XXXI.—On Inheritance of Hair and Eye Colour.

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Some time ago, in a paper published by the Royal Anthropological Institute, I applied a Mendelian analysis to that part of the observations made by the late Dr Beddoe (1) which refers to the colour of the hair. In that paper (2) I showed that these observations obeyed in a highly remarkable degree the law referred to, and that this result held from the north of Scotland, through the whole of England, Ireland, France, and Germany, to the south of Italy. At that time I was unable to make any application to the observations on eye colour also published in the same work, but I have now succeeded in completing the analysis.

The whole depends on a theorem of population stability which may be easily proved.

Let the population consist of a mixture of two races having two characters such as hair colour and eye colour inherited according to the Mendelian law of segregation. Let these qualities be denoted by (BB), (*bb*) for the hair, and (DD), (*dd*) for the eyes. Then the population may be considered given by

$$a^2\left|\begin{matrix}\mathrm{DD}\\\mathrm{BB}\end{matrix}\right| + 2ab\left|\begin{matrix}\mathrm{DD}\\\mathrm{B}b\end{matrix}\right| + b^2\left|\begin{matrix}\mathrm{DD}\\bb\end{matrix}\right| + 2ac\left|\begin{matrix}\mathrm{D}d\\\mathrm{BB}\end{matrix}\right| + (2ad+2bc)\left|\begin{matrix}\mathrm{D}d\\\mathrm{B}b\end{matrix}\right|$$
$$+ 2bd\left|\begin{matrix}\mathrm{D}d\\bb\end{matrix}\right| + c^2\left|\begin{matrix}dd\\\mathrm{BB}\end{matrix}\right| + 2cd\left|\begin{matrix}dd\\\mathrm{B}b\end{matrix}\right| + d^2\left|\begin{matrix}dd\\bb\end{matrix}\right|^* \quad . \quad (1)$$

If this population mate freely, and if all matings possess equal fertility, the relationship of the constants required for a stable population depends on whether coupling exists or not.

The meaning of the term "coupling" may be easily seen from a consideration of the different units in the above expression. It will be noticed that every term of the expression except that in the middle has either two eye units or two hair units the same. It is thus impossible when division takes

* The factors outside the brackets are the proportional numbers of each variety. The simple case is: if x(A, A) mate at random with itself and with $y(a, a)$ and all subsequent matings are equally probable, the stable population is given

$$x^2(\mathrm{A, A}) + 2xy(\mathrm{A}, a) + y^2(a, a).$$

Cf. p. 462.

place for anything else to occur than that two constantly linked pairs are given off in equal numbers. Thus $\left|\begin{array}{c} d\mathrm{D} \\ bb \end{array}\right|$ can only divide into $\left|\begin{array}{c} d \\ b \end{array}\right.$ and $\left|\begin{array}{c} \mathrm{D} \\ b \end{array}\right.$. But when we consider the case of $\left|\begin{array}{c} \mathrm{D}d \\ \mathrm{B}b \end{array}\right|$ other things may easily happen. If D have a greater affinity for B than for b, then we may have more of the element $\left|\begin{array}{c} \mathrm{D} \\ \mathrm{B} \end{array}\right.$ given off than of the element $\left|\begin{array}{c} \mathrm{D} \\ b \end{array}\right.$.* But here also there is a necessary arithmetical relationship between the different elements resulting, and if n $\left|\begin{array}{c} \mathrm{D} \\ \mathrm{B} \end{array}\right.$ elements occur for one $\left|\begin{array}{c} \mathrm{D} \\ b \end{array}\right.$ it follows that there will also be n $\left|\begin{array}{c} d \\ b \end{array}\right.$ for one $\left|\begin{array}{c} d \\ \mathrm{B} \end{array}\right.$ even although the attraction of D for B might be different from that of d for b.

If the population (1) mate freely and if m $\left|\begin{array}{c} \mathrm{D} \\ \mathrm{B} \end{array}\right.$ occur with n $\left|\begin{array}{c} \mathrm{D} \\ b \end{array}\right.$ (where $m+n=\cdot 5$ and $2(ad+bc)$ is denoted by h), the next generation will be given by

$$\left.\begin{array}{l} (a^2+ab+ac+mh)^2 \left|\begin{array}{c} \mathrm{DD} \\ \mathrm{BB} \end{array}\right| + 2(a^2+ab+ac+mh)(b^2+ab+bd+nh) \left|\begin{array}{c} \mathrm{DD} \\ \mathrm{B}b \end{array}\right| \\ \quad + (b^2+ab+bd+nh)^2 \left|\begin{array}{c} \mathrm{DD} \\ bb \end{array}\right| + 2(a^2+ab+ac+mh)(c^2+ac+dc+nh) \left|\begin{array}{c} \mathrm{D}d \\ \mathrm{BB} \end{array}\right| \\ \quad + \left\{\begin{array}{l} 2(a^2+ab+ac+mh)(d^2+db+cd+mh) \\ +2(b^2+ab+bd+nh)(c^2+ac+cd+nh) \end{array}\right\} \left|\begin{array}{c} \mathrm{D}d \\ \mathrm{B}b \end{array}\right| \\ \quad + 2(b^2+ab+bd+nh)(d^2+bd+dc+mh) \left|\begin{array}{c} \mathrm{D}d \\ bb \end{array}\right| + (c^2+ac+cd+nh)^2 \left|\begin{array}{c} dd \\ \mathrm{BB} \end{array}\right| \\ \quad + 2(c^2+ac+cd+nh)(d^2+db+cd+mh) \left|\begin{array}{c} dd \\ \mathrm{B}b \end{array}\right| + (a^2+db+cd+mh)^2 \left|\begin{array}{c} dd \\ bb \end{array}\right| \end{array}\right\} \quad (2)$$

This has exactly the same form as that from which it is derived, but the relative proportions of the different classes may be different. If the population is stable we have as the sufficient conditions,

$$\frac{(a^2+ab+ac+mh)^2}{a^2} = \frac{(b^2+ba+bd+nh)^2}{b^2} = \frac{(c^2+cd+ca+nh)^2}{c^2} = \frac{(d^2+db+dc+mh)^2}{d^2},$$

as all the similar relationships hold if these are true.

Taking the first equation

$$\frac{(a^2+ab+ac+mh)^2}{a^2} = \frac{(b^2+ba+bd+nh)^2}{b^2},$$

* The assumption made here is that there is no special mortality or instability among the pairs which are actually formed.

we have, since the positive root must be taken,

$$a+b+c+\frac{mh}{a}=b+a+d+\frac{nh}{b},$$

or

$$c+\frac{mh}{a}=d+\frac{nh}{b},$$

or

$$abc+2mabd+2mb^2c=abd+2na^2d+2nabc,$$

$$\text{since } h=2(ad+bc);$$

$$=abd+(1-2m)a^2d+(1-2m)abc,$$

$$\text{since } 2(m+n)=1;$$

or

$$2m(abd+b^2c+a^2d+abc)=abd+a^2d,$$

or

$$2m(a+b)(ad+bc)=(a+b)ad$$

$$2m(ad+bc)=ad$$

$$2mbc=(1-2m)ad$$

$$=2nad.$$

The other equations also reduce to this, so that

$$\frac{ad}{bc}=\frac{m}{n}$$

is the criterion of stability if coupling exists. If there is no coupling,

$$m=n \quad \text{and} \quad ad=bc.$$

Some remarks may be made in this place concerning the meaning of coupling. It has two forms: either each unit has a special attraction for the corresponding unit originally associated with it, or on the other hand for the one with which it has come in contact when hybridisation occurs. The theory at present advanced by Mendelian biologists makes in my notation $\frac{m}{n}=2^p-1$ when p is a positive integer. I confess that I cannot follow the arguments on which this is based. The facts seem to me much more in line with the conditions of stability in chemical solutions. If there be a solution, say, of Na_2SO_4 and HCl, the relative proportions of the four possible substances depend on the rate at which the reactions between Na_2SO_4 and HCl and between NaCl and H_2SO_4 take place. Denoting these respectively by n and m, if the amount of these four substances be respectively a, d, b, c, equilibrium will exist if $nad=mbc$. Or, in other words, the equation of chemical equilibrium is the same as that of the stability of the population considered. The advantage of this method of looking at the matter is that it implies no special values of m and n. Short, therefore, of some fundamental reason for the value $\frac{m}{n}=2^p-1$, it is better to consider that other values may be possible and that facts on one side or the other are at present of more importance than theories. The only difference in this case is that either $\left|\begin{matrix}D\\B\end{matrix}\right. + \left|\begin{matrix}d\\b\end{matrix}\right.$ or $\left|\begin{matrix}D\\b\end{matrix}\right. + \left|\begin{matrix}B\\d\end{matrix}\right.$ must exist; thus four different compounds cannot all appear together, but if an average of a large number of examples is taken the result must be the same.

Referring back to the expression for a freely mating population, we see that the fact that it forms a perfect square is not a sufficient criterion of stability. All that is stable is the relation of the eyes alone or of the hair alone. Thus, taking formula (2) and summing each line as regards number, we have for the total of the first line, or the terms containing (DD),

$$(a^2+ab+ac+mh)^2+2(a^2+ab+ac+mh)(b^2+ab+bd+nh)+(b^2+ab+bd+nh)^2,$$

or

$$(a^2+ab+ac+mh+b^2+ab+bd+nh)^2,$$

or

$$(a^2+ab+ac+ad+b^2+ab+bc+bd)^2,$$

since $m+n=\cdot 5$
and $h=2ad+2bc$;

or

$$(a+b)^2(a+b+c+d)^2;$$

the second line, *i.e.* the terms containing (Dd), is equal to

$$2(a+b)(c+d)(a+b+c+d)^2,$$

and the third to

$$(c+d)^2(a+b+c+d)^2,$$

and the proportions of the original population (1) are exactly maintained.

Shortly written as before shown, the general formula may be denoted by

$$\left\{ a \left|\begin{matrix}D\\B\end{matrix}\right. + b \left|\begin{matrix}D\\b\end{matrix}\right. + c \left|\begin{matrix}d\\B\end{matrix}\right. + d \left|\begin{matrix}d\\b\end{matrix}\right. \right\}^2 .$$

This is the typical stable Mendelian population without coupling if $ad=bc$; if coupling exists, $\frac{ad}{bc}=\frac{m}{n}$ is the criterion, and stability in the population is only established after many generations.

Suppose equal numbers of two populations mix and mating is free: suppose also that the coupling ratio is 7, one actually found by Bateson and Punnett (3). Then if mating is free the first generation will be given by the ratio

$$\left|\begin{matrix}DD\\BB\end{matrix}\right| + 2\left|\begin{matrix}Dd\\Bb\end{matrix}\right| + \left|\begin{matrix}dd\\bb\end{matrix}\right|.$$

With a ratio of 7 the next generation will be represented by

$$\left\{ 8\left|\begin{matrix}D\\B\end{matrix}\right. + 7\left|\begin{matrix}D\\B\end{matrix}\right. + \left|\begin{matrix}D\\b\end{matrix}\right. + \left|\begin{matrix}d\\B\end{matrix}\right. + 7\left|\begin{matrix}d\\b\end{matrix}\right. + 8\left|\begin{matrix}d\\b\end{matrix}\right. \right\}^2,$$

or

$$\left\{15\left|\begin{matrix}\mathrm{D}\\ \mathrm{B}\end{matrix}\right. + \left|\begin{matrix}\mathrm{D}\\ b\end{matrix}\right. + \left|\begin{matrix}d\\ \mathrm{B}\end{matrix}\right. + 15\left|\begin{matrix}d\\ b\end{matrix}\right.\right\}^2,$$

which when expanded gives

							Total
	225	DD BB	+30	DD Bb	+1	DD bb	256
	30	Dd BB	+452	Dd Bb	+30	Dd bb	512
	1	dd BB	+30	dd Bb	+225	dd bb	256
Total		256		512		256	

Hence $\frac{m}{n} = 225$ in place of 7.

The subsequent matings can be easily calculated by the application of the form in expression (2). The first term is $(225+15+15+\frac{7}{16}\cdot 452)^2$, and the rest are found likewise.

Applying the process seriatim with suitable approximations we have the successive values of $\frac{ad}{bc}$ given in Table I.

Table I.

	Value of $\frac{ad}{bc}$ or $\frac{m}{n}$.
After first generation . . .	225
,, second ,, . . .	56
,, third ,, . . .	27
,, fourth ,, . . .	19
,, fifth ,, . . .	14
,, sixth ,, . . .	11
,, seventh ,, . . .	9·6
,, eighth ,, . . .	8·6
,, ninth ,, . . .	8·3

It is thus seen that stability is attained only after a considerable number of generations in a free-mating population if coupling exists.

It is possible to introduce a shortened notation. In all circumstances these populations after one generation consist of numbers which are those of a perfect square. If we write this in the following way we can at once proceed to the full expression with little trouble.

Thus

$$\left|\begin{array}{cc} a\left|\begin{array}{c}\mathrm{D}\\ \mathrm{B}\end{array}\right. & b\left|\begin{array}{c}\mathrm{D}\\ b\end{array}\right. \\[2ex] c\left|\begin{array}{c}d\\ \mathrm{B}\end{array}\right. & d\left|\begin{array}{c}d\\ b\end{array}\right. \end{array}\right|^2 \quad \text{or more shortly} \quad \left|\begin{array}{cc} a & b \\[2ex] c & d \end{array}\right|$$

denotes

$$\left|\begin{array}{cccccc} a^2 & \left|\begin{array}{c}\mathrm{DD}\\ \mathrm{BB}\end{array}\right| & 2ab & \left|\begin{array}{c}\mathrm{DD}\\ \mathrm{B}b\end{array}\right| & b^2 & \left|\begin{array}{c}\mathrm{DD}\\ bb\end{array}\right| \\[2ex] 2ac & \left|\begin{array}{c}\mathrm{D}d\\ \mathrm{BB}\end{array}\right| & \begin{array}{c}2ad\\ +2bc\end{array} & \left|\begin{array}{c}\mathrm{D}d\\ \mathrm{B}b\end{array}\right| & 2bd & \left|\begin{array}{c}\mathrm{D}d\\ bb\end{array}\right| \\[2ex] c^2 & \left|\begin{array}{c}dd\\ \mathrm{BB}\end{array}\right| & 2cd & \left|\begin{array}{c}dd\\ \mathrm{B}b\end{array}\right| & d^2 & \left|\begin{array}{c}dd\\ bb\end{array}\right| \end{array}\right|$$

Each of the four sides of the complete expression is the square of the corresponding terms of the contracted expression, and the term in the middle the sum of twice the product of the diagonal elements.

One or two other examples of the rate at which stability is approached in one generation are shown in the following table:—

Table II., showing Rate of Approximation to a Stable Population.

Form of Population.	Value of $\frac{m}{n}$.	Value of $\frac{ad}{bc}$ at Commencement.	Value of $\frac{ad}{bc}$ after One Generation.
$\left\lvert\begin{array}{cc}9 & \cdot5\\ 1 & 1\end{array}\right\rvert^2$	9	18	15·6
$\left\lvert\begin{array}{cc}11 & 1\\ 1 & 1\end{array}\right\rvert^2$	9	11	10·5
$\left\lvert\begin{array}{cc}10 & 1\\ 1 & 1\end{array}\right\rvert^2$	9	10	9·76
$\left\lvert\begin{array}{cc}\sqrt{10} & 1\\ 1 & \sqrt{10}\end{array}\right\rvert^2$	9	10	9·67
$\left\lvert\begin{array}{cc}9\cdot5 & 1\\ 1 & 1\end{array}\right\rvert^2$	9	9·5	9·46
$\left\lvert\begin{array}{cc}5 & 1\\ 1 & 1\end{array}\right\rvert^2$	3	5	4·1
$\left\lvert\begin{array}{cc}4 & 1\\ 1 & 1\end{array}\right\rvert^2$	3	4	3·7

Before quoting any examples of Dr Beddoe's figures it will be well to state clearly his hair and eye categories. He recognises five types of hair colour. The meanings of these types seem to me as follows:—

(1) *Jet black.*—This is a true single hue, and persons possessing this colour of hair are with few exceptions those who possess two distinct pure-black elements in the gametes. The exceptions, so far as I have seen, are a few persons who have one red and one jet-black element. In manhood this may resemble jet black very closely, but the colour of the hair on the body usually shows some trace of the ruddy pigment. These are, however, so few in number that they do not disturb the calculation.

(2) *Dark hair.*—This is really a mixture consisting of one jet-black element and one element of either medium hair or fair hair. Black is thus imperfectly dominant.

(3) *Brown hair.*—This consists of those who are true brown or medium and of those who possess one brown element and one fair element.

(4) *Fair hair.*—This again is a pure pigment, the person possessing it having two fair elements.

(5) *Red hair.*—In this group are included the pure reds, the mixtures of red and fair hair, and the mixtures of red and brown hair.

For purposes of analysis it is necessary to combine the last three classes.

Eyes are more difficult.* Dr Beddoe recognises three classes:—

(1) *Light eyes.*—This includes, in my opinion, the pure blue, the grey or pale yellow, and the mixture of these. All are distinct varieties, and can be distinguished with fair accuracy after a certain amount of practice.

(2) *Mixed eyes.*—This class contains a certain proportion of those eyes which are a mixture of the shades of eye just mentioned and of the chocolate and dark-yellow eyes.

(3) *Dark eyes.*—This class contains all the pure-dark eyes and I think the pure-yellow eyes, as, on account of the manner in which the dark pigment of the back of the iris frequently shows both internally and externally, these may look dark except on careful inspection. It also contains many eyes which a moment's careful inspection would show to be either mixed dark and grey or dark and blue eyes. The latter types of eye are much more common than the true dark or chocolate eye. That they have not been more definitely distinguished is somewhat surprising.

It is obvious from what has been said that the last two classes must at least in the first instance be placed together.

We thus have six equations to determine four unknown quantities. The success of this fitting must be the test of the truth of these statements.

* See Appendix.

As an example, the figures Dr Beddoe obtained by observation in the town of Caen are given. The numbers are as follows:—

	Light, Medium and Red Hair.	Dark Hair.	Jet-black Hair.	Total.
Light eyes	149·5* (149·5)	27 (27·15)	1·5* (1·11)	178
Mixed and dark eyes	51·5*	93·5*	16	161
Total	201 (201)	120·5* (120·0)	17·5* (17·9)	339

In this case $a^2=149·5$ and $(a+b)^2=178$. This gives on solution $a=12·23$ and $b=1·11$, so that we obtain $2ab=27·15$ as against 27 found and $b^2=1·23$ as against 1·5 found. Whether we regard the 1·5 as really one individual or two, the fit is exceedingly good. The same process applied to the total gives $(a+c)^2=201$ and $(a+b+c+d)^2=339$, so that $(a+c)=14·18$ and $(b+d)=4·23$, which give $2(a+c)(b+d)=120$ as against 120·5 and $(b+d)^2=17·9$ as against 17·5 found.

This example illustrates how the race mixture can be analysed and the closeness with which the numbers accord with such distribution of the population as is given by the Mendelian theory. Such complete correspondence is of course rare. Another example almost equally good is that of Bradford. Here the numbers are even larger, the sample of the population observed numbering 1400 persons. In this case the theoretical numbers are printed in brackets above the actual:—

	Light, Medium and Red Hair.	Dark Hair.	Jet-black Hair.	Total.
Light eyes	(663) 663	(117·8) 117	(5·24) 6	786
Total (all eyes)	(968) 968	(392·4) 387	(39·6) 45	1400

The method of testing the suitability of such fitting is that given by Professor Pearson (4). The differences are taken between each theoretical and actual number; these are squared, divided by the corresponding theoretical number, and summed. In the case of the totals this is equal to $\frac{(5·4)^2}{39·6}+\frac{(5·4)^2}{392·4}$ or ·81.

* Where ·5 occurs, the indications were so nearly equal that the individual was recorded half in one class and half in another.

This sum is denoted by the symbol χ^2; the value of P or the probability that the fit might be worse is then obtained from the published tables (5). For the above figures $P = \cdot67$; that is, if 1400 persons were observed by random sampling 100 times, in 67 of these a worse fit might be expected than that found. In the case of the upper line the fit is practically perfect.

In what follows, the figures relating to Scotland are chiefly used. Concerning the suitability of these it may be remarked that (excluding Glasgow and Edinburgh, where the recent Irish immigrations have introduced a large element unassimilable on account of the difference in religion, and which therefore fulfil none of the conditions necessary to the application of the present theory), Dr Beddoe made observations in 43 localities in which the characteristics of the hair and eyes were noted in more than 150 persons.

If 43 cases are noted at random, the number of good fits and bad fits may easily be calculated from the probability table already referred to. We find χ^2 should be less than unity in ·393 of the cases; greater than unity and less than two in ·239; greater than two and less than three in ·125; and greater than three in the remainder, namely, ·223. The following table is divided into two classes—the towns with the larger districts, and the country districts. It is seen that the number expected not only is realised but largely exceeded; in other words, except for the fact that the number of towns and large districts in which χ^2 is greater than three is twice that expected, the number of small values of χ^2 is much in excess of that required. The exception is to be expected as into these towns specially the immigration has been much the greatest in recent years.

Table III., showing the Distribution of the Forty-three Districts in Scotland according to the Actual Findings and the Theoretical Proportions expected by the Theory of Chance.

Values of χ^2.		0–1.	1–2.	2–3.	3–.
Towns and large districts	Actual	6	2	...	5
	Theoretical	5·1	3·2	1·9	2·9
Small districts	Actual	22	3	2	3
	Theoretical	11·7	7·1	4·4	6·7
Total	Actual	28	5	2	8
	Theoretical	11·7	10·3	6·3	9·7

For comparison of hair and eyes a further selection has been made. Only those towns and districts in which χ^2 is less than unity have been analysed, as it is only in those we can expect sufficient freedom of mating to allow of the degree of coupling being determined.

These number twenty-seven in all. An analysis has been made in the manner already indicated. The values of a and b have been determined from the numbers showing the combinations of hair and light eyes, and the values of $a+c$ and $b+d$ similarly from the totals of each colour of hair when all eyes are grouped together. The four elements of the population are thus found, and a, b, c, d being thus known, the ratio of ad to bc may be calculated and the degree of coupling of the eyes and hair known.

In the adjoining table these values are given. The numbers of persons observed and the probable proportions in which the present population is derived from the three great races of Europe are given for comparison :—

Table IV., showing the Constitution of the Population in Different Districts in Scotland, with Dr Beddoe's Reference Numbers.

	No. of Persons.	Teutonic Race.	Alpine Race.	Mediterranean Race.	$\frac{D}{B}$	$\frac{D}{b}$	$\frac{d}{B}$	$\frac{d}{b}$	$R=\frac{m}{n}$	$X^2 < 1$
15. Beauly, etc.	170	47	32	21	7·3	1·3	·7	·7	5·6	<1
16. Inverness town	200	32	42	26	6·5	1·4	·9	1·2	6·2	<1
18. „ district	500	38	39	23	7·0	1·3	·7	1·0	7·7	<1
19. Keith, etc.	200	36	40	24	6·7	1·18	·62	1·13	9·8	<1
20B. Forres	210	37	46	17	7·4	·9	·9	·8	7·3	<1
30. Kirkcaldy, etc.	300	44	39	17	7·4	·9	·9	·8	7·3	<1
34. Perth	665	42	36	22	7	1	·83	1·17	9·8	<1
37. Auchterarder	180	43	33	24	6·8	1·1	·8	1·28	9·9	2·5
38. Forteviot	300	42	35	23	6·4	·8	1·37	1·43	8·3	1·2
40. Callander	150	37	37	26	7·0	1·6	·4	1·0	10·9	<1
47. Breadalbane	199	41	30	29	6·5	1·6	·69	1·3	8·8	<1
51. Athol	290	39	39	22	7·1	1·1	·73	1·07	9·5	<1
57. Great Glen	200	44	38	18	7·5	1·2	·66	·64	6·0	<1
72. Ayr	500	42	36	22	7·2	1·2	·69	·99	9·1	2·3
73. Maybole	250	39	41	20	7·4	1	·67	·93	10·3	<1
74. Sanquhar	200	36	41	23	9·7	1·7	1·25	1·76	7·8	<1
76. Upper Galloway	250	38	41	21	6·99	1·17	·93	·91	5·8	<
78. Dumfries	200	39	42	19	6·9	·9	1·25	·95	4·6	<1
86. Leith, etc.	200	46	37	17	7·8	·7	·48	1·02	23·3	<1
88. Dunbar	150	42	44	14	9·6	·9	·89	·85	10·2	1·2
89. Midlothian	300	54	32	14	7·8	·8	·76	·64	8·2	<1
90. Newhaven	176	52	33	15	9·7	·8	1·48	1·32	10·2	7
100. Duns	230	48	42	10	7·4	5	1·65	·45	4·1	<1
109. Jedburgh	150	44	39	17	8·8	·6	1·27	1·57	18·1	<1
115. Rulewater, etc.	180	47	38	15	10·1	·8	1·32	1·2	11·4	<1
116. Teviotdale	272	44	40	16	7·45	·7	1	·83	8·8	<1
117. Langholm	200	44	42	14	7·33	·61	1·24	·82	7·8	<1

It is seen on inspection that in these twenty-seven cases the degree of successful fitting when the persons with light eyes are considered is exceedingly good. In twenty-two cases χ^2 is less than unity as against 10·6 expected; but as there must be some correlation between the two sets of results, the great excess is not unexpected. In two cases χ^2 is between 1 and 2, and in two between 2 and 3. In two of the latter cases the presence of a single individual would make the fit good, and only one individual could be expected considering the small numbers observed. It may be taken, then, that in these twenty-seven districts at the present moment the conditions for the applications of the theory may be held to exist.

In the table just given the value of the ratio $ad:bc$ is stated in each case. For convenience it will in future be noted by the letter R. It has a wide range of variation in value. The lowest value is 4·1 and the highest 23·3; but of the twenty-one different values eighteen lie between 7 and 11. The mean is 9·14, and the probable error of this $\pm$·48. A number, however, such as the ratio at present considered has for each individual observation a very high probable error. I have been unable to evaluate the expression for the probable error of R in terms of the frequencies, and it is difficult to make a reliable estimate of this; but by an application of the formula given by Mr Udny Yule (6) for the probable error of the same ratio in the fourfold division, it must be large. The average number of observations in each case does not much exceed two hundred, and, taking this value and making a rough estimate, it would seem that the probable error when $R=9$ is 2. That is to say, that in half the cases R should lie between 7 and 11. As we have just seen, two-thirds lie in this interval. When these ratios are considered from the point of view of the median it is found that the latter lies almost exactly in the same place. As small values of the ratio are just as likely to arise from emigration as large values from immigration, it therefore seems probable that the number 9 approximately represents the value of the ratio. The only value which is possible on the current theory of Mendelism is 7, namely 2^3-1. The observations do not favour this value, so that the latter cannot be taken with reasonable probability.

Leaving Scotland for further verification, it seems best to take only large numbers. Dr Beddoe gives eight instances in which the criteria demanded in Scotland approximately hold, and in which the numbers observed are upwards of four hundred. These are collected in Table V., p. 469.

The mean value of R in the case of these towns is 9·4, with a probable error of $\pm$·67, so that they show no certain difference from the result obtained. If anything, they render the value 7 obtained by the Mendelians less probable. In the absence of other evidence we may take it that R ≒ 9, and that if it differs much from that, it is in excess rather than in defect.

As the result of these calculations it is seen that if we take collectively all those with light eyes and distribute them according to the colour of the hair, the number of those with dark hair is always equal to twice the product of the square roots of the numbers of those possessing light hair and black hair. The proportions in which the eyes are divided among the different types of hair show also that something mathematically equivalent to coupling takes place with apparent uniformity. This is the Mendelian law, and the evidence seems to me sufficient to prove that something at least analogous to segregation takes place. Whether the actual mechanism is Mendelian or not, it is evident that any other theory which seeks support must lead to the same numerical relationship.

Table V., showing the Values of R in several Large Towns and Districts.

Reference to *Races of Britain.*	Place.	Number of Observations.	R.
Page 162	Manchester	475	9
„ 179	St Austell	850	8·6
„ 180	Truro	500	10·3
„ 183	Gloucester	500	10
„ 177	Chippenham	650	6·8
„ 163	Bradford	1400	8·4
„ 199	Bourges	420	10·8
„ 212	Vienna	1700	10·8

We now come to the discussion of mixed and dark eyes. Light eyes have been shown to fulfil the necessary conditions for Mendelian inheritance, but the other groups evidently have some different significance. This is best understood by referring again to Expression (1), or

$$\begin{array}{|c|c|c|c|c|c|}
a^2 & \begin{array}{c} bb \\ \mathrm{BB} \end{array} & 2ab & \begin{array}{c} bb \\ \mathrm{BD} \end{array} & b^2 & \begin{array}{c} bb \\ \mathrm{DD} \end{array} \\
2ac & \begin{array}{c} bd \\ \mathrm{BB} \end{array} & (2ad+2bc) & \begin{array}{c} bd \\ \mathrm{BD} \end{array} & 2bd & \begin{array}{c} bd \\ \mathrm{DD} \end{array} \\
c^2 & \begin{array}{c} dd \\ \mathrm{BB} \end{array} & 2cd & \begin{array}{c} dd \\ \mathrm{BD} \end{array} & d & \begin{array}{c} dd \\ \mathrm{DD} \end{array}
\end{array}$$

which is stable if $\mathrm{R}=\dfrac{ad}{bc}$.

The ratios of mixed eyes to dark eyes in each class of hair are therefore

$$\frac{2ac}{c^2}, \quad \frac{2ad+2bc}{2cd}, \quad \frac{2bd}{d^2},$$

or

$$2, \quad 1+\frac{bc}{ad}, \quad \frac{2bc}{ad},$$

cancelling and multiplying by $\frac{c}{a}$,

or

$$2R, \quad R+1, \quad 2.$$

This ratio is evidently independent of the relative proportions of the different elements of the population. If $R=9$, which is the value it approximates to in the majority of cases, this ratio becomes 9 : 5 : 1. Nine districts in Scotland have values of R approximating to 9; they range from 8·2 to 9·9. The relative proportions of mixed and of dark eyes are given in the following table:—

Table VI.—Percentage Mixed and Dark Eyes associated with each Class of Hair.

Eyes.	Light Hair.		Dark Hair.		Black Hair.	
	Mixed.	Dark.	Mixed.	Dark.	Mixed.	Dark.
Selected districts	6·7	6·7	7·3	11·5	·74	3·14
All districts	6·9	6·3	6·3	12·6	·82	2·56

It is a matter of observation that many mixed eyes are classed as dark, and it seems reasonable to suppose that a fixed proportion are so classed; but the figures given by the selected districts cannot be adapted to the ratios given above by transferring the same proportion from each group of mixed eyes to the corresponding group of dark eyes which we have shown takes place. The numbers, however, in the last group, that of black hair, are small, and the error of the ratio, which is approximately 1 in 4, may be large.

The second group of ratios—*i.e.* that for the whole twenty-seven groups—is more nearly in accord with the supposition that a fixed proportion of the mixed eyes are called dark; but it would seem probable that with each change of the constitution of the gamete as regards hair colour a mixed eye tends to assume a darker hue to the casual observer, though it may well be that this is due as much to the colour of the eyelashes as of the eye itself. In fact, the difference to be explained is not so great but that it might be accounted for on this supposition.

One other point requires to be considered. In this paper fair-haired and medium-haired persons have been classed together, and the question arises as to the effect this may have on the relative proportions of mixed and dark eyes, as it might well be that a mixed grey and chocolate eye and a mixed blue and chocolate eye would impress an observer differently. Personal observation renders it probable that the latter is more often classed as dark, and the figures bear out this observation. The proportions are shown in the accompanying table

Table VII.

Ratio of Fair to Medium Hair.	Light Hair.		Ratio.	Dark Hair.		Ratio.	Black Hair.		Ratio.
	Mixed Eyes.	Dark Eyes.		Mixed Eyes.	Dark Eyes.		Mixed Eyes.	Dark Eyes.	
$>1{\cdot}2$	5·8	5·1	1·13	5·4	11·5	·47	·95	2·60	·37
$>1<1{\cdot}2$	6·9	6·5	1·06	5·6	11·2	·50	·84	2·24	·37
<1	6·2	6·8	·91	5·7	11·3	·50	·74	2·91	·25

It is to be noted that the ratio of mixed to dark eyes tends in the groups of light hair and black hair to decrease with the decrease of light hair and to remain constant in the group of dark hair. From such facts no certain inferences can be drawn, but the suggestion is that a mixed blue and chocolate eye is somewhat darker on the average than a mixed grey and chocolate eye.

Conclusions.

(1) Many of Dr Beddoe's populations are stable in a Mendelian sense. Though this does not necessarily imply that the theory as stated by Mendel is the only explanation of the arithmetical proportions found, any other theory claiming to explain the facts of heredity must also explain these relative proportions.

(2) That linkage between hair colour and eye colour exists. The coupling factor is more likely to be 9 than 7, and therefore does not agree with the present Mendelian theory. It is quite possibly to be explained on the analogy of chemical equilibrium.

(3) That it is possible that the colour of the hair has, in addition to this, some other effect in altering the colour of the eyes; but the evidence is not sufficient to prove this, and it may be only due to the fact that dark eyelashes tend to lend a darker appearance to eyes than would be found justified on a more careful examination.

(4) A further result of the analysis made in this paper is that Dr Beddoe's figures give no suggestion of the presence of any race in this country which had different hair and eye relationships from those pertaining to the three races generally considered to form the basis of the European population. This, of course, does not exclude the possibility of an older race surviving in sufficient numbers to form a considerable part of the British population; but, so far as the survey is valid, this race must have had a hair and eye complex closely allied to one or other of the hair and eye complexes considered in this paper.

APPENDIX.

On the Categories of Eye Colour, with a Record of One Observation.

Eye colour is the subject of much controversy. I am personally of the opinion that all categories that have been described are very imperfect. In the first place, apart from actual colour, the pigment of the posterior layer of the iris may be seen at times with more or less prominence along the inner and outer edges of the iris, often causing the eye to appear darker than the colour alone would permit.

Again, mixed eyes are of two kinds—those in which the pigment is (1) diffuse and (2) discrete, that is, in spots; but as far as my observations go, I have never seen pigment in the eyes of children which was not present in the eyes of one or other of the parents. In mixed eyes the pigment tends to collect more markedly near the inner edge of the iris, so that in a mixed chocolate and grey eye we may have both the chocolate and the grey pigment in the inner part, and the outer edge simulating a blue eye.

Of actual types of pure as distinct from mixed eyes I recognise four:—

(1) The pure blue eye, in which there is no pigment in the iris, such grey as appears being due to strands of connective tissue.

(2) The grey or pale yellow, in which there is always visible pigment present in little masses, quite distinct from definite strands of connective tissue.

(3) The deep yellow eye, a more or less rare form, not much exceeding 1 per cent. of the adult population as seen in Glasgow.

(4) The dark-brown or chocolate eye, of which the shades vary, but in all of which the iris is sensibly the same colour from the inner margin to the outer.

All these types of eyes may be found mixed, and as regards eyes the population may be taken as given by

$$m^2(a, a) + n^2(b, b) + p^2(c, c) + q^2(d, d) + 2mn(a, b) + 2mp(a, c) \\ + 2mq(a, d)2np(b, c) + 2nq(b, d) + 2pq(c, d).$$

Now, some of these types are very difficult to distinguish, especially in children. Of the varieties which are very difficult to distinguish are: (1) the mixture of yellow and grey from the mixture of chocolate and grey, a small amount of chocolate pigment being not unlike yellow; and (2) the mixture of chocolate and grey from the mixture of chocolate and blue, the connective tissue of the latter simulating grey pigment when masked by a veil of chocolate pigment.

Last summer I examined a school of nearly one hundred children in Skye, a school where the population may be considered free-mating and uncontaminated by immigration. As each child was shown to me I stated to an amanuensis my decision concerning the eye colour, and the numbers are as follows:—

Class 1.	Pure blue	12
,, 2.	Pure grey	9
,, 3.	Dark yellow	1
,, 4.	Chocolate	4
,, 5.	Mixed blue and grey	23
,, 6.	Blue and yellow	2
,, 7.	Blue and chocolate	1
,, 8.	Grey and yellow	18
,, 9.	Grey and chocolate	18
,, 10.	Yellow and chocolate	3

The difficulties above mentioned show themselves at once; but if classes 3, 4 and 10 be combined, and if classes 6, 7, 8, and 9 be also combined, we have the following figures:—

	Actual Figures.	Theoretical* Proportions.
Pure blue	12	12·39
Mixed grey and blue	23	21·54
Pure grey	9	9·36
Mixed blue or grey and chocolate or yellow	39	38·95
Chocolate and yellow	8	8·76
Pure and mixed	91	91·00

These results are too close to be wholly chance, but as it is a solitary instance they are advanced with diffidence. They are, however, in complete accordance with those given in the preceding notes on "Inheritance of Hair and Eye Colour."

* Fitted by the method of least squares.

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The Theory of Probable Error and its Application to Vital Statistics,

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1. WITH the increase of the use of statistics in public health it is becoming increasingly important that an accurate knowledge of the processes by which results are arrived at should be in the hands of all working with figures. The theory of error was originally developed in connection with games of chance, further developed to suit the requirements of astronomy, and contemporaneously applied from a different point of view to the construction of life tables. In recent years these two applications have converged, till it is now possible to apply many results deduced from the theory of chance to the discussion of problems which could formerly only be attacked by the method of finite differences.

2. Modern mathematical analysis has developed very specially three branches. It has greatly extended the application of the method of curve-fitting to smooth observations. It has brought into use a large number of methods for calculating the correlation between different qualities. It has also concerned itself largely with the discussion of probable error. It is this last branch I intend to treat chiefly to-day.

3. This subject falls naturally into four divisions:

- I. The error due to random selection;
- II. The assumptions on which the mathematical proofs are based and the modifications required;
- III. The influence of experimental error; and
- IV. The method of testing how far theory and observation agree.

I.

4. The subject of probable error due to random sampling is as a rule dismissed in public health text books with a simple statement of Poisson's Formula, or with a treatment which almost wholly neglects the limitations of its application. The actual mathematics, however, required for its understanding is not very advanced. The general theorem which is of most importance can be found proved in any elementary text book of Algebra, and is as follows. If p be the chance of an event happening and q that of it failing to happen so that $(p + q) = 1$, that is, either the event happens or it fails, then in n trials the chance of its happening $(n - m)$

times and failing m times is given by the $(m+1)^{th}$ term of the binomial expansion

$$\text{of } (p+q)^n$$

$$\text{or of } p^n + np^{n-1}q + \frac{n(n-1)}{1.2}p^{n-2}q^2 + \frac{n(n-1)(n-2)}{1.2.3}p^{n-3}q^3 \ldots.$$

$$\text{that is } \frac{n(n-1) \ldots\ldots (n-m+1)}{1.2 \ldots . m} p^{n-m} q^m$$

If $p = q$ this expression is symmetrical, and the chances of the event happening m times is the same as that of it failing m times, the formula in this case becoming

$$(\tfrac{1}{2} + \tfrac{1}{2})^n$$

It is to be noted that as $p + q$ is equal to unity $(p+q)^n$ is also equal to unity, and that if we have M cases the distribution is given by $M(p+q)^n$.

5. Many distributions are described very approximately by one or other of these formulæ. Thus stature, head breadth, head length, cephalic Index, etc., are very closely represented by

$$(\tfrac{1}{2} + \tfrac{1}{2})^n$$

while such cases as the number of persons suffering from enteric fever at each age period, etc., are described by the formula $(p+q)^n$. But these distributions are not as a rule used in the forms above given. Certain curves which can be calculated much more simply have been found to represent these formulæ very closely.

Thus $(\frac{1}{2} + \frac{1}{2})^n$ is represented

by $y = y_1 e^{-\frac{x^2}{2\sigma^2}}$, commonly called the "Normal Curve of Error,"

and $(p+q)^n$ by $y = y_o z^p e^{-\gamma x}$ known to statisticians as Type III.

6. The method in which the form $(\frac{1}{2} + \frac{1}{2})^n$ arises is of special interest. It is commonly derived from the analogy of coin tossing. Only heads or tails can occur, and the chance of either is equal. Thus, if we toss a single coin a large number of times, in the end approximately equal proportions of heads and tails will ensue. If we toss two coins together a large number of times, two heads or two tails will each occur once, and a head and a tail twice, approximately, out of every four times the coins are spun. If n coins be spun the chance of each combination of heads and tails is given by the terms of the binomial expression

$$(\tfrac{1}{2} + \tfrac{1}{2})^n.$$

It is to be noted here that the chances are quite independent of each other, as a head or a tail is equally probable at each separate experiment. If a head in excess denote a positive error, and a tail a negative, we find that not only are the errors independent, but positive and negative errors of like size occur with equal frequency. But there is no necessity in nature for the odds to be equal on both sides. If we take a six-sided die, for instance, six can only be thrown once on the average for five times the other numbers are thrown. If we take n dice, then the proportion in which the sixes will turn up are given by the terms of the expression

$$\left(\tfrac{1}{6} + \tfrac{5}{6}\right)^n$$

n sixes turning up only once in 6^n times.

7. Certain quantities are specially important. The mean of the observations is one of these, this being regularly used in all statistical work for purposes of comparison. The next most important is the standard deviation, which is the square root of the second moment taken round a vertical line through the mean, and which is equivalent in dynamics to the radius of gyration.

The mean may be defined as the average value of the quantities considered. It is obtained by multiplying the size of each unit by the number of times it occurs, taking the sum of all such values and dividing this sum by the total number of units considered. Thus, if the size a occurs m times, and the size b, n times, the mean is given by

$$\frac{ma + nb}{m + n}$$

If more sizes exist, and the sum be denoted by Σ as usual, then the mean is given by

$$\frac{\Sigma\, ma}{\Sigma\, m}$$

8. In the case of $(p + q)^n$ the mean can readily be found. Suppose the expression expanded as before, and suppose that the frequency value p^n corresponds to the value of the size h, and $p^{n-1}q$ to the value $(h + a)$, etc., where a is the increase of value in passing from one term to the next, then we have at once, as corresponding to the expression Σma,

$$p^n h + np^{n-1} q(h + a) + \frac{n(n-1)}{1 \cdot 2} p^{n-2} q^2 (h + 2a) + \ldots .$$

which equals

$$p^n h + n p^{n-1} q h + \frac{n(n-1)}{1 \cdot 2} p^{n-2} q^2 h + \ldots .$$

$$+ n a p^{n-1} q + n(n-1) a p^{n-2} q^2 + \frac{n(n-1)(n-2)}{1 \cdot 2} a p^{n-3} q^3 + \ldots .$$

$$= h(p + q)^n + n a q\, (p + q)^{n-1}$$

$$\therefore \text{ Mean} = \frac{h(p + q)^n + n a q\, (p + q)^{n-1}}{(p + q)^n}$$

$$= h + n a q \quad \text{since } (p + q) = 1.$$

The mean may obviously be calculated as a distance from any origin; it is usual, however, in practice, to calculate it from some point in the middle of the series of observations, as will be presently shown.

9. In a similar way the second moment is calculated. This is usually denoted by μ_2. In this case we multiply the terms by h^2, $(h+a)^2$, ... instead of by h, $h+a$ ..... This gives for the separate terms

$$\mu_2(p+q)^n = p^n h^2 + np^{n-1}q(h+a)^2 + \frac{n(n-1)}{1.2}p^{n-2}q(h+2a)^2 + \dots$$
$$= h^2(p+q)^n$$
$$+ 2ahnq(p+q)^{n-1}$$
$$+ a^2nq\left\{p^{n-1} + (n-1)\,2p^{n-2}q + \frac{(n-1)(n-2)}{1.2}3\,p^{n-3}q^2 + \dots\right.$$

As the last expression is equal to

$$a^2nq\,(p+q)^{n-1}$$
$$+ a^2n(n-1)q^2(p+q)^{n-2}$$
$$\mu_2 = h^2 + 2ahn\,q + a^2nq + a^2n(n-1)q^2$$

This is the second moment taken about a vertical line at distance $h + naq$ from the centre of gravity.

10. Supposing now that the origin is at the centre of gravity instead of the position formerly assumed, it follows that $h + naq = 0$. If we substitute then $h = -naq$ in the formula for the second moment, we have as the value of that moment round a vertical line through the centre of gravity or the mean

$$h^2 + 2ahnq + a^2nq + a^2n^2q^2 - a^2nq^2 \text{ when } h = -anq$$
$$= a^2nq - a^2nq^2$$
$$= a^2nq(1-q)$$
$$= a^2npq$$

The standard deviation, usually denoted by σ, is equal to the square root of this, and is therefore $a\sqrt{npq}$, or $\sqrt{npq}$ if a be taken as unity, as is usually done. In general to calculate the second moment round the ordinate through the centre of gravity, which for shortness is called "centroid verticle," the distance of the mean from some suitable origin and the second moment round the same origin are calculated. If these are denoted by ν_1 and ν_2 respectively, then $\sigma^2 = \nu_2 - \nu_1^2$, which is easily seen to be the case by a modification of the proof given above, for if the last formula hold

$$\sigma^2 = h^2 + 2ahnq + a^2nq^2 + a^2n(n-1)q^2 - (h+anq)^2$$
$$= a^2nq(1-q)$$
$$= a^2npq, \text{ as already found.}$$

11. As an example, take the number of deaths in each series of

one hundred cases of scarlet fever. Here out of thirty instances the deaths ranged from 0 to 6, as seen below.

No. of Deaths.	No. of Instances.	Multipliers.		
0	1	−3	−3	9
1	6	−2	−12	24
2	6	−1	−6	6
3	9	0	−21	39
4	4	1	4	4
5	3	2	6	12
6	1	3	3	9
	30		13	25
			−21	39
			−8	64

The origin has been taken at 3 deaths and the abscissæ measured positively and negatively from this point. The products for the first and second moments are then found and added together, having regard to sign. So that we have

$$\nu_1 = -\frac{8}{30}$$

$$\nu_2 = \frac{64}{30}$$

$$\text{So that } \sigma^2 = \frac{64}{30} - \left(\frac{8}{30}\right)^2$$

$$= 2{\cdot}062$$

$$\text{or } \sigma = 1{\cdot}436$$

Since $\nu_1 = -\frac{8}{30}$, the mean number of deaths in each hundred cases is equal to $3 - \frac{8}{30} = 2{\cdot}73$, since 3 deaths has been chosen as the point of origin. This is in general much the simplest way of calculating the mean and the standard deviation.

12. The significance of the standard deviation can be best seen when two normal curves of equal area are compared. This is shown in the diagram. Both these curves relate to the same number of cases, N. The equation of the first is

$$y = \frac{N}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and of the second,

$$y = \frac{N}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}}$$

The standard deviation of the first is unity, and of the second two. It is seen at once that a much greater variation of values takes place in the second than in the first, or that a very much smaller proportion of cases having the mean properties is found. In other words, the larger the standard deviation the less likely it is that the mean value obtained from the observations represents a large proportion of the values.

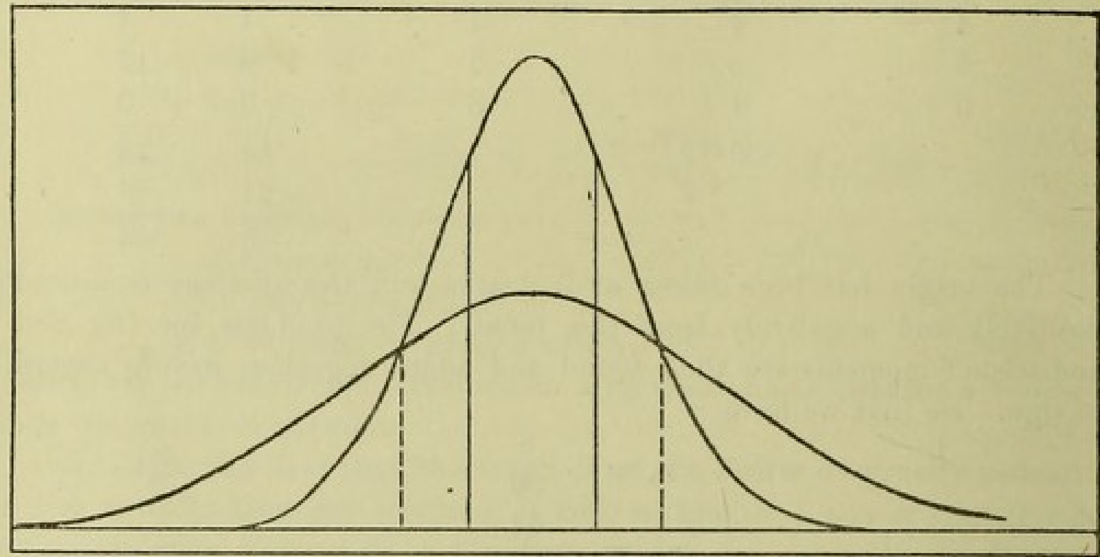

Diagram illustrating the meaning of probable error. The two curves shown have the same area, but the standard deviation of the lower is twice that of the upper. The continuous vertical lines divide the upper curve into parts so that the centre area is equal to the sum of the two external portions, and the chain lines do the same for the flatter curve, showing that the greater the standard deviation the greater the probable error.

13. The definition of the term "probable error" can now be given. It has been determined by this use of the "normal" curve to describe the variations due to error in observation. If we divide the area of the curve into three portions, *viz.*, one limited by two ordinates each equidistant from the middle line, and two portions external to these ordinates; so that the area of the central portion is equal to twice the area of either of the external portions, it can be calculated that the distance of the ordinates from the middle line is $\cdot 67449\ \sigma$. This is termed the probable error, and it signifies that the chances of an observation falling into the central portion or into one or other of the external portions are equal.

14. When the curve is asymmetrical, that is, when it is derived from $(p + q)^n$, where p is not equal to q, the standard deviation still has a significance as indicating the degree of "scatter," but it can no longer be

used to measure the deviation on both sides of the mean. The mode or most probable value is now no longer coincident with the mean, but lies more or less to one side of it.

15. The probable error has in itself little practical use, since no inference can be drawn where the odds are equal. The common rule is to take three times the probable error as indicating the point at which a conclusion may be taken as fairly probable, but it is better to avoid using the term "probable error" and consider only the standard deviation, which, as twice the standard deviation is almost exactly equal to three times the probable error, occasions no change of argument, and only a small change of nomenclature. In this connection the standard deviation may well be called the "standard error," as is done by Mr. Yule. In the accompanying table (Table I.) are shown the chances of the observation lying within the area of the curve limited by distances from the mid-line $\pm \cdot 5\sigma$, $\pm \sigma$, $\pm 1 \cdot 5\sigma$, etc., and the approximate values of the chance of failure in fractions. If the standard deviation itself is used the odds are two to one in favour of the actual figures lying within the area bounded by $y = \pm \sigma$; if twice the standard deviation be taken, the usual limit, these rise to 21 to 1, while if three times the standard deviation be used they rise to 369 to 1. Even in the latter instance, however, the odds must not be considered overwhelming.

TABLE I.—*Showing the Chances that the Actual Observation lies within* $y = \pm n\sigma$.

n	Chances of success.	Approximate chances of failure.
·5	·3829	$\frac{2}{3}$
1·0	·6827	$\frac{1}{3}$
1·5	·8664	$\frac{1}{7}$
2·0	·9545	$\frac{1}{22}$
2·5	·9876	$\frac{1}{81}$
3·0	·9973	$\frac{1}{370}$

16. Before making application of what has been said, it will be well to observe more particularly what happens when samples of a population are drawn: (1) theoretically, and (2) in actual instances.

The theory of chance gives us two formulæ. If we draw M samples of, say, r individuals from the very large population, the proportion p of which consist of one class, and the proportion q of the remainder, then the numbers in which the different proportions of p and q occur are given by the several terms of the expression,

$$M(p+q)^r$$

If, however, the number from which samples may be drawn is limited say to n, p and q remaining as before, the proportions of the different populations are given by the terms of

$$1+\frac{r}{1}\cdot\frac{qn}{pn-r+1}+\frac{r(r-1)}{1\cdot 2}\,\frac{qn(qn-1)}{(pn-r+1)\,(pn-r+2)}+\cdots$$

17. These are abstract problems, but such problems appear regularly in public health statistics. I have had a series of these investigated. In the case of scarlet fever, Belvidere Hospital (1900–1912); scarlet fever, Ruchill Hospital (1909–1912); enteric fever, Belvidere and Ruchill Hospitals (1900–1912); and diphtheria, Ruchill Hospital, (1909–1912), all the cases have been tabulated in consecutive groups of 100 to 200 cases (Table II.). In each of these groups the even numbers

TABLE II.—*Showing the Number of Deaths in Parallel Series of Cases Chosen so that the Alternate Cases fall into Different Groups.*

Scarlet Fever,* Belvidere, 1900—1908. 4,700 cases.				Enteric Fever,* Belvidere and Ruchill, 1900—1912, 1,900 cases.		Diphtheria,† Ruchill. 1909—1912, 1,800 cases.		Scarlet Fever,† Ruchill, 1909—1912, 2,900 cases.	
1	–	2	1	12	8	10	13	4	3
2	–	1	2	11	6	7	7	5	3
1	3	3	2	10	10	6	9	1	2
1	3	3	–	10	6	10	4	2	3
3	–	2	6	11	14	13	10	3	4
3	–	2	2	14	5	8	12	1	3
3	1	–	1	9	9	14	11	6	5
1	–	2	4	6	5	9	9	2	4
2	1	1	4	15	5	11	12	3‡	1‡
5	4	–	3	4	6			2	1
1	3	3	2	9	10			1	–
3	2	2	3	10	6			4	2
1	4	–	1	9	19			3	3
4	1	2	2	10	14			3	2
3	2	–	1	10	15			5	1
2	7	1	1	10	7				
3	2	2	1	10	5				
3	3	3	1	8	8				
1	4	4	5	7	7				
3	3	7	4	6	7				
1	3	2	1						
2	1	2	1						
3	2	1	2						
3	4								

* Each figure denotes the number of deaths in 50 cases.
† Each figure denotes the number of deaths in 100 cases.
‡ Each 80 cases.

of the hospital register have been kept separate from the odd numbers. It is thus possible to compare the mortalities of groups of cases admitted at the same time and selected from each other only after an interval of years by a method as absolutely fair as seems possible.

18. Take as a first instance the drawing of samples of 50 from a general population which is divided into two portions in the proportions of 18 to 82. The result of M trials is given by the terms of the expansion

$$M\left(\frac{82}{100}+\frac{18}{100}\right)^{50}$$

$$\text{Or } M\left\{\left(\frac{82}{100}\right)^{50}+50\left(\frac{82}{100}\right)^{49}\frac{18}{100}+\frac{50\cdot49}{1\cdot2}\left(\frac{82}{100}\right)^{48}\left(\frac{18}{100}\right)^{2}+\ldots\ldots\right.$$

This expansion includes the case of the groups of enteric fever where the mortality has been on the average 18 per cent. Thirty-eight groups of fifties occur. The numbers of these groups with each definite number

TABLE III.—*Showing the Actual Number of Groups of* 50 *Cases of Enteric Fever (Belvidere and Ruchill Hospitals) which contain a Definite Number of Deaths compared with Expectation.*

Number of Deaths.	Number of Groups with x Deaths.	Number of Groups Expected Theoretically.
0	–	·–
1	–	·–
2	–	·1
3	·–	·4
4	1	1·0
5	4	2·1
6	5	3.5
7	3	4·5
8	4	5·4
9	3	5·5
10	9	5·0
11	2	4·0
12	1	2·8
13	.–	1·8
14	3	1·0
15	2	.5
16	·–	·26
17	·–	·12
18	·–	·05
19	1	·02
20	1	·01
21	·–	·00

of deaths are given in the adjoining table and compared with the numbers obtained from the expression given above. The fit is not good, but if they be grouped in larger classes, namely 0-3 deaths, 4-7 deaths, etc., the theoretical and the actual numbers show a good correspondence. (Table IV.)

TABLE IV.

No. of Deaths.	Actual.	Theoretical.
0—3	0	·5
4—7	13	11·1
8—11	18	19·9
12—15	6	6·1
16—19	1	·47
	38	38·07

19. It is not at all clear, however, that we should draw from an infinite class. The consecutive cases of scarlet fever in Ruchill number 2,960 with 82 deaths. Taking these numbers the relative proportions of samples of 100 of different constitution can easily be calculated (par. 16). These numbers are shown below and compared with those acutually found. (Table V.)

TABLE V.—*Scarlet Fever, Ruchill Hospital.*

Number of Deaths.	Number Expected.	Number Found.	Difference.	Difference Squared Divided by Theoretical Numbers.
0	1·83	1	·83	·38
1	5·34	6	·66	·08
2	7·41	6	1·41	·27
3	6·85	9	2·15	·70
4	4·66	4	·66	·10
5	2·47	3	·53	1·13
6	1·06	1	·06	·00
7	·38	—	—	—
8	·11	—	—	—
	30·11	30·00	—	2·64

It is a very good fit considering the small number of the observations. A similar table is given for Belvidere. Here in the earlier period the mortality was much higher, 203 deaths taking place in 4,700 cases. The variation in groups of 50 cases is considered. The actual and theoretical figures are given in Table VI.

TABLE VI.—*Scarlet Fever, Belvidere Hospital.*

Number of Deaths.	Theoretical.	Actual	Difference.	Difference Squared Divided by Theoretical Numbers.
0	10·4	10	·4	·00
1	23·4	25	1·6	·11
2	25·8	23	2·8	·30
3	18·7	22	3·3	·59
4	9·8	9	·8	·07
5	4·0	2	2·0	1·00
6	1·3 } 1.7	1 } 3	1·3	1·00
7	·4	2		
	93·8	94		3·07

These figures show a very fair correspondence between fact and theory. Possibly the fit might be better or might be worse with larger numbers, for in the figures as given there is a correlation between high numbers of deaths or low numbers of deaths in the corresponding fifties or hundreds to be expected as the fevers vary somewhat in severity from period to period, but the numbers are not sufficient to determine the amount definitely.

20. We are now in a position to explain the proof of the chief theorem in probable error as applied to vital statistics. The problem is: if we have a population of N individuals consisting of s groups $y_1, y_2, \ldots y_s$, to find the standard deviation of the group y_p. The chance of one individual being drawn from this group is evidently $\frac{y_p}{N}$ and likewise the chance of his not being drawn is $1-\frac{y_p}{N}$. If then m individuals have been selected by chance the proportional distributions will be represented, as has been seen before, by the terms of the expansion of $\left\{\frac{y_p}{N} + \left(1-\frac{y_p}{N}\right)\right\}^m$ the standard deviation of which is $\sqrt{m\frac{y_p}{N}\left(1-\frac{y_p}{N}\right)}$. Now we do not know the ratio of y_p to N. All that is known is the ratio which the samples of these quantities bear to each other. We may, however, assume, subject to subsequent investigation, that these ratios are for practical purposes identical, keeping in mind that this at present is only an assumption. If then y'_p denotes the actual number of y_p found, we have the standard deviation of the error of y_p since the total number of observations is m, represented by $\sqrt{m\frac{y_p'}{m}\left(1-\frac{y_p'}{m}\right)}$ or suppressing the accents by $\sqrt{\frac{y_p(m-y_p)}{m}}$

21. To make clearer the meaning of the formula just found it is applied to the example given before (par. 19) regarding the actual groups found of 4,700 cases of scarlet fever. The mean death-rate and the standard deviation of these, calculated as in the example (par. 11) are found to be $M = 2{\cdot}16$ and $\sigma = 1{\cdot}475$. With the formula just given

$$\sigma = \sqrt{\frac{2{\cdot}16 \quad (50 - 2{\cdot}16)}{50}}$$
$$= \quad 1{\cdot}437$$

It must be noted that the grouping is quite asymmetric, nearly twice as many cases occurring on the one side of the mean as on the other, so that a smaller value of the death-rate than that given by the mean is twice as probable as one that is larger. A second application is made to deaths in each series of 100 cases of scarlet fever seen in Ruchill Hospital. Here, as we have already seen (par. 11) $\sigma = 1{\cdot}436$, the average number of deaths per hundred is 2·73, so that we have by the formula of standard error

$$\sigma = \sqrt{\frac{2{\cdot}73 \quad (100 - 2{\cdot}73)}{100}}$$
$$= \sqrt{2{\cdot}658}$$
$$= \quad 1{\cdot}630$$

or in this case the actual range found is considerably less than that expected theoretically.

22. The method in which the standard error varies can best be observed by considering actual figures. In the two next tables (Tables VII.–VIII.) two sets of values of the standard error are given. The first values given are the absolute values. Thus, if from the column showing the number of cases 5,000 is chosen, the corresponding value of the standard error when the death-rate is 5 per cent. is seen to be 15·411; a 5 per cent. mortality in 5,000 cases means 250 deaths. Twice the standard error is 30·8, so

TABLE VII.—*Showing the Value of the Standard Error* of the Number of Deaths *for Different Percentage Death-Rates when the Number of Cases Increases.*

No. of Cases.	Percentage Mortality.									
	1 p.c.	2 p.c.	3 p.c.	4 p.c.	5 p.c.	10 p.c.	20 p.c.	30 p.c.	40 p.c.	50 p.c.
100	·995	1·400	1·706	1·960	2·179	3·000	4·000	4·583	4·899	5·000
500	2·225	3·131	3·815	4·382	4·874	6·708	8·944	10·247	11·045	11·180
1,000	3·146	4·427	5·394	6·197	6·892	9·487	12·649	14·491	15·492	15·811
5,000	7·036	9·900	12·063	13·586	15·411	21·331	28·284	32·404	34·641	35·355
10,000	9·95	14·00	17·06	19·60	21·79	30·000	40·000	45·83	48·99	50·00

TABLE VIII.—*Showing the Value of the Standard Error* of the Percentage Death-Rate *when the Number of Cases Increases.*

No. of Cases.	Percentage Mortality.									
	1 p.c.	2 p.c.	3 p.c.	4 p.c.	5 p.c.	10 p.c.	20 p.c.	30 p.c.	40 p.c.	50 p.c.
100	·995	1·400	1·706	1·960	2·179	3·000	4·000	4·583	4·899	5·000
500	·445	·626	·763	·872	·975	1·341	1·788	2·050	2·209	2·236
1,000	·315	·443	·539	·620	·689	·949	1·265	1·449	1·549	1·581
5,000	·121	·198	·241	·271	·308	·426	·566	·648	·693	·707
10,000	·100	·140	·171	·196	·218	·300	·400	·458	·490	·500

that the odds are 21 to 1 that the real number of deaths lies between 220 and 281. The figure in the same place in the second table is ·308. This is the standard error of the percentage death-rate, or the odds are again 21 to 1 that the true percentage death-rate lies between 4·382 and 5·616 if the death-rate is based on 5,000 cases.

23. Several facts are easily seen in considering these tables. First the standard error increases with the increasing percentage mortality, rising from ·995 in the first row when the percentage is unity, to 5·000 when the percentage is 50; but relatively to the percentage itself it steadily decreases. Twice the standard error when the percentage is 3 gives values $3 \pm 3{\cdot}4$ for the limits, which will be exceeded once in every 22 times, while the same limits when the percentage is 40 are $40 \pm 9{\cdot}798$. The first instance tells us little; while the last suggests that a severe mortality must be the rule.

24. It is also to be noted that very large numbers give little more certainty than more moderate numbers. Considering the last two rows in Table VIII., it is seen that the standard error is only reduced by about 15 per cent. when the mortality is 1 per cent., and about 29 per cent. when the mortality is 50 per cent. as the numbers increase from 5,000 to 10,000.

TABLE IX.—*Showing the Number of Deaths in each Series of 200 or 400 Cases.*

Scarlet Fever,* Belvidere. Even.	Odd.	Scarlet Fever,* Ruchill. Even.	Odd.	Enteric Fever,† Belvidere and Ruchill. Even.	Odd.	Diphtheria,† Ruchill. Even.	Odd.
15	7	12	11	43	30	13	16
21	24	12	16	40	33	21	13
19	22	10	4	38	27	20	22
15	18	11‡	6‡	39	55		
9	17						

* Each figure is the deaths in 400 cases. † Each figure is the deaths in 200 cases.
‡ Each figure is the deaths in 300 cases.

25. One more table is given to show how the death-rate may actually vary in fairly large numbers. In each instance 400 or 200 cases are compared with 400 or 200 parallel cases, the first the even number in the registers and the second the odd; the differences are surprising. 200, in the case of enteric fever or of diphtheria, is a large number; 400, in the instance of scarlet fever, not a small number, yet had any treatment been the subject of investigation and the alternate cases taken, the one for treatment and the other for control, very erroneous conclusions could easily have been advanced.

26. One more formula may be given without discussion. It is the standard error of the mean. The proof involves principles not discussed in this paper, but the result can easily be understood. If h be the mean of the number of observations and σ the standard deviation of these observations, h and σ being calculated as described in par. 11, then the standard error of the mean $= \frac{\sigma}{\sqrt{m}}$, when m is the total number of observations.

This signifies that if we are comparing the means of two series of observations, no conclusion can be deemed even moderately definite unless the differences between the two means is greater than $\frac{2\sigma}{\sqrt{m}}$

II.

In the proof of the standard error given in par. 20 an assumption was made, namely, that the proportions of the sample which had been found by random selection might be considered as equivalent to those existing in the general population. Now in this case all the information in our possession can be stated mathematically by saying that an event has happened m times, and failed n times. From this the probable constitution of the universe must be deduced. This problem was first considered by Bayes, and the solution is known as Bayes' Theorem. The proof is difficult, but the formula is easily understood. If m deaths and n recoveries have taken place, the different populations from which these may have been drawn have the relative probabilities given by the areas of the successive strips of the curve.

$$y = x^m (1-x)^n$$

or if the total chance be denoted by unity, the chance of each type of population existing

$$\text{is} \qquad \frac{x^m (1-x)^n \, dx}{\int_0^1 x^m (1-x)^n \, dx}$$

For practical purposes the ordinates at any points roughly give the relative probabilities. Thus, if 3 represent the number of deaths found in 100 cases, and 97 the number of recoveries, the probabilities that the general population possesses *1* per cent., *2* per cent., *3* per cent. death-rates, etc., are given by substituting these values for m, n and x respectively in the above formula, and are:—

$$(\cdot 01)^3 \quad (1-\cdot 01)^{97}$$
$$(\cdot 02)^3 \quad (1-\cdot 02)^{97}$$
$$(\cdot 03)^3 \quad (1-\cdot 03)^{97}, \text{ etc.}$$

The figures are given in the adjoining table (Table X). For comparison the values obtained by the same formula where $m=30$ and $n=970$ (figures which express the same death-rate in a larger number of cases) are arranged in a parallel column:—

TABLE X.—*Showing the Chances of Each Constitution of the Population when the Sample Contains* (1) *3 of one kind* (a) *to 97 of the other* (b) *and* (2) *30 of* (a) *to 970 of* (b).

Percentage of (a) in the Constitution of the Sample.	(1) Relative Size of Ordinates of $x^3(1-x)^{97}$	(2) Relative size of Ordinates of $x^{30}(1-x)^{970}$
0	·70	·000
1	3·77	·000
1·5	—	·083
2	11·28	3·342
2·5	—	18·548
3	14·06	30·235
3·5	—	20·711
4	12·37	10·085
4·5	—	3·135
5	8·63	·228
6	5·35	·002
7	3·01	—
8	1·62	—
9	·78	—
10	·36	—
11	·16	—
12	·07	—

It is seen, in the first instance, that a population constituted so as to possess a four per cent. mortality is about equally as probable as one constituted to possess a three per cent. mortality, and that a two per cent. mortality is only a little less probable. It is also evident that populations with mortalities of five and six per cent. will occur once in every seven and twelve times respectively. Little can therefore be surmised concerning the constitution of the population from information based on one hundred observations. When one thousand cases, however, are considered, the range is much less, as is seen in the table. The probability rises to nearly $\frac{2}{3}$rds, but even here a population with a four per cent. constitution will occur more than once for every three times the sample represents the population accurately.

This problem is distinctly different from that considered in the previous pages. It seeks to find the constitution of the general population from the sample, that previously considered to find the probable constitution of the sample when the type of the population is known. The standard deviations are therefore different. In the present instance

$$\sigma^2 = \frac{(m+1)\ (n+1)\ (m+n)^2}{(m+n+3)\ \ (m+n+2)^2}$$

which is larger than the corresponding value of the standard error of a sample, most markedly so when m and n are small, but very closely approximating when the values are greater. Thus, for $m=3$ $n=97$ the values of the two standard errors are 1·91 and 1·71 respectively, while for $m=30$ $n=970$ the corresponding figures are 5·46 and 5·39.

With 100 cases the limits given by twice the standard error are $3 \pm 3{\cdot}8$, with 1,000 cases $30 \pm 10{\cdot}92$, giving a range of percentage 0 to 6·8 in the former case and 1·9 to 4·1 in the latter. Such are the limitations of the assumption on which the proof given in par. 20 depends.

The two theorems may, however, be combined. The proof is given by Prof. Pearson in the *Philosophical Magazine*, Mar., 1907: the results only concern us here.

If m and n are the numbers of each kind found in the sample, and if the next sample number q, then the standard deviation with samples of number q is given by

$$\sigma^2 = \frac{q\ (m+1)\ \ (n+1)\ \ (m+n+q+2)}{(m+n+2)^2\ (m+n+3)}$$

If $q = m + n$, *i.e.*, if the standard deviation of m or n in samples numbering $m+n$ is desired, then

$$\sigma^2 = \frac{2(m+n)\ (m+1)\ (n+1)\ (m+n+1)}{(m+n+2)^2\ (m+n+3)}$$

which is approximately equal to $\frac{2mn}{m+n}$ if m and n be large, or a formula similar to Poisson's is arrived at, though the two are not really comparable, as they have been obtained on quite different premises.

If q be unequal to $m+n$, but both numbers large, the formula becomes

$$\sigma^2 = q^2 \frac{mn}{(m+n)^2} \left\{ \frac{1}{m+n} + \frac{1}{q} \right\}$$

This shows that if $m+n$ be small compared with q, the standard deviation does not become smaller with the increase of q, or we cannot predict a large from a small sample, but only the opposite, the latter reason explaining why the standard deviations obtained experimentally in the earlier part of this paper (par. 21) are in so close accord with the ordinary theory.

The formulæ in this section, however, are those which should be used to check the validity of conclusions drawn from figures.

An example will make this easier to understand. Let the first series of cases be 100, namely, 40 of one class and 60 of the other, and let it be desired to find the standard error in a second series of 1,000 cases.

Here $m = 40, \quad n = 60, \quad q = 1{,}000$

so that
$$\sigma^2 = \frac{1{,}000^2 \times 40 \times 60}{100^2} \left\{ \frac{1}{100} + \frac{1}{1{,}000} \right\}$$
$$= 2{,}640$$

or $\sigma = 51{\cdot}4$

In 1,000 cases 400 cases are to be expected if the proportions of the first sample are preserved so that $400 \pm 51{\cdot}4 \times 2$ are the limits, as before described. Were there 1,000 cases in the first sample with 600 of one type and 400 of the other the standard error would be given by

$$\sigma^2 = \frac{1{,}000^2 \times 400 \times 600}{1{,}000^2} \left\{ \frac{1}{1{,}000} + \frac{1}{1{,}000} \right\}$$
$$= 480$$

or $\sigma = 21{\cdot}9$

or the standard error is less than one-half of that in the previous instance.

Apart from tables these formulæ are of difficult application, as the distributions are so markedly skew or asymmetrical, the variation in one direction being much greater than in the other, and the mean not giving the most probable value. Actual calculation in individual cases is very laborious.

III.

The errors due to random selection of the population have been fully discussed, but there is yet one other type of error which is not usually given sufficient weight in actual statistical work, that is, the error due to imperfection of technique. This appears in a variety of ways, in the case of the astronomer in the slight difference between one observation and another, in the case of the marksman in the number of inners or outers he makes in comparison with the number of bulls eyes. All human measurements are liable to a certain amount of error variable in different individuals, but in the end more or less describable in each separate individual by some definite law.

Public health statistics seem at first sight comparatively free from such errors. We have a certain number of deaths and each of these represents a fact. But even weekly death-rates are far from certain apart altogether from the random selection of deaths in each week. As five days are allowed for the certification of a death, a few wet days at the end of a week may throw many certifications from one week into the next. Even such a comparatively simple matter as the number of deaths from a disease has a large experimental error. Out of the 2,960 cases of scarlet fever which were treated in Ruchill Hospital as before described, 82 deaths occurred. Of these, 5 could not definitely be ascribed to the fever, occurring associated with conditions which themselves were not likely to be fatal, but which the double disease made specially dangerous. This is a fair number as the standard error of 82 deaths in 2,960 cases is 8·9, so that the experimental error is more than half of this. On such large numbers, however, it may be neglected. In the groups each of 200 cases, however, it appears in the following manner and might easily lead to false reasoning.

Total Deaths.	Experimental Error.
8	1
11	1
4	1
3	1
6	1

I feel convinced that this is an under and not an over estimate.

In some examples which have been before me lately in another branch of science, the experimental error is for each separate observation very large, and this case is worth considering. If we have a limited number of observations, each of which we know is open to a large experimental error, we may consider the matter in this way. Let the observations have values $x_1\ x_2\ x_3\ .\ .\ .\ x_n$, let the mean be h.

$$\text{So that } h = \frac{1}{n}(x_1 + x_2 + x_3 + \dots\dots \quad x_n)$$

let the standard deviation of each term be σ, and let it be required to know the standard deviation of h.

This is found to be $\frac{\sigma}{\sqrt{m}}$, the same result as that given in par. 26: σ, however, is not the "standard error" due to random sampling, but the "standard error" of the experimental error, the result being derived by a different process of reasoning. It is easily seen that a difference exists, for if we have two groups of quantities with the same disposition in statistical series, one which can be measured exactly and the other only inexactly, the error of the mean from random sampling will be the same in both cases, but the experimental error will greatly differ. This part of the subject I intend to develop more fully later when I am in possession of the requisite public health data.

IV.

The theorem of this section is due to Prof. Pearson and furnishes a very useful criterion as to whether groups of statistics really fulfil certain conditions with reasonable probability. The proof of the theorem is very difficult and need not even be outlined, but the application, given a table of the function, is very easy. The method of calculation is as follows :— If there be an actual distribution and a theoretical distribution, the difference of the values actual and theoretical of each term is to be taken, squared and divided by the corresponding theoretical value. These values are then summed. This sum is denoted by χ^2. A table of the function is then consulted. In this the value of P is tabulated according to the values of χ^2 and of n the number of terms compared, P being the probability that in a certain number of trials a worse fit than the theoretical values will be found. In paragraph 19 two examples have been given. In Table V. we find that $\chi^2 = 2{\cdot}64$. The number of terms compared is seven. We then consult the table and find $\chi^2 = 2$ gives $P = {\cdot}920$ and $\chi^2 = 3$, $P = {\cdot}809$ whence $\chi^2 = 2{\cdot}64$ gives approximately $P = {\cdot}849$, or in 849 random trials out of one thousand a worse fit between theory and observation would occur. In the second sample $\chi^2 = 3{\cdot}07$. Hence the value of $P = {\cdot}8$ approximately, or in 8 trials out of 10 a worse result would be found. In other words, theory and observation are in good correspondence.

A third example is taken from one of my old hospital reports, 1903-4. The question to be ascertained was if there was any special day or series of days on which children sickened from scarlet fever. The days of the week on which 907 children at school ages sickened during the months of August and September, 1901-1904 were tabulated. These months were chosen as being the epidemic months, and also the months immediately after the holidays, when many susceptible children go to the school for the first time. If there be any special evidence of school infection, it should be seen in a variation in the numbers sickening on different days, as the schools do not meet on either Saturday or Sunday. The figures are given in the adjoining table in which also the application of the method is shown. The theoretical value to be tested here is obviously the mean number of all the cases namely $\frac{1}{7} \times 907$ or $129{\cdot}6$.

TABLE XI.—*Showing Days of Sickening in 907 Cases of Scarlet Fever.*

	No. of Cases.	Theoretical Value.	Difference.	(Difference)²	(Difference)² / Theoretical Value.
Sunday ...	124	129·6	−5·6	31·36	·24
Monday ...	143	129·6	13·4	179·56	1·38
Tuesday ...	117	129·6	-12·6	158·76	1·22
Wednesday ...	134	129·6	4·4	19·36	·15
Thursday ...	120	129·6	−9·6	92·16	·71
Friday ...	143	129·6	13·4	179·56	1·38
Saturday ...	126	129·6	−3·6	12·96	·10
Total ...	907	907	0·0	673·72	5·18

Thus $\chi^2 = 5{\cdot}18$ or $P = {\cdot}522$ or in half the trials made as much divergence would be found, so that there is little evidence from this source that scarlatina is spread by schools.

As tables of this function are rather inaccessible, I have constructed a short table from a diagram in my note-book. It does not profess to be more than a first approximation, and it is constructed on this principle. If the probability be less than ·1 it is of not much practical public health use. The probabilities have, therefore, been given in the top line in values from ·1, ·2, · · · to ·9. The number of instances compared is in the first verticle column. The value of χ^2, which gives each of these values, is tabulated. We can then see at a glance the probability of the result. As the value of χ^2 and the value of n are known, the probability can at once be placed between two adjacent decimals of unity, which is quite close enough for all practical purposes.

TABLE XII.—*Showing the Values of χ^2 for certain Values of P and n.*

N	Values of *P*.								
	·9	·8	·7	·6	·5	·4	·3	·2	·1
3	—	—	·7	1·0	1·2	1·8	2·4	3·4	4·6
4	—	1·00	1·4	1·9	2·4	2·9	3·6	4·6	6·1
5	1·0	1·7	2·2	2·8	3·4	4·1	4·9	5·9	7·7
6	1·6	2·3	3·0	3·6	4·3	5·1	6·0	7·3	9·2
7	2·2	3·0	3·8	4·6	5·3	6·2	7·2	8·6	10·5
8	2·9	3·8	4·7	5·4	6·3	7·3	8·4	9·7	12·0
9	3·5	4·6	5·5	6·4	7·4	8·4	9·5	11·0	13·2
10	4·2	5·4	6·4	7·4	8·4	9·4	10·6	12·2	14·6
12	5·6	7·0	8·2	9·3	10·4	11·5	12·8	14·5	17·2
14	7·0	8·7	9·9	11·1	12·3	13·6	15·1	16·9	19·6
16	8·6	10·3	11·8	13·0	14·4	15·7	16·3	19·3	22·3