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To Dr. Plarr
With the author's respects

4.

ON THE

EFFECT OF THE INTERNAL FRICTION OF FLUIDS

ON THE

MOTION OF PENDULUMS.

FROM THE TRANSACTIONS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.
VOL. IX. PART II.

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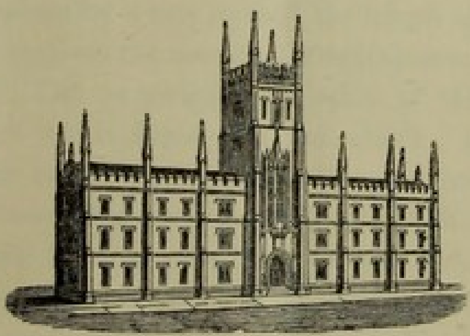

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M.DCCC.LI.

II. *On the Effect of the Internal Friction of Fluids on the Motion of Pendulums.* *By* G. G. Stokes, M.A., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *December* 9, 1850.]

The great importance of the results obtained by means of the pendulum has induced philosophers to devote so much attention to the subject, and to perform the experiments with such a scrupulous regard to accuracy in every particular, that pendulum observations may justly be ranked among those most distinguished by modern exactness. It is unnecessary here to enumerate the different methods which have been employed, and the several corrections which must be made, in order to deduce from the actual observations the result which would correspond to the ideal case of a simple pendulum performing indefinitely small oscillations in vacuum. There is only one of these corrections which bears on the subject of the present paper, namely, the correction usually termed the *reduction to a vacuum.* On account of the inconvenience and expense attending experiments in a vacuum apparatus, the observations are usually made in air, and it then becomes necessary to apply a small correction, in order to reduce the observed result to what would have been observed had the pendulum been swung in a vacuum. The most obvious effect of the air consists in a diminution of the moving force, and consequent increase in the time of vibration, arising from the buoyancy of the fluid. The correction for buoyancy is easily calculated from the first principles of hydrostatics, and formed for a considerable time the only correction which it was thought necessary to make for reduction to a vacuum. But in the year 1828 Bessel, in a very important memoir in which he determined by a new method the length of the seconds' pendulum, pointed out from theoretical considerations the necessity of taking account of the inertia of the air as well as of its buoyancy. The numerical calculation of the effect of the inertia forms a problem of hydrodynamics which Bessel did not attack; but he concluded from general principles that a fluid, or at any rate a fluid of small density, has no other effect on the time of very small vibrations of a pendulum than that it diminishes its gravity and increases its moment of inertia. In the case of a body of which the dimensions are small compared with the length of the suspending wire, Bessel represented the increase of inertia by that of a mass equal to k times the mass of the fluid displaced, which must be supposed to be added to the inertia of the body itself. This factor k he determined experimentally for a sphere a little more than two inches in diameter, swung in air and in water. The result for air, obtained in a rather indirect way, was $k = 0{\cdot}9459$, which value Bessel in a subsequent paper increased to 0·956. A brass sphere of the above size having been swung in water with two different lengths of wire in succession gave two values of k, differing a little from each other, and equal to only about two-thirds of the value obtained for air.

The attention of the scientific world having been called to the subject by the publication of Bessel's memoir, fresh researches both theoretical and experimental soon appeared. In order to examine the effect of the air by a more direct method than that employed by Bessel, a large vacuum apparatus was erected at the expense of the Board of Longitude, and by means of this apparatus Captain (now Colonel) Sabine determined the effect of the air on the time of vibration of a particular invariable pendulum. The results of the experiments are contained in a memoir read before the Royal Society in March 1829, and printed in the *Philosophical Transactions* for that year. The mean of eight very consistent experiments gave 1·655 as the factor by which for that pendulum the old correction for buoyancy must be multiplied in order to give the whole correction on account of the air. A very remarkable fact was discovered in the course of these experiments. While the effects of air at the atmospheric pressure and under a pressure of about half an atmosphere were found to be as nearly as possible proportional to the densities, it was found that the effect of hydrogen at the atmospheric pressure was much greater, compared with the effect of air, than corresponded with its density. In fact, it appeared that the ratio of the effects of hydrogen and air on the times of vibration was about 1 to $5\frac{1}{4}$, while the ratio of the densities is only about 1 to 13. In speaking of this result Colonel Sabine remarks, "The difference of this ratio from that shewn by experiment is greater than can well be ascribed to accidental error in the experiment, particularly as repetition produced results almost identical. May it not indicate an inherent property in the elastic fluids, analogous to that of viscidity in liquids, of resistance to the motion of bodies passing through them, independently of their density? a property, in such case, possessed by air and hydrogen gas in very different degrees; since it would appear from the experiments that the ratio of the resistance of hydrogen gas to that of air is more than double the ratio following from their densities. Should the existence of such a distinct property of resistance, varying in the different elastic fluids, be confirmed by experiments now in progress with other gases, an apparatus more suitable than the present to investigate the ratio in which it is possessed by them, could scarcely be devised: and the pendulum, in addition to its many important and useful purposes in general physics, may find an application for its very delicate, but, with due precaution, not more delicate than certain, determinations, in the domain of chemistry." Colonel Sabine has informed me that the experiments here alluded to were interrupted by a cause which need not now be mentioned, but that as far as they went they confirmed the result of the experiments with hydrogen, and pointed out the existence of a specific action in different gases, quite distinct from mere variations of density.

Our knowledge on the subject of the effect of air on the time of vibration of pendulums has received a most valuable addition from the labours of the late Mr Baily, who erected a vacuum apparatus at his own house, with which he performed many hundreds of careful experiments on a great variety of pendulums. The experiments are described in a paper read before the Royal Society on the 31st of May 1832. The result for each pendulum is expressed by the value of n, the factor by which the old correction for buoyancy must be multiplied in order to give the whole effect of the air as deduced from observation. Four spheres, not quite $1\frac{1}{2}$ inch in diameter, gave as a mean $n = 1{\cdot}864$, while three spheres, a little more than 2 inches in diameter, gave only 1·748. The latter were nearly of the same size as those with which Bessel, by a different method, had obtained $k = 0{\cdot}946$ or $0{\cdot}956$, which corresponds to $n = 1{\cdot}946$ or $1{\cdot}956$. Among the "Additional Experiments" in the latter part of Baily's paper, is a set in which the pendulums consisted of plain cylindrical rods. With these pendulums it was found that n regularly increased, though according to an unknown law, as the diameter of the rod decreased. While a brass tube $1\frac{1}{2}$ inch in diameter gave n equal to about 2·3, a thin rod or thick wire only 0·072 inch in diameter gave for n a value as great as 7·530.

Mathematicians in the meanwhile were not idle, and several memoirs appeared about this time, of which the object was to determine from hydrodynamics the effect of a fluid on the motion of a pendulum. The first of these came from the pen of the celebrated Poisson. It was read before the French Academy on the 22nd of August 1831, and is printed in the 11th Volume of the Memoirs. In this paper, Poisson considers the case of a sphere suspended by a fine wire, and oscillating in the air, or in any gas. He employs the ordinary equations of motion of an elastic fluid, simplified by neglecting the terms which involve the square of the velocity; but in the end, in adapting his solution to practice, he neglects, as insensible, the terms by which alone the action of an elastic differs from that of an incompressible fluid, so that the result thus simplified is equally applicable to fluids of both classes. He finds that when insensible quantities are neglected $n = 1{\cdot}5$, so that the mass which we must suppose added to that of the pendulum is equal to half the mass of the fluid displaced. This result does not greatly differ from the results obtained experimentally by Bessel in the case of spheres oscillating in water, but differs materially from the result he had obtained for air. It agrees pretty closely with some experiments which had been performed about fifty years before by Dubuat, who had in fact anticipated Bessel in shewing that the time of vibration of a pendulum vibrating in a fluid would be affected by the inertia of the fluid as well as by its density. Dubuat's labours on this subject had been altogether overlooked by those who were engaged in pendulum experiments; probably because such persons were not likely to seek in a treatise on hydraulics for information connected with the subject of their researches. Dubuat had, in fact, rather applied the pendulum to hydrodynamics than hydrodynamics to the pendulum.

In the *Philosophical Magazine* for September 1833, p. 185, is a short paper by Professor Challis, on the subject of the resistance to a ball pendulum. After referring to a former paper, in which he had shewn that no sensible error would be committed in a problem of this nature by neglecting the compressibility of the fluid even if it be elastic, Professor Challis, adopting a particular hypothesis respecting the motion, obtains 2 for the value of the factor n for such a pendulum. This mode of solution, which is adopted in several subsequent papers, has given rise to a controversy between Professor Challis and the Astronomer Royal, who maintains the justice of Poisson's result.

In a paper read before the Royal Society of Edinburgh on the 16th of December 1833, and printed in the 13th Volume of the Society's *Transactions*, Green has determined from the common equations of fluid motion the resistance to an ellipsoid performing small oscillations without rotation. The result is expressed by a definite integral; but when two of the principal axes of the ellipsoid become equal, the integral admits of expression in finite terms, by means of circular or logarithmic functions. When the ellipsoid becomes a sphere, Green's result reduces itself to Poisson's.

In a memoir read before the Royal Academy of Turin on the 18th of January 1835, and printed in the 37th Volume of the memoirs of the Academy, M. Plana has entered at great length into the theory of the resistance of fluids to pendulums. This memoir contains, however, rather a detailed examination of various points connected with the theory, than the determination of the resistance for any new form of pendulum. The author first treats the case of an incompressible fluid, and then shews that the result would be sensibly the same in the case of an elastic fluid. In the case of a ball pendulum, the only one in which a complete solution of the problem is effected, M. Plana's result agrees with Poisson's.

In a paper read before the Cambridge Philosophical Society on the 29th of May 1843, and printed in the 8th Volume of the *Transactions*, p. 105, I have determined the resistance to a ball pendulum oscillating within a concentric spherical envelope, and have pointed out the source of an error into which Poisson had fallen, in concluding that such an envelope would have no effect. When the radius of the envelope becomes infinite, the solution agrees with that which Poisson had obtained for the case of an unlimited mass of fluid. I have also investigated the increase of resistance due to the confinement of the fluid by a distant rigid plane. The same paper contains likewise the calculation of the resistance to a long cylinder oscillating in a mass of fluid either unlimited, or confined by a cylindrical envelope, having the same axis as the cylinder in its position of equilibrium. In the case of an unconfined mass of fluid, it appeared that the effect of inertia was the same as if a mass equal to that of the fluid displaced were distributed along the axis of the cylinder, so that $n = 2$ in the case of a pendulum consisting of a long cylindrical rod. This nearly agrees with Baily's result for the long $1\frac{1}{2}$ inch tube; but, on comparing it with the results obtained with the cylindrical rods, we observe the same sort of discrepancy between theory and observation as was noticed in the case of spheres. The discrepancy is, however, far more striking in the present case, as might naturally have been expected, after what had been observed with spheres, on account of the far smaller diameter of the solids employed.

A few years ago Professor Thomson communicated to me a very beautiful and powerful method which he had applied to the theory of electricity, which depended on the consideration of what he called *electrical images*. The same method, I found, applied, with a certain modification, to some interesting problems relating to ball pendulums. It enabled me to calculate the resistance to a sphere oscillating in presence of a fixed sphere, or within a spherical envelope, or the resistance to a pair of spheres either in contact, or connected by a narrow rod, the direction of oscillation being, in all these cases, that of the line joining the centres of the spheres. The effect of a rigid plane perpendicular to the direction of motion is of course included as a particular case. The method even applies, as Professor Thomson pointed out to me, to the uncouth solid bounded by the exterior segments of two intersecting spheres, provided the exterior angle of intersection be a submultiple of two right angles. A set of corresponding problems, in which the spheres are replaced by long cylinders, may be solved in a similar manner. These results were mentioned at the meeting of the British Association at Oxford in 1847, and are noticed in the volume of reports for that year, but they have not yet been published in detail.

The preceding are all the investigations that have fallen under my notice, of which the object was to calculate from hydrodynamics the resistance to a body of given form oscillating as a pendulum. They all proceed on the ordinary equations of the motion of fluids. They all fail to account for one leading feature of the experimental results, namely, the increase of the factor n with a decrease in the dimensions of the body. They recognize no distinction between the action of different fluids, except what arises from their difference of density.

In a conversation with Dr Robinson about seven or eight years ago on the subject of the application of theory to pendulums, he noticed the discrepancy which existed between the results of theory and experiment relating to a ball pendulum, and expressed to me his conviction that the discrepancy in question arose from the adoption of the ordinary theory of fluid motion, in which the pressure is supposed to be equal in all directions. He also described to me a remarkable experiment of Sir James South's which he had witnessed. This experiment has not been published, but Sir James South has kindly allowed me to mention it. When a pendulum is in motion, one would naturally have supposed that the air near the moving body glided past the surface, or the surface past it, which comes to the same thing if the relative motion only be considered, with a velocity comparable with the absolute velocity of the surface itself. But on attaching a piece of gold leaf to the bottom of a pendulum, so as to stick out in a direction perpendicular to the surface, and then setting the pendulum in motion, Sir James South found that the gold leaf retained its perpendicular position just as if the pendulum had been at rest; and it was not till the gold leaf carried by the pendulum had been removed to some distance from the surface, that it began to lag behind. This experiment shews clearly the existence of a tangential action between the pendulum and the air, and between one layer of air and another. The existence of a similar action in water is clearly exhibited in some experiments of Coulomb's which will be mentioned in the second part of this paper, and indeed might be concluded from several very ordinary phenomena. Moreover Dubuat, in discussing the results of his experiments on the oscillations of spheres in water, notices a slight increase in the effect of the water corresponding to an increase in the time of vibration, and expressly attributes it to the *viscosity* of the fluid.

Having afterwards occupied myself with the theory of the friction of fluids, and arrived at general equations of motion, the same in essential points as those which had been previously obtained in a totally different manner by others, of which, however, I was not at the time aware, I was desirous of applying, if possible, these equations to the calculation of the motion of some kind of pendulum. The difficulty of the problem is of course materially increased by the introduction of internal friction, but as I felt great confidence in the essential parts of the theory, I thought that labour would not be ill-bestowed on the subject. I first tried a long cylinder, because the solution of the problem appeared likely to be simpler than in the case of a sphere. But after having proceeded a good way towards the result, I was stopped by a difficulty relating to the determination of the arbitrary constants, which appeared as the coefficients of certain infinite series by which the integral of a certain differential equation was expressed. Having failed in the case of a cylinder, I tried a sphere, and presently found that the corresponding differential equation admitted of integration in finite terms, so that the solution of the problem could be completely effected. The result, I found, agreed very well with Baily's experiments, when the numerical value of a certain constant was properly assumed; but the subject was laid aside for some time. Having afterwards attacked a definite integral to which Mr Airy had been led in considering the theory of the illumination in the neighbourhood of a caustic, I found that the method which I had employed in the case of this integral would apply to the problem of the resistance to a cylinder, and it enabled me to get over the difficulty with which I had before beeen baffled. I immediately completed the numerical calculation, so far as was requisite to compare the formulæ with Baily's experiments on cylindrical rods, and found a remarkably close agreement between theory and observation. These results were mentioned at the meeting of the British Association at Swansea in 1848, and are briefly described in the volume of reports for that year.

The present paper is chiefly devoted to the solution of the problem in the two cases of a sphere and of a long cylinder, and to a comparison of the results with the experiments of Baily and others. Expressions are deduced for the effect of a fluid both on the time and on the arc of vibration of a pendulum consisting either of a sphere, or of a cylindrical rod, or of a combination of a sphere and a rod. These expressions contain only one disposable constant, which has a very simple physical meaning, and which I propose to call the *index of friction* of the fluid. This constant we may conceive determined by one observation, giving the effect of the fluid either on the time or on the arc of vibration of any one pendulum of one of the above forms, and then the theory ought to predict the effect both on the time and on the arc of vibration of all such pendulums. The agreement of theory with the experiments of Baily on the time of vibration is remarkably close. Even the rate of decrease of the arc of vibration, which it formed no part of Baily's object to observe, except so far as was necessary for making the small correction for reduction to indefinitely small vibrations, agrees with the result calculated from theory as nearly as could reasonably be expected under the circumstances.

It follows from theory that with a given sphere or cylindrical rod the factor n increases with the time of vibration. This accounts in a good measure for the circumstance that Bessel obtained so large a value of k for air, as is shewn at length in the present paper; though it unquestionably arose in a great degree from the increase of resistance due to the close proximity of a rigid plane to the swinging ball.

I have deduced the value of the index of friction of water from some experiments of Coulomb's on the decrement of the arc of oscillation of disks, oscillating in water in their own plane by the torsion of a wire. When the numerical value thus obtained is substituted in the expression for the time of vibration of a sphere, the result agrees almost exactly with Bessel's experiments with a sphere swung in water.

The present paper contains one or two applications of the theory of internal friction to problems which are of some interest, but which do not relate to pendulums. The resistance to a sphere moving uniformly in a fluid may be obtained as a limiting case of the resistance to a ball pendulum, provided the circumstances be such that the square of the velocity may be neglected. The resistance thus determined proves to be proportional, for a given fluid and a given velocity, not to the surface, but to the radius of the sphere; and therefore the accelerating force of the resistance increases much more rapidly, as the radius of the sphere decreases, than if the resistance varied as the surface, as would follow from the common theory. Accordingly, the resistance to a minute globule of water falling through the air with its terminal velocity depends almost wholly on the internal friction of air. Since the index of friction of air is known from pendulum experiments, we may easily calculate the terminal velocity of a globule of given size, neglecting the part of the resistance which depends upon the square of the velocity. The terminal velocity thus obtained is so small in the case of small globules such as those of which we may conceive a cloud to be composed, that the apparent suspension of the clouds does not seem to present any difficulty. Had the resistance been determined from the common theory, it would have been necessary to suppose the globules much more minute, in order to account in this way for the phenomenon. Since in the case of minute globules falling with their terminal velocity the part of the resistance depending upon the square of the velocity, as determined by the common theory, is quite insignificant compared with the part which depends on the internal friction of the air, it follows that were the pressure equal in all directions in air in the state of motion, the quantity of water which would remain suspended in the state of cloud would be enormously diminished. The pendulum thus, in addition to its other uses, affords us some interesting information relating to the department of meteorology.

The fifth section of the first part of the present paper contains an investigation of the effect of the internal friction of water in causing a series of oscillatory waves to subside. It appears from the result that in the case of the long swells of the ocean the effect of friction is insignificant, while in the case of the ripples raised by the wind on a small pool, the motion subsides very rapidly when the disturbing force ceases to act.

PART I.

ANALYTICAL INVESTIGATION.

SECTION I.

Adaptation of the general equations to the case of the fluid surrounding a body which oscillates as a pendulum. General laws which follow from the form of the equations. Solution of the equations in the case of an oscillating plane.

1. In a paper "*On the Theories of the Internal Friction of Fluids in Motion, &c.*," which the Society did me the honour to publish in the 8th Volume of their *Transactions*, I have arrived at the following equations for calculating the motion of a fluid when the internal friction of the fluid itself is taken into account, and consequently the pressure not supposed equal in all directions:

$$\frac{dp}{dx} = \rho\left(X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz}\right) + \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)$$
$$+ \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

with two more equations which may be written down from symmetry. In these equations u, v, w are the components of the velocity along the rectangular axes of x, y, z; X, Y, Z are the components of the accelerating force; p is the pressure, t the time, ρ the density, and μ a certain constant depending on the nature of the fluid.

The three equations of which (1) is the type are not the general equations of motion which apply to a heterogeneous fluid when internal friction is taken into account, which are those numbered 10 in my former paper, but are applicable to a homogeneous incompressible fluid, or to a homogeneous elastic fluid subject to small variations of density, such as those which accompany sonorous vibrations. It must be understood to be included in the term *homogeneous* that the temperature is uniform throughout the mass, except so far as it may be raised or lowered by sudden condensation or rarefaction in the case of an elastic fluid. The general equations contain the differential coefficients of the quantity μ with respect to x, y, and z; but the equations of the form (1) are in their present shape even more general than is required for the purposes of the present paper.

These equations agree in the main with those which had been previously obtained, on different principles, by Navier, by Poisson, and by M. de Saint-Venant, as I have elsewhere observed*. The differences depend only on the coefficient of the last term, and this term vanishes in the case of an incompressible fluid, to which Navier had confined his investigations.

The equations such as (1) in their present shape are rather complicated, but in applying

* Report on recent researches in Hydrodynamics. Report of the British Association for 1846, p. 16.

them to the case of a pendulum they may be a good deal simplified without the neglect of any quantities which it would be important to retain. In the first place the motion is supposed very small, on which account it will be allowable to neglect the terms which involve the square of the velocity. In the second place, the nature of the motion that we have got to deal with is such that the compressibility of the fluid has very little influence on the result, so that we may treat the fluid as incompressible, and consequently omit the last terms in the equations. Lastly, the forces X, Y, Z are in the present case the components of the force of gravity, and if we write

$$p + \Pi + \rho \int (X dx + Y dy + Z dz)$$

for p, we may omit the terms X, Y, Z.

If z' be measured vertically downwards from a horizontal plane drawn in the neighbourhood of the pendulum, and if g be the force of gravity, $\int (X dx + Y dy + Z dz) = gz'$, the arbitrary constant, or arbitrary function of the time if it should be found necessary to suppose it to be such, being included in Π. The part of the whole force acting on the pendulum which depends on the terms $\Pi + g\rho z'$ is simply a force equal to the weight of the fluid displaced, and acting vertically upwards through the centre of gravity of the volume.

When simplified in the manner just explained, the equations such as (1) become

$$\left.\begin{aligned} \frac{dp}{dx} &= \mu \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \rho \frac{du}{dt}, \\ \frac{dp}{dy} &= \mu \left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) - \rho \frac{dv}{dt}, \\ \frac{dp}{dz} &= \mu \left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right) - \rho \frac{dw}{dt}, \end{aligned}\right\} \quad \text{. . . . . . .} \quad (2)$$

which, with the equation of continuity,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \quad \text{. . . . .} \quad (3)$$

are the only equations which have to be satisfied at all points of the fluid, and at all instants of time.

In applying equations (2) to a particular pendulum experiment, we may suppose μ constant; but in order to compare experiments made in summer with experiments made in winter, or experiments made under a high barometer with experiments made under a low, it will be requisite to regard μ as a quantity which may vary with the temperature and pressure of the fluid. As far as the result of a single experiment*, which has been already mentioned, performed with a single elastic fluid, namely air, justifies us in drawing such a general conclusion, we may assert that for a given fluid at a given temperature μ varies as ρ.

2. For the formation of the equations such as (1), I must refer to my former paper;

* The first of the experiments described in Col. Sabine's paper, in which the gauge stood as high as 7 inches, leads to the same conclusion; but as the vacuum apparatus had not yet been made stanch it is perhaps hardly safe to trust this experiment in a question of such delicacy.

but it will be possible, in a few words, to enable the reader to form a clear idea of the meaning of the constant μ.

Conceive the fluid to move in planes parallel to the plane of xy, the motion taking place in a direction parallel to the axis of y. The motion will evidently consist of a sort of continuous sliding, and the differential coefficient $\frac{dv}{dz}$ may be taken as a measure of the rate of sliding. In the theory it is supposed that in general the pressure about a given point is compounded of a normal pressure, corresponding to the density, which being normal is necessarily equal in all directions, and of an oblique pressure or tension, altering from one direction to another, which is expressed by means of linear functions of the nine differential coefficients of the first order of u, v, w with respect to x, y, z, which define the state of relative motion at any point of the fluid. Now in the special case considered above, if we confine our attention to one direction, that of the plane of xy, the total pressure referred to a unit of surface is compounded of a normal pressure corresponding to the density, and a tangential pressure expressed by $\mu \frac{dv}{dz}$, which tends to reduce the relative motion.

In the solution of equations (2), μ always appears divided by ρ. Let $\mu = \mu' \rho$. The constant μ' may conveniently be called the *index of friction* of the fluid, whether liquid or gas, to which it relates. As regards its dimensions, it expresses a moving force divided by the product of a surface, a density, and the differential coefficient of a velocity with respect to a line. Hence μ' is the square of a line divided by a time, whence it will be easy to adapt the numerical value of μ' to a new unit of length or of time.

3. Besides the general equations (2) and (3), it will be requisite to consider the equations of condition at the boundaries of the fluid. For the purposes of the present paper there will be no occasion to consider the case of a free surface, but only that of the common surface of the fluid and a solid. Now, if the fluid immediately in contact with a solid could flow past it with a finite velocity, it would follow that the solid was infinitely smoother with respect to its action on the fluid than the fluid with respect to its action on itself. For, conceive the elementary layer of fluid comprised between the surface of the solid and a parallel surface at a distance h, and then regard only so much of this layer as corresponds to an elementary portion dS of the surface of the solid. The impressed forces acting on the fluid element must be in equilibrium with the effective forces reversed. Now conceive h to vanish compared with the linear dimensions of dS, and lastly let dS vanish*. It is evident that the conditions of equilibrium will ultimately reduce themselves to this, that the oblique pressure which the fluid element experiences on the side of the solid must be equal and opposite to the pressure which it experiences on the side of the fluid. Now if the fluid could flow past the solid with a finite velocity, it would follow that the tangential pressure

* To be quite precise it would be necessary to say, Conceive h and dS to vanish together, h vanishing compared with the linear dimensions of dS; for so long as dS remains finite we cannot suppose h to vanish altogether, on account of the curvature of the elementary surface. Such extreme precision in unimportant matters tends, I think, only to perplex the reader, and prevent him from entering so readily into the spirit of an investigation.

called into play by the continuous sliding of the fluid over itself was no more than counteracted by the abrupt sliding of the fluid over the solid. As this appears exceedingly improbable *a priori*, it seems reasonable in the first instance to examine the consequences of supposing that no such abrupt sliding takes place, more especially as the mathematical difficulties of the problem will thus be materially diminished. I shall assume, therefore, as the conditions to be satisfied at the boundaries of the fluid, that the velocity of a fluid particle shall be the same, both in magnitude and direction, as that of the solid particle with which it is in contact. The agreement of the results thus obtained with observation will presently appear to be highly satisfactory. When the fluid, instead of being confined within a rigid envelope, extends indefinitely around the oscillating body, we must introduce into the solution the condition that the motion shall vanish at an infinite distance, which takes the place of the condition to be satisfied at the surface of the envelope.

To complete the determination of the arbitrary functions which would be contained in the integrals of (2) and (3), it would be requisite to put $t = 0$ in the general expressions for u, v, w, obtained by integrating those equations, and equate the results to the initial velocities supposed to be given. But it would be introducing a most needless degree of complexity into the solution to take account of the initial circumstances, nor is it at all necessary to do so for the sake of comparison of theory with experiment. For in a pendulum experiment the pendulum is set swinging and then left to itself, and the first observation is not taken till several oscillations have been completed, during which any irregularities attending the initial motion would have had time to subside. It will be quite sufficient to regard the motion as already going on, and limit the calculation to the determination of the simultaneous periodic movements of the pendulum and the surrounding fluid. The arc of oscillation will go on slowly decreasing, but it will be so nearly constant for several successive oscillations that it may be regarded as strictly such in calculating the motion of the fluid; and having thus determined the resultant action of the fluid on the solid we may employ the result in calculating the decrement of the arc of oscillation, as well as in calculating the time of oscillation. Thus the assumption of periodic functions of the time in the expressions for u, v, w will take the place of the determination of certain arbitrary functions by means of the initial circumstances.

4. Imagine a plane drawn perpendicular to the axis of x through the point in the fluid whose co-ordinates are x, y, z. Let the oblique pressure in the direction of this plane be decomposed into three pressures, a normal pressure, which will be in the direction of x, and two tangential pressures in the directions of y, z, respectively. Let P_1 be the normal pressure, and T_3 the tangential pressure in the direction of y, which will be equal to the component in the direction of x of the oblique pressure on a plane drawn perpendicular to the axis of y. Then by the formulæ (7), (8) of my former paper, and (3) of the present,

$$P_1 = p - 2\mu \frac{du}{dx}, \quad \ldots \ldots \ldots \quad (4)$$

$$T_3 = -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right). \quad \ldots \ldots \ldots \quad (5)$$

These formulæ will be required in finding the resultant force of the fluid on the pendulum, after the motion of the fluid has been determined in terms of the quantities by which the motion of the pendulum is expressed.

5. Before proceeding to the solution of the equations (2) and (3) in particular cases, it will be well to examine the general laws which follow merely from the dimensions of the several terms which appear in the equations.

Consider any number of similar systems, composed of similar solids oscillating in a similar manner in different fluids or in the same fluid. Let $a, a', a''\ldots$ be homologous lines in the different systems; $T, T', T''\ldots$ corresponding times, such for example as the times of oscillation from rest to rest. Let x, y, z be measured from similarly situated origins, and in corresponding directions, and t from corresponding epochs, such for example as the commencements of oscillations when the systems are beginning to move from a given side of the mean position.

The form of equations (2), (3) shews that the equations being satisfied for one system will be satisfied for all the systems provided

$$u \propto v \propto w, \; x \propto y \propto z, \text{ and } p \propto \frac{\mu u}{x} \propto \frac{\rho u x}{t}.$$

The variations $x \propto y \propto z$ merely signify that we must compare similarly situated points in inferring from the circumstance that (2), (3) are satisfied for one system that they will be satisfied for all the systems. If $c, c', c''\ldots$ be the maximum excursions of similarly situated points of the fluids

$$u \propto \frac{c}{T}, \quad x \propto a, \quad t \propto T,$$

and the sole condition to be satisfied, in addition to that of geometrical similarity, in order that the systems should be dynamically similar, becomes

$$\frac{a^2}{T} \propto \frac{\mu}{\rho} \text{ or } \propto \mu'. \qquad (6)$$

This condition being satisfied, similar motions will take place in the different systems, and we shall have

$$p \propto \frac{\rho a c}{T^2}. \qquad (7)$$

It follows from the equations (4), (5), and the other equations which might be written down from symmetry, that the pressures such as P_1, T_3 vary in the same manner as p, whence it appears from (7) that the resultant or resultants of the pressures of the fluids on the solids, acting along similarly situated lines, which vary as pa^2, vary as ρa^3 and cT^{-2} conjointly. In other words, these resultants in two similar systems are to one another in a ratio compounded of the ratio of the masses of fluid displaced, and of the ratio of the maximum accelerating effective forces belonging to similarly situated points in the solids.

6. In order that two systems should be similar in which the fluids are confined by envelopes that are sufficiently narrow to influence the motion of the fluids, it is necessary that

the envelopes should be similar and similarly situated with respect to the solids oscillating within them, and that their linear dimensions should be in the same ratio as those of the oscillating bodies. In strictness, it is likewise necessary that the solids should be similarly situated with respect to the axis of rotation. If however two similar solids, such as two spheres, are attached to two fine wires, and made to perform small oscillations in two unlimited masses of fluid, and if we agree to neglect the effect of the suspending wires, and likewise the effect of the rotation of the spheres on the motion of the fluid, which last will in truth be exceedingly small, we may regard the two systems as geometrically similar, and they will be dynamically similar provided the condition (6) be satisfied. When the two fluids are of the same nature, as for example when both spheres oscillate in air, the condition of dynamical similarity reduces itself to this, that the times of oscillation shall be as the squares of the diameters of the spheres.

If, with Bessel, we represent the effect of the inertia of the fluid on the time of oscillation of the sphere by supposing a mass equal to k times that of the fluid displaced added to the mass of the sphere, which increases its inertia without increasing its weight, we must expect to find k dependant on the nature of the fluid, and likewise on the diameter of the sphere. Bessel, in fact, obtained very different values of k for water and for air. Baily's experiments on spheres of different diameters, oscillating once in a second nearly, shew that the value of k increases when the diameter of the sphere decreases. Taking this for the present as the result of experiment, we are led from theory to assert that the value of k increases with the time of oscillation; in fact, k ought to be as much increased as if we had left the time of oscillation unchanged, and diminished the diameter in the ratio in which the square root of the time is increased. It may readily be shewn that the value of k obtained by Bessel's method, by means of a long and short pendulum, is greater than what belongs to the long pendulum, much more, greater than what belongs to the shorter pendulum, which oscillated once in a second nearly. The value of k given by Bessel is in fact considerably larger than that obtained by Baily, by a direct method, from a sphere of nearly the same size as those employed by Bessel, oscillating once in a second nearly.

The discussion of the experiments of Baily and Bessel belongs to Part II. of this paper. They are merely briefly noticed here to shew that some results of considerable importance follow readily from the general equations, even without obtaining any solution of them.

7. Before proceeding to the problems which mainly occupy this paper, it may be well to exhibit the solution of equations (2) and (3) in the extremely simple case of an oscillating plane.

Conceive a physical plane, which is regarded as infinite, to be situated in an unlimited mass of fluid, and to be performing small oscillations in the direction of a fixed line in the plane. Let a fixed plane coinciding with the moving plane be taken for the plane of yz, the axis of y being parallel to the direction of motion, and consider only the portion of fluid which lies on the positive side of the plane of yz. In the present case, we must evidently have $u = 0$, $w = 0$; and p, v will be functions of x and t, which have to be determined. The equation (3) is satisfied identically, and we get from (2), putting $\mu = \mu'\rho$,

$$\frac{dp}{dx} = 0, \quad \frac{dv}{dt} = \mu' \frac{d^2 v}{dx^2}. \quad \ldots \ldots \ldots \quad (8)$$

The first of these equations gives p = a constant, for it evidently cannot be a function of t, since the effect of the motion vanishes at an infinite distance from the plane; and if we include this constant in Π, we shall have $p = 0$. Let V be the velocity of the plane itself, and suppose

$$V = c \sin nt. \quad \ldots \ldots \quad (9)$$

Putting in the second of equations (8)

$$v = X_1 \sin nt + X_2 \cos nt, \quad \ldots \ldots \quad (10)$$

we get

$$nX_1 = \mu' \frac{d^2 X_2}{dx^2}, \quad nX_2 = -\mu' \frac{d^2 X_1}{dx^2} = -\frac{\mu'^2}{n} \frac{d^4 X_2}{dx^4}. \quad \ldots \quad (11)$$

The last of these equations gives

$$X_2 = \epsilon^{-\sqrt{\frac{n}{2\mu'}}x} \left(A \sin \sqrt{\frac{n}{2\mu'}}x + B \cos \sqrt{\frac{n}{2\mu'}}x\right) + \epsilon^{\sqrt{\frac{n}{2\mu'}}x} \left(C \sin \sqrt{\frac{n}{2\mu'}}x + D \cos \sqrt{\frac{n}{2\mu'}}x\right).$$

Since X_2 must not become infinite when $x = \infty$, we must have $C = 0$, $D = 0$. Obtaining X_1 from the first of equations (11), and substituting in (10), we get

$$v = \epsilon^{-\sqrt{\frac{n}{2\mu'}}x} \left\{-A \sin \left(nt - \sqrt{\frac{n}{2\mu'}}x\right) + B \cos \left(nt - \sqrt{\frac{n}{2\mu'}}x\right)\right\}.$$

Now by the equations of conditions assumed in Art. 3, we must have $v = V$ when $x = 0$, whence

$$v = c\epsilon^{-\sqrt{\frac{n}{2\mu'}}x} \sin \left(nt - \sqrt{\frac{n}{2\mu'}}x\right). \quad \ldots \ldots \quad (12)$$

To find the normal and tangential components of the pressure of the fluid on the plane, we must substitute the above value of v in the formulæ (4), (5), and after differentiation put $x = 0$. P_1, T_3 will then be the components of the pressure of the solid on the fluid, and therefore $-P_1$, $-T_3$, those of the pressure of the fluid on the solid. We get

$$P_1 = 0, \quad T_3 = c\rho \sqrt{\frac{n\mu'}{2}} (\sin nt + \cos nt) = \rho \sqrt{\frac{n\mu'}{2}} \left(V + \frac{1}{n}\frac{dV}{dt}\right). \quad . \quad (13)$$

The force expressed by the first of these terms tends to diminish the amplitude of the oscillations of the plane. The force expressed by the second has the same effect as increasing the inertia of the plane.

8. The equation (12) shews that a given phase of vibration is propagated from the plane into the fluid with a velocity $\sqrt{(2\mu' n)}$, while the amplitude of oscillation decreases in geometric progression as the distance from the plane increases in arithmetic. If we suppose the time of oscillation from rest to rest to be one second, $n = \pi$; and if we suppose $\sqrt{\mu'} = .116$ inch, which, as will presently be seen, is about its value in the case of air, we get for the velocity of propagation .2908 inch per second nearly. If we enquire the distance from the plane at which the amplitude of oscillation is reduced to one half, we have only to put $\sqrt{\frac{n}{2\mu'}}x = \log_\epsilon 2$, which gives, on the same suppositions as before respecting numerical values, $x = .06415$ inch nearly.

For water the value of μ' is a good deal smaller than for air, and the corresponding value of x smaller likewise, since it varies *cæteris paribus* as $\sqrt{\mu'}$. Hence if a solid of revolution of large, or even moderately large, dimensions be suspended by a fine wire coinciding with the axis of revolution, and made to oscillate by the torsion of the wire, the effect of the fluid may be calculated with a very close degree of approximation by regarding each element of the surface of the solid as an element of an infinite plane oscillating with the same linear velocity.

For example, let a circular disk of radius a be suspended horizontally by a fine wire attached to the centre, and made to oscillate. Let r be the radius vector of any element of the disk, measured from its centre, θ the angle through which the disk has turned from its mean position. Then in equation (13), we must put $V = r\dfrac{d\theta}{dt}$, whence

$$T_3 = \rho\sqrt{\frac{n\mu'}{2}}\, r\left(\frac{d\theta}{dt} + \frac{1}{n}\frac{d^2\theta}{dt^2}\right).$$

The area of the annulus of the disk comprised between the radii r and $r + dr$ is $4\pi r dr$, both faces being taken, and if G be the whole moment of the force of the fluid on the disk, $G = -4\pi\int_0^a r^2 T_3 dr$, whence

$$G = -\pi\rho a^4\sqrt{\frac{n\mu'}{2}}\left(\frac{d\theta}{dt} + \frac{1}{n}\frac{d^2\theta}{dt^2}\right).$$

Let $M\gamma^2$ be the moment of inertia of the disk, and let n_1 be what n would become if the fluid were removed, so that $-n_1^2 M\gamma^2\theta$ is the moment of the force of torsion. Then when the fluid is present the equation of motion of the disk becomes

$$\left(M\gamma^2 + \pi\rho a^4\sqrt{\frac{\mu'}{2n}}\right)\frac{d^2\theta}{dt^2} + \pi\rho a^4\sqrt{\frac{n\mu'}{2}}\frac{d\theta}{dt} + n_1^2 M\gamma^2\theta = 0, \quad . \quad . \quad . \quad (14)$$

or, putting for shortness

$$\pi\rho a^4\sqrt{\frac{\mu'}{2n}} = 2\beta M\gamma^2,$$

$$(1 + 2\beta)\frac{d^2\theta}{dt^2} + 2n\beta\frac{d\theta}{dt} + n_1^2\theta = 0,$$

which gives, neglecting β^2,

$$\theta = \theta_0\epsilon^{-n\beta t}\sin(nt + a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where

$$n = n_1(1 - \beta).$$

The observation of n and n_1, or else the observation of n and of the decrement of the arc of oscillation, would enable us to determine β, and thence μ'. The values of β determined in these two different ways ought to agree.

There would be no difficulty in obtaining a more exact solution, in which the decrement of the arc of oscillation should be taken into account in calculating the motion of the fluid, but I pass on to the problems, the solution of which forms the main object of this paper.

Section II.

Solution of the equations in the case of a sphere oscillating in a mass of fluid either unlimited, or confined by a spherical envelope concentric with the sphere in its position of equilibrium.

9. Suppose the sphere suspended by a fine wire, the length of which is much greater than the radius of the sphere. Neglect for the present the action of the wire on the fluid, and consider only that of the sphere. The motion of the sphere and wire being supposed to take place parallel to a fixed vertical plane, there are two different modes of oscillation possible. We have here nothing to do with the rapid oscillations which depend mainly on the rotatory inertia of the sphere, but only with the principal oscillations, which are those which are observed in pendulum experiments. In these principal oscillations the centre of the sphere describes a small arc of a curve which is very nearly a circle, and which would be rigorously such, if the line joining the centre of gravity of the sphere and the point of attachment of the wire were rigorously in the direction of the wire. In calculating the motion of the fluid, we may regard this arc as a right line. In fact, the error thus introduced would only be a small quantity of the second order, and such quantities are supposed to be neglected in the investigation. Besides its motion of translation, the sphere will have a motion of rotation about a horizontal axis, the angular motion of the sphere being very nearly the same as that of the suspending wire. This motion, which would produce absolutely no effect on the fluid according to the common theory of hydrodynamics, will not be without its influence when friction is taken into account; but the effect is so very small in practical cases that it is not worth while to take it into account. For if a be the radius of the sphere, and l the length of the suspending wire, the velocity of a point in the surface of the sphere due to the motion of rotation will be a small quantity of the order $a l^{-1}$ compared with the velocity due to the motion of translation. In finding the moment of the pressures of the fluid on the pendulum, forces arising from these velocities, and comparable with them, have to be multiplied by lines which are comparable with a, l, respectively. Hence the moment of the pressures due to the motion of rotation of the sphere will be a small quantity of the order $a^2 l^{-2}$, compared with the moment due to the motion of translation. Now in practice l is usually at least 20 or 30 times greater than a, and the whole effect to be investigated is very small, so that it would be quite useless to take account of the motion of rotation of the sphere.

The problem, then, reduces itself to this. The centre of a sphere performs small periodic oscillations along a right line, the sphere itself having a motion of translation simply: it is required to determine the motion of the surrounding fluid.

10. Let the mean position of the centre of the sphere be taken for origin, and the direction of its motion for the axis of x, so that the motion of the fluid is symmetrical with respect to this axis. Let ϖ be the perpendicular let fall from any point on the axis of x, q

the velocity in the direction of ϖ, ω the angle between the line ϖ and the plane of xy. Then p, u, and q will be functions of x, ϖ, and t, and we shall have

$$v = q\cos\omega, \quad w = q\sin\omega, \quad y = \varpi\cos\omega, \quad z = \varpi\sin\omega,$$

whence

$$\varpi^2 = y^2 + z^2, \quad \omega = \tan^{-1}\frac{z}{y}.$$

We have now to substitute in equations (2) and (3), and we are at liberty to put $\omega = 0$ after differentiation. We get

$$\frac{d}{dy} = \cos\omega\frac{d}{d\varpi} - \frac{\sin\omega}{\varpi}\frac{d}{d\omega}, \quad = \frac{d}{d\varpi} \text{ when } \omega = 0,$$

$$\frac{d^2}{dy^2} = \frac{d^2}{d\varpi^2} \text{ when } \omega = 0,$$

$$\frac{d}{dz} = \sin\omega\frac{d}{d\varpi} + \frac{\cos\omega}{\varpi}\frac{d}{d\omega}, \quad = \frac{1}{\varpi}\frac{d}{d\omega} \text{ when } \omega = 0,$$

$$\frac{d^2}{dz^2} = \frac{1}{\varpi}\frac{d}{d\varpi} + \frac{1}{\varpi^2}\frac{d^2}{d\omega^2} \text{ when } \omega = 0,$$

whence we obtain

$$\frac{dp}{dx} = \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{d\varpi^2} + \frac{1}{\varpi}\frac{du}{d\varpi}\right) - \rho\frac{du}{dt}, \quad \ldots\ldots \quad (16)$$

$$\frac{dp}{d\varpi} = \mu\left(\frac{d^2q}{dx^2} + \frac{d^2q}{d\varpi^2} + \frac{1}{\varpi}\frac{dq}{d\varpi} - \frac{q}{\varpi^2}\right) - \rho\frac{dq}{dt}, \quad \ldots\ldots \quad (17)$$

$$\frac{du}{dx} + \frac{dq}{d\varpi} + \frac{q}{\varpi} = 0. \quad \ldots\ldots \quad (18)$$

Eliminating p from (16) and (17), and putting for μ its equivalent $\mu'\rho$, we get

$$\mu'\frac{d}{d\varpi}\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi}\right)u - \mu'\frac{d}{dx}\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi} - \frac{1}{\varpi^2}\right)q - \frac{d}{dt}\left(\frac{du}{d\varpi} - \frac{dq}{dx}\right) = 0,$$

$$\text{or} \quad \left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi} - \frac{1}{\varpi^2} - \frac{1}{\mu'}\frac{d}{dt}\right)\left(\frac{du}{d\varpi} - \frac{dq}{dx}\right) = 0. \quad \ldots \quad (19)$$

By virtue of (18), $\varpi(ud\varpi - q\,dx)$ is an exact differential. Let then

$$\varpi(ud\varpi - q\,dx) = d\psi. \quad \ldots\ldots \quad (20)$$

Expressing u and q in terms of ψ, we get

$$\frac{du}{d\varpi} - \frac{dq}{dx} = \frac{1}{\varpi}\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi}\right)\psi.$$

Substituting in (19), and operating separately on the factor ϖ^{-1}, we obtain

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{d^2\varpi} - \frac{1}{\varpi}\frac{d}{d\varpi} - \frac{1}{\mu'}\frac{d}{dt}\right)\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi}\right)\psi = 0. \quad \ldots \quad (20')$$

Since the operations represented by the two expressions within parentheses are evidently convertible, the integral of this equation is

$$\psi = \psi_1 + \psi_2,^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where ψ_1, ψ_2 are the integrals of the equations

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi}\right)\psi_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi} - \frac{1}{\mu'}\frac{d}{dt}\right)\psi_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

11. By means of the last three equations, the expression for dp obtained from (16) and (17) is greatly simplified. We get, in the first place,

$$\frac{1}{\rho}\frac{dp}{dx} = \left\{\mu'\left(\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} + \frac{1}{\varpi}\frac{d}{d\varpi}\right) - \frac{d}{dt}\right\}\frac{1}{\varpi}\frac{d\psi}{d\varpi}: \quad . \quad . \quad . \quad . \quad (24)$$

but by adding together equations (22) and (23), and taking account of (21), we get

$$\frac{d^2\psi}{dx^2} = -\frac{d^2\psi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\psi}{d\varpi} + \frac{1}{\mu'}\frac{d\psi_2}{dt}.$$

On substituting in (24), it will be found that all the terms in the right-hand member of the equation destroy one another, except those which contain $\frac{d\psi}{dt}$ and $\frac{d\psi_2}{dt}$, and the equation is reduced to

$$\frac{dp}{dx} = -\frac{\rho}{\varpi}\frac{d^2\psi_1}{dt\,d\varpi}.$$

The equation (17) may be reduced in a similar manner, and we get finally

$$dp = \frac{\rho}{\varpi}\left(\frac{d^2\psi_1}{dt\,dx}d\varpi - \frac{d^2\psi_1}{dt\,d\varpi}dx\right), \quad . \quad . \quad . \quad . \quad . \quad (25)$$

which is an exact differential by virtue of (22).

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* If we denote for shortness the operation

$$\frac{d^2}{dx^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi}$$

by D, our equation becomes

$$D\left(D - \frac{1}{\mu'}\frac{d}{dt}\right)\psi = 0,$$

which gives by the separation of symbols

$$\psi = \left\{D\left(D - \frac{1}{\mu'}\frac{d}{dt}\right)\right\}^{-1} 0$$

$$= \left(\frac{1}{\mu'}\frac{d}{dt}\right)^{-1}\left\{\left(D - \frac{1}{\mu'}\frac{d}{dt}\right)^{-1} - D^{-1}\right\}0, \quad . \quad . \quad (a)$$

so that $\frac{d\psi}{dt}$ is composed of two parts, which are separately the integrals of (22), (23). Hence we have for the integral of (20′) $\psi = \psi_1 + \psi_2 + \Psi$, Ψ being a function of x and ϖ without t which satisfies the equation $D^2\Psi = 0$. For the equations (22), (23) will not be altered if we put $\int\psi_1 dt$, $\int\psi_2 dt$ for ψ_1, ψ_2, the arbitrary functions which would arise from the integration with respect to t being supposed to be included in Ψ. The function Ψ, which taken by itself can only correspond to steady motion, is excluded from the problem under consideration by the condition of periodicity. But we may even, independently of this condition, regard (21) as the complete integral of (20′), provided we suppose included in (21) terms which would be obtained by supposing ψ at first to vary slowly with the time, employing the integrals of (22) and (23), and then making the rate of variation diminish indefinitely. By treating the symbolical expression in the right-hand member of equation (a) as a vanishing fraction, $\frac{d}{dt}$ being supposed to vanish, we obtain in fact $D^{-2}0$; so that under the convention just mentioned the function Ψ may be supposed to be included in $\psi_1 + \psi_2$. The same remarks will apply to the equation in Section III. which answers to (20′).

12. Passing to polar co-ordinates, let r be the radius vector drawn from the origin, θ the angle which r makes with the axis of x, and let R be the velocity along the radius vector, Θ the velocity perpendicular to the radius vector: then

$$x = r\cos\theta,\quad \varpi = r\sin\theta,\quad u = R\cos\theta - \Theta\sin\theta,\quad q = R\sin\theta + \Theta\cos\theta.$$

Making these substitutions in (20), (22), (23), and (25), we obtain

$$r\sin\theta\,(Rr\,d\theta - \Theta\,dr) = d\psi, \qquad (26)$$

$$\frac{d^2\psi_1}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi_1}{d\theta}\right) = 0, \qquad (27)$$

$$\frac{d^2\psi_2}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi_2}{d\theta}\right) - \frac{1}{\mu'}\frac{d\psi_2}{dt} = 0, \qquad (28)$$

$$dp = \frac{\rho}{r\sin\theta}\left(\frac{d^2\psi_1}{dt\,dr}\,r\,d\theta - \frac{1}{r}\frac{d^2\psi_1}{dt\,d\theta}\,dr\right). \qquad (29)$$

We must now determine ψ_1 and ψ_2 by means of (27) and (28), combined with the equations of condition. When these functions are known, p will be obtained by integrating the exact differential which forms the right-hand member of (29), and the velocities R, Θ, if required, will be got by differentiation, as indicated by equation (26). Formulæ deduced from (4) and (5) will then make known the pressure of the fluid on the sphere.

13. Let ξ be the abcissa of the centre of the sphere at any instant. The conditions to be satisfied at the surface of the sphere are that when $r = r_1$, the radius vector of the surface, we have

$$R = \cos\theta\frac{d\xi}{dt},\qquad \Theta = -\sin\theta\frac{d\xi}{dt}.$$

Now r_1 differs from a by a small quantity of the first order, and since this value of r has to be substituted in functions which are already small quantities of that order, it will be sufficient to put $r = a$. Hence, expressing R and Θ in terms of ψ, we get

$$\frac{d\psi}{dr} = a\sin^2\theta\frac{d\xi}{dt},\quad \frac{d\psi}{d\theta} = a^2\sin\theta\cos\theta\frac{d\xi}{dt},\quad \text{when } r = a. \qquad (30)$$

When the fluid is unlimited, it will be found that certain arbitrary constants will vanish by the condition that the motion shall not become infinite at an infinite distance in the fluid. When the fluid is confined by an envelope having a radius b, we have the equations of condition

$$\frac{d\psi}{dr} = 0,\quad \frac{d\psi}{d\theta} = 0,\quad \text{when } r = b. \qquad (31)$$

14. We must now, in accordance with the plan proposed in Section I., introduce the condition that the function ψ shall be composed, so far as the time is concerned, of the circular functions $\sin nt$ and $\cos nt$, that is, that it shall be of the form $P\sin nt + Q\cos nt$, where P

and Q are functions of r and θ only. An artifice, however, which has been extensively employed by M. Cauchy* will here be found of great use. Instead of introducing the circular functions $\sin nt$ and $\cos nt$, we may employ the exponentials $\epsilon^{\sqrt{-1}nt}$, and $\epsilon^{-\sqrt{-1}nt}$. Since our equations are linear, and since each of these exponential functions reproduces itself at each differentiation, it follows that if all the terms in any one of our equations be arranged in two groups, containing as a factor $\epsilon^{\sqrt{-1}nt}$ in one case, and $\epsilon^{-\sqrt{-1}nt}$ in the other, the two groups will be quite independent, and the equations will be satisfied by either group separately. Hence it will be sufficient to introduce one of the exponential functions. We shall thus have only half the number of terms to write down, and half the number of arbitrary constants to determine that would have been necessary had we employed circular functions. When we have arrived at our result, it will be sufficient to put each equation under the form $U + \sqrt{-1}\,V = 0$, and throw away the imaginary part, or else throw away the real part and omit $\sqrt{-1}$, since the system of quantities U, and the system of quantities V must separately satisfy the equations of the problem. Assuming then

$$\frac{d\xi}{dt} = c\epsilon^{\sqrt{-1}nt}, \qquad \psi = \epsilon^{\sqrt{-1}nt}P,$$

we have to determine P as a function of r and θ.

15. The form of the equations of condition (30) points out $\sin^2\theta$ as a factor of P, and since the operation $\sin\theta\frac{d}{d\theta}\frac{1}{\sin\theta}\frac{d}{d\theta}$ performed on the function $\sin^2\theta$ reproduces the same function with a coefficient -2, it will be possible to satisfy equations (27) and (28) on the supposition that $\sin^2\theta$ is a factor of ψ_1 and ψ_2*. Assume then

$$\psi_1 = \epsilon^{\sqrt{-1}nt}\sin^2\theta\, f_1(r), \qquad \psi_2 = \epsilon^{\sqrt{-1}nt}\sin^2\theta\, f_2(r).$$

Putting for convenience

$$n\sqrt{-1} = \mu' m^2, \quad \dots\dots\dots \quad (32)$$

and substituting in (27) and (28), we get

$$f_1''(r) - \frac{2}{r^2}f_1(r) = 0, \quad \dots\dots\dots \quad (33)$$

$$f_2''(r) - \frac{2}{r^2}f_2(r) - m^2 f_2(r) = 0. \quad \dots\dots \quad (34)$$

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* When this operation is performed on the function $\sin\theta\frac{dY_i}{d\theta}$, the function is reproduced with a coefficient $-i(i+1)$. Y_i here denotes a Laplace's coefficient of the i^{th} order, which contains only one variable angle, namely θ. Now ψ may be expanded in a series of quantities of the general form $\sin\theta\frac{dY_i}{d\theta}$. For, since we are only concerned with the differential coefficients of ψ with respect to r and θ, we have a right to suppose ψ to vanish at whatever point of space we please. Let then $\psi = 0$ when $r = a$ and $\theta = 0$. To find the value of ψ at a distance r from the origin, along the axis of x positive, it will be sufficient to put $\theta = 0$, $d\theta = 0$ in (26), and integrate from $r = a$ to r, whence $\psi = 0$. To find the value of ψ at the same distance r along the axis of x negative, it will be sufficient to leave r constant, and integrate $d\psi$ from $\theta = 0$ to $\theta = \pi$. Referring to (26), we see that the integral vanishes, since the total flux across the surface of the sphere whose radius is r must be equal to zero. Hence ψ vanishes when $\theta = 0$ or $= \pi$, and it appears from (26) that when θ is very small or very nearly equal to π, ψ varies ultimately as $\sin^2\theta$ for given values of r and t. Hence $\psi\operatorname{cosec}\theta$, and therefore $\int\psi\operatorname{cosec}\theta\,d\theta$, is finite even when $\sin\theta$ vanishes, and therefore $\int\psi\operatorname{cosec}\theta\,d\theta$ may be expanded in a series of Laplace's coefficients, and therefore ψ itself in a series of quantities of the form $\sin\theta\frac{dY_i}{d\theta}$. It was somewhat in this way that I first obtained the form of the function ψ.

The equations of condition (30), (31) become, on putting $f(r)$ for $f_1(r) + f_2(r)$,

$$f'(a) = ac, \quad f(a) = \tfrac{1}{2}a^2c, \quad \ldots \quad (35)$$

$$f'(b) = 0, \quad f(b) = 0. \quad \ldots \quad (36)$$

We may obtain p from (29) by putting for ψ_1 its value $\epsilon^{\mu' m^2 t}\sin^2\theta f_1(r)$, replacing after differentiation $2f_1(r)$ by its equivalent $r^2 f_1''(r)$, and then integrating. It is unnecessary to add an arbitrary function of the time, since any such function may be supposed to be included in Π. We get

$$p = -\rho\mu' m^2 \epsilon^{\mu' m^2 t}\cos\theta\, f_1'(r). \quad \ldots \quad (37)$$

16. The integration of the differential equation (33) does not present the least difficulty, and (34) comes under a well known integrable form. The integrals of these equations are

$$\left.\begin{aligned} f_1(r) &= \frac{A}{r} + Br^2, \\ f_2(r) &= C\epsilon^{-mr}\left(1 + \frac{1}{mr}\right) + D\epsilon^{mr}\left(1 - \frac{1}{mr}\right), \end{aligned}\right\} \quad \ldots \quad (38)$$

and we have to determine A, B, C, D by the equations of condition.

The solution of the problem, in the case in which the fluid is confined by a spherical envelope, will of course contain as a particular case that in which the fluid is unlimited, to obtain the results belonging to which it will be sufficient to put $b = \infty$. As, however, the case of an unlimited fluid is at the same time simpler and more interesting than the general case, it will be proper to consider it separately.

Let $+m$ denote that square root of $\mu'^{-1} n\sqrt{-1}$ which has its real part positive; then in equations (38) we must have $D = 0$, since otherwise the velocity would be infinite at an infinite distance. We must also have $B = 0$, since otherwise the velocity would be finite when $r = \infty$, as appears from (26). We get then from the equations of condition (35)

$$A = \tfrac{1}{2}a^3c + \frac{3a^2c}{2m}\left(1 + \frac{1}{ma}\right), \quad C = -\frac{3ac}{2m}\epsilon^{ma},$$

whence

$$\xi = \frac{c}{\mu' m^2}\epsilon^{\mu' m^2 t}, \quad \ldots \quad (39)$$

$$\psi = \tfrac{1}{2}a^2c\,\epsilon^{\mu' m^2 t}\sin^2\theta\left\{\left(1 + \frac{3}{ma} + \frac{3}{m^2a^2}\right)\frac{a}{r} - \frac{3}{ma}\left(1 + \frac{1}{mr}\right)\epsilon^{-m(r-a)}\right\}, \quad . \quad (40)$$

$$p = \tfrac{1}{2}\rho a c \mu' m^2\left(1 + \frac{3}{ma} + \frac{3}{m^2a^2}\right)\epsilon^{\mu' m^2 t}\cos\theta\,\frac{a^2}{r^2}. \quad \ldots \quad (41)$$

17. The symbolical equations (40), (41) contain the solution of the problem, the motion of the sphere being defined by the symbolical equation (39). If we wish to exhibit the actual results by means of real quantities alone, we have only to put the right-hand members

of equations (39), (40), (41) under the form $U + \sqrt{-1}\,V$, and reject the imaginary part. Putting for shortness

$$\sqrt{\frac{n}{2\mu'}} = \nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

we have $m = \nu(1 + \sqrt{-1})$, and we obtain

$$\xi = \frac{c}{n} \sin nt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$\psi = \tfrac{1}{2} a^2 c \sin^2\theta \left\{ \left[\left(1 + \frac{3}{2\nu a}\right) \cos nt + \frac{3}{2\nu a}\left(1 + \frac{1}{\nu a}\right) \sin nt \right] \frac{a}{r} \right.$$
$$\left. - \frac{3}{2\nu a} \epsilon^{-\nu(r-a)} \left[\cos(nt - \nu r + \nu a) + \left(1 + \frac{1}{\nu r}\right) \sin(nt - \nu r + \nu a) \right] \right\}, \quad . \quad (44)$$

$$p = -\tfrac{1}{2} \rho a c n \left\{ \left(1 + \frac{3}{2\nu a}\right) \sin nt - \frac{3}{2\nu a}\left(1 + \frac{1}{\nu a}\right) \cos nt \right\} \cos\theta \,.\, \frac{a^2}{r^2}. \quad . \quad . \quad . \quad . \quad (45)$$

The reader will remark that the ξ, ψ, p of the present article are not the same as the ξ, ψ, p of the preceding. The latter are the imaginary expressions, of which the real parts constitute the former. It did not appear necessary to change the notation.

When $\mu' = 0$, $\nu = \infty$, and ψ reduces itself to

$$\frac{a^3 c}{2r} \sin^2\theta \cos nt, \quad \text{or} \quad \frac{a^3}{2r} \sin^2\theta \frac{d\xi}{dt}.$$

In this case we get from (26)

$$R = a^3 \frac{d\xi}{dt} \frac{\cos\theta}{r^3}, \quad \Theta = \tfrac{1}{2} a^3 \frac{d\xi}{dt} \frac{\sin\theta}{r^3},$$

and $R\,dr + \Theta\, r\, d\theta$ is an exact differential $d\phi$ where

$$\phi = -\tfrac{1}{2} a^3 \frac{d\xi}{dt} \frac{\cos\theta}{r^2},$$

which agrees with the result deduced directly from the ordinary equations of hydrodynamics*.

18. Let us now form the expression for the resultant of the pressures of the fluid on the several elements of the surface of the sphere. Let P_r be the normal, and T_θ the tangential, component of the pressure at any point in the direction of a plane drawn perpendicular to the radius vector. The formulæ (4), (5) are general, and therefore we may replace x, y in these formulæ by x', y', where x', y' are measured in any two rectangular directions we please. Let the plane of $x'\,y'$ pass through the axis of x and the radius vector, and let the axis of x' be inclined to that of x at an angle ϑ, which after differentiation is made equal to θ. Then P_1, T_3 will become P_r, T_θ, respectively. We have

$$u' = R\cos(\theta - \vartheta) - \Theta \sin(\theta - \vartheta), \quad v' = R \sin(\theta - \vartheta) + \Theta\cos(\theta - \vartheta),$$

* See Camb. Phil. Trans. Vol. VIII. p. 119.

and when $\theta = \vartheta$

$$\frac{d}{dx'} = \frac{d}{dr}, \quad \frac{d}{dy'} = \frac{d}{rd\theta},$$

$$\frac{du'}{dx'} = \frac{dR}{dr}, \quad \frac{du'}{dy'} = \frac{dR}{rd\theta} - \frac{\Theta}{r}, \quad \frac{dv'}{dx'} = \frac{d\Theta}{dr};$$

whence

$$P_r = p - 2\mu\frac{dR}{dr}, \quad T_\theta = -\mu\left(\frac{dR}{rd\theta} + \frac{d\Theta}{dr} - \frac{\Theta}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad (46)$$

In these formulæ, suppose r put equal to a after differentiation. Then P_r, T_θ will be the components in the direction of r, θ of the pressure of the sphere on the fluid. The resolved part of these in the direction of x is

$$P_r \cos\theta - T_\theta \sin\theta,$$

which is equal and opposite to the component, in the direction of x, of the pressure of the fluid on the sphere. Let F be the whole force of the fluid on the sphere, which will evidently act along the axis of x. Then, observing that $2\pi a^2 \sin\theta d\theta$ is the area of an elementary annulus of the surface of the sphere, we get

$$F = 2\pi a^2 \int_0^\pi (-P_r \cos\theta + T_\theta \sin\theta)_a \sin\theta d\theta, \quad . \quad . \quad . \quad . \quad (47)$$

the suffix a denoting that r is supposed to have the value a in the general expressions for P_r and T_θ.

The expression for F may be greatly simplified, without employing the solution of equations (27), (28), by combining these equations in their original state with the equations of condition (30). We have, in the first place, from (26)

$$R = \frac{1}{r^2 \sin\theta}\frac{d\psi}{d\theta}, \quad \Theta = -\frac{1}{r\sin\theta}\frac{d\psi}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Now the equations (30) make known the values of ψ and $\frac{d\psi}{dr}$, and of their differential coefficients of all orders with respect to θ, when $r = a$. When the expressions for R and Θ are substituted in (46), the result will contain only one term in which the differentiation with respect to r rises to the second order. But we get from (21), (27), (28)

$$\frac{d^2\psi}{dr^2} = -\frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d\psi}{d\theta}\right) + \frac{1}{\mu'}\frac{d\psi_2}{dt},$$

and the second of equations (30) gives the value for $r = a$ of the first term in the right-hand member of the equation just written. We obtain from (48) and (30)

$$\left(\frac{dR}{dr}\right)_a = 0,$$

$$\left(\frac{dR}{rd\theta}\right)_a = -\frac{\sin\theta}{a}\frac{d\xi}{dt} = \left(\frac{\Theta}{r}\right)_a,$$

$$\left(\frac{d\Theta}{dr}\right)_a = -\frac{1}{\mu' a \sin\theta}\left(\frac{d\psi_2}{dt}\right)_a.$$

Substituting in (47), and writing $\mu'\rho$ for μ, we get

$$F = 2\pi a \int_0^\pi \left\{ - a p_a \cos\theta + \rho \left(\frac{d\psi_2}{dt}\right)_a \right\} \sin\theta d\theta.$$

With respect to the first term in this expression, we get by integration by parts

$$\int p \cos\theta \sin\theta \, d\theta = \tfrac{1}{2} \sin^2\theta \, . \, p - \tfrac{1}{2} \int \sin^2\theta \, \frac{dp}{d\theta} d\theta.$$

The first term vanishes at the limits. Substituting in the second term for $\frac{dp}{d\theta}$ the expression got from (29), and putting $r = a$, we get

$$\int_0^\pi p_a \cos\theta \sin\theta \, d\theta = -\tfrac{1}{2}\rho \frac{d}{dt} \int_0^\pi \left(\frac{d\psi_1}{dr}\right)_a \sin\theta \, d\theta.$$

Substituting in the expression for F, we get

$$F = \pi\rho a \frac{d}{dt} \int_0^\pi \left\{ a \left(\frac{d\psi_1}{dr}\right)_a + 2(\psi_2)_a \right\} \sin\theta \, d\theta. \quad . \quad . \quad . \quad . \quad (49)$$

19. The above expression for F, being derived from the general equations (27), (28), combined with the equations of condition (30), holds good, not merely when the fluid is confined by a spherical envelope, but whenever the motion is symmetrical about an axis, and that, whether the motion of the sphere be or be not expressed by a single circular function of the time. It might be employed, for instance, in the case of a sphere oscillating in a direction perpendicular to a fixed rigid plane.

When the fluid is either unconfined, or confined by a spherical envelope concentric with the sphere in its position of equilibrium, the functions ψ_1, ψ_2 consist, as we have seen, of $\sin^2\theta$ multiplied by two factors independent of θ. If we continue to employ the symbolical expressions, which will be more convenient to work with than the real expressions which might be derived from them, we shall have

$$\epsilon^{\sqrt{-1}\,nt} f_1(r), \qquad \epsilon^{\sqrt{-1}\,nt} f_2(r),$$

for these factors respectively. Substituting in (49), and performing the integration with respect to θ, we get

$$F = \tfrac{4}{3}\pi\rho a n \sqrt{-1} \left\{ a f_1'(a) + 2 f_2(a) \right\} \epsilon^{\sqrt{-1}\,nt}. \quad . \quad . \quad . \quad . \quad (50)$$

20. Consider for the present only the case in which the fluid is unlimited. The arbitrary constants which appear in equations (38) were determined for this case in Art. 16. Substituting in (50) we get

$$F = -\tfrac{2}{3}\pi\rho a^3 c n \sqrt{-1} \left(1 + \frac{9}{ma} + \frac{9}{m^2a^2}\right) \epsilon^{\sqrt{-1}\,nt}.$$

Putting for m its value $\nu(1+\sqrt{-1})$, and denoting by M' the mass of the fluid displaced by the sphere, which is equal to $\frac{4}{3}\pi\rho a^3$, we get

$$F=-M'cn\left\{\left(\frac{1}{2}+\frac{9}{4\nu a}\right)\sqrt{-1}+\frac{9}{4\nu a}\left(1+\frac{1}{\nu a}\right)\right\}\epsilon^{\sqrt{-1}nt};$$

whence

$$F=-\left(\frac{1}{2}+\frac{9}{4\nu a}\right)M'\frac{d^2\xi}{dt^2}-\frac{9}{4\nu a}\left(1+\frac{1}{\nu a}\right)M'n\frac{d\xi}{dt}. \quad \ldots \ldots \quad (51)$$

Since $\sqrt{-1}$ has been eliminated, this equation will remain unchanged when we pass from the symbolical to the real values of F and ξ.

Let τ be the time of oscillation from rest to rest, so that $n\tau=\pi$, and put for shortness k, k' for the coefficients of M' in (51); then

$$\nu=\sqrt{\frac{\pi}{2\mu'\tau}}, \quad k=\frac{1}{2}+\frac{9}{4\nu a}, \quad k'=\frac{9}{4\nu a}\left(1+\frac{1}{\nu a}\right). \quad \ldots \quad (52)$$

The first term in the expression for the force F has the same effect as increasing the inertia of the sphere. To take account of this term, it will be sufficient to conceive a mass kM' collected at the centre of the sphere, adding to its inertia without adding to its weight. The main effect of the second term is to produce a diminution in the arc of oscillation: its effect on the time of oscillation would usually be quite insensible, and must in fact be neglected for consistency's sake, because the motion of the fluid was determined by supposing the motion of the sphere permanent, which is only allowable when we neglect the square of the rate of decrease of the arc of oscillation.

If we form the equation of motion of the sphere, introducing the force F, and then proceed to integrate the equation, we shall obtain in the integral an exponential $\epsilon^{-\delta t}$ multiplying the circular function, δ being half the coefficient of $\frac{d\xi}{dt}$ divided by that of $\frac{d^2\xi}{dt^2}$. Let M be the mass of the sphere, $M\gamma^2$ its moment of inertia about the axis of suspension, then

$$nk'M'(l+a)^2=2\delta\left\{M\gamma^2+kM'(l+a)^2\right\}.$$

In considering the diminution of the arc of oscillation, we may put $l+a$ for γ. During i oscillations, let the arc of oscillation be diminished in the ratio of A_0 to A_i, then

$$\log_\epsilon\frac{A_0}{A_i}=i\tau\delta=\frac{\pi i}{2}\frac{k'M'}{M+kM'}. \quad \ldots \ldots \quad (53)$$

For a given fluid and a given time of oscillation, both k and k' increase as a decreases. Hence it follows from theory, that the smaller be the sphere, its density being supposed given, the more the time of oscillation is affected, and the more rapidly the arc of oscillation diminishes, the alteration in the rate of diminution of the arc due to an alteration in the radius of the sphere being more conspicuous than the alteration in the time of oscillation.

21. Let us now suppose the fluid confined in a spherical envelope. In this case, we have

to determine the four arbitrary constants which appear in (38) by the four equations (35) and (36). We get, in the first place,

$$\frac{A}{a} + Ba^2 + C\epsilon^{-ma}\left(1+\frac{1}{ma}\right) + D\epsilon^{ma}\left(1-\frac{1}{ma}\right) = \tfrac{1}{2}a^2c, \quad . \quad . \quad . \quad . \quad (54)$$

$$-\frac{A}{a} + 2Ba^2 - C\epsilon^{-ma}\left(ma+1+\frac{1}{ma}\right) + D\epsilon^{ma}\left(ma-1+\frac{1}{ma}\right) = a^2c, \quad . \quad . \quad (55)$$

$$\frac{A}{b} + Bb^2 + C\epsilon^{-mb}\left(1+\frac{1}{mb}\right) + D\epsilon^{mb}\left(1-\frac{1}{mb}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$-\frac{A}{b} + 2Bb^2 - C\epsilon^{-mb}\left(mb+1+\frac{1}{mb}\right) + D\epsilon^{mb}\left(mb-1+\frac{1}{mb}\right) = 0. \quad . \quad . \quad (57)$$

Putting a^2cK for $af_1'(a) + 2f_2(a)$, which is the quantity that we want to find, we get from (38) and (54)

$$K = 1 - \frac{3A}{a^3c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Eliminating in succession B from (54) and (55), from (56) and (57), and from (54) and (56), we shall obtain for the determination of A, C, D three equations which remain unchanged when a and b are interchanged, and the signs of A, C, and D changed. Hence $-A$, $-C$, $-D$ are the same functions of b and a that A, C, D are of a and b. It will also assist in the further elimination to observe that C and D are interchanged when the sign of m is changed. The result of the elimination is

$$K = 1 - \frac{3b}{2m^2a^2} \cdot \frac{\eta(a,b) - \eta(b,a)}{12mab + \zeta(a,b) + \zeta(b,a)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

the functions ζ, η being defined by the equations

$$\left.\begin{aligned} \eta(a,b) &= (m^2a^2 + 3ma + 3)\,(m^2b^2 - 3mb + 3)\,\epsilon^{m(b-a)}, \\ \zeta(a,b) &= \{b\,(m^2b^2 - 3mb + 3) - a\,(m^2a^2 + 3ma + 3)\}\,\epsilon^{m(b-a)}. \end{aligned}\right\} \quad . \quad . \quad (60)$$

It turns out that K is a complicated function of m and ab^{-1}, and the algebraical expressions for the quantities which answer to k and k' in Art. 20 would be more complicated still, because $\nu(1+\sqrt{-1})$ would have to be substituted for m in (60) and (59), and then K reduced to the form $-k + \sqrt{-1}\,k'$. To obtain numerical results from these formulæ, it would be best to substitute the numerical values of a, b, and ν in (60) and (59), and perform the reduction of K in figures.

22. If the distance of the envelope from the surface of the sphere be at all considerable, the exponential $\epsilon^{\nu(b-a)}$, which arises from $\epsilon^{m(b-a)}$, will have so large a numerical value that we may neglect the terms in the numerator and denominator of the fraction in the expression for K which contain $\epsilon^{-\nu(b-a)}$, as well as the term in the denominator which is free from exponentials, in comparison with the terms which contain $\epsilon^{\nu(b-a)}$. Thus, if $b-a$ be two inches, τ one second, and $\sqrt{\mu'} = .116$, we have $\epsilon^{\nu(b-a)} = 2424000000$, nearly; and if $b-a$ be only an

inch or half an inch, we have still the square or fourth root of the above quantity, that is, about 49234 or 222, for the value of that exponential. Hence, in practical cases, the above simplification may be made, which will cause the exponentials to disappear from the expression for K. We thus get

$$K = 1 - \frac{3b}{2m^2a^2}\,\frac{(m^2a^2 + 3ma + 3)\,(m^2b^2 - 3mb + 3)}{b\,(m^2b^2 - 3mb + 3) - a\,(m^2a^2 + 3ma + 3)}. \quad . \quad . \qquad (61)$$

If we assume

$$3\nu a + 3 + (2\nu^2a^2 + 3\nu a)\sqrt{-1} = A'(\cos\alpha + \sqrt{-1}\,\sin\alpha),$$

$$-3\nu b + 3 + (2\nu^2b^2 - 3\nu b)\sqrt{-1} = B'(\cos\beta + \sqrt{-1}\,\sin\beta),$$

$$bB'\cos\beta - aA'\cos\alpha = C'\cos\gamma,$$

$$bB'\sin\beta - aA'\sin\alpha = C'\sin\gamma,$$

we get from (61)

$$K = 1 + \frac{3b\sqrt{-1}}{4\nu^2a^2}\cdot\frac{A'B'}{C'}\left\{\cos(\alpha+\beta-\gamma) + \sqrt{-1}\,\sin(\alpha+\beta-\gamma)\right\},$$

whence

$$\left.\begin{aligned} k &= \frac{3bA'B'}{4\nu^2a^2C'}\sin(\alpha+\beta-\gamma) - 1,\\ k' &= \frac{3bA'B'}{4\nu^2a^2C'}\cos(\alpha+\beta-\gamma)\,; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (62)$$

and, as before, kM' is the imaginary mass which we must conceive to be collected at the centre of the sphere, in order to allow for the inertia of the fluid, and $-k'M'n\dfrac{d\xi}{dt}$ the term in F on which depends the diminution in the arc of oscillation.

23. If we suppose $\mu' = 0$, and therefore $m = \infty$, we get from (61)

$$K = -\frac{b^3 + 2a^3}{2(b^3 - a^3)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (63)$$

and, in this case, k is the same as with K sign changed, and $k' = 0$, which agrees with the result obtained directly from the ordinary equations of hydrodynamics*. If, on the other hand, we make $b = \infty$, we arrive at the results already obtained in Art. 20. In both these cases it becomes rigorously exact to neglect in the expression for $K - 1$ given by (59) all the terms which are not multiplied by $\epsilon^{\nu(b-a)}$.

If the effect of the envelope be but small, which will generally be the case, it will be convenient to calculate k and k' from the formulæ (52), which apply to the case in which $b = \infty$, and then add corrections Δk, $\Delta k'$ due to the envelope. We get from (61)

$$\Delta k - \sqrt{-1}\,\Delta k' = \frac{3}{2m^2a}\,\frac{(m^2a^2 + 3ma + 3)^2}{b\,(m^2b^2 - 3mb + 3) - a\,(m^2a^2 + 3ma + 3)}, \quad . \qquad (64)$$

* See Camb. Phil. Trans. Vol. VIII. p. 120.

which may be treated, if required, as the equation (61) was treated in the preceding article. If, however, we suppose m large, and are content to retain only the most important term in (64), we get simply

$$\Delta k = \frac{3a^3}{2(b^3 - a^3)}, \quad \Delta k' = 0, \quad \ldots \ldots \quad (65)$$

so that the correction for the envelope may be calculated as if the fluid were destitute of friction.

SECTION III.

Solution of the equations in the case of an infinite cylinder oscillating in an unlimited mass of fluid, in a direction perpendicular to its axis.

24. Suppose a long cylindrical rod suspended at a point in its axis, and made to oscillate as a pendulum in an unlimited mass of fluid. The resistance experienced by any element of the cylinder comprised between two parallel planes drawn perpendicular to the axis will manifestly be very nearly the same as if the element belonged to an infinite cylinder oscillating with the same linear velocity. For an element situated very near either extremity of the rod, the resistance thus determined would, no doubt, be sensibly erroneous; but as the diameter of the rod is supposed to be but small in comparison with its length, it will be easily seen that the error thus introduced must be extremely small.

Imagine then an infinite cylinder to oscillate in a fluid, in a direction perpendicular to its axis, so that the motion takes place in two dimensions, and let it be required to determine the motion of the fluid. The mode of solution of this problem will require no explanation, being identical in principle with that which has been already adopted in the case of a sphere. In the present instance the problem will be found somewhat easier, up to the formation of the equations analogous to (33) and (34), after which it will become much more difficult.

25. Let a plane drawn perpendicular to the axis of the cylinder be taken for the plane of xy, the origin being situated in the mean position of the axis of the cylinder, and the axis of x being measured in the direction of the cylinder's motion. The general equations (2), (3) become in this case

$$\left.\begin{aligned} \frac{dp}{dx} &= \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right) - \rho\frac{du}{dt}, \\ \frac{dp}{dy} &= \mu\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}\right) - \rho\frac{dv}{dt}, \end{aligned}\right\} \quad \ldots \ldots \quad (66)$$

$$\frac{du}{dx} + \frac{dv}{dy} = 0. \quad \ldots \ldots \quad (67)$$

By virtue of (67), $u\,dy - v\,dx$ is an exact differential. Let then

$$u\,dy - v\,dx = d\chi. \quad \ldots \ldots \quad (68)$$

Eliminating p by differentiation from the two equations (66), and expressing u and v in terms of χ in the resulting equation, we get

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}-\frac{1}{\mu'}\frac{d}{dt}\right)\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\chi=0, \quad \ldots \quad (69)$$

and, as before

$$\chi=\chi_1+\chi_2, \quad \ldots \quad (70)$$

where

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\chi_1=0, \quad \ldots \quad (71)$$

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}-\frac{1}{\mu'}\frac{d}{dt}\right)\chi_2=0. \quad \ldots \quad (72)$$

We get from (66) and (68)

$$dp=\mu'\rho\,dx\,.\frac{d}{dy}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}-\frac{1}{\mu'}\frac{d}{dt}\right)\chi-\mu'\rho\,dy\,.\frac{d}{dx}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}-\frac{1}{\mu'}\frac{d}{dt}\right)\chi,$$

which becomes by means of (70), (71), and (72)

$$dp=\rho\left(\frac{d^2\chi_1}{dt\,dx}\,dy-\frac{d^2\chi_1}{dt\,dy}\,dx\right). \quad \ldots \quad (73)$$

26. Passing to polar co-ordinates r, θ, where θ is supposed to be measured from the axis of x, we get from (68), (71), (72), and (73)

$$Rr\,d\theta-\Theta\,dr=d\chi, \quad \ldots \quad (74)$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}\right)\chi_1=0, \quad \ldots \quad (75)$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2}-\frac{1}{\mu'}\frac{d}{dt}\right)\chi_2=0, \quad \ldots \quad (76)$$

$$dp=\rho\frac{d}{dt}\left(\frac{d\chi_1}{dr}\,r\,d\theta-\frac{d\chi_1}{r\,d\theta}\,dr\right), \quad \ldots \quad (77)$$

R, Θ in (74) being the velocities along and perpendicular to the radius vector.

27. Let a be the radius of the cylinder; and as before let the cylinder's motion be defined by the equation

$$\frac{d\xi}{dt}=c\epsilon^{\sqrt{-1}nt}=c\epsilon^{\mu'm^2t}; \quad \ldots \quad (78)$$

then we have for the equations of condition which relate to the surface of the cylinder

$$\left.\begin{aligned} R&=\frac{d\chi}{r\,d\theta}=\cos\theta\,\frac{d\xi}{dt}=c\cos\theta\,\epsilon^{\mu'm^2t},\\ \Theta&=-\frac{d\chi}{dr}=-\sin\theta\,\frac{d\xi}{dt}=-c\sin\theta\,\epsilon^{\mu'm^2t}.\end{aligned}\right\}\text{when } r=a. \quad \ldots \quad (79)$$

The general equations (75), (76), as well as the equations of condition (79), may be satisfied by taking

$$\chi_1 = \epsilon^{\mu' m^2 t} \sin\theta\, F_1(r), \qquad \chi_2 = \epsilon^{\mu' m^2 t} \sin\theta\, F_2(r). \quad . \quad . \quad . \quad . \quad (80)$$

Substituting in (75), (76), and (79), we get

$$F_1''(r) + \frac{1}{r} F_1'(r) - \frac{1}{r^2} F_1(r) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

$$F_2''(r) + \frac{1}{r} F_2'(r) - \frac{1}{r^2} F_2(r) - m^2 F_2(r) = 0, \quad . \quad . \quad . \quad . \quad (82)$$

$$F_1(a) + F_2(a) = ac, \qquad F_1'(a) + F_2'(a) = c, \quad . \quad . \quad . \quad . \quad (83)$$

besides which we have the condition that the velocity shall vanish at an infinite distance.

28. The integral of (81) is

$$F_1(r) = \frac{A}{r} + Br. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

The integral of (82) cannot be obtained in finite terms.

To simplify the latter equation, assume $F_2(r) = F'_3(r)$. Substituting in (82), and integrating once, we get

$$F_3''(r) + \frac{1}{r} F_3'(r) - m^2 F_3(r) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

It is unnecessary to add an arbitrary constant, because such a constant, if introduced, might be got rid of by writing $F_3(r) + C$ for $F_3(r)$.

To integrate (85) by series according to ascending powers of r, let us first, instead of (85), take the equation formed from it by multiplying the second term by $1 - \delta$. Assuming in this new equation $F_3(r) = A_{,}x^{\alpha} + B_{,}x^{\beta} + \ldots$, and determining the arbitrary indices α, β...and the arbitrary constants $A_{,}$, $B_{,}$...so as to satisfy the equation, we get

$$F_3(r) = A_{,} \left\{1 + \frac{m^2 r^2}{2(2-\delta)} + \frac{m^4 r^4}{2\,.\,4(2-\delta)(4-\delta)} + \ldots\right\}$$

$$+ A_{,,} r^{\delta} \left\{1 + \frac{m^2 r^2}{2(2+\delta)} + \frac{m^4 r^4}{2\,.\,4(2+\delta)(4+\delta)} + \ldots\right\}$$

$$= (A_{,} + A_{,,} + A_{,,}\delta \log r) \left\{1 + \frac{m^2 r^2}{2^2} + \frac{m^4 r^4}{2^2\,.\,4^2} + \ldots\right\}$$

$$+ \tfrac{1}{2}(A_{,} - A_{,,})\,\delta \left\{\frac{m^2 r^2}{2^2} S_1 + \frac{m^4 r^4}{2^2\,.\,4^2} S_2 + \frac{m^6 r^6}{2^2\,.\,4^2\,.\,6^2} S_3 + \ldots\right\}$$

$$+ \text{ terms involving } \delta^2, \delta^3 \ldots$$

In this expression

$$S_i = 1^{-1} + 2^{-1} + 3^{-1} \ldots + i^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

Putting now

$$A_{,} = C - A_{,,}, \qquad A_{,,} = D\delta^{-1},$$

substituting in the above equation, and then making δ vanish, we get

$$F_3(r) = (C + D \log r)\left(1 + \frac{m^2 r^2}{2^2} + \frac{m^4 r^4}{2^2 . 4^2} + \ldots\right)$$

$$- D\left(\frac{m^2 r^2}{2^2} S_1 + \frac{m^4 r^4}{2^2 . 4^2} S_2 + \frac{m^6 r^6}{2^2 . 4^2 . 6^2} S_3 + \ldots\right). \quad . \quad . \quad . \quad (87)$$

The series in this equation are evidently convergent for all values of r, however great; but, nevertheless, they give us no information as to what becomes of $F_3(r)$ when r becomes infinite, and yet one relation between C and D has to be determined by the condition that $F_3(r)$ shall not become infinite with r.

The equation (85) may be integrated by means of descending series combined with exponentials, by assuming $F_3(r) = \epsilon^{\pm mr}(A_{,} r^{\alpha} + B_{,} r^{\beta} \ldots)$. I have already given the integral in this form in a paper, *On the numerical calculation of a class of definite integrals and infinite series**. The result is

$$F_3(r) = C' \epsilon^{-mr} r^{-\frac{1}{2}} \left\{1 - \frac{1^2}{2 . 4mr} + \frac{1^2 . 3^2}{2 . 4(4mr)^2} - \frac{1^2 . 3^2 . 5^2}{2 . 4 . 6(4mr)^3} + \ldots\right\}$$

$$+ D' \epsilon^{mr} r^{-\frac{1}{2}} \left\{1 + \frac{1^2}{2 . 4mr} + \frac{1^2 . 3^2}{2 . 4(4mr)^2} + \frac{1^2 . 3^2 . 5^2}{2 . 4 . 6(4mr)^3} + \ldots\right\}. \quad . \quad (88)$$

These series, although ultimately divergent in all cases, are very convenient for numerical calculation when the modulus of mr is large. Moreover they give at once $D' = 0$ for the condition that $F_3(r)$ shall not become infinite with r, and therefore we shall be able to obtain the required relation between C and D, provided we can express D' as a function of C and D.

29. This may be effected by means of the integral of (85) expressed by definite integrals. This form of the integral is already known. It becomes, by a slight transformation,

$$F_3(r) = \int_0^{\frac{\pi}{2}} \{C'' + D'' \log(r \sin^2 \omega)\} (\epsilon^{mr \cos \omega} + \epsilon^{-mr \cos \omega})\, d\omega, \quad . \quad . \quad (89)$$

C'', D'' being the two arbitrary constants. If we expand the exponentials in (89), and integrate the terms separately, we obtain, in fact, an expression of the same form as (87). This transformation requires the reduction of the definite integral

$$P_i = \int_0^{\frac{\pi}{2}} \cos^{2i} \omega \log \sin \omega \, d\omega.$$

If we integrate by parts, integrating $\cos \omega \log \sin \omega d\omega$, and differentiating $\cos^{2i-1}\omega$, we shall make P_i depend on P_{i-1}. Assuming $P_0 = Q_0$, $P_1 = \frac{1}{2} Q_1 \ldots$, and generally

$$P_i = \frac{1 . 3 \ldots (2i - 1)}{2 . 4 \ldots 2i} Q_i,$$

* Camb. Phil. Trans. Vol. IX. p. 182.

we get

$$Q_i = Q_0 - \{2^{-1} + 4^{-1} \ldots + (2i)^{-1}\} \frac{\pi}{2} = \frac{\pi}{2} \log(\tfrac{1}{2}) - \frac{\pi}{4} S_i.^*$$

The equivalence of the expressions (87) and (89) having been ascertained, in order to find the relations between C, D and C'', D'', it will be sufficient to write down the two leading terms in (87) and (89), and equate the results. We thus get

$$C + D \log r = \pi C'' + \pi D'' \log r + 2\pi D'' \log(\tfrac{1}{2}),$$

whence

$$C = \pi C'' + 2\pi \log(\tfrac{1}{2}) . D'', \quad D = \pi D''. \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

There remains the more difficult step of finding the relation between D' and C'', D''. For this purpose let us seek the ultimate value of the second member of equation (89) when r increases indefinitely. In the first place we may observe that if Ω, Ω' be two imaginary quantities having their real parts positive, if the real part of Ω be greater than that of Ω', and if r be supposed to increase indefinitely, $\epsilon^{\Omega r}$ will ultimately be incomparably greater than $\epsilon^{\Omega' r}$, or even than $\log r . \epsilon^{\Omega' r}$, or, to speak more precisely, the modulus of the former expression will ultimately be incomparably greater than the modulus of either of the latter. Hence, in finding the ultimate value of the expression for $F_3(r)$ in (89), we may replace the limits 0 and $\frac{1}{2}\pi$ of ω by 0 and ω_1, where ω_1 is a positive quantity as small as we please, which we may suppose to vanish after r has become infinite. We may also, for the same reason, omit the second of the exponentials. Let $\cos\omega = 1 - \lambda$, so that

$$\sin^2\omega = 2\lambda\left(1 - \frac{\lambda}{2}\right), \quad d\omega = \frac{d\lambda}{\sqrt{2\lambda - \lambda^2}} = \left(1 + \frac{\lambda}{4} + \ldots\right)\frac{d\lambda}{\sqrt{(2\lambda)}};$$

then the limits of λ will be 0 and λ_1, where $\lambda_1 = 1 - \cos\omega_1$. Since $\log\left(1 - \frac{\lambda}{2}\right)$ ultimately vanishes, and $1 + \frac{\lambda}{4} + \ldots$ becomes ultimately 1, we get from (89)

$$\text{limit of } F_3(r) = \epsilon^{mr} \times \text{limit of } \int_0^{\lambda_1} (C'' + D'' \log 2\lambda r)\, \epsilon^{-m\lambda r} \frac{d\lambda}{\sqrt{(2\lambda)}}.\dagger$$

If now we put $\lambda = \lambda' r^{-1}$, we shall have 0 and $\lambda_1 r$ for the limits of λ', and the second of these becomes infinite with r. Hence

$$\text{limit of } F_3(r) = (2r)^{-\frac{1}{2}} \epsilon^{mr} \int_0^\infty (C'' + D'' \log 2\lambda')\, \epsilon^{-m\lambda'} \lambda'^{-\frac{1}{2}} d\lambda'. \quad . \quad . \quad (91)$$

Now $\int_0^\infty \epsilon^{-x} x^{-\frac{1}{2}} dx = \pi^{\frac{1}{2}}$, and if we differentiate both sides of the equation

$$\int_0^\infty \epsilon^{-x} x^{s-1} dx = \Gamma(s)$$

---

* A demonstration by Mr Ellis of the theorem

$$\int_0^{\frac{\pi}{2}} \log \sin\theta\, d\theta = \frac{\pi}{2} \log(\tfrac{1}{2})$$

due to Euler will be found in the 2nd volume of the Cambridge Mathematical Journal, p. 282, or in Gregory's Examples, p. 484.

† The word *limit* is here used in the sense in which $f(r)$ may be called the limit of $\phi(r)$ when the ratio of $\phi(r)$ to $f(r)$ is ultimately a ratio of equality, though $f(r)$ and $\phi(r)$ may vanish or become infinite together, in which case the limit of $\phi(r)$, according to the usual sense of the word *limit*, would be said to be zero or infinity.

with respect to s, and after differentiation put $s = \frac{1}{2}$, we get

$$\int_0^\infty \epsilon^{-x} x^{-\frac{1}{2}} \log x \, dx = \Gamma'(\tfrac{1}{2}).$$

Putting $x = m\lambda'$ in these equations we get

$$\int_0^\infty \epsilon^{-m\lambda'} \lambda'^{-\frac{1}{2}} d\lambda' = \pi^{\frac{1}{2}} m^{-\frac{1}{2}}, \quad \int_0^\infty \epsilon^{-m\lambda'} \lambda'^{-\frac{1}{2}} \log \lambda' \, d\lambda' = m^{-\frac{1}{2}} \{\Gamma'(\tfrac{1}{2}) - \pi^{\frac{1}{2}} \log m),$$

where that value of $m^{-\frac{1}{2}}$ is to be taken which has its real part positive. Substituting in (91) we get

$$\text{limit of } F_3(r) = \left(\frac{\pi}{2mr}\right)^{\frac{1}{2}} \epsilon^{mr} \{C'' + \left(\pi^{-\frac{1}{2}} \Gamma' \tfrac{1}{2} - \log \frac{m}{2}\right) D''\}.$$

Comparing with (88) we get

$$D' = \left(\frac{\pi}{2m}\right)^{\frac{1}{2}} \{C'' + \left(\pi^{-\frac{1}{2}} \Gamma' \tfrac{1}{2} - \log \frac{m}{2}\right) D''\}. \quad . \quad . \quad . \quad . \quad (92)$$

30. We are now enabled to find the relation between C and D arising from the condition that the motion of the fluid shall not become infinitely great at an infinite distance from the cylinder. The determination of the arbitrary constants A, B, C, D will present no further difficulty. We must have $B = 0$, since otherwise the velocity would be finite at an infinite distance, and then the two equations (83), combined with the relation above mentioned, will serve to determine A, C, D. The motion of the fluid will thus be completely determined, the functions $F_1(r)$, $F_3(r)$ being given by (84) and (87). When the modulus of mr is large, the series in (87), though ultimately hypergeometrically convergent, are at first rapidly divergent, and in calculating the numerical value of $F_3(r)$ in such a case it would be far more convenient to employ equation (88). The employment of this equation for the purpose would require the previous determination of the constant C'. It will be found however that in calculating the resultant pressure of the fluid on the cylinder, which it is the main object of the present investigation to determine, a knowledge of the value of C' will not be required, and that, even though the equation (88) be employed.

Putting $D' = 0$ in (92), and eliminating C'' and D'' between the resulting equation and the two equations (90), we get

$$C = (\log \frac{m}{8} - \pi^{-\frac{1}{2}} \Gamma' \tfrac{1}{2}) D; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

and we get from (83) and (84), observing that $F_2(r) = F_3'(r)$, and that $B = 0$,

$$\frac{A}{a} + F_3'(a) = ac, \quad -\frac{A}{a} + aF_3''(a) = ac, \quad . \quad . \quad . \quad . \quad (94)$$

whence

$$\frac{a^2c + A}{a^2c - A} = \frac{aF_3''(a)}{F_3'(a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

This equation will determine A, because if $F_3(a)$ be expressed by (87) the second member of

(95) will only contain the ratio of C to D, which is given by (93), and if $F_3(a)$ be expressed by (88) C' will disappear, inasmuch as $D' = 0$.

31. Let us now form the expression for the resultant of the forces which the fluid exerts on the cylinder. Let F be the resultant of the pressures acting on a length dl of the cylinder, which will evidently be a force acting in the direction of the axis of x; then we get in the same way as the expression (47) was obtained

$$F = a\,dl \int_0^{2\pi} (-P_r \cos\theta + T_\theta \sin\theta)_a \, d\theta, \quad . \quad . \quad . \quad . \quad . \quad (96)$$

and P_r, T_θ are given in terms of R and Θ by the same formulæ (46) as before. When the right-hand members of these equations are expressed in terms of χ, there will be only one term in which the differentiation with respect to r rises to the second order, and we get from (70), (75), and (76)

$$\frac{d^2\chi}{dr^2} = -\frac{1}{r}\frac{d\chi}{dr} - \frac{1}{r^2}\frac{d^2\chi}{d\theta^2} + \frac{1}{\mu'}\frac{d\chi_2}{dt}.$$

We get from this equation and the equations of condition (79)

$$\left(\frac{dR}{dr}\right)_a = \frac{1}{a}\left(\frac{d\chi}{d\theta}\right)_a - \frac{1}{a^2}\left(\frac{d^2\chi}{dr\,d\theta}\right)_a = 0,$$

$$\left(\frac{dR}{r\,d\theta}\right)_a = \frac{1}{a^2}\left(\frac{d^2\chi}{d\theta^2}\right)_a = -\frac{\sin\theta}{a}\frac{d\xi}{dt} = \frac{\Theta}{a},$$

$$\left(\frac{d\Theta}{dr}\right)_a = -\left(\frac{d^2\chi}{dr^2}\right)_a = \frac{1}{a}\left(\frac{d\chi}{dr}\right)_a + \frac{1}{a^2}\left(\frac{d^2\chi}{d\theta^2}\right)_a - \frac{1}{\mu'}\left(\frac{d\chi_2}{dt}\right)_a = -\frac{1}{\mu'}\left(\frac{d\chi_2}{dt}\right)_a.$$

Hence

$$F = a\,dl \int_0^{2\pi} \left\{ -p_a \cos\theta + \rho\left(\frac{d\chi_2}{dt}\right)_a \sin\theta \right\} d\theta. \quad . \quad . \quad . \quad . \quad (97)$$

We get by integration by parts

$$\int p_a \cos\theta\, d\theta = p_a \sin\theta - \int \left(\frac{dp}{d\theta}\right)_a \sin\theta\, d\theta.$$

The first term vanishes at both limits; and putting for $\frac{dp}{d\theta}$ its value given by (77), and substituting in (97), we get

$$F = \rho a\,dl \frac{d}{dt}\int_0^{2\pi} \left\{ a\left(\frac{d\chi_1}{dr}\right)_a + (\chi_2)_a \right\} \sin\theta\, d\theta,$$

or

$$F = \pi\rho a\,dl \,.\, n\sqrt{-1}\,\{aF_1'(a) + F_3'(a)\}\,\epsilon^{\sqrt{-1}\,nt}.$$

Observing that $F_3'(a)$ or $F_2(a) = ac - F_1(a)$ from (83), and that $F_1(a) = Aa^{-1}$, where A is given by (95), and putting M' for $\pi\rho a^2 dl$, the mass of the fluid displaced, we get

$$F = M'cn\sqrt{-1}\left\{1 - 2\frac{aF_3''(a) - F_3'(a)}{aF_3''(a) + F_3'(a)}\right\}\epsilon^{\sqrt{-1}\,nt},$$

which becomes by means of the differential equation (85) which F_3 satisfies

$$F = -M'cn\sqrt{-1}\left\{1 - \frac{4F_3'(a)}{m^2 a F_3(a)}\right\}\epsilon^{\sqrt{-1}\,nt}. \quad . \quad . \quad . \quad . \quad (98)$$

Let

$$1 - \frac{4F_3'(a)}{m^2 a F_3(a)} = k - \sqrt{-1}\,k', \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

where k and k' are real, then, as before, $kM'\dfrac{d^2\xi}{dt^2}$ will be the part of F which alters the time of oscillation, and $k'M'n\dfrac{d\xi}{dt}$ the part which produces a diminution in the arc of oscillation.

When μ' vanishes, m becomes infinite, and we get from (88) and (99), remembering that $D'=0$; $k=1$, $k'=0$, a result which follows directly and very simply from the ordinary equations of hydrodynamics*.

32. Every thing is now reduced to the numerical calculation of the quantities k, k', of which the analytical expressions are given. The series (87) being always convergent might be employed in all cases, but when the modulus of ma is large, it will be far more convenient to employ a series according to descending powers of a. Let us consider the ascending series first.

Let $2\mathfrak{m}$ be the modulus of ma; then

$$ma = 2\mathfrak{m}\,\epsilon^{\frac{\pi}{4}\sqrt{-1}}, \quad \mathfrak{m} = \frac{a}{2}\sqrt{\frac{n}{\mu'}} = \frac{a}{2}\sqrt{\frac{\pi}{\mu'\tau}}, \quad . \quad . \quad . \quad (100)$$

τ being as before the time of oscillation from rest to rest. Substituting in (99) the above expression for ma, we get

$$k - \sqrt{-1}\,k' = 1 + \frac{\sqrt{-1}\,aF_3'(a)}{\mathfrak{m}^2 F_3(a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

Putting for shortness

$$\log_\epsilon 4 + \pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2}) = -\Lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

we get from (87) and (93)

$$\frac{1}{D}F_3(a) = (\log\mathfrak{m} + \Lambda + \frac{\pi}{4}\sqrt{-1})\,(1 + \frac{\mathfrak{m}^2}{1^2}\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2^2} - \frac{\mathfrak{m}^6}{1^2.2^2.3^2}\sqrt{-1} + \ldots)$$

$$- \left(\frac{\mathfrak{m}^2}{1^2}S_1\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2^2}S_2 - \frac{\mathfrak{m}^6}{1^2.2^2.3^2}S_3\sqrt{-1} + \ldots\right),$$

$$\frac{1}{D}aF_3'(a) = 1 + \frac{\mathfrak{m}^2}{1^2}\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2^2} - \ldots + 2\left(\log\mathfrak{m} + \Lambda + \frac{\pi}{4}\sqrt{-1}\right)\left(\frac{\mathfrak{m}^2}{1}\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2} - \ldots\right)$$

$$-2\left(\frac{\mathfrak{m}^2}{1}S_1\sqrt{-1} - \frac{\mathfrak{m}^4}{1^2.2}S_2 - \frac{\mathfrak{m}^6}{1^2.2^2.3}S_3\sqrt{-1} + \ldots\right).$$

* See Camb. Phil. Trans. Vol. VIII. p. 116.

Let

$$\left.\begin{array}{ll}\dfrac{\mathfrak{m}^2}{1}-\dfrac{\mathfrak{m}^6}{1^2.2^2.3}+\ldots=M_0, & \dfrac{\mathfrak{m}^4}{1^2.2}-\dfrac{\mathfrak{m}^8}{1^2.2^2.3^2.4}+\ldots=M_e,\\ \dfrac{\mathfrak{m}^2}{1^2}-\dfrac{\mathfrak{m}^6}{1^2.2^2.3^2}+\ldots=M_0', & \dfrac{\mathfrak{m}^4}{1^2.2^2}-\dfrac{\mathfrak{m}^8}{1^2.2^2.3^2.4^2}+\ldots=M_e',\\ \dfrac{\mathfrak{m}^2}{1}S_1-\dfrac{\mathfrak{m}^6}{1^2.2^2.3}S_3+\ldots=N_0, & \dfrac{\mathfrak{m}^4}{1^2.2}S_2-\dfrac{\mathfrak{m}^8}{1^2.2^2.3^2.4}S_4+\ldots=N_e,\\ \dfrac{\mathfrak{m}^2}{1^2}S_1-\dfrac{\mathfrak{m}^6}{1^2.2^2.3^2}S_3+\ldots=N_0', & \dfrac{\mathfrak{m}^4}{1^2.2^2}S_2-\dfrac{\mathfrak{m}^8}{1^2.2^2.3^2.4^2}S_4+\ldots=N_e',\end{array}\right\} \quad . \quad (103)$$

$$\log_\epsilon \mathfrak{m} + \Lambda = L: \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

then substituting in (101), changing the sign of $\sqrt{-1}$, and arranging the terms, we get

$$k+\sqrt{-1}\,k'=1+\frac{2}{\mathfrak{m}^2}\,\frac{-LM_0+\frac{\pi}{4}M_e-\frac{1}{2}M_0'+N_0+\left\{\frac{\pi}{4}M_0+LM_e-\frac{1}{2}(1-M_e')-N_e\right\}\sqrt{-1}}{-\frac{\pi}{4}M_0'+L(1-M_e')+N_e'+\left\{-LM_0'-\frac{\pi}{4}(1-M_e')+N_0'\right\}\sqrt{-1}} \quad (105)$$

33. Before going on with the calculation, it will be requisite to know the numerical value of the transcendental quantity Λ. Now

$$\pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2})=(\Gamma\tfrac{1}{2})^{-1}\Gamma'(\tfrac{1}{2})=\frac{d}{ds}\log\Gamma(s)=\frac{d}{ds}\log\Gamma(1+s)-\frac{1}{s},\ \text{ for } s=\tfrac{1}{2},$$

and the value of $\dfrac{d}{ds}\log\Gamma(1+s)$ may be got at once from Legendre's table of the common logarithms of $\Gamma(1+s)$, in which the interval of s is .001. Putting l_s for the tabular number corresponding to s, we have

$$\frac{d}{ds}\log\Gamma(1+s)=1000\log_\epsilon 10\left\{\Delta l_s-\tfrac{1}{2}\Delta^2 l_s+\tfrac{1}{3}\Delta^3 l_s-\tfrac{1}{4}\Delta^4 l_s+\ldots\right\}.$$

For $s=\frac{1}{2}$

$$\Delta l_s=+16050324,\quad \Delta^2 l_s=+405620,\quad \Delta^3 l_s=-359,\quad \Delta^4 l_s=+6,^*$$

the last figure being in each case in the 12th place of decimals. We thus get

$$\pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2})=-1.9635102,\quad \Lambda=+.5772158. \quad . \quad . \quad . \quad . \quad . \quad (106)$$

34. When $\mathfrak{m}$ is large, it will be more convenient to employ series according to descending powers of a. Observing that the general term of $F_3(a)$ as given by (88), in which $D'=0$, is

$$(-1)^i C'\epsilon^{-ma}\frac{[1.3\ldots(2i-1)]^2}{2.4\ldots 2i\,(4ma)^i a^{\frac{1}{2}}},$$

---

* These numbers are copied from De Morgan's Differential and Integral Calculus, p. 588.

we get for the general term of $F_3'(a)$

$$(-1)^{i-1} C'\epsilon^{-ma} \frac{[1.3...(2i-3)]^2}{2.4...(2i-2)(4ma)^{i-1}a^{\frac{1}{2}}} \left\{ m\frac{(2i-1)^2}{2i.4ma} - \frac{2i-1}{2a} \right\},$$

and the expression within brackets is equivalent to

$$-\frac{(2i-1)(2i+1)}{8ia},$$

whence

$$aF_3'(a) = C'\epsilon^{-ma} ma^{\frac{1}{2}} \left\{ -1 - \frac{1.3}{2.4ma} + \frac{1^2.3.5}{2.4(4ma)^2} - ... \right\},$$

and we find by actual division

$$\frac{aF_3'(a)}{F_3(a)} = -ma - \tfrac{1}{2} + \tfrac{1}{8}(ma)^{-1}...$$

35. When many terms are required, the calculation of the coefficients may be facilitated in the following manner.

Assuming $aF_3'(a) = v(a) F_3(a)$, we have

$$F_3'(a) = a^{-1} v(a) F_3(a), \quad F_3''(a) = \{a^{-1} v'(a) - a^{-2} v(a) + a^{-2}(va)^2\} F_3(a).$$

Substituting in the differential equation (85) which F_3 has to satisfy, we get

$$av'(a) + \{v(a)\}^2 - m^2a^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Assuming

$$v(a) = -ma + A_0 + A_1(ma)^{-1} + A_2(ma)^{-2} + ..., \quad . \quad . \quad . \quad (108)$$

and substituting in the above equation, we get

$$-ma - 1A_1(ma)^{-1} - 2A_2(ma)^{-2} - 3A_3(ma)^{-3}...$$

$$+ \{-2ma + A_0 + A_1(ma)^{-1} + ...\} \{A_0 + A_1(ma)^{-1} + ...\} = 0,$$

which gives on equating coefficients, $A_0 = -\frac{1}{2}$, and for $i > 0$

$$2A_{i+1} = -iA_i + A_0A_i + A_1A_{i-1} ... + A_iA_0,$$

or, assuming to avoid fractions,

$$A_i = 2^{-2i-1} B_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

$$B_{i+1} = -2iB_i + B_0B_i + B_1B_{i-1} ... + B_iB_0, \quad . \quad . \quad . \quad . \quad (110)$$

a formula by means of which the coefficients B_1, B_2, B_3... may be readily calculated one after another. We get

$$\left. \begin{array}{l} B_0 = -1, \quad B_1 = +1, \quad B_2 = -4, \quad B_3 = +25, \quad B_4 = -208, \quad B_5 = +2146, \\ B_6 = -26368, \quad B_7 = +375733, \quad B_8 = -6092032. \end{array} \right\} \quad . \quad (111)$$

We get now from (100), (101), (108), and (109)

$$k - \sqrt{-1}\, k' = 1 + 2\epsilon^{-\frac{\pi}{4}\sqrt{-1}} \mathfrak{m}^{-1} - \tfrac{1}{2}B_0\epsilon^{-2\frac{\pi}{4}\sqrt{-1}} \mathfrak{m}^{-2} - \frac{1}{2^4} B_1 \epsilon^{-3\frac{\pi}{4}\sqrt{-1}} \mathfrak{m}^{-3}..., \quad (112)$$

whence if we calculate

$$u_1 = 2\mathfrak{m}^{-1}, \quad u_2 = -\tfrac{1}{2}B_0\mathfrak{m}^{-2}, \quad u_3 = \tfrac{1}{16}B_1\mathfrak{m}^{-3}\ldots, \quad u_i = (-1)^{i+1}\tfrac{1}{2}B_{i-2}\,8^{-i+2}\mathfrak{m}^{-i},$$

we shall have, changing the sign of $\sqrt{-1}$ in (112), and writing 8 for $\epsilon^{\frac{\pi}{4}\sqrt{-1}}$,

$$\left.\begin{aligned} k + \sqrt{-1}\,k' &= 1 + u_1 8 + u_2 8^2 - u_3 8^3 + u_4 8^4 - u_5 8^5 + \ldots \\ k &= 1 + \sqrt{\tfrac{1}{2}}\,u_1 + \sqrt{\tfrac{1}{2}}\,u_3 - u_4 + \sqrt{\tfrac{1}{2}}\,u_5 - \sqrt{\tfrac{1}{2}}\,u_7 + u_8 - \sqrt{\tfrac{1}{2}}\,u_9\ldots \\ k' &= \sqrt{\tfrac{1}{2}}\,u_1 + u_2 - \sqrt{\tfrac{1}{2}}\,u_3 + \sqrt{\tfrac{1}{2}}\,u_5 - u_6 + \sqrt{\tfrac{1}{2}}\,u_7 - \sqrt{\tfrac{1}{2}}\,u_9\ldots \end{aligned}\right\} \quad (113)$$

If l_1, $l_2\ldots$ be the common logarithms of the coefficients of $\mathfrak{m}^{-1}$, $\mathfrak{m}^{-2}\ldots$ in the last two of the formulæ (113),

$$\begin{aligned} &l_1 = .1505150; && l_4 = \bar{2}.4948500; && l_7 = \bar{2}.3646348; \\ &l_2 = \bar{1}.6989700; && l_5 = \bar{2}.2371251; && l_8 = \bar{2}.7019316; \\ &l_3 = \bar{2}.6453650; && l_6 = \bar{2}.4046734; && l_9 = \bar{2}.6017045; \end{aligned}$$

and if the logarithms of the coefficients of $\mathfrak{m}^{-1}$, $\mathfrak{m}^{-2}\ldots$ in $u_1, u_2\ldots$ be required, it will be sufficient to add .1505150 to the 1st, 3rd, 5th, &c. of the logarithms above given.

36. It will be found that when $\mathfrak{m}$ is at all large, the series (113) are at first convergent, and afterwards divergent, and in passing from convergent to divergent the quantities u_i become nearly equal for several successive terms. If after having calculated i terms of the first of the series (113) we wish to complete the series by a formula involving the differences of u_i, we have

$$\begin{aligned} u_i 8^i - u_{i+1}8^{i+1} + u_{i+2}8^{i+2} - \ldots &= 8^i\{1 - 8(1+\Delta) + 8^2(1+\Delta)^2 - \ldots\}\,u_i \\ &= 8^i\{1 + 8(1+\Delta)\}^{-1}u_i \\ &= \frac{8^i}{1+8}\left\{1 - \frac{8}{1+8}\Delta + \left(\frac{8}{1+8}\right)^2\Delta^2 - \ldots\right\}u_i, \end{aligned}$$

and $1 + 8 = 1 + \cos\dfrac{\pi}{4} + \sqrt{-1}\sin\dfrac{\pi}{4} = 2\cos\dfrac{\pi}{8}\,\epsilon^{\frac{\pi}{8}\sqrt{-1}}$, $\quad 8(1+8)^{-1} = \tfrac{1}{2}\sec\dfrac{\pi}{8}\,.\,\epsilon^{\frac{\pi}{8}\sqrt{-1}}$,

so that the quantities to be added to k, k', are

$$\left.\begin{aligned} &\text{to } k,\ (-1)^i\tfrac{1}{2}\sec\frac{\pi}{8}\left\{\cos\frac{2i-1}{8}\pi\,.\,u_i - \tfrac{1}{2}\sec\frac{\pi}{8}\cos\frac{2i}{8}\pi\,.\,\Delta u_i \right. \\ &\qquad\qquad \left. + \left(\tfrac{1}{2}\sec\frac{\pi}{8}\right)^2\cos\frac{2i+1}{8}\pi\,.\,\Delta^2 u_i\ldots\right\} \\ &\text{to } k',\ (-1)^i\tfrac{1}{2}\sec\frac{\pi}{8}\left\{\sin\frac{2i-1}{8}\pi\,.\,u_i - \tfrac{1}{2}\sec\frac{\pi}{8}\sin\frac{2i}{8}\pi\,.\,\Delta u_i \right. \\ &\qquad\qquad \left. + \left(\tfrac{1}{2}\sec\frac{\pi}{8}\right)^2\sin\frac{2i+1}{8}\pi\,.\,\Delta^2 u_i\ldots\right\} \end{aligned}\right\} \quad (114)$$

37. The following table contains the values of the functions k and k' calculated for 40 different values of $\mathfrak{m}$. From $\mathfrak{m} = .1$ to $\mathfrak{m} = 1.5$ the calculation was performed by means

of the formula (105); the rest of the table was calculated by means of the series (113). In the former part of the calculation, six places of decimals were employed in calculating the functions M_0, &c. given by (103). The last figure was then struck out, and five-figure logarithms were employed in multiplying the four functions M_0, M'_0, M_e, and $1 - M'_e$ by $\frac{\pi}{4}$, and by L, as well as in reducing the right-hand member of (105) to the form $k + \sqrt{-1}\,k'$. The terms of the series (113) were calculated to five places of decimals. That these series are sufficiently convergent to be employed when $\mathfrak{m} = 1·5$, might be presumed from the numerical values of the terms, and is confirmed by finding that they give $k = 1·952$, and $k' = 1·153$. For $\mathfrak{m} = 1·5$ and a few of the succeeding values, the second and third of the series (113) were summed directly as far as $\mathfrak{m}^{-5}$ inclusively, and the remainders were calculated from the formulæ (114). Two columns are annexed, which give the values of $\mathfrak{m}^2 k$ and $\mathfrak{m}^2 k'$, and exhibit the law of the variation of the two parts of the force F, when the radius of the cylinder varies, the nature of the fluid and time of oscillation remaining unchanged. Four significant figures are retained in all the results.

$\mathfrak{m}$	k	k'	$\mathfrak{m}^2 k$	$\mathfrak{m}^2 k'$	$\mathfrak{m}$	k	k'	$\mathfrak{m}^2 k$	$\mathfrak{m}^2 k'$
0	∞	∞	0	0	2·1	1·677	·7822	7·395	3·450
·1	19·70	48·63	·1970	·4863	2·2	1·646	·7421	7·966	3·592
·2	9·166	16·73	·3666	·6691	2·3	1·618	·7059	8·557	3·734
·3	6·166	9·258	·5549	·8332	2·4	1·592	·6730	9·168	3·877
·4	4·771	6·185	·7633	·9896	2·5	1·568	·6430	9·799	4·019
·5	3·968	4·567	·9920	1·142	2·6	1·546	·6154	10·45	4·160
·6	3·445	3·589	1·240	1·292	2·7	1·526	·5902	11·12	4·303
·7	3·082	2·936	1·510	1·439	2·8	1·507	·5669	11·81	4·444
·8	2·812	2·477	1·800	1·585	2·9	1·489	·5453	12·52	4·586
·9	2·604	2·137	2·110	1·731	3·0	1·473	·5253	13·25	4·728
1·0	2·439	1·876	2·439	1·876	3·1	1·457	·5068	14·01	4·870
1·1	2·306	1·678	2·790	2·021	3·2	1·443	·4895	14·78	5·012
1·2	2·194	1·503	3·160	2·164	3·3	1·430	·4732	15·57	5·154
1·3	2·102	1·365	3·552	2·307	3·4	1·417	·4581	16·38	5·296
1·4	2·021	1·250	3·961	2·450	3·5	1·405	·4439	17·21	5·437
1·5	1·951	1·153	4·389	2·595	3·6	1·394	·4305	18·06	5·580
1·6	1·891	1·069	4·841	2·739	3·7	1·383	·4179	18·93	5·721
1·7	1·838	·9965	5·312	2·880	3·8	1·373	·4060	19·82	5·863
1·8	1·791	·9332	5·804	3·024	3·9	1·363	·3948	20·73	6·005
1·9	1·749	·8767	6·314	3·165	4·0	1·354	·3841	21·67	6·145
2·0	1·711	·8268	6·845	3·307	∞	1	0	∞	∞

The numerical calculation by means of the formulæ (103), (104), (105) becomes very laborious when many values of the functions are required. The difficulty of the calculation increases with the value of $\mathfrak{m}$ for two reasons, first, the calculation of the functions M_0, &c. becomes longer, and secondly, the moduli of the numerator and denominator of the fraction in the right-hand member of (105) go on decreasing, so that greater and greater accuracy is required in the calculation of the functions M_0, &c., and of the products LM_0, &c., in order to ensure a given degree of accuracy in the result. The calculation by the descending series (113) is on the contrary very easy.

It will be found that the first differences of $\mathfrak{m}^2 k'$ and of $\mathfrak{m}^2 (k-1)$ are nearly constant, except near the very beginning of the table. Hence in the earlier part of the table the value of k or k' for a value of $\mathfrak{m}$ not found in the table will be best got by finding $\mathfrak{m}^2 k - \mathfrak{m}^2$ or $\mathfrak{m}^2 k'$ by interpolation, and thence passing to the value of k or k'. Very near the beginning of the table, interpolation would not succeed, but in such a case recourse may be had to the formulæ (103), (104), (105), the calculation of which is comparatively easy whem $\mathfrak{m}$ is small. It did not seem worth while to extend the table beyond $\mathfrak{m} = 4$, because where $\mathfrak{m}$ is greater than 4, the series (113) are so rapidly convergent that k and k' may be calculated to a sufficient degree of accuracy with extreme facility.

38. Let us now examine the progress of the functions k and k'.

When $\mathfrak{m}$ is very small, we may neglect the powers of $\mathfrak{m}$ in the numerator and denominator of the fraction in the right-hand member of equation (105), retaining only the logarithms and the constant terms. We thus get

$$k + \sqrt{-1}\, k' = 1 - \frac{\mathfrak{m}^{-2}\sqrt{-1}}{L - \frac{\pi}{4}\sqrt{-1}},$$

whence

$$\mathfrak{m}^2 (k-1) = \frac{\frac{\pi}{4}}{L^2 + \left(\frac{\pi}{4}\right)^2}, \qquad \mathfrak{m}^2 k' = \frac{-L}{L^2 + \left(\frac{\pi}{4}\right)^2}, \quad . \quad . \quad . \quad . \quad (115)$$

L being given by (102) and (104), or (104) and (106). When $\mathfrak{m}$ vanishes, L, which involves the logarithm of $\mathfrak{m}^{-1}$, becomes infinite, but ultimately increases more slowly than if it varied as $\mathfrak{m}$ affected with any negative index however small. Hence it appears from (115), that $k-1$ and k' are expressed by $\mathfrak{m}^{-2}$ multiplied by two functions of $\mathfrak{m}$ which, though they ultimately vanish with $\mathfrak{m}$, decrease very slowly, so that a considerable change in $\mathfrak{m}$ makes but a small change in these functions. Now when the radius a of the cylinder varies, everything else remaining the same, $\mathfrak{m}$ varies as a, and in general the parts of the force F on which depend the alteration in the time of vibration, and the diminution in the arc of oscillation, vary as $a^2 k$, $a^2 k'$, respectively. Hence in the case of a cylinder of small radius, such as the wire used to support a sphere in a pendulum experiment, a considerable change in the radius of the cylinder produces a comparatively small change in the part of the alteration in the time and arc of vibration which is due to the resistance experienced by the wire. The simple formulæ (115) are accurate enough for the fine wires usually employed in such experiments if the theory itself be applicable; but reasons will presently be given for regarding the application of the theory to such fine wires as extremely questionable.

From $\mathfrak{m} = \cdot 3$ or $\cdot 4$ to the end of the table, the first differences of each of the functions $\mathfrak{m}^2(k-1)$ and $\mathfrak{m}^2 k'$ remain nearly constant. Hence for a considerable range of values of $\mathfrak{m}$, each of the functions may be expressed pretty accurately by $A + B\mathfrak{m}$. When $\mathfrak{m}$ is at all large, the first two terms in the 2nd and 3rd of the formulæ (113) will give k and k' with considerable accuracy, because, independently of the decrease of the successive quantities $\mathfrak{m}^{-1}$, $\mathfrak{m}^{-2}$, $\mathfrak{m}^{-3}$..., the coefficients of $\mathfrak{m}^{-1}$ and $\mathfrak{m}^{-2}$ are considerably larger than those of several of the succeeding powers. If we neglect in these formulæ the terms after u_2, we get

$$k = 1 + \sqrt{2}.\mathfrak{m}^{-1}, \quad k' = \sqrt{2}.\mathfrak{m}^{-1} + \tfrac{1}{2}\mathfrak{m}^{-2}.$$

It may be remarked that these approximate expressions, regarded as functions of the radius a, have precisely the same form as the exact expressions obtained for a sphere, the coefficients only being different.

Section IV.

Determination of the motion of a fluid about a sphere which moves uniformly with a small velocity. Justification of the application of the solutions obtained in Sections II. *and* III. *to cases in which the extent of oscillation is not small in comparison with the radius of the sphere or cylinder. Discussion of a difficulty which presents itself with reference to the uniform motion of a cylinder in a fluid.*

39. Let a sphere move in a fluid with a uniform velocity V, its centre moving in a right line; and let the rest of the notation be the same as in Section II. Conceive a velocity equal and opposite to that of the sphere impressed both on the sphere and on the fluid, which will not affect the relative motion of the sphere and fluid, and will reduce the determination of the motion of the fluid to a problem of steady motion. Then we have for the equations of condition

$$R = 0, \qquad \Theta = 0, \text{ when } r = a; \quad \ldots \ldots \quad (116)$$

$$R = -V\cos\theta, \quad \Theta = V\sin\theta, \text{ when } r = \infty. \quad \ldots \ldots \quad (117)$$

The equations of condition, as well as the equations of motion, may be satisfied by supposing ψ to have the form $\sin^2\theta f(r)$. We get from (20′), by the same process as that by which (33), (34) were obtained,

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f(r) = 0, \quad \ldots \ldots \quad (118)$$

the only difference being that in the present case the equation (20′) cannot be replaced by the two (22), (23), which become identical, inasmuch as the velocity of the fluid is independent of the time.

The integral of (118) is

$$f(r) = Ar^{-1} + Br + Cr^2 + Dr^4, \quad \ldots \ldots \quad (119)$$

which gives

$$R = \frac{1}{r^2 \sin\theta} \frac{d\psi}{d\theta} = 2\cos\theta\,(Ar^{-3} + Br^{-1} + C + Dr^2),$$

$$\Theta = -\frac{1}{r\sin\theta}\frac{d\psi}{dr} = \sin\theta\,(Ar^{-3} - Br^{-1} - 2C - 4Dr^2).$$

The first of the equations of condition (117) requires that

$$D = 0, \qquad C = -\tfrac{1}{2}V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

It is particularly to be remarked that inasmuch as the two arbitrary constants C, D are determined by the first of the conditions (117), none remain whereby to satisfy the second. Nevertheless it happens that the second of these conditions leads to precisely the same equations (120) as the first. The equations of condition (116) give

$$A = -\tfrac{1}{4}Va^3, \qquad B = \tfrac{3}{4}Va;$$

whence

$$\psi = \tfrac{1}{4}Va^2\left(-\frac{2r^2}{a^2} + \frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta,^* \quad . \quad . \quad . \quad . \quad . \quad (121)$$

$$R = -V\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\cos\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

$$\Theta = V\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

If now we wish to obtain the solution of the problem in its original shape, in which the sphere is in motion and the fluid at rest, except so far as it is disturbed by the sphere, we have merely to add $V\cos\theta$, $-V\sin\theta$, $\frac{1}{2}Vr^2\sin^2\theta$ to the expressions for R, Θ, ψ. We get from (121)

$$\psi = \tfrac{1}{4}Va^2\left(\frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (124)$$

40. Let us now return to the problem of Section II.; let us suppose the time of oscillation to increase indefinitely, and examine what equation (40) becomes in the limit.

When τ becomes infinite, n, and therefore m, vanishes; the expression within brackets in (40) takes the form $\infty - \infty$, and its limiting value is easily found by the ordinary methods. We must retain the m^2 in the coefficient of t, because t is susceptible of unlimited increase. We get in the limit

$$\psi = \tfrac{1}{4}a^2 c\epsilon^{\mu' m^2 t}\left(\frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

If now we put V for $\dfrac{d\xi}{dt}$, the velocity of the sphere, we get from (39), $c\epsilon^{\mu' m^2 t} = V$. After substituting in (125), the equation will remain unchanged when we pass from the symbolical to the real values of ψ and V, and thus (125) will be reduced to (124).

* I have already had occasion, in treating of another subject, to publish the solution expressed by this equation, which I had obtained as a limiting case of the problem of a ball pendulum. See *Philosophical Magazine* for May 1848, p. 343.

41. It appears then that by supposing the rate of alteration of the velocity of the sphere to decrease indefinitely, we obtain from the solution of the problem of Section II. the same result as was obtained in Art. 39, by treating the motion as steady. As yet, however, the method of Art. 40 is subject to a limitation from which that of Art. 39 is free. In obtaining equation (40), it was supposed that the maximum excursion of the sphere was small in comparison with its radius. Retaining this restriction while we suppose τ to become very large, we are obliged to suppose c to become very small, so that the velocity of the sphere is not merely so small that we may neglect terms depending upon its square, a restriction to which Art. 39 is also subject, but so extremely small that the space passed over by the sphere in even a long time is small in comparison with its radius.

We have seen, however, that on supposing τ very large in (40) we obtain a result identical with (124), not merely a result with which (124) becomes identical when the restriction above mentioned is introduced. This leads to the supposition that the solution expressed by (40) is in fact more general than would appear from the way in which it was obtained. That such is really the case may be shewn by a slight modification of the analysis. Instead of referring the fluid to axes fixed in space, refer it to axes originating at the centre of the sphere, and moveable with it. In the general equations of motion, the terms which contain differential coefficients taken with respect to the co-ordinates will remain unchanged, inasmuch as they represent the very same limiting ratios as before: it is only those in which differentiation with respect to t occurs that will be altered. If $\frac{d'}{dt}$ be the symbol of differentiation with respect to t when the fluid is referred to the moveable axes, we shall have

$$\frac{d}{dt} = \frac{d'}{dt} - \frac{d\xi}{dt}\frac{d}{dx};$$

but the terms arising from $\frac{d\xi}{dt}\frac{d}{dx}$ are of the order of the square of the velocity, and are therefore to be neglected. Hence the general equations have the same form whether the fluid be referred to the fixed or moveable axes. But on the latter supposition the equations of condition (30) become rigorously exact. Hence equation (40) gives correctly the solution of the problem, independently of the restriction that the maximum excursion of the sphere be small compared with its radius, provided we suppose the polar co-ordinates r, θ measured from the centre of the sphere in its actual, not its mean position. Similar remarks apply to the problem of the cylinder. Moreover, in the case of a sphere oscillating within a concentric spherical envelope, it is not necessary, in order to employ the solution obtained in Section II., that the maximum excursion of the sphere be small compared with its radius; it is sufficient that it be small compared with the radius of the envelope.

These are points of great importance, because the excursions of an oscillating sphere in a pendulum experiment are not by any means extremely small compared with the radius of the sphere; and in the case of a narrow cylinder, such as the suspending wire, so far from the maximum excursion being small compared with the radius of the cylinder, it is, on the contrary, the radius which is small compared with the maximum excursion.

42. Let us now return to the case of the uniform motion of a sphere. In order to obtain directly the expression for the resistance of the fluid, it would be requisite first to find p, then to get P_r and T_θ from (46), or at least to get the values of these functions for $r = a$, and lastly to substitute in (47) and perform the integration. We should obtain p by integrating the expression for dp got from (16) and (17). It would be requisite first to express u and q in terms of ψ, then to transform the expression for dp so as to involve polar co-ordinates, and then substitute for ψ its value given by (121); or else to express the right-hand member of (121) by the co-ordinates x, ϖ, and substitute in the expression for dp*. We have seen, however, that the results applicable to uniform motion may be deduced as limiting cases of those which relate to oscillatory motion, and consequently, we may make use of the expression for F already worked out. Writing V for $c\epsilon^{\sqrt{-1}nt}$ in the first equation of Art. 20, expressing m in terms of n, and then making n vanish, we get

$$-F = 6\pi\mu'\rho a V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

and $-F$ is the resistance required.

This equation may be employed to determine the terminal velocity of a sphere ascending or descending in a fluid, provided the motion be so slow that the square of the velocity may be neglected. It has been shewn experimentally by Coulomb †, that in the case of very slow motions, the resistance of a fluid depends partly on the square and partly on the first power of the velocity. The formula (126) determines, in the particular case of a sphere, that part of the whole resistance which depends on the first power of the velocity, even though the part which depends on the square of the velocity be not wholly insensible.

It is particularly to be remarked, that according to the formula (126), the resistance varies not as the surface but as the radius of the sphere, and consequently the quotient of the resistance divided by the mass increases in a higher ratio, as the radius diminishes, than if the resistance varied as the surface. Accordingly, fine powders remain nearly suspended in a fluid of widely different specific gravity.

43. When the motion is so slow that the part of the resistance which depends on the square of the velocity may be neglected, we have, supposing V to be the terminal velocity,

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* The equations (16), (17) give, after a troublesome transformation to polar co-ordinates,

$$\frac{dp}{dr} = \frac{\mu}{r^2 \sin\theta} \frac{d}{d\theta} \left(\frac{d^2}{dr^2} + \frac{\sin\theta}{r^2} \frac{d}{d\theta} \frac{1}{\sin\theta} \frac{d}{d\theta} - \frac{\rho}{\mu} \frac{d}{dt} \right) \psi, \quad (a)$$

$$\frac{dp}{d\theta} = -\frac{\mu}{\sin\theta} \frac{d}{dr} \left(\frac{d^2}{dr^2} + \frac{\sin\theta}{r^2} \frac{d}{d\theta} \frac{1}{\sin\theta} \frac{d}{d\theta} - \frac{\rho}{\mu} \frac{d}{dt} \right) \psi. \quad (b)$$

The expression for dp got from these equations is an exact differential by virtue of the equation which determines ψ; and in the problems considered in Section II. and in the present Section ψ has the form $\Psi \sin^2\theta$, where Ψ is independent of θ. Hence we get from (b), by integrating partially with respect to θ,

$$p = \mu \cos\theta \frac{d}{dr} \left(\frac{d^2}{dr^2} - \frac{2}{r^2} - \frac{\rho}{\mu} \frac{d}{dt} \right) \Psi. \quad (c)$$

It is unnecessary to add an arbitrary function of r, because if $\lambda(r)$ be such a function which we suppose added to the right-hand member of (c), we must determine λ by substituting in (a). The resulting expression for $\lambda'(r)$ cannot contain θ, inasmuch as the expression for dp is an exact differential, but it is composed of terms which all involve $\cos\theta$ as a factor, and therefore we know, without working out, that these terms must destroy one another. Hence $\lambda(r)$ must be constant, or at most be a function of t, which we may suppose included in Π. $\lambda(r)$ will in fact be equal to zero if Π be the equilibrium pressure at the depth at which $\int g dz'$ vanishes.

† *Mémoires de l'Institut*, Tom. III. p. 246.

$-F = \frac{4}{3}\pi g(\sigma - \rho)a^3$, where g is the force of gravity, and σ, which is supposed greater than ρ, the density of the sphere. Substituting in (126) we get

$$V = \frac{2g}{9\mu'}\left(\frac{\sigma}{\rho} - 1\right)a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

Let us apply this equation to determine the terminal velocity of a globule of water forming part of a cloud. Putting $g = 386$, $\mu' = (.116)^2$, an inch being the unit of length, and supposing $\sigma\rho^{-1} - 1 = 1000$, in order to allow a little for the rarity of the air at the height of the cloud, we get $V = 6372 \times 1000a^2$. Thus, for a globule the one thousandth of an inch in diameter, we have $V = 1.593$ inch per second. For a globule the one ten thousandth of an inch in diameter, the terminal velocity would be a hundred times smaller, so as not to amount to the one sixtieth part of an inch per second.

We may form a very good judgment of the magnitude of that part of the resistance which varies as the square of the velocity, and which is the only kind of resistance that could exist if the pressure were equal in all directions, by calculating the numerical value of the resistance according to the common theory, imperfect though it be. It follows from this theory that if h be the height due to the velocity V, the resistance is to the weight as $3\rho h$ to $8\sigma a$. For $V = 1.593$ inch per second, the resistance is not quite the one four hundredth part of the weight; and for a sphere only the one ten thousandth of an inch in diameter, moving with the velocity calculated from the formula (127), the ratio of the resistance to the weight would be ten times as small. The terminal velocities of the globules calculated from the common theory would be 32.07 and 10.14 inches per second, instead of only 1.593 and .01593 inch. It appears then that the apparent suspension of the clouds is mainly due to the internal friction of air.

44. The resistance to the globule has here been determined as if the globule were a solid sphere. In strictness, account ought to be taken of the relative motion of the fluid particles forming the globule itself. Although it may readily be imagined that no material change would thus be made in the numerical result, it may be worth while to point out the mode of solution of the problem. Suppose the globule preserved in a strictly spherical shape by capillary attraction, which will very nearly indeed be the case. Conceive a velocity equal and opposite to that of the globule impressed both on the globule and on the surrounding fluid, which will reduce the problem to one of steady motion. Let ψ_1, &c. refer to the fluid forming the globule, and assume $\psi_1 = f_1(r)\sin^2\theta$. Then we get on changing the constants in (119)

$$f_1(r) = A_1r^{-1} + B_1r + C_1r^2 + D_1r^4.$$

The arbitrary constants A_1, B_1 vanish by the condition that the velocity shall not become infinite at the centre. There remain the two arbitrary constants C_1, D_1 to be determined, in addition to those which appeared in the former problem. But we have now four instead of two equations of condition which have to be satisfied at the surface of the sphere, which are that

$$R = 0, \quad R_1 = 0, \quad \Theta = \Theta_1, \quad T_\theta = T_{1\theta}, \quad \text{when } r = a. \quad . \quad . \quad . \quad (128)$$

We shall thus have the same number of arbitrary constants as conditions to be satisfied. Now $T_{1\theta}$ will involve μ_1 as a coefficient, just as T_θ involves $\mu'\rho$ or μ; and μ_1, which refers to water, is much larger than μ, which refers to air, although μ' is larger than μ_1'. Hence the results will be nearly the same as if we had taken $\mu_1 = \infty$, or regarded the sphere as solid.

If, however, instead of a globule of liquid descending in a gas we have a very small bubble ascending in a liquid, we must not treat the bubble as a solid sphere. We may in this case also neglect the motion of the fluid forming the sphere, but we have now arrived at the other extreme case of the general problem, and the two equations of condition which have to be satisfied at the surface of the sphere are that $R = 0$ and $T_\theta = 0$ when $r = a$, instead of $R = 0$ and $\Theta = 0$, when $r = a$.

The equation of condition $T_\theta = 0$ which applies to a bubble, as well as the fourth of equations (128), will not be the true equations, if forces arising from internal friction exist in the superficial film of a fluid which are of a different order of magnitude from those which exist throughout the mass. At the end of the memoir already referred to, Coulomb states that in very slow motions the resistance of bodies not completely immersed in a liquid is much greater than that of bodies wholly immersed, and promises to communicate a second memoir in continuation of the first. This memoir, so far as I can find out, has never appeared. Should the existence of such forces in the superficial film of a liquid be made out, the results deduced from the theory of internal friction will be modified in a manner analogous to that in which the results deduced from the common principles of hydrostatics are modified by capillary attraction. It may be remarked that we have nothing to do with forces of this kind in considering the motion of pendulums in air, or even in considering the oscillations of a sphere in water, except as regards the very minute fraction of the whole effect which relates to the resistance experienced by the suspending wire in the immediate neighbourhood of the free surface.

It may readily be seen that the effect of a set of forces in the superficial film of a liquid offering a peculiar resistance to the relative motion of the particles would be, to make the resistance of a gas to a descending globule agree still more clearly with the result obtained by regarding the globule as solid, while the resistance experienced by an ascending bubble would be materially increased, and made to approach to that which would belong to a solid sphere of the same size without mass, or more strictly, with a mass only equal to that of the gas forming the bubble. Possibly the determination of the velocity of ascent of very small bubbles may turn out to be a good mode of measuring the amount of friction in the superficial film of a liquid, if it be true that forces of this kind have any existence. But any investigation relating to such a subject would at present be premature.

45. Let us now attempt to determine the uniform motion of a fluid about an infinite cylinder. Employing the notation of Section III, and reducing the problem to one of steady motion as in Art. 39, we obtain the same equations of condition (116), (117), as in the case of the sphere. Assuming $\chi = \sin\theta F(r)$, and substituting in the equation obtained from (69) by transforming to polar co-ordinates and leaving out the terms which involve $\frac{d}{dt}$, we get

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 F(r) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

The integral of this equation may readily be obtained by multiplying the last term of the operating factor by $(1+\delta)^2$, integrating the transformed equation, and then making δ vanish. It is

$$F(r) = Ar^{-1} + Br + Cr\log r + Dr^3, \quad . \quad . \quad . \quad . \quad . \quad (130)$$

which gives

$$R = \frac{d\chi}{rd\theta} = (Ar^{-2} + B + C\log r + Dr^2)\cos\theta,$$

$$\Theta = -\frac{d\chi}{dr} = (Ar^{-2} - B - C - C\log r - 3Dr^2)\sin\theta.$$

The first of the equations of condition (117) requires that

$$C = 0, \qquad D = 0, \qquad B = -V,$$

which also satisfies the second. We have thus only one arbitrary constant left whereby to satisfy the two equations of condition (116), and the same value of A will not satisfy these two equations.

46. It appears then that the supposition of steady motion is inadmissible. It will be remembered that, in the case of the sphere, the solution of the problem was only possible because it so happened that the values of two arbitrary constants determined by satisfying the first of the equations of condition (117) satisfied also the second, which indicates that the solution was to a certain extent tentative. We have evidently a right to conceive a sphere or infinite cylinder to exist at rest in an infinite mass of fluid also at rest, to suppose the sphere or cylinder to be then moved with a uniform velocity V, and to propose for determination the motion of the fluid at the end of the time t. But we have no right to assume that the motion approaches a permanent state as t increases indefinitely. We may follow either of two courses. We may proceed to solve the general problem in which the sphere or cylinder is supposed to move from rest, and then examine what results we obtain by supposing t to increase indefinitely, or else we may assume *for trial* that the motion is steady, and proceed to inquire whether we can satisfy all the conditions of the problem on this supposition. The former course would have the disadvantage of requiring a complicated analysis for the sake of obtaining a comparatively simple result, and it is even possible that the solution of the problem might baffle us altogether; but if we adopt the latter course, we must not forget that the equations with which we work are only provisional.

It might be objected that the impossibility of satisfying the conditions of the problem on the hypothesis of steady motion arose from our assumption that $\sin\theta$ was a factor of χ, the other factor being independent of θ. This however is not the case. For, for given values of r and t, χ is a finite function of θ from $\theta = 0$ to $\theta = \pi$. We have a right to suppose χ to vanish at any point of the axis of x positive that we please; and if we suppose χ to vanish at one such point, it may be shewn as in the note to Art. 15, that χ will vanish at all points of the axis of x positive or x negative. Hence χ may be expanded in a convergent series of sines of θ and its multiples; and since χ and its derivatives with respect to θ alter continuously with θ, the expansions of the derivatives will be got by direct differentiation*. This being

* See a paper *On the Critical Values of the Sums of Periodic Series.* Camb. Phil. Trans. Vol. VIII. p. 533.

true for all other pairs of values of r and t, χ can in general be expanded in a convergent series of sines of θ and its multiples; but the coefficients, instead of being constant, will be functions of r and t, or in the particular case of steady motion, functions of r alone. Now a very slight examination of the general equations will suffice to shew that the coefficients of the sines of the different multiples of θ remain perfectly independent throughout the whole process, and consequently had we employed the general expansion, we should have been led to the very same conclusions which have been deduced from the assumed form of χ.

47. If we take the impossibility of the existence of a limiting state of motion, which has just been established, in connexion with the results obtained in Section III., we shall be able to understand the general nature of the motion of the fluid around an infinite cylinder which is at first at rest, and is then moved on indefinitely with a uniform velocity.

The fluid being treated as incompressible, the first motion which takes place is impulsive. Since the terms depending on the internal friction will not appear in the calculation of this motion, we may employ the ordinary equations of hydrodynamics. The result, which is easily obtained, is

$$Rdr + \Theta r d\theta = d\phi, \quad \text{where } \phi = -\frac{Va^2}{r}\cos\theta^* \quad . \quad . \quad . \quad . \quad . \quad (131)$$

As the cylinder moves on, it carries more and more of the fluid with it, in consequence of friction. For the sake of precision, let the quantity carried by the element dl of the cylinder be defined to be that which, moving with the velocity V, would have the same momentum in the direction of the motion that is actually possessed by the elementary portion of fluid which is contained between two parallel infinite planes drawn perpendicular to the axis of the cylinder, at an interval dl, the particles composing which are moving with velocities that vary from V to zero in passing from the surface outwards. The pressure of the cylinder on the fluid continually tends to increase the quantity of fluid which it carries with it, while the friction of the fluid at a distance from the cylinder continually tends to diminish it. In the case of a sphere, these two causes eventually counteract each other, and the motion becomes uniform. But in the case of a cylinder, the increase in the quantity of fluid carried continually gains on the decrease due to the friction of the surrounding fluid, and the quantity carried increases indefinitely as the cylinder moves on. The rate at which the quantity carried is increased, decreases continually, because the motion of the fluid in the neighbourhood of the cylinder becomes more and more nearly a simple motion of translation equal to that of the cylinder itself, and therefore the rate at which the quantity of fluid carried is increased would become smaller and smaller, even were no resistance offered by the surrounding fluid.

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* According to these equations, the fluid flows past the surface of the cylinder with a finite velocity. At the end of the small time t' after the impact, the friction has reduced the velocity of the fluid in contact with the cylinder to that of the cylinder itself, and the tangential velocity alters very rapidly in passing from the surface outwards. At a small distance s from the surface of the cylinder, the relative velocity of the fluid and the cylinder, in a direction tangential to the surface, is a function of the independant variables t', s, which vanishes with s for any given value of t', however small, but which for any given value of s, however small, approaches indefinitely to the quantity determined by (131) as t vanishes. The communication of lateral motion is similar to the communication of temperature when the surface of a body has its temperature instantaneously raised or lowered by a finite quantity.

The correctness of this explanation is confirmed by the following considerations. Suppose that $F(r)$ had been given by the equation

$$F(r) = Ar^{-1} + Br + Cr^{1-\delta} + Dr^3$$

instead of (130), δ being a small positive quantity. On this supposition it would have been possible to satisfy all the equations of condition, and therefore steady motion would have been possible. By determining the arbitrary constants, and substituting in χ, we should have obtained

$$\psi = aV\left\{-\frac{\delta}{2-\delta}\frac{a}{r} - \frac{r}{a} + \frac{2}{2-\delta}\left(\frac{r}{a}\right)^{1-\delta}\right\}\sin\theta,$$

$$R = V\left\{-\frac{\delta}{2-\delta}\left(\frac{a}{r}\right)^2 - 1 + \frac{2}{2-\delta}\left(\frac{r}{a}\right)^{-\delta}\right\}\cos\theta,$$

$$\Theta = V\left\{-\frac{\delta}{2-\delta}\left(\frac{a}{r}\right)^2 + 1 - \frac{2(1-\delta)}{2-\delta}\left(\frac{r}{a}\right)^{-\delta}\right\}\sin\theta.$$

Since δ is supposed to be extremely small, it follows from these expressions that when r is not greater than a moderate multiple of a, the velocities R, Θ are extremely small; but, however small be δ, we have only to go far enough from the cylinder in order to find velocities as nearly equal to $-V\cos\theta$, $+V\sin\theta$ as we please. But the distance from the cylinder to which we must proceed in order to find velocities R, Θ which do not differ from their limiting values $-V\cos\theta$, $+V\sin\theta$ by more than certain given quantities, increases indefinitely as δ decreases. Hence, restoring to the fluid and the cylinder the velocity V, we see that in the neighbourhood of the cylinder the motion of the fluid does not sensibly differ from a motion of translation, the same as that of the cylinder itself, while the distance to which the cylinder exerts a sensible influence in disturbing the motion of the fluid increases indefinitely as δ decreases.

48. When we have formed the equations of motion of a fluid on any particular dynamical hypothesis, it becomes a perfectly definite mathematical problem to determine the motion of the fluid when a given solid, initially at rest as well as the fluid, is moved in a given manner, or to discuss the character of the analytical solution in any extreme case proposed. It is quite another thing to enquire how far the principles which furnished the mathematical data of the problem may be modified in extreme cases, or what will be the nature of the actual motion in such cases. Let us regard in this point of view the case considered in the preceding article as a mathematical problem. When the quantity of fluid carried with the cylinder becomes considerable compared with the quantity displaced, it would seem that the motion must become unstable, in the sense in which the motion of a sphere rolling down the highest generating line of an inclined cylinder may be said to be unstable. But besides the instability, it may not be safe in such an extreme case to neglect the terms depending on the square of the velocity, not that they become unusually large in themselves, but only unusually large compared with the terms retained, because when the relative motions of neighbouring portions of the fluid become very small, the tangential pressures which arise from friction become very small likewise.

Now the general character of the motion must be nearly the same whether the velocity of the cylinder be constant, or vary slowly with the time, so that it does not vary materially when the cylinder passes through a space equal to a small multiple of its radius. To return to the problem considered in Section III., it would seem that when the radius of the cylinder is very small, the motion which would be expressed by the formulæ of that Section would be unstable. This might very well be the case with the fine wires used in supporting the spheres employed in pendulum experiments. If so, the quantity of fluid carried by the wire would be diminished, portions being continually left behind and forming eddies. The resistance to the wire would on the whole be increased, and would moreover approximate to a resistance which would be a function of the velocity. Hence, so far as depends on the wire, the arc of oscillation would be more affected by the resistance of the air than would follow from the formulæ of Section III. Whether the effect on the time of oscillation would be greater or less than that expressed by the formulæ is difficult to say, because the increase of resistance would tend to increase the effect on the time of vibration, while on the other hand the approximation of the law of resistance to that of a function of the velocity would tend to diminish it.

SECTION V.

On the effect of internal friction in causing the motion of a fluid to subside. Application to the case of oscillatory waves.

49. We have already had instances of the effect of friction in causing a gradual subsidence in the motion of a solid oscillating in a fluid; but a result may easily be obtained from the equations of motion in their most general shape, which shews very clearly the effect of friction in continually consuming a portion of the work of the forces acting on the fluid.

Let P_1, P_2, P_3 be the three normal, and T_1, T_2, T_3 the three tangential pressures in the direction of three rectangular planes parallel to the co-ordinate planes, and let D be the symbol of differentiation with respect to t when the particle and not the point of space remains the same. Then the general equations applicable to a heterogeneous fluid, (the equations (10) of my former paper,) are

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} = 0, \quad . \quad . \quad . \quad . \quad . \quad (132)$$

with the two other equations which may be written down from symmetry. The pressures P_1, &c. are given by the equations

$$P_1 = p - 2\mu\left(\frac{du}{dx} - \delta\right), \quad T_1 = -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right), \quad . \quad . \quad . \quad (133)$$

and four other similar equations. In these equations

$$3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

At the end of the time t let V be the *vis viva* of a limited portion of the fluid, occupying the space which lies inside the closed surface S, and let $V + DV$ be the *vis viva* of the same mass at the end of the time $t + Dt$. Then

$$V = \iiint \rho \, (u^2 + v^2 + w^2) \, dx \, dy \, dz,$$

$$DV = 2Dt \iiint \rho \left(u \frac{Du}{Dt} + v \frac{Dv}{Dt} + w \frac{Dw}{Dt} \right) dx \, dy \, dz, \quad . \quad . \quad . \quad (135)$$

the triple integrals extending throughout the space bounded by S. Substituting now for $\frac{Du}{Dt}$, &c. their values given by the equations of the system (132), we get

$$DV = 2Dt \iiint \rho \, (uX + vY + wZ) \, dx \, dy \, dz$$

$$- 2Dt \iiint \left\{ u \left(\frac{dP_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} \right) + v \left(\frac{dP_2}{dy} + \frac{dT_1}{dz} + \frac{dT_3}{dx} \right) \right.$$

$$\left. + w \left(\frac{dP_3}{dz} + \frac{dT_2}{dx} + \frac{dT_1}{dy} \right) \right\} dx \, dy \, dz. \quad . \quad . \quad . \quad . \quad . \quad (136)$$

The first part of this expression is evidently twice the work, during the time Dt, of the external forces which act all over the mass. The second part becomes after integration by parts

$$- 2Dt \iint (uP_1 + vT_3 + wT_2) \, dy \, dz - 2Dt \iint (vP_2 + wT_1 + uT_3) \, dz \, dx$$

$$- 2Dt \iint (wP_3 + uT_2 + vT_1) \, dx \, dy$$

$$+ 2Dt \iiint \left\{ \frac{du}{dx} P_1 + \frac{dv}{dy} P_2 + \frac{dw}{dz} P_3 + \left(\frac{dv}{dz} + \frac{dw}{dy} \right) T_1 + \left(\frac{dw}{dx} + \frac{du}{dz} \right) T_2 \right.$$

$$\left. + \left(\frac{du}{dy} + \frac{dv}{dx} \right) T_3 \right\} dx \, dy \, dz.$$

The double integrals in this expression are to be extended over the whole surface S. If dS be an element of this surface, l', m', n' the direction-cosines of the normal drawn outwards at dS, we may write $l'dS$, $m'dS$, $n'dS$ for $dydz$, $dzdx$, $dxdy$. The second part of DV thus becomes

$$- 2Dt \iint \{ u (l'P_1 + m'T_3 + n'T_2) + v (m'P_2 + n'T_1 + l'T_3) + w (n'P_3 + l'T_2 + m'T_1) \} \, dS.$$

The coefficients of u, v, w in this expression are the resolved parts, in the direction of x, y, z, of the pressure on a plane in the direction of the elementary surface dS, whence it appears that the expression itself denotes twice the work of the pressures applied to the surface of the portion of fluid that we are considering.

On substituting for P_1, &c. their values given by the equations (133), (134), we get for the last part of DV

$$+ 2Dt \iiint p \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) dx \, dy \, dz$$

$$- 2Dt \iiint \mu \left\{ 2 \left(\frac{du}{dx} \right)^2 + 2 \left(\frac{dv}{dy} \right)^2 + 2 \left(\frac{dw}{dz} \right)^2 - \frac{2}{3} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right)^2 \right.$$

$$\left. + \left(\frac{dv}{dz} + \frac{dw}{dy} \right)^2 + \left(\frac{dw}{dx} + \frac{du}{dz} \right)^2 + \left(\frac{du}{dy} + \frac{dv}{dx} \right)^2 \right\} dx \, dy \, dz.$$

In this expression p denotes, in the case of an elastic fluid, the pressure statically corresponding to the density which actually exists about the point whose co-ordinates are x, y, z, and the part of the expression which contains p denotes twice the work converted into *vis viva* in consequence of internal expansions, and arising from the forces on which the elasticity depends. The last part of the expression is essentially negative, or at least cannot be positive, and can only vanish in one very particular case. It denotes the *vis viva* consumed, or twice the work lost in the system during the time dt, in consequence of internal friction. According to the very important theory of Mr Joule, which is founded on a set of most striking and satisfactory experiments, the work thus apparently lost is in fact converted into heat, at such a rate, that the work expressed by the descent of 772 lbs through one foot, supplies the quantity of heat required to raise 1 lb. of water through 1^0 of Fahrenheit's thermometer.

50. The triple integral containing μ can only vanish when the differential coefficients of u, v, w satisfy the five following equations,

$$\left.\begin{array}{c} \dfrac{du}{dx}=\dfrac{dv}{dy}=\dfrac{dw}{dz}, \\ \dfrac{dv}{dz}+\dfrac{dw}{dy}=0, \qquad \dfrac{dw}{dx}+\dfrac{du}{dz}=0, \qquad \dfrac{du}{dy}+\dfrac{dv}{dx}=0. \end{array}\right\} \quad . \quad . \quad . \quad (137)$$

These equations give immediately the following expressions for the differentials of u, v, w, in which the co-ordinates alone are supposed to vary, the time being constant:

$$\left.\begin{array}{l} du=\delta dx-\omega'''dy+\omega''dz, \\ dv=\delta dy-\omega'dz\ +\omega'''dx, \\ dw=\delta dz-\omega''dx\ +\omega'dy. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

In these equations $\delta, \omega', \omega'', \omega'''$ are certain functions of which the forms are defined by the equations (138), but need not at present be considered. It follows from equations (138) that the motion of each element of the fluid within the surface S is compounded of a motion of translation, a motion of rotation, and a motion of dilatation alike in all directions. So far as regards the first two kinds of motion, the fluid element moves like a solid, and of course there is nothing to call internal friction into play. For the reasons stated in my former paper, I was led to assume that a motion of dilatation alike in all directions, (which of course can only exist in the case of an elastic fluid,) has no effect in causing the pressure to differ from the statical pressure corresponding to the actual density, that is, in occasioning a violation of the functional relation commonly supposed to exist between the pressure, density, and temperature. The reader will observe that this is a totally different thing from assuming that a motion of dilatation has no effect on the pressure at all.

When the fluid is incompressible $\delta=0$, and it may be proved without difficulty that $\omega', \omega'', \omega'''$ are constant, that is to say, constant so far as the co-ordinates are concerned. In this case we get by integrating equations (137)

$$\left.\begin{array}{l} u=a-\omega'''y+\omega''z, \\ v=b-\omega'z\ +\omega'''x, \\ w=c-\omega''x\ +\omega'y. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

Hence, in the case of an incompressible fluid, unless the whole mass comprised within the surface S move together like a solid, there cannot fail to be a certain portion of *vis viva* lost by internal friction. In the case of an elastic fluid, the motion which may take place without causing a loss of *vis viva* in consequence of friction is somewhat more general, and corresponds to velocities $u + \Delta u$, $v + \Delta v$, $w + \Delta w$, where u, v, w are the same as in (139), and

$$\Delta u = \delta x + 2(\alpha x + \beta y + \gamma z)\, x - \alpha (x^2 + y^2 + z^2),$$

with similar expressions for Δv and Δw. In these expressions α, β, γ are three constants symmetrically related to x, y, z, and δ is a constant which has the same relation to each of the co-ordinates *.

51. By means of the expression given in Art. 49, for the loss of *vis viva* due to internal friction, we may readily obtain a very approximate solution of the problem: To determine the rate at which the motion subsides, in consequence of internal friction, in the case of a series of oscillatory waves propagated along the surface of a liquid.

Let the vertical plane of xy be parallel to the plane of motion, and let y be measured vertically downwards from the mean surface; and for simplicity's sake suppose the depth of the fluid very great compared with the length of a wave, and the motion so small that the square of the velocity may be neglected. In the case of motion which we are considering, $udx + vdy$ is an exact differential $d\phi$ when friction is neglected, and

$$\phi = c\epsilon^{-my} \sin (mx - nt), \quad \ldots \ldots \quad (140)$$

where c, m, n are three constants, of which the last two are connected by a relation which it is not necessary to write down. We may continue to employ this equation as a near approximation when friction is taken into account, provided we suppose c, instead of being constant, to be a parameter which varies slowly with the time. Let V be the *vis viva* of a given portion of the fluid at the end of the time t, then

$$V = \rho c^2 m^2 \iiint \epsilon^{-2my}\, dx\, dy\, dz. \quad \ldots \ldots \quad (141)$$

But by means of the expression given in Art. 49, we get for the loss of *vis viva* during the time dt, observing that in the present case μ is constant, $w = 0$, $\delta = 0$, and $udx + vdy = d\phi$, where ϕ is independent of z,

$$4\mu dt \iiint \left\{ \left(\frac{d^2\phi}{dx^2}\right)^2 + \left(\frac{d^2\phi}{dy^2}\right)^2 + 2\left(\frac{d^2\phi}{dx\, dy}\right)^2 \right\} dx\, dy\, dz,$$

which becomes, on substituting for ϕ its value,

$$8\mu c^2 m^4\, dt \iiint \epsilon^{-2my}\, dx\, dy\, dz.$$

But we get from (141) for the decrement of *vis viva* of the same mass arising from the variation of the parameter c

$$-\, 2\rho m^2 c \frac{dc}{dt}\, dt \iiint \epsilon^{-2my}\, dx\, dy\, dz.$$

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* (*See Note* C *at the end.*)

Equating the two expressions for the decrement of *vis viva*, putting for m its value $2\pi\lambda^{-1}$, where λ is the length of a wave, replacing μ by $\mu'\rho$, integrating, and supposing c_0 to be the initial value of c, we get

$$c = c_0 \epsilon^{-\frac{16\pi^2\mu' t}{\lambda^2}}.$$

It will presently appear that the value of $\sqrt{\mu'}$ for water is about 0·0564, an inch and a second being the units of space and time. Suppose first that λ is two inches, and t ten seconds. Then $16\pi^2\mu' t\lambda^{-2} = 1·256$, and $c : c_0 :: 1 : 0·2848$, so that the height of the waves, which varies as c, is only about a quarter of what it was. Accordingly, the ripples excited on a small pool by a puff of wind rapidly subside when the exciting cause ceases to act.

Now suppose that λ is 40 fathoms or 2880 inches, and that t is 86400 seconds or a whole day. In this case $16\pi^2\mu' t\lambda^{-2}$ is equal to only 0·005232, so that by the end of an entire day, in which time waves of this length would travel 574 English miles, the height would be diminished by little more than the one two hundredth part in consequence of friction. Accordingly, the long swells of the ocean are but little allayed by friction, and at last break on some shore situated at the distance of perhaps hundreds of miles from the region where they were first excited.

52. It is worthy of remark, that in the case of a homogeneous incompressible fluid, whenever $udx + vdy + wdz$ is an exact differential, not only are the ordinary equations of fluid motion satisfied*, but the equations obtained when friction is taken into account are satisfied likewise. It is only the equations of condition which belong to the boundaries of the fluid that are violated. Hence any kind of motion which is possible according to the ordinary equations, and which is such that $udx + vdy + wdz$ is an exact differential, is possible likewise when friction is taken into account, provided we suppose a certain system of normal and tangential pressures to act at the boundaries of the fluid, so as to satisfy the equations of condition. The requisite system of pressures is given by the system of equations (133). Since μ disappears from the general equations (1), it follows that p is the same function as before. But in the first case the system of pressures at the surface was $P_1 = P_2 = P_3 = p$, $T_1 = T_2 = T_3 = 0$. Hence if ΔP_1 &c. be the additional pressures arising from friction, we get from (133), observing that $\delta = 0$, and that $udx + vdy + wdz$ is an exact differential $d\phi$,

$$\Delta P_1 = -2\mu\frac{d^2\phi}{dx^2}, \qquad \Delta P_2 = -2\mu\frac{d^2\phi}{dy^2}, \qquad \Delta P_3 = -2\mu\frac{d^2\phi}{dz^2}, \quad . \quad (142)$$

$$\Delta T_1 = -2\mu\frac{d^2\phi}{dy\,dz}, \qquad \Delta T_2 = -2\mu\frac{d^2\phi}{dz\,dx}, \qquad \Delta T_3 = -2\mu\frac{d^2\phi}{dx\,dy}. \quad . \quad (143)$$

Let dS be an element of the bounding surface, l', m', n' the direction-cosines of the normal drawn outwards, ΔP, ΔQ, ΔR the components in the direction of x, y, z of the additional

* It is here supposed that the forces X, Y, Z are such that $Xdx + Ydy + Zdz$ is an exact differential.

pressure on a plane in the direction of dS. Then by the formulæ (9) of my former paper applied to the equations (142), (143) we get

$$\Delta P = -2\mu\left\{l'\frac{d^2\phi}{dx^2} + m'\frac{d^2\phi}{dx\,dy} + n'\frac{d^2\phi}{dx\,dz}\right\}, \quad . \quad . \quad . \quad . \quad (144)$$

with similar expressions for ΔQ and ΔR, and ΔP, ΔQ, ΔR are the components of the pressure which must be applied at the surface, in order to preserve the original motion unaltered by friction.

53. Let us apply this method to the case of oscillatory waves, considered in Art. 51. In this case the bounding surface is nearly horizontal, and its vertical ordinates are very small, and since the squares of small quantities are neglected, we may suppose the surface to coincide with the plane of xz in calculating the system of pressures which must be supplied, in order to keep up the motion. Moreover, since the motion is symmetrical with respect to the plane of xy, there will be no tangential pressure in the direction of z, so that the only pressures we have to calculate are ΔP_2 and ΔT_3. We get from (140), (142), and (143), putting $y = 0$ after differentiation,

$$\Delta P_2 = -2\mu m^2 c \sin(mx - nt), \qquad \Delta T_3 = 2\mu m^2 c \cos(mx - nt). \quad . \quad . \quad (145)$$

If u_1, v_1 be the velocities at the surface, we get from (140), putting $y = 0$ after differentiation,

$$u_1 = mc \cos(mn - nt), \qquad v_1 = -mc \sin(mx - nt). \quad . \quad . \quad . \quad (146)$$

It appears from (145) and (146) that the oblique pressure which must be supplied at the surface in order to keep up the motion is constant in magnitude, and always acts in the direction in which the particles are moving.

The work of this pressure during the time dt corresponding to the element of surface $dx\,dz$, is equal to $dx\,dz\,(\Delta T_3 . u_1 dt + \Delta P_1 . v_1 dt.)$ Hence the work exerted over a given portion of the surface is equal to

$$2\mu m^3 c^2 dt \iint dx\,dz.$$

In the absence of pressures ΔP_2, ΔT_3 at the surface, this work must be supplied at the expence of *vis viva*. Hence $4\mu m^3 c^2 dt \iint dx\,dz$ is the *vis viva* lost by friction, which agrees with the expression obtained in Art. 51, as will be seen on performing in the latter the integration with respect to y, the limits being $y = 0$ to $y = \infty$.

PART II.

COMPARISON OF THEORY AND EXPERIMENT.

SECTION I.

Discussion of the Experiments of Baily, Bessel, Coulomb, and Dubuat.

54. THE experiments discussed in this Section will be taken in the order which is most convenient for discussion, which happens to be almost exactly the reverse of the chronological order. I commence with the experiments of the late Mr Baily, which are described in the *Philosophical Transactions* for 1832, in a memoir entitled "On the Correction of a Pendulum for the Reduction to a Vacuum: together with Remarks on some anomalies observed in Pendulum experiments."

The object of these experiments was, to determine by actual observation the correction to the time of vibration due to the presence of the air in the case of a great number of pendulums of various forms. This was effected by placing each pendulum in succession in a vacuum apparatus, by which means the pendulum, without being dismounted, could be swung alternately under the full atmospheric pressure, and in air so highly rarefied as nearly to approach to a vacuum. The paper, as originally presented to the Royal Society, contained the results obtained with 41 pendulums, the same body being counted as a different pendulum when swung in a different manner. Out of these, 14 are of such forms as to admit of comparison with theory. An addition to the paper contains the results obtained with 45 pendulums more, of which 24 admit of comparison with theory. The details of these additional experiments are omitted, the results only being given.

Baily has exhibited the results obtained with the several pendulums in each of two ways, first, by the value of the factor n by which the correction for buoyancy must be multiplied in order to amount to the whole effect of the air as given by observation, and, secondly, by the weight of air which must be conceived to be attached to the centre of gyration of the pendulum, adding to its inertia without adding to its weight, in order that the increased inertia, combined with the buoyancy of the air, may account for the whole effect observed. I shall uniformly write $\mathfrak{n}$ for Baily's n, in order to distinguish it from the n of Part I. of the present paper, which has a totally different meaning. In the case of a pendulum oscillating in air, it will be sufficient, unless the pendulum be composed of extremely light materials, to add together the effects of buoyancy and inertia. Hence if the pendulum consist of a sphere attached to a fine wire of which the effect is neglected, or else of a uniform cylindrical rod, we may suppose $\mathfrak{n} = 1 + k$, where k is the factor so denoted in Part I.; so that if M' be the mass of air displaced, kM' will be the mass which we must suppose collected at the centre of the sphere, or distributed uniformly along the axis of the cylinder, in order to express the effect of the inertia of the air. The second mode of exhibiting the effect of the air was suggested by Mr Airy, and is better adapted than the former for investigating the effect of the several pieces of which a pendulum of complicated form is composed. Since the value of the factor $\mathfrak{n}$ and that of the weight of air are merely two different expressions for the result of the same experiment, it would be sufficient to compare either with the result calculated from theory. In some cases, however, I have computed both. In almost all the calculations I have employed 4-figure logarithms. The experimental result is sometimes exhibited to four figures, but no reliance can be placed on the last. In fact, in the best observations, the mean error in different determinations of $\mathfrak{n}$ for the same pendulum appears to have been about the one-hundredth part of the whole, and that it should be so small, is a proof of the extreme care with which the experiments must have been performed.

55. I commence with the 13th set of experiments—*Results with plain cylindrical rods*—page 441. This set contains three pendulums, each consisting of a long rod attached to a knife-edge apparatus. The result obtained with each pendulum furnishes an equation for the determination of μ', and the theory is to be tested by the accordance or discordance of the values so obtained. The principal steps of the calculation are contained in the following table.

Determination of $\sqrt{\mu'}$ *by means of* Baily's *experiments with plain cylindrical rods.*

Pendulum rod.	No.	Diameter $2a$.	Time of vibration τ.	$\mathfrak{n}$ by experiment.	Correction for confined space (by theory.)	Deduced value of k by experiment.	Corresponding value of $\mathfrak{m}$.	Resulting value of $\sqrt{\mu'}$.
Copper, 58·8 inches long	21	0·410	1·0136	2·932	− 0·009	1·923	1·5445	0·1166
Brass, 56·4	43	0·185	0·9933	4·083	− 0·002	3·081	0 7000	0·1175
Steel, 56·4	44	0·072	0·9933	7·530		6·530	0·2822	0·1134

In this table the first column explains itself. The next contains the reference number. In the case of the copper rod I have replaced 42 by 21, under which number the details of the experiment will be found. The diameters of the rods are expressed in decimals of an inch. The time of vibration of the pendulum No. 21 may be got from the tables at the end of Baily's memoir, which contain the details of the experiments. Nos. 43 and 44 belong to the "additional experiments," of which all the details are suppressed. Baily has not even given the times of vibration, not having been aware of the circumstance, indicated by the theory of this paper, that the factor $\mathfrak{n}$ and the weight of air which must be conceived as dragged by the pendulum are functions of the time of vibration. Accordingly, in the cases of the pendulums Nos. 43 and 44, and in all similar cases, I have calculated the time of vibration by the ordinary formulæ of dynamics. In calculating τ, I have added 1·55 inch, the length of the shank of the knife-edge apparatus, to the length of the rods. The result so obtained is abundantly accurate enough for my purpose. Had the rod, retaining its actual length, been supposed to begin directly at the knife-edge, the error thence resulting in the value of τ, or rather the correspoding error in the calculated value of $\mathfrak{n}$ or k, might just have been sensible. The fifth column in the above table is copied from Baily's table. The next contains a small correction necessary to reduce the value of $\mathfrak{n}$ got from observation to what would have been got from observations made in an unlimited mass of fluid. It is calculated from the formula

$2a^2(b^2 - a^2)^{-1}$ or $2a^2b^{-2}$ nearly, which is obtained from the ordinary equations of hydrodynamics, and therefore it cannot be regarded as more than a rude approximation. It will be useful, however, as affording an estimate of the magnitude of the effect produced by confining the air. The diameter of the vacuum tube (whether external or internal is not specified) is stated to have been six inches and a half, whence $2b = 6{\cdot}5$. The values of k given in the next column are obtained by applying the correction for confined space to Baily's values of $\mathfrak{n}$, and subtracting unity. The value of $\mathfrak{m}$ corresponding to each value of k was got by interpolation from the table near the end of Section III. of the former part of this paper. For $k = 1{\cdot}923$ the interpolation is easy. The value 3·081 happens to be almost exactly found in the table. For $k = 6{\cdot}530$, a remark already made will be found to be of importance, namely, that the first differences of $\mathfrak{m}^2(k-1)$ are nearly constant. The last column contains the value of $\sqrt{\mu'}$ obtained from the equation

$$\mathfrak{m} = \frac{a}{2}\sqrt{\frac{\pi}{\mu'\tau}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (147)$$

which contains the definition of $\mathfrak{m}$.

It will be observed that the three values of $\sqrt{\mu'}$ are nearly identical. Of course any theory professing to account for a set of experiments by means of a particular value of a disposable constant, when applied to the experiments would lead to nearly the same numerical value of the constant if the experiments were made under nearly the same circumstances. But in the present case the circumstances of the experiments are widely different. The diameter of the steel rod is little more than the sixth part of that of the copper rod, and the value of k obtained by experiment for the steel rod is more than three times as great as that obtained for the copper rod. It is a simple consequence of the ordinary theory of hydrodynamics that in the case of a long rod oscillating in an unlimited fluid $k = 1$, and we see that this value of k must be multiplied, in round numbers, by 2, by 3, and by $6\frac{1}{2}$, in order to account for the observed effect. The value 1·5445 of $\mathfrak{m}$ is so large that the descending series comes into play in the calculation of the function k, while 0·2822 is so small that the ascending series are rapidly convergent. Hence the near agreement of the values of $\sqrt{\mu'}$ deduced from the three experiments is a striking confirmation of the theory. The mean of the three is 0·1158, but of course the last figure cannot be trusted. I shall accordingly assume as the value of the square root of the index of friction of air in its average state of pressure, temperature, and moisture

$$\sqrt{\mu'} = 0{\cdot}116.$$

It is to be remembered that $\sqrt{\mu'}$ expresses a length divided by the square root of a time, and that the numerical value above given is adapted to an English inch as the unit of length, and a second of mean solar time as the unit of time.

56. I now proceed to compare the observed values of $\mathfrak{n}$ with those calculated from theory with the assumed value of $\sqrt{\mu}$. I begin with the same cylindrical rods as before, together with the long brass tubes Nos. 35 to 38. The diameter of this tube was 1·5 inch, and its length 56 inches. The ends were open, but as the included air was treated by Mr Baily in the reduction of his experiments as if it formed part of the pendulum, we may regard

the pendulum as a solid rod. The tube was furnished with six agate planes, represented in the wood-cut at page 417, which rested on fixed knife-edges. The pendulums Nos. 35, 36, 37, and 38 consisted of the same tube swung on the planes marked A, C, a, c. In air the pendulum swung at the rate of about 90080 vibrations in a day, so that $\tau = 0{\cdot}9596$ nearly. The values of $\mathfrak{n}$ obtained with the end planes A, c were slightly though sensibly greater than the values obtained with the mean planes C, a. I shall suppose the mean of the four values of $\mathfrak{n}$, namely 2·290, to be the result of the experiments. In the following table the difference between the theoretical and experimental values of $\mathfrak{n}$ is exhibited both by decimals and as a fractional part of the former of these values.

Baily's results with a long brass tube and with long cylindrical rods.

No.	$2a$	$\mathfrak{m}$	k	Add for confined space.	Total $\mathfrak{n}$ by theory.	$\mathfrak{n}$ by experiment.	Difference.
35 to 38	1·5	5·849	1·242	0·122	2·364	2·290	$-$ 0·074, or $-\frac{1}{31}$
21 or 42	0·410	1·555	1·917	0·009	2·926	2·932	+ 0·006, or $+\frac{1}{489}$
43	0·185	0·7089	3·055	0·002	4·057	4·083	+ 0·026, or $+\frac{1}{157}$
44	0·072	0·2759	6·670		7·670	7·530	$-$ 0·140, or $-\frac{1}{54}$

It will be seen at once how closely the experiments are represented by theory. The largest proportionate difference occurs in the case of the brass tube, and even that is less than one thirtieth. A glance at Baily's wood-cut at page 417 will shew that the six planes with which the tube was furnished caused the whole figure to deviate sensibly from the cylindrical form. Moreover the resistance experienced by each element of the cylinder has been calculated by supposing the element in question to belong to an infinite cylinder oscillating with the same linear velocity, and the resistance thus determined must be a little too great in the immediate neighbourhood of the ends of the cylinder, where the free motion of the air is less impeded than it would be if the cylinder were prolonged. Lastly, the correction for confined space is calculated according to the ordinary equations of hydrodynamics, and on that account, as well as on account of the abrupt termination of the cylinder, will be only approximate. The small discrepancy between theory and observation, as well as the small difference (amounting to about the 1-83rd of the whole) detected by experiment between the results obtained with the extreme planes and those obtained with the mean planes, may reasonably be attributed to some such causes as those just mentioned. In the case of the steel rod or wire, the difference between theory and observation may be altogether removed by supposing a very small error to have existed in the measurement of the diameter of the rod. Since, as we have seen, the observation is satisfied by $\mathfrak{m} = 0{\cdot}2822$, and (147) gives $a \propto \mathfrak{m}$ when μ' and τ are constant, it is sufficient, in order to satisfy the experiment, to increase the diameter of the rod in the ratio of 0·2759 to 0·2822, or to suppose an error of only 0.0017 inch in defect to have existed in the measurement of the diameter.

57. I proceed next to the experiments on spheres attached to fine wires. The pendulums of this construction comprise four $1\frac{1}{2}$-inch spheres, Nos. 1, 2, 3, and 4; three 2-inch

spheres, Nos. 5, 6, and 7; and one 3-inch sphere, No. 66. Nos. 8 and 9 are the same spheres as Nos. 5 and 7 respectively, swung by suspending the wire over a cylinder instead of attaching it to a knife-edge apparatus. As this mode of suspension was not found very satisfactory, and the results are marked by Baily as doubtful cases, I shall omit the pendulums Nos. 8 and 9, more especially as with reference to the present inquiry they are merely repetitions of Nos. 5 and 7.

In the case of a sphere attached to a fine wire of which the effect is neglected, and swung in an unconfined mass of fluid, we have by the formulæ (52)

$$k = \frac{1}{2} + \frac{9}{2a}\sqrt{\frac{\mu' \tau}{2\pi}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

$2a$ being in this case the diameter of the sphere. Before employing this formula in the comparison of theory and experiment, it will be requisite to consider two corrections, one for the effect of the wire, the other for the effect of the confinement of the air by the sides of the vacuum tube.

I have already remarked at the end of Section IV., Part I., that the application of the formulæ of Section III. to the case of such fine wires as those used in pendulum experiments is not quite safe. Be that as it may, these formulæ will at any rate afford us a good estimate of the probable magnitude of the correction.

Let l be the length, a_1 the radius, V_1 the volume of the wire, V the volume of the sphere, I the moment of inertia of the pendulum, I' that of the air which we may conceive dragged by it, H the sum of the elements of the mass of the pendulum multiplied by their respective vertical distances below the axis of suspension, H' the same for the air displaced, σ the density of the air. Then the length of the isochronous simple pendulum is IH^{-1} in vacuum, and $(I + I')(H - H')^{-1}$ in air, and the time of vibration is increased by the air in the ratio of $I^{\frac{1}{2}}H^{-\frac{1}{2}}$ to $(I + I')^{\frac{1}{2}}(H - H')^{-\frac{1}{2}}$, or, on account of the smallness of σ, in the ratio of 1 to $1 + \frac{1}{2}(I'I^{-1} + H'H^{-1})$ nearly. Now $\frac{1}{2}H'H^{-1}$ is the correction for buoyancy, and therefore

$$\mathfrak{n} - 1 = \frac{I'}{H'} \cdot \frac{H}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (149)$$

We have also, if k_1 be the value of the function k of Section III., Part I.,

$$I' = k\sigma V(l + a)^2 + \frac{1}{3}k_1\sigma V_1 l^2, \qquad H' = \sigma V(l + a) + \frac{1}{2}\sigma V_1 l, \quad . \quad . \quad (150)$$

and $HI^{-1} = (l + a)^{-1}$ very nearly. Substituting in (149), expanding the denominator, and neglecting V_1^2, we get

$$\mathfrak{n} - 1 = k + \frac{1}{3}\frac{V_1}{V}k_1\left(\frac{l}{l + a}\right)^2 - \frac{1}{2}\frac{V_1}{V}k\frac{l}{l + a}.$$

Now V_1 is very small compared with V, and it is only by being multiplied by the large factor k_1 that it becomes important. We may then, without any material error, replace the last term in the above equation by $\frac{1}{3}V_1V^{-1}l^2(l + a)^{-2}$, and if λ be the length of the isochronous simple pendulum, we may suppose $l + a = \lambda$, and replace $l^2 (l + a)^{-2}$ by $1 - 2a\lambda^{-1}$, since a is small compared with λ. We thus get, putting $\Delta\mathfrak{n}$ for the correction due to the wire,

$$\Delta\mathfrak{n} = \frac{1}{3}\frac{V_1}{V}\left(1 - \frac{2a}{\lambda}\right)(k_1 - 1).$$

Substituting for $k_1 - 1$ from (115), and for $\mathfrak{m}$ from (147), in which equations, however, k_1, a_1 must be supposed to be written for k, a, expressing V_1, V in terms of the diameters of the wire and sphere, and neglecting as before a^2 in comparison with λ^2, we get

$$\Delta\mathfrak{n} = \frac{(2\lambda - 3 \times 2a)\,\mu'\tau}{\left\{L^2 + \left(\frac{\pi}{4}\right)^2\right\}(2a)^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

where

$$-L = \log_e \frac{4}{2a_1}\sqrt{\frac{\mu'\tau}{\pi}} - 0\cdot5772. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

It is by these formulæ that I have computed the correction for the wire in the following table. In the experiments, the time of oscillation was so nearly one second that it is sufficient in the formulæ (148), (151), and (152) to put $\tau = 1$, and take λ for the length of the seconds pendulum, or 39·14 inches.

With respect to the correction for confined space, it seems evident that the vacuum tube must have impeded the free motion of the air, and consequently increased the resistance experienced by the pendulum when it was swung in air, and that the increase of resistance caused by the cylindrical tube must have been somewhat less than that which would have been produced by a spherical envelope of the same radius surrounding the sphere. The effect of a spherical envelope has been investigated in Section II., Part I.; but as we are obliged at last to have recourse to estimation, it is needless to be very precise in calculating the increase of resistance due to such an envelope, and we may accordingly employ the expression obtained from the ordinary theory of hydrodynamics. According to this theory, the increase of the factor k, which is due to the envelope, is equal to $\frac{3}{2}a^3(b^3 - a^3)^{-1}$, or $\frac{3}{2}a^3 b^{-3}$ nearly, when b is large compared with a. The increase due to a cylindrical envelope whose axis is vertical, and consequently perpendicular to the direction of oscillation of the sphere, may be estimated at about two-thirds of the increase due to a spherical envelope of the same diameter. I have accordingly taken $+ a^3 b^{-3}$ for the correction for confined space, and have supposed $2b = 6\cdot5$ inches.

The diameter of the wire employed in the pendulums Nos. 1, 2, 3, 5, 6, and 7, is stated to have been about the $\frac{1}{70}$th of an inch, and that of the wire employed with the heavy brass sphere No. 66, about 0·023 inch. The ivory sphere No. 4 was swung with a fine wire weighing rather more than half a grain. Taking the weight at half a grain, and the specific gravity of silver at 10·5, we have for this wire $2a_1 = 0\cdot00251$ nearly. The diameters of the three brass spheres in the following table are taken from page 447 of Baily's memoir. The several parts of which, according to theory, $\mathfrak{n}$ is composed, are exhibited separately.

Baily's results with spheres suspended by fine wires.

No. and kind.	Diameter of sphere. $2a$	Diameter of wire. $2a_1$	n By theory.						n By experiment.	Difference.
			For buoyancy.	For inertia on common theory.	Additional for inertia on account of internal friction.	Correction for wire.	Correction for confined space (estimated.)	Total.		
$1\frac{1}{2}$-INCH SPHERES.										
No. 1, Platina.	1·44	0·01429	1	0·5	0·289	0·035	0·011	1·835	1·881	$+ 0·046$, or $+ \frac{1}{40}$
No. 2, Lead.	1·46	0·01429	1	0·5	0·285	0·035	0·011	1·831	1·871	$+ 0·040$, or $+ \frac{1}{46}$
No. 3, Brass.	1·465	0·01429	1	0·5	0·284	0·035	0·011	1·830	1·834	$+ 0·004$, or $+ \frac{1}{457}$
No. 4, Ivory.	1·46	0·00251	1	0·5	0·285	0·016	0·011	1·812	1·872	$+ 0·060$, or $+ \frac{1}{30}$
2-INCH SPHERES.										
No. 5, Lead.	2·06	0·01429	1	0·5	0·202	0·012	0·032	1·746	1·738	$- 0·008$, or $- \frac{1}{218}$
No. 6, Brass.	2·065	0·01429	1	0·5	0·202	0·012	0·032	1·746	1·751	$+ 0·005$, or $+ \frac{1}{349}$
No. 7, Ivory.	2·06	0·01429	1	0·5	0·202	0·012	0·032	1·746	1·755	$+ 0·009$, or $+ \frac{1}{194}$
3-INCH SPHERE.										
No. 66, Brass.	3·030	0·023	1	0·5	0·137	0·005	0·101	1·743	1·748	$+ 0·005$, or $+ \frac{1}{349}$

The mean error in different determinations of n for the same sphere was about 0·01 or 0·02, and this does not include errors arising from small errors in specific gravities, &c. Hence, if we except the spheres Nos. 1, 2, and 4, the discrepancies between theory and experiment are altogether insignificant. In considering the confirmation thence arising to the theory, it must be borne in mind that the theory did not furnish a single disposable constant, inasmuch as $\sqrt{\mu'}$ was already determined from the experiments with cylindrical rods. The result obtained with the brass sphere No. 3 happens to agree almost exactly with theory. However, as the results obtained with this sphere exhibited some anomalies, it seems best to exclude it from consideration. The value of n, then, which belongs to a $1\frac{1}{2}$ inch sphere, appears to exceed by a minute quantity the value deduced from theory. The difference is indeed so small that it might well be attributed to errors of observation, were it not that all the spheres tell the same tale. Thus the error + 0·046 in the case of the platina sphere corresponds to an error of less than the fortieth part of a second in the observation of an interval of time amounting to $4\frac{1}{2}$ hours. If the apparent defect, amounting to about 0·04 or 0·05, in the theoretical result be real, it may be attributed with probability to an error in the correction for the wire. This would be no objection to the theory, for it will be remembered that the theory itself indicated the probable failure of the formulæ generally applicable to a long cylinder when the cylinder comes to be of such extreme fineness as the wires employed in pendulum experiments.

58. The preceding experiments of Baily's are the most important for the purposes of the present paper, inasmuch as they were performed on pendulums of simple and very different forms; but there still remain three sets of experiments, the fourteenth, fifteenth, and sixteenth, in which the pendulum consisted of a combination of a sphere and a rod, so that the results can be compared with theory. The details of these experiments being suppressed, I

have been obliged to calculate the time of oscillation from the ordinary formulæ of dynamics, but the results will no doubt be accurate enough for the purpose required. In all the calculations I have supposed the rod to reach up to the axis of suspension, and have consequently added 1·55 inch (the length of the shank of the knife-edge apparatus) to the length of the rod, and have added to the weight of the rod a quantity bearing to the whole weight the ratio of 1·55 inch to the whole length.

In the case of the spheres attached to the ends of the rods (sets 14 and 16) the process of calculation is as follows. Let l be the length of the rod increased by 1·55 inch, W_1 its weight, increased as above explained, a the radius and W the weight of the sphere, λ the length of the isochronous simple pendulum. Then supposing the masses of the rod and sphere to be respectively distributed along the axis, and collected at the centre, which will be quite accurate enough for the present purpose, and putting α for the ratio of a to l, we have by the ordinary formula

$$\lambda = \frac{\frac{1}{3} W_1 + (1+\alpha)^2 W}{\frac{1}{2} W_1 + (1+\alpha) W} l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

whence τ, the time of vibration, is known. The formula (148) then gives k, which applies to the sphere, and (147) gives $\mathfrak{m}$, the a in this formula being the radius of the rod, from whence k_1, which applies to the rod, may be got by interpolation from the table in Part I. Let Δk, Δk_1 be the corrections which must be applied to k, k_1 on account of the confined space of the vacuum apparatus, and let S_1, S be the specific gravities of the rod and sphere respectively; then we get by means of the formulæ (149), (150)

$$\mathfrak{n} - 1 = \frac{\frac{1}{3}(k_1 + \Delta k_1)\dfrac{W_1}{S_1} + (1+\alpha)^2 (k + \Delta k)\dfrac{W}{S}}{\frac{1}{3} W_1 + (1+\alpha)^2 W} \times \frac{\frac{1}{2} W_1 + (1+\alpha) W}{\frac{1}{2}\dfrac{W_1}{S_1} + (1+\alpha)\dfrac{W}{S}}. \quad . \quad (154)$$

The first of the two factors connected by the sign × in this equation is equal to $\sigma^{-1} I' I^{-1}$, and if we want to calculate the weight of air which we must conceive attached to the centre of gyration of the pendulum in order to allow for the inertia of the air, we have only to multiply the factor just mentioned by σ and by the weight of the whole pendulum. The following table contains the comparison of theory and experiment in the case of the 14th set. The rods here mentioned are the same as those which composed the pendulums Nos. 21, 43, and 44, and the spheres are the three brass spheres of Nos. 3, 5, and 66. It appears from p. 432 of Baily's paper that his results are all reduced to a standard pressure and temperature, on the supposition that the effect of the air on the time of vibration is proportional to its density. The theory of the present paper shews that this will only be the case if μ' be constant, which however there is reason for supposing it to be when the pressure alone varies. Be that as it may, no material error can be produced by reducing the observations in this way, because the difference of density in any pair of experiments did not much differ from the density of air at the standard pressure and temperature. The standard pressure and temperature taken were 29·9218 inches of mercury and 32° F, and the assumed specific gravity of air at this pressure and temperature was the 1-770th of that of water, so that in the calculations from theory it is to be supposed that $\sigma^{-1} = 770$.

If w be the weight of the whole pendulum, w' that of the air which we must suppose attached to the pendulum at its centre of gyration in order to express the effect of the inertia of the air, S the *vibrating* specific gravity of the pendulum, the effects of buoyancy and inertia are as σS^{-1} to $w'w^{-1}$; but they are also as 1 to $\mathfrak{n}-1$, according to the definition of the factor $\mathfrak{n}$, and therefore

$$w' = (\mathfrak{n} - 1)\frac{\sigma}{S}w, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

a formula which may be employed to calculate w' when $\mathfrak{n}$ is known.

Baily's results with spheres at the ends of long rods.

No. and nature of pendulum.	Value of $\mathfrak{n}$.			Weight of adhesive air, in grains.		
	By theory.	By experiment.	Difference.	By theory.	By experiment.	Difference.
No. 45 – $1\frac{1}{2}$-inch sphere with copper rod.	2·525	2·458	– 0·067, or – $\frac{1}{38}$	4·863	4·564	– 0·299, or – $\frac{1}{16}$
No. 46 – 2-inch sphere with ditto.	2·202	2·234	+ 0·032, or + $\frac{1}{69}$	5·005	5·076	+ 0·071, or + $\frac{1}{70}$
No. 47 – 3-inch sphere with ditto.	1·957	1·873	– 0·084, or – $\frac{1}{23}$	7·071	6·425	– 0·646, or – $\frac{1}{11}$
No. 48 – $1\frac{1}{2}$-inch sphere with brass rod.	2·375	2·356	– 0·019, or – $\frac{1}{125}$	1·447	1·417	– 0·030, or – $\frac{1}{48}$
No. 49 – 2-inch sphere with ditto.	2·060	1·982	– 0·078, or – $\frac{1}{26}$	2·135	1·973	– 0·162, or – $\frac{1}{13}$
No. 50 – 3-inch sphere with ditto.	1·631	1·933?	+ 0·302.?	4·411	4·868 ?	+ 0·457 ?
No. 51 – $1\frac{1}{2}$-inch sphere with steel rod.	2·099	2·344?	+ 0·245 ?	0·682	0·834 ?	+ 0·152 ?
No. 52 – 2-inch sphere with ditto.	1·920	1·793	– 0·127, or – $\frac{1}{15}$	1·457	1·259	– 0·198, or – $\frac{1}{7}$
No. 53 – 3-inch sphere with ditto.	1·781	1·759	– 0·022, or – $\frac{1}{81}$	3·742	3·670	– 0·072, or – $\frac{1}{52}$

With respect to the two experiments marked ? Baily remarks, "These two experiments (with the pendulums Nos. 50 and 51) are very unsatisfactory; and are marked as such in my journal. It was consequently my intention to have repeated them: but the subject was overlooked till it was too late. I should propose their being rejected altogether." If these two experiments be struck out, it will be seen that the differences between theory and experiment are very small, especially when the difficulty of this set of experiments is considered, arising from the frequency of the coincidences with the mean solar clock.

59. On account of the difficulty which Baily experienced in obtaining accurate results with the long rods and spheres attached, he divided the brass and steel rods near the centre of oscillation, and after having cut off an inch from each portion inserted the spheres where the rods had been divided. The results thus obtained constitute the 15th set of experiments. He afterwards removed the lower segments of the rods, and obtained the results contained in the 16th set. I shall give the computation of the latter set first, inasmuch as the formulæ to be employed are exactly the same as those required for the 14th set. The experiments belonging to this set in which the spheres were swung with iron wires have already been computed under the head of spheres attached to fine wires.

Baily's results with the spheres at the end of the short rods.

No. and nature of pendulum.	Value of n.			Weight of adhesive air.		
	By theory.	By experiment.	Difference.	By theory.	By experiment.	Difference.
No. 60 – $1\frac{1}{2}$-inch sphere with brass rod.	2·149	2·198	+ 0·049, or + $\frac{1}{44}$	1·011	1·047	+ 0·036, or + $\frac{1}{28}$
No. 61 – 2-inch sphere with ditto.	1·879	1·901	+ 0·022, or + $\frac{1}{85}$	1·619	1·513	– 0·106, or – $\frac{1}{15}$
No. 62 – 3-inch sphere with ditto.	1·787	1·830	+ 0·043, or + $\frac{1}{42}$	3·970	4·202	+ 0·232, or + $\frac{1}{17}$
No. 63 – $1\frac{1}{2}$-inch sphere with steel rod.	1·960	1·904	– 0·056, or – $\frac{1}{35}$	0·570	0·537	– 0·033, or – $\frac{1}{16}$
No. 64 – 2-inch sphere with ditto.	1·796	1·785	– 0·011, or – $\frac{1}{163}$	1·239	1·227	– 0·012, or – $\frac{1}{103}$
No. 65 – 3-inch sphere with ditto.	1·758	1·779	+ 0·021, or + $\frac{1}{84}$	3·609	3·720	+ 0·111, or + $\frac{1}{32}$

Here again the differences between theory and experiment are extremely small. In the case of the pendulum No. 61, Baily's two results 1·901 and 1·513 appear to be inconsistent, as not agreeing with the formula (155).

60. The following table contains the values of τ, k, and k_1 deduced from the given data, and employed in the calculations of which the results are contained in the two preceding tables. It is added, partly to facilitate a comparison of the circumstances of the different experiments, partly to assist in the re-computation of any of the experiments, or the detection of any numerical error which I may have committed. I may here observe that I have not, generally speaking, re-examined the calculations, except where an error was apparent, but that each step requiring addition, subtraction, multiplication, or division, was checked immediately after it was performed. I have not thought it requisite to check in this manner the taking of logarithms or antilogarithms out of a table.

Values of τ, k, and k_1 employed in the calculation of the theoretical results employed in the two preceding tables.

Sphere.	Rod.	Long rods.				Short rods.			
		No.	τ	k	k_1	No.	τ	k	k_1
$1\frac{1}{2}$-inch	copper	45	1·090	0·7968	1·951				
2-inch	copper	46	1·158	0·7170	1·981				
3-inch	copper	47	1·227	0·6523	2·010				
$1\frac{1}{2}$-inch	brass	48	1·155	0·8055	3·222	60	0·9517	0·7772	3·012
2-inch	brass	49	1·198	0·7207	3·264	61	0·9806	0·7005	3·042
3-inch	brass	50	1·222	0·6520	3·288	62	0·9982	0·6373	3·062
$1\frac{1}{2}$-inch	steel	51	1·190	0·8099	7·272	63	0·9868	0·7824	6·649
2-inch	steel	52	1·199	0·7208	7·299	64	0·9954	0·7021	6·679
3-inch	steel	53	1·231	0·6525	7·396	65	1·0030	0·6377	6·714

The corrections for confined space employed are, for the spheres, (Δk), 0·0115, 0·0321, 0·1013; and for the rods, (Δk_1), 0·009, 0·002, 0·000. These corrections are to be added to the values of k, k_1 given in the preceding table before going on with the calculation.

61. In the 14th set of experiments, the weight of adhesive air due to the spheres alone has been computed by Baily by subtracting from the whole weight, as given by observation, the weight due to the rods as given by the 13th set of experiments, taking account of the change of weight corresponding to the change in the position of the centre of gyration, the point at which the air is supposed to be attached. According to theory, this process is not legitimate, inasmuch as the weight dragged by a rod in a function of the time of vibration, which is altered when a sphere is attached to the end of the rod. But in the 15th set of experiments the spheres did not materially affect the time of vibration, inasmuch as they were inserted nearly at the centre of oscillation of the rods, and therefore in this case the process is legitimate. Accordingly, I think it is a sufficient comparison between theory and experiment in the case of the 15th set, to compare the weights of air due to the spheres alone, as calculated by Baily, with the weights calculated according to the theory of this paper with the assumed value of $\sqrt{\mu'}$. I have exhibited separately the weight corresponding to the correction for confined space, in order to enable the reader to form an estimate of the extent to which the results may be affected by the uncertainty relating to the amount of this correction.

Weights of air dragged by the spheres alone, as deduced from Baily's results with the spheres at the centre of oscillation of the long rods.

By Theory.				By Experiment.			
	$1\frac{1}{2}$-inch sphere.	2-inch sphere.	3-inch sphere.		$1\frac{1}{2}$-inch sphere.	2-inch sphere.	3-inch sphere.
In free air	0·431	1·060	3·002	From exper[ts] with brass rod	0·446	1·180	3·382
Additional for confined space	0·006	0·048	0·476	From exper[ts] with steel rod	0·405	1·039	3·371
Total	0·437	1·108	3·478	Mean	0·425	1·109	3·377
Diff[ce]. th. & exp., as decimal	−0·012	+0·001	−0·101	Diff[ce], as fraction of the whole	$-\frac{1}{36}$	$+\frac{1}{1108}$	$-\frac{1}{34}$

62. I pass now to Bessel's experiments described in his memoir entitled *Untersuchungen über die Länge des einfachen Sekundenpendels*, which is printed among the memoirs of the Academy of Sciences of Berlin for the year 1826. The object of this memoir was to determine the length of the seconds' pendulum by a new method, which consisted in swinging the same sphere with wires of two different lengths, the difference of lengths being measured with extreme precision. In the calculation, the absolute length of the simple pendulum isochronous with either the long or the short compound pendulum was regarded as unknown, but the difference of the two as known, and this difference, combined with the observed times of oscillation, is sufficient for the determination of the quantity sought. Nothing more would have been required if the pendulums had been swung in a vacuum; but inasmuch as they were swung in air, a further correction was necessary to reduce the observations to a vacuum. Since it is necessary to take into account the inertia of the air, as well as its buoyancy, in reducing the observations to a vacuum, Bessel sought to determine by experiment the value of

the factor k, of which the meaning has been already explained. The value of this factor, as Bessel remarked, will depend upon the form of the body; but he does not seem, at least in his first memoir, to have contemplated the possibility of its depending on the time of oscillation, and consequently he supposed it to have the same value for the long as for the short pendulum. When the factor k is introduced, the equation obtained from the known difference of length of the two simple pendulums contains two unknown quantities, namely k, and the length of the seconds' pendulum. To obtain a second equation, Bessel made another set of experiments, in which the brass sphere was replaced by an ivory sphere, having as nearly as possible the same diameter. The results obtained with the ivory sphere furnished a second equation, in which k appeared with a much larger coefficient, on account of the lightness of ivory compared with brass. The two equations determined the two unknown quantities.

Let λ be the length of the seconds' pendulum, t_1, t_2 the times of oscillation of the brass sphere when swung with the short wire and long wire respectively, l_1, l_2 the lengths of the corresponding simple pendulums, corrected for everything except the inertia of the air, m the mass of the sphere, m_1 the mass of the fluid displaced; then

$$\lambda t_1^2 \left(1 + \frac{m_1}{m} k\right)^{-1} = l_1;$$

or, since m_1 is so small that we may neglect m_1^2,

$$\lambda t_1^2 \left(1 - \frac{m_1}{m} k\right) = l_1.$$

The long pendulum furnishes a similar equation, and the result obtained from the brass sphere is

$$\lambda (t_2^2 - t_1^2)\left(1 - \frac{m_1}{m} k\right) = l_2 - l_1, \quad \ldots \ldots \quad (156)$$

since $l_2 - l_1$ is the quantity which is regarded as accurately known. The ivory sphere in like manner furnishes the equation

$$\lambda (t'^2_2 - t'^2_1)\left(1 - \frac{m_1'}{m'} k\right) = l'_2 - l'_1, \quad \ldots \ldots \quad (157)$$

where the accented letters refer to that sphere. The equation for the determination of k results from the elimination of λ between the equations (156) and (157).

Now, according to the theory of this paper, the factor k has really different values for the long and short pendulums. Let k_1 refer to the short, and k_2 to the long pendulum with the brass sphere, k_1' to the short, and k_2' to the long pendulum with the ivory sphere. Then

$$\lambda t_1^2 \left(1 - \frac{m_1}{m} k_1\right) = l_1, \quad \lambda t_2^2 \left(1 - \frac{m_1}{m} k_2\right) = l_2,$$

and therefore

$$l_2 - l_1 = \lambda t_2^2 \left(1 - \frac{m_1}{m} k_2\right) - \lambda t_1^2 \left(1 - \frac{m_1}{m} k_1\right). \quad \ldots \ldots \quad (158)$$

In the equation resulting from the elimination of λ between (156) and (157), let the values of $l_2 - l_1$ and $l_2' - l_1'$ got from (158) and the similar equation relating to the ivory sphere be substituted. The result is

$$(t_2^2 - t_1^2)(1 - \frac{m_1}{m}k)\{t_2'^2(1 - \frac{m_1}{m'}k'_2) - t_1'^2(1 - \frac{m_1}{m'}k'_1)\}$$

$$= (t_2'^2 - t_1'^2)(1 - \frac{m_1}{m'}k)\{t_2^2(1 - \frac{m_1}{m}k_2) - t_1^2(1 - \frac{m_1}{m}k_1)\}.$$

This equation is of the form

$$P + Qm_1 + Rm_1^2 = P' + Q'm_1 + R'm_1^2,$$

and $P = P'$, and Rm_1^2, $R'm_1^2$ may be neglected, so that the equation is reduced to $Q = Q'$. It is now no longer necessary to distinguish between t_2 and t_2', and between t_1 and t_1', which may be supposed equal. Also $m : m' :: S : S'$, where S, S' are the specific gravities of the brass and ivory spheres respectively. Substituting in the equation $Q = Q'$, and solving with respect to k, we get

$$k = \frac{t_2^2(Sk'_2 - S'k_2) - t_1^2(Sk'_1 - S'k_1)}{(t_2^2 - t_1^2)(S - S')} \quad . \quad . \quad . \quad . \quad . \quad (159)$$

This equation contains the algebraical definition of that function k of which the numerical value is determined by combining, in Bessel's manner, the results obtained with the four pendulums. Since the equation is linear so far as regards k, k_1, &c., we may consider separately the different parts of which these quantities are composed, and add the results. For the part which relates to the spheres, regarded as suspended by infinitely fine wires, we have $k'_2 = k_2$ and $k'_1 = k_1$, since the radii of the two spheres were equal, or at least so nearly equal that the difference is insensible in the present enquiry. We get then from (159)

$$k = \frac{t_2^2 k_2 - t_1^2 k_1}{t_2^2 - t_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

which gives

$$\frac{k - k_1}{t_2^2} = \frac{k - k_2}{t_1^2} = \frac{k_2 - k_1}{t_2^2 - t_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (161)$$

Since $t_2 > t_1$ and $k_2 > k_1$, the equations (161) shew that the value of k determined by Bessel's method is greater than the factor which relates to the short pendulum, which was a seconds' pendulum nearly, and even greater than that which relates to the long pendulum, as has been already remarked in Art. 6.

If k_s be the factor relating to either sphere oscillating once in a second, and if the effect of the confinement of the air be neglected, we have from the formula (148)

$$k_1 - \tfrac{1}{2} : k_2 - \tfrac{1}{2} : k_s - \tfrac{1}{2} :: t_1^{\frac{1}{2}} : t_2^{\frac{1}{2}} : 1,$$

and in Bessel's experiments $t_1 = 1{\cdot}001$, $t_2 = 1{\cdot}721$, $2a = 2{\cdot}143$ in English inches. We thus get from either of the equations (160) or (161), on substituting $0{\cdot}116$ for $\sqrt{\mu'}$, $k = 0{\cdot}786$. The value of the factor k_s, which relates to a sphere of the same size, swung as a seconds' pendulum, is only $0{\cdot}694$, and k_1 may be regarded as equal to k_s. The formula (148) gives $k_2 = 0{\cdot}755$.

63. We have next to investigate the correction for the wire. The effect of the inertia of the air set in motion by the wire was altogether neglected by Bessel, and indeed it would have been quite insensible had the parts of the correction for inertia due to the wire and to the sphere, respectively, been to each other in nearly the same ratio as the parts of the correction for buoyancy. Baily, however, was led to conclude from his experiments that the effect of the wire was probably not altogether insignificant, and the theory of this paper leads, as we have seen, to the result that the factor $\mathfrak{n}$ is very large in the case of a very fine wire.

The ivory sphere in Bessel's experiments was swung with a finer wire than the brass sphere. It was for this reason that I did not from the first suppose $k'_1 = k_1$ and $k'_2 = k_2$. Let Δk, Δk_1 &c. be the corrections due to the wire. The values of Δk_1, Δk_2, $\Delta k'_1$, $\Delta k'_2$, may be got from the formula (151), in which it is to be remembered that λ denotes the length of the isochronous simple pendulum, not, as in Bessel's notation, the length of the seconds' pendulum. It is stated by Bessel (p. 131), that the wire used with the brass sphere weighed 10·95 Prussian grains in the case of the long pendulum, and 3·58 grains in the case of the short. This gives 7·37 grains for the weight of one toise or 72 French inches. The weight of one toise of the wire employed with the ivory sphere was 6·28 – 2·04 or 4·24 grains (p. 141). The specific gravity of the wire was 7·6 (p. 40), and the weight of a cubic line (French) of water is about 0·1885 grain. From these data it results that the radii of the wires were 0·003867 and 0·002933 inch English. The formula (147) gives $\mathfrak{m}$, whence L is known from (152). The lengths of the isochronous simple pendulums were about 39·20 inches for the short pendulum, and 116·94 for the long. On substituting the numerical values we get from (151), since $k_1 = \mathfrak{n}_1 - 1$ and $k_2 = \mathfrak{n}_2 - 1$,

$$\Delta k_1 = 0{\cdot}0107, \quad \Delta k_2 = 0{\cdot}0286, \quad \Delta k'_1 = 0{\cdot}0090, \quad \Delta k'_2 = 0{\cdot}0244.$$

The specific gravities of the two spheres were about 8·190 and 1·794, whence we get from (159) $\Delta k = 0{\cdot}0308$, or 0·031 nearly.

The value of k deduced by Bessel from his experiments was 0·9459 or 0·946 nearly, which in a subsequent paper he increased to 0·956. In this paper he contemplates the possibility of its being different in the cases of the long and of the short pendulum, and remarks with justice that no sensible error would thence result in the length of the seconds' pendulum, as determined by his method, but that the factor k would belong to the system of the two pendulums.

The following is the result of the comparison of theory and experiment in the case of Bessel's experiments on the oscillations of spheres in air.

Value of k belonging to the system of a long and a short pendulum, as determined experimentally by Bessel	0·956
Value deduced from theory, including the correction for the wire, but not the correction for confined space..............................	0·817
difference	+ 0·139

I cannot find that Bessel has stated exactly the distance of the centre of the sphere from the back of the frame within which it was swung, but if we may judge by the sketch of the whole apparatus which is given in Plate I., and by a comparison of figs. 2 and 3, Plate II., it must have been very small, that is to say, a small fraction of the radius of the sphere *. If so, although the exact calculation of the correction for confined space would form a problem of extreme difficulty, it may be shewn from theoretical considerations that the correction would be by no means insensible, so that it might wholly or in part account for the difference + 0·139 between the results of theory and observation. It is, however, not improbable, for a reason which has been already mentioned, that the theoretical correction for the wire is not quite exact.

64. The experiments performed by Bessel on a sphere vibrating in water will be more conveniently considered after the discussion of some experiments of Coulomb's, to which I now proceed. These experiments are contained in a memoir entitled *Expériences destinées à déterminer la cohérence des fluides et les lois de leur résistance dans les mouvements très-lents*, which will be found in the 3rd Volume of the *Mémoires de l'Institut*, p. 246. The experiments which I shall first consider are those which relate to the oscillations of disks suspended in water with their planes horizontal. In these experiments the disk operated upon was attached to the lower extremity of a vertical cylinder of copper, not quite half an inch in diameter, the axis of which passed through the centre of the disk. The cylinder was suspended by a fine wire attached to its upper extremity. The under portion of the cylinder, together with the attached disk, were immersed in water, the disk at the bottom of the cylinder being immersed to the depth of 4 or 5 centimetres below the surface. The upper portion carried a horizontal metallic graduated disk, by means of which the arc of oscillation could be read off, and which, on account of its size and weight, mainly determined the inertia of the system, so that the time of oscillation in the different experiments was nearly the same. The observations were taken as follows. The whole system was turned very slowly round by applying the hands of the graduated disk, taking care not to derange the vertical position of the suspending wire. The arc through which the system had been turned was read by means of the graduation, or rather the system was turned through an arc previously fixed on; the system was then left to itself, and the arc again read off to a certain number of oscillations. Thus it was the decrement of the arc of oscillation that was observed; the time of oscillation was indeed also observed, but only approximately, for the sake of determining a subsidiary quantity required in the calculation. Indeed, it will be easily seen that the experiments were not adapted to determine the effect of the fluid on the time of oscillation. The decrement of arc so determined had to be corrected for the effect of the imperfect elasticity of the wire, and of the resistance of the air against the graduated disk, and of the water against the portion of the copper cylinder immersed. The amount of the correction was determined by repeating the observation when the lower disk had been removed.

It appeared from the experiments, *first*, that with the same disk immersed, the successive

* The measurement of either of Bessel's figures, figs. 5 or 6, Plate II. gives 1·53 inch for the distance of the centre of the sphere from the surface of the broad iron bar which formed the back of the frame, the surface of the bar being supposed truly vertical; and the measurement of fig. 2 giving 2·06 inches for the diameter of the sphere, it appears that the distance of the surface of the sphere from the surface of the bar was barely equal to half the radius of the sphere.

amplitudes of oscillation decreased in geometric progression; *secondly*, that with different disks the moment of the resisting force was proportional to the fourth power of the radius. From these laws Coulomb concluded that each small element of any one of the disks experienced a resistance varying as the area of the element multiplied by its linear velocity. It should be observed that Coulomb was only authorized by his experiments to assert this law to be true in the case of oscillations of given period, inasmuch as the time of oscillation was nearly the same in all the experiments.

Let a be the radius of the disk in the fluid, τ the time of oscillation, θ the angular displacement of the disk, measured from its mean position, I the moment of inertia of the whole system; and let $1 : 1 - m$ be the ratio in which the arc of oscillation is diminished in one oscillation. According to the formula (15) we have

$$\epsilon^{-n\beta t}$$

for the factor which expresses the ratio of the arc of oscillation at the end of the time t to the initial arc. At the end of one oscillation $t = \tau$, and the value of the above factor is $1 - m$, which is given by observation. Putting for β its value, in which $M\gamma^2 = I$, and $n\tau = \pi$, we get

$$\log_\epsilon (1 - m) = -\frac{\rho a^4}{I}\sqrt{\frac{\pi^3 \mu' \tau}{8}}. \quad . \quad . \quad . \quad . \quad . \quad (162)$$

Let T be the time of oscillation, and I_0 the moment of inertia, when the under disk is removed: then $I = I_0 \tau^2 T^{-2}$. Also if M be the mass and R the radius of the large graduated disk, we have $I_0 = \frac{1}{2} MR^2$, neglecting, as Coulomb did, the rotatory inertia of the copper cylinder. Substituting in (162), we get

$$\log_\epsilon (1 - m)^{-1} = 2^{-\frac{1}{2}} \pi^{\frac{3}{2}} \rho \mu'^{\frac{1}{2}} \tau^{-\frac{3}{2}} T^2 a^4 R^{-2} M^{-1}. \quad . \quad . \quad . \quad . \quad (163)$$

Let W be the weight of the disk in grammes. Then the mass of the disk is equal to that of W cubic centimetres or $1000\,W$ cubic millimetres of water. Hence $M = 1000\,\rho W$, a millimetre being the unit of length. Substituting in (163), and solving with respect to $\sqrt{\mu'}$, we get

$$\sqrt{\mu'} = 1000 \times 2^{\frac{1}{2}} \log_\epsilon 10 \,.\, \pi^{-\frac{3}{2}} W R^2 T^{-2} a^{-4} \tau^{\frac{3}{2}} \log_{10} (1 - m)^{-1}, \quad . \quad . \quad . \quad (164)$$

and the same value of $\sqrt{\mu'}$ ought to result from different experiments.

The weight of the disk is stated to have been 1003 grammes, and its diameter 271 millimetres, and it made 4 oscillations in 91 seconds. Hence $W = 1003$, $R = 135{\cdot}5$, $T = 22{\cdot}75$. The last three factors in (164) vary from one experiment to another. After making experiments with three disks of different radii attached to the copper cylinder, Coulomb made another set with nothing attached, for the purpose of eliminating the effect of the imperfect elasticity of the wire. The following table contains the data furnished by experiment, together with the value of $\sqrt{\mu'}$ deduced from the several experiments. The latter is reduced to the decimal of an English inch, by including $\bar{2}{\cdot}5952$ (the logarithm of the ratio of a millimetre to an inch) in the logarithm of the constant part of the 2nd member of equation (164).

Determination of the value of $\sqrt{\mu'}$ for water from Coulomb's experiments on the decrement of the arc of oscillation of disks, oscillating in their own plane by the force of torsion.

No.	Diameter of disk $2a$ in millimetres.	Time of four oscillations 4τ.	$\log_{10}(1-m)^{-1}$	Resulting value of $\sqrt{\mu'}$ in inches.
1	195	97	0·0568	0·05519
2	140	92	0·021	0·05716
3	119	91	0·0135	0·05436
4	0	91	0·0058	

In correcting the results of the first three experiments for the imperfect elasticity of the wire, Coulomb calculated the values of m given by the four experiments, and subtracted the value given by the fourth from each of the others. But it is at the same time easier and more exact to subtract the value of $\log(1-m)^{-1}$ given by the fourth experiment from that given by each of the others. For if

$$-2c\frac{d\theta}{dt}, \qquad -2c'\frac{d\theta}{dt}$$

be the moments of two forces, each varying as the velocity, divided by the moment of inertia, the factors by which the initial arc of oscillation must be multiplied to get the arc at the end of the time t, first, when the two forces act together, secondly, when the second force acts alone, are

$$\epsilon^{-(c+c')t}, \qquad \epsilon^{-c't},$$

respectively, and that, whether the time t be great or small. Hence if we subtract the logarithm of the second factor from that of the first we shall get the logarithm of the factor due to the action of the first force alone. But if we put each factor under the form $1-m$, and subtract the m of the second factor from the m of the first, we shall not get the m due to the first force alone, unless t be small enough to allow of our neglecting the squares of ct and $c't$, or at least the product $ct \cdot c't$. In truth, when $t=\tau$, the quantities m are sufficiently small to be treated in Coulomb's manner without any material error, since the corrected values of $\log(1-m)$, obtained in the two ways, would only differ in the 4th place of decimals.

The numbers given in the last column of the above table were calculated from the formula (164), on substituting for $\log(1-m)^{-1}$ the numbers found in the first three lines of the 4th column, corrected by subtracting 0·0058. The mean of the three results is 0·05557, but the three experiments are not equally valuable for the determination of $\sqrt{\mu'}$. For the three numbers from which $\sqrt{\mu'}$ was deduced are 0·0510, 0·0152, 0·0077, and a given error in the first of these numbers would produce a smaller error in $\sqrt{\mu'}$ than that which would be produced by the same error in the second, still more, than that which would be produced by the same error in the third. If we multiply the three values of $\sqrt{\mu'}$ by 510, 152, and 77, respectively, and divide the sum of the products by 510 + 152 + 77 or 739, we get 0.05551. We may then take 0.555 as the result of the experiments. Assuming $\sqrt{\mu'} = 0.0555$ we have

$\log(1-m)^{-1}$	from experiment	0.0568 in No. 1,	0.021 in No. 2,	0.0135 in No. 3,
....	from theory	0.0571	0.0206	0.0137
	difference	− 0.0003	+ 0.0004	− 0.0002

65. So far the accordance of the theoretical and observed results is no very searching test of the truth of the theory. For, in fact, the theory is involved in the result only so far as this, that it shews that the resistance experienced by a given small element of a disk oscillating in a given period varies as the linear velocity; since the difference of periods in Coulomb's experiments was so small that the effects thence arising would be mixed up with errors of observation. This law is so simple that it might very well result from theories differing in some essential particulars from the theory of this paper. But should the numerical value of $\sqrt{\mu'}$ determined by Coulomb's experiments on disks be found to give results in accordance with theory in totally different cases, then the theory will receive a striking confirmation. Before proceeding to the discussion of other experiments, there are one or two minute corrections to be applied to the value of $\sqrt{\mu'}$ given above, which it will be convenient to consider.

In the first place, the result obtained in Art. 8 is only approximate, the approximation depending upon the circumstance that the diameter of the revolving body is large compared with a certain line determined by the values of μ' and τ. In the particular case in which the revolving solid is a circular disk, it happens that the approximate solution satisfies the general equations exactly, except so far as relates to the abrupt termination of the disk at its edge*. In consequence of this abrupt termination, the fluid annuli in the immediate neighbourhood of the edge are more retarded by the action of the surrounding fluid than they would have been were the disk continued, and consequently the resistance experienced by the disk in the immediate neighbourhood of its edge is actually a little greater than that given by the formula. I have not investigated the correction due to this cause, but it would doubtless be very small.

In the second place, the formula (15) is adapted to an indefinite succession of oscillations, whereas Coulomb did not turn the disk through an angle greater than the largest intended to be observed, and suffer one or two oscillations to pass before the observation commenced, but took for the initial arc that at which the disk had been set by the hand. Probably the disk was held in this position for a short time, so that the fluid came nearly to rest. If so, the resulting value of $\sqrt{\mu'}$, as may readily be shewn, would be a little too small. For in the course of an indefinite series of oscillations, the disk, in its forward motion, carries a certain quantity of fluid with it, and this fluid, in consequence of its inertia, tends to preserve its motion. Hence, when the disk, having attained its maximum displacement in the positive direction, begins to return, it finds the fluid moving in such a manner as to oppose its return,

* (*See Note A at the end.*)

and therefore it experiences a greater resistance than if it had started from the same position with the fluid at rest. In fact, it appears from the expression for G in Art. 8, that the moment of the resistance vanishes, in passing from negative to positive, not when the disk has reached the end of its excursion in the positive direction, but the eighth part of a period earlier. Hence, had the observation commenced during a series of oscillations, a larger initial arc would have been necessary, to overcome the greater resistance, in order to produce, after a given number of oscillations, the same final arc as that actually observed. I have investigated the correction to be applied on account of this cause, and find it to be about + 0.009, but I must refer to a note for the demonstration, in order not to interrupt the present discussion *. I shall assume then, in the following comparisons, that for water

$$\sqrt{\mu'} = 0.0564,$$

the units being the same as before, namely, an English inch and a second. That μ' is independent of the pressure of the fluid, or at least very nearly so, appears from an experiment of Coulomb's, in which it was found that the decrement of the arc of oscillation of a disk oscillating in water was the same in an exhausted receiver as under the full atmospheric pressure.

I will here mention another experiment of Coulomb's which bears directly on one part of the theory. On covering the disk with a thin coating of tallow, the resistance was found to be the same as before; and even when the tallow was sprinkled with powdered sandstone, by means of a sieve, the increase of resistance was barely sensible. This strikingly confirms the correctness of the equations of condition assumed to hold good at the surface of a solid.

66. I will now compare the formula (148) with the results obtained by Bessel for the oscillations of the brass sphere in water, which will be found at page 65 of his memoir. This sphere was suspended so as to be immersed in the water contained in a large vessel, and was swung with two different lengths of wire, the same as those employed for the experiments in air. The times of oscillation were 1·9085 second for the long pendulum, and 1·1078 for the short. The results are

	Long pendulum.	Short pendulum.
k, by experiment	0·648	0·602
k, by theory	0·631	0·600
difference	+ 0·017	+ 0·002

The depth to which the spheres were immersed is not stated, but it was probably sufficient to render the effect of the free surface small, if not insensible. The vessel was three feet in diameter, and the water 10 inches deep, so that unless the spheres were suspended near the bottom, which is not likely to have been the case, the effect of the limitation of the fluid by the sides of the vessel must have been but trifling. The agreement of theory and observation, as will be seen, is very close.

67. In the same memoir which contains the experiments on disks, Coulomb has given the results of some experiments in which the disk immersed in the fluid was replaced by a

* (*See Note* B *at the end.*)

long narrow cylinder, placed with its axis horizontal and its middle point in the prolongation of the axis of the vertical copper cylinder. In these experiments, the arcs did not decrease in geometric progression, as would have been the case if the resistance had varied as the velocity; but it was found that the results of observation could be satisfied by supposing the resistance to vary partly as the first power, and partly as the square of the velocity. In Coulomb's notation, $1 : 1 - m$ denotes the ratio in which the arc of oscillation would be altered after one oscillation, if the part of the resistance varying as the square of the velocity were destroyed. The several experiments performed with the same cylinder were found to be sufficiently satisfied by the formula deduced from the above-mentioned hypothesis respecting the resistance, when suitable numerical values were assigned to two disposable constants m and p, of which p related to the part of the resistance varying as the square of the velocity.

Conceive the cylinder divided into elementary slices by planes perpendicular to its axis. Let r be the distance of any slice from the middle point, θ the angle between the actual and the mean positions of the axis, dF that part of the resistance experienced by the slice which varies as the first power of the velocity. Then calculating the resistance as if the element in question belonged to an infinite cylinder moving with the same linear velocity, we have by the formulæ of Art. 31

$$dF = k'M'n\frac{d\xi}{dt}, \quad \text{where } M' = \pi\rho a^2 dr, \quad \frac{d\xi}{dt} = r\frac{d\theta}{dt}.$$

If G be the moment of the resistance, l the whole length of the cylinder, we have, putting $n = \pi\tau^{-1}$,

$$G = \frac{\pi^2 k'\rho a^2 l^3}{12\tau}\frac{d\theta}{dt};$$

whence

$$\log_e (1 - m)^{-1} = \frac{\pi^2 k'\rho a^2 l^3}{24 I}, \quad . \ . \ . \ . \ . \ . \ . \ (165)$$

I being the moment of inertia.

Expressing I in terms of the same quantities as in the case of the disk, we get from (147) and (165)

$$\log_{10} (1 - m)^{-1} = \log_{10}\epsilon \, . \frac{\pi\mu' T l^3}{3R^2} . \frac{g\rho}{W} . \frac{T}{\tau} . \mathfrak{m}^2 k', \quad . \ . \ . \ (166)$$

and $g\rho$ is the weight of a cubic millimetre of water, or the 1000th part of a gramme. The numerical values of μ', T, R, W have been already given, but μ' must be reduced from square inches to square millimetres. The cylinders, of which three were tried in succession, had all the same length, namely, 249 millimetres. Their circumferences, calculated from their weights and expressed in millimetres, were 21.1, 11.2, and 0.87, and the time of four oscillations was 92^s, 91^s, 91^s. The values of $\mathfrak{m}$ calculated from these data by means of the formula (147) are 0.4332, 0.2312, and 0.01796. For the first and second of these values, $\mathfrak{m}^2 k'$ may be obtained by interpolation from the table given in Part I.; for the third it will be sufficient to employ the second of the formulæ (115).

The following are the results :

	Cylinder, No. 1.	No. 2.	No. 3.
m, by experiment	0.0400	0.0260	0.0136
m, by theory................	0.0413	0.0291	0.0113
Difference	− 0.0013	− 0.0031	+ 0.0023

The differences between the results of theory and experiment are perhaps as small as could reasonably be expected, when it is considered that, notwithstanding the delicate nature of the experiments, the numerical values of two constants, m and p, had to be deduced from their results.

68. This memoir of Coulomb's contains also a notice of a set of experiments with disks and cylinders in which the water was replaced by oil. The experiments with disks shewed that with a given disk the arc of oscillation decreased in geometric progression, and that with different disks the moments of the resistances were as the fourth powers of the diameters. The absolute resistances were greater than in the case of water in the ratio of about 17·5 to 1. The details of Coulomb's experiments on cylinders oscillating in oil are entirely omitted. It is merely stated that on making the same cylinders as before, or shorter cylinders when the resistance was too great, oscillate in oil, it was found, conformably with the results obtained with planes, that the coherence of oil was to that of water as 17 to 1. The coherence is here supposed to be measured by that part of the resistance which is proportional to the first power of the velocity. On making a rough calculation of the ratio of the resistances to cylinders oscillating in oil and in water, on the supposition that $\sqrt{\mu'}$ for oil is to $\sqrt{\mu'}$ for water as 17·5 to 1, as would follow from the experiments on disks if the difference of the specific gravities of the two fluids be neglected, I found that the ratio in question ought to have been somewhere about 100 to 1, instend of only 17 to 1. It would seem from this that the theory of the present paper is not applicable to oil; but fresh experiments would be required before this point can be considered as established, on account of the theoretical doubt respecting the application of the formulæ of Section III. Part I., to extremely fine cylinders, especially in cases in which μ' is large, so that $\mathfrak{m}$ is very small. It would be interesting to make out whether what I have called internal friction is or is not of the same nature as viscosity. Coulomb and Dubuat apply the term *viscosity* to that property of water by virtue of which certain effects are produced which have been shewn in this paper to be perfectly explicable on the theory of internal friction; whereas Poisson, in one of his memoirs, expressly asserts that the terms in the equations of motion which result from what has been called in this paper internal friction belong to perfect fluids, and have nothing to do with viscosity*. Poisson does not give the slightest hint as to the grounds on which he rested his opinion.

69. I come now to the experiments of Dubuat, which are contained in an excellent work of his entitled ***Principes d'Hydraulique***, of which the second edition was published in 1786.

* *Journal de l'Ecole Polytechnique*, Tom. XIII. p. 95.

The first edition does not contain the experiments in question. Dubuat justly remarked that the time of oscillation of a pendulum oscillating in a fluid is greater than it would be in vacuum, not only on account of the buoyancy of the fluid, which diminishes the moving force, but also on account of the mass of fluid which must be regarded as accompanying the pendulum in its motion; and even determined experimentally the mass of fluid which must be regarded as carried by the oscillating body in the case of spheres and of several other solids. Thus Dubuat anticipated by about forty years the discovery of Bessel; but it was not until after the appearance of Bessel's memoir that Dubuat's labours relating to the same subject attracted attention.

Dubuat's method was as follows. Imagine a body suspended by a fine thread or wire and swung in vacuum, and let a be the length of the pendulum, reckoned from the centre of suspension to the centre of oscillation. Now imagine the same body swung in a fluid, in which its apparent weight is p, so that if P denote the weight of fluid displaced, the true weight of the body will be $p + P$. Since the moving force is diminished in the ratio of $p + P$ to p, if the inertia of the body were all that had to be overcome, it would be necessary to diminish the length of the pendulum in the same ratio, in order to preserve the same time of oscillation. But since the mass in motion consists not only of the mass of the body itself, but also of that of the fluid which it carries with it, the pendulum must be shortened still more, in order that the time of oscillation may be unaltered. Let l be the length of the pendulum so shortened, and $\mathfrak{n}$ (which for the same reason as before I write instead of Dubuat's n,) a factor greater than unity, such that $p + \mathfrak{n}P$ is the weight of the mass in motion; then

$$l = \frac{ap}{p + \mathfrak{n}P}, \qquad \text{whence } \mathfrak{n} = \frac{p}{P}\left(\frac{a}{l} - 1\right). \qquad . \quad . \quad . \quad (167)$$

Dubuat's experiments on this subject consist of 44 experiments on spheres oscillating in water, (Tom. II. p. 236); 31 experiments on other solids oscillating in water, (p. 246); and 3 experiments on spheres oscillating in air, (p. 283). The following table contains a comparison of the formula (148) with Dubuat's results for spheres oscillating in water. The value of $\sqrt{\mu'}$ employed in the calculation is 0.0564 inch English, or 0.05291 inch French.

Dubuat's experiments on spheres oscillating in water.

		n		
	τ	calc.	obs.	diff.
Sphere of lead. Diameter 1·0113 inches. Weight in water 2102 grains.	$\frac{1}{2}$	1·633	1·502	−·131
	1	1·687	1·502	−·185
	2	1·766	1·522	−·244
	3	1·825	1·620	−·205
Sphere of glass. Diameter 2·645 inches. Weight in water 574 grains.	2	1·602	1·518	−·084
	4	1·644	1·569	−·075
	6	1·676	1·598	−·078
Same sphere weighing in water 2102 grains.	1	1·572	1·515	−·057
	2	1·602	1·516	−·086
	3	1·624	1·523	−·101
	4	1·644	1·546	−·098
Same sphere weighing in water 4204 grains.	1	1·572	1·537	−·035
	2	1·602	1·523	−·079
	3	1·624	1·524	−·100
	4	1·644	1·538	−·106
Same sphere weighing in water 9216 grains.	$\frac{1}{2}$	1·551	1·449	−·102
	1	1·572	1·372	−·200
	2	1·602	1·494	−·108
	3	1·624	1·494	−·130

		n		
	τ	calc.	obs.	diff.
Sphere of wood. Diameter 4·076 inches. Weight in water 2102 grains.	2	1·566	1·507	−·059
	3	1·581	1·547	−·034
	4	1·593	1·547	−·046
	6	1·614	1·567	−·057
Same sphere weighing in water 4204 grains.	1	1·547	1·375	−·172
	2	1·566	1·456	−·110
	3	1·581	1·525	−·056
	4	1·593	1·557	−·036
	6	1·614	1·549	−·065
Same sphere weighing in water 9216 grains.	1	1·547	1·57	+·023
	2	1·566	1·553	−·013
	3	1·581	1·59	+·009
	4	1·593	1·583	−·010
Another sphere of wood. Diameter $6\frac{2}{3}$ inches. Weight in water 2102 grains.	3	1·549	1·27	−·279
	4	1·557	1·394	−·163
	6	1·570	1·487	−·083
	9	1·585	1·566	−·019
	12	1·599	1·569	−·030
	18	1·621	1·565	−·056
Same sphere weighing in water 3204 grains.	10·85	1·594	1·634	+·040
Same sphere weighing in water 4204 grains.	3	1·549	1·651	+·102
	4	1·557	1·627	+·070
	6	1·570	1·654	+·084
	9	1·585	1·664	+·079
	12	1·599	1·674	+·075

70. If we strike out the experiments with the large sphere, which cannot well be compared with theory for a reason which will be explained further on, it will be observed that in seven out of the eight groups of experiments left, the signs in the last column are regularly *minus*. The preponderance of negative errors could be destroyed by using a much smaller value of $\sqrt{\mu'}$ in the reduction. We have seen, however, that the value of $\sqrt{\mu'}$ deduced from Coulomb's experiments on the decrement of the arc of oscillation of disks satisfied almost exactly Bessel's observations of the time of oscillation of a sphere about two inches in diameter oscillating in water. The very small errors which remained in this case had both the sign +, whereas in Dubuat's experiments on the 1-inch and $2\frac{1}{2}$ inch spheres, the errors, which are far larger, have all the sign −. Since the experiments of Dubuat and Bessel, though made under similar circumstances, do not lead to the same result, it is of course impossible for any theory to

satisfy them both. The numbers in the last column of the preceding table are, however, far too regular to be attributable to mere fortuitous errors of observation. If we suppose Bessel's results to have been nearly exact, there must have been something in the mode either of making or of reducing Dubuat's experiments which caused a tendency to error in one direction.

With respect to the reduction of the experiments it may be observed that the length l was measured from the centre of oscillation, whereas in the formula (148) it is supposed that the mass of which the weight is kP or $(\mathfrak{n} - 1) P$ is collected at the centre of the sphere. If h be the distance of the centre of the sphere from the axis of suspension, the observed value of $\mathfrak{n} - 1$ ought in strictness to be increased in the ratio of h^2 to l^2, or the calculated value diminished in the ratio of l^2 to h^2, before comparing the results of theory and experiment. In the case of the loaded spheres especially, the theoretical value of $\mathfrak{n}$ would thus be a little diminished; but except in a very few cases, in which either l or $a - l$ is small, the diminution is hardly worth considering. After having been for a good while at a loss to account for the regular occurrence of rather large negative errors, the following occurred to me as the probable solution of the difficulty.

When a pendulum oscillates in water, the arc of oscillation rapidly decreases; this rapid diminution forms in fact the grand difficulty in experiments of this kind. In Dubuat's experiments, it will be remembered, the suspending thread was lengthened or shortened till the time of oscillation was an exact number of seconds, or occasionally half a second. Now, it is probable that the observer occasionally gave the suspending thread a slight push as the pendulum was commencing its return, in order to keep the oscillations going for a sufficient time to allow of tolerable precision in rendering the time of oscillation equal to what it ought to be. If so, these pushes would slightly accelerate the oscillations, and therefore cause the length of thread fixed on by observation to be a little too great, which would make the effect of the water in retarding the oscillations appear a little too small. On inspecting the table of differences, it may be observed that sometimes when the same sphere differently loaded is swung in the same time as before, the numbers in the table of differences are altered more than appears to be attributable to merely fortuitous errors of observation. This accords very well with the conjecture just mentioned, and seems difficult to account for in any other way, inasmuch as everything relating to the fluid must have been almost exactly the same in the two cases.

The occurrences of positive differences in the case of the large wooden sphere may be accounted for by the limitation of the fluid mass by the sides and bottom of the vessel, and by the free surface, which, except in the case of very short oscillations, would have much the same effect as a rigid plane, inasmuch as it would be preserved almost exactly horizontal by the action of gravity. The vessel which contained the water was 51 inches long and 17 broad, the water was 14 inches deep, and the spheres were plunged to about 3 inches below the surface, so that the effect of the confinement of the fluid mass would have been quite sensible in the case of such large spheres. If it be objected that the same sphere gave negative differences in the case of the first group of experiments, it must be observed, that when the apparent weight of so large a sphere was only 2102 French grains, the resistance would quickly have caused the oscillations to subside if an extraneous force had not frequently been applied.

71. In Dubuat's experiments on spheres oscillating in air, the lightness of the fluid was compensated by the extreme lightness of the spheres, which were composed, the first two of paper, and the third of gold-beater's skin. In the following table the diameter $2a$ of the sphere is expressed in French inches. The value of $\sqrt{\mu'}$ employed in the reduction is the same as was before used in the reduction of observations made in air, namely 0.116 inch English, or 0.1088 inch French.

Dubuat's experiments on light spheres oscillating in air.

No.	$2a$	τ	n calc.	n obs.	Diff.
337	4·0416	1·51	1·61	1·51	− 0·10
338	6·625	1·84	1·57	1·63	+ 0·06
339	17·25	3·625	1·53	1·54	+ 0·01

The differences certainly appear very small when the delicacy of the experiments and the simplicity of the apparatus employed are considered.

72. The only comparison yet made in this section between theory and observation in the case of pendulum experiments, consists in comparing the observed times of vibration with the results calculated with an assumed value of $\sqrt{\mu'}$. But according to theory we ought to be able, without assigning a particular value to any new disposable constant, to calculate the rate of decrease of the arc of vibration. I have not met with any experiments made with a view of investigating the decrease in the arc of vibration in the case of extremely small vibrations, such as those employed in pendulum experiments. The experiments of Newton and others, in which the arc of vibration was so large that the resistance depended mainly on the square of the velocity, would be quite useless for my purpose. The pendulum experiments of Bessel and Baily contain however the requisite information, or at least some portion of it, for the arcs are registered for the sake of giving the data for calculating the small reduction to indefinitely small vibrations.

In Bessel's experiments the arc is registered for the end of equal intervals of time during the motion. The number of such registrations in one experiment amounts in some cases to eleven, and is never less than three. So far the observations are just what are wanted; but there are other causes which prevent an exact comparison between theory and experiment. In the first place the spheres were swung so close to the back of the frame that the increase of resistance due to the confinement of the air must have been very sensible. In the second place the effect of the wire must have been very sensible, especially in the case of the long pendulum. For the table of Section III. Part I., shews that for the wire (for which $\mathfrak{m}$ is very small) the value of k' is much larger than that of k, whereas for spheres of the size of those employed, when the time of oscillation is only one or two seconds, k' is a good deal smaller than k. Hence, if the formulæ of that section applied to such fine wires, the effect of the wire on the arc of vibration would be much greater than its effect on the time of vibration, and therefore would be quite sensible. But it has been shewn in Section IV., that the effect of the wire in diminishing the arc of vibration is probably greater than would be given by the formula, and therefore the uncertainty depending on the wire is likely to amount to a very sensible fraction of the whole amount. Again, since Bessel's experiments were all made in air, no data are afforded whereby to eliminate the portion of the observed result which was due to friction at the point of support, imperfect elasticity of the wire, or gradual dissipation of vis viva by communication of motion to the supporting frame. Moreover in the case of the long pendulum the observations were made with rather too large arcs, for the law of the decrease of the arc of vibration deviated sensibly from that of a geometric progression. In Baily's experiments, only the initial and final arcs are registered, and not even those in the case of the "additional experiments." Hence these experiments do not enable us to make out whether it would be sufficiently exact to suppose the decrease to take place in geometric progression. Moreover, the final arc was generally so small, that a small error committed in the measurement of it would cause a very sensible error in the rate of decrease concluded from the experiment. For these reasons it would be unreasonable to expect a near accordance between the formulæ and the results of the experiments of Bessel and Baily. Still, the formulæ might be expected to give a result in defect, and yet not so much in defect as not to form a large portion of the result given by observation. On this account it will not be altogether useless to compare theory and observation with reference to the decrement of the arc of vibration.

73. Let us first consider the case of a sphere suspended by a fine wire. Let the notation be the same as was used in investigating the expression for the effect of the air on the time of vibration, except that the factors k', k'_1 come in place of k, k_1. Considering only that part of the resistance which affects the arc of vibration, we have for the portions due respectively to the sphere and to the element of the wire whose length is ds, and distance from the axis of suspension s,

$$k'M'n(l+a)\frac{d\theta}{dt}, \qquad k'_1\frac{M'_1}{l}ds\,.\,ns\frac{d\theta}{dt},$$

and if we take the moment of the resistance, and divide by twice the moment of inertia, the coefficient of $\frac{d\theta}{dt}$ in the result, taken negatively, and multiplied by t, will be the index of ϵ in the expression for the arc. Hence if α_0 be the initial arc of vibration, and α_t the arc at the end of the time t

$$\log_\epsilon \alpha_0 - \log_\epsilon \alpha_t = \frac{k'M'(l+a)^2 + \frac{1}{3}k_1'M_1'l^2}{M(l+a)^2 + \frac{1}{3}M_1l^2}\cdot\frac{\pi t}{2\tau}, \quad . \quad . \quad . \quad (168)$$

$M'(l+a)^2$ being as before taken for the moment of inertia of the sphere, which will be abundantly accurate enough. If then we put $\mathfrak{l}$ for the Napierian logarithm of the ratio of the arc at the beginning to the arc at the end of an oscillation, we must put $t = \tau$ in (168), whence, neglecting the effect of the wire, we obtain

$$\mathfrak{l} = \frac{\pi k'}{2} \cdot \frac{\sigma}{S} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

If now $\Delta k'$ be the correction to be applied to k' in this formula on account of the wire, since k', k_1' are combined together in the expression for the arc just as k, k_1 in the expression for the time, we get

$$\Delta k' = \frac{k_1'}{k_1} \Delta k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

and the approximate formulæ (115) give

$$\Delta k' = -\frac{4L}{\pi} \Delta k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

whence the numerical value of $\Delta k'$ is easily deduced from that of Δk, which has been already calculated. We get also from (52)

$$k' = k - \tfrac{1}{2} + \tfrac{4}{9}(k - \tfrac{1}{2})^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

whence k' may be readily deduced from k, which has been already calculated.

74. Before comparing these formulæ with Bessel's experiments, it will be proper to enquire how far the latter are satisfied by supposing the arcs of oscillation to decrease in geometric progression. In Bessel's tables the arc is registered in the column headed μ. This letter denotes the number of French lines read off on a scale placed behind the wire, and a little above the sphere, and is reckoned from the position of instantaneous rest of the wire on one side of the vertical to the corresponding position on the other side. The distance of the scale from the axis of suspension being given, as well as the correction to be applied to μ on account of parallax, the arc of oscillation may be readily deduced. However, for our present purpose, any quantity to which the arc is proportional will do as well as the arc itself, and μ, though strictly proportional to the tangent of the arc, may be regarded as proportional to the arc itself, inasmuch as the initial arc usually amounted to only about 50′ on each side of the vertical.

Now we may form a very good judgment as to the degree of accuracy of the geometric formula by comparing the arc observed in the middle of an experiment with the geometric mean of the initial and final arcs. I have treated in this way Bessel's experiments, Nos. 1, 2, 3, 4, and 5. Each of these is in fact a group of six experiments, four with the long pendulum and two with the short, so that the whole consists of 20 experiments with the long pendulum, and 10 with the short. In the case of the long pendulum, the observed value of μ regularly fell short of the calculated value, and that by a tolerably constant quantity. The mean difference amounted to 0.688 line, and the mean error in this quantity to 0.109. This mean error was not due entirely to errors of observation, or variations in the state of the air, &c., but partly also to slight variations in the initial arc, larger differences usually accompanying larger initial arcs. The initial arc usually corresponded to $\mu = 39$ or 40 lines, and the final to $\mu = 15$

or 16 lines. In the case of the short pendulum, the differences in 8 cases out of 10 had the same sign as before. The mean difference was 0.025, and the mean error 0.043. The arcs of oscillation were nearly the same as before; but inasmuch as the axis of suspension was nearer to the scale than before, the initial value of μ was only about 12 or 13 lines, and the final value about 7 lines. When the results of some of the experiments were laid down on paper, by abscissæ taken proportional to the times and ordinates to the logarithms of μ, it was found that in the case of the long pendulum the line so drawn was decidedly curved, the concavity being turned toward the side of the positive ordinates. The curvature of the line belonging to the short pendulum could hardly be made out, or at least separated from the effects of errors of observation. The experiments 9, 10, 11, having been treated numerically in the same way as the experiments 1—5, led to much the same result. In the 16 experiments with the ivory sphere and short pendulum contained in the experiments Nos. 12, 13, 14, and 15, the excess of the calculated over the observed value of μ was more apparent, the mean excess amounting to 0.129. The reason of this probably was, that the observations with the ivory sphere were made through a somewhat wider range of arc than those with the brass sphere.

It appears then that at least in the case of the long pendulum a correction is necessary, in order to clear the observed decrease in the arc of oscillation from the effect of that part of the resistance which increases with the arc more rapidly than if it varied as the first power of the velocity, and so to reduce the observed rate of decrease to what would have been observed in the case of indefinitely small oscillations.

75. In Coulomb's experiments it appeared that the resistance was composed of two terms, one involving the first power, and the other the square of the velocity. If we suppose the same law to hold good in the present case, and denote the amplitude of oscillation at the end of the time t, measured as an angle, by a, we shall obtain

$$\frac{da}{dt} = -Aa - Ba^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

where A and B are certain constants. We must now endeavour to obtain A from the results of observation. Since the substitution for a of a quantity proportional to a will only change the constant B in (173), and the numerical value of this constant is not required for comparison with theory, we may substitute for a the number of lines read off on the scale as entered in Bessel's tables in the columns headed μ.

I have employed four different methods to obtain A from the observed results. The one I am about to give is the shortest of the four, and is sufficiently accurate for the purpose.

The equation (173) gives after dividing by a

$$\frac{d \log a}{dt} = -A - Ba. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

Now, as has been already observed, the arcs of vibration decrease nearly in geometric progression. If this law were strictly true, we should have

$$a = a_0 \left(\frac{a_2}{a_0}\right)^{\frac{t}{T}}, \quad \ldots \ldots \quad (175)$$

where a_0 denotes the initial and a_2 the final arc, and T denotes the whole time of observation. We may, without committing any material error, substitute this value of a in the last term of (174). The magnitude of the error we thus commit is not to be judged of merely by the smallness of B. The approximate expression (175) is rather to be regarded as a well-chosen formula of interpolation, and in fact $T^{-1} \log_\epsilon (a_0 a_2^{-1})$ differs very sensibly from A. Making now this substitution in (174), integrating, and after integration restoring a in the last term by means of (175), we get

$$\log a = -At - \frac{BTa}{\log a_2 - \log a_0} + C, \quad \ldots \ldots \quad (176)$$

C being an arbitrary constant. To determine the three constants A, B, C, let a_1 be the arc observed at the middle of the experiment, apply the last equation to the arcs a_0, a_1, a_2, and take the first and second differences of each member of the equation. Let Δ_1 denote the sum of the two first differences, so that $\Delta_1 t$ is the same thing as T. Then we may take for the two equations to determine A and B

$$\Delta_1 \log a_0 = -A\Delta_1 t - \frac{B\Delta_1 t \,.\, \Delta_1 a_0}{\Delta_1 \log a_0}; \qquad \Delta^2 \log a_0 = -\frac{B\Delta_1 t \,.\, \Delta^2 a_0}{\Delta^2 \log a_0}.$$

Eliminating B, and passing from Napierian to common logarithms, which will be denoted by Log., we get

$$A = \frac{-\Delta_1 \operatorname{Log} a_0}{\operatorname{Log} \epsilon \,.\, \Delta_1 t} \left\{1 - \frac{\Delta^2 \operatorname{Log} a_0 \,.\, \Delta_1 a_0}{\Delta_1 \operatorname{Log} a_0 \,.\, \Delta^2 a_0}\right\}. \quad \ldots \quad (177)$$

If we suppose the part of $-\dfrac{da}{dt}$ which does not vary as the first power of a to be $a^2 \phi'(a)$ instead of Ba^2, we shall get in the same way

$$A = \frac{-\Delta_1 \operatorname{Log} a_0}{\operatorname{Log} \epsilon \,.\, \Delta_1 t} \left\{1 - \frac{\Delta^2 \operatorname{Log} a_0 \,.\, \Delta_1 \phi(a_0)}{\Delta_1 \operatorname{Log} a_0 \,.\, \Delta^2 \phi(a_0)}\right\}. \quad \ldots \ldots \quad (178)$$

76. I have not attempted to deduce evidence for or against the truth of equation (173) from Bessel's experiments. The approximate formula (175) so nearly satisfied the observations, that almost any reasonable formula of interpolation which introduced one new disposable constant would represent the experiments within the limits of errors of observation. It may be observed, that the factor outside the brackets in equations (177) and (178) is the first approximate value of A got by using only the initial and final arcs, and supposing the arcs to decrease in geometric progression. In the case of the long pendulum, the value of A, corrected in accordance with the formula (178), would be very sensibly different according as we supposed $\phi(a)$ to be equal to Ba, in which case (178)

would reduce itself to (177), or equal to Ba^2. In the case of the long pendulum with the brass sphere, the corrected value of A, deduced from the formula (177), was equal to about 0.77 of the first approximate value.

I have not considered it necessary to go through all Bessel's experiments, as it was not to be expected that the formula should account for the whole observed decrement. I have only taken four experiments for each kind of pendulum, namely, I. a, b, e, and f for the long pendulum with the brass sphere; I. c and d and II. c and d for the short pendulum with the brass sphere; XII. a, b, c, and d for the long pendulum with the ivory sphere, and XII. a', b', c', and d' for the short pendulum with the ivory sphere. The formula (177) gave the following results. First case, $\text{Log}\,\epsilon\,.\,\tau A = .0000759$; mean error $= .0000020$. Second case, $\text{Log}\,\epsilon\,.\,\tau A = .0000504$; mean error $= .0000075$. Third case, $\text{Log}\,\epsilon\,.\,A = .000631$; mean error $= .000046$. Fourth case, $\text{Log}\,\epsilon\,.\,A = .000167$; mean error $= .000074$. Now $\mathfrak{l} = \tau A$, and therefore, to get the values of $\mathfrak{l}$ deduced from experiment, it will be sufficient to divide the numbers above given by the modulus of the common system of logarithms. The theoretical value of $\mathfrak{l}$ will be got from (169), if we add to k' the correction $\Delta k'$ depending upon the wire. The following are the results:

	long p. brass s.	short p. brass s.	long p. ivory s.	short p. ivory s.
1000000 $\mathfrak{l}$ for sphere alone in an unlimited mass of fluid, by theory	67	50	298	222
additional for wire	27	9	114	39
	94	59	412	261
1000000 $\mathfrak{l}$ by experiment	175	116	1453	384

It appears then that the calculated rate of decrease of the arc amounts on the average to about half the rate deduced from observation. This is about what we might have expected, considering the various circumstances, all tending materially to augment the rate of decrease, which were not taken into account in the calculation.

77. Of Baily's pendulums I have compared the following with theory in regard to the decrement of the arc of vibration. No. 1 (the $1\frac{1}{2}$-inch platina sphere), experiments 1 to 8; No. 3 (the brass $1\frac{1}{2}$-inch sphere), experiments 9 to 16; No. 6 (the 2-inch brass sphere), experiments 33 to 40; No. 21 (the 0.410 inch long copper cylindrical rod), experiments 109 to 112; and No. 35—38 (the $1\frac{1}{2}$-inch long brass tube), experiments 167 to 174. I have not thought it worth while to compute the results obtained with the other $1\frac{1}{2}$-inch and 2-inch spheres, inasmuch as they were of the same size as the brass spheres, and moreover the observation of the decrement of the arc was not the object Baily had in view in making the experiments. The 3-inch sphere, and all the other cylindrical rods and combinations of cylindrical rods and spheres, belong to the "additional experiments" for which the arcs are not given.

The mode of performing the calculation will best be explained by an example. Take, for instance, the pair of experiments Nos. 1 and 2. In No. 1 the total interval was 4.22 hours, the initial arc was $0^\circ.77$, the final arc $0^\circ.29$, the mean height of the barometer 30.24 inches, and the temperature about $38\frac{1}{2}^\circ F$. The difference of the common logarithms of the initial and final arcs is 0.424, and this divided by the total interval gives 0.1005 for the difference of logarithms for one hour. The second experiment, treated in a similar way, gives 0·0352, which expresses the effect of friction at the point of support, communication of motion to the support itself, &c., together with the resistance of highly rarefied air at a pressure of only 0.97 inch of mercury. Since we have reason to believe that μ' is independent of the density, we may get the effect of air at a pressure of 30.24 – 0.97 or 29.27 inches of mercury by subtracting 0.0352 from 0.1005, which gives 0.0653. Reducing to 29 inches of mercury for convenience of comparison, we get 0.0649. Each pair of experiments is to be treated in the same way. Since the temperature was nearly the same in the experiments made with the same pendulum, we may suppose it constant, and equal to the mean of the temperatures in the experiments made under the full atmospheric pressure. The experiments reduced consist of four pair for each pendulum, except No. 21, for which only two pair were performed. The following are the results. For the $1\frac{1}{2}$-inch platina sphere 0.0644, mean error 0.0044. For the $1\frac{1}{2}$-inch brass sphere 0.180, mean error 0.024. For the 2-inch brass sphere 0.094, mean error 0.013. For the copper rod 0.486, mean error 0.113. For the brass tube the results were 0.145, 0.363, 0.338, 0.305. Rejecting the first result as anomalous, and taking the mean of the others, we get 0.335, mean error 0.030. To obtain I from the mean results above given we have only to divide by 3600 times the modulus, and multiply by τ, and for the experiments with spheres we may suppose $\tau = 1$.

The mode of calculating I from theory in the case of a sphere suspended by a fine wire has already been explained. For the sake of exhibiting separately the effect of the wire, I will give one intermediate step in the calculation.

	1.44 inch sphere.	1.46 inch sphere.	2.06 inch sphere.
k', for sphere alone...............	0.326	0.320	0.220
$\Delta k'$, the correction for the wire...	0.130	0.130	0.045
Total, to be substituted in (169)..	0.456	0.450	0.265

The formula (168), which applies to a sphere suspended by a wire, will be applicable to a long cylindrical rod if we suppose $M = 0$. Hence the same formula (169) that has been used for a sphere may be applied to a cylindrical rod if we suppose k' to refer to the rod. For the copper rod $k' = 1.107$, and for the tube $k' = 0.2561$. The following are the results for the three spheres and two cylinders.

	No. 1.	No. 3.	No. 6.	No. 21.	Nos. 35—38.
1000000I, from experiment...	41	115	60	315	206
. . . . . from theory	39	106	60	237	156
Difference.........	+ 2	+ 9	0	+ 78	+ 50

It appears that the experiments with spheres are satisfied almost exactly. The differences between the results of theory and observation are much larger in the case of the long cylinders. Large as these differences appear, they are hardly beyond the limits of errors of observation, though they would probably be far beyond the limits of errors of observation in a set of experiments performed on purpose to investigate the decrement of the arc of vibration. It was to be expected beforehand that the results of calculation would fall short of those of observation, inasmuch as only two arcs were registered in each experiment, so that no data were afforded for eliminating the effect of that part of the resistance which did not vary as the first power of the velocity.

78. I have now finished the comparison between theory and experiment, but before concluding this Section I will make a few general remarks.

When a new theory is started, it is proper to enquire how far the theory does violence to the notions previously entertained on the subject. The present theory can hardly be called new, because the partial differential equations of motion were given nearly thirty years ago by Navier, and have since been obtained, on different principles, by other mathematicians; but the application of the theory to actual experiment, except in some doubtful cases relating to the discharge of liquids through capillary tubes, and the determination of the numerical value of the constant μ', are, I believe, altogether new. Let us then, in the first instance, examine the magnitude of the tangential pressure which we are obliged by theory to suppose capable of existing in air or water.

For the sake of clear ideas, conceive a mass of air or water to be moving in horizontal layers, in such a manner that each layer moves uniformly in a given horizontal direction, while the velocity increases, in going upwards, at the rate of one inch per second for each inch of ascent. Then the sliding in the direction of a horizontal plane is equal to unity, and therefore the tangential pressure referred to a unit of surface is equal to μ or $\mu'\rho$. The absolute magnitude of this unit sliding evidently depends only on the arbitrary unit of time, which is here supposed to be a second. In the case supposed, it will be easily seen that the particles situated at one instant in a vertical line are situated at the expiration of one second in a straight line inclined at an angle of 45^0 to the horizon. Equating the tangential pressure $\mu'\rho$ to the normal pressure due to a height h of the fluid, we get $h = g^{-1}\mu'$, g being the force of gravity. Putting now $g = 386$, $\mu' = (0.116)^2$ for air, $\mu' = (0.0564)^2$ for water, we get $h = 0.00003486$ inch for air, and $h = 0.000008241$ inch for water, or about the one thirty-thousandth part of an inch for air, and less than the one hundred-thousandth part of an inch for water. If we enquire what must be the side of a square in order that the total tangential pressure on a horizontal surface equal to that square may amount to one grain, supposing the density of air to be to that of water as 1 to 836, and the weight of a cubic inch of water to be 252.6 grains, we get 25 feet 8 inches for air, and 1 foot 10 inches for water. It is plain that the effect of such small forces may well be insignificant in most cases.

79. In a former paper I investigated the effect of internal friction on the propagation of sound, taking the simple case of an indefinite succession of plane waves*. It appeared that the effect consisted partly in a gradual subsidence of the motion, and partly in a diminution of the velocity of propagation, both effects being greater for short waves than for long. The second effect, as I there remarked, would be contrary to the result of an experiment of

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* Camb. Phil. Trans. Vol. VIII. p. 302.

M. Biot's, unless we supposed the term expressing this effect to be so small that it might be disregarded. I am now prepared to calculate the numerical value of the term in question, and so decide whether the theory is or is not at variance with the result of M. Biot's experiment.

According to the expression given in the paper just mentioned, we have for the proportionate diminution in the velocity of propagation

$$\frac{8\pi^2\mu'^2}{9\lambda^2 V^2},$$

λ being the length of a wave, and V the velocity of sound. To take a case as disadvantageous as possible, suppose λ only equal to one inch, which would correspond to a note too shrill to be audible to human ears. Taking the velocity of sound in air at 1000 feet per second, there results for the common logarithm of the expression above written $\overline{11}.0428$, so that a wave would have to travel near 100000000000 inches, or about 1578000 miles, before the retardation due to friction amounted to one foot. It is plain that the introduction of internal friction leaves the theory of sound just as it was, so far as the velocity of propagation is concerned, at least if the sound be propagated in free air.

The effect of friction on the intensity of sound depends on the first power of μ'. In the case of an indefinite succession of plane waves, it appears that during the time t the amplitude of vibration is diminished in the ratio of 1 to ϵ^{-ct}, and therefore the intensity in the ratio of 1 to ϵ^{-2ct}, where

$$c = \frac{8\pi^2\mu'}{3\lambda^2}.$$

Putting $\lambda = 1$ and $t = 1$ we get 1 to 0·4923, or 2 to 1 nearly, for the ratio in which the intensity is altered during one second in the case of a series of waves an inch long. The rate of diminution decreases very rapidly as the length of wave increases, so that in the case of a series of waves one foot long the intensity is altered in one second in the ratio of 1 to 0·995095, or 201 to 200 nearly. It appears then that in all ordinary cases the diminution of intensity due to friction may be neglected in comparison with the diminution due to divergence. If we had any accurate mode of measuring the intensity of sound it might perhaps be just possible, in the case of shrill sounds, to detect the effect of internal friction in causing a more rapid diminution of intensity than would correspond to the increase of distance from the centre of divergence.

Section II.

Suggestions with reference to future experiments.

80. I am well aware that the mere proposal of experiments does not generally form a subject fit to be brought before the notice of a scientific society. Nevertheless, as it frequently happens in the division of labour that one person attends more to the theoretical, another to the experimental investigation of some branch of science, it is not always useless for the theorist to point out the nature of the information which it would be most important to obtain from experiment. I hope, therefore, that I may be permitted to offer a few hints with reference to experiments in which the theory of the internal friction of fluids is concerned. I shall omit all details, since they would properly come in connexion with the experiments.

Experiments with which the theory of internal friction in fluids has more or less to do may be performed for either of the following objects: first, to test still further the truth of the theory; secondly, to determine the index of friction of various gases, liquids, or solutions; to investigate the dependance of the index of friction of a gas on its state of pressure, temperature, and moisture; or to endeavour to make out the law according to which the index of friction of a mixture of gases depends upon the indices of friction of the separate gases; thirdly, to measure the length of the seconds' pendulum, or its variation from one part of the earth's surface to another.

81. *First object.* The theory has been already put to a pretty severe test by means of the experiments of Baily and others. Nevertheless there are some uncertainties in the comparison of theory and experiment arising from the influence of modifying causes of which the effect could only be estimated from theory, and yet was not so small as to be merged in errors of observation. Moreover, experiments on the decrement of the arc of vibration are almost wholly wanting. The following system of pendulums, meant to be swung in air and in vacuum, would afford a very good test of the theory.

No. 1. A 2-inch or $1\frac{1}{2}$-inch sphere swung with a fine wire.

No. 2. A very small sphere swung with the same kind of wire.

No. 3. A long cylindrical rod, a few tenths of an inch in diameter.

No. 4. A cylinder only three or four inches long, of the same diameter as No. 3, swung with the same kind of wire as No. 1.

The vacuum tube ought to be of sufficient size to render the estimated correction for confined space less than, or at most comparable with, errors of observation. The vacuum apparatus used by Col. Sabine would do very well. If the vacuum tube be not of sufficient size, it ought to admit of removal, and to be removed when the pendulums are swung in air.

In all the experiments the arc of oscillation ought to be carefully observed several times during the motion, the observation of the arc being quite as important for the purposes of theory as the observation of the time. Indeed, if it should be inconvenient to observe the time, the observation merely of the arc would be very valuable as a test of theory. In that case an approximate value of the time of oscillation in air would be required.

In the system proposed, Nos. 1 and 3 are the principal pendulums, Nos. 2 and 4 are introduced for the sake of making certain small corrections to the results of Nos. 1 and 3. No. 2 is meant for the elimination from No. 1 of the effect of the wire, and No. 4 for the elimination from No. 3 of the effect of the resistance experienced by a small portion of the rod near its end. The times of vibration of the four pendulums ought to be nearly the same, although for that purpose slightly different lengths of wire would be required in Nos. 1, 2, and 4.

It follows from theory that for a given pendulum the factor n is a function of the time of vibration. This is a result which seems to have been hardly so much as suspected by those who were engaged in pendulum experiments, or at most to have been mentioned as a mere possibility*, and therefore it might be thought advisable to verify it by direct experiment. For my own part I regard it as so intimately connected with the fundamental principles of the theory, that if the theory be confirmed in other respects I think this result may be accepted on the strength of theory alone. The direct comparison with experiment would be inconvenient, because it would require a clock which kept excellent time, and yet admitted of being adjusted so as to make widely different numbers of vibrations in a day. The result could, however, be confirmed indirectly by observing the arc of vibration, an observation which is as easy with one time of vibration as with another.

82. *Second object.* According to theory, the index of friction may be deduced from experiments either on the arc or on the time of vibration. It must be left to observation to decide which give the more consistent results. Should the results obtained from the arc appear as trustworthy as those obtained from the time, it would apparently be much the easiest way of determining μ' for an elastic fluid to observe the arc, because no particular accuracy would then be required in the observation of time. As to the form of the pendulum, a cylindrical rod would apparently be the best if only a single pendulum were employed. The observation of the arc seems the only practicable way of determining the influence of temperature on the index of friction, unless the pendulum be extremely light, or unless the observer be content with the limited range of temperature which may be procured by making observations at different times of year. For in an apparatus artificially heated or cooled, it would be difficult to prevent small unknown variations of temperature, which would cause variations in the rate of vibration, in consequence of the expansion and contraction of the pendulum; and these variations would vitiate the result of the experiment, so far as the time of vibration is concerned, because the effect of the gas on the time of vibration is deduced from the small difference between two large quantities which are directly observed. But the effect of the gas on the arc of vibration produces by far the greater part of the whole diminution observed, and therefore small fluctuations of temperature would not be of much consequence, except so far as they might

* It should be observed however that in a subsequent memoir (*Astronomische Nachrichten*, No. 223, p. 106), Bessel deduced from other experiments that the value of k was larger for the long than for the short pendulum.

occasion gentle currents; and even then would not be very important, because the forces thence arising would not be periodic, and dependent upon the phase of vibration of the pendulum.

The grand difficulty which besets the observation of the time of vibration of a pendulum oscillating in a liquid consists in the rapidity with which the oscillations subside. The best form of a pendulum to oscillate in a liquid would be a sphere suspended by a fine wire. The vessel containing the liquid and the sphere immersed in it ought to be so large as to render the correction for confined space insensible. But the index of friction of a liquid would probably be better determined by experiments more of the nature of those of Coulomb, or perhaps by the slow discharge of liquids through narrow tubes.

Among the gases for which μ' ought to be determined experimentally should be mentioned coal-gas, on account of the practical application which it appears possible to make of the result to the laying down of gas-pipes. The calculation of the resistance in a circular pipe is very simple, and is given in Art. 9 of my former paper. According to the equations of condition assumed in the present paper we must put $U = 0$, U denoting in that article the velocity close to the surface. It appears that the pressure spent in overcoming friction varies as the mean velocity divided by the square of the diameter of the pipe, or as the rate of supply divided by the fourth power of the diameter. This goes on the supposition that the motion is sufficiently slow to allow of our neglecting the pressure which may be spent in producing eddies, in comparison with that spent in overcoming what really constitutes internal friction.

83. *Third object.* With respect to experiments for determining the length of the seconds' pendulum, the theory of internal friction rather enables us to calculate for certain forms of pendulum the correction due to the inertia of the air than points out any particular mode of performing the experiments. Even the ordinary theory of hydrodynamics points out the importance of removing all obstacles to the free motion of the air in the neighbourhood of the pendulum if we would calculate from theory the whole correction for reduction to a vacuum.

Since the theoretical solution has been obtained in the case of a long cylindrical rod, or of such a rod combined with a sphere, we may regard a pendulum formed in this manner, and which is convertible in air, as also convertible in vacuum, for it is of small consequence whether the pendulum be or be not really convertible in vacuum, provided that if it be not we know the correction to be applied in consequence.

Note A, *Article* 65.

Let us apply the general equations (2), (3) to the fluid surrounding a solid of revolution which turns about its axis, with either a uniform or a variable motion, supposing the fluid to have been initially either at rest, or moving in annuli about the axis of symmetry.

In the first place we may observe, that the fluid will always move in annuli about the axis of symmetry. For let P be any point of space, and L any line passing through P, and lying in a plane drawn through P and through the axis of symmetry; and at the end of the time t let u' be the velocity at P resolved along L. Now consider a second case of motion, differing from the first in having the angular velocity of the solid and the initial velocity of the fluid reversed, every thing else being the same as before. It follows from symmetry, that at the end of the time t the velocity at P resolved along L will be equal to u', since the motion of the solid and the initial motion of the fluid, which form the data of the one problem, differ from the corresponding quantities in the other problem only as regards the distinction between one way round and the other way round, which has no relation to the distinction between to and fro in the direction of a line lying in a plane passing through the axis of rotation. But since all our equations are linear as regards the velocity, it follows that in the second problem the velocity will be the same as in the first, with a contrary sign, and therefore the velocity at P in the direction of the line L will be equal to $-u'$. Hence $u' = -u'$, and therefore $u' = 0$, and therefore the whole motion takes place in annuli about the axis of rotation.

Let the axis of rotation be taken for the axis of z; let ω be the angle which a plane passing through this axis and through the point P makes with the plane of xy, and let v' be the velocity at P. Then

$$u = -v' \sin \omega, \qquad v = v' \cos \omega, \qquad w = 0,$$

and all the unknown quantities of the problem are functions of t, z, and ϖ, where $\varpi = \sqrt{(x^2 + y^2)}$. Substituting in equations (2) the above values of u, v, and w, and after differentiation putting $\omega = 0$, as we are at liberty to do, we get

$$\frac{dp}{d\varpi} = 0, \qquad \frac{dp}{dz} = 0,$$

$$\mu \left(\frac{d^2 v'}{dz^2} + \frac{d^2 v'}{d\varpi^2} + \frac{1}{\varpi} \frac{dv'}{d\varpi} - \frac{v'}{\varpi^2} \right) = \rho \frac{dv'}{dt} \quad . \quad . \quad . \quad . \quad . \quad (179)$$

The first two of these equations give $p =$ a constant, or rather $p =$ a function of t, which for the same reason as in Art. 7 we have a right to suppose to be equal to zero. The third equation combined with the equations of condition serves to determine v'.

Now in the particular case of an oscillating disk, the equation (179) becomes according to the mode of approximation adopted in Art. 8

$$\mu \frac{d^2 v'}{dz^2} = \rho \frac{dv'}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

which in fact is the same as the second of the equations (8). The solution thus obtained is as we have seen

$$v' = \varpi f(z, t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

f denoting a function the form of which there is no need to write down, which satisfies (180) when written for v'. Now it will be seen at once that the expression (181) satisfies the exact equation (179), and therefore the approximate solution obtained by the method of Art. 8 is in fact exact, except so far as regards the termination of the disk at its edge, which is what it was required to prove.

Passing from semi-polar to polar co-ordinates, by putting $z = r\cos\theta$, $\varpi = r\sin\theta$, we get from (179), after writing $\mu'\rho$ for μ',

$$\frac{d^2v'}{dr^2} + \frac{2}{r}\frac{dv'}{dr} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dv'}{d\theta}\right) - \frac{v'}{r^2\sin^2\theta} = \frac{1}{\mu'}\frac{dv'}{dt} \quad . \quad . \quad . \quad . \quad (182)$$

Suppose now the solid to be a sphere, having its centre at the origin. Let a be its radius, δ its angular velocity, and suppose the fluid initially at rest. Then v' is to be determined from the general equation (182) and the equations of condition

$$v' = 0 \text{ when } t = 0, \qquad v' = a\delta\sin\theta \text{ when } r = a, \qquad v' = 0 \text{ when } r = \infty.$$

All these equations are satisfied by supposing

$$v' = v''\sin\theta,$$

v'' being a function of r and t only. We get from (182)

$$\frac{d^2v''}{dr^2} + \frac{2}{r}\frac{dv''}{dr} - \frac{2v''}{r^2} = \frac{1}{\mu'}\frac{dv''}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

If we suppose δ constant, v'' will tend indefinitely to become constant as t increases indefinitely, and in the limit $\dfrac{dv''}{dt} = 0$, whence we get from (183) and the equations of condition $v'' = a\delta$ when $r = a$, $v'' = 0$ when $r = \infty$,

$$v'' = \frac{\delta a^3}{r^2}, \qquad v' = \frac{\delta a^3}{r^2}\sin\theta.$$

This is the solution alluded to in Art. 8 of my paper *On the Theories of the Internal Friction of Fluids in motion, &c.*

Note B, *Article* 65.

Let us resume the problem of Art. 7, but instead of the motion of the plane being periodic, let us suppose that the plane and fluid are initially at rest, and that the plane is then moved with a constant velocity V, and let the notation be the same as in Art. 7.

The general equations (8) remain the same as before, but the equations of condition become in this case

$$v = 0 \text{ when } t = 0 \text{ from } x = 0 \text{ to } x = \infty,$$

$$v = V \text{ when } x = 0 \text{ from } t = 0 \text{ to } t = \infty.$$

By Fourier's theorem and another theorem of the same kind, v may be expanded between the limits 0 and ∞ of x in the following form:

$$v = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos \alpha x \cos \alpha x' \phi(x', t)\, dx' d\alpha + \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin \alpha x \sin \alpha x' \psi(x', t)\, dx' d\alpha. \quad (184)$$

In fact, v could be expanded by means of either of these expressions separately, and of course can be expanded in an infinite number of ways by the sum of the two. If however v had been expanded by means of the first expression alone, its derivatives with respect to x could not have been obtained by differentiating under the integral signs, inasmuch as the derivatives of an odd order do not vanish when $x = 0$, but would have been given by certain formulæ which I have investigated in a former paper.* A similar remark applies to the second expansion, in consequence of the circumstance that v itself and its derivatives of an even order do not vanish with x. But by combining the two expansions we may obtain the derivatives of v, up to any order i that we please to fix on, by merely differentiating under the integral signs. For we may evidently express the finite function v, and that in an infinite number of ways, as the sum of two finite functions $\phi(x, t)$, $\psi(x, t)$ which like v vanish when $x = \infty$, and which are such that the odd derivatives of the first, and the even derivatives of the second, up to the order i, as well as $\psi(x, t)$ itself, vanish when $x = 0$. Substituting now in the second equation (8) the expression for v given by (184), we see that the equation is satisfied provided

$$\frac{d\phi}{dt} + \mu'\alpha^2\phi = 0, \qquad \frac{d\psi}{dt} + \mu'\alpha^2\psi = 0.$$

These equations give

$$\phi(x', t) = \chi(x')\,\epsilon^{-\mu'\alpha^2 t}, \qquad \psi(x', t) = \sigma(x')\,\epsilon^{-\mu'\alpha^2 t},$$

where χ, σ denote two new arbitrary functions. Substituting in (184), and then passing to the first of the equations of condition, we get

$$0 = \chi(x) + \sigma(x),$$

whence $\sigma(x) = -\chi(x)$ and

$$v = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos \alpha(x + x')\,\epsilon^{-\mu'\alpha^2 t}\chi(x')\, dx' d\alpha$$

$$= \frac{1}{\sqrt{\pi\mu' t}}\int_0^\infty \epsilon^{-\frac{(x'+x)^2}{4\mu' t}}\chi(x')\, dx'. \quad . \quad . \quad . \quad . \quad . \quad (185)$$

The second of the equations of condition requires that

$$V = \frac{1}{\sqrt{\pi\mu' t}}\int_0^\infty \epsilon^{-\frac{x'^2}{4\mu' t}}\chi(x')\, dx' = \frac{2}{\sqrt{\pi}}\int_0^\infty \epsilon^{-s^2}\chi(2s\sqrt{\mu' t})\, ds.$$

* On the critical values of the sums of periodic series. Camb. Phil. Trans. Vol. VIII. p. 533.

Since the second member of this equation must be independent of t, we get $\chi(x') =$ a constant, and this constant must be equal to V, since

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \epsilon^{-s^2}\,ds = 1.$$

Substituting in (185) we get

$$v = \frac{V}{\sqrt{\pi\mu' t}}\int_0^\infty \epsilon^{-\frac{(x+x')^2}{4\mu' t}}\,dx'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

For the object of the present investigation nothing is required but the value of $\frac{dv}{dx}$ for $x = 0$, which we may denote by $\left(\frac{dv}{dx}\right)_0$. We get from (186)

$$\left(\frac{dv}{dx}\right)_0 = -\frac{V}{\sqrt{\pi\mu' t}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

Now suppose the plane to be moved in any manner, so that its velocity at the end of the time t is equal to $f(t)$. We may evidently obtain the result for this case by writing $f'(t')\,dt'$ for V, and $t - t'$ for t in (187), and integrating with respect to t'. We thus get

$$\left(\frac{dv}{dx}\right)_0 = -\frac{1}{\sqrt{\pi\mu'}}\int_{-\infty}^t f'(t')\frac{dt'}{\sqrt{t-t'}} = -\frac{1}{\sqrt{\pi\mu'}}\int_0^\infty f'(t-t_1)\frac{dt_1}{\sqrt{t_1}}. \quad . \quad (188)$$

To apply this result to the case of an oscillating disk, let $r\frac{d\theta}{dt} = rF(t)$ be the velocity of any annulus, and G the moment of the whole force of the fluid on the disk. Then

$$G = 4\pi\mu'\rho\int_0^a r^2\left(\frac{dv_0}{dx}\right)_0 dr;$$

and $\left(\frac{dv}{dx}\right)_0$ will be got from (188) by substituting $rF(t)$ for $f(t)$. We find thus

$$G = -\sqrt{\pi\mu'}\cdot\rho a^4\int_0^\infty F'(t-t_1)\frac{dt_1}{\sqrt{t_1}}. \quad . \quad . \quad . \quad . \quad (189)$$

If we suppose the angular velocity of the disk to be expressed by $A\sin nt$, where A is constant, we must put $F(t) = A\sin nt$ in (189), and we should then get after integration the same expression for G as was obtained in Art. 8 by a much simpler process. Suppose, however, that previously to the epoch from which t is measured the disk was at rest, and that the subsequent angular velocity is expressed by $A_t\sin nt$, where A_t is a slowly varying function of t. Then

$$F(t) = 0 \text{ when } t < 0, \qquad F(t) = A_t\sin nt \text{ when } t > 0.$$

On substituting in (189) we get

$$G = -\sqrt{\pi\mu'}\cdot\rho a^4 n\int_0^t A_{t-t_1}\cos n(t-t_1)\frac{dt_1}{\sqrt{t_1}}. \quad . \quad . \quad . \quad (190)$$

Now treating A_t as a slowly varying parameter, we get from a formula given by Mr Airy, and obtained by the method of the variation of parameters,

$$\frac{dA_t}{dt} = \frac{G}{I} \sin nt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (191)$$

where I denotes the moment of inertia. In the expression for G we may replace A_{t-t_1} under the integral sign by A_t outside it, because A_t is supposed to vary so slowly that A_{t-t_1} does not much differ from A_t while t_1 is small enough to render the integral of importance. Making this simplification and substituting in (191) we get

$$\frac{dA_t}{A_t dt} = - c \sin nt \int_0^t \cos n(t - t_1) \frac{dt_1}{\sqrt{t_1}}, \quad . \quad . \quad . \quad . \quad . \quad (192)$$

where $c = \sqrt{(\pi\mu')} \cdot \rho a^4 n I^{-1}$. If then A_0 be the initial and A the final value of A_t, we get from (192)

$$\log \frac{A_0}{A} = c \int_0^t \left\{ \sin nt \int_0^t \cos n(t - t_1) \frac{dt_1}{\sqrt{t_1}} \right\} dt. \quad . \quad . \quad . \quad . \quad (193)$$

Let now $A_0 + \Delta A_0$ be what A_0 would become if, while the final arc A and the whole time t remained the same, the motion had been going on for an indefinite time before the epoch from which t is measured, in which case the superior limit in the integral involved in the expression for G would have been ∞ in place of t. Then

$$\log \frac{A_0 + \Delta A_0}{A} = c \int_0^t \left\{ \sin nt \int_0^\infty \cos n(t - t_1) \frac{dt_1}{\sqrt{t_1}} \right\} dt, \quad . \quad . \quad (194)$$

whence by subtracting, member from member, equation (193) from equation (194), we get

$$\log \frac{A_0 + \Delta A_0}{A_0} = c \int_0^t \left\{ \sin nt \int_t^\infty \cos n(t - t_1) \frac{dt_1}{\sqrt{t_1}} \right\} dt,$$

which becomes after integration by parts

$$\log \frac{A_0 + \Delta A_0}{A_0} = \frac{c}{4n} \left\{ \sqrt{\frac{\pi}{2n}} - 2\sqrt{t} \cdot \cos nt - \cos 2nt \int_t^\infty \cos nt \frac{dt}{\sqrt{t}} \right.$$

$$\left. + (2nt - \sin 2nt) \int_t^\infty \sin nt \frac{dt}{\sqrt{t}} \right\}. \quad . \quad (195)$$

Now t is supposed to be very large: in Coulomb's experiments in fact 10 oscillations were observed, so that $nt = 10\pi$. But when t is at all large the two integrals

$$\int_t^\infty \cos nt \frac{dt}{\sqrt{t}}, \qquad \int_t^\infty \sin nt \frac{dt}{\sqrt{t}}$$

can be expressed under the forms

$$- P \sin nt + Q \cos nt, \qquad P \cos nt + Q \sin nt,$$

where

$$P = n^{-1} t^{-\frac{1}{2}} - 1 \,.\, 3 \,.\, 2^{-2} n^{-3} t^{-\frac{5}{2}} + \ldots, \qquad Q = 1 \,.\, 2^{-1} n^{-2} t^{-\frac{3}{2}} - 1 \,.\, 3 \,.\, 5 \,.\, 2^{-3} n^{-4} t^{-\frac{7}{2}} + \ldots,$$

series which are at first rapidly convergent, and which enable us to calculate the numerical values of the integrals with extreme facility. These expressions were first given by M. Cauchy, in the case of Fresnel's integrals, to which the integrals just written are equivalent. They may readily be obtained by integration by parts, though it is not thus that they were demonstrated by M. Cauchy. If now the above expressions be substituted for the integrals in (195) the terms containing $t^{\frac{1}{2}}$ destroy each other, and for general values of t the most important term after the first contains $t^{-\frac{1}{2}}$. Since however t is supposed to correspond to the end of an oscillation, so that nt is a multiple of π, the coefficient of this term vanishes, and the most important term that actually remains contains only $t^{-\frac{3}{2}}$. Hence neglecting insensible quantities we get from (195)

$$\log \frac{A_0 + \Delta A_0}{A_0} = \frac{c}{4n}\sqrt{\frac{\pi}{2n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

We get from (194) by performing the integrations

$$\log \frac{A_0 + \Delta A_0}{A} = c\sqrt{\frac{\pi}{2n}}\int_0^t \sin nt\,(\cos nt + \sin nt)\,dt$$

$$= \frac{c}{4n}\sqrt{\frac{\pi}{2n}}\left\{2nt + 1 - \cos 2nt - \sin 2nt\right\},$$

which becomes since nt is a multiple of π

$$\log \frac{A_0 + \Delta A_0}{A} = \frac{c}{4n}\sqrt{\frac{\pi}{2n}}.2nt. \quad . \quad . \quad . \quad . \quad . \quad (197)$$

We get from (196) and (197)

$$2nt \log \frac{A_0 + \Delta A_0}{A_0} = \log \frac{A_0 + \Delta A_0}{A} = \log \frac{A_0 + \Delta A_0}{A_0} + \log \frac{A_0}{A},$$

whence

$$\log \frac{A_0 + \Delta A_0}{A_0} = (2nt - 1)^{-1} \log \frac{A_0}{A}, \quad . \quad . \quad . \quad . \quad . \quad (198)$$

and the same relation exists between the common logarithms of the arcs, which are proportional to the Napierian logarithms. Now $\text{Log}\,A_0 - \text{Log}\,A$ is the quantity immediately deduced from experiment, and $\text{Log}\,(A_0 + \Delta A_0) - \text{Log}\,A_0$ is the correction to be applied, in consequence of the circumstance that the motion began from rest. Instead of applying the proportionate correction $+ (2nt - 1)^{-1}$ to the difference of the logarithms, we may apply it to the deduced value of $\sqrt{\mu'}$, which is proportional to the difference of the logarithms. In Coulomb's experiments 10 oscillations were observed, and therefore $2nt = 20\pi$, and $(2nt - 1)^{-1} = 0.01617$, and the uncorrected value of $\sqrt{\mu'}$ being 0.0555, we get 0.0009 for the correction, giving $\sqrt{\mu'} = 0.0564$.

NOTE C. *Article* 50.

The results mentioned in this article were originally given without demonstration; but as the mode in which they were obtained is short, and by no means obvious, I have thought it advisable to add the demonstrations.

In order that the right-hand members of equations (138) may be perfect differentials, we must have

$$\frac{d\delta}{dy}+\frac{d\omega'''}{dx}=0,\quad \frac{d\delta}{dz}+\frac{d\omega'}{dy}=0,\quad \frac{d\delta}{dx}+\frac{d\omega''}{dz}=0,\ldots\quad (a)$$

$$\frac{d\delta}{dz}-\frac{d\omega''}{dx}=0,\quad \frac{d\delta}{dx}-\frac{d\omega'''}{dy}=0,\quad \frac{d\delta}{dy}-\frac{d\omega'}{dz}=0,\ldots\quad (b)$$

$$\frac{d\omega''}{dy}+\frac{d\omega'''}{dz}=0,\quad \frac{d\omega'''}{dz}+\frac{d\omega'}{dx}=0,\quad \frac{d\omega'}{dx}+\frac{d\omega''}{dy}=0.\ldots\quad (c)$$

The equations (*c*) give

$$\frac{d\omega'}{dx}=0,\quad \frac{d\omega''}{dy}=0,\quad \frac{d\omega'''}{dz}=0.\ldots\quad (d)$$

In the particular case in which $\delta = 0$, the equations (*a*), (*b*), and (*d*) give

$$d\omega'=0,\quad d\omega''=0,\quad d\omega'''=0,$$

and therefore ω', ω'', and ω''' are constant as stated in Art. 50. In the general case the equations (*a*), (*b*), and (*d*) give for the differentials of ω', ω'', and ω''' the following expressions:

$$\left.\begin{aligned} d\omega' &= -\frac{d\delta}{dz}dy+\frac{d\delta}{dy}dz,\\ d\omega'' &= -\frac{d\delta}{dx}dz+\frac{d\delta}{dz}dx,\\ d\omega''' &= -\frac{d\delta}{dy}dx+\frac{d\delta}{dz}dy. \end{aligned}\right\}\ldots\quad (e)$$

In order that the right-hand members of these equations may be perfect differentials, we must have

$$\frac{d^2\delta}{dydz}=0,\quad \frac{d^2\delta}{dzdx}=0,\quad \frac{d^2\delta}{dxdy}=0,\ldots\quad (f)$$

$$\frac{d^2\delta}{dy^2}+\frac{d^2\delta}{dz^2}=0,\quad \frac{d^2\delta}{dz^2}+\frac{d^2\delta}{dx^2}=0,\quad \frac{d^2\delta}{dx^2}+\frac{d^2\delta}{dy^2}=0,$$

and therefore

$$\frac{d^2\delta}{dx^2}=0,\quad \frac{d^2\delta}{dy^2}=0,\quad \frac{d^2\delta}{dz^2}=0.\ldots\quad (g)$$

The equations (f), (g) give

$$d\frac{d\delta}{dx} = 0, \qquad d\frac{d\delta}{dy} = 0, \qquad d\frac{d\delta}{dz} = 0,$$

so that $\frac{d\delta}{dx}$, $\frac{d\delta}{dy}$, and $\frac{d\delta}{dz}$ are constant. Substituting in (e) and integrating, and then substituting in (138) the resulting expressions for ω', ω'', and ω''', and integrating again, we shall obtain the results given in Art. 50.

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