Ks. VI. N. VIII. G., Tm.
2. *E. oblongum* (Grev.) Ralfs. I. Ep. III. Ew., Mp., Pu. IV. Ks. VII. D., Gp. VIII. G., H., Tm.
3. *E. crassum* (Bréb.) Kütz. III. C., Th. VII. D. VIII. H., W.
   Var. *scrobiculatum* Lund. VI. N.
4. *E. ventricosum* Lund. III. Th.
5. *E. humerosum* Ralfs. VIII. H.
6. *E. affine* Ralfs. III. C.
7. *E. ampullaceum* Ralfs. III. C., Dj.. Th. VII. D.
8. *E. insigne* Hass. III. C. VII. D.
9. *E. didelta* (Turp.) Ralfs. III. Bi., C., E., Pt., Th. IV. Ks. VI. N. VII. D., Gp. VIII. H.
   Forma *scrobiculata* Nordst. VI. N.
10. *E. cuneatum* Jenner. III. Th.
11. *E. ansatum* Ehrenb. I. Ep. III. Bi., E., Pu., Th. IV. Ks. VII. Gp. VIII. G., H., L., T., W.
12. *E. sinuosum* Lenorm. III. Th. VI. N. VII. D.
13. *E. pectinatum* Bréb. I. Ep. III. Bi., Mp., Th. (with zygospores). IV. Ks. VI. N. VII. D. VIII. G. (with zygospores), H., L., T., Tm., W.
14. *E. gemmatum* (Bréb.) Ralfs. VII. D. VIII. H.
15. *E. rostratum* Ralfs. VII. D. VIII. G.
16. *E. elegans* (Bréb.) Kütz. I. Ep. III. Bi., Pu., Th. IV. Ks. VI. N. (with zygospores). VII. D., Gp. VIII. G., H.
    Var. *bidentatum* (Näg.) Jacobs. III. Ew., Mp., Pu. V. Go. VI. N. VII. Gp. VIII. L.

17. *E. inerme* (Ralfs) Lund. III. Th. VII. D.
18. *E. erosum* Lund. var. *notabile* West. I. Ep. III. Bi. VIII. L.
19. *E. pyramidatum* West. III. Th. VIII. W.
20. *E. insulare* (Wittr.) Roy. VI. N.
21. *E. binale* (Turp.) Ehrenb. I. Ep. III. Bi., C, E., M., Mp., Pu., Th. IV. Ks. VI. N. (with zygospores). VII. D., Gp. VIII. G., H., Hl., K., L., R., W.
    Forma *minor* West. III. Pu.
    Var. *elobatum* Lund. III. Ew.
22. *E. denticulatum* (Kirchn.) Gay. III. Bi., C., E., Ew., Mp., Th. IV. Ks. VI. N. VII. D. VIII. L.

Genus *Micrasterias* Ag.

1. *M. mucronata* (Rabenh.). III. C., Th.
2. *M. americana* (Ehrenb.) Ralfs. III. Mp.
3. *M. angulosa* Hantzsch. III. Th. VI. N.
4. *M. denticulata* Bréb. III. C., E., Ew., Th. (with zygospores). IV. Ks. VII. D. VIII. H. (with zygospores), Hl., R., Tm., W.
5. *M. rotata* (Grev.) Ralfs. III. Pu. IV. Ks. VII. D., Gp. VIII. L., R., Tm.
6. *M. Thomasiana* Arch. III. C., Th.
7. *M. radiosa* Ralfs. III. Mp.
8. *M. papillifera* (Bréb.). III. Th. VII. D. VIII. L.
9. *M. truncata* (Corda) Bréb. III. Bi., C., E., Th. VII. D. VIII. G., Hl., W.
    Var. *Bahusiensis* Wittr. VI. N.
10. *M. Jenneri* Ralfs. III. Th. VI. N.
    Var. *simplex* West. VI. N.

Genus *Xanthidium* Ehrenb.

1. *X. armatum* (Bréb.) Rabenh. III. C., E., Th. VII. D. VIII. H., R., W.
2. *X. fasciculatum* Ehrenb. III. Ew., Mp. VIII. H.
3. *X. cristatum* Bréb. VI. N. VIII. G., H.
4. *X. antilopæum* (Bréb.) Kütz. I. Ep. III. Mp., Th. (with zygospores). VI. N. VII. D. VIII. G., H.
5. *X. Smithii* Arch. var. *variabile* Nordst. III. Dj. (with zygospores), E., Th.
6. *X. concinnum* Arch. (*Arthrodesmus hexagonus* Boldt).

The typical form is rather more deeply constricted than the var. *Boldtianum*; there are two small mucros at the apical angles, but only one at the lateral angles, this character being well seen in vertical view. (Pl. VI. fig. 15.)

Long. 9–9·5 μ; lat. s. mucr. 9·5–10·5 μ; lat. isthm. 2·5–3 μ; crass. 7 μ. III. Pu., Th.

Var. *Boldtianum* West (*Arthrodesmus hexagonus* Boldt forma Boldt). III. Th.

Genus *Cosmarium* Corda.

1. *C. quadratum* Ralfs. I. Ep. III. Bh., Mp., Pu., Th., Wi. IV. Ks. V. Go. VI. N. VIII. G., T., W.
2. *C. Nymannianum* Grun. III. Th.
3. *C. Hammeri* Reinsch. VIII. G.
4. *C. granatum* Bréb. I. Ep. III. Fg., Rp. IV. Ks. VI. N. VII. D. VIII. K.

 Var. *subgranatum* Nordst. I. Ep. III. Rp. VII. S.

5. *C. trilobulatum* Reinsch. VI. N.
6. *C. tetragonum* (Näg.) Arch. I. Ep.
7. *C. Cucumis* Corda. I. Ep. III. Ea., Mp., Pu. V. Go. VIII. H., P.

*8. *C. subcucumis* Schmidle (in Berichte der Naturf. Gesellsch. Freiburg, Bd. vii., Heft 1, p. 98, t. iv. f. 20–22).

The forms seen were relatively longer and the sinus was rather more open. There were two large pyrenoids in each semi-cell. The relative proportion of breadth to length in Schmidle's specimens was 1:1·42; that of ours was 1:1·79.

Long. 64–77 μ; lat. 38–44 μ; lat. isthm. 13·5–18 μ; crass. 25–27 μ. III. Ew. (abundant Feb. 1894), Wi.

9. *C. Ralfsii* Bréb. III. C. VII. D.
10. *C. pachydermum* Lund. III. M., Mp. VIII. Tm.
11. *C. canaliculatum* West & G. S. West. V. Go.
12. *C. pyramidatum* Bréb. III. Bi., C., Rp., Th. VII. D., Gp. VIII. G.
13. *C. pseudopyramidatum* Lund. III. Th. VII. S.
14. *C. variolatum* Lund. III. Th. IV. N.

*15. *C. ocellatum* B. Eichl. & Gutw. (in Rospr. Akad. Umiej. Krakow. Wydz. matem.-prz., ser. ii. tom. viii. vol. xxviii. (1895) p. 164, t. iv. f. 7.

 Var. INCRASSATUM var. n. (Pl. VI. fig. 12.)

 Var. sinu semper aperto; semicellulis apicibus subtruncatis, in centro incrassatis et luteo-fuscis, cum scrobiculis parvis 3–6; a latere visis circularibus; a vertice visis ellipticis, incrassatis, et leviter tumidis, ad medium utrobique.

 Long. 28–30 μ; lat. 24–26 μ; lat. isthm. 5·5–6·5 μ; crass. 14·5–15 μ. III. Th. VI. N.

The thickened area in the centre of the semi-cells is always more or less coloured, and possesses a central larger scrobiculation surrounded by a variable number of somewhat smaller ones.

16. *C. nitidulum* De Not. III. Th.
17. *C. subtumidum* Nordst. III. Dj. IV. Ks. VIII. H.

*18. *C. Klebsii* Gutw. VI. N.

19. *C. Phaseolus* Bréb. II. Wh. III. Bi., Dj., E., Fg., Mp., Th. VII. S. VIII. G., T.
    Forma *minor.*
    Long. 21 μ; lat. 18 μ; lat. isthm. 5·5 μ; crass. 10 μ; VII. Gp.
20. *C. fontigenum* Nordst. III. Pu.
    Long. 22 μ; lat. 25 μ; lat. isthm. 6 μ; crass. 11·5 μ.
21. *C. tetrachondrum* Lund. III. Mp.
22. *C. Scenedesmus* Delp. I. Ep. III. Th.
23. *C. pseudoprotuberans* Kirchn. I. Ep. III. Th. (with zygospores). VI. N.
24. *C. rectangulare* Grun. III. C., Dj., E.
25. *C. inconspicuum* West & G. S. West. III. M. (with zygospores).
26. *C. bioculatum* Bréb. II. U. III. Bi., E., Ew., Mp., Pu. (with zygospores), Th., Wi. VIII. W.
    Diam. zygosp. s. spin. 17–19 μ; c. spin. 26–28 μ. (Pl. VI. fig. 17.)
    Var. *depressum* Schaar. I. Ep. IV. Ks.
    Var. HIANS var. n. (Pl. VI. fig. 24.)
    Var. sinu apertiori extrorsum, marginibus inferioribus semicellularum convexis, apicibus rectis vel levissime retusis; membrana distincte punctata.
    Long. 17–19 μ; lat. 15–18 μ; lat. isthm. 3·5–4 μ; crass. 7·5–8·5 μ. III. Pu., Th.

This comes nearest to *C. bioculatum* var. *excavatum* Gutw., but is proportionately shorter; the apex of the sinus is subacute; the apices of the cells are straight or very slightly retuse; and the membrane is punctate.

*27. *C. aspherosporum* Nordst., var. *strigosum* Nordst. III. Pu.
    Long. 10·5–11 μ; lat. 9 5 μ; lat. isthm. 3·5 μ; crass. 5 μ.
28. *C. tinctum* Ralfs. I. Ep. III. Di., E., Ew., Pu., Th., Wi. IV. Ks. VI. N. VII. D., Gp. VIII. G., H., T.
29. *C. succisum* West. III. Pu.
30. *C. abbreviatum* Racib. I. Ep. II. U.
31. *C. pygmæum* Arch. III. Mp.
32. *C. truncatellum* Perty. III. C.
    Long. 9·5–10·5 μ; lat. 12·5–14·5 μ; lat. isthm. 5·5 μ; crass. 5 μ. (Pl. VII. fig. 3.)
*33. *C. geometricum* West & G. S. West. III. Pu.
*34. *C. hexangulare* Nordst. III. Th.
*35. *C. Heimerlii* West & G. S. West. (*C. minutissimum* Heimerl non Arch.) III. Mp., Th. IV. Ks.
36. C. SPHAGNICOLUM sp. n. (Pl. VI. figs. 13, 14.)
    C. minutissimum, paullo latius quam longum, modice con-

strictum, sinu brevi et aperto; semicellulæ subtrapeziformes, angulis superioribus oblique truncatis, lateribus divergentibus sursum, apicibus latis, rectis vel levissime retusis; a latere visæ subcirculares; a vertice visæ elliptico-oblongæ, polis subacutis, medio utrobique leviter tumidæ, polos versus utrobique papilla instructæ; pyrenoidibus singulis.

Long. 10·5–11·5 μ; lat. 11–13·5 μ; lat. isthm. 5–5·5 μ; crass. 6·5 μ. III. Th.

We also have this minute species in immense quantity among *Sphagnum* from Mossdale Moor, Widdale Fell, N. Yorks. It is most nearly related to some forms of *C. Heimerlii* in vertical view, but the front view is quite distinct. The figures are all taken from Yorkshire specimens.

37. *C. Regnellii* Wille. I. Ep. II. Ha. III. Bi. VII. S.
38. *C. impressulum* Elfv. I. Ep. II. U. III. Fg., Mp. VII. S. VIII. Tm.
39. *C. venustum* (Bréb.) Arch. VI. N. VII. D. VIII. G., K., L.
    Var. *majus* Wittr. III. Th. VI. N.
40. *C. læve* Rabenh. I. Ep. IV. Hc. VIII. T.
    Var. *septentrionale* Wille. I. Ep. II. Wh. III. Pu.
41. *C. Meneghinii* Bréb. General.
    Forma *octangularis* Wille. II. Wh. III. B., Di., M., Mp., Wi. IV. Ks. V. Go. VIII. G., P., T.
    Var. *Wollei* Lagerh. VII. D.

*42. *C. difficile* Lütkem. III. Th. VI. N. VIII. T.
    *Var. *sublæve* Lütkem. VIII. T., W.

*43. *C. umbilicatum* Lütkem. II. Wh., abundant.

44. *C. concinnum* (Rabenh.) Reinsch. III. Pu., Th.
45. *C. obliquum* Nordst. III. C.
46. *C. Regnesii* Reinsch. III. Th. (with zygospores), Pu., abundant. VI. N.
47. *C. montanum* Schmidle (*C. Pseudoregnesii* West & G. S. West). I. Ep.

*48. *C. Novæ Semliæ* Wille var. *sibiricum* Boldt. VI. N., very abundant, June 1897.

49. *C. cymatonotophorum* West. III. Th. VI. N.

*50. *C. Sinostegos* Schaarsm. var. *obtusius* Gutw. III. Pu.

51. C. ADOXUM sp. n. (Pl. VII. fig. 24.)

C. minutissimum, paullo longius quam latum, profunde constrictum, sinu lineari; semicellulæ late pyramidato-trapeziformes, angulis basalibus oblique rotundo-truncatis et leviter divergentibus, lateribus (superioribus) subrectis vel levissime retusis, apicibus late truncatis rectisque; a vertice visæ ellipticæ, polis rotundatis, cum papilla prominente ad medium utrobique.

Long. 10–11 μ; lat. 9·5 μ; lat. isthm. 3 μ; crass. 5 μ. VI. N.

This minute species is nearest to *C. Sinostegos* Schaars. var. *obtusius* Gutw., but differs in being proportionately longer, in the more rounded and more rectangular basal angles, and in the rounded poles of the vertical view. It is very distinct from typical *C. Sinostegos*.

52. *C. substriatum* Nordst. III. Dj., Fg., Fl., Mp., Rp. V. Go. VI. N. VII. S. VIII. H.
53. *C. notabile* Bréb. VIII. T., W.
 *Forma *media* Gutw. VI. N.
 Long. 25 μ; lat. 16 μ; lat. isthm. 9 μ.
54. *C. crenatum* Ralfs. III. Di., Ra., Wi. VIII. H., P., T.
55. *C. subcrenatum* Hantzsch. I. Ep. II. Wh. III. Wt.
56. *C. hexalobum* Nordst. var. *minus* Roy et Biss. I. Ep.
57. *C. undulatum* Corda. III. Di., Ew.
58. *C. cymatopleurum* Nordst. var. *tyrolicum* Nordst. VIII. Tm.
59. *C. tetraophthalmum* (Kütz.) Menegh. III. B., Th. IV. Ks. VI. N. VIII. G., T.
 Var. *Lundellii* Wittr. I. Ep. III. Fg., Rp. VIII. G., T.
60. *C. ovale* Ralfs. VI. N.
 *Var. *subglabrum* West & G. S. West. VI. N.
61. *C. Brebissonii* Menegh. III. Th. IV. Ks. VII. D. VIII. H., R., W.
 Forma *erosa* West. III. Ew. VIII. T., W.
62. *C. latum* Bréb. II. Wh.
63. *C. margaritatum* (Lund.) Roy et Biss. III. Fg., Th., very abundant. VI. N.
64. *C. conspersum* Ralfs. IV. Ks. VIII. T.
65. *C. margaritiferum* (Turp.) Menegh. II. Ki. III. C., Rp., Wi. IV. Ks. VII. D. VIII. T.
66. *C. reniforme* (Ralfs) Arch. III. Mp., Th. IV. Ks. V. Go. VIII. K., T.
67. *C. Portianum* Arch. III. Mp., Pu., Th. VII. D. VIII. G.
 *Var. *orthostichum* Schmidle. III. Pu.
68. *C. amœnum* Bréb. III. Th. VIII. H., L.
69. *C. Logiense* Biss. I. Rp. VII. S.
70. *C. punctulatum* Bréb. II. U., Wh. III. C., Pu., Wi. IV. Ks. VII. D., S. VIII. T.
71. *C. subpunctulatum* Nordst. III. Rp., Th. VI. N. VIII. H., T.

Numerous specimens of this species were observed from the New Forest, all of which had rather more prominent central granules than the original examples from New Zealand. (Pl. VI. fig. 19.)

Long. 33 μ; lat. 31 μ; lat. isthm. 9 μ; crass. 18 μ.
Var. *Börgesenii* West. III. Fl., Mp.

C. trachypleurum Lund. var. *minus* Racib. forma Borge (in Bih. till K. Sv. Vet.-Akad. Handl., Bd. 19, Afd. iii. p. 28, t. ii. fig. 30) is *C. subpunctulatum* var. *Börgesenii*, and not a form of *C. trachypleurum*.

72. *C. sphalerostichum* Nordst. III. Th. VII. Gp. VIII. T., W.

73. *C. abruptum* Lund. VIII. H.

74. *C. Blyttii* Wille. III. Bi., Th.

Var. NOVÆ SYLVÆ var. n. (Pl. VI. fig. 10.)

Var. paullo major, oblongo-subrectangularis, crenis lateralibus prominentibus et truncatis, serie singula granulorum (10) intra margines, in centro semicellularum cum annulo granulorum 5, eo superiori maximo.

Long. 22 μ; lat. 17·5 μ; lat. isthm. 5·5 μ; crass. 10 μ. VI. N.

This variety was in abundance, and is nearest to *C. Blyttii* subsp. *Hoffii* Börgesen, from which it differs in being proportionately longer, in its more pronounced lateral crenations, and in the central granules of the semi-cells. In comparing our variety with Börgesen's, we have taken his figure, which is almost as broad as long; his measurements indicate that it is 1⅓ times longer than broad, and are evidently incorrect. His figure is also under a wrongly stated magnification.

75. *C. Bœckii* Wille. III. Mp., Rp., Wi. IV. Hc. VI. N. VIII. H.

76. *C. subcostatum* Nordst. I. Ep. II. Ha., U. III. M. (very abundant), 1894, Mp., Pu., Th. IV. Hc.

77. C. FASTIDIOSUM sp. n. (Pl. VI. fig. 11.)

C. parvum, paullo longius quam latum, profunde constrictum, sinu angusto lineari extremo ampliato; semicellulæ angulari-semicirculares, angulis basalibus rotundatis, apicibus late truncatis, in margine laterali unoquoque granulis 7 (vel 8), apicibus glabris, granulis minutis paucis intra margines subconcentrice ordinatis, in centro granulis majoribus depressis 3–4; a latere visæ globosæ; a vertice visæ ellipticæ; ad medium utrobique cum granulis latis depressis tribus, polis granulatis cum granulis minutis in seriebus transversis ordinatis, in centro glabro; pyrenoidibus singulis magnis.

Long. 37–38·5 μ; lat. 33–36 μ; lat. isthm. 11 μ; crass. 21 μ. V. Go., frequent.

This species approaches *C. subreniforme* Nordst., *C. subcostatum* Nordst., and *C. Bœckii* Wille, but differs sufficiently from all of them.

*78. *C. alatum* Kirchn. VII. S.

79. *C. Gregorii* Roy et Biss. III. Pu. VII. S.

80. *C. Kjellmani* Wille subsp. *grande* Wille. III. Ra.

*81. *C. Nathorstii* Boldt. III. Mp.

82. *D. formosulum* Hoff. II. Ha., Ru., Wh. III. Ew, Pu., Ra., Wt. IV. Hc.

83. *C. quinarium* Lund. III. Mp. VI. N.

84. *C. præmorsum* Bréb. II. Wh. III. Ew. IV. Ks. VII. S. VIII. T.

85. *C. Botrytis* (Bory) Menegh. General and abundant; zygospores from Harefield, Middlesex; Frensham Great Pond, and Mill-pond E. of Chapel Wood, Surrey.

Var. *mediolæve* West. III. Ew. IV. Hc.

Var. *tumidum* Wolle. I. Ep. V. Go. (with zygospores in abundance).

This variety does not differ from the type sufficiently to warrant its being placed as a species (*C. subbotrytis* Schmidle). Moreover, the zygospores are precisely similar to those of the typical form.

86. *C. Turpinii* Bréb. (Inclus. *C. Turpinii* var. *Lundellii* Gutw.). III. Mp., Ra., Rp. VII. S.

87. *C. eboracense* West. III. Rp.

88. *C. Quasillus* Lund. III. Ew., Wi.

89. *C. Corbula* Bréb. I. Ep. III. C., Ew., abundant.

90. *C. Sportella* Bréb. I. Ep.

91. *C. biretum* Bréb. II. Wh., pure gathering. III. Di., Ew., Wk.

92. *C. Broomei* Thwaites. III. Mp.

*93. *C. subbroomei* Schmidle. III. Mp.

Long. 42 μ; lat. 38 μ; lat. isthm. 12 μ; crass. 23 μ. (Pl. VI. fig. 20.)

We are a little uncertain exactly where to place this plant; it is certainly not *C. Broomei* Thwaites, nor is it *C. pseudobroomei* Wolle, and is a species we have not previously observed. It appears to be nearer to Schmidle's species (in Ber. d. Nat. Gesellsch. Freiburg, Bd. vii. Heft i. p. 104, t. v. f. 22–24) than any other, but the granules are more numerous, and those in the centre of the semi-cells do not project so prominently when seen in vertical view.

94. *C. confusum* Cooke var. *regularius* Nordst. III. Bi., Mp., Pu., Th. (with zygospores). VI. N. VII. D. VIII. G., H.

Subsp. *ambiguum* West. III. Pu., Th.

95. *C. ochthodes* Nordst. I. Ep. III. Ew., Mp., Wi. V. Go. VIII. G., H., Tm.

96. *C. orthostichum* Lund. III. Th. VI. N.

*97. *C. Ungerianum* (Näg.) De Bary var. SUBTRIPLICATUM var n. (Pl. VI. fig. 21.)

Var. semicellulis oblongo-rectangularibus, angulis superioribus rotundatis.

Long. 67 μ; lat. 54 μ; lat. isthm. 22 μ; crass. 36 μ. III. Mp., frequent.

This variety is of the same dimensions as the typical form, and has the same arrangement of granules, but the outward form of the semi-cells is characteristically subrectangular. At first we considered it might be a large variety of *C. triplicatum* Wolle; but on comparison with specimens of that species, we found that, although the form was the same, yet the English plant was of larger size, and the arrangement of the granules was quite different (but like that in *C. Ungerianum*).

98. *C. cœlatum* Ralfs. I. Ep. IV. Ks. VII. Gp.

99. *C. commissurale* Bréb. II. Ha.

100. *C. ornatum* Ralfs. III. Mp., Th. (with zygospores). VI. N. (with zygospores). VII. D. VIII. G., H., L.

101. *C. cristatum* Ralfs. III. Th.

102. *C. speciosum* Lund. I. Ep.

103. *C. isthmium* West. III. Pu.

104. *C. orbiculatum* Ralfs. III. Bi. VI. N.

105. *C. contractum* Kirchn. III. Pu.

106. *C. ellipsoideum* Elfv. III. Th.

* Var. *minus* Racib. III. Th.

Long. 18 μ; lat. 13·5 μ.

107. *C. arctoum* Nordst. VI. N.

A small and rather irregular form. Long. 14 μ; lat. 19·5 μ; lat. isthm. 7 μ; crass. 7·5 μ.

108. *C. pseudarctoum* Nordst. III. Ew.

*109. *C. subarctoum* (Lagerh.) Racib. III. Ew.

Long. 15–17 μ; lat. 13·5 μ; lat. isthm. 8·5 μ; crass. 8·5 μ.

110. *C. globosum* Buln. VII. D., abundant.

*Var. *minus* Hansg. I. Ep.

This was in great abundance. and most nearly approaches the form figured by Nordstedt in Ofvers. K. Vet.-Akad. Förh., 1872, No. 6, t. vii. f. 25; and as Hansgirg includes Nordstedt's form in his variety, we place it under this variety. It is smaller than the typical form, with depressed apices similar to those of *C. pseudarctoum*, but the constriction is different from that in the latter species, and the vertical view is elliptical.

Long. 17–20 μ; lat. 13–14·5 μ; lat. isthm. 9·5 μ; crass. 9 μ.

111. *C. moniliforme* (Turp.) Ralfs. III. B., E., Th. VII. D. VIII. G., R., W.

A form of this was observed from the New Forest with obovate semi-cells and subtruncate apices; long. 41 μ; lat. 20 μ; lat. isthm. 7 μ. Many examples were seen, and they constantly retained the same characters.

Forma *panduriformis* Heimerl. III. Th. VIII. W.

112. *C. connatum* Bréb. III. Pu., Th. VI. N.
113. *C. pseudoconnatum* Nordst. VI. N.
*114. *C. gonioides* West and G. S. West. III. Th.

The specimens were rather smaller than those originally described, and the vertical view was not perfectly circular.

Long. 14·5–15·5 μ; lat. 7·5 μ.

115. *C. Cucurbita* Bréb. III. Bi., C., E., Th. (with zygospores). IV. Ks. VII. D., Gp. VIII. Hl., R.. T.
Diam. zygosp. 30–32 μ. (Pl. VI. fig. 26.)
116. *C. obcuneatum* West. VII. D., abundant.
117. *C. Thwaitesii* Ralfs. II. Ru. III. Mp.
118. *C. annulatum* (Näg.) De Bary. VIII. T.
119. *C. elegantissimum* Lund. III. Th. VI. N.

Genus *Cosmocladium* Bréb.

1. *C. saxonicum* De Bary. III. Th.

This is referred with hesitation to this species; it always occurred in twos or in fours within a very firm gelatinous investment, and the cells were not arranged in one plane. No trace of connecting filaments could be observed in preserved specimens. It is rather smaller than the published dimensions of this species, but the cells were elliptic-reniform, and it otherwise seemed to agree.

Long. 15–17 μ; lat. 13·5–14·5 μ; lat. isthm. 4·5 μ; crass. 8–9 μ.

Genus *Spondylosium* Bréb.

1. *S. pulchellum* Arch. VIII. G.

Genus *Staurastrum* Meyen.

1. *S. connatum* (Lund.) Roy et Biss. VII. D. VIII. G.
2. *S. dejectum* Bréb. I. Ep. III. Bi., C., E., Pu., Th. VI. N. VII. D. VIII. G., H., R., T., W. Subsp. *Tellamii* West. VIII. G.
3. *S. apiculatum* Bréb. III. C., Th. (with zygospores). VIII. T.
4. *S. mucronatum* Ralfs. IV. Ks.
5. *S. Dickiei* Ralfs. III. E., Fg., Pu. (with zygospores), Th. IV. Ks. VI. N. (with zygospores).
Forma *punctata* West. III. Th.
Var. *semicirculare* W. B. Turner. III. Th.
6. *S. glabrum* (Ehrenb.) Ralfs. III. Bi.
7. *S. brevispinum* Bréb. III. Th. VI. N.
8. *S. cuspidatum* Bréb. I. Ep. III. M., Pu., Th. IV. Ks. V. Go. VI. N. (with zygospores). VII. D.

Var. *divergens* Nordst. IV. Ks. (with zygospores).

Var. *maximum* West. I. Ep.

9. *S. O'Mearii* Arch. III. Pu., Th. VIII. H.

10. *S. sibiricum* Borge, forma *trigona* West and G. S. West. III. Dj.

11. *S. lanceolatum* Arch. III. Th., abundant. VI. N., abundant (with zygospores). VIII. H.

Var. *compressum* West. VI. N.

12. *S. corniculatum* Lund. III. Th. (with zygospores). VI. N.

13. S. TRACHYTITHOPHORUM sp. n. (Pl. VI. fig. 22.)

S. parvum, tam longum quam latum, profunde constrictum, sinu aperto et subrectangulari; semicellulæ late ellipticæ, polis mamillatis et seorsum curvatis, annulis duobus granulorum minutorum circa polum unumquemque; a vertice visæ triangulares, lateribus subrectis (levissime convexis), angulis leviter productis, cum annulis duobus granulorum minutorum.

Long. 30–34 μ; lat. 29–32·5 μ; lat. isthm. 10·5–11·5 μ. III. Th.

This species seems to be quite distinct from any other.

14. *S. lunatum* Ralfs. I., Ep. V. Go. VI. N.

Var. *subarmatum* West. I. Ep.

*15. *S. tunguscanum* Boldt. III. Pu.

16. *S. denticulatum* (Näg.) Arch. I. Ep.

17. *S. Avicula* Bréb. VII. D. VIII. G.

Var. *subarcuatum* (Wolle) West. I. Ep. VI. N.

18. *S. furcatum* (Ehrenb.) Bréb. (*S. spinosum* Ralfs). III. Th. VIII. H.

Var. *subsenarium* W. VII. D.

19. *S. armigerum* Bréb. (*S. pseudofurcigerum* Reinsch). III. Bi., Pu., We. VI. N. VII. D., Gp.

20. *S. arcuatum* Nordst. I. Ep. VI. N. VIII. G.

21. *S. Reinschii* Roy. III. C., Dj., E., Th. VIII. R.

22. *S. rostellum* Roy et Biss. VI. N.

Var. EROSTELLUM var. n. (Pl. VI. fig. 18.)

Var. minor, profundius constricta, spina magna prope constrictionem nulla; a vertice visum triangulare, angulis rotundatioribus.

Long. s. spin. 19·5 μ; lat. s. spin. 19·5 μ; lat. isthm. 6·5 μ. III. Th.

Although the character giving the name to this species is absent, still the number, size, and arrangement of the spines is exactly the same; the form of the semi-cells also would be just the same if the constriction were less deep.

23. *S. Hystrix* Ralfs. III. Di., Dj., Th.

24. *S. teliferum* Ralfs. III. Bi., Pu., Th. VII. D., Gp. VIII. G., H., L., W.

25. *S. gladiosum* W. B. Turn. VIII. G.
    Long. s. spin. 41 μ; c. spin. 51 μ; lat. s. spin. 37·5 μ; c. spin. 48 μ; lat. isthm. 12 μ.
26. *S. polytrichum* Perty (*S. Pringsheimii* Reinsch). III. Ew., Pu., Th. VI. N.
27. *S. saxonicum* Buln. VI. N.
28. *S. pilosum* (Näg.) Arch. I. Ep. III. Bi., C., Ew., Mp., Th., Wi. IV. Ks. V. Go. VI. N. VII. D. VIII. H., R., T., W.
29. *S. hirsutum* (Ehrenb.) Bréb. I. Ep. III. Dj., Ew., We., Wi. IV. Ks. VII. D.
30. *S. muticum* Bréb. I. Ep. III. Th., Wi. VII. S. VIII. G.
    *Var. *minus* Wolle. VI. N.
    Long. 17·5 μ; lat. 16·5 μ; lat. isthm. 4·5 μ.
31. *S. orbiculare* (Ehrenb.) Ralfs. III. Bi., C., Ew., Th. VI. N. VII. D., Gp. VIII. G., H., L., T.
    Var. *depressum* Roy et Biss. III. C., Wt. VI. N.
32. *S. cosmarioides* Nordst. VI. N.
33. *S. Bieneanum* Rabenh. I. Ep. V. Go. (with zygospores). VI. N.

Some of the New Forest examples had from two to three apiculi on each semi-cell, sometimes near the angles and sometimes between them.

Var. *ellipticum* Wille. III. Pu. IV. Ks. VI. N.

34. *S. pachyrhynchum* Nordst. III. Th., frequent. VI. N., June 1897, frequent.
    Long. 38–41 μ; lat. 35–39 μ; lat. isthm. 9·5–10·5 μ.
35. *S. punctulatum* Bréb. Frequent; zygospores from "Devil's Jumps" near Frensham.
6. *S. pygmæum* Bréb. III. Bi., C., M., Mp. VII. S. VIII. H., P., T.
37. *S. turgescens* De Not. III. Wi.
38. *S. muricatum* Bréb. I. Ep. III. C., Dj., Wi. VI. N.
39. *S. pyramidatum* West. IV. Ks.
40. *S. spongiosum* Bréb. III. M., Mp., Th. VII. D. VIII. H.
    Var. *perbifidum* West. III. Th. (3- and 4-ended).
41. *S. Meriani* Reinsch. VIII. T.
42. *S. alternans* Bréb. I. Ep. III. Th. (with zygospores). IV. Ks. V. Go. VII. D., S. VIII. W.
43. *S. dilatatum* Ehrenb. III. Bi. VI. N. VII. D. VIII. H., L.
    Var. *obtusilobum* De Not. III. Pu. V. Go.
44. *S. subpygmæum* West. III. Th., frequent.
45. *S. tumidum* Bréb. III. Th. VI. N. VII. D. VIII. W.
46. *S. brachiatum* Ralfs. III. Dj., E., Pu., Th. (with zygospores). VII. D. VIII. R., W.
47. *S. læve* Ralfs. III. Th., very abundant.

48. *S. inconspicuum* Nordst. III. Dj., Th. VI. N. VII. D. VIII. R.
49. S. NODOSUM sp. n. (Pl. VI. fig. 23.)

S. minutum, circiter tam longum quam latum (cum processibus), profunde constrictum, sinu aperto lateribus subrectis et apice acuto; semicellulæ late rectangulari-oblongæ, angulis superioribus in processus breves uninodulosos divergentes truncatos productis, angulis inferioribus subrectangularibus, apicibus concavis; a vertice visæ triangulares, lateribus concavis, processibus truncatis binodulosis; membrana glabra.

Long. s. proc. 11 μ, c. proc. 21 μ; lat. c. proc. 19 μ; lat. isthm. 5 μ. III. Th.

This differs from *S. inconspicuum* Nordst. in its marked and deep constriction, and in its triradiate vertical view with much smaller body.

50. *S. micron* West & G. S. West. III. Pu., abundant.
51. *S. iotanum* Wolle. III. Pu.
52. *S. tetracerum* Ralfs. III. Pu., Th. IV. Ks. VI. N. VII. D. VIII. G., H., W.

Forma *trigona*. III. Th. VI. N. VII. D. VIII. W.

Forma *tetragona*. VIII. R.

Var. VALIDUM var. n. (Pl. VI. fig. 25.)

Var. corpore longiori, processibus validioribus non attenuatis et 5-nodulosis.

Long. s. proc. 18 μ, c. proc. 42 μ; lat. s. proc. 13 μ, c. proc. 37 μ; lat. isthm. 5 μ. III. Mp.

53. *S. hexacerum* (Ehrenb.) Wittr. (*S. tricorne* Ralfs). II. U. III. C., Di., Dj., Ew., Fg., M., Mp., Ra. IV. Ks. VI. N. VII. S. VIII. G., L.

Var. β Ralfs. VIII. H., K.

54. *S. cyrtocerum* Bréb. III. Ew., M., Mp., Th. VI. N. VII. D. VIII. G., L.
55. *S. inflexum* Bréb. I. Ep. III. Bi., C., Ew., Mp., Rp., Th., Wi. IV. Ks. VI. N. VII. D.
56. *S. polymorphum* Bréb. I. Ep. III. C., E., Th., Wi. VIII. H., Tm.
57. *S. crenulatum* (Näg.) Delp. II. U. III. Th. IV. Ks. V. Go.

*58. *S. Heimerlianum* Lütkem. var. *spinulosum* Lütkem. III. Th.

59. *S. gracile* Ralfs. I. Ep. III. Ew., Th. (with zygospores). V. Go. VI. N. VII. D., S.

Zygospora globosa, spinis obsessa; spinis dilatatis ad basin et dichotome furcatis ad apicem.

Diam. zygosp. s. spin. 32 μ, c. spin. 60 μ. (Pl. VI. fig. 27.)

Var. *nanum* Wille. III. Bi., Mp. V. Go. VI. N.

60. *S. paradoxum* Meyen. III. Bi., E., Th. V. Go. VI. N. VIII. G., Hl.
    Forma *parva* West. III. Th. IV. Ks. VI. N.
61. *S. oxyacanthum* Arch. III. Th.
62. *S. controversum* Bréb. III. Bi., Pu. VI. N. VIII. H.
63. *S. aculeatum* (Ehrenb.) Menegh. III. Th. VI. N. VIII. W.
64. *S. vestitum* Ralfs. III. Th. VI. N.
    Var. *semivestitum* West. III. Pu. VIII. G.
65. *S. anatinum* Cooke & Wills. VI. N.
66. *S. pseudosebaldi* Wille. III. Th.
67. *S. Arachne* Ralfs. III. Th. VI. N.
68. *S. proboscideum* (Bréb.) Arch. I. Ep. III. Mp.
69. *S. asperum* Bréb. III. Th. VI. N. VII. D.
70. *S. subscabrum* Nordst. VI. N.
71. *S. sexcostatum* Bréb. III. Ew.
    Subsp. *productum* West. VI. N.
72. *S. margaritaceum* (Ehrenb.) Menegh. I. Ep. III. Bi., C., Dj. (up to 9-ended; with zygospores), Ea., Th., Wo. VI. N. VII. Gp. VIII. R.
    Var. SUBCONTORTUM var. n. (Pl. VII. figs. 15–17.)
        Var. cellulis a vertice visis 6–7 radiatis, processibus truncatis et curvatis ut in *S. cyrtocero*.
        Long. 26 μ; lat. 25–27 μ; lat. isthm. 9 μ. III. Dj.
    Var. ROBUSTUM var. n. (Pl. VII. fig. 14.)
        Var. validior, semicellulis late ellipticis (sine proc.), ad basin processuum non contractis; a vertice visis quinqueradiatis, cum verruca parva emarginata ad basin processuum utrobique.
        Long. 25·5 μ; lat. 27 μ; lat. isthm. 8 μ. II. U.

This variety approaches *S. ornatum* (Boldt) W. B. Turn. (*S. margaritaceum* var. *ornatum* Boldt), but has much shorter and stouter processes. It may be also compared with *S. foliatum* W. B. Turn., but Turner's figure is too indistinct to admit of a detailed comparison.

73. *S. furcigerum* Bréb. I. Ep. III. M. VI. N.
74. *S. eustephanum* (Ehrenb.) Ralfs. VI. N.

Genus *Arthrodesmus* Ehrenb.

1. *A. bifidus* Bréb. III. Pu., Th. VIII. R.
    Var. *truncatus* West. III. Th. IV. Ks.
2. *A. octocornis* Ehrenb. III. E., Rw., Pu. (with zygospores), Th. VI. N. VIII. H.
    Diam. zygosp. s. spin. 15 μ, c. spin. 28 μ. (Pl. I. fig. 16.)
3. *A. Incus* (Bréb.) Hass. III. C. (with zygospores), Dj., E. (with zygospores), Th. VII. D. VIII. H., L., R., W.
    Var. SUBQUADRATUS var. n. (Pl. VII. fig. 20.)

Var. minus constricta, semicellulis subquadratis.

Long. s. spin. 15–17 μ; lat. s. spin. 11 μ; long. spin. 5·5–7 μ; lat. isthm. 7·5 μ; crass. 8 μ. III. C.

4. *A. Ralfsii* West. I. Ep. III. Bi., Ew., Pu., Wi.
5. *A. convergens* Ehrenb. III. Mp., Th. VI. N.

Genus *Onychonema* Wallich.

1. *O. filiforme* (Ehrenb.) Roy et Biss. VIII. G.

Genus *Sphærozosma* Corda.

1. *S. vertebratum* (Bréb.) Ralfs. III. M.

Var. LATIUS var. n. (Pl. VI. fig. 7.)

Var. cellulis latioribus (1⅔-plo latioribus quam longis), sinu angustiori profundiorique, dorso convexiori.

Long. 15–16 μ; lat. 25–27 μ; lat. isth. 5·5–7·5 μ. III. Ew.

2. *S. excavatum* Ralfs. I. Ep. III. C., E., Pu. (with zygospores), Th. (with zygospores), Wi. VI. N. (with zygospores). VII. D., Gp. VIII. T., W.
3. *S. granulatum* Roy et Biss. I. Ep. III. Mp., Pu.

*4. *S. Wallichii* Jacobs. var. ANGLICUM var. n. (Pl. VI. fig. 6.)

Var. apicibus semicellularum subarctis, sinu minore, granulis 2 vel 3 ad margines laterales semicellarum, cum granulis sparsis trans semicellulas irregulariter ordinatis.

Long. 10–11·5 μ; lat. 10·5–11 μ; lat. isthm. 6 μ; crass. 5·5 μ. VI. N. abundant, July 1897.

Genus *Desmidium* Ag.

1. *D. cylindricum* Grev. III. Th. VI. N.
2. *D. Swartzii* Ag. III. Mp., Th. VIII. H., W.

Genus *Gymnozyga* Ehrenb.

1. *G. moniliformis* Ehrenb. III. Bi., C., Th. VI. N. VIII. H., R.

Genus *Hyalotheca* Kütz.

1. *H. mucosa* (Dillw.) Ehrenb. III. C. Th. VIII. G. W.
2. *H. dissiliens* (Sm.) Bréb. Frequent; zygospores from Bisley, Chobham, Esher West-end, and Thursley Commons, Surrey; Keston Common, Kent; Roughtor Moor, Cornwall.

Forma *tridentula* Nordst. III. Ew. VIII. H., L.

Var. *hians* Wolle. III. Ew. (pure gathering, April 1895).

*3. *H. neglecta* Racib. III. Th. VI. N. (with zygospores).

Long. cell. 28–34·5 μ; lat. cell. 11·5–13 μ.

This species was in great abundance from Thursley Common, Surrey, and from the New Forest, Hants. It occurs in long filaments, which are easily overlooked on account of their resemblance to some confervaceous algæ; the filaments have a very wide gelatinous sheath (up to 54 μ diam.).

There is an almost imperceptible constriction at the middle of the cells, and on each side of it a very minute inflation, often causing the widest part of the cell to be in the middle. In a gathering made in August 1897, from a bog in the New Forest, it was obtained with zygospores.

Zygosporæ globosæ, glabræ, nonnunquam mamillatæ ad polos oppositos. Diam. zygosp. 23–28 μ.

Some filaments showed the formation of aplanospores, breaking up into separate cells during their formation. These spores were elliptical or oblong-elliptical, with acutely rounded poles, and when mature had a strong yellowish punctate membrane. Long. aplanosp. 20–31 μ; lat. 9·5–11 μ.

Ord. *Protococcoideæ.*

Fam. **Volvocineæ.**

Genus *Volvox* Ehrenb.

1. *V. globator* (L.) Ehrenb. I. Ep. II. U.

Genus *Eudorina* Ehrenb.

1. *E. elegans* Ehrenb. II. Ki. III., Ew., M., Mp., Pu., Wo. V. Go.

Genus *Pandorina* Bory.

1. *P. morum* (Müll.) Bory. General.

Genus *Gonium* Müller.

1. *G. pectorale* Müller. I. Ep. III. B., Ew., Mp., Ra., W., Wt. V. Go.

Genus *Tetragonium* West & G. S. West.

1. *T. lacustre* West & G. S. West. III. E.

Genus *Chlamydomonas* Ehrenb.

*1. *C. Kleinii* Schmidle. II. U. III. B., C., E., Wi.

Fam. **Palmellaceæ.**

Sub-fam. Cœnobieæ.

Genus *Hydrodictyon* Roth.

1. *H. reticulatum* (L.) Lagerh. I. River Lea (J. H. Vanstone, 1893).

Genus *Cœlastrum* Näg.

1. *C. sphæricum* Näg. Very frequent.
2. *C. microporum* Näg. III. C., Ra., Rp. IV. Hc.
3. *C. cambricum* Arch. IV. Ks.
4. *C. cubicum* Näg. IV. Ks.
5. *C. verrucosum* Reinsch. III. E.

Genus *Sorastrum* Kütz.

1. *S. spinulosum* Näg. VI. N. VIII. H.

*2. *S.* (?) *simplex* Wille. VIII. K.

Genus *Pediastrum* Meyen.

1. *P. Boryanum* (Turp.) Menegh. Common.
   Var. *granulatum* (Kütz.) A. Br. I. Ep. III. Fg., Mp. IV. Ks. VII. S.
2. *P. glanduliferum* Bennett. III. Bi.
   Arms delicate, faintly rough, converging and often crossing one another; diam. cell. peripher. 22–27 μ.
3. *P. constrictum* Hass. II. U. III. Ra., Rp.
4. *P. duplex* Meyen (*P. pertusum* Kütz.). II. Ha., Wh. III. Bi., C., Fl., M., Pu., Ra., Rp., Wt. IV. Ks.
5. *P. integrum* Näg. III. Fg., Pu.
6. *P. angulosum* (Ehrenb.) Menegh. III. Ra., Wt.
7. *P. tetras* (Ehrenb.) Ralfs. I. Ep. II. U. III. Bi., Fl., Mp., Pu., Ra. IV. Hc. Ks. V. Go. VI. N. VII. D., S. VIII. G.

Genus *Crucigenia* Morren.

1. *C. rectangularis* (Näg.) A. Br. III. Bi., Mp., Ra. V. Go. VII. S. VIII. H., K., L.

This was noticed from Frensham, Surrey, with families of 128 cells.

Sub-fam. Pseudocœnobieæ.

Genus *Sciadium* A. Br.

1. *S. Arbuscula* A. Br. I. Ep. II. Ha. III. B., M., Wi. IV. Hc., Ks.

Abundant on Wimbledon Common, Surrey, in the autumn of 1892, and again in the autumn of 1894. Also exceptionally fine from a pond on Hayes Common, Kent.

Genus *Ophiocytium* Näg.

1. *O. cochleare* (Eich.) A. Br. General.
2. *O. majus* Näg. III. E., Mp.

Genus *Mischococcus* Näg.

1. *M. confervicola* Näg. II. Ru.

Sub-fam. Rhaphidieæ.

Genus *Dactylococcus* Näg.

1. *D. Debaryanus* Reinsch. II. Ki.
2. *D. bicaudatus* A. Br. var. exilis var. n. (Pl. VII. fig. 18.)
   Var. cellulis multo minoribus angustioribusque, apicibus in cornibus longioribus et tenuioribus productis; contentus chlorophyllosus cellularum cum granulis numerosis parvis.
   Long. c. corn. 17–36 μ; long. corn. 9·5–15 μ; lat. 3·5–5 μ. I. Ep., in ditches.
3. D. dispar sp. n. (Pl. VII. fig. 19.)
   D. cellulis parvis, diametro triplo quadruplove longioribus, fusiformibus, obliquis et sæpe sublunatis, apice altero acuto, apice altero acutissimo vel producto; contentus chlorophyllosus cellularum homogeneus (rare cum granulis magnis paucis).
   Long. 8·5–21 μ; lat. 2–5·7 μ. III. Do. On old wood.

The form of the cells at once distinguishes this species from *D. infusionum* Näg. and *D. obtusus* Lagerh. The nearest species is probably *D. rhaphidioides* Hansg.

Genus *Scenedesmus* Meyen.

1. *S. bijugatus* (Turp.) Kütz. (*S. obtusus* Meyen). I. Ep. III. Ew., Ra., Rp., Wi. IV. Ks.
2. *S. alternans* Reinsch. II. Th. IV. Ks. VIII. T.
   Var. *apiculatus* West. VI. N.
3. S. granulatus sp. n. (Pl. VII. figs. 1, 2.)
   S. cellulis plerumque quaternis, oblongis, diametro circiter 3-plo longioribus, polis conicis, in seriem rectam conjunctis; membrana cellularum granulata, granulis minutis in seriebus tribus longitudinaliter ordinatis.
   Long. cell. 20–21 μ; lat. cell. 6–6·5 μ. III. Rp.

   Compare with *S. Hystrix* Lagerh. and *S. aculeolatus* Reinsch.
4. *S. denticulatus* Lagerh. I. Ep.
   Var. *linearis* Hansg. II. U. III. Ew., Fg., M., Mp., Pu. IV. Ks. V. Go.
5. *S. quadricauda* (Turp.) Bréb. General and abundant.
   Var. *abundans* Kirchn. II. U. III. Mp., Pu., Ra. IV. Ks. VI. N.
6. *S. antennatus* Bréb. III. Pu. IV. Ks.
7. *S. obliquus* (Turp.) Kütz. (*S. acutus* Meyen). Frequent.
   Var. *dimorphus* (Kütz.) Rabenh. II. U. III. E., Fh., Pu.

Genus *Rhaphidium* Kütz.

1. *R. polymorphum* Fresen. var. *falcatum* (Corda) Rabenh. General and abundant.
   Var. *aciculare* (A. Br.) Rabenh. General.

Var. *duplex* (Kütz.) Rabenh. III. Fh.

Var. TUMIDUM var. n. (Pl. VII. fig. 8.)

Var. cellulis solitariis vel subsolitariis, curvatis, in medio inflatis, apicibus acutissimis.

Long. 61–73 μ; lat. 4·5–6·5 μ. III. Pu., abundant.

Var. MIRABILE var. n. (Pl. VII. figs. 9–13.)

Var. cellulis semper solitariis, longissimis, variabiliter curvatis (nonnunquam sigmoideis), apicibus acutissimis. Long. usque 117 μ; lat. 2–3·5 μ.

III. Wi.; in a pond amongst *Conferva*, July 1894; in same pond, Jan. 1895.

The regularly curved forms of this variety are with difficulty distinguished from a minute *Closterium*. At or near the middle of the cells is a clear space, the chromatophores being *completely* interrupted, and towards the apices there is often present a clear space containing one minute moving corpuscle, *but beyond this clear space there is sometimes a portion of the green cell-contents.* The cell-contents are very irregular, and frequently much vacuolated. In a remark concerning the moving granules present towards the apices of the cells in the genus *Closterium*, Archer states (Quart. Journ. Micr. Sc., ii. 1862, pp. 257–8) that "In *Ankistrodesmus* there are no such granules." The latter genus is synonymous with *Rhaphidium*, and this plant was undoubtedly a *Rhaphidium*, and one which possessed these moving granules.

2. *R. convolutum* (Corda) Rabenh. III. Rp.

Genus *Selenastrum* Reinsch.

1. *S. Bibraeanum* Reinsch. IV. Ks.
*2. *S. gracile* Reinsch. III. Pu.

Genus *Lagerheimia* (De Toni) Chodat.

*1. *L. genevensis* Chodat. III. Mp.

Genus *Tetraëdron* Kütz.

1. *T. minimum* (A. Br.) Hansg. III. Fg., Mp., Th. IV. Ks.

This was in large quantity from a pond on Keston Common, Kent. Many specimens showed the reproduction as described and figured by Lagerheim (Stud. Arktisc. Cryptog. I., Tetraëdron u. Euastropsis, Tromsö Mus. Aarsh. 17, 1894). Cf. Pl. VII. figs. 21–23.

2. *T. trigonum* (Näg.) Hansg. I. Ep. III. Bi., Mp.
3. *T. tetragonum* (Näg.) Hansg. III. Di.
4. *T. caudatum* (Corda) Hansg. (*Polyedrium pentagonum* Reinsch). III. Fg.

Forma *incisa* Lagerh. II. U. III. Ra. IV. Ks.

5. *T. regulare* Kütz. (*Polyedrium tetraëdricum* Näg.). III. Mp., Pu., Th.
6. *T. enorme* (Ralfs) Hansg. I. Ep. III. Fg. VI. N.
7. T. HORRIDUM sp. n. (Pl. VII. figs. 4, 5.)

T. magnum, irregulariter tetragonum, pentagonum, hexagonum, vel suboblongatum, compressum, angulis acutis vel rotundatis (rare submamillatis); membrana firma, ad angulos incrassata et spinis validis rectis curvatisque (nonnunquam bi- vel trifurcatis) singulis vel geminatis vel pluribus præditis, etiam spinis brevibus paucis inter angulos dispositis; a latere visum ellipticum vel oblongum.

Diam. s. spin. 27–42 μ, c. spin. 33–61 μ; crass. 19–21 μ. I. Ep. III. Pt.

This species differs from *T. irregulare* (Reinsch) De Toni in being more irregular, in not having the angles produced, and in the irregular nature and disposition of the spines.

Sub-fam. CHARACIEÆ.

Genus *Characium* A. Br.

*1. *C. Nägelii* A. Br. II. Wh.
*2. *C. Pringsheimii* A. Br. III. B., M. VIII. G.
3. *C. Sieboldii* A. Br. VI. N.
4. *C. heteromorphum* (Reinsch) (*Hydrianum heteromorphum* Reinsch). Common.
5. *C. ornithocephalum* A. Br. III. Fg.
*6. *C. longipes* Rabenh. II. U.

Sub-fam. TETRASPOREÆ.

Genus *Schizochlamys* A. Br.

1. *S. gelatinosa* A. Br. I. Ep. III. C.
2. *S. delicatula* West. III. Bi., Ew., Mp. VII. D.

Genus *Palmodactylon* Näg.

1. *P. subramosum* Näg. I. Ep. III. Ew. V. Go.
2. *P. varium* Näg. II. Ha.

Genus *Apiocystis* Näg.

1. *A. Brauniana* Näg. II. Ha. III. Ew., M., Wt. VII. S.

Genus *Tetraspora* Link.

1. *T. gelatinosa* (Vauch.) Desv. III. B., Ew.
2. *T. lubrica* (Roth) Ag. IV. Ks. VIII. near Bodmin.

Genus *Palmella* Lyngb.

1. *P. mucosa* Kütz. II. Wh. III. E.
2. *P. hyalina* Bréb. IV. Ks.

Sub-fam. Dictyosphærieæ.

Genus *Dimorphococcus* A. Br.

1. *D. lunatus* A. Br. III. Th., very rare.

Genus *Tetracoccus* West.

1. *T. botryoides* West. I. Ep. III. Ra., VII. L.

Genus *Botryococcus* Kütz.

1. *B. Braunii* Kütz. I. Ep. III. Bi., C., Mp., Pu., Th. IV. Ks. V. Go. VII. D., Gp. VIII. W.

Ineffigiata gen. n.

Cellulæ submagnæ, irregulariformæ, angulari-oblongæ, angulari-ellipticæ vel subtrapezicæ, familias valde irregulares libere natantes cellularum 2–8 formantes; cellulæ cum processibus brevibus irregularibus et spinis paucis valde variabilibus, irregulariter curvatis et irregulariter dispositis; contentus chlorophyllosus cellularum viridissimus et granulas multas includens.

1. I. neglecta sp. unica.
   Character idem ac generis.
   Diam. cell. sine spin. et proc. 21–40 μ; long. spin. et proc. 5·5–19 μ; diam. fam. 76–115 μ. III. Ew., Mp., Pu., Th. V. Go.

We have long had this alga under observation as something which required investigating. The cells are rather large with very irregular outlines, and each possesses a few irregularly disposed processes or spines. These spines are straight or curved, of very variable length, and are often somewhat subcapitate or otherwise irregular; they are also the chief agents in uniting the cells into irregular families. The exact structure of the cells and families is exceedingly difficult to observe, owing to their extreme irregularity and the opacity of the densely packed granulose chlorophyll.

It is more or less abundant throughout the British Islands and the United States of America, and often occurs in quantity in tanks, water-barrels, &c. We find it frequently in gatherings of the smaller algæ, and can find no mention of anything approaching it.

It may be regarded as nearest to *Botryococcus* Kütz.

The name was suggested by its indefinite misshapen form and it evident neglect by algologists.

Genus *Dictyosphærium* Näg.

1. *D. Ehrenbergianum* Näg. I. Ep. II. Wh. III. E., M., Mp., Ra., Rp., Wi. IV. Hc. V. Go. VI. N. VIII. L.
2. *D. reniforme* Buln. I. Ep.

Sub-fam. Glœocystideæ.

Genus *Nephrocytium* Näg.

1. *N. Agardhianum* Näg. IV. Ks.
2. *N. Nägelii* Grun. III. Ea., M.
3. *N. ecdysiscepanum* West & G. S. West. V. Go.
4. *N. lunatum* West. III. Th.

Genus *Oocystis* Näg.

1. *O. solitaria* Wittr. I. Ep. III. Bi., E., Ew., Fg., Rp., Th. IV. Ks. VI. N. VII. S. VIII. G., T., W.
   Forma *major* Wille. III. Th.
2. *O. crassa* Wittr. III. Mp. VIII. L.
3. *O. gigas* Arch. var. *incrassata* West. VIII. G.
4. *O. Novæ Semliæ* Wille. V. Go.
   Forma *major* Wille. VI. N.
   Var. *maxima* West. VI. N.
5. *O. elliptica* West. III. Mp.
6. *O. asymmetrica* West. III. Bi., C., Mp.
7. *O. panduriformis* West. VIII. G.

Genus *Glœocystis* Näg.

1. *G. gigas* (Kütz.) Lagerh. [*Chlorococcum gigas* (Kütz.) Grun.; *Glœocystis ampla* (Kütz.) Rabenh.] I. Ep. II. Ha., No., Wh. III. B., M., Mp. IV. Ks. V. Go. VIII. G., L., R., W.
   Var. *maxima* West. III. E.
2. *G. vesiculosa* Näg. I. Ep. II. Ha., Ki., Ru. III. Ea., M., Mp., Rp., Th., Wi. IV. Hc., Ks. VI. N. VIII. P.
3. *G. infusionum* (Schrank). [*Chlorococcum infusionum* (Schrank) Menegh.] III. B., Ew., Wi. VII. D.
4. *G. regularis* nob. (*Chlorococcum regulare* West). I. Ep. III. Bi., Ew., Th. VII. D. VIII. Hl., L.

Genus *Kirchneriella* Schmidle.

1. *K. obesa* (West) Schmidle (*Selenastrum obesum* West). III. Bi., Fg., Mp., Pu.

Genus *Urococcus* Kütz.

1. *U. insignis* (Hass.) Kütz. III. Bi., C., E., Th. VII. S. VIII. T.

Sub-fam. Protococcaceæ.

Genus *Stichococcus* Näg.

1. *S. bacillaris* Näg. II. Ki., Pi. III. Nr. Pt., Wi., Boxhill.
2. *S. flaccidus* (Kütz.) Gay. III. B., Rp.
*3. *S. dissectus* Gay. III. Damp walls about London.

Genus *Eremosphæra* De Bary.

1. *E. viridis* De Bary. I. Ep. III. Bi., C., E., Th. IV. Ks. VII. D., Gp. VIII. R.

Genus *Pleurococcus* Menegh.

1. *P. vulgaris* Menegh. Everywhere abundant.
2. *P. nimbatus* Wildem. [*Tetracoccus nimbatus* (Wildem.) Schmidle]. IV. Ks.

Genus *Trochiscia* Kütz.

1. *T. aciculifera* (Lagerh.) Hansg. VI. N.

*2. *T. aspera* (Reinsch) Hansg. III. Wi.

*3. *T. stagnalis* Hansg. III. C.

*4. *T. reticularis* (Reinsch) Hansg. IV. Ks.

Genus *Protococcus* Ag.

1. *P. viridis* Ag. II. No. III. Esher, Dorking.

Class MYXOPHYCEÆ.

Ord. *Hormogoneæ.*

Sub-ord. Heterocysteæ.

Fam. **Rivulariaceæ.**

Genus *Gloiotrichia* J. Ag.

1. *G. Pisum* (Ag.) Thur. III. Th. V. Go.

Fam. **Sirosiphoniaceæ.**

Genus *Hapalosiphon* Näg.

1. *H. hibernicus* West & G. S. West. VI. N.

*2. *H. intricatus* West. III. Pu.

Genus *Stigonema* Ag.

1. *S. ocellatum* (Dillw.) Thur. VI. N. VIII. T.

Fam. **Scytonemaceæ.**

Genus *Scytonema* Ag.

1. *S. cincinnatum* (Kütz.) Thur. VI. On stones in stream near Lyndhurst, New Forest (G. Massee).
   Crass. fil. 20–23 μ; crass. trich. 14–16 μ.
2. *S. figuratum* Ag. III. Ew.

Genus *Tolypothrix* Kütz.

1. *T. tenuis* Kütz. I. Ep. III. Fg.
2. *T. lanata* (Desv.) Wartm. III. Fg.

Fam. **Nostoceæ.**

Genus *Nostoc* Vauch.

1. *N. microscopicum* Carm. III. M., Mp. V. Go.

Genus *Anabæna* Bory.

1. *A. inæqualis* (Kütz.) Born. et Flah. I. Ep. (spores up to 5-seriate).
2. *A. oscillarioides* Bory. V. Go.

Genus *Aphanizomenon* Morren.

1. *A. flos-aquæ* (L.) Ralfs. III. M.

Genus *Cylindrospermum* Kütz.

1. *C. stagnale* (Kütz.) Born. et Flah. III. Pu.

Sub-ord. Homocysteæ.

Fam. **Camptotricheæ.**

Ammatoidea gen. n.

Plantæ filamentosæ et epiphyticæ; fila longa e serie regulari singula cellularum formata, haud ramosa, prope medium subito et acute genuflexa circa ramos hospitis, extremitates ambas versus gradatim et gradatim attenuata, flexuosa et vaginata; vagina firma, arcta, et lamellosa, achroa vel luteo-fusca in partibus vetustis; trichomata ad extremitates ambas in pilum longum attenuata, ad dissepimenta constricta; cellulæ subquadratæ (vel diametro breviores) prope medium filamentorum, extremitates versus gradatim longiores (diametro usque ad 6-plo longiores); protoplasma ærugineum, granulosum; fila secundaria ubi genuflexa, in filis primariis sedentes et iis similes.

1. A. Normanii sp. unica. (Pl. VII. figs. 25–28.)

Character idem ac generis.

Crass. fil. ad med. 5·5–12·5 μ; crass. trich. 3·5–5·5 μ.

VII. Dartmoor (coll. T. Norman), epiphytic on *Batrachospermum moniliforme.*

The long attenuation of both extremities of the filaments readily distinguishes this genus from all others and places this plant in the Camptotricheæ. The filaments are very different from those of the only other genus (*Camptothrix*) of this order, and at first sight much resemble those of a *Calothrix.* The false ramification is most peculiar, the geniculate flexure of one filament being merely applied to the inner lamellæ of the sheath of another, and included in the outer lamellæ of the sheath. The plant is attached to its host by the medium acute flexure of the filaments around either one of the strands of the axis or one of the small lateral branches (*vide* fig. 25).

The discovery of this genus necessitates the modification of the Fam. *Camptotricheæ* as follows:—

Fam. Camptotricheæ. Fila epiphytica et aquatica, brevia vel longa, vaginata, haud ramosa, irregulariter flexuosa, *extremitates ambas versus attenuata,* serie singula cellularum intra vaginam unamquamque formata, vaginis delicatis vel firmis, achrois vel luteo-fuscis.

Gen. 1. *Camptothrix.* Fila brevissima, e serie singula cellularum irregularium formata, extremitates versus paullo attenuata, submoniliformia; vagina delicata et hyalina; protoplasma cellularum homogeneum.

Gen. 2. *Ammatoidea.* Fila longa, e serie singula cellularum regularium formata, subito genuflexa, in medio et extremitates versus longissime attenuata; vagina firma et sæpe luteo-fusca; protoplasma cellularum granulosum.

Fam. Vaginarieæ.

Genus *Schizothrix* Kütz.

1*. *S. vaginata* (Näg.) Gomont.
Articuli sæpius diametro trichomatis paullo longiores. Crass. trich. 2·5–3 μ. VI. N.

Genus *Dasyglœa* Thwaites.

1. *D. amorpha* Berkeley. III. Th. (a form with thinner trichomes). VI. N.

Fam. Lyngbyeæ.

Sub-fam. Lyngbyoideæ.

Genus *Lyngbya* Ag.

*1. *L. putealis* Montagne. III. Rp.
2. *L. Martensiana* Menegh. III. Dj.
*3. *L. ærugineo-cœrulea* (Kütz.) Gomont. III. Kingston to Esher.
*4. *L. versicolor* (Wartm.) Gomont. III. Dorking.
5. *L. ochracea* (Kütz.) Thur. III. Cr.

Sub-fam. Oscillatorioideæ.

Genus *Phormidium* Kütz.

*1. *P. molle* (Kütz.) Gomont. III. Rp. (on *Myriophyllum*).
*2. *P. foveolarum* (Mont.) Gomont. III. In chalk pit, Dorking.
*3. *P. tenue* (Menegh.) Gomont. II. Pi. III. Bi., Fg., Wi. IV. Hc.
4. *P. autumnale* (Ag.) Gomont. III. Wi. Esher, Thames Ditton, Kingston.

Genus *Oscillatoria* Vauch.

*1. *O. prolifica* (Grev.) Gomont. III. Ra.
2. *O. princeps* Vauch. III. Th. IV. Ks. VIII. H.
3. *O. limosa* Ag. (*Frœlichii* Kütz.) Frequent.
4. *O. irrigua* Kütz. III. Ew., Th. IV. Ks.
*5. *O. simplicissima* Gomont. III. Wi.
6. *O. tenuis* Ag. II. Wh. VI. N.
7. *O. amphibia* Ag. (*O. tenerrima* Kütz.) II. Wh. III. E., Th., Wi.
*8. *O. angustissima* West and G. S. West. III. Wi.
9. *O. splendida* Grev. IV. Ks.
   Var. *attenuata* West and G. S. West. II. No.
*10. *O. formosa* Bory. IV. Ks.

Genus *Spirulina* Turp.

1. *S. major* (Kütz. (*S. oscillarioides* Kütz.). I. Ep. II. Ru. III. Pu. VIII. H., W.

Ord. *Chroococcoideæ.*

Fam. **Chroococcaceæ.**

Genus *Glœothece* Näg.

1. *G. confluens* Näg. III. Th.
2. *G. linearis* Näg. III. Th. VI. N. VII. D. VIII. T.
3. *G. cystifera* (Hass.) Rabenh. III. Mp.

Genus *Aphanothece* Näg.

1. *A. microscopica* Näg. III. Fg., Th. IV. Ks. VII. D. VIII. W.
2. *A. saxicola* Näg. III. Fg., Th. VII. S. VIII. W.
   Var. *violacea* West. VI. N.
3. *A. prasina* A. Br. III. Fg., Th., Wt.

Genus *Synechococcus* Näg.

1. *S. major* Schroet. (*S. crassus* Arch.). I. Ep. IV. Ks.
2. *S. æruginosus* Näg. VI. N. VII. D. VIII. T.

Genus *Glaucocystis* Itzigsh.

1. *G. Nostochinearum* Itzigsh. I. Ep. III. Fg., Th. IV. Ks. VII. D. VIII. L.

Genus *Merismopedia* Meyen.

1. *M. glauca* (Ehrenb.) Näg. III. C., E., Ra., Rp., Th., Wt. VII. D., Tq. VIII. H., L., T., W.
2. *M. violacea* (Bréb.) Kütz. II. U. III. F. VIII. K.
3. *M. æruginea* Bréb. V. Go.

Genus *Tetrapedia* Reinsch.

1. *T. Reinschiana* Arch. I. Ep. II. Wh. III. M., Mp., Rp., Th. V. Go.

Genus *Gomphosphæria* Kütz.

1. *G. aponina* Kütz. III. Fg., Th. IV. Ks. V. Go.

Genus *Clathrocystis* Henfrey.

1. *C. æruginosa* (Kütz.) Henfrey. III. Fg.

Genus *Polycystis* Kütz.

1. *P. marginata* (Menegh.) Richter. II. U. III. Mp.
*2. *P. elabens* (Bréb.) Kütz. VI. N.
*3. *P. flos-aquæ* Wittr. III. Th. VI. N.

Genus *Aphanocapsa* Näg.

1. *A. pulchra* (Kütz.) Rabenh. III. Th.

Genus *Porphyridium* Näg.

1. *P. cruentum* (Ag.) Näg. III. Clapham, Esher.

Genus *Chroococcus* Näg.

1. *C. minor* (Kütz.) Näg. III. C., Th. IV. Ks. VII. D. VIII. L., W.
2. *C. cohærens* (Bréb.) Näg. III. Dj. VI. N. VIII. H.
3. *C. turgidus* (Kütz.) Näg. I. Ep. III. C., Ew., Th. IV. Ks. VIII. K., T.
4. *C. pallidus* Näg. III. Th.
5. *C. rufescens* (Bréb.) Näg. VIII. W.

Summary.

	Genera.	Species.*
Class FLORIDEÆ.		
Fam. *Batrachospermeæ* .	1	2
Class CHLOROPHYCEÆ.		
Ord. **Confervoideæ Heterogamæ.**		
Fam. *Coleochætaceæ* . .	1	3
Fam. *Œdogoniaceæ* . .	2	21
Ord. **Siphoneæ.**		
Fam. *Vaucheriaceæ* . .	1	6
Fam. *Hydrogastraceæ* .	1	1

* Excluding varieties and forms.

	Genera.	Species.
Ord. **Confervaceæ Isogamæ.**		
Fam. *Ulvaceæ*	2	2
Fam. *Ulotrichaceæ.*		
Sub-fam. Ulotricheæ .	3	8
Sub-fam. Chætophoreæ	5	11
Sub-fam. Conferveæ .	2	6
Fam. *Chroolepidaceæ* .	1	2
Fam. *Cladophoraceæ* . .	1	3
Ord. **Conjugatæ.**		
Fam. *Zygnemaceæ.*		
Sub-fam. Mesocarpeæ .	2	10
Sub-fam. Zygnemeæ .	3	22
Fam. *Desmidiaceæ* . .	23	333
Ord. **Protococcoideæ.**		
Fam. *Volvocineæ* . .	6	6
Fam *Palmellaceæ.*		
Sub-fam. Cœnobieæ .	5	16
Sub-fam. Pseudocœnobieæ	3	4
Sub-fam. Rhaphidieæ .	6	22
Sub-fam. Characieæ .	1	6
Sub-fam. Tetrasporeæ .	5	9
Sub-fam. Dictyosphærieæ	5	6
Sub-fam. Glœocystideæ	6	17
Sub-fam. Protococcaceæ	5	11
Class MYXOPHYCEÆ.		
Ord. **Hormogoneæ.**		
Sub-ord. Heterocysteæ.		
Fam. *Rivulariaceæ* . .	1	1
Fam. *Sirosiphoniaceæ* . .	2	3
Fam. *Scytonemaceæ* . .	2	4
Fam. *Nostoceæ* . . .	4	5
Sub-ord. Homocysteæ.		
Fam. *Camptotricheæ* . .	1	1
Fam. *Vaginarieæ* . .	2	2
Fam. *Lyngbyeæ.*		
Sub-fam. Lyngbyoideæ	1	5
Sub-fam. Oscillatorioideæ	3	1
Ord. **Chroococcoideæ.**		
Fam. *Chroococcaceæ* . .	12	25
Total . .	118	588

APPENDIX.

The following notes of localities for Freshwater Algæ in Middlesex (M.) and Surrey (S.) were made with the idea of preparing an 'Algologia Metropolitana,' a scheme which circumstances have compelled me to abandon. This short list may, however, be of use in stimulating others to note the forms of life which may be found close to their own doors. It is, as far as I am aware, almost the first record of Freshwater Algæ gathered within the limits of London itself.

ALFRED W. BENNETT.

Coleochæte scutata Bréb. S. Kew Gardens.

Vaucheria sessilis DC. M. Botanic Gardens, Regent's Park; S. Kew Gardens.

Var. cæspitosa Vauch. S. Brown's Millpond, Waddon.

Sphæroplea annulina Ag. S. Kew Gardens, occasional.

Microspora vulgaris Rabh. M. Regent's Canal, Camden Town.

Cladophora flavescens Ag. M. Botanic Gardens, Regent's Park.

„ fracta Ktz. M. Regent's Canal, Camden Town.

Spirogyra nitida Lk. M. Regent's Canal, Camden Town. S. Brown's Millpond, Waddon.

Penium digitus Bréb. S. Brown's Millpond, Waddon.

Closterium lunula Nitzsch. M. Regent's Canal, Camden Town.

„ Leibleinii Ktz. S. Kew Gardens.

Docidium clavatum Ktz. S. Kew Gardens.

Euastrum didelta Rlfs. S. Brown's Millpond, Waddon.

Tetmemorus granulatus Rlfs. S. Brown's Millpond, Waddon.

Micrasterias denticulata Bréb. S. Brown's Millpond, Waddon.

Arthrodesmus Incus Hass. M. Regent's Canal, Camden Town.

Eudorina elegans Ehrb. S. Brown's Millpond, Waddon.

Hydrodictyon reticulatum Lag. S. Tank, Kew Gardens; abundant in some years.

Pediastrum Boryanum Men. M. Regent's Canal, Camden Town. S. In the Wandle, Beddington.

„ tetras Rlfs. M. Regent's Canal, Camden Town.

„ sp. Cœnobe quite compact; cells deep green, about the size of those of *P. tetras*, each with a single central conspicuous pyrenoid; marginal cells very variable in shape, each with two colourless horns proceeding from the margin, or from an angle very near the margin; tips of horns slightly thickened, but not bidentate. M. Regent's Canal, Camden Town.

Scenedesmus obliquus Ktz. M. Regent's Park.

„ quadricauda Bréb. M. Regent's Canal, Camden Town.

Tetraëdron longispinum Perty. S. Kew Gardens.

Protococcus viridis Ag. M. Regent's Canal, Camden Town.

Anabæna oscillarioides Bor. *var.* elongata. M. Regent's Park.

Oscillatoria limosa Ag. M. Botanic Gardens, Regent's Park. S. Wandle, Beddington.

„ tenuis Ag. M. Botanic Gardens, Regent's Park. S. Wandle, Beddington.

Chroöcoccus cohærens Näg. M. Botanic Gardens, Regent's Park.

Microcystis protogenita Rbh. M. Regent's Canal, Camden Town.

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"Cloud Photography conducted under the Meteorological Council at the Kew Observatory." By Lieut.-General R. Strachey, R.E., F.R.S., and G. M. Whipple, Superintendent of the Observatory. Received April 23, 1891.

In 1878 the Meteorological Council decided upon undertaking a series of experiments with the view of attempting by means of photography to obtain a record of the height and velocity of the clouds, as indicating the movements of the upper parts of the atmosphere. For this purpose a plain cubical camera was constructed, with its optical axis directed to the zenith, and a number of pictures of clouds were thus obtained. The results were so far satisfactory as to establish the possibility of identifying points in the clouds which would admit of the calculation of their height with considerable precision. But, owing to the small field of view of the lens made use of, it was found that the opportunities of photographing clouds in this manner were of somewhat rare occurrence, and it was therefore decided, on the proposal of Captain Abney, to whom the Meteorological Council is indebted for his valuable advice throughout the course of these experiments, to construct two cameras so arranged as to enable them to be directed to any part of the sky, and thus to photograph clouds in all positions.

For this purpose the cameras were fitted with theodolite mountings, provided with altitude and azimuth circles. The dark slides for carrying the sensitised plates were fitted with glass plates, upon which cross lines indicating the position of the optical axis were etched. These lines were photographed simultaneously with the clouds, and the readings of the divided circles, recorded at the time of exposure, thus supplied the altitude and azimuth of the point of the cloud covered by the intersection of the cross lines at that moment.

From a photographic picture of a series of staves erected at known angular intervals, a scale of angular distances was obtained, by means of which the azimuth and altitude of any point in the cloud picture could be deduced from those of the intersection of the cross lines.

Arrangements were made for erecting these cameras at the extremities of a base of known length (800 yards), between which an electrical communication was established.

Spring shutters were placed over the lenses, which could be liberated and again closed, at the will of the observer, by the passage of an electric current, so as to expose the plates for any desired interval of time.

Captain Abney also, after numerous trials, devised a suitable formula for an emulsion for coating the plates, as special precautions were found to be necessary in order to obtain good cloud photographs.

Captain Abney thus describes the photographic process he proposed:—"My attention has been once more directed to the best photographic process to employ for the delineation of the clouds, a certain inconvenience having attached to the use of collodion-emulsion, which at first I had not foreseen. I had then recourse to gelatine plates, but the manner in which they are ordinarily prepared induces a sensitiveness which becomes unmanageable, even when a diaphragm with a small aperture is used in the lenses. The great desideratum in the plates appears to be that a small variation in the intensity of the light proceeding from the sky or cloud shall produce a great contrast in the intensity of the developed image. A very rapid plate does not answer for this purpose; hence I tried several modifications. The process which at present has given the best results is as follows:—

"150 grains of bromide of ammonium and 10 grains of iodide of potassium are dissolved in 3 oz. of water, to which 80 grains of Nelson's No. 1 photographic gelatine and 80 grains of Coignet's gelatine have been added. This is dissolved by the aid of heat, and 200 grains of silver nitrate dissolved in $1\frac{1}{2}$ oz. of water are added. The whole is warmed to 100° F. for five minutes, and allowed to set after being poured out in a flat dish. The emulsion thus produced is washed (in the usual manner) from the soluble salts, and is then re-melted and plates coated and dried, as is customary in the gelatine process.

"This formula gives very constant results, and great contrasts of image are obtained by careful development."

The years 1881 to 1884 were passed in working out the details of the arrangements above described, and in 1885, after numerous preliminary trials, it was resolved to erect the two cameras at the Kew Observatory. One was placed on the roof of the Observatory building, and the other on a stand in the Old Deer Park, 800 yards from the other, on the road leading to the Observatory from Richmond; and a telegraph cable carrying two insulated copper wires of low resistance, buried a few inches below the surface of the ground, was laid between the two stands. Switches, attached to telephones as well as to an electric battery, were fixed to these stands, and wires were arranged on the cameras, so that the observers could either communicate with one another, or work the exposing shutters of the two cameras at will.

Operations for the determination of cloud height and motion were then carried out on suitable occasions, as follows :—The two observers, termed for convenience A and B, proceeded to their respective stations, each provided with a box containing half-a-dozen dark slides charged with sensitised plates, and also an adjusted watch. The cameras were set up on the pedestals, levelled, and the connecting wires joined up. Locking plates of peculiar construction were provided, which ensured that the zero points in azimuth of both cameras were exactly directed to the same point of the horizon.

The observer at A, when he saw B had reached his station and placed his camera on the pedestal ready for use, attracted B's attention by means of a flag waved overhead, and directed him through the telephone to set the instantaneous shutter of his camera, setting that of his own camera at A at the same time. A then, making use of the push, sent a current of electricity through the two cameras, which should liberate both shutters at the same instant of time. An enquiry was immediately made through the telephone of B, and, if the reply assured A that the shutters were working satisfactorily, the observers proceeded to the second stage of the observation, which was as follows :—

A carefully examined the sky and, selecting a suitable cloud, directed the sights on his camera towards it, making a convenient setting of the horizontal and vertical circles, which he then read off. He then told B to set his camera to the same azimuth and altitude, and insert a loaded plate-holder in its groove, repeating the circle readings to ensure accuracy, and also at the same time to set his shutter. A, whilst directing B through the telephone, conducted the same series of operations at his own instrument, so that, as soon as B telephoned that he was ready for action, A switched the battery on to the line, and, watching the cloud for a favourable instant, touched the push, whereby the two plates were exposed simultaneously, the instant of the exposure being recorded by both observers in their respective note-books. They then quickly exchanged their plate-holders for others containing fresh plates, and again set the shutters, so that by the time sixty or seventy seconds had elapsed since the first exposure was made they were ready for a second, which was carried out as before under the directions of A, both observers again noting the time. After this, A, having switched on the telephones, enquired of B if he had obtained the two pictures. If the reply was in the affirmative, he was directed to read both his circles, and to enter the readings, with the times of the two exposures and the numbers of the plate-holders in his book, A doing the same for his own instrument.

Having deposited the plate-holders in the light-tight carrying box, another charged pair were taken, and a fresh cloud in another part of the sky selected, and the operations already detailed were repeated, until the stock of charged holders was exhausted.

The observers then, by means of the telephones, again compared their watches, and noting their differences, if any, sighted their cameras on each other, and read their mutual bearings and altitudes. This was done in order to be sure no displacement had taken place in either the orientation or level of the instruments. They then unlocked the stands, dismounted the cameras, and put them away in the lockers of the pedestals, ready for use on another occasion, conveying the plates to the photographic laboratory for development and subsequent treatment.

From time to time, the empty plate-holders were taken out, the lenses directed to each other, and settings made and circles read with the view of determining the true bearings of the fiducial lines before described, from which the angular position of the cloud-points dealt with were obtained.

On removal of the exposed plates from the holders, the dates of the observation having been written on each of the films in pencil, as well as a register number, development proceeded. This was conducted in a wooden tray with a glass bottom specially adapted to hold four plates. The two A's and two B's forming one set of pictures were usually selected for simultaneous development, in order that the negatives obtained might possess the same degree of intensity. Before hydrokinone became an article of commerce, a solution of pyrogallic acid or sulphate of iron was employed as the developing agent, but, since 1889, Edwards's hydrokinone developer has been employed by preference, as being less liable to produce fogged plates.

Owing to the efforts of the Kew observers being chiefly directed to photographing high cirrus clouds, very careful and slow development was required, to produce satisfactory negatives, and it has been generally necessary to continue the operation for about forty minutes to bring out a successful result. In some cases of very thin filmy cirrus, the so-called mare's tail clouds, the development occupied $1\frac{1}{2}$ hours, before the picture appeared.

For discussion of the photographs, in most cases prints were made of the negatives by the ordinary albuminised paper process.

Various methods of obtaining the heights and velocity of motion of the clouds from the photographs thus made have been attempted. The computation by the ordinary trigonometrical formulæ from the azimuths and altitudes derived by measurement of a series of points in the clouds, properly identified in the sets of pictures, is very tedious, and a graphical method was suggested by Sir G. Stokes, which, though very ingenious, was found to be troublesome in practice, and was not persevered in.

From the nature of the process employed, the indefinite outlines of the clouds, and their incessant change of form, complicated by the effects of perspective distortion on an irregular and ill-defined surface, it is necessarily impossible to identify cloud-points in the different pictures with much precision or make exact measurements; and approximate results, therefore, are all that can be sought for. The object of the enquiry is chiefly to determine the velocity of movement of clouds at varying heights above the earth's surface and to obtain the heights of those observed at the greatest elevations, which appear as cirrus.

If A and B are the azimuths of any point in a cloud, and Z_a and Z_b the zenith distances, observed respectively at A and B, the ends of the base β, then the distances, measured in a horizontal plane passing through the base, D_a, D_b from A and B respectively of the point vertically under the cloud-point will be

$$D_a = \beta \frac{\sin (B)}{\sin (A-B)}, \qquad D_b = \beta \frac{\sin (A)}{\sin (A-B)},$$

and H, the height of the cloud-point above the horizontal plane passing through the base, will be

$$H = \beta \frac{\sin (B)}{\sin (A-B) \tan Z_a} = \beta \frac{\sin (A)}{\sin (A-B) \tan Z_b}.$$

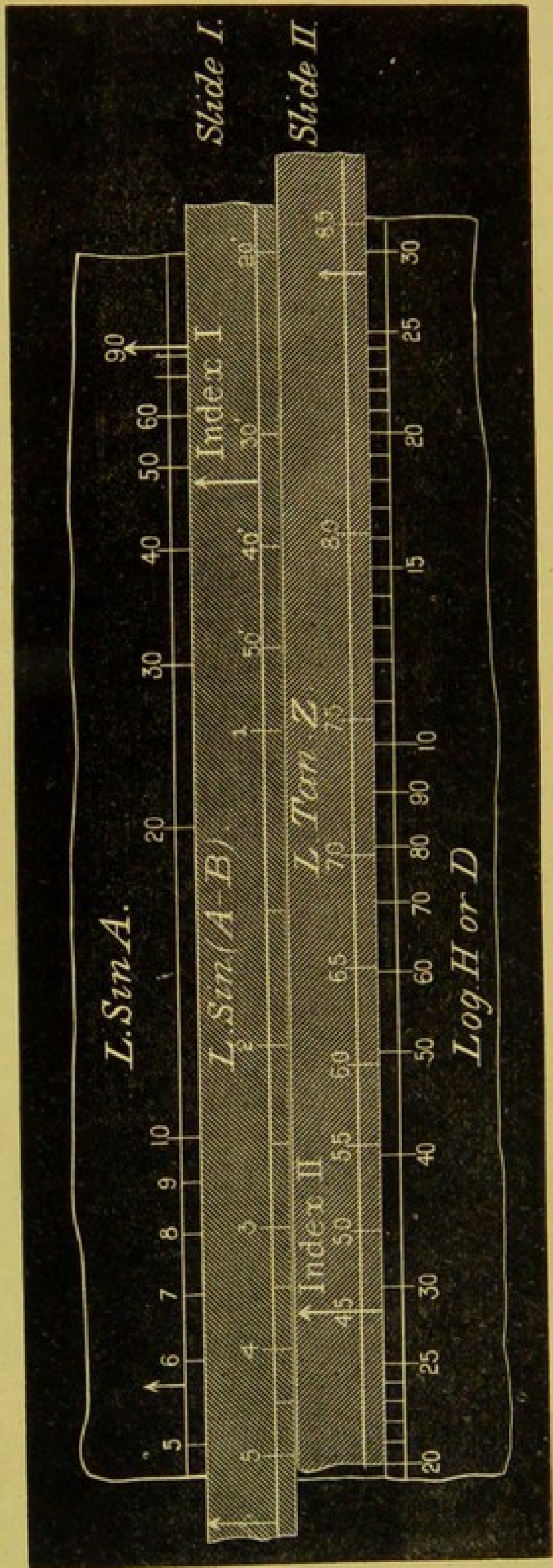
These values are readily found by means of a slide-rule constructed as shown below. The graduations of the upper scale of the fixed rule are log sines; those of the lower scale of the fixed rule logs of numbers, the log of 2400 feet, the length of the base, coinciding with log sin 90°.

The upper sliding rule No. I is graduated with log sines of small angles on the same scale as the first rule, the point marked with index No. I indicating log sine 5° 44′ 27″, which is 9·00000, or 0° 34′ 23″, which is 8·00000.

The lower sliding rule No. II is graduated with log tangents Z, the point marked with index No. II, corresponding to log tan 45°, and on the same scale as the sines.

To apply the rule, bring index No. I of the slide-rule No. I opposite the angle A on the upper fixed scale. Then bring the index No. II of the slide-rule No. II opposite to the angle A − B on the slide-rule No. I.

Fig. 1.



Opposite the index No. II, or tan 45°, will be found on the lower fixed scale the distance D_b, in feet; and opposite to the angle Z_b will be found on the same scale the height of the cloud in feet. By a similar process will be found the distance D_a and a height of the cloud determined from Z_a.

The position of the point vertically under the selected cloud-point will be determined with sufficient accuracy graphically, by the intersection of the two distances measured from the ends of a line drawn to represent the base.

The repetition of this process for the second set of photographs will in like manner give the position of the cloud-point after the interval elapsed between the taking of the two sets of pictures, and the distance travelled being measured on the diagram, the velocity can be found, and the direction of motion will be shown in relation to the direction of the base.

Irrespective of the laborious nature of this process, it was found that the angles on which it was based were often so small that the results obtained were inconsistent and unreliable.

In 1890, therefore, it was decided to try another method of observing, which would admit of much simpler treatment. This was to fix the cameras so that the optical axes were directed to the zenith, and to photograph clouds which passed across the field of view which is comprised within a circle described at an angular distance of about 15° round the zeniths of the two stations. The defect of this method is that it very materially limited the scope of operations, and reduced the opportunities of taking pictures to a comparatively small number, for it was found that a large proportion of the clouds which seemed apparently favourable for photographing when viewed by reflected solar light incident upon them at oblique angles became almost invisible when observed directly overhead. This was notably the case with cirrus, some forms of which, especially those possessing the nature of cirro-stratus, appear as practically structureless masses when seen in this position. But notwithstanding these drawbacks, some of which, it is hoped, may be obviated, the advantages of this method of observing seem to be sufficient to lead to its adoption in preference to any other yet suggested.

To adapt the cameras for work in this manner, both altitude and azimuth circles were permanently clamped, rendering them immovable in both vertical and horizontal planes, and the locking plates were shifted on the pedestals, so that, while the fiducial lines on the pictures intersect at the zenith, the direction of one of them is that of the line joining the two stations, or the base, the other being at right angles to it.

With the object of ensuring the proper adjustment of the optical axes of the cameras, a tripod stand 12 feet in height was made, which was temporarily erected immediately over them. A plummet was suspended directly above the lens-centre, from the point of intersection of two horizontal wires fixed at right angles to one another, one of them being carefully made to coincide in direction with the line joining the two cameras.

The charged dark slides, which are separately numbered, so that the correction for each of them may be ascertained and recorded, are then successively placed in the camera and photographs taken of the cross wires overhead, the pictures of which should coincide with the fiducial lines of the camera, the position of which is as nearly as possible adjusted to secure this coincidence. The photographs thus made are preserved, to supply data for correcting the negatives for any error of the fiducial lines, should the slides not be properly adjusted so as to secure the coincidence before spoken of.

Assuming, as may be done without objection for this purpose, that the cloud surface photographed and the earth's surface at the place of observation are in parallel planes, distances measured on the photographs from the intersection of the fiducial lines will represent tangents of angles measured from the zenith to radius equal to the height of the cloud.

Again, if a pair of photographs made simultaneously at the extremities of the base are superimposed one on the other, so that the forms of the clouds coincide, which they will do accurately if the pictures are properly placed, then the line joining the intersections of the cross lines will represent, both in magnitude and direction, the line joining the zeniths of the two ends of the base, from which the observations are made, or the base itself.

If the adjustments before described have been satisfactorily made, the base, as thus indicated, should obviously fall on one pair of the fiducial lines, which, when the photographs are superimposed, should also coincide; otherwise, if the fiducial lines in the two pictures are made to coincide, then the separation of points properly identified in the pictures will be the measure of the parallax or angle subtended by the base at such points.

A scale of angular distance having been prepared as before explained, the parallax thus measured may at once be converted into angular measure, and the height of the cloud is given by the equation

$$H = \beta/\tan\pi,$$

where π is the angular parallax.

In like manner, if two photographs taken from the same point with an interval of time between them be superimposed, so that the cloud pictures coincide, the line joining the intersections of the cross lines will represent in magnitude and direction the movement or drift of the cloud, and the velocity in miles per hour will be found from the equation

$$V = \frac{\delta}{p} \times \frac{\beta}{5280} \times \frac{3600}{t''},$$

where δ and p are the drift and parallax as measured on the photographs, and t the interval in seconds between the pictures being taken.

The method of reduction of the photographs first adopted and employed during the early part of the past summer was as follows :—Prints were made on albuminised paper of the set of four pictures, two taken at each end of the base with an interval of time between them, and they were mounted on stout cards in order to avoid the usual curling up of the paper. When necessary, new fiducial lines were then drawn in the proper direction through the points that had been ascertained to represent the corrected position of the lines of reference as before described, and these lines were extended to the margins of the cards.

If possible, five or six cloud-points were then selected in each print, capable of satisfactory identification. A sheet of paper was next procured, larger than the pictures, and lines intersecting at right angles were drawn across it. Punctures were then made, by means of a needle, through all the selected cloud-points in the four pictures, which were successively placed over the reference sheet (termed hereafter the receiver), so that the fiducial lines upon the pictures coincided with the lines drawn upon the receiver, thereby ensuring the points of intersection being directly superimposed, and, by means of a needle passed through the pricked holes, the marked cloud-points were transferred to the receiver.

This having been done in turn for all the four pictures of the set, the points thus pricked off were joined by inked lines, those obtained from the pair of pictures taken simultaneously being drawn in black ink, and those from the other pair in red, by which a series of parallelograms was formed, equal in number to the number of points selected for treatment.

The black lines or sides of these parallelograms then represented the parallax of the several cloud-points, being proportional in length to the tangent of the angle subtended by the base line at the altitude of the cloud, whilst the red lines forming the other two sides of the quadrilaterals represented on the same scale the drift of the cloud during the interval which elapsed between the taking of the two sets of pictures.

The measurement of these black and red lines provided the means already explained of determining the height of the clouds and the rate of their motion, the direction being given by the inclination of the two lines, of which the black one represented the base.

In dealing with the direction of the drift when thus obtained from positive prints, it has to be remembered that by the printing the right and left of the pictures are transposed, so that the east is on the left and the west on the right in a picture the top of which is directed to the north.

The necessary measurements were made on a scale of millimeters, and the computations carried out by the help of logarithms.

The operations thus described have lately been much abbreviated in various ways. First, it has been found possible to carry out the superposition of the pictures by means of the negatives only, and to work without either employing positives or depending on the identification of a few selected points whose positions were transferred to a receiver.

A frame has been constructed which carries the glass negative plates upon sliders in grooves running in parallel planes, one immediately over the other, but arranged so as to travel at right angles to one another, the lower moving towards and away from the observer, whilst the upper traverses from right to left. A mirror, either a silvered or an opal plate, is employed to reflect the light of the sky upwards to the eye through the negative photograph when the apparatus is placed upon a table in front of a well-lighted window. Stray or diffused light is excluded by placing a box, darkened on its inner surface, over the negatives, and the observer views the combination through a tube fixed perpendicularly upon the top of the box. The two photographs to be compared are placed one in each of the sliding frames, which are first so adjusted that the fiducial lines which follow the direction of the base pass exactly over one another. Next, the bottom or backwards-and-forwards slider is moved until the cloud pictures, say a pair marked A and B, are seen to coincide, and the distance between the intersections of the cross lines on the two plates representing the zenith points, which is the parallax, is then measured by means of a pair of compasses; but a scale could readily be fixed on the slides from which the parallax could be read off without measurement.

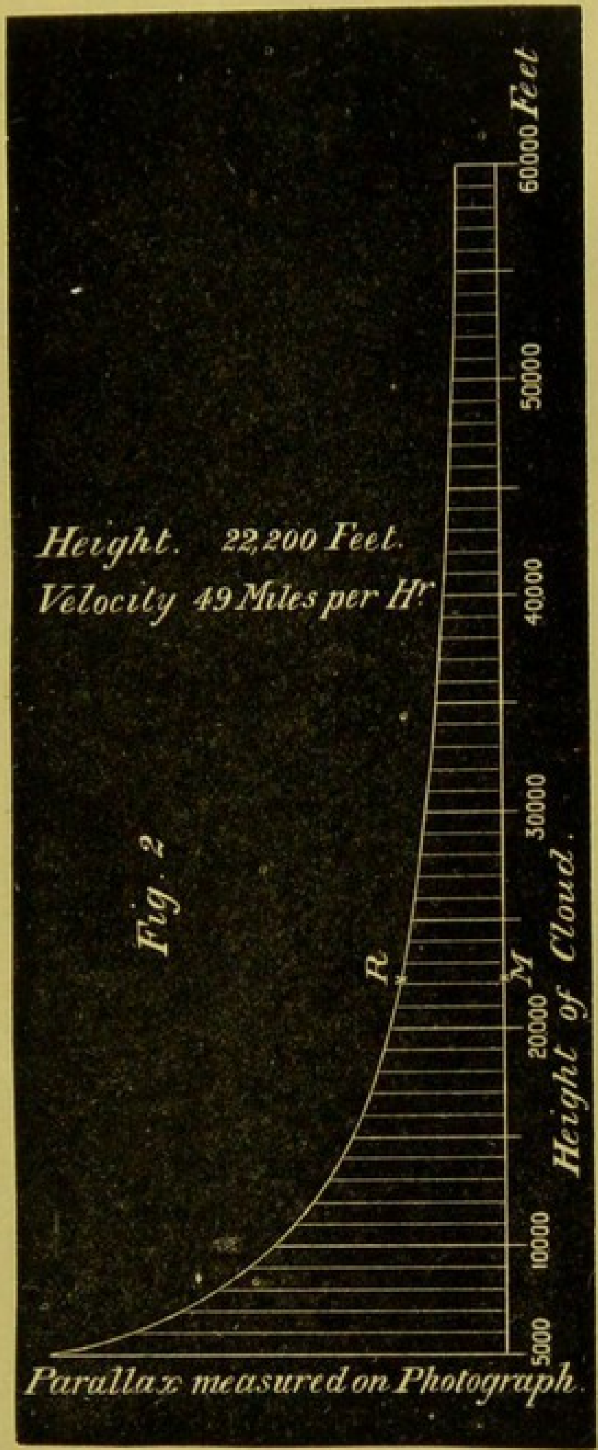
In order to avoid calculations, a standard curve has been drawn (see fig. 2), from which the height of the cloud may at once be graphically determined from the distance between the intersections of the cross lines or parallax of the base as thus measured.

On the axis of abscissæ of this curve are marked off the heights on a scale which makes 2400 feet, the length of the base, equal to the focal distance of the camera, and at regular intervals along this line ordinates are drawn of the length, as measured on the photographs, of the parallax corresponding to the several heights. Through the extremities of these ordinates a curved line is drawn, which gives the locus of the equation

$$h = p \cot \pi,$$

the lengths h and p being both expressed on the scale just mentioned.

The same operations are next performed with pictures A_2 and B_2,



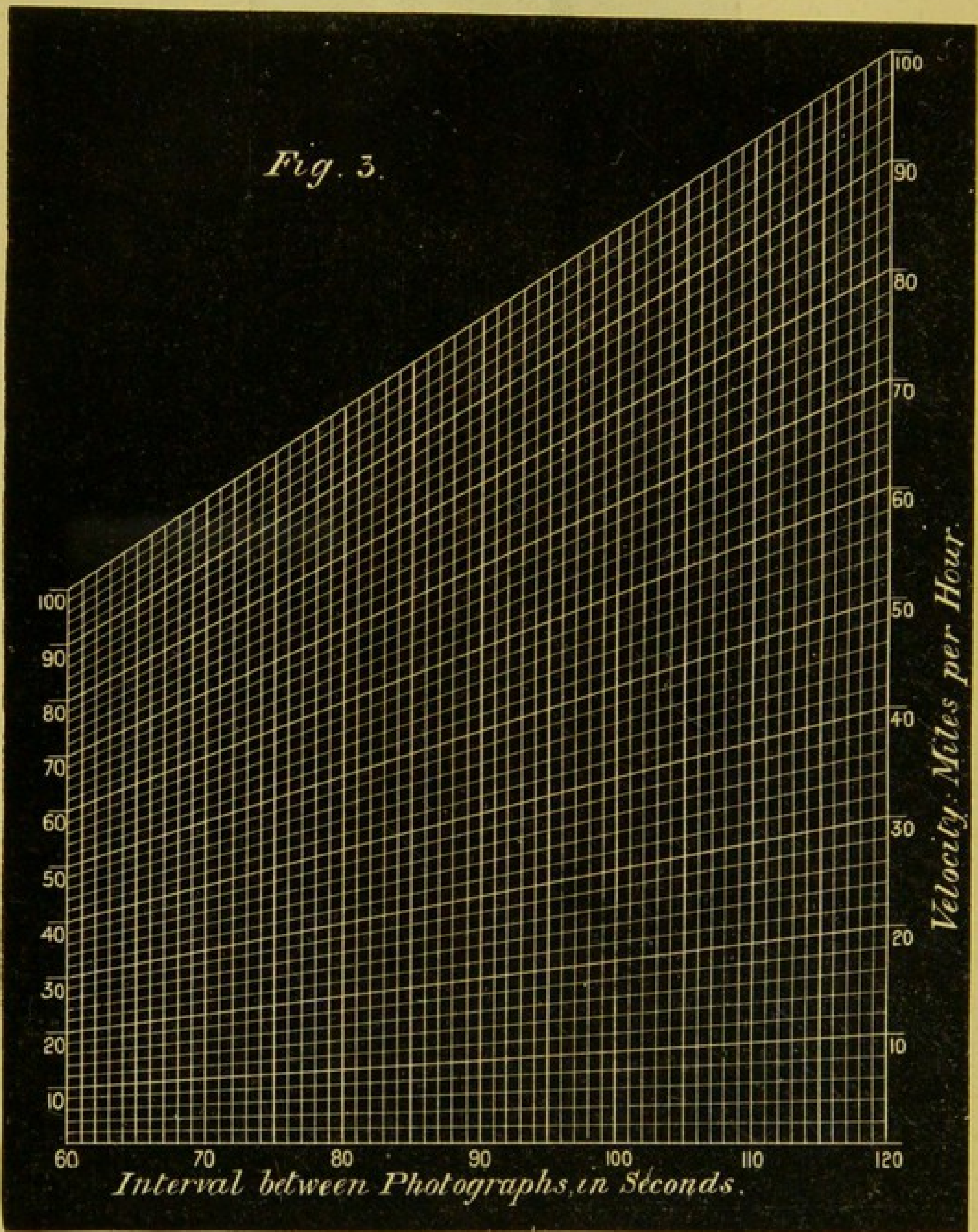
and a second value of the cloud height is obtained, which serves to confirm or modify the first determination.

Then pictures A_1 and A_2 are placed in the frame, and the images superimposed and made to coincide as before, but now the distance separating the zenith of the two pictures, which will be termed the drift, will indicate the space the cloud has moved during the interval between the taking of the two pictures; and the angle which the line joining the zeniths makes with the line of base gives the direction in which the drift has taken place.

From the length of the drift measured upon the plates as above, the velocity of motion may easily be obtained by a graphical method.

As before stated, the velocity in miles per hour is

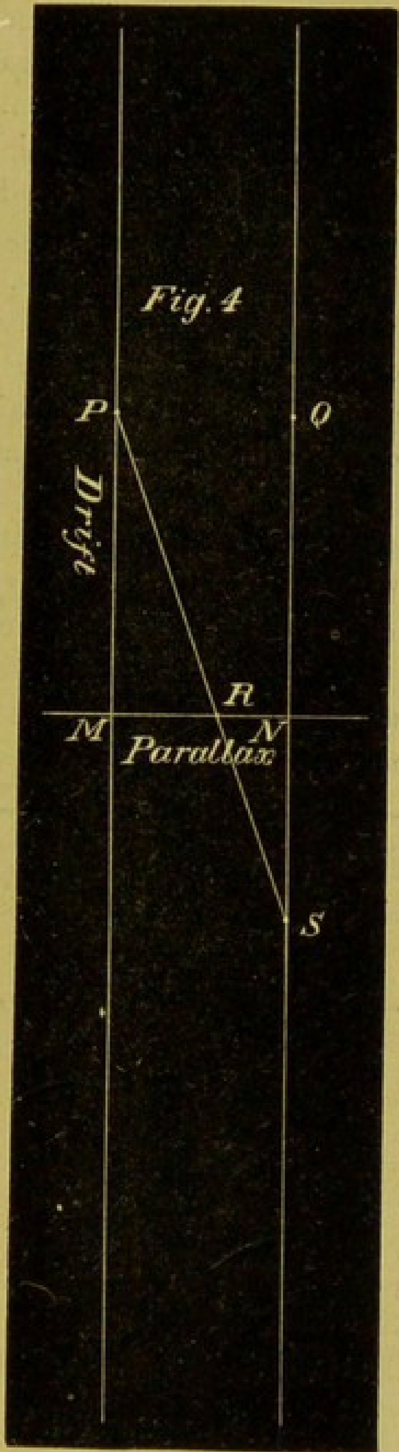
$$V = \frac{\delta}{p} \cdot \frac{\beta}{5280} \cdot \frac{3600}{t''}$$



To obtain the value of V graphically, proceed as follows:—

Draw a horizontal line on which will be represented equal time-intervals from 0 to 120 seconds, see fig. 3. Erect vertical lines at all the points between 60 and 120 seconds, which will include all the time intervals between the pictures likely to occur in practice. On the first of these verticals mark off any convenient length to represent

1 mile, and divide it into 60 equal parts, and from the zero point on the horizontal line draw radiating lines through the points of division, extending to the vertical at 120 seconds. This constitutes a scale of proportional velocities from 0 to 60 miles per hour, and may be extended to any higher velocity. Next (see fig. 4) draw two parallel



vertical lines at a distance apart equal to the length of the base, 2400 feet, on the scale before assumed to represent 1 mile, and draw a horizontal line intersecting the other two at right angles at points M and N.

Then mark off the length of drift δ upwards on each of the two vertical lines from M and N at points P and Q; and the length of the parallax p, on the horizontal line from M towards N, at a point R. Join P, R, intersecting the vertical through N at S. Then QS represents the drift on the scale assumed to represent 1 mile. Let this be marked off upwards on the vertical line drawn on the scale of proportional velocities, fig. 3, from the seconds division corresponding to the time interval between the pictures, and the velocity of drift will be indicated by the radiating line nearest to the mark thus made.

The scales above described for the graphical determination of the cloud heights and velocities are engraved and printed on sheets of paper, which, after the computations are completed by their aid, will serve as convenient records of the observations.

After a little practice, the whole of the processes requisite for these determinations from the glass plate-negatives of a complete set of four pictures will not exceed 20 minutes. Quite sufficient accuracy is ensured, and the labour and risk of error arising from the use of tables is entirely avoided.

Although the cameras now in use only embrace a circle of angular diameter of about 30°, trials have been made with a lens which gives satisfactory pictures of double that extent, which is probably as much as could be desired.

The following is a list of the determinations made during the past year by the methods now described:—

Date.	Height.	Velocity.	Direction.	Surface.	
				Velocity.	Direction.
1890.	miles.	miles.		miles.	
July 10	1·29	7·27	N.W.	10	N.W.
„ 16	5·20	45·80	S.W.	5	S.W.
	5·47	41·39	S.W.	5	S.W.
„ 16	8·39	64·61	S.W.	5	S.W.
	6·34	49·16	S.W.	5	S.W.
August 26	2·87	15·19	S.S.E.	15	S.W.
„ 26	1·64	20·19	S.S.E.	15	S.W.
„ 29	1·97	13·70	W.S.W.	7	N.
„ 29	1·93	13·28	W.S.W.	7	N.
September 9	6·87	42·40	W.	3	W.S.W.
„ 9	6·29	45·18	W.	3	W.S.W.
„ 10	7·22	42·00	N.	8	W.N.W.
„ 17	2·60	25·90	S.S.W.	10	S.E.
	2·66	19·90	S.S.W.	10	S.E.
„ 17	2·87	19·70	S.S.W.	10	S.E.
	2·27	22·00	S.S.W.	10	S.E.
„ 18	4·60	54·40	S.W.	16	S.
„ 18	4·60	53·10	S.W.	16	S.
„ 23	1·72	5·30	S.W.	5	S.
„ 23	1·71	6·40	S.W.	5	S.

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THE THEORY OF RELATIVITY.

BY

HENRY L. BROSE, M.A. (Oxon.), B.Sc. (Adel.)

INTRODUCTION.

1. THE MECHANICAL THEORY OF RELATIVITY
2. THE " SPECIAL " THEORY OF RELATIVITY.
3. THE GENERAL THEORY OF RELATIVITY.

PREFATORY NOTE.

The following pages represent the substance of a lecture which was given in the Clarendon Laboratory for the Science Colloquium. Friends who were present expressed the wish that the notes might be made available to a wider circle, and it is acting on this suggestion that I am venturing to have them published in pamphlet form. The object is to give a non-mathematical description of some of the leading ideas of the theory of relativity and to emphasise the successive steps in its development.

I wish to express my great indebtedness to Einstein's writings above all (the enunciations and definitions are taken from him), as well as to those of Weyl, Laue and Erwin Freundlich: a booklet by Freundlich entitled Die Grundlagen der Einsteinschen Gravitations theorie* is in course of publication by the Cambridge University Press, and will be found to fulfil the requirements of those who wish to avoid the cumbersome mathematics.

The philosophical aspects of relativity are clearly presented in a booklet by Schlick (of Rostock), Raum und Zeit in der gegenwärtigen Physik,** an English version of which is also to be issued shortly. The last chapter of

* The Foundations of Einstein's Theory of Gravitation.
** Space and Time in Present-Day Physics.

his book discusses in detail how far the simultaneity of two events corresponds with the space-time coincidence of two elements of sensation. This question would exceed the bounds of an introductory sketch intended for the general reader.

My thanks are due to Dr. J. S. Haldane, F.R.S., for several helpful suggestions, and to Mr. H. W. Davies, M.B., B.S., for reading the proof sheets.

HENRY L. BROSE.

Christ Church, Oxford.

THE THEORY OF RELATIVITY.

INTRODUCTORY.

Physics, being a science of observation which seeks to arrange natural phenomena into a consistent scheme by using the methods and language of mathematics, has to inquire whether the assumptions implied in any branch of mathematics used for this purpose are legitimate in its sphere or whether they are merely the outcome of convention or have been built up from abstract notions containing foreign elements. The use of a unit length as an unalterable measure or of a time-division has been accepted in traditional mechanics without any inconsistency manifesting itself in general until the field of electrodynamics became accessible to investigators and rendered a re-examination of the foundations of our modes of measurement necessary: it is upon these that the whole science of mathematical physics rests. The road of advance of all science is in like manner conditioned by the inter-play of observations and notions, each assisting the other in giving us a clearer view of Nature regarded purely as a physical reality. The discovery of additional phenomena presages a still greater unification, revealing new relations and exposing new differences; the ultimate aim of physics would seem to consist in reaching perfect separation and distinctness of detail simultaneously with perfect co-ordination of the whole: "the all-embracing harmony of the world is the true source of beauty and is the real truth," as Poincaré has expressed it. The noblest task of co-ordinating *all* knowledge falls to the lot of philosophy.

A principle which has proved fruitful in one sphere of physics suggests that its range may be extended into others ; nowhere has this led to more successful results than in the increasing generalisation which has characterised the advance of the principle of relativity. This advance is marked by three stages, quite distinct, however, in the nucleus of their growth, yet the later one successively includes the *results* of the earlier. Relativity first makes its appearance as a governing principle in Newtonian or Galilean mechanics: difficulties arising out of the study of the phenomena of radiations led to a new enunciation of the principle upon another basis by Einstein in 1905, which comprised the phenomena of both mechanics and radiation : this will be referred to as the " special " principle of relativity to distinguish it from the " general " principle of relativity enunciated by Einstein in 1915, which applied to all physical phenomena and every kind of motion ; the latter theory also led to a new theory of gravitation.

1. The Mechanical Theory of Relativity.

In order to arrive at the precise significance of the principle of relativity in the form in which it held sway in classical mechanics, we must briefly analyse the terms which will be used to express it. Mechanics is usually defined as the science which describes how the " position of bodies in ' space ' alters with the ' time.' " We shall for the present only discuss the term " position," which also involves " distance," leaving time and space to be dealt with later when we have to consider the meaning of physical simultaneity.

Modern pure geometry starts out from certain conceptions such as "point," "straight line" and "plane," which were originally abstracted from natural objects and which are implicitly defined by a number of irreducible and independent axioms; from these a series of propositions is deduced by the application of logical rules which we feel compelled to regard as legitimate. The great similarity which exists between geometrically constructed figures and objects in Nature has led people erroneously to regard these propositions as true: but the truth of the propositions depends on the truth of the axioms from which the propositions were logically derived. Now empirical truth implies *exact* correspondence with reality. But *pure* geometry by the very nature of its genesis excludes the test of truth. There are no geometrical points or straight lines in Nature, nor geometrical surfaces; we only find coarse approximations which are helpful in representing these mathematically conceived elements.

If, however, certain principles of mechanics are conjoined with the axioms of geometry, we leave the realm of pure geometry and obtain a set of propositions which may be verified by comparison with experience, but only within limits, viz., in respect to numerical relations, for again no exact correspondence is possible, merely a superposition of geometrical points with places occupied by matter. Our idea of the form of space is derived from the behaviour of matter, which indeed conditions it. Space itself is amorphous, and we are at liberty to build up any geometry we choose for the purpose of making empirical content fit into it. Neither Euclidean,—nor any of the forms of metageometry has any claim to precedence: we may select whichever is the more convenient for a consistent description of physical phenomena, and which requires a minimum of auxiliary hypotheses to express the laws of physical nature.

Applied geometry is thus to be treated as a branch of physics. We are accustomed to associate two points on a straight line with two marks on a (practically) rigid body: when once we have chosen an arbitrary, rigid body of reference, we can discuss motions or events mechanically by using the body as the seat of a set of axes of co-ordinates. The use of the rule and compasses gives us a physical interpretation of the distance between points and enables us to state this distance by measurement numerically, inasmuch as we may fix upon an arbitrary unit of length and count how often it has to be applied end to end to occupy the distance between the points. Every description in space of the scene of an event or of the positon of a body consists in designating a point or points on a rigid body imagined for the purpose, which coincides with the spot at which the event takes place or the object is situated. We ordinarily choose as our rigid body a portion of the earth or a set of axes attached to it.

Now Newton's (or Galilei's) law of motion states that a body which is sufficiently far removed from all other bodies continues in its state of rest or uniform motion in a straight line. This holds very approximately for the fixed stars. If, however, we refer the motion of the stars to a set of axes fixed to the earth, the stars describe circles of immense radius, that is, for such a system of reference the law of inertia only holds approximately. Hence we are led to the definition of Galilean systems of co-ordinates. **A Galilean system is one, the state of motion of which is such that the law of inertia holds for it.** It follows naturally that Newtonian or Galilean mechanics is only valid for such Galilean or inertial systems of co-ordinates, *i.e.*, in formulating expressions for the motion of bodies we must choose some such system at an immense distance where the Newtonian law would hold. It will be noticed that this is an abstraction, and that such a system is merely postulated by the law of motion. It is the foundation of classical mechanics, and hence also of the first or " mechanical " principle of relativity.

If we suppose a crow flying in a straight line at uniform velocity with respect to the earth diagonally over a train likewise moving uniformly and rectilinearly with respect to the earth (since motion is change of position we must specify our rigid body of reference, viz., the earth), then an observer in the train would also see the crow flying in a straight line but with a different uniform velocity, judged from a system of co-ordinates attached to the train. We may consider both the train and the earth to be carriers of inertial systems as we are only dealing with small distances. We can then formulate the mechanical principle of relativity as follows :—

If a body be moving uniformly and rectilinearly with respect to a co-ordinate system K then it will likewise move uniformly and rectilinearly with respect to a second co-ordinate system K^1, provided that the latter be moving uniformly and rectilinearly with respect to the first system K.

In our illustration, the crow represents the body, K is the earth, and K^1 is carried by the train.

Or, we may say that if K be an inertial system then K^1, which moves uniformly and rectilinearly with respect to K, is also an inertial system. Hence, since the laws of Newtonian mechanics are based on inertial systems, it follows that all such systems are equivalent for the description of the laws of mechanics : no one system amongst them is unique, and we cannot define absolute motion or rest ; any systems moving with mutual rectilinear uniform motion may be regarded as being at rest. Mathematically, this means that **the laws of mechanics remain unchanged in form for any transformation from one set of inertial axes to another.**

The development of electrodynamics and the phenomena of radiation generally showed, however, that the laws of radiation in one inertial system did not preserve their form when referred to another inertial system : K and K^1 were no longer equivalent for the description of phenomena such as that of light passing through a moving medium. This meant that either there was a unique inertial system enabling us to define absolute motion and rest in nature, or that we would have to build up a theory of relativity, *not* on the inertial law and inertial systems, but on some new foundation which would bring it about that the form of *all* physical laws would be preserved in passing from one system of reference to another.

This dilemma arose out of the conflicting results of two experiments, viz., Fizeau's and Michelson and Morley's.

Fizeau's experiment was designed to determine whether the velocity of light through moving liquid media was different from that through a stationary medium, *i.e.*, whether the motion of the liquid caused a drag on the aether, which it would do if the mechanical law of relativity held for light phenomena, for then the light ray would be in the same position as a swimmer travelling upstream or downstream respectively.*

No "aether-drag" was, however, detected, only a fraction of the velocity of the liquid seemed to be added to the velocity of light (c) under ordinary conditions and this fraction depends on the refractive index of the liquid and had previously been calculated by Fresnel: for a vacuum this fraction vanishes.

This result seemed to favour the hypothesis of a fixed aether, as was supported by Fresnel and Lorentz. But a fixed aether implies that we should be able to detect absolute motion, that is, motion with respect to the aether.

Arguing from this, let us consider an observer in the liquid moving with it. *If* there is a fixed aether, he should find a lesser value for the velocity of light (*i.e.*, $< c$) owing to his own velocity in the same direction or *vice versâ* in the opposite direction.

But we on the earth are in the position of the observer in the liquid since we revolve around the sun at the rate of approximately 30 kilometres per second (*i.e.*, $\frac{c}{10,000}$), and we are subject to a translatory motion of about the same magnitude: hence we should be able to detect a change in the velocity of

* It is well known that it takes a swimmer longer to travel a certain distance up and down stream than to swim across the stream and back an equal distance.

light due to our change of motion through the aether. These considerations gave rise to Michelson and Morley's experiment.

Michelson and Morley attempted to detect motion relative to the supposedly fixed aether by the interference of two rays of light, one travelling in the direction of motion of the earth's velocity, the other travelling across this direction of motion.

No change in the initial interference bands was, however, observed when the position of the instrument was changed, although such an effect was easily within the limits of accuracy of the experiment. Many modifications of the experiment likewise failed to demonstrate the presence of any "aether-wind."

To account for these negative results as contradicting deductions from Fizeau's experiment, Fitzgerald and, later, independently, Lorentz suggested the theory that bodies automatically contract when moving through the aether, and since our measuring scales contract in the same ratio, we are unable to detect this alteration in length; this effect would lead us always to get the same result for the velocity of light. This contraction-hypothesis agrees well with the electrical theory of matter and may be attributed to changes in the electro-magnetic forces, acting between particles, which determine the equilibrium of a so-called rigid body.

Thus Michelson and Morley's experiment seems to prove that the principle of relativity of mechanics also holds for radiation effects, that is, it is impossible to determine absolute motion through the aether or space: this implies that there is no unique system of co-ordinates. It conflicts with Fizeau's result and seems to indicate the existence of a "moving aether," *i.e.*, an aether which is carried along by moving bodies, as was upheld by Stokes and Hertz. Lord Rayleigh pointed out that if the contraction-hypothesis of Lorentz and Fitzgerald were true, isotropic bodies ought to become anisotropic on account of the motion of the earth, and that consequently, phenomena of double refraction should make their appearance. Experiments which he conducted himself with carbon bisulphide and others carried out by Bruce with water and glass produced a negative result.

II. The "Special" Theory of Relativity.

Einstein, in the special theory of relativity, surmounts these difficulties by doing away with the aether (as a substance) and assumes that light-signals project themselves *as such* through space. Faraday had already long ago expressed the opinion that the field in which radiations take place must not be founded upon considerations of matter, but rather that matter should be regarded as singularities or places of a singular character in the field. We may retain the name "aether" for the field as long as we do not regard it as composed of matter of the kind we know. Einstein arrives at these conclusions by critically examining our notions of space and time or of distance and simultaneity

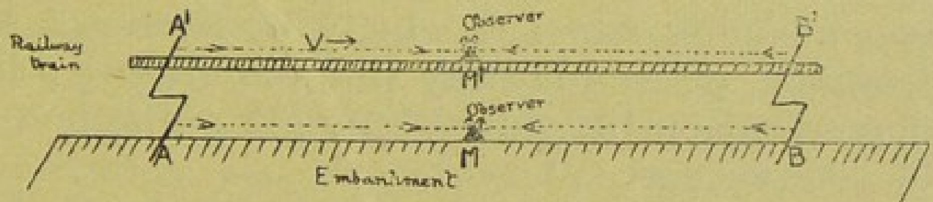
We know what simultaneity (time-coincidence of two events) means for our consciousness, but in making use of the idea of simultaneity in physics, we must be able to prove by actual experiment or observation that two events are simultaneous according to some definition of simultaneity. **A conception only has meaning for the physicist if the possibility of verifying that it agrees with actual experience is given.** In other words, we must have a definition of simultaneity which gives us an immediate means of proving by experiment that, *e.g.*, two lightning-strokes at different places occur simultaneously for an observer situated somewhere between them or not. Whenever measurements are undertaken in physics two points are made to coincide, whether they be marks on a scale and on an object, or whether they be cross-wires in a telescope which have been made to coincide with a distant object and angular measurements made; *coincidence is the only exact mode of observation and lies at the bottom of all physical measurements.* The same importance attaches to simultaneity, which is coincidence in time. It is to be noted that no definition will be made for simultaneity occurring at (practically) one point: for this case psychological simultaneity must be accepted as the basis: the necessity for a physical definition only arises when two events happening at great distances apart are to be compared as regards the moment of their happening. We cannot do more than reduce the simultaneity of two events happening a great distance apart to simultaneity referred to a single observer at one point: this would satisfy the requirements of physics.

Einstein, accepting Michelson and Morley's result, introduced the convention in 1905 that light is propagated with a constant velocity (= *c*. *i.e.*, 300,000 km. per sec. approximately) in vacuo in all directions, and he then makes use of light-signals to connect up two events in time.

He illustrates his line of argument roughly by assuming two points, A and B, very far apart on a railway embankment and an observer at M midway between A and B, provided with a contrivance such as two mirrors inclined at 90° and adjusted so that light from A and B would be reflected into his vertical line of sight (Fig. 1).

Two events such as lightning-strokes are then to be defined as simultaneous for the observer at M if rays of light from them reach the observer at the same moment (psychologically) : *i.e.*, if he sees the strokes in his mirror-contrivance simultaneously.

Next suppose that a very long train is moving with very great uniform velocity along the embankment, and that the lightning-strokes pass through the two corresponding points A^1 and B^1 of the train thus :



Fig, 1.

The question now arises : Are the two lightning-strokes at A and B, which are simultaneous with respect to the embankment also simultaneous with respect to the moving train ? It is quite clear that as M^1 is moving towards B^1 and away from A^1, the observer at M^1 (mid-point of A^1B^1) will receive the ray emitted from B^1 sooner than that emitted from A^1 and he would say that the lightning-stroke at B or B^1 occurred earlier than the one at A or A^1. Hence our condition of simultaneity is not satisfied and we are forced to the conclusion that events which are simultaneous for one rigid body of reference (the embankment) are not simultaneous for another body of reference (the train) which is in motion with regard to the first rigid body of reference. This establishes the relativity of simultaneity.

This is, of course, only an elementary example of a very special case of the regulation of clocks by light-signals. It may be asked how the mid-point M is found: one might simply fix mirrors at A and B and by flashing light-signals from points between A and B ascertain by trial, the point (M) at which the return-flashes are observed simultaneously: this makes M the mid-point between the "*time*"-distance from A and B on the embankment.

The relativity of simultaneity states that every rigid body of reference (co-ordinate system) has its own time: a time-datum only has meaning when the body of reference is specified, or we may say that simultaneity is dependent on the state of motion of the body of reference.

Similar reasoning applies in the case of the distance between two points on a rigid body. The length of a rod is defined as the distance, measured by (say) a metre-rule, between the two points which are occupied *simultaneously* by the two ends. Since simultaneity, as we have just seen, is relative, the distance between two points, since they depend on a simultaneous reading of two events, is also relative, and length only has a meaning if the body of reference is likewise specified: any change of motion entails a corresponding change of length: we cannot detect the change since our measures alter in the same ratio. **Length is thus a relative conception, and only reveals a relation between the observer and an object :** the "actual" length of a body in the sense we usually understand it does not exist : there is no meaning in the term. The length of a body measured parallel to its direction of motion will always yield a greater result when judged from a system attached to it than from any other system. These few remarks may suffice to indicate the relativity of distance.

In classical mechanics it had always been assumed that the time which elapsed between the happening of two events, and also the distance between two points of a rigid body were independent of the state of motion of the body of reference : these hypotheses must, as a result of the relativity of simultaneity and distance, be rejected. We may now ask whether a mathematical relation between the place and time of occurrence of various events is possible, such that every ray of light travels with the same constant velocity c whichever rigid body of reference be chosen, *e.g.*, such that the rays measured by an observer either in the train or on the embankment travel with the same apparent velocity.

In other words, if we assume the constancy of propagation of light in vacuo for two systems, K and K^1 moving uniformly and rectilinearly with respect to one another, what are the values of the co-ordinates x^1, y^1, z^1, t^1 of an event with respect to K^1, if the values x, y, z, t of the same event with respect to K are given ?

It is easy to arrive at this so-called Lorentz-Einstein transformation, *e.g.*, in the case where K^1 is moving relative to K parallel to K's x axis with uniform velocity v.

$$x^1 = \frac{x - vt.}{\sqrt{1 - v^2/c^2}} \qquad y^1 = y \qquad z^1 = z$$

$$t^1 = \frac{t - v/c^2\, x}{\sqrt{1 - v^2/c^2}}$$

If we put $x = ct$, then we find that $\frac{x^1}{t^1}$ reduces to c.

$i.e., c = \frac{x}{t} = \frac{x^1}{t^1}$ is the same for both systems.

and the condition of the constancy of c, the velocity of light in vacuo, is preserved.

If $\sqrt{1 - v^2/c^2}$ is to be real, then v cannot be greater than c., *i.e.*, c is the limiting or maximum velocity in nature and has thus a universal significance.

If we imagine c to be infinitely great in comparison with v (and this will be the case for all ordinary velocities, such as those which occur in mechanics), the equations of transformation degenerate into :

$$x^1 = x - vt. \quad y^1 = y \quad z^1 = z \quad t^1 = t$$

This is the familiar Galilean transformation which holds for the " mechanical " principle of relativity. We see that the Lorentz-Einstein transformation covers both mechanical and radational phenomena.

The special theory of relativity may now be enunciated as follows :—**All systems of reference which are in uniform rectilinear motion with regard to one another can be used for the description of physical events with equal justification.** That is, if physical laws assume a particularly simple form when referred to any particular system of reference, they will preserve this form when they are transformed to any other co-ordinate system which is in uniform rectilinear motion relatively to the first system. The mathematical significance of the Lorentz-Einstein equations of transformation is that the expression for the infinitesimal length of arc ds

$$(\text{viz., } ds^2 = dx^2 + dy^2 + dz^2 - c^2\, dt^2).$$

in the space-time * manifold x, y, z, t, preserves its form for all systems moving uniformly and rectilinearly with respect to one another.

Interpreted geometrically this means that the transformation is conformal in imaginary space of four dimensions. Moreover, the time-co-ordinate enters into physical laws in exactly the same way as the three space-co-ordinates, *i.e*, we may regard time spatially as a fourth dimension of space. This has been very beautifully worked out by Minkowski, whose premature loss is deeply to be regretted. It may be fitting to recall some remarks of Bergson in his *Time and Free Will.* He there states that "time is the medium in which conscious states form discrete series: this time is nothing but space, and pure duration is something different." Again, "what we call measuring time is nothing but counting simultaneities; owing to the fact that our consciousness has organised the oscillations of a pendulum as a whole in memory, they are first preserved and afterwards disposed in a series: in a word, we create for them a fourth dimension of space, which we call homogeneous time, and which enables the move-

* A continuous manifold may be defined as any continuum of elements such that a single element is defined by n continuously variable magnitudes.

ment of the pendulum, although taking place at one spot, to be continually set in juxtaposition to itself. Duration thus assumes the illusory form of a homogeneous medium and the connecting link between these two terms, space and duration, is simultaneity, which might be defined as the intersection of time and space." Minkowski calls the space-time-manifold "world" and each point (event) "world-point."

The results achieved by the special theory of relativity may be tabulated as follows :—

1. It gives a consistent explanation of Fizeau's and Michelson and Morley's experiment.
2. It leads mathematically at once to the value suggested by Fresnel and experimentally verified by Fizeau for the velocity of a beam of light through a moving refracting medium without making any hypothesis about the physical nature of the liquid.
3. It gives the contraction in the direction of motion for electrons moving with high speed, without requiring any artificial hypothesis such as that of Lorentz and Fitzgerald to explain it.
4. It satisfactorily explains aberration, *i.e.*, the influence of the relative motion of the earth to the fixed stars upon the direction of motion of the light which reaches us.
5. It accounts for the influence of the radial component of the light which reaches us, as shown by a slight displacement of the spectral lines of the light which reaches us from the

stars when compared with the position of the same lines as produced by an earth source.

6. It gives the expression for the increase of inertia, owing to the addition of (apparent) electromagnetic inertia of a charged body in motion.

The last result, however, introduces an anomaly inasmuch as the inertial mass of a quickly-moving body increases, but not the gravitational mass, *i.e.*, there is an increase of inertia without a corresponding increase of weight asserting itself. One of the most firmly established facts in all physics is hereby transgressed. This weakness (which has now been overcome) in the theory suggested a new basis for a more general theory of relativity, viz., that proposed by Einstein in 1915.

III. The General Theory of Relativity.

We have seen that the first or "mechanical" theory of relativity was built up on the notion of inertial systems as deduced from the law of inertia; the "special" theory of relativity was built up on the universal significance and invariance of c, the velocity of light in vacuo; the third or general form of relativity is to be founded on the principle of the equality of inertial and gravitational mass and in contradistinction to the other two is to hold not only for systems moving uniformly and rectilinearly with respect to one another, but for all systems whatever their motion; *i.e.*, physical laws are to preserve their form for any arbitrary transformaton of the variables from one system to another.

Mass enters into the formulae of the older physics in two forms: (1) Force = inertial mass multiplied by the acceleration. (2) Force = gravitational mass multiplied by the intensity of the field of gravitation.

$$\text{or: } p = m.a \quad p = m.^1g$$

$$i.e., \quad a = \frac{m^1}{m} g$$

Observation tells us that for a given field of gravitation the acceleration is independent of the nature and state of a body; this means that the proportionality between the two characteristic masses (inertial and gravitational) must be the same for all bodies. By a suitable choice of units we can make the factor of proportionality unity, *i.e.*, $m = m^1$.

This fact had been noticed in classical mechanics, but not interpreted.

Eötvös in 1891 devised an experiment to test the law of the equality of inertial and gravitational mass: he argued that if the centre of inertia of a heterogeneous body did not coincide with the centre of gravity of the same body, the centrifugal forces acting on the body due to the earth's rotation acting at the centre of inertia would not, when combined with the gravitational forces acting at the centre of gravitational mass, resolve into a single resultant, but that a torque or turning couple would exist which would manifest itself, if the body were suspended by a very delicate torsionless thread or filament. His experiment disclosed that the law of proportionality of inertial and gravitational mass is obeyed with extreme accuracy:

fluctuations in the ratio could only be less than a twenty-millionth.

Einstein hence assumes the exact validity of the law, and asserts that **inertia and gravitation are merely manifestations of the same quality cf a body according to circumstances.** As an illustration of the purport of this equivalence he takes the case of an observer enclosed in a box in free space (*i.e.*, gravitation is absent) to the top of which a hook is fastened. Some agency or other pulls this hook (and together with it the box) with a constant force. To an observer outside, not being pulled, the box will appear to move with constant acceleration upwards, and finally acquire an enormous velocity. But how would the observer in the box interpret the state of affairs? He would have to use his legs to support himself and this would give him the sensation of weight. Objects which he is holding in his hands and releases will fall relatively to thefloor with acceleration, for the acceleration of the box will no longer be communicated to them by the hand; moreover, all bodies will "fall" to the floor with the same acceleration. The observer in the box, whom we suppose to be familiar with gravitational fields, will conclude that he is situated in a uniform field of gravitation: the hook in the ceiling will lead him to suppose that the box is suspended at rest in the field and will account for the box not falling in the field. Now the interpretation of the observer in the box and the observer outside, who is not being pulled, are equally justifiable and valid, as long as the equality of inertial and gravitational mass is maintained.

We may now enunciate **Einstein's Principle of Equivalence: Any change which an ob-**

server perceives in the passage of an event to be due to a gravitational field would be perceived by him exactly in the same way, if the gravitational field were not present and provided that he—the observer—make his system of reference move with the acceleration which was characteristic of the gravitation at his point of observation.

It might be concluded from this that one can always choose a rigid body of reference such that, with respect to it, no gravitational field exists, *i.e.*, the gravitational field may be eliminated ; this, however, only holds for particular cases. It would be impossible, for example, to choose a rigid body of reference such that the gravitational field of the earth with respect to it vanishes entirely.

The principle of equivalence enables us theoretically to deduce the influence of a gravitational field on events, the laws of which are known for the special case in which the gravitational field is absent.

We are familiar with space-time-domains, which are approximately Galilean when referred to an appropriate rigid body of reference. If we refer such a domain to a rigid body of reference K^1 moving irregularly in any arbitrary fashion, we may assume that a gravitational field varying both with respect to time and to space is present for K^1 : the nature of this field depends on the choice of the motion of K^1. This enabled Einstein to discover the laws which a gravitational field itself satisfies. It is important to notice that Einstein does not seek to build up a model to explain gravitation but merely proposes a theory of motions. His equations describe the motion of any body in terms of co-ordinates of the space-time manifold, making use of the interchangeability and equivalence implied in relativity. He does not discuss forces as such ; they are, after all, as Karl Pearson states "arbitrary conceptual measures of motion without any perceptual equivalent." They are simply intermediaries which have been inserted between matter and motion from analogy with our muscular sense.

A direct consequence of the application of the Principle of Equivalence in its general form is that the velocity of light varies for different gravitational fields and is only constant for uniform fields (this does not contradict the special theory of relativity, which was built up for uniform fields, and only makes it a special case of this much more general theory of relativity). But change of velocity implies refraction, *i.e.*, a ray of light must have a curved path in passing through a variable field of gravitation. This affords a very valuable test of the truth of the theory, since a star, the rays from which pass very near the sun before reaching us, would have to appear displaced (owing to the stronger gravitational field around the sun), in comparison with its relative position when the sun is in another part of the heavens : this effect can only be investigated during a total eclipse of the sun, when its light does not overpower the rays passing close to it from the star in question. The calculated curvature is, of course, exceedingly small (1·7 seconds of arc), but nevertheless should be observable.

The motion of the perihelion of Mercury, discovered by Leverrier, which long proved an insuperable obstacle regarded in the light

We shall return to this test at the conclusion of the chapter.

of Newtonian mechanics, is immediately accounted for by the general theory of relativity ; this is a very remarkable confirmation of the theory.

Before we finally enunciate the general theory of relativity, it is necessary to consider a special form of acceleration, viz., rotation. Let us suppose a space-time-domain (referred to a rigid body K) in which the first Newtonian Law holds, *i.e.*, a Galilean field : we shall suppose a second rigid body of reference K^1 to be rotating uniformly with respect to K, say a plane disc rotating in its plane with constant angular velocity. An observer situated on the disc near its periphery will experience a force radially outwards, which is interpreted by an external observer at rest relatively to K as centrifugal force, due to the inertia of the rotating observer. But according to the principle of equivalence the rotating observer is justified in assuming himself to be at rest, *i.e.*, the disc to be at rest. He regards the force acting on him as an effect of a particular sort of gravitational field (in which the field vanishes at the centre and increases as the distance from the centre outwards). This rotating observer, who considers himeslf at rest, now performs experiments with clocks and measuring-scales in order to be able to define time—and space—data with reference to K^1. It is easy to show that if, of two clocks which go at exactly the same rate when relatively at rest in the Galilean field K one be placed at the centre of the rotating disc and one at the circumference, the latter will continually lose time as compared with the former.

Secondly, if an observer at rest in K measure the radius and circumference of the rotating disc, he will obtain the same value for the radius as when the disc is at rest, but since, when he measures the circumference of the disc, the scale lies along the direction of motion, it suffers contraction, and, consquently will divide more often into the circumference than if the scale and the disc were at rest. (The circumference does not change, of course, in rotation.) That is, he would get a value greater than π for the ratio $\frac{\text{circumference}}{\text{diameter}}$. This means that Euclidean geometry does not hold for the observer making his observations on the disc, and we are obliged to use co-ordinates which will enable his results to be expressed consistently. Gauss invented a method for the mathematical treatment of any continua whatsoever, in which measure-relations ("distance" of neighbouring points) are defined. Just as many numbers (Gaussian or curvilinear coordinates) are assigned to each point as the continuum has dimensions. The allocation of numbers is such that the uniqueness of each point is preserved and that numbers whose difference is infinitely small are assigned to infinitely near points. This Gaussian or curvilinear system of co-ordinates is a logical generalisation of the Cartesian system. It has the great advantage of also being applicable to non-Euclidean continua, but only in the cases in which infinitesimal portions of the continuum considered are of the Euclidean form. This calls to mind the remarks made at the commencement of this sketch about the validity of geometrical theorems. It seems as though the miniature view that we can take of straight lines and points in space led to a firm belief in the universal significance of Euclidean geometry. When we deal with light phenomena which range to enormous distances, we find that we are not justified in confining ourselves to Euclidean geometry; the "straightest" line in the time-space-manifold is "curved." We must therefore choose that geometry which, expressed analytically, enables us to describe observed phenomena most simply: it is clear that for even large finite portions of space the non-Euclidean geometry chosen must practically coincide with Euclidean geometry.

We now see that the general theory of relativity cannot admit that *all rigid* bodies of reference K, K^1, etc., are equally justifiable for the description of the general laws underlying the phenomena of physical nature, since it is, in general, not possible to make use of rigid bodies of reference for space-time descriptions of events in the manner of the special theory of relativity. Using Gaussian co-ordinates, *i.e.*, labelling each point in space with four arbitrary numbers in the way specified above (three of these correspond to three space dimensions and one to time), the general principle of relativity may be enunciated thus:—

All Gaussian four-dimensional co-ordinate-systems are equally applicable for formulating the general laws of physics. This carries the principle of relativity, *i.e.*, of equivalence of systems to an extreme limit.

With regard to the relativity of rotations, it may be briefly mentioned that centrifugal forces can, according to the general theory of relativity, only be due to the presence of other bodies. This will be better understood by imagining an isolated body poised in space; there could be no meaning in saying that it rotated, for there would be nothing to which such a rotation could be referred : classical mechanics, however, asserts that, in spite of the absence of other bodies, centrifugal forces would manifest themselves : this is denied by the general theory of relativity. No experimental test has hitherto been devised which could be carried out practically to give a decision in favour of either theory.

A favourable opportunity for detecting the slight curvature of light rays (which is predicted by the general theory) when passing in close vicinity to the sun occurred during the total eclipse of the 29th May, 1919. The results, which were made public at the meeting of the Royal Society on 6th November following, were reported as fully confirming the theory.

In addition to the slight motion of Mercury's perihelion, there is still a third test which is based upon a shift of the spectral line, as a result of an application of Doppler's principle ; this has not yet led to a conclusive experimental result. Recent experiments by Dr. Freundlich, which are to be made known in the next issue of the *Physikalische Zeitschrift* will throw light on this question.

It is hoped that this short sketch of the main threads in the development of the subject will help to stimulate interest in a theory which revolutionises the views of classical physics and discloses the intimate connection between science and philosophy.

1. Note on Non-Euclidean Geometry.

In practical geometry we do not actually deal with straight lines, but only with distances, *i.e.*, with finite parts of straight lines, yet we feel irresistibly impelled to form some conception of the parts of a straight line which vanish into inconceivably distant regions. We are accustomed to imagining that a straight line may be produced to an infinite distance in either direction, yet in our mathematical reasoning we find that in order to preserve consistency (in Euclid), we may only allocate to this straight line *one* point at infinity: we say that two straight lines are parallel when they cut at a point at infinity *i.e.*, this point is at an infinite distance from an arbitrary starting-point on either straight line, and is reached by moving forwards or backwards on *either*.

Many attempts have been made, without success, to deduce Euclid's "axiom of parallels" which asserts that only one straight line can be drawn parallel to another straight line through a point outside the latter, from the other axioms: it finally came to be recognised that this axiom of parallels was an unnecessary assumption, and that one could quite well build up other geometries by making other equally justified assumptions.

If we consider a point, P, outside a straight line, L (Fig. 2) to send out rays in all directions, then, starting from the perpendicular position Pa_1 we find that the more obliquely the ray falls on L the further does the point of intersection a_n travel along L to the left (say). Our experience teaches us that the ray and L have one point in common. There is no justifiable reason, however, for asserting, as Euclid's axiom does, that for a final infinitely small increase of the angle $a_1 Pa_n$ (*i.e.*, additional turn of Pa_n about P) a_n suddenly bounds off to infinity along L, *i.e.*, a_n, the point of intersection leaves finitude to disappear into so-called "infinity," and that, for a further infinitesimal increase, a_n, reappears at infinity at the other end of L to the right of α_1.

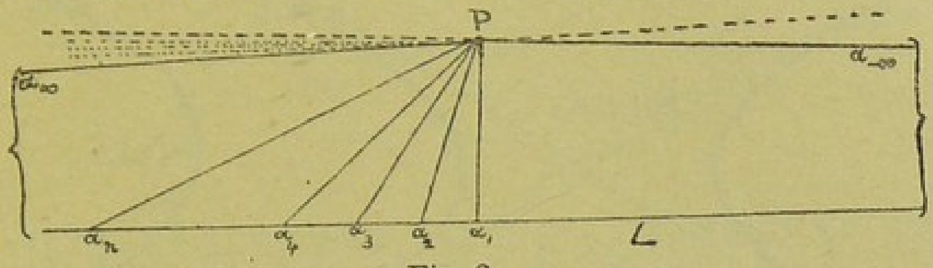


Fig. 2.

One might equally well assume, as Lobatschewsky did, that Pa_{∞} and $Pa_{-\infty}$ form an angle which differs ever so slightly from two right angles, and that there are an infinite number of other straight lines included between these two positions (as indicated by the dotted lines in the figure), which do not cut L at all, Lobatschewsky (and also Bolyai) built up an entirely consistent geometry on this latter assumption.

Riemann later abolished the assumption of infinite length of a straight line, and assumed that in travelling along a straight line sufficiently far one finally arrives at the starting point again without having encountered any limit or barrier. This means that our space is regarded as being finite but *unbounded*.* Thus in Riemann's case there is *no* parallel line to L for a_n never leaves L; there is no a_{∞}. This geometry was called by Klein elliptical geometry (and includes spherical geometry as a special case). He calls Euclidean geometry parabolic (Fig. 3)

* *E.g.*, the surface of a sphere cuts a finite volume out of space but particles sliding on the surface nowhere encounter boundaries or barriers. This is a three-dimensional analogon to the *four* dimensional space-time-manifold of Minowski. It does not mean that the universe is enclosed by a spherical shell, as was supposed by the ancients. We cannot form a *picture* of the corresponding result in the four-dimensional continuum in which, according to the general theory of relativity, we live.

for the branches of a parabola continue to recede from one to another, and yet in order

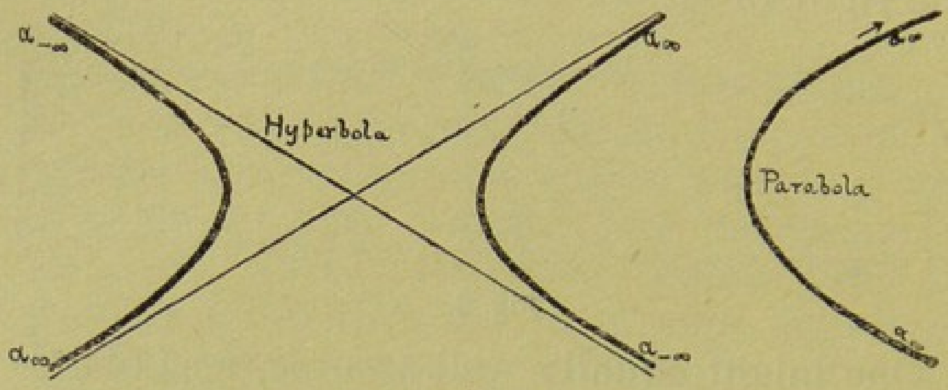


Fig. 3.

to obtain consistent results in its formulae we are obliged only to assign one point at infinity to it, just as to the Euclidean straight line. Lobatschewsky's geometry is similarly called hyperbolic (Fig. 4) since a hyperbola has two points at infinity, corresponding in analogy to the two points at infinity at which the two parallels through a point external to a straight line cut the latter.

The fact that one is obliged to renounce Euclidean geometry in the general theory of relativity leads to the conclusion that our space is to be regarded as finite but unbounded* : it is curved, as Einstein expresses it, like the faintest of ripples on a surface of water : this point is discussed in detail by Schlick in his book mentioned in the preface.

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A CHAPTER

ON

FRESNEL'S THEORY OF DOUBLE REFRACTION.

INTENDED FOR THE USE OF STUDENTS IN THE UNIVERSITY.

BY

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Mr GRIFFIN's tract on Double Refraction has been for some time quite out of print. The following pages are published with a view to supply the deficiency thus caused. It is hoped that they may serve as a useful companion to the latter part of the Astronomer Royal's treatise on the Undulatory Theory of Light.

W. S. A.

CAMBRIDGE,
April, 1870.

FRESNEL'S

THEORY OF DOUBLE REFRACTION.

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1. FRESNEL'S Theory of Double Refraction supposes that the phenomena of light are produced by the vibrations of particles of ether under the influence of their mutual attractions.

The hypothesis is first made that the particles of ether are arranged in such a manner that each of them is in stable equilibrium under the influence of the attractions of the others. Let $-R$ be the potential of all the system of particles with respect to a point. Then the resolved parts of the force on the particle at this point parallel to axes arbitrarily assumed will be, if (x, y, z) be the co-ordinates of the point, $\frac{dR}{dx}, \frac{dR}{dy}, \frac{dR}{dz}$ respectively, tending towards the origin. Hence we have

$$\frac{dR}{dx}=0,\ \frac{dR}{dy}=0,\ \frac{dR}{dz}=0 \ldots\ldots\ldots\ldots\ldots (1).$$

Let the single particle at x, y, z be displaced to a point

$$x+u,\ y+v,\ z+w,$$

while all the other particles remain at rest. Then if we suppose u, v, w so small that we may neglect their squares and higher powers, the force on this displaced particle parallel to the axes will be

$$\frac{dR}{dx}+u\frac{d^2R}{dx^2}+v\frac{d^2R}{dx\,dy}+w\frac{d^2R}{dz\,dx}$$

$$\frac{dR}{dy}+u\frac{d^2R}{dx\,dy}+v\frac{d^2R}{dy^2}+w\frac{d^2R}{dy\,dz}$$

$$\frac{dR}{dz}+u\frac{d^2R}{dz\,dx}+v\frac{d^2R}{dy\,dz}+w\frac{d^2R}{dz^2}$$

respectively.

Of these the first term in each vanishes by (1), and putting

$$\frac{d^2R}{dx^2}=A,\ \frac{d^2R}{dy^2}=B,\ \frac{d^2R}{dz^2}=C,\ \frac{d^2R}{dy\,dz}=A',\ \frac{d^2R}{dz\,dx}=B',\ \frac{d^2R}{dx\,dy}=C'$$

we get, if X, Y, Z denote the forces parallel to the axes on the displaced particle,

$$\left.\begin{aligned} X &= Au + C'v + B'w \\ Y &= C'u + Bv + A'w \\ Z &= B'u + A'v + Cw \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2).$$

Now if we construct the quadric whose equation is

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy = 1 \ldots (3)$$

it is easily seen that the direction of the resultant force whose components are X, Y, Z is perpendicular to the plane which bisects all chords of the surface (3) parallel to the direction of displacement of the particle; for the equation of this plane is

$$\xi(Au + C'v + B'w) + \eta(C'u + Bv + A'w) + \zeta(B'u + A'v + Cw) = 0.$$

The resultant force on the particle will therefore not usually coincide with the direction of its displacement; and if we suppose the particle free to move under the action of this force it will not usually return to its old position. There will be however three directions of displacement with which the directions of the force of restitution will coincide, namely the directions of the three principal axes of the surface (3).

If these directions be taken as axes of co-ordinates the equation (3) reduces to

$$Ax^2 + By^2 + Cz^2 = 1,$$

and the equations (2) reduce to

$$X = Au,\ Y = Bv,\ Z = Cw.$$

Now it is evident that if u, v, w are all positive, X, Y, Z must all tend towards the origin, since the equilibrium is stable, and A, B, C must be all positive. They are usually denoted by the letters a^2, b^2, c^2. The equation (3) thus becomes

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

This surface is usually called the ellipsoid of elasticity, and its axes the axes of elasticity. It is assumed that the directions of these axes and the values of a, b, c are constant throughout the medium. A medium in which a, b, c are all or any of them different is called a crystal. If all are unequal it is called a biaxal crystal. If two of them are equal and the third different it is called a uniaxal crystal.

If the particle be displaced parallel to the axis of x and the other particles be undisturbed it will oscillate in a time $\frac{2\pi}{a}$, for its motion is given by the equation $\frac{d^2u}{dt^2} = -a^2u$.

2. It is then assumed that under these circumstances a particle so displaced will draw an adjacent particle into a precisely similar state of displacement, and that this again will draw the next, and so on; that thus a series of vibrations will be propagated through the medium, the velocity of propagation being connected with the constant a and the wave length by the simple relation

$$\frac{\lambda}{v} = \frac{2\pi}{a} \text{ or } v = \frac{\lambda}{2\pi} \cdot a.$$

For it is supposed that the wave travels over a wave length while one particle performs a complete oscillation.

If the particle be displaced through a space p in a direction inclined at angles (α, β, γ) to the axes of elasticity, the forces on it parallel to the axes are

$$a^2p\cos\alpha,\ a^2p\cos\beta,\ a^2p\cos\gamma,$$

respectively, and the force on it in the direction of displacement will be $$p\,(a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma),$$
and for its motion in that direction we have therefore

$$\frac{d^2p}{dt^2} = -p\,(a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma).$$

If therefore the motion in that direction alone be considered the time of the particle's oscillation will be

$$\frac{2\pi}{\sqrt{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma}};$$

and if a wave of such vibrations can be propagated through the medium its velocity of propagation would as above be proportional to $$\sqrt{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma}.$$

But if r be the central radius vector of the ellipsoid of elasticity drawn in the direction of this displacement, we have

$$\frac{1}{r^2} = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma.$$

Hence the velocity of propagation of the wave corresponding to any given direction of displacement, if such a wave exist, is inversely proportional to the central radius vector of the ellipsoid of elasticity drawn in that direction.

3. At this point it will be well to notice the important *assumption* made. The force on any particle is made to depend on its absolute displacement, and is supposed to be the same as if the other particles were undisplaced. It is evident that the real force will depend on the displacement of the particle relative to the surrounding particles, and quite a different equation of motion from that given above will arise. A particular case of the investigation is given in Airy's *Undulatory Theory of Optics*, Art. 103, and the general problem has been discussed by Cauchy. The results derived by Cauchy do not probably express the truth much better than those of Fresnel, nor can the hypothesis of the vibrations of mutually attracting particles be in any case accepted as a complete explanation of the phenomena of double refraction.

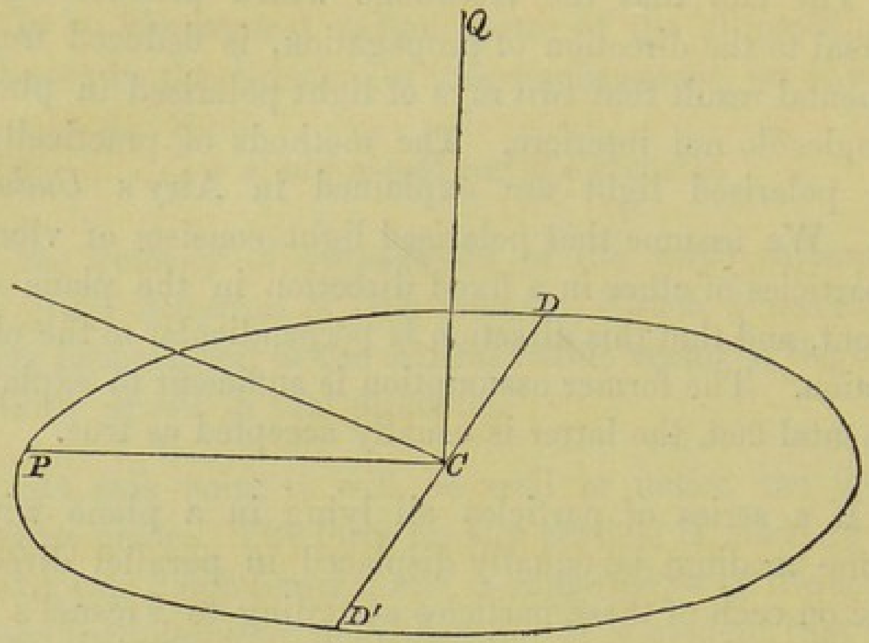
4. In considering the propagation of light through media of any kind, it is necessary to examine not the motion of one particle alone, but to imagine a series of particles simultaneously vibrating similarly. The most simple hypothesis that can be made is that all the similarly displaced particles at any instant lie in a plane, the case ordinarily called a plane wave. It is evident that by the combination of a number of such plane waves we can represent any other form of wave.

A plane wave of light consists of vibrations of the particles of ether in the plane of the wave front, the displacements and velocities of all the particles in that plane being parallel and equal. This wave is propagated with a velocity which in a crystalline medium depends, as above explained, on the direction of the displacement of the particles.

5. The fact that the vibrations which produce light are transversal to the direction of propagation, is deduced from the experimental result that two rays of light polarised in planes at right angles do not interfere. The methods of practically producing polarised light are explained in Airy's *Undulatory Theory*. We assume that polarised light consists of vibrations of the particles of ether in a fixed direction in the plane of the wave front, and that this direction is perpendicular to the plane of polarisation. The former assumption is sufficient to explain the experimental fact, the latter is usually accepted as true.

6. If a series of particles all lying in a plane within a crystalline medium be equally displaced in parallel directions, the force on each of these particles according to Fresnel's hypothesis will not usually be in the direction of displacement, or even in the plane. It may happen however that the resolved part of this force in the plane may coincide with the direction of displacement; and we will prove presently that there are two directions of displacement for which this is the case. If the particles be displaced in either of these directions the force perpendicular to the plane will produce vibrations perpendicular to that plane, which do not therefore produce light, the other parts of the force will cause each particle to oscillate equally in the plane front, and will thus produce a wave of light, if we assume that the particles oscillating in this plane immediately put in motion those in a contiguous parallel plane. The velocity of propagation of the wave will also, by what has preceded, be inversely proportional to the radius vector of the ellipsoid of elasticity drawn in the direction of the displacement.

7. Suppose that DPD' represents the central section of the ellipsoid of elasticity by a plane parallel to the wave front, and let C be its centre, CP be the direction of displacement, CD the diameter of the section conjugate to CP, and CQ the diameter of the ellipsoid conjugate to the plane PCD. Then the force of restitution is perpendicular to the plane QCD, since this is the plane to which CP is conjugate, and if the resolved part of this force in the plane of the wave front coincide with CP, we must have CP and CD at right angles, or CP must be an axis of the section



DPD′. Hence the two directions of vibration with which the resolved part of the corresponding force in the plane coincides are the axes of the section of the ellipsoid of elasticity by the plane front, and the velocities of propagation of the corresponding waves are inversely proportional to the lengths of those axes.

8. If the equation of the plane front at first be

$$lx + my + nz = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and λ, μ, ν the direction cosines of either axis of the section, we have (Aldis, *Solid Geom.* Art. 56) the equations

$$\left.\begin{aligned} l\lambda + m\mu + n\nu = 0 \\ \frac{l}{\lambda}(b^2 - c^2) + \frac{m}{\mu}(c^2 - a^2) + \frac{n}{\nu}(a^2 - b^2) = 0 \end{aligned}\right\} \ldots\ldots (2)$$

to determine λ, μ, ν, the direction cosines of the lines of displacement; and if v be the velocity of propagation of either wave we have to determine v the equation

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \;\ldots\ldots\ldots\ldots (3).$$

(Aldis' *Solid Geom.* Art. (56), Formula (10), altering a^2, b^2, c^2, r^2 into $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, $\frac{1}{v^2}$ respectively.)

If all the particles in the plane (1) be displaced in any other direction than either of those given by (2), these displacements can be resolved into two, one in each of those directions, and there will then result two sets of vibrations travelling with the velocities given by (3). Hence if at any instant there be a series of particles in the plane (1) vibrating equally in parallel directions, after a unit of time vibrations will be excited in the two planes

$$lx + my + nz = v_1,$$

$$lx + my + nz = v_2,$$

v_1 v_2 being the values of v obtained from (3). Also each of these sets of vibrations will compose a wave of *polarised* light, the planes of polarisation being perpendicular to the two lines whose direction cosines are given by (2).

9. If the envelope of the plane

$$lx + my + nz = v \text{ .... (1)},$$

be investigated, where l, m, n, v are connected by the equations

$$l^2 + m^2 + n^2 = 1 \text{ .... (2)},$$

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \text{ .... (3)},$$

we shall obtain the equation of a surface which all the wave fronts touch after a unit of time, in whatever direction the original wave front may have been situated.

Differentiating (1), (2) and (3) we have

$$xdl + ydm + zdn - dv = 0,$$

$$ldl + mdm + ndn = 0,$$

$$\frac{ldl}{v^2-a^2} + \frac{mdm}{v^2-b^2} + \frac{ndn}{v^2-c^2} - vdv\left\{\frac{l^2}{(v^2-a^2)^2} + \frac{m^2}{(v^2-b^2)^2} + \frac{n^2}{(v^2-c^2)^2}\right\} = 0.$$

Whence, using indeterminate multipliers we obtain

$$x + Al + \frac{Bl}{v^2 - a^2} = 0 \text{ .... (4)},$$

$$y + Am + \frac{Bm}{v^2 - b^2} = 0 \text{ .... (5)},$$

$$z + An + \frac{Bn}{v^2 - c^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$1 + Bv\left\{\frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2}\right\} = 0 \ldots (7).$$

Multiplying the first three of these equations by l, m, n respectively and adding, we get,

$$v + A = 0.$$

Transposing the third terms of these same equations, squaring and adding, we get, if $r^2 = x^2 + y^2 + z^2$,

$$r^2 + 2Av + A^2 = B^2\left\{\frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2}\right\};$$

$$\therefore\ r^2 - v^2 = -\frac{B}{v}, \text{ by } (7);$$

$$\left.\begin{aligned} \therefore \text{ by } (4)\ x &= vl\left\{1 + \frac{r^2 - v^2}{v^2 - a^2}\right\} = lv\,\frac{r^2 - a^2}{v^2 - a^2} \\ (5)\ y &= mv\,\frac{r^2 - b^2}{v^2 - b^2} \\ (6)\ z &= nv\,\frac{r^2 - c^2}{v^2 - c^2} \end{aligned}\right\} \ldots (8).$$

Again, from (4), (5), and (6) multiplying them by x, y, z respectively and adding

$$r^2 + Av + B\left(\frac{lx}{v^2 - a^2} + \frac{my}{v^2 - b^2} + \frac{nz}{v^2 - c^2}\right) = 0,$$

or $$r^2 - v^2 + \frac{B}{v}\left\{\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2}\right\} = 0, \text{ by } (8).$$

Whence, putting for B its value, the equation required becomes

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1.$$

This can be reduced into a different form, for by multiplying by $r^2 \equiv x^2 + y^2 + z^2$ it becomes

$$\frac{r^2x^2}{r^2 - a^2} - x^2 + \frac{r^2y^2}{r^2 - b^2} - y^2 + \frac{r^2c^2}{r^2 - c^2} - z^2 = 0,$$

or $$\frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2} = 0.$$

The equation of the wave surface can also be deduced in the following manner.

The perpendicular on any tangent plane to the surface being inversely proportional to a principal axis of the parallel central section of the ellipsoid of elasticity, it follows that the polar reciprocal of the wave surface with reference to the origin is an *apsidal surface* of this ellipsoid. Whence by Salmon, *Solid Geometry*, Art. 463, the wave surface is also an apsidal surface of the reciprocal surface of the ellipsoid of elasticity, that is of the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ .... } (9).$$

From this property its equation can be easily deduced by eliminating l, m, n between the equation

$$\frac{a^2l^2}{r^2-a^2}+\frac{b^2m^2}{r^2-b^2}+\frac{c^2n^2}{r^2-c^2}=0,$$

which gives the lengths of the axes of the section of (9) by the plane whose equation is $lx+my+nz=0$, and the equations

$$x=lr,\ y=mr,\ z=nr,$$

whence we get

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0,$$

where $$r^2=x^2+y^2+z^2.$$

10. If with the different points of the original wave front as centres we describe a series of equal wave surfaces it is evident that the plane

$$lx+my+nz=v$$

will touch them all. That is, the new wave front may be regarded as the envelope of these wave surfaces. This is analogous to the case of propagation of light through a homogeneous medium, in which case the wave surfaces are spheres. We may also fairly suppose that the point in which the wave surface having any given point of the original wave front as centre touches the second wave front, is the point at which the disturbance in the second wave front is produced by the disturbance at the given point of the first front, and the line joining these points is the direction of the *ray* proceeding from the first point. A ray must be considered as a small portion of a wave separated from the rest. The existence of such rays must be accepted as a fact; the theoretical explanation of the separation of a portion of a wave from the rest is very difficult.

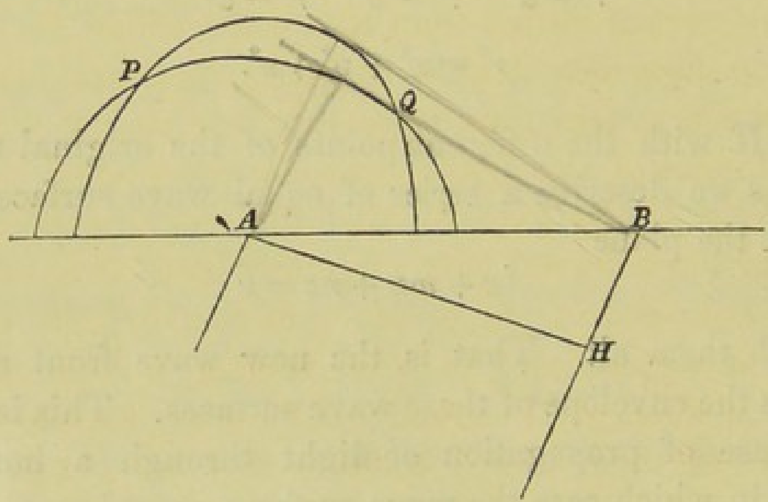
It is not difficult to see that the reciprocal ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

has important properties relating to the *ray velocities* analogous to those which the ellipsoid of elasticity possesses with relation to wave velocities. These the student can develope for himself.

11. If a wave of light be incident from vacuum into a double refracting medium, we may suppose the vibration of each point of the incident wave to produce after a time, a vibration at some point of the wave surface described with the point of incidence as centre.

Let the plane of the paper be the plane of incidence, and let AH be the trace of the front of the wave on the plane of the paper, AB the trace of the face of the crystal. Also let PQ be the



wave surface to some point of which the disturbance produced by A has arrived when the disturbance at H has reached B. The vibrations at intermediate points will have reached points of wave surfaces similar and similarly situated to PQ, but successively diminishing in size. Any plane drawn through B perpendicular to the plane of the paper touching the surface PQ will touch all these other surfaces and will be a front of the refracted wave. There can be two such planes drawn, and thus one incident wave will produce two refracted waves. The corresponding refracted rays will be obtained by joining A with the points of contact of these planes with PQ.

12. The preceding Article gives the refracted rays when a ray passes from any homogeneous medium into a double refracting crystal. The following construction applies when a ray passes from any medium into any other.

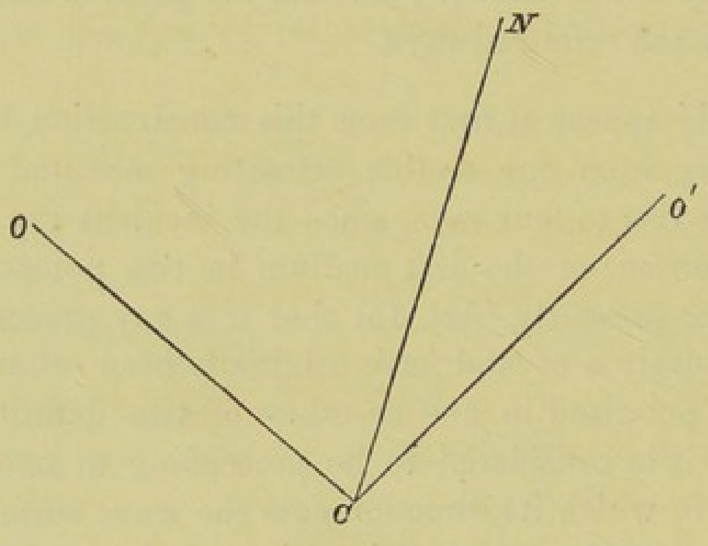
With the point of incidence of the ray on the common surface of the media as centre, describe in the second medium the half of the wave surface belonging to each medium. Produce the incident ray to cut the surface belonging to the first medium, and at the point of intersection draw a tangent plane. This tangent plane will cut the bounding plane of the media in a straight line. Through this line draw tangent planes to the wave surface of the second medium. The lines joining the point of contact of these tangent planes to the point of incidence of the ray will be the refracted rays.

It would appear at first from this construction that a single ray passing from one double refracting medium into another would give rise to four rays, since the incident ray would meet the wave surface of the first medium in two points. We shall see however presently (Art. 15) that if a ray proceeding in any direction within a crystal have originally been refracted from air, it must be polarised in one or other of two definite planes according as it is considered to be proceeding to one or other of the points in which its direction cuts the wave surface; and thus if the given ray be polarised in either of these planes we must only take one of the points as the point to which the incident ray corresponds. If the given ray be either unpolarised or polarised in any other plane it must have arisen from two rays of common light, and must be considered to consist of two rays polarised in the required planes travelling with different velocities. We should in this case expect four rays, which the construction would give. The construction includes the last article as a particular case.

13. Returning to Art. 8, we see that for all ordinary positions of the wave front, there are two velocities of propagation of the wave. These two will be equal if the wave front coincide with one of the circular sections of the ellipsoid of elasticity, and in that case, whatever be the direction of the vibrations in the plane of the front, only one wave will be propagated. The two lines perpendicular to these positions of the wave front are called the optic axes of the crystal, or the lines of *equal wave velocity*.

The planes of polarisation of the two rays corresponding to any given wave front are connected with the optic axes by a very simple relation, which we will now investigate.

Let CN be the normal to the wave front, CO, CO' the optic axes of the crystal. Then the planes of polarisation of the two rays are planes which contain CN and the axes of the sec-



tion of the ellipsoid of elasticity by a plane perpendicular to CN. This section will evidently cut the circular section perpendicular to CO in a line perpendicular to the plane OCN. Similarly it will cut the other circular section in a line perpendicular to the plane $O'CN$. Hence the radii of the section by the wave front perpendicular to the planes OCN, $O'CN$ are equal and therefore they are equally inclined to the axes of the section. The planes of polarisation of the two rays are therefore planes through CN bisecting the angles between the planes OCN and $O'CN$.

14. Again let v_1, v_2 be the velocities of the two waves corresponding to the same wave front. We can express these velocities in terms of the angles OCN and $O'CN$, as follows.

The equation of the ellipsoid of elasticity being

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \quad\text{.................. (1)},$$

the equations of the planes of circular section are

$$x\sqrt{a^2 - b^2} \pm z\sqrt{b^2 - c^2} = 0 \quad\text{.......... ...(2)},$$

and that of the wave front is

$$lx + my + nz = 0 \quad\text{.................... (3)}.$$

Hence if we denote the angles OCN, $O'CN$ by θ, θ' respectively, we have

$$\left.\begin{aligned} \cos\theta &= \frac{l\sqrt{a^2 - b^2} - n\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}} \\ \cos\theta' &= \frac{l\sqrt{a^2 - b^2} + n\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}} \end{aligned}\right\} \quad\text{......... (4)},$$

$$\left.\begin{aligned} \therefore (\cos\theta' + \cos\theta)\sqrt{a^2 - c^2} &= 2l\sqrt{a^2 - b^2} \\ (\cos\theta' - \cos\theta)\sqrt{a^2 - c^2} &= 2n\sqrt{b^2 - c^2} \end{aligned}\right\} \quad\text{...... (5)}.$$

Again v_1, v_2 are the roots of the equation

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0.$$

Hence

$$\begin{aligned} v^2 + v_2^2 &= l^2(b^2 + c^2) + m^2(c^2 + a^2) + n^2(a^2 + b^2) \\ &= a^2 + c^2 - l^2(a^2 - b^2) + n^2(b^2 - c^2) \quad \text{since } l^2 + m^2 + n^2 = 1 \\ &= a^2 + c^2 - (a^2 - c^2)\cos\theta\cos\theta' \text{ by (4)} \quad\text{................(6)}. \end{aligned}$$

$$\begin{aligned} v_1^2 v_2^2 &= l^2b^2c^2 + m^2c^2a^2 + n^2a^2b^2, \\ &= a^2c^2 - c^2l^2(a^2 - b^2) + n^2a^2(b^2 - c^2); \end{aligned}$$

$$\therefore 4v_1^2v_2^2 = 4a^2c^2 - (a^2-c^2)\{c^2(\cos\theta'+\cos\theta)^2 - a^2(\cos\theta'-\cos\theta)^2\} \text{ by (5)},$$

$$= 4a^2c^2 + (a^2-c^2)^2(\cos^2\theta+\cos^2\theta') - 2(a^4-c^4)\cos\theta\cos\theta' \ldots (7).$$

Hence squaring (6) and subtracting (7) we get

$$(v_1^2 - v_2^2)^2 = (a^2-c^2)^2\{1 - \cos^2\theta - \cos^2\theta' + \cos^2\theta\cos^2\theta'\},$$

$$= (a^2-c^2)^2 \sin^2\theta \sin^2\theta' ;$$

$$\therefore v_1^2 \sim v_2^2 = (a^2-c^2)\sin\theta\sin\theta' \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

From (6) and (8) we easily deduce by adding and subtracting

$$\left.\begin{aligned} v_1^2 &= a^2\sin^2\frac{\theta\pm\theta'}{2} + c^2\cos^2\frac{\theta\pm\theta'}{2} \\ v_2^2 &= a^2\sin^2\frac{\theta\mp\theta'}{2} + c^2\cos^2\frac{\theta\mp\theta'}{2} \end{aligned}\right\} \ldots\ldots (9).$$

The results of equations (8) and (9) are easily seen to coincide with those deduced in a different manner in Salmon's *Solid Geometry*, Art. 245. Analogous results can be obtained for *ray velocities* from the reciprocal ellipsoid. (Lloyd, *Wave Theory of Light*, Art. 186.)

15. The formulæ of the last article enable us to determine completely the circumstances of the vibrations of the two rays corresponding to the same wave front in the crystal. They do not however determine the plane of polarisation if we are only given the direction in which the *ray* proceeds within the crystal. For this purpose we must revert to the wave surface of Art. 9.

Let a ray meet the wave surface at the point x, y, z, let l, m, n be the direction cosines of the normal to the wave front to which the ray belongs, and λ, μ, ν the direction cosines of the direction of vibration of the particles in the ray. Then we have, if v be the corresponding wave velocity,

$$v^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2,$$

where $$\frac{l}{\lambda}(b^2-c^2) + \frac{m}{\mu}(c^2-a^2) + \frac{n}{\nu}(a^2-b^2) = 0,$$

and $$l\lambda + m\mu + n\nu = 0.$$

Whence eliminating n we get

$$l\left\{\frac{\nu}{\lambda}(b^2-c^2)-\frac{\lambda}{\nu}(a^2-b^2)\right\}+m\left\{\frac{\nu}{\mu}(c^2-a^2)-\frac{\mu}{\nu}(a^2-b^2)\right\}=0,$$

or $$\frac{l}{\lambda}\{b^2-\lambda^2a^2-\mu^2b^2-\nu^2c^2\}-\frac{m}{\mu}\{a^2-\nu^2c^2-\mu^2b^2-\lambda^2a^2\}=0\,;$$

$$\therefore\ \frac{l}{\lambda(a^2-v^2)}=\frac{m}{\mu(b^2-v^2)}=\frac{n}{\nu(c^2-v^2)}\text{ by symmetry.}$$

These equations determine λ, μ, ν in terms of v.

Combining these results with the equations (8) of Art. (9), we easily obtain

$$\frac{\lambda(r^2-a^2)}{x}=\frac{\mu(r^2-b^2)}{y}=\frac{\nu(r^2-c^2)}{z},$$

which give the direction of vibration in the ray proceeding to any given point (x, y, z).

A geometrical interpretation can be given to these equations. The co-ordinates of the foot of the perpendicular on the tangent plane to the wave surface at x, y, z are with our previous notation, lv, mv, nv, and the direction cosines of the line joining this point with the point of contact are proportional to

$$x-lv,\ y-mv,\ z-nv.$$

But we have by equations (8) of Art. (9),

$$lv=x\frac{v^2-a^2}{r^2-a^2};$$

$$\therefore\ x-lv=\frac{x(r^2-v^2)}{r^2-a^2}.$$

Similarly $$y-mv=\frac{y(r^2-v^2)}{r^2-b^2},$$

$$z-nv=\frac{z(r^2-v^2)}{r^2-c^2}.$$

Hence λ, μ, ν are proportional to $x-lv$, $y-mv$, $z-nv$, or the direction of the vibration constituting any ray is the projection of the ray on the tangent plane to the wave surface at the point where it meets it. The plane of polarisation is of course perpendicular to this. The plane of polarisation of a ray proceeding in a double refracting medium is usually taken to be the plane containing the normal to the wave front and perpendicular to the direction of vibration.

This result may be otherwise obtained. If λ, μ, ν are the direction cosines of the direction of displacement of a particle, those of the resultant force are proportional to $a^2\lambda$, $b^2\mu$, $c^2\nu$. Hence the direction of displacement is perpendicular to the tangent plane drawn to the reciprocal ellipsoid at the point where the line of the resultant force meets it. From this, since the wave surface is the apsidal surface of the reciprocal ellipsoid, it follows by Arts. 461, 462 of Salmon's *Solid Geometry* that the direction of displacement, the direction of the resultant force, the normal to the wave front and the ray all lie in one plane. Hence the direction of displacement is the projection of the radius vector on the tangent plane to the wave surface.

It also follows that the ray and the direction of the resultant force are at right angles.

16. The wave surface has thus been shown to possess the following properties.

Its tangent planes give the positions of the wave fronts after a given time, and the perpendiculars on those tangent planes from the centre represent the *velocities of wave fronts* in different directions.

Its radii vectores from the centre to the points of contact are the directions of the rays corresponding to the different wave fronts, and the lengths of these radii vectores represent the corresponding *ray velocities*.

The projections of these radii on the tangent planes give the directions of the vibrations in the corresponding rays.

17. The equation of the wave surface becomes by multiplying up

$$x^2(r^2-b^2)(r^2-c^2)+y^2(r^2-c^2)(r^2-a^2)+z^2(r^2-a^2)(r^2-b^2)$$
$$=(r^2-a^2)(r^2-b^2)(r^2-c^2).$$

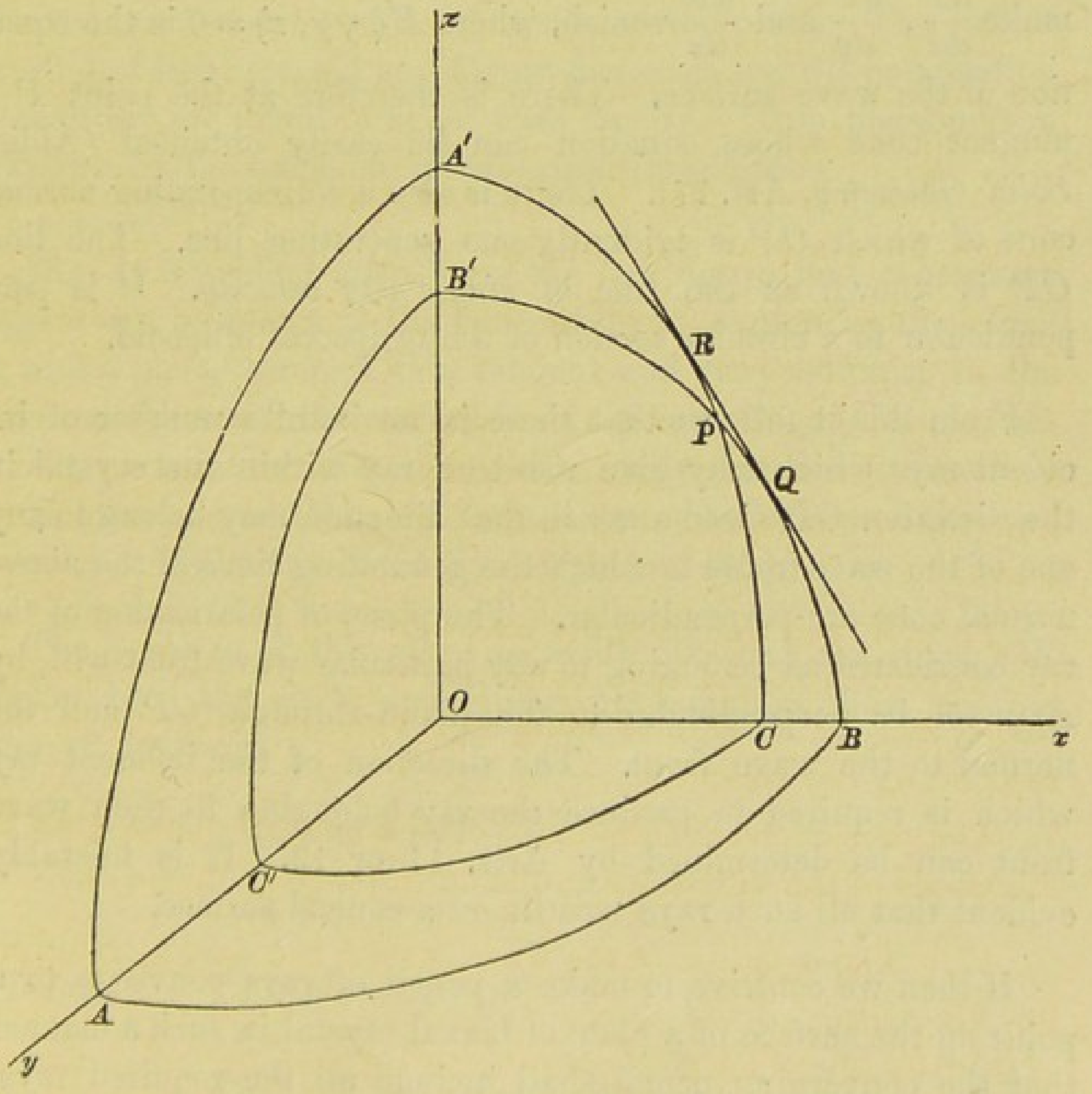
Its trace on the plane of xy is obtained by putting $z=0$; we then get

$$(x^2+y^2-c^2)\{x^2(x^2+y^2-b^2)+y^2(x^2+y^2-a^2)\}$$
$$=(x^2+y^2-a^2)(x^2+y^2-b^2)(x^2+y^2-c^2)$$

whence $$x^2+y^2=c^2,$$

or $$a^2x^2+b^2y^2=a^2b^2.$$

That is, the trace consists of a circle whose radius is c and an ellipse whose axes are $2b$ and $2a$ in the directions Ox and Oy respectively. Similarly the trace on the plane of yz consists of a circle whose radius is a and an ellipse whose axes are $2c$ and $2b$; and the trace on the plane of zx, of a circle of radius b and an ellipse whose axes are $2a$ and $2c$.



The figure represents these traces on the supposition that a, b, c are in descending order of magnitude.

The figure shows that the two curves in the plane of zx cut at a point P whose co-ordinates can be obtained from the two equations

$$x^2 + z^2 = b^2,$$

$$a^2x^2 + c^2z^2 = a^2c^2.$$

Whence $$x^2 = c^2\frac{a^2 - b^2}{a^2 - c^2}, \quad z^2 = a^2\frac{b^2 - c^2}{a^2 - c^2}.$$

It will be found that these values combined with $y = 0$ satisfy the conditions for a singular point on the surface, for they make $\frac{dF}{dx}$, $\frac{dF}{dy}$ and $\frac{dF}{dz}$ vanish, where $F(x, y, z) = 0$ is the equation of the wave surface. There is therefore at the point P a tangent cone whose equation can be easily obtained (Aldis, *Solid Geometry*, Art. 77). There is also a corresponding normal cone of which OP is evidently one generating line. The line OP is known as the line of *single ray velocity*. It is perpendicular to a circular section of the reciprocal ellipsoid.

From this it follows that there is an infinite number of incident rays which may give rise to a ray within the crystal in the direction OP, since a ray in that direction may belong to any one of the wave fronts to which the generating lines of the above normal cone are perpendicular. The plane of polarisation of the ray considered as belonging to any particular wave front will, by Art. 15, be perpendicular to the plane through OP and the normal to the wave front. The direction of the incident ray which is required to produce the ray belonging to each wave front can be determined by Arts. 11 or 12. It is tolerably evident that all such rays will lie on a conical surface.

If then we contrive to make a pencil of rays converge to a point on the surface of a plate of biaxal crystal in such a manner that the converging pencil shall include all the required rays, and by some means limit the direction in which light can pass through the crystal from the point of incidence, to the direction OP there will be an assemblage of rays proceeding all in that one direction. These rays, on arriving at the other face of the crystal, will be all differently refracted, since they correspond to different wave fronts, and there will thus be produced on emergence a hollow diverging cone of rays, each ray of the cone having a different plane of polarisation.

This is found to be experimentally the fact. The limitation that the rays shall only pass in one direction, is effected by placing two thin plates of metal, with a small hole perforated in each, in such positions on the two sides of the plate of crystal that the line joining them shall coincide with the line of single ray velocity, a pencil of light is then made to converge to a point at the hole in one of the pieces of metal and the light received on a screen on the other side of the plate. The size of the ring of light formed at different distances and the polarisation of each ray are found to agree with theory. This phenomenon is known by the name of *external conical refraction.*

18. It is evident again from the last figure that a common tangent can be drawn to the two curves of section in the plane zx, and a plane through this tangent and perpendicular to the plane zx will touch the surface in two points Q and R. It can be shown however that this plane really touches the surface along a curve, and that this curve is a circle of which RQ is the diameter. (Salmon's *Solid Geometry*, Art. 465.)

The equation of this plane is easily obtained, for since QR touches both the circle and ellipse, its equation must assume either of the forms

$$lx + nz = b\sqrt{l^2 + n^2},$$

or

$$lx + nz = \sqrt{l^2c^2 + n^2a^2}.$$

Whence $$l^2c^2 + n^2a^2 = (l^2 + n^2)\, b^2:$$

$$\therefore\ l^2(b^2 - c^2) = n^2(a^2 - b^2) \text{ or } \frac{l^2}{a^2 - b^2} = \frac{n^2}{b^2 - c^2} = \frac{l^2 + n^2}{a^2 - c^2}.$$

Hence the plane through QR perpendicular to the plane zx has for its equation

$$x\sqrt{a^2-b^2} \pm z\sqrt{b^2-c^2} = b\sqrt{a^2-c^2} \text{ .... (1).}$$

It is therefore perpendicular to an optic axis, which is also *a priori* evident, since the rays from O to Q and R correspond to the same wave front.

The equations (8) of Art. 9 give the co-ordinates of the point of contact of any tangent plane

$$lx + my + nz = v.$$

For the co-ordinates of the point of contact of (1), confining ourselves to the upper sign and putting

$$l = \sqrt{\frac{a^2-b^2}{a^2-c^2}},\ m = 0,\ n = \sqrt{\frac{b^2-c^2}{a^2-c^2}},\ v = b,$$

the second of those equations reduces to an identity, and the first and third give

$$x^2 + y^2 + z^2 - a^2 = -\frac{\sqrt{(a^2-b^2)(a^2-c^2)}}{b}x,$$

$$x^2 + y^2 + z^2 - c^2 = \frac{\sqrt{(b^2-c^2)(a^2-c^2)}}{b}z.$$

Which two spheres by their intersection determine a circle at every point of which the plane (1) touches the wave surface.

19. If therefore a ray of light be incident on a biaxal crystal in such a direction that the front of the refracted wave shall be the plane (1) of the last article, there will be produced a cone of rays proceeding to all points of the circle of contact, the planes of polarisation of all these rays being different, the direction of vibration of the ether in any ray being parallel to the chord of the circle joining its extremity with Q, by Art. 15. Hence the plane of polarisation of the ray at Q is the plane zx, while that of the ray at R is perpendicular to the plane zx. The plane of polarisation therefore turns through a right angle,

while the point of incidence of the ray sweeps half round the circle of contact. All these rays when incident on the second face of the plate of crystal will emerge parallel, since they all belong to one wave front, and we shall thus obtain a hollow cylinder of light.

The limitation of the direction of light is made by placing two plates of metal with a small hole in each, one of them being in contact with the crystal, the other at some distance from it, and allowing only the ray which has passed through both these to enter the crystal. If the line joining these be experimentally adjusted to the right position, a cylindrical pencil of rays will be found to issue from the plate. This cylinder is very small unless the piece of crystal be of considerable thickness. The phenomenon may be made more conspicuous by receiving the light on a lens of short focal length which will convert the cylinder into a hollow cone of light which may either be received on a screen or by the eye. If this pencil be viewed through a Nichol's prism, the polarisation of its different rays is found to agree with the theory. This phenomenon is known as *internal conical refraction.*

20. The preceding investigations apply to the most general case of double refraction, namely that in which all three constants a, b, c are unequal. If all three become equal, the ellipsoid of elasticity becomes a sphere, the wave surface two coincident spheres, and double refraction ceases. If however only two are equal, as a and b, double refraction will still exist. In this case the ellipsoid of elasticity becomes a spheroid, and its two sets of circular sections coincide. There is thus only one optic axis, which is called the axis of the crystal, and such crystals are, as before stated, called uniaxal crystals.

The equation (3) of Art. 8

$$l^2(v^2-b^2)(v^2-c^2)+m^2(v^2-c^2)(v^2-a^2)+n^2(v^2-a^2)(v^2-b^2)=0,$$

becomes

$$(v^2-a^2)\{v^2-(l^2+m^2)c^2-n^2a^2\}=0,$$

whence

$$v^2=a^2, \text{ or } v^2=a^2n^2+c^2(l^2+m^2).$$

So that of the two waves corresponding to a given direction of front, one has a constant velocity a, and the other has a velocity

$$\sqrt{a^2 \cos^2\theta + c^2 \sin^2\theta},$$

where θ is the angle between the normal to the wave front and the axis of the crystal.

The equation of the wave surface reduces to the form

$$x^2(r^2-a^2)(r^2-c^2)+y^2(r^2-a^2)(r^2-c^2)+z^2(r^2-a^2)^2=(r^2-c^2)(r^2-a^2)^2,$$

whence

$$r^2 - a^2 = 0, \text{ or } x^2 + y^2 + z^2 = a^2,$$

or $$(x^2+y^2)(r^2-c^2)+z^2(r^2-a^2)=(r^2-a^2)(r^2-c^2),$$

which reduces to

$$a^2(x^2+y^2)+c^2z^2=a^2c^2.$$

Hence the wave surface reduces to a sphere and a spheroid.

The formula of Art. 15 will still give the direction of vibration of the ether for the ray which proceeds to a point on the spheroid, which is usually called the extraordinary ray. This direction will easily be seen to lie in a plane containing the axis of z, and the plane of polarisation of this ray is thus perpendicular to this plane. The direction of vibration of the ray which proceeds to a point of the sphere will be easily seen to be perpendicular to the plane through the ray and the axis of z, and this plane is therefore its plane of polarisation.

21. Fresnel's Theory, of which we have endeavoured to explain the main features, is undoubtedly not a sound *dynamical Theory.* It has however the great merit of representing accurately the facts of double refraction as far as experiment at present has tested them, and in one instance has led to the discovery of facts (the conical refractions) previously unobserved. It is also probably better adapted for students than any other of the theories yet promulgated. Those who take an interest in these theories can refer to Cauchy's *Exercises de Mathématiques*, Tomes III. and IV., MacCullagh's papers in the *Transactions of the Royal Irish Academy*, and should certainly consult Green's papers in Vol. VII. of the *Cambridge Philosophical Transactions.* A full account and comparison of these and other theories will

be found in the *Report to the British Association*, in 1862, by one of the highest authorities on the subject, Professor Stokes. After careful examination of them all he expresses his own opinion that the true dynamical theory of Double Refraction has yet to be found.

Probably when the Newton of Physical Optics has succeeded in linking together all the phenomena of Light into one continuous chain, the name of Fresnel will yet be remembered with a reverence akin to that which astronomers feel for Copernicus and Kepler.

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From the PHILOSOPHICAL MAGAZINE for October 1871.

ON A CLASS OF DEFINITE INTEGRALS.

BY

J. W. L. GLAISHER, B.A., F.R.A.S., F.C.P.S.

THE Theory of Definite Integrals, strictly speaking, is confined within a very small compass; in fact it can scarcely be said that there exists a Theory of Definite Integrals in the same sense as we speak of the Theory of Equations, the Theory of Curves, &c.: the integrals are evaluated, but their properties are not, as a rule, studied. The majority of works having the title are devoted to the evaluation of Integrals by different isolated methods, which, though well adapted for the purpose and interesting, are not connected together as parts of a theory. A similar want of system holds with regard to the integrals themselves. Many have been evaluated on account of their use in physics, a greater number on account of their intrinsic interest or elegance, more still as examples of different and frequently highly ingenious modes of evaluation; while in not a few cases it is hard to see what inducement tempted their authors to spend time over them. The subject of Definite Integrals, regarded as a science, is still rather in the observational than the theoretical state; *i. e.* the results have more resemblance to detached facts observed than to a chain of facts connected by a theory; and so it must for a long time remain.

Any one who takes up a memoir or work on the subject (such as Meyer's *Theorie der bestimmten Integrale*, published during the present year) must at once notice that, except in a few cases such as the Elliptic Functions, the Gamma Function, &c., which are usually considered separately, a number of definite integrals are proved equal to certain quantities, without any indication being apparent why those given should have been preferred to others omitted, or, in the absence of any properties of the functions, for what purpose they were evaluated. The fact seems to be that every definite integral, not so complicated or unsymmetrical in form as to be absolutely destitute of interest, which admits of finite expression, or of expression in a tolerable simple series, has been evaluated; and the gaps which occur are due to the integrals omitted not being so expressible. Thus, for example, $\int_0^\infty \frac{\cos bx\, dx}{a^2+x^2} = \frac{\pi}{2a} e^{-ab}$; but $\int_0^\infty \frac{\sin bx\, dx}{a^2+x^2}$ is not expressible in terms of ordinary functions. The number of what may be called principal integrals evaluable is not large; and that this is the case is not remarkable, when it is considered that there is scarcely a function which cannot be thrown into the form of a definite integral, while for the evaluation of the latter we can only employ combinations of algebraical, circular, logarithmic, and exponential quantities. For the advance of the subject, therefore, the introduction of new fundamental functions is a necessity; and Schlömilch, in order to evaluate the second of the integrals written above and some few others of allied form, made use of the functions known as the sine-integral, cosine-integral, and exponential-integral. As soon as it appeared that these were suitable primary functions, a large number of definite integrals, previously inexpressible, were reduced to dependence on them, their properties were investigated, and Tables constructed of their numerical values.

This procedure was truly scientific, and has extended the limits of the science; and a similar course must continue to be pursued, not only with the view of increasing the number of integrals which, if need be, could be calculated numerically, but also for the sake of making the subject more systematic and homogeneous in form, as well as connecting the different results with more completeness and unity.

The fact also, previously alluded to, of the power of definite integrals as a means of expressing other functions (such as solutions of algebraical and differential equations &c.), points to the value of a good classification accompanied by full numerical Tables.

The chief point of importance, therefore, is the choice of the elementary functions; and this is a work of some difficulty. One function, however, viz. the integral $\int_x^\infty e^{-x^2}dx$, well known for its use in physics, is so obviously suitable for the purpose, that, with the exception of receiving a name and a fixed notation, it may almost be said to have already become primary. I propose, therefore, in the present communication to investigate some of the most important integrals evaluable by its means, and several connected results—and in a subsequent communication, after noticing a few of the principal physical results involving it, to describe the Tables that have been calculated of its numerical values, and supplement them by a Table with different arguments, which is nearly completed.

As it is necessary that the function should have a name, and as I do not know that any has been suggested, I propose to call it the *Error-function*, on account of its earliest and still most important use being in connexion with the theory of Probability, and notably the theory of Errors, and to write

$$\int_x^\infty e^{-x} dx = \text{Erf}\, x. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

We then have the following results obtained by obvious transformations:

$$\int_x^\infty e^{-a^2x^2}dx = \frac{1}{a}\operatorname{Erf} ax, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\int_x^\infty e^{-ax}\frac{dx}{\sqrt{x}} = \frac{2}{\sqrt{a}}\operatorname{Erf}\sqrt{ax}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\int_x^\infty e^{-a(x+b)^2}dx = \frac{1}{\sqrt{a}}\operatorname{Erf}(x+b)\sqrt{a}\,; \quad . \quad . \quad . \quad . \quad (4)$$

also $\operatorname{Erf} 0 = \frac{1}{2}\sqrt{\pi}$, so that

$$\int_0^x e^{-x^2}dx = \tfrac{1}{2}\sqrt{\pi} - \operatorname{Erf} x,$$

$$\int_0^x e^{-ax}\frac{dx}{\sqrt{x}} = \sqrt{\frac{\pi}{a}} - \frac{2}{\sqrt{a}}\operatorname{Erf}\sqrt{ax}, \text{ \&c.}$$

We know that

$$\int_0^\infty e^{-cx^2}dx = \frac{\sqrt{\pi}}{2\sqrt{c}},$$

whence

$$\int_0^\infty e^{-c(x^2+a^2)}dx = \frac{\sqrt{\pi}}{2}\frac{e^{-a^2c}}{\sqrt{c}};$$

whence, integrating with regard to c,

$$\int_0^\infty \frac{e^{-c(x^2+a^2)}}{x^2+a^2}dx = \frac{\sqrt{\pi}}{a}\operatorname{Erf} a\sqrt{c}$$

from (3), and therefore

$$\int_0^\infty \frac{e^{-cx^2}}{x^2+a^2}dx = \frac{\sqrt{\pi}}{a}e^{a^2c}\operatorname{Erf} a\sqrt{c}. \quad . \quad . \quad . \quad (5)$$

This result is not new; it is obtained, though in a different manner, in De Morgan's 'Diff. and Int. Calc.' p. 676.

From it we can deduce, by Boole's theorem

$$\int_{-\infty}^\infty \phi\left(x-\frac{\alpha}{x}\right)dx = \int_{-\infty}^\infty \phi(x)\,dx,$$

α being positive (Phil. Trans. 1857, p. 780), that

$$\int_0^\infty \frac{x^2e^{-c\left(x^2+\frac{\alpha^2}{x^2}\right)}dx}{x^4+(a^2-2\alpha)x^2+\alpha^2} = \frac{\sqrt{\pi}}{a}e^{a^2c-2\alpha c}\operatorname{Erf} a\sqrt{c}, \quad . \quad (6)$$

or, taking $x=\frac{1}{z}$, changing the values of the constants, and writing x for z,

$$\int_0^\infty \frac{e^{-c\left(x^2+\frac{\alpha^2}{x^2}\right)}dx}{x^4+(a^2-2\alpha)x^2+\alpha^2} = \frac{\sqrt{\pi}}{a\alpha}e^{a^2c-2\alpha c}\operatorname{Erf} a\sqrt{c}. \quad . \quad (7)$$

Putting $\alpha = \frac{a^2}{2}$, we have, as particular cases,

$$\int_0^\infty \frac{x^2 e^{-c\left(x^2+\frac{\alpha^2}{x^2}\right)}}{x^4+\alpha^2}\,dx = \sqrt{\frac{\pi}{2\alpha}}\,\text{Erf}\,\sqrt{2\alpha c}, \quad . \quad (8)$$

$$\int_0^\infty \frac{e^{-c\left(x^2+\frac{\alpha^2}{x^2}\right)}}{x^4+\alpha^2}\,dx = \sqrt{\frac{\pi}{2\alpha^3}}\,\text{Erf}\,\sqrt{2\alpha c}. \quad . \quad . \quad (9)$$

The former result can be verified by integrating both sides of the equation

$$\int_0^\infty e^{-c\left(x^2+\frac{\alpha^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}\frac{e^{-2\alpha c}}{\sqrt{c}}$$

with respect to c.

The integral $\int_0^\infty e^{-(ax^2+bx+c)}\,dx$ is simply expressible in terms of the error-function; for the former

$$= e^{-c+\frac{b^2}{4a}}\int_0^\infty e^{-a\left(x+\frac{b}{2a}\right)^2}dx = \frac{1}{\sqrt{a}}\,e^{\frac{b^2-4ac}{4a}}\,\text{Erf}\,\frac{b}{2\sqrt{a}}. \quad . \quad (10)$$

Integrate both sides of the well-known equation

$$\int_0^\infty e^{-a^2x^2}\cos 2rx\,dx = \frac{\sqrt{\pi}}{2a}\,e^{-\frac{r^2}{a^2}}. \quad . \quad . \quad . \quad (11)$$

with regard to r between the limits r and 0, and we have

$$\int_0^\infty e^{-a^2x^2}\frac{\sin 2rx}{2x}\,dx = \frac{\sqrt{\pi}}{2a}\cdot\left\{a\frac{\sqrt{\pi}}{2} - a\,\text{Erf}\,\frac{r}{a}\right\};$$

that is,

$$\int_0^\infty e^{-a^2x^2}\sin 2rx\,\frac{dx}{x} = \frac{\pi}{2} - \sqrt{\pi}\,\text{Erf}\,\frac{r}{a}. \quad . \quad . \quad . \quad . \quad (12)$$

By differentiating (11) with regard to r and dividing by r, we obtain

$$\int_0^\infty e^{-a^2x^2}\frac{\sin 2rx}{r}\,.\,x\,dx = \frac{\sqrt{\pi}}{2a^3}\,e^{-\frac{r^2}{a^2}}.$$

From this we deduce, by integrating with regard to r,

$$\int_0^\infty xe^{-a^2x^2}\,\text{Si}(2rx)\,dx = \frac{\pi}{4a^2} - \frac{\sqrt{\pi}}{2a^2}\,\text{Erf}\,\frac{r}{a}. \quad . \quad (13)$$

De Morgan (Diff. and Int. Calc. p. 675) obtained a formula which, when slightly altered in form and generalized, may be written

$$\int_0^\infty \frac{e^{-cx^2}\cos 2bx}{a^2+x^2}\,dx$$

$$= \frac{\sqrt{\pi}}{2a}\,e^{a^2c}\left\{e^{-2ab}\,\text{Erf}\left(a\sqrt{c}-\frac{b}{\sqrt{c}}\right) + e^{2ab}\,\text{Erf}\left(a\sqrt{c}+\frac{b}{\sqrt{c}}\right)\right\}; \quad (14)$$

and differentiating with regard to b, we have

$$\int_0^\infty \frac{e^{-cx_2}\sin 2bx}{a^2+x^2}\,x\,dx$$

$$= \frac{\sqrt{\pi}}{2}e^{a^2c}\left\{e^{-2ab}\operatorname{Erf}\left(a\sqrt{c}-\frac{b}{\sqrt{c}}\right)-e^{2ab}\operatorname{Erf}\left(a\sqrt{c}+\frac{b}{\sqrt{c}}\right)\right\}. \quad (15)$$

Among integrals of less interest may be noticed

$$\int_0^\infty (\cos px - \sin px)\log\left(1+\frac{q^2}{x^2}\right)\frac{dx}{\sqrt{x}}$$

$$= \frac{\pi}{\sqrt{2p}}\left\{\sqrt{\pi} - 2\operatorname{Erf}\sqrt{pq}\right\}, \quad . \quad (16)$$

and

$$\int_0^{\frac{\pi}{2}} e^{-q^2(\tan x+\cot x)}\sqrt{\tan x}\,dx = \sqrt{2\pi}\operatorname{Erf} q\sqrt{2}, \quad . \quad (17)$$

obtained from results given in De Haan's *Nouvelles Tables*, No. 11, Table 178, and No. 9, Table 276.

There are several simple formulæ involving the function under the integral sign. Thus from

$$\int_0^\infty \frac{e^{-ax}}{\sqrt{a}}\,dx = \frac{1}{a^{\frac{3}{2}}}$$

we deduce

$$\int_0^\infty \operatorname{Erf}\sqrt{ax}\,\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{a}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

from

$$\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a},$$

by integrating between limits, we find

$$\int_0^\infty (\operatorname{Erf} ax - \operatorname{Erf} bx)\frac{dx}{x} = \frac{\sqrt{\pi}}{2}\log\frac{b}{a}; \quad . \quad . \quad . \quad (19)$$

and from

$$\int_0^\infty e^{-c^2x^2-\frac{a^2}{x^2}}\,dx = \frac{\sqrt{\pi}}{2c}e^{-2ac}$$

we deduce

$$\int_0^\infty xe^{-c^2x^2}\operatorname{Erf}\frac{a}{x}\,dx = \frac{\sqrt{\pi}}{4c^2}e^{-2ac}. \quad . \quad . \quad . \quad . \quad (20)$$

By the aid of Fourier's theorem, that if

$$f(r) = \sqrt{\frac{2}{\pi}}\int_0^\infty \phi(x)\sin rx\,dx,$$

then

$$\phi(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(r)\sin rx\,dr,$$

we derive from (12), which may be written

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-\frac{x^2}{4a^2}}}{x}\sin rx\,dx=\sqrt{\frac{\pi}{2}}-\sqrt{2}\,\text{Erf}\,ar,$$

the result

$$\int_0^\infty \sin rx\,dr-\frac{2}{\sqrt{\pi}}\int_0^\infty \text{Erf}\,ar\sin rx\,dr=\frac{e^{-\frac{x^2}{4a^2}}}{x},$$

in which it is easy to see that we may legitimately put $\cos\infty=0$ and obtain

$$\int_0^\infty \text{Erf}\,ar\sin rx\,dr=\frac{\sqrt{\pi}}{2x}\left(1-e^{-\frac{x^2}{4a^2}}\right). \quad . \quad . \quad (21)$$

This result can also be obtained independently and in a more simple manner by integrating

$$\int_0^\infty e^{-a^2x^2}\sin rx\,x\,dx=\frac{r\sqrt{\pi}}{4a^3}e^{-\frac{r^2}{4a^2}}$$

with regard to a.

The well-known formula

$$\int_0^\infty e^{-a^2x}\cos bx\,dx=\frac{a^2}{a^4+b^2}$$

affords, on integration,

$$\int_0^\infty \text{Erf}\,a\sqrt{x}\cos bx\,\frac{dx}{\sqrt{x}}=\int_a^\infty \frac{a^2}{a^4+b^2}$$

$$=\frac{1}{2\sqrt{2b}}\left\{\pi-\frac{1}{2}\log\frac{a^2-a\sqrt{2b}+b}{a^2+a\sqrt{2b}+b}-\tan^{-1}\left(a\sqrt{\frac{2}{b}}+1\right)\right.$$

$$\left.-\tan^{-1}\left(a\sqrt{\frac{2}{b}}-1\right)\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and similarly from

$$\int_0^\infty e^{-a^2x}\sin bx\,dx=\frac{b}{a^4+b^2}$$

we deduce

$$\int_0^\infty \text{Erf}\,a\sqrt{x}\sin bx\,\frac{dx}{\sqrt{x}}$$

$$=\frac{1}{2\sqrt{2b}}\left\{\frac{1}{2}\log\frac{a^2-a\sqrt{2b}+b}{a^2+a\sqrt{2b}+b}+\tan^{-1}\left(\frac{\sqrt{2b}}{a}+1\right)\right.$$

$$\left.+\tan^{-1}\left(\frac{\sqrt{2b}}{a}-1\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Many other formulæ could, no doubt, be found; but the above probably include the most simple cases. When the constants have imaginary values assigned to them, the results suggest the real values of the integrals; for example, write bi ($i=\sqrt{-1}$ as usual) for b in

$$\int_0^\infty e^{-ax^2-2bx}\,dx=\frac{1}{\sqrt{a}}\,e^{\frac{b^2}{a}}\,\mathrm{Erf}\,\frac{b}{\sqrt{a}},$$

and there results

$$\int_0^\infty e^{-ax^2}(\cos 2bx-i\sin 2bx)dx=\frac{1}{\sqrt{a}}\,e^{-\frac{b^2}{a}}\,\mathrm{Erf}\,\frac{bi}{\sqrt{a}},$$

and
$$\mathrm{Erf}\,\frac{bi}{\sqrt{a}}=\frac{\sqrt{\pi}}{2}-\int_0^{\frac{bi}{\sqrt{a}}}e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}-i\int_0^{\frac{b}{\sqrt{a}}}e^{x^2}\,dx,$$

whence

$$\int_0^\infty e^{-ax^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2\sqrt{a}}\,e^{-\frac{b^2}{a}},$$

and
$$\int_0^\infty e^{-ax^2}\sin 2bx\,dx=\frac{1}{\sqrt{a}}\,e^{-\frac{b^2}{a}}\int_0^{\frac{b}{\sqrt{a}}}e^{x^2}\,dx.$$

The former result is well known; and the latter is easily verified by differentiating with respect to b and forming a differential equation, from which the value of the integral can be determined.

As another example, writing bi for b in (14), and noticing that

$$\int_{-\infty}^\infty e^{-cx^2}\sin 2bx\,dx=0,$$

we find

$$\int_{-\infty}^\infty \frac{e^{-cx^2\pm 2bx}}{a^2+x^2}\,dx$$

$$=\frac{\sqrt{\pi}}{a}e^{a^2c}\left\{e^{2abi}\,\mathrm{Erf}\left(a\sqrt{c}+\frac{bi}{\sqrt{c}}\right)+e^{-2abi}\,\mathrm{Erf}\left(a\sqrt{c}-\frac{bi}{\sqrt{c}}\right)\right\}. \quad (24)$$

Now
$$\mathrm{Erf}\left(a\sqrt{c}+\frac{bi}{\sqrt{c}}\right)-\mathrm{Erf}\,(a\sqrt{c})=-\int_{a\sqrt{c}}^{a\sqrt{c}+\frac{bi}{\sqrt{c}}}e^{-x^2}\,dx$$

$$=-\frac{e^{-a^2c}}{\sqrt{c}}\int_0^{bi}e^{-\frac{x^2}{c}-2ax}\,dx=\frac{e^{-a^2c}}{i\sqrt{c}}\int_0^b e^{\frac{b^2}{c}-2abi}\,db,$$

whence

$$\mathrm{Erf}\left(a\sqrt{c}+\frac{bi}{\sqrt{c}}\right)+\mathrm{Erf}\left(a\sqrt{c}-\frac{bi}{\sqrt{c}}\right)=2\,\mathrm{Erf}\,a\sqrt{c}$$
$$-2\,\frac{e^{-2a^2c}}{\sqrt{c}}\int_0^b e^{\frac{b^2}{c}}\sin 2ab\,db,$$

$$\mathrm{Erf}\left(a\sqrt{c}+\frac{bi}{\sqrt{c}}\right)-\mathrm{Erf}\left(a\sqrt{c}-\frac{bi}{\sqrt{c}}\right)=2\,\frac{e^{-a^2c}}{i\sqrt{c}}\int_0^b e^{\frac{b^2}{c}}\cos 2\;ab\,db$$

and we have

$$\int_{-\infty}^{\infty} \frac{e^{-cx^2 \pm 2bx}}{a^2+x^2}\,dx$$

$$= 2\,\frac{\sqrt{\pi}}{a}\left\{ e^{a^2c}\,\mathrm{Erf}\,(a\sqrt{c})\,\cos 2ab + \frac{\sin 2ab}{\sqrt{c}}\int_0^b e^{\frac{b^2}{c}}\cos 2ab\,db \right.$$

$$\left. - \frac{\cos 2ab}{\sqrt{c}}\int_0^b e^{\frac{b^2}{c}}\sin 2ab\,db \right\}, \quad . \quad (25)$$

a result which can be verified by forming the differential equation

$$\frac{d^2y}{db^2} + 4a^2y = 4\int_{-\infty}^{\infty} e^{-cx^2 \pm 2bx}\,dx = 4e^{\frac{b^2}{c}}\sqrt{\frac{\pi}{c}},$$

y denoting the integral, which, on integrating and determining the constants by the considerations that the result must be independent of the sign of b, and that when $b=0$ it must equal

$$\frac{2\sqrt{\pi}e^{a^2c}}{a}\,\mathrm{Erf}\,(a\sqrt{c}),$$

gives the same expression for y.

The result (25) may be written in another form for

$$\mathrm{Erf}\left(a\sqrt{c}+\frac{bi}{\sqrt{c}}\right) = \int_{a\sqrt{c}}^{\infty} e^{-\left(x+\frac{bi}{\sqrt{c}}\right)^2}dx$$

$$= e^{\frac{b^2}{c}}\int_{a\sqrt{c}}^{\infty} e^{-x^2}\left(\cos\frac{2bx}{\sqrt{c}} - i\sin\frac{2bx}{\sqrt{c}}\right)dx,$$

so that

$$\mathrm{Erf}\left(a\sqrt{c}+\frac{bi}{\sqrt{c}}\right) + \mathrm{Erf}\left(a\sqrt{c}-\frac{bi}{\sqrt{c}}\right) = 2e^{\frac{b^2}{c}}\int_{a\sqrt{c}}^{\infty} e^{-x^2}\cos\frac{2bx}{\sqrt{c}}\,dx,$$

$$\mathrm{Erf}\left(a\sqrt{c}-\frac{bi}{\sqrt{c}}\right) - \mathrm{Erf}\left(a\sqrt{c}+\frac{bi}{\sqrt{c}}\right) = 2ie^{\frac{b^2}{c}}\int_{a\sqrt{c}}^{\infty} e^{-x^2}\sin\frac{2bx}{\sqrt{c}}\,dx.$$

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ON A CLASS OF DEFINITE INTEGRALS.

PART II.

BY

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BEFORE noticing the applications and Tables of the Error-function referred to in my previous communication on the subject, it seems desirable to supplement the integrals already obtained by several additional formulæ.

The integral $\int_0^x e^{-u^2}du$ is so frequently used that it is convenient to have a separate notation for it apart from its value $\frac{1}{2}\sqrt{\pi}-\text{Erf}\,x$. Denoting this integral, therefore, by $\text{Erfc}\,x$ (*i. e.* the *Error-function-complement* of x)*, we have

$$\int_x^\infty e^{-u^2}\,du=\text{Erf}\,x,\quad \int_0^x e^{-u^2}\,du=\text{Erfc}\,x,$$

and

$$\text{Erf}\,x+\text{Erfc}\,x=\frac{\sqrt{\pi}}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

x being supposed positive. If x be negative, since

$$\text{Erf}\,(-x)=-\text{Erf}\,x,$$

the relation is

$$\text{Erf}\,x+\text{Erfc}\,x=-\frac{\sqrt{\pi}}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

This enables us to express in a simple manner some of the integrals previously obtained; for instance, (12) may be written

$$\int_0^\infty e^{-a^2x^2}\sin 2rx\frac{dx}{x}=\sqrt{\pi}\,\text{Erfc}\left(\frac{r}{a}\right).$$

In his *Exercices de Mathématiques,* 1827, Cauchy has given the theorem

* This notation is in harmony with that adopted in the case of the sine and cosine; the cosine of x is the sine of the complement of x (not the complement of $\sin x$), while $\text{Erfc}\,x$ (in which the letter standing for complement is at the end of the word) denotes the complement of $\text{Erf}\,x$. Similarly in the case of the sine-integral, it would be convenient to write $\text{Sic}\,x$ for $\frac{1}{2}\pi-\text{Si}\,x$.

$$e^{-x^2}+e^{-(x-a)^2}+e^{-(x+a)^2}+e^{-(x-2a)^2}+e^{-(x+2a)^2}+\ldots$$
$$=\frac{2\sqrt{\pi}}{a}\left\{\frac{1}{2}+e^{-\frac{\pi^2}{a^2}}\cos\frac{2\pi x}{a}+e^{-\frac{4\pi^2}{a^2}}\cos\frac{4\pi x}{a}+\ldots\right\}; \quad (28)$$

from which, by writing $2a$ for a, and subtracting the original series from the double of the series so formed, we obtain*

$$e^{-x^2}-e^{-(x-a)^2}-e^{-(x+a)^2}+e^{-(x-2a)^2}+e^{-(x+2a)^2}-\ldots$$
$$=\frac{2\sqrt{\pi}}{a}\left\{e^{-\frac{\pi^2}{4a^2}}\cos\frac{\pi x}{a}+e^{-\frac{9\pi}{4a^2}}\cos\frac{3\pi x}{a}+\ldots\right\}. \quad . \quad (29)$$

Integrating these series between the limits x and 0, we find

$$\text{Erfc}\, x+\text{Erfc}\,(x-a)+\text{Erfc}\,(x+a)+\ldots$$
$$=\frac{x\sqrt{\pi}}{a}+\frac{1}{\sqrt{\pi}}\left\{e^{-\frac{\pi^2}{a^2}}\sin\frac{2\pi x}{a}+\frac{1}{2}e^{-\frac{4\pi^2}{a^2}}\sin\frac{4\pi x}{a}+\ldots\right\}, \quad (30)$$

$$\text{Erfc}\, x-\text{Erfc}\,(x-a)-\text{Erfc}\,(x+a)+\ldots$$
$$=\frac{2}{\sqrt{\pi}}\left\{e^{-\frac{\pi^2}{4a^2}}\sin\frac{\pi x}{a}+\frac{1}{3}e^{-\frac{9\pi^2}{a^2}}\sin\frac{3\pi x}{a}+\ldots\right\}. \quad (31)$$

These formulæ, however, involve an ambiguity; for on the left-hand side the terms at a great distance from the commencement of the series take the form $\frac{\sqrt{\pi}}{2}(1-1+1-\ldots)$; and if we substitute $\frac{\sqrt{\pi}}{2}-\text{Erf}\, x$ for $\text{Erfc}\, x$ &c., we still introduce the same indeterminate series. The difficulty would not be avoided by integrating between the limits ∞ and x instead of x and 0; for then, on the right-hand side, we should introduce $\sin\infty$ into both formulæ, and in addition an infinite term in (30).

We should, however, from the results of similar inquiries, be inclined to suspect that in point of fact we must, with the exception of the first term, take the terms in pairs, so as not to end with a term $\text{Erfc}\,(x\pm na)$ without including also $\text{Erfc}\,(x\mp na)$ (n infinite); and the following independent investigation will show that this is the case.

From the integral (21) of the previous paper we find that

$$\int_0^\infty \text{Erf}\, x \sin\frac{2n\pi x}{a}\,dx=\frac{a}{4n\sqrt{\pi}}\left(1-e^{-\frac{n^2\pi^2}{a^2}}\right),$$

whence

$$\int_0^a\{\text{Erf}\, x+\text{Erf}\,(x-a)+\text{Erf}\,(x+a)+\ldots\}\sin\frac{2n\pi x}{a}dx$$
$$=\frac{a}{2n\sqrt{\pi}}\left(1-e^{-\frac{n\,^2\pi}{n^2}}\right);$$

* The series (29) is given by Sir W. Thomson (Quarterly Journal of Mathematics, vol. i. p. 316); and (31) is deduced by integration; the ambiguity, however, is not noticed.

and therefore, by Fourier's theorem, between the limits 0 and a of x,

$$\operatorname{Erf} x+\operatorname{Erf}(x-a)+\operatorname{Erf}(x+a)+\ldots$$
$$=\frac{1}{\sqrt{\pi}}\left\{\sin\frac{2\pi x}{a}+\frac{1}{2}\sin\frac{4\pi x}{a}+\ldots-e^{-\frac{\pi^2}{a^2}}\sin\frac{2\pi x}{a}\right.$$
$$\left.-\frac{1}{2}e^{-\frac{4\pi^2}{a^2}}\sin\frac{4\pi x}{a}-\ldots\right\}$$
$$=\frac{1}{2}\sqrt{\pi}-\frac{x\sqrt{\pi}}{a}-\frac{1}{\sqrt{\pi}}\left\{e^{-\frac{\pi^2}{a^2}}\sin\frac{2\pi x}{a}+\frac{1}{2}e^{-\frac{4\pi^2}{a^2}}\sin\frac{4\pi x}{a}+\ldots\right\} \quad (32)$$

on replacing the former trigonometrical series by its sum $\frac{1}{2}\left(\pi-\frac{2\pi x}{a}\right)$.

We can either deduce from this equation or prove independently that

$$\operatorname{Erf} x-\operatorname{Erf}(x-a)-\operatorname{Erf}(x+a)+\ldots$$
$$=\frac{1}{2}\sqrt{\pi}-\frac{2}{\sqrt{\pi}}\left\{e^{-\frac{\pi^2}{4a^2}}\sin\frac{\pi x}{a}+\frac{1}{3}e^{-\frac{9\pi^2}{a^2}}\sin\frac{3\pi x}{a}+\ldots\right\}. \quad (33)$$

If x be negative in (32) or (33), the sign of the constant $\frac{1}{2}\sqrt{\pi}$ must, of course, be changed.

Putting $x=\frac{1}{2}a$ in (33), we find

$$\operatorname{Erf} a-\operatorname{Erf} 3a+\operatorname{Erf} 5a-\ldots$$
$$=\frac{1}{4}\sqrt{\pi}-\frac{1}{\sqrt{\pi}}\left(e^{-\frac{\pi^2}{16a^2}}-\frac{1}{3}e^{-\frac{9\pi^2}{16a^2}}+\ldots\right), \quad (34)$$

writing a for $\frac{1}{2}a$.

This formula, (34), I have verified numerically to seven decimal places when $a=\frac{1}{2}$.

In (29) put $x=0$, and we have

$$e^{-a^2}-e^{-4a^2}+e^{-9a^2}-\ldots=\frac{1}{2}-\frac{\sqrt{\pi}}{a}\left\{e^{-\frac{\pi^2}{4a^2}}+e^{-\frac{9\pi^2}{a^2}}+\ldots\right\}. (35)$$

Integrate this between the limits x and 0, and we obtain an interesting formula connecting the error-function and the exponential-integral, viz.

$$\operatorname{Erfc} x-\tfrac{1}{2}\operatorname{Erfc} 2x+\tfrac{1}{3}\operatorname{Erfc} 3x-\ldots$$
$$=\frac{1}{2}x+\frac{\sqrt{\pi}}{2}\left\{\operatorname{Ei}\left(-\frac{\pi^2}{4x^2}\right)+\operatorname{Ei}\left(-\frac{9\pi^2}{4x^2}\right)+\ldots\right\}, \quad (36)$$

since

$$\int_0^x \frac{e^{-\frac{n^2}{a^2}}}{a}\,da=\int_{\frac{1}{x}}^{\infty} e^{-n^2a^2}\frac{da}{a}=\frac{1}{2}\int_{\frac{1}{x^2}}^{\infty} e^{-n^2 a}\frac{da}{a}=-\frac{1}{2}\operatorname{Ei}\left(-\frac{n^2}{x^2}\right).$$

Putting $x=0$ in (28), we find

$$\frac{1}{2}+e^{-a^2}+e^{-4a^2}+\ldots=\frac{\sqrt{\pi}}{a}\left\{\frac{1}{2}+e^{-\frac{\pi^2}{a^2}}+e^{-\frac{4\pi^2}{a^2}}+\ldots\right\}, \quad (37)$$

which on integration between limits gives

$$\frac{a-b}{2}+(\text{Erf}\,b-\text{Erf}\,a)+\frac{1}{2}(\text{Erf}\,2b-\text{Erf}\,2a)+\ldots=\frac{\sqrt{\pi}}{2}\log\frac{a}{b}$$
$$+\frac{\sqrt{\pi}}{2}\left\{\text{Ei}\left(-\frac{\pi^2}{b^2}\right)-\text{Ei}\left(-\frac{\pi^2}{a^2}\right)+\text{Ei}\left(-\frac{4\pi^2}{b^2}\right)-\text{Ei}\left(-\frac{4\pi^2}{a^2}\right)+\ldots\right\}$$

and many similar formulæ could, no doubt, be obtained.

It is a matter of some importance in regard to the use of the function to be enabled to replace it, when the argument is imaginary, by a real integral; this may, of course, be done in many ways, but probably the most convenient forms will be found to be those deduced from the integral

$$\int_0^\infty e^{-ax^2-2bx}\,dx=\frac{1}{\sqrt{a}}\,e^{\frac{b^2}{a}}\,\text{Erf}\,\frac{b}{\sqrt{a}}.$$

Writing in this integral $a(\cos\alpha+i\sin\alpha)$ and $b(\cos\beta+i\sin\beta)$ for a and b respectively, there results

$$\int_0^\infty e^{-ax^2\cos\alpha-2bx\cos\beta}\{\cos(ax^2\sin\alpha+2bx\sin\beta)$$
$$-i\sin(ax^2\sin\alpha+2bx\sin\beta\}\,dx$$

$$=\frac{1}{\sqrt{a}}\,e^{\frac{b^2}{a}\cos(2\beta-\alpha)}\left[\cos\left\{\frac{b^2}{a}\sin(2\beta-\alpha)-\frac{1}{2}\alpha\right\}\right.$$
$$\left.+i\sin\left\{\frac{b^2}{a}\sin(2\beta-\alpha)-\frac{1}{2}\alpha\right\}\right]\text{Erf}\,\frac{b}{\sqrt{a}}\left\{\cos\left(\beta-\frac{1}{2}\alpha\right)\right.$$
$$\left.+i\sin\left(\beta-\frac{1}{2}\alpha\right)\right\}. \quad (38)$$

Numerous special forms are deducible for $\text{Erf}(\text{A}+\text{B}i)$ by giving particular or zero values to a, b, α, and β; $a\cos\alpha$, however, must not be made negative.

Two of the most simple forms are given at the end of the previous paper; from them we see that, to form a Table of the error-function for arguments of the form $a+bi$, it would be necessary to tabulate

$$e^{-a^2}\int_0^b e^{x^2}\sin 2ax\,dx \text{ and } e^{-a^2}\int_0^b e^{x^2}\cos 2ax\,dx,$$

or

$$e^{b^2}\int_a^\infty e^{-x^2}\sin 2bx\,dx \text{ and } e^{b^2}\int_a^\infty e^{-x^2}\cos 2bx\,dx.$$

The Table would be of double entry, and its calculation would entail more labour than the importance of the function at present merits. By the comparison of the two forms for

$$\text{Erf}\,(a+bi) \pm \text{Erf}\,(a-bi)$$

just referred to, we have incidentally* (putting $c=1$) the two theorems

$$e^{b^2}\int_a^\infty e^{-x^2}\sin 2bx\,dx = e^{-a^2}\int_0^b e^{x^2}\cos 2ax\,dx, \quad . \quad . \quad (39)$$

$$e^{b^2}\int_a^\infty e^{-x^2}\cos 2bx\,dx + e^{-a^2}\int_0^b e^{x^2}\sin 2ax\,dx = \text{Erf}\,a. \quad . \quad (40)$$

Denoting for brevity $\frac{1}{\sqrt{a}}e^{\frac{b^2}{4a}}\text{Erf}\frac{b}{2\sqrt{a}}$ by u, we see from (10) that $\left(-\frac{d}{da}\right)^n u = \left(\frac{d}{db}\right)^{2n} u$, which on taking $2\sqrt{a}=\alpha$, and replacing α by a, assumes the form

$$\left(\frac{d}{db}\right)^{2n} e^{\frac{b^2}{a^2}}\text{Erf}\frac{b}{a} = a\left(-\frac{2d}{ada}\right)^n \frac{1}{a}e^{\frac{b^2}{a^2}}\text{Erf}\frac{b}{a},$$

or, as we may write it after an obvious transformation,

$$\left(\frac{d}{db}\right)^{2n} e^{a^2b^2}\,\text{Erf}\,ab = \frac{1}{a}\left(2a^3\frac{d}{da}\right)^n ae^{a^2b^2}\,\text{Erf}\,ab, \quad . \quad . \quad (41)$$

a good instance of a result obtained at once from a definite integral, but which it would not be easy to prove otherwise.

Among miscellaneous formulæ may be noticed

$$\int_0^\infty e^{-\frac{a^2}{x^2}}\,\text{Erf}\,(cx)\frac{dx}{x} = -\frac{\sqrt{\pi}}{2}\,\text{Ei}\,(-2ac) = \int_0^\infty e^{-a^2x^2}\,\text{Erf}\left(\frac{c}{x}\right)\frac{dx}{x} \quad (42)$$

$$\int_0^\infty (\text{Erf}\,ax - \text{Erf}\,bx)\cos 2rx\,\frac{dx}{x} = \frac{\sqrt{\pi}}{4}\left\{\text{Ei}\left(-\frac{r^2}{a^2}\right) - \text{Ei}\left(-\frac{r^2}{b^2}\right)\right\} \quad (43)$$

$$\int_0^\infty \text{Erf}\sqrt{c(x^2+a^2)}\,\frac{dx}{\sqrt{(x^2+a^2)}} = -\frac{\sqrt{\pi}}{4}\,\text{Ei}\,(-a^2c), \quad . \quad . \quad (44)$$

deduced by integration from the formula intermediate to (19) and (20), from (11), and from a formula intermediate to (4) and (5) respectively.

From No. 17, Table 267, and No. 1, Table 266 of De Haan's *Nouvelles Tables d'Intégrales définies,* we deduce by dividing by p (p^2 having been previously written for p in the latter integral), and integrating with respect to p between the limits ∞ and q,

* On page 301, bottom line but one, e^{-2a^2c} should be e^{-a^2c}.

that

$$\int_0^\infty \frac{\sin(2a+1)x}{\sin x}\,\text{Ei}(-q^2x^2)dx$$

$$= -2\sqrt{\pi}\left\{\frac{1}{2q} + \text{Erfc}\frac{1}{q} + \frac{1}{2}\text{Erfc}\frac{2}{q}\ldots + \frac{1}{a}\text{Erfc}\frac{a}{q}\right\}, \quad (45)$$

and

$$\int_0^\infty \frac{\text{Ei}(-q^2x^2)\,dx}{1-2r\cos x+r^2}$$

$$= -\frac{4}{1-r^2}\left\{\frac{1}{8q} + r\,\text{Erfc}\frac{1}{2q} + \frac{r^2}{2}\,\text{Erfc}\frac{2}{2q} + \ldots\right\}, \quad (46)$$

the latter series extending to infinity.

A method similar to one frequently given for the evaluation of $\int_0^\infty e^{-x^2}dx$ enables us to express the product of two error-functions as a single integral for

$$\text{Erfc}\,a\,\text{Erfc}\,b = \int_0^a\int_0^b e^{-x^2-y^2}\,dx\,dy,$$

which on transforming to polar coordinates becomes

$$= \frac{1}{2}\left\{\int_0^{\tan^{-1}\frac{b}{a}}(1-e^{-a^2\sec^2\theta})\,d\theta + \int_{\tan^{-1}\frac{b}{a}}^{\frac{1}{2}\pi}(1-e^{-b^2\text{cosec}^2\theta})\,d\theta\right\}$$

$$= \frac{\pi}{4} - \frac{1}{2}ae^{-a^2}\int_0^b \frac{e^{-x^2}dx}{a^2+x^2} - \frac{1}{2}be^{-b^2}\int_0^a\frac{e^{-x^2}dx}{b^2+x^2} \quad . \quad . \quad . \quad (47)$$

after a couple of obvious transformations.

Taking $a=b$, we find

$$(\text{Erfc}\,a)^2 = \frac{\pi}{4} - e^{-a^2}\int_0^1\frac{e^{-a^2x^2}dx}{1+x^2},$$

whence

$$\int_0^1\frac{e^{-a^2x^2}dx}{1+x^2} = e^{a^2}\{\sqrt{\pi}\,\text{Erf}\,a - 2(\text{Erf}\,a)^2\}. \quad . \quad . \quad (48)$$

The equation (48) is not new, being only a simple transformation of one given by Raabe in 1847*. Raabe shows that

$$\int_0^\infty \frac{e^{-ax^2}}{1+x^2}dx = e^{\frac{1}{2}a}\sqrt{\pi}\left\{\frac{1}{2}\sqrt{\pi}e^{\frac{1}{2}a} - \sqrt{f(a)}\right\},$$

* "Ueber Producte und Potenzen bestimmter einfacher Integral-Ausdrücke, durch mehrfache dargestellt," Crelle's *Journal*, vol. xlviii. p. 137. See also De Haan's *Nouvelles Tables*, No. 2. T. 29, and No. 5. T. 80.

where

$$f(a)=a+\left(1-\frac{1}{3}\right)\frac{a^2}{1.2}+\left(1-\frac{1}{3}+\frac{1}{5}\right)\frac{a^3}{1.2.3}+\dots.$$

Comparing this result with (5), we find

$$\overline{\text{Erfc}}\,x=e^{-\frac{1}{2}x^2}\sqrt{f(x^2)}. \quad \dots\dots \quad (49)$$

The forms assumed by (5) and (44) when x is taken $=\tan\theta$ are worthy of attention from their simplicity; we have

$$\int_0^{\frac{1}{2}\pi} e^{-a\tan^2\theta}\,d\theta=\sqrt{\pi}e^a\,\text{Erf}\,\sqrt{a}, \quad \dots \quad (50)$$

and

$$\int_0^{\frac{1}{2}\pi}\text{Erf}\,(a\sec\theta)\frac{d\theta}{\cos\theta}=-\frac{\sqrt{\pi}}{4}\,\text{Ei}\,(-a^2). \quad (51)$$

Writing x^2 for x in (18), we find the area of the curve $y=\text{Erf}\,ax$, viz.

$$\int_0^\infty \text{Erf}\,(ax)\,dx=\frac{1}{2a}. \quad \dots\dots \quad (52)$$

For the numerical calculation of the values of the error-function, there are three series, viz.

$$\text{Erfc}\,x=x-\frac{x^3}{3}+\frac{1}{1.2}\frac{x^5}{5}-\frac{1}{1.2.3}\frac{x^7}{7}+\dots, \quad \dots \quad (53)$$

$$\text{Erfc}\,x=xe^{-x^2}\left(1+\frac{2x^2}{1.3}+\frac{(2x^2)^2}{1.3.5}+\frac{(2x^2)^3}{1.3.5.7}+\dots\right), \quad (54)$$

$$\text{Erf}\,x=\frac{e^{-x^2}}{2x}\left(1-\frac{1}{2x^2}+\frac{1.3}{(2x^2)^2}-\frac{1.3.5}{(2x^2)^3}+\dots\right), \quad (55)$$

and the continued fraction

$$\text{Erf}\,x=\frac{e^{-x^2}}{2x}\,\frac{1}{1+}\,\frac{1}{2x^2+}\,\frac{2}{1+}\,\frac{3}{2x^2+}\,\frac{4}{1+\&\text{c.}}. \quad \dots \quad (56)$$

The formula (34) might also be of use in the calculation of the integral.

The discontinuity in the values of the constants and other points connected with the series (55), when x is of the form $a+bi$, are fully discussed by Professor Stokes in a memoir "On the Discontinuity of Arbitrary Constants which appear in Divergent Developments," Camb. Phil. Trans. vol. x.

Applications of the Error-function to Physics &c.

The work which first gave the error-function an importance of the first rank in physics was Kramp's *Analyse des Réfractions Astronomiques et Terrestres,* Strasbourg, 1798. Kramp shows (page 36) that "all the great problems of astronomical and terrestrial refraction depend on the integration of the differential equation"

$$d\text{R} = -\frac{\omega \text{Y} v\, dv}{c\sqrt{(1-v^2-2\omega+2\omega\text{Y})}}, \quad . \quad . \quad . \quad (57)$$

in which dR denotes an element of the curvature of the path of the ray (viz. $\frac{ds}{\rho}$, in the usual notation, ρ being the radius of curvature), $v = \frac{a \sin \text{A}}{y}$ (a being the earth's radius, A the angle of incidence of the ray at the surface of the earth, and y the radius vector from the centre of the earth to a point of the ray's path), Y the density of the air $= e^{\frac{v - \sin \text{A}}{c}}$, while c and ω (which are very small) are independent of Y, and also therefore of v.

The limits of integration are a and ∞ for y, and therefore sin A and 0 for v.

To effect the integration, Kramp expands the right-hand side of (57) in a series proceeding according to ascending powers of ω, viz.

$$d\text{R} = -\frac{\omega \text{Y} v dv}{c(1-v^2)^{\frac{1}{2}}} - \frac{\omega^2 \text{Y}(1-\text{Y}) v dv}{c(1-v^2)^{\frac{3}{2}}} - \frac{3}{2} \cdot \frac{\omega^3 \text{Y}(1-\text{Y})^2 v dv}{c(1-v^2)^{\frac{5}{2}}} - \ldots,$$

and treats each term separately.

To integrate $\frac{\text{Y} v dv}{(1-v^2)^{n+}}$, put $ct = 1 - v$, and we obtain

$$e^{\frac{\text{K}}{c}} c^{-n+\frac{1}{2}} \int_{\frac{\text{K}}{c}}^{\infty} (2t - ct^2)^{-n-\frac{1}{2}} (1-ct) e^{-1} dt,$$

K being written for $1 - \sin \text{A}$ (Kramp, p. 118).

In the problem of refraction we may neglect powers of c and retain only the first term of the integral, so that we are only concerned with the integral

$$\int_{\frac{\text{K}}{c}}^{\infty} t^{-n-\frac{1}{2}} e^{-t} dt;$$

that is, with

$$\int_x^{\infty} t^{-2n} e^{-t^2} dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

x being written for $\sqrt{\left(\frac{1-\sin A}{c}\right)}$. By integration by parts, it is easily shown that (58) is reduced to dependence on the error-function according to the formula

$$(-)^n \int_x^\infty t^{-2n} e^{-t^2} dt = \frac{2^n}{1.3\ldots(2n-1)} \text{Erf}\, x$$

$$- \frac{2^{n-1}}{1.3\ldots(2n-1)} \left\{ \frac{1}{x} - \frac{1}{2x^3} + \frac{1.3}{4x^5} - \ldots \pm \frac{1.3\ldots(2n-3)}{2^{n-1}\, x^{2n-1}} \right\}^{*},$$

so that for the calculation of refraction a Table of the values of Erf x was necessary.

In 1805 the fourth volume of Laplace's *Mécanique Céleste* was published. Chapters I. and II. of Livre X. are devoted to Astronomical and Terrestrial Refraction; and the error-function necessarily occupies a conspicuous place in the investigation. On page 285 † Laplace investigates the continued fraction (56).

To give in a short space any account of the applications of the function to the Theory of Probabilities would be impossible; but the results are so well known that a detailed description of them would be superfluous. In the Theory of Errors of Observations the function is of paramount importance; and this fact is the justification of the name and symbol by which it has been here denoted. Laplace's great work on the subject, the *Théorie analytique des Probabilités,* was published in 1812; but most of the results had previously appeared in various memoirs. The law of facility to which Laplace's, Poisson's, Gauss's, in fact all investigations lead, is represented by e^{-cx^2}, so that the probability of a single error of observation lying between x and $x+dx$ is $\left(\frac{c}{\pi}\right)^{\frac{1}{2}} e^{-cx^2}dx$, and the chance of an error lying between p and q is

$$\left(\frac{c}{\pi}\right)^{\frac{1}{2}} \int_p^q e^{-cx^2} dx = \pi^{-\frac{1}{2}}(\text{Erf}\, p\sqrt{c} - \text{Erf}\, q\sqrt{c}).$$

It is unnecessary to notice the papers in which the theory of errors and the method of least squares are discussed. De Morgan's treatise in the *Encyclopædia Metropolitana* contains an analysis of Laplace's work; and Todhunter's 'History of Probabilities' supplies a commentary to it. References will be found in a memoir by Ellis, Camb. Phil. Trans. vol. viii. p. 204.

* Kramp, p. 133.
† The reference is to the National Edition, 1845.

Todhunter remarks (p. 486 of his 'History') that in a memoir of 1783 Laplace pointed out the use of tabulating the integral $\int e^{-t^2}dt$ for different limits.

In the Theory of the Conduction of Heat the function occupies a place of great importance. Thus Fourier proves* that if one extremity of a bar be kept at a constant temperature equal to unity, the initial temperature at all points in the rest of the bar being zero, then the temperature v at time t will be given by the formula

$$v=e^{-cx}-\frac{e^{-cx}}{\sqrt{\pi}}\operatorname{Erf}\left(\sqrt{ht}-\frac{x}{2\sqrt{kt}}\right)+\frac{e^{cx}}{\sqrt{\pi}}\operatorname{Erf}\left(\sqrt{ht}+\frac{x}{2\sqrt{kt}}\right),$$

c being written for $\sqrt{\frac{h}{k}}$.

On page 514 of the memoir of Fourier's just referred to is given the proposition from which Sir W. Thomson has deduced his results with regard to the secular cooling of the earth†. Fourier's problem is, that if in a solid extending to infinity in all directions the temperature has initially two different constant values, viz. unity and zero on the two sides of a certain infinite plane, then at time t the temperature v will be given by the equation

$$v=\frac{2}{\sqrt{\pi}}\operatorname{Erf}\frac{x}{2\sqrt{kt}}.$$

Problems dependent for their solution on the error-function are also discussed by Fourier, *loc. cit.* p. 516 *et seqq.*, and by Riemann, pp. 124, 164, &c. of his *Partielle Differentialgleichungen*, &c. (Braunschweig, 1869). On page 169 Riemann proves‡ that

$$\int_a^b e^{-x^2-\frac{c}{x^2}}dx=\tfrac{1}{2}e^{2c}\left\{\operatorname{Erf}\left(a+\frac{c}{a}\right)-\operatorname{Erf}\left(b+\frac{c}{b}\right)\right\}$$

$$+\tfrac{1}{2}e^{-2c}\left\{\operatorname{Erf}\left(a-\frac{c}{a}\right)-\operatorname{Erf}\left(b-\frac{c}{b}\right)\right\}. \quad . \quad . \quad (59)$$

A particular case of this elegant theorem was discovered by Boole§ in 1849; his result may be written

* *Mémoires de l'Institut*, t. iv. (1819 and 1820), p. 508.

† Thomson and Tait's 'Natural Philosophy,' vol. i. p. 717, where other references are given.

‡ Riemann has left out the factor $\frac{1}{2}$ on the right-hand side.

§ Cambridge and Dublin Mathematical Journal, vol. iv. p. 18.

$$\int_0^{\sqrt{c}} e^{-x^2-\frac{c^2}{x^2}}dx = \frac{\sqrt{\pi}}{4}e^{-2c} - \tfrac{1}{2}e^{2c}\,\text{Erf}\,(2\sqrt{c}),$$

agreeing with (59).

It need scarcely be remarked that the function owes its importance in the Theory of Heat to the fact of $u = \text{Erf}\,\dfrac{x}{2\sqrt{kt}}$ being an integral of

$$\frac{du}{dt} = k\frac{d^2u}{dx^2}.$$

From its uses in physics, it will be evident that the error-function may fairly claim at present to rank in importance next to the trigonometrical and logarithmic functions.

Tables of the Error-function.

For large values of x the formula (55) converges rapidly for some distance, and affords a means of calculating $\text{Erf}\,x$ and $e^{x^2}\,\text{Erf}\,x$ with ease; and for small values the formula (53) gives $\text{Erfc}\,x$ conveniently. For intermediate values the three formulæ (53), (54), and (55) are all inappropriate. It was this difficulty of calculation which induced Kramp to compute his Tables. Speaking of the series (53) and (55), he remarks:—"Les deux séries laissent donc entr'elles un vuide pour l'évaluation exacte de l'intégrale, qui peut aller depuis $x=\frac{1}{4}$ jusqu'à $x=4$, et que nous ne pouvons remplir par aucune des méthodes connues. Comme la connoissance de cette intégrale est absolument essentielle pour le calcul des réfractions qui approchent de l'horizon; comme elle est également indispensable dans l'analyse de hasards; comme de plus la solution d'une infinité d'équations différentielles revient à cette même intégrale, j'ai cru qu'il valoit la peine d'en calculer la table depuis $x=0{\cdot}01$ jusqu'à $x=3{\cdot}00$" (*Traité des Réfractions*, p. 134). The Tables which Kramp calculated are three in number. Table I. gives $\text{Erf}\,x$ from $x=0$ to $x=1{\cdot}24$ to eight decimal places, from $x=1{\cdot}24$ to $x=1{\cdot}50$ to nine places, from $x=1{\cdot}50$ to $x=2{\cdot}00$ to ten places, from $x=2{\cdot}00$ to $x=3{\cdot}00$ to eleven places, the intervals throughout being ·01. Differences as far as Δ^3 are added from $x=0$ to $x=1{\cdot}61$, and as far as Δ^4 from $x=1{\cdot}61$ to $x=3{\cdot}00$. Table II. contains $\log_{10}(\text{Erf}\,x)$ from $x=0$ to $x=3{\cdot}00$ at intervals of ·01 to seven figures. First and second differences are added. Table III. gives $\log_{10}(e^{x^2}\,\text{Erf}\,x)$ from $x=0$ to $x=3{\cdot}00$ at intervals of ·01 to seven figures. First and second differences are added. Table I. was calculated by means of the formula

$$\operatorname{Erf}(x+h)-\operatorname{Erf}x=-he^{-x^2}\left\{1-xh+\frac{2x^2-1}{3}h^2-\frac{2x^3-3x}{6}h^3\right.$$
$$\left.+\frac{4x^4-12x^2+3}{30}h^4-\dots\right\}, \quad \dots \quad (60)$$

obtained at once by Taylor's theorem; h was $=\cdot01$, and the term in h^5 (*i. e.* the last written above) was small enough to be neglected. Kramp does not state what value he started from in applying the differences, or what means of verification he adopted. In all cases where a Table is constructed by means of differences, the last value should be calculated independently, and then the agreement of the two values would verify all the preceding portion of the Table. This, however, Kramp does not appear to have done, though it would not, as Kramp intimates, have been impossible. To calculate Erfc 3 from (53) would have been a heavy, but by no means exceptionally heavy piece of work. The remark that the "void" cannot be filled up by any known method is not now true. Laplace's continued fraction gives with very little trouble Erf 3 = 000 019 577 193 ... This value of Erf 3 differs in the tenth and eleventh figures from Kramp's result obtained by differences; so that it is probable that a portion of his Table is incorrect in the last two figures. I have not yet examined the cause of the error so as to be able to state whether it is due to an inaccuracy in the calculation of a difference, or to an accumulation of small errors in the differences; but I hope shortly to be able to make a complete examination of Kramp's Table.

The next Tables of the functions which were published are due to Bessel; they occupy pages 36, 37 of the *Fundamenta Astronomiæ*, Regiomonti, 1818. Bessel remarks that Kramp's Table does not go beyond $x=3$, and that in a few cases the function might be wanted for larger arguments; he therefore tabulates $e^{x^2}\operatorname{Erf}x$ for the argument $\log_{10}x$ from 0 to 1 (so that the limits of x are 1 and 10). Bessel gives two Tables. Table I. contains $\log_{10}(e^{x^2}\operatorname{Erf}x)$ from $x=0$ to $x=1$, at intervals of ·01, to seven figures, with first and second differences, and is the same as Kramp's*. Table II. gives $\log_{10}(e^{x^2}\operatorname{Erf}x)$, corresponding to the argument $\log_{10}x$, the arguments increasing from 0 to 1 at intervals of ·01. First and second differences are added. It is not stated how this Table was calculated; the values correspond-

* Bessel probably recomputed the Table, as several of his values differ from Kramp's in the last figure.

ing to arguments between 0 and 3 were probably interpolated from Kramp's Table, and the rest calculated from (55).

A portion* of Legendre's *Traité des Fonctions Elliptiques* (Paris, 1826) is devoted to the discussion of indefinite integrals of the form $\int\left(\log\frac{1}{x}\right)^{a-1}dx$. Legendre writes

$$\Gamma(a,\,x)=\int_0^x\left(\log\frac{1}{x}\right)^{a-1}dx\,; \quad . \quad . \quad . \quad . \quad (61)$$

so that

$$\Gamma(a,\,e^{-x})=\int_x^\infty e^{-v}v^{a-1}dv, \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and therefore

$$\tfrac{1}{2}\Gamma(\tfrac{1}{2},\,e^{-x^2})=\text{Erf.}\ x.$$

Two Tables are given (pp. 520 and 521). Table I. contains 2 Erf x from $x=0$ to $x=\cdot 50$ at intervals of ·01, to ten decimal places. The results in this Table should be double of those in the corresponding Table of Kramp's. I have not yet examined if this is the case throughout. This Table was calculated from (53). Table II. gives $\int_0^x dx\left(\log\frac{1}{x}\right)^{a-1}$ from $x=\cdot 80$ to $x=0\cdot 00$ at intervals of ·01; that is, it gives $2\,\text{Erf}\left(\log\frac{1}{x}\right)^{\frac{1}{2}}$ from $x=0$ to $x=\cdot 80$. This Table begins about where Table I. ends; so that together they extend from Erf 0 to about Erf (2·14). It was calculated by quadratures, with the exception of the last five or six values, which were obtained from the continued fraction.

At the end of the *Berliner Astronomisches Jahrbuch* for 1834 is a paper by Encke, on the method of least squares, to which are appended two Tables. Table I. gives $\frac{2}{\sqrt{\pi}}$ Erfc x from $x=0$ to $x=2\cdot 00$ at intervals of ·01, to seven places, with first and second differences. Table II. gives $\frac{2}{\sqrt{\pi}}$ Erfc (ρx) from $x=0$ to $x=3\cdot 40$ at intervals of ·01, and from $x=3\cdot 40$ to $x=5\cdot 00$ at intervals of ·1, to five decimal places, with first difference, ρ being determined by the equation $\frac{2}{\sqrt{\pi}}$ Erfc $\rho=\frac{1}{2}$, so that $\rho=\cdot 476\,9360$. The use of this Table is explained by De Morgan

* Chapter xvii. vol. ii.

(*Encyc. Metropol.* "Theory of Probabilities," p. 451). De Morgan remarks that he does not know whether this Table was calculated independently or depends on Kramp's Table; but on p. 269 of the *Jahrbuch* Encke says that Table I. was deduced from Bessel's Table of $\int e^{-x^2}dx$ in the *Fundamenta.* Bessel, as we have seen, tabulated $\log_{10}(e^{x^2}\,\mathrm{Erf}\,x)$; and if Encke's Table was derived from Bessel's, it must have been by interpolation from his Table II. It is more probable, however, that it was derived from Kramp's Table I., from which it can be deduced at once. Table II., Encke states, was formed by interpolation, and is probably founded on Table I. Both these Tables, as well as Kramp's Tables I. and II., are reprinted at the end of De Morgan's article, previously referred to, in the *Encyclopædia Metropolitana.*

The Table accompanying the present paper gives Erf x from $x=3{\cdot}00$ to $x=3{\cdot}50$ to eleven places, from $x=3{\cdot}50$ to $x=4{\cdot}00$ to thirteen places, from $x=4{\cdot}00$ to $x=4{\cdot}50$ to fourteen places, subject to certain qualifications with regard to the accuracy of the last figure, which will be stated further on. The values were calculated by means of the same difference-formula, viz. (60), that Kramp used. Separate Tables of $\log_{10} e^{-x^2}(=-x^2\mu$, μ being the modulus) and of

$$\log_{10}\left\{h-h^2x+\frac{2x^2-1}{3}h^3-\frac{2x^3-3x}{6}h^4\right\}$$

were formed. The second of these Tables was differenced throughout, and the gradual change of the differences from ·0000430 to ·0000427 afforded a very good test of its accuracy. A Table of $\mathrm{Erf}\,(x+h)-\mathrm{Erf}\,x$ was then deduced from the two subsidiary Tables, and was differenced throughout as far as Δ^3; and the regularity of these last differences proved the correctness of the Table. Erf 3 was calculated, as previously mentioned, from the continued fraction and the differences subtracted from it, till Erf 3·5 was obtained $=$·000 000 658 5487... The correct value obtained from the continued fraction was

$$\mathrm{Erf}\,3{\cdot}5={\cdot}000\,000\,658\,553\,76\ldots,$$

so that eleven figures are the same in both; it is probable, therefore, that the last figure in the values from $x=3{\cdot}00$ to $x=3{\cdot}50$ is nowhere in error by so much as a unit. As an additional verification, Erf 3·2 was calculated from the continued fraction and found $=$·000 005 340 191 . . ., which agrees with the number given in the Table. Erf 4 was calculated from the continued fraction and found $= \cdot 000\,000\,013\,663\,189\ldots$; the differences corresponding to argument between 4·0 and 3·5 were then added, and Erf 3·5 was obtained $= \cdot 000\,000\,658\,553\,74\ldots$, agreeing with the value found by direct calculation for Erf 3·5 as far as the thirteenth figure. The values from $x=3\cdot5$ to $x=4\cdot0$ were therefore entered in the Table to thirteen decimal places; and it is not probable that any is in error by more than a unit in the last place in any of the values. Starting again from 4·0, the differences corresponding to arguments between 4·0 and 4·5 were subtracted, and Erf 4·5 was thus obtained $= \cdot 000\,000\,000\,174\,250\ldots$ By direct calculation from the continued fraction Erf 4·5 was found $= \cdot 000\,000\,000\,174\,237\,6\ldots$ As these values only differ by unity in the fourteenth place, the portion of the Table from $x=4\cdot0$ to $x=4\cdot5$ was given to fourteen places, it being understood that the last figure may be in error to the extent of one or even two units.

From the above description it will be seen that the mode of verification adopted formed a very perfect test of the accuracy of the Table. I rather regret now that I did not tabulate $e^{x^2}\,\mathrm{Erf}\,x$ in preference to Erf x. When a function whose value is numerically small enters into analysis, the term involving it can usually be neglected, unless it contains also a very large factor, which is usually an exponential. It is also a matter of observation that when a function is expansible in a descending series multiplied by an exponential, this factor points out the factor which it will be convenient to multiply the function by previously to tabulation; thus from (55) we see that $e^{x^2}\,\mathrm{Erf}\,x$ is a function which only becomes infinitely small for very large values of x. Similarly when x is large, it is preferable to tabulate $e^{-x}\,\mathrm{Ei}\,x$ and $e^{x}\,\mathrm{Ei}(-x)$ in place of Ei x and Ei $(-x)$. This deficiency in the case of the error-function I hope to supply by forming a Table of $e^{x^2}\,\mathrm{Erf}\,x$ for arguments above 3.

Since Erfc $x=\frac{1}{2}\sqrt{\pi}-\mathrm{Erf}\,x$, the value of $\frac{1}{2}\sqrt{\pi}$ to fourteen places of decimals is printed on the same page as the Tables, to facilitate the deduction of Erfc x from Erf x.

TABLE of the Values of the Error-function.

x.	Erf x.	x.	Erf x.	x.	Erf x.
	·0000		·000000		·0000000
3·00	1957719	3·51	6123403	4·01	1258171
3·01	1837943	3·52	5692599	4·02	1158358
3·02	1725164	3·53	5291082	4·03	1066256
3·03	1618995	3·54	4916932	4·04	0981285
3·04	1519069	3·55	4568356	4·05	0902910
3·05	1425037	3·56	4243670	4·06	0830633
3·06	1336570	3·57	3941298	4·07	0763993
3·07	1253354	3·58	3659762	4·08	0702562
3·08	1175095	3·59	3397679	4·09	0645945
3·09	1101510	3·60	3153753	4·10	0593775
3·10	1032335	3·61	2926773	4·11	0545712
3·11	0967319	3·62	2715602	4·12	0501441
3·12	0906224	3·63	2519181	4·13	0460673
3·13	0848824	3·64	2336513	4·14	0423136
3·14	0794907	3·65	2166671	4·15	0388582
3·15	0744272	3·66	2008786	4·16	0356780
3·16	0696729	3·67	1862045	4·17	0327517
3·17	0652097	3·68	1725689	4·18	0300596
3·18	0610207	3·69	1599008	4·19	0275834
3·19	0570898	3·70	1481339	4·20	0253062
3·20	0534019	3·71	1372062	4·21	0232124
3·21	0499426	3·72	1270601	4·22	0212878
3·22	0466984	3·73	1176414	4·23	0195189
3·23	0436566	3·74	1088998	4·24	0178936
3·24	0408050	3·75	1007881	4·25	0164003
3·25	0381324	3·76	0932626	4·26	0150288
3·26	0356279	3·77	0862823	4·27	0137692
3·27	0332816	3·78	0798089	4·28	0126128
3·28	0310837	3·79	0738068	4·29	0115512
3·29	0290254	3·80	0682429	4·30	0105769
3·30	0270982	3·81	0630862	4·31	0096829
3·31	0252941	3·82	0583078	4·32	0088627
3·32	0236056	3·83	0538809	4·33	0081104
3·33	0220255	3·84	0497804	4·34	0074206
3·34	0205472	3·85	0459830	4·35	0067880
3·35	0191644	3·86	0424671	4·36	0062082
3·36	0178713	3·87	0392123	4·37	0056768
3·37	0166622	3·88	0362000	4·38	0051899
3·38	0155319	3·89	0334126	4·39	0047438
3·39	0144754	3·90	0308338	4·40	0043352
3·40	0134883	3·91	0284485	4·41	0039610
3·41	0125660	3·92	0262427	4·42	0036184
3·42	0117045	3·93	0242032	4·43	0033048
3·43	0109000	3·94	0223178	4·44	0030178
3·44	0101488	3·95	0205753	4·45	0027552
3·45	0094476	3·96	0189651	4·46	0025149
3·46	0087931	3·97	0174776	4·47	0022952
3·47	0081824	3·98	0161036	4·48	0020942
3·48	0076126	3·99	0148347	4·49	0019105
3·49	0070811	4·00	0136632	4·50	0017425
3·50	0065855				

$$\text{Erfc}\,x = \tfrac{1}{2}\sqrt{\pi} - \text{Erf}\,x.$$

$$\tfrac{1}{2}\sqrt{\pi} = ·886\ 226\ 925\ 452\ 75\ \ldots$$

ON FOURIER'S (DOUBLE-INTEGRAL) THEOREM.

BY

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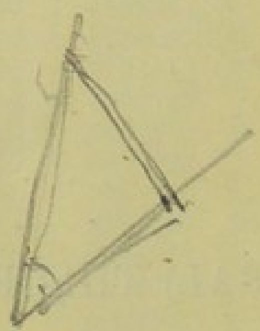

ON FOURIER'S (DOUBLE-INTEGRAL) THEOREM.

ONE of the clearest and most convincing proofs of Fourier's Theorem, viz.:

$$\phi(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} da\,dv \cos(a-x)v\phi(a) \text{ .... (1)},$$

is that given by Boole in Vol. XXI. of the *Irish Transactions*.* Starting from the function $\tan^{-1}\frac{a-x}{k}$ (k infinitesimal and positive), which $=\frac{1}{2}\pi$ if $x<a$, 0 if $x=a$, and $-\frac{1}{2}\pi$ if $x>a$, Boole followed the discontinuity through a series of transformations of the function until finally the latter assumed the form (1). The theorem is thus proved essentially from first principles, and the origin of its chief characteristic, viz. the capacity for exhibiting discontinuity, is, as it were, traced from its source. In this note I propose to follow Boole's principle, only instead of starting with the discontinuity in $\tan^{-1}\frac{a-x}{k}$ to take as the basis of the investigation the integral

$$\left.\begin{aligned}\int_0^{\infty}\frac{\sin av}{v}\,dv, \text{ which} &= \tfrac{1}{2}\pi \text{ if } a \text{ be positive}\\ &= 0, \text{ if } a \text{ be zero}\\ &= -\tfrac{1}{2}\pi, \text{ if } a \text{ be negative}\end{aligned}\right\}\text{....(2).}$$

The proof will thus be rendered shorter and somewhat clearer as the infinitesimal k enters in a manner that renders the functions involving it more easily conceivable; on the

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* *On the Analysis of Discontinuous Functions*, p. 126.

other hand, the theorem will not be deduced so entirely from first principles.

From (2) it follows immediately that $\frac{2}{\pi}\int_0^\infty \frac{\sin(a-x)v}{v}\,dv=1$, if $x<a$; $=0$ if $x=a$; and $=-1$, if $x>a$, and therefore

$$\frac{1}{\pi}\int_0^\infty \left[\frac{\sin(a+h-x)\,v}{v} - \frac{\sin(a-x)\,v}{v}\right] dv$$

$$\begin{aligned} &= 1, \text{ if } x>a \text{ and } <a+h \\ &= \tfrac{1}{2}, \text{ if } x=a, \text{ or } =a+h \\ &= 0, \text{ if } x<a, \text{ or } >a+h, \end{aligned}$$

the expression in square brackets under the integral sign is, when h is made infinitesimal, $=h\cos(a-x)\,v$; multiply then both sides by $\phi(a)$, which, if x lies between a and $a+h$, is the same as $\phi(x)$, and we have

$$\begin{aligned} \frac{1}{\pi}\int_0^\infty \cos(a-x)\,vdv\,.\,h\phi(a) &= \phi(x), \text{ if } x>a \text{ and } <a+h \\ &= \tfrac{1}{2}\phi(x), \text{ if } x=a, \text{ or } =a+h \\ &= 0, \text{ if } x<a, \text{ or } >a+h. \end{aligned}$$

Similarly

$$\frac{1}{\pi}\int_0^\infty \cos(a+h-x)\,vdv\,.\,h\phi(a+h) = \phi(x),$$

if $x>a+h$ and $<a+2h$, &c.,

$$\frac{1}{\pi}\int_0^\infty \cos(a+2h-x)\,vdv\,.\,h\phi(a+2h) = \phi x,$$

if $x>a+2h$ and $<a+3h$, &c.,

&c. &c.

Now the value of

$$h\,\{\cos(a-x)\,v\phi(a) + \cos(a+h-x)\,v\phi(a+h)\ldots + \cos(a+nh-x)\,v\phi(a+nh)\},$$

is $\int_p^q \cos(a-x)\,v\phi(a)\,da$, (taking $a=p$, $a+nh=q$) and we therefore have by addition

$$\left.\begin{aligned} \frac{1}{\pi}\int_p^q\int_0^\infty \cos(a-x)\,v\phi(a)\,dadv &= \phi(x), \text{ if } x>p \text{ and } <q \\ &= \tfrac{1}{2}\phi(x), \text{ if } x=p, \text{ or } =q \\ &= 0, \text{ if } x<p, \text{ or } >q \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

By taking p and q infinite this becomes (1). It will be noticed that if x be taken on the borders of two of the elementary integrals, viz. if $x = a + ih$ (i an integer), then the one integral gives $\frac{1}{2}\phi(x)$, and the next $\frac{1}{2}\phi(x)$ also, so that in the summation we always have $\phi(x)$, if x lies between p and q.

The above method gives Fourier's theorem in the form (1), and the investigation is as correct as the result. Several of the integrals used are not however rigorously true, thus the statement

$$\frac{h}{\pi}\int_0^\infty \cos(a-x)\,v\,dv = 1, \text{ if } x \text{ lies between } a \text{ and } a+h,$$

can only be regarded as correct in a certain sense, as we must allow that the integral is indeterminate. The same remark applies to the final result (1), and Boole pointed out that the true form of the theorem was

$$\phi(x) = \int_{-\infty}^\infty \int_0^\infty da\,dv\,e^{-kv}\cos(a-x)\,v\phi(a) \;\ldots\ldots\; (4),$$

in the limit when k is zero, and in the *Messenger of Mathematics*, First Series, Vol. v., p 238, I remarked that, from his investigation, it followed that it was also necessary to add the condition, that the upper limit of the integral having v for its subject, must be of a higher grade of infinity than k^{-1}, so that if v_1 be written for the infinity of the upper limit of the integration with regard to v, kv_1 must be infinite. It will now be shown that the investigation in this paper proves the theorem rigorously in this form. The ambiguity made its appearance through replacing

$$\int_0^\infty \left\{\frac{\sin(a+h-x)\,v}{v} - \frac{\sin(a-x)\,v}{v}\right\} dv \;\ldots\ldots\; (5),$$

by $h\int_0^\infty \cos(a-x)\,v\,dv$. Properly (5) is equal to

$$h\int_0^\infty \cos(a-x)\,v\,dv - \frac{h^2}{\lfloor 2}\int_0^\infty v\sin(a-x)\,v\,dv + \ldots \;(6),$$

and since the second term is of the form $h\,(0\times\infty)$, we have no right to neglect it compared to h. To acquire this right we must make some assumption or convention that shall render $\int_0^\infty v\sin(a-x)\,v\,dv$, &c. finite, and this will be effected if we start with the integral $\int_0^\infty e^{-kv}\frac{\sin av}{v}\,dv$ (k infinitesimal and such that k^{-1} is of a lower grade than the ∞ of the limit) instead of $\int_0^\infty \frac{\sin av}{v}\,dv$. Such an alteration of form produces no change in the value of the integral as the factor introduced does not influence at all the elements of the integral until v is greater than k^{-1}, when $\frac{\sin av}{v}$ is indefinitely small. (6) then takes the form

$$h\int_0^\infty e^{-kv}\cos(a-x)\,vdv - \frac{h^2}{\underline{|2}}\int_0^\infty e^{-kv}v\sin(a-x)\,vdv + \ldots,$$

and, writing c for $a-x$ for brevity,

$$\int^v e^{-kv}v\sin cv\,dv \text{ is of the form } e^{-kv}\frac{A+Bkv}{(k^2+c^2)^2},$$

so that when v is made infinite this result is prevented from becoming infinite by the factor e^{-kv} or $e^{-\infty}$, since kv is infinite at the limit; and similarly for the terms in h^3, &c. as the factor e^{-kv} always occurs. Or the same thing may be seen thus: since the ∞ of the limit is the largest infinity we are concerned with, we may regard it as an absolute infinity, viz. take the result of integration between 0 and ∞, and subsequently make k equal to zero: $\int_0^\infty e^{-kv}\cos cv.v^n dv$ then depends on $\frac{d^n}{dk^n}\left(\frac{k}{c^2+k^2}\right)$, which is evidently finite when $k=0$. We thus have a complete investigation of the theorem in the form (4), which is, I believe, the only form in which it is universally true. If, for instance, in (1) we were to take $\phi(x)=x$, the integral would certainly be infinite and indeterminate, if the integral-signs are to be taken in their ordinary acceptations; and even for $\phi(x)=1$ the integral is indeterminate. All the methods, therefore, that profess to prove (1) universally must, I imagine, be deficient in rigour, and that such is the case I have found in all the proofs I have examined.

It may be remarked, that the proof in this paper is in no way a reversal of the verification known by the name of Deflers.* The investigation afforded by the latter would be as follows:

$$\phi(x) = \frac{1}{\pi}\phi(x)\int_{-\infty}^\infty \frac{\sin z}{z}\,dz,$$

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* De Morgan's *Diff. and Int. Calc.*, p. 619.

replace x on the right-hand side by $x+\frac{z}{\alpha}$, α being infinite, and we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin z}{z}\,\phi\left(x+\frac{z}{\alpha}\right)dz,$$

which

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha\,(a-x)}{a-x}\,\phi\,(a)\,da\,\ldots\ldots\ldots\ldots(7),$$

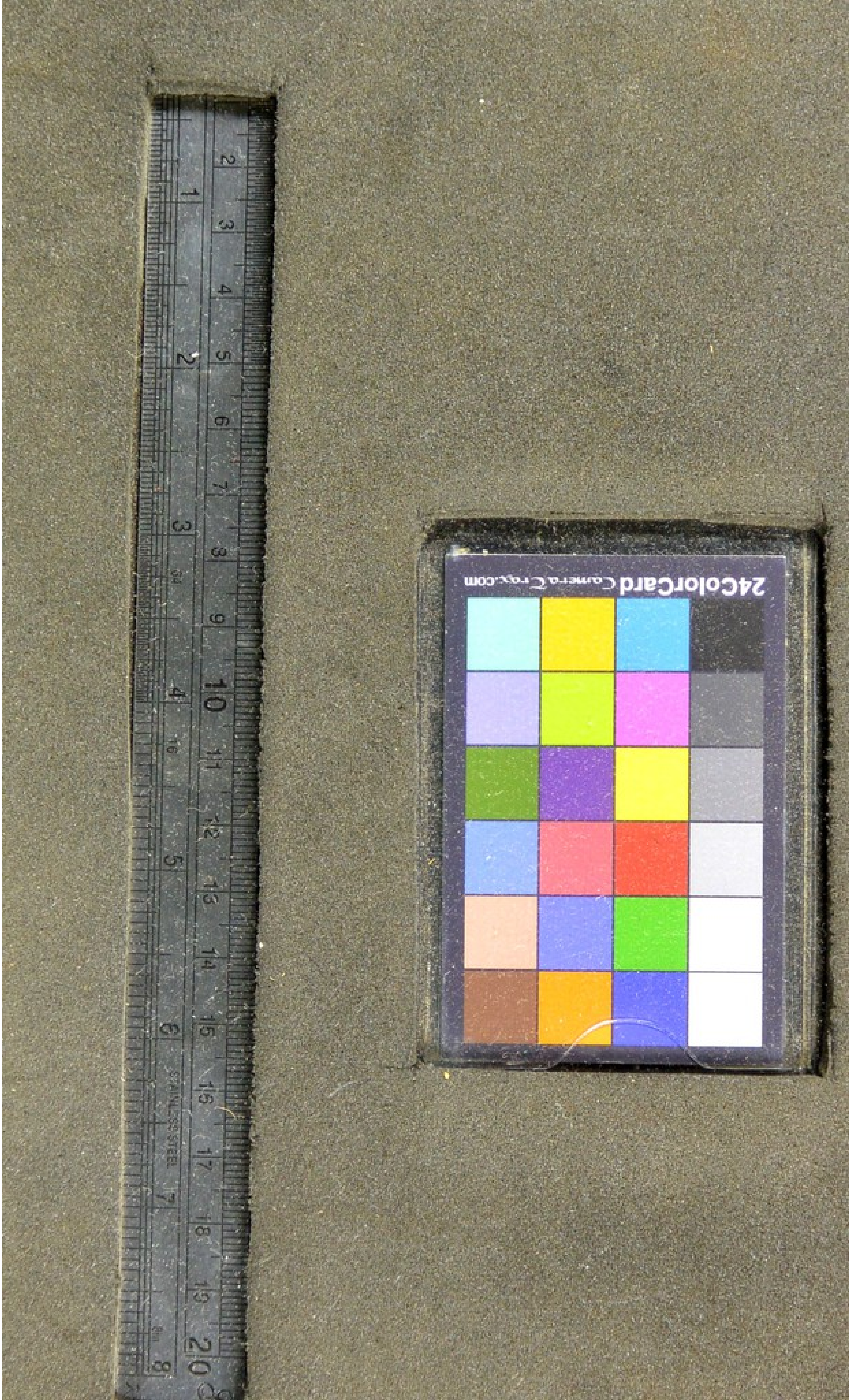
when transformed by assuming $a=x+\frac{z}{\alpha}$, and since

$$\int_0^{\alpha}\cos\,(a-x)\,v\,dv=\frac{\sin\alpha\,(a-x)}{a-x}\,\ldots\ldots\ldots\ldots(8),$$

(7) becomes

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\alpha}\cos\,(a-x)\,v\,\phi\,(a)\,da\,dv\ldots\ldots\ldots\ldots(9).$$

This is quite different from the method used in this paper; the integrations being effected in the reverse order. Deflers's process is rendered more rigorous by taking α an infinity of a lower grade than the limit for z, (so that the limits for a are $\pm\infty$) and by introducing a factor e^{-kv} under the integral sign in (8). The form (3) for the case of x between p and q follow from (9) merely by regarding $\phi\,(x)$ as a function that vanishes unless x lies between p and q; but the method does not bring out the points relating to the discontinuity so clearly or fully.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