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THE ORIGINAL AND ACTUAL FLUIDITY

OF

THE EARTH AND PLANETS.

BY THE

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THE communication which is here offered to the Academy contains a brief examination of the three following questions:

1st. Whether the nebular hypothesis of LAPLACE affords an explanation of the equality of the mean movements of rotation and revolution of the moon and other satellites.

2nd. Whether the evidence of the original fluidity of the earth and planets, afforded by their observed figures, is satisfactory with respect to all the planets.

3rd. Whether we possess, from the data afforded by astronomy, sufficient knowledge of the structure of the interior of the earth to enable us to draw conclusions respecting it, which are of geological value.

The answer which I have given to each of these questions is in the negative, and the object I have had in view in offering this communication will be accomplished, if it should in any way assist inquirers in estimating at their just value speculations relating to the original condition of the earth. The importance of such speculations has been, I believe, greatly overrated, and they have been too readily applied to the explanation of some geological facts, for which other and more probable causes can be assigned; such as the changes of climate which have taken place on the surface of the earth, and the increase of temperature as we descend below its surface. I have, therefore, examined these questions with the view of proving that, if we confine ourselves to the facts which we certainly know respecting the earth and planets, neither the nebular hypothesis, nor the hypothesis of the internal fluidity of the earth, is entitled to take a place in the list of positive facts.

three unequal diameters, the least being the axis of revolution, and the greatest being directed towards the central body.

The components of attraction of the fluid mass upon a particle at its surface* are

$$Ax, \quad By, \quad Cz;$$

where

$$A = \frac{3Mf}{a^3} L, \quad B = \frac{3Mf}{a^3} \frac{d.\lambda L}{d\lambda}, \quad C = \frac{3Mf}{a^3} \frac{d.\lambda' L}{d\lambda'},$$

M denoting the mass of the fluid, a the least semi-axis of the ellipsoid, f the dynamical measure of attraction of two units of mass at the unit-distance,

$$L = \int_0^1 \frac{u^2 du}{\sqrt{1+\lambda^2 u^2}\sqrt{1+\lambda'^2 u^2}},$$

$$\lambda^2 = \frac{b^2 - a^2}{a^2}, \quad \lambda'^2 = \frac{c^2 - a^2}{a^2};$$

the equation of the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let the axis of rotation be the axis of x, and the axis of y be directed towards the central body; if u denote the distance from the centre of the sun to any particle of the planet, δ the distance between the centres of the sun and planet, E the mass of the central body, and ω the angular velocity; it is easy to see that the equation of the surface, deduced from hydrostatical principles, will be

$$\left(A + \frac{fE}{u^3}\right) x dx + \left(By - \frac{fE}{\delta^2} + \frac{fE(\delta - y)}{u^3} - \omega^2 y\right) dy + \left(C - \omega^2 + \frac{fE}{u^3}\right) z dz = 0;$$

but, neglecting small quantities,

$$\frac{E}{u^3} = \frac{E}{\delta^3}\left(1 + \frac{3y}{\delta}\right), \text{ and } \frac{E}{\delta^2} - \frac{E(\delta - y)}{u^3} = \frac{2Ey}{\delta^3},$$

therefore, the equation of the surface becomes

* Vid. DUHAMEL, "Cours de Mecanique," Tom. I. p. 198.

$$\left(A+\frac{fE}{\delta^3}\right)xdx+\left(B-\frac{2fE}{\delta^3}-\omega^2\right)ydy+\left(C+\frac{fE}{\delta^3}-\omega^2\right)zdz=0.$$

Combining this with the assumed equation,

$$\frac{xdx}{a^2}+\frac{ydy}{b^2}+\frac{zdz}{c^2}=0,$$

we find the following equations of condition,

$$A+\frac{fE}{\delta^3}=(1+\lambda'^2)\left(C+\frac{fE}{\delta^3}-\omega^2\right);$$

$$A+\frac{fE}{\delta^3}=(1+\lambda^2)\left(B-\frac{2fE}{\delta^3}-\omega^2\right).$$

Substituting in these the values of A, B, C, and making

$$3\phi=\frac{E}{M}\,\frac{a^3}{\delta^3},\qquad 3\varepsilon=\frac{\omega^2 a^3}{fM},$$

we obtain

$$L+\phi=(1+\lambda'^2)\left(\frac{d\,.\,\lambda' L}{d\lambda'}+\phi-\varepsilon\right);$$

$$L+\phi=(1+\lambda^2)\left(\frac{d\,.\,\lambda L}{d\lambda}-2\phi-\varepsilon\right). \tag{1}$$

But since $\phi=\varepsilon$, by the third law of KEPLER, equations (1) become simply

$$L+\phi=(1+\lambda'^2)\,\frac{d\,.\,\lambda' L}{d\lambda'},$$

$$L+\phi=(1+\lambda^2)\left(\frac{d\,.\,\lambda L}{d\lambda}-3\phi\right). \tag{2}$$

If the definite integral L be expanded, it becomes

$$L=\frac{1}{3}-\left(\frac{1}{2}\cdot\frac{\lambda^2+\lambda'^2}{5}\right)+\left(\frac{1\,.\,3}{2\,.\,4}\,\frac{\lambda^4+\lambda'^4}{7}+\frac{1\,.\,1}{2\,.\,2}\,\frac{\lambda^2\lambda'^2}{7}\right)-\&c.,$$

substituting this value in (2), and neglecting quantities higher than λ^2, λ'^2, we find,

$$\lambda^2=30\phi,\qquad \lambda'^2=\tfrac{15}{2}\phi\,;$$

or if ϵ, ϵ', denote the ellipticities of the principal sections, passing through the greatest and least diameter, and mean and least, respectively; since $\lambda^2 = 2\epsilon$, $\lambda'^2 = 2\epsilon'$, we obtain finally for the ellipticities of the principal sections

$$\epsilon = 5\,\frac{E}{M}\frac{a^3}{\delta^3}, \qquad \epsilon' = \tfrac{5}{4}\cdot\frac{E}{M}\cdot\frac{a^3}{\delta^3}; \tag{3}$$

from which it appears that the ellipticity of the section passing through the greatest and least diameters is four times greater than the ellipticity of the section passing through the mean and least diameters.

If the planet be supposed to revolve on its axis with an angular rotation different from that of its revolution round the central body, the equality $\phi = \varkappa$ will no longer subsist, and we should therefore use equations (1) to determine the ellipticities of the principal sections. The result is

$$\epsilon = \tfrac{15}{4}\,(3\phi + \varkappa), \qquad \epsilon' = \tfrac{15}{4}\,\varkappa; \tag{4}$$

ϕ and $\varkappa$ being the quantities already defined, and depending on the central body and rotation of the planet respectively. If the central body be supposed so remote as to produce no effect on the figure of the planet, then $\phi = 0$, which renders the ellipticities equal, and corresponds to the figure of revolution assumed by the planet, if acted on only by its own attraction, and the centrifugal force caused by its rotation.* If, therefore, we suppose the spheroid of revolution, whose ellipticity is $\epsilon = \frac{15}{4}\varkappa$, described, having the axis of rotation for its least diameter, the effect produced by the attraction of the central body will be measured by the shape and magnitude of the couche included between this spheroid of rotation and the ellipsoid which forms the actual surface of the planet. The friction between this couche and the interior spheroid, which would constitute the surface of the planet, if the central body ceased to exist, will tend to render the motions of rotation and revolution of the planet equal to each other, and when the difference of these motions has fallen within the narrow limits indicated by analysis, will destroy the libration produced by the action of the central body in rendering those motions exactly equal. It may be proved by simple geometrical considerations, that if the planet separates from the central body, as a nodular or annular mass, without much friction, that

* Vid. POISSON, "Traité de Mecanique," Tom. II. p. 544.

its times of rotation and revolution at the period of separation will be nearly equal; and since we have no reason to assume any difference in the mode in which the planets and satellites were thrown off from the central mass, we may suppose, in order to render our calculations possible, that at the period of separation, the movements of rotation and revolution were so nearly equal as to justify us in using equations (3) instead of (4). Equations (4) might be used as well as (3), but require an additional hypothesis as to the time of rotation of the planet; but as this hypothesis should be the same for the planets and satellites, the generality of the reasoning is not affected by the use of equations (3). In these equations, the only quantity which is unknown is a, the radius of the planet or satellite at the time of its separation. We may obtain a value for a, in terms of the actual radius of the planet and its past and present moments of inertia, by the ordinary principles of mechanics; and if we assume as the measure of contraction of each planet the ratio which its original time of rotation bears to its actual time of rotation, we can calculate the value of ϵ and ϵ' for each planet and satellite. It will be shown afterwards, that the amount of contraction thus assumed is much too small for the planets which are attended with satellites, and probably for all the planets; but it will be useful to make the calculation upon this supposition in the first instance.

Let i, I denote the former and present moment of inertia of the planet, supposed homogeneous; a, $\mathbf{a}$, its former and present radius, and n the number of rotations contained in one revolution; then

$$a^5 : \mathbf{a}^5 :: i : I :: n : 1,$$

therefore,

$$a : \mathbf{a} :: \sqrt[5]{n} : 1,$$

or,

$$a^3 = \mathbf{a}^3 \sqrt[5]{n^3};$$

and substituting this value in (3), we find

$$\epsilon = 5 \frac{E}{M} \frac{\mathbf{a}^3}{\delta^3} \sqrt[5]{n^3}. \qquad (5)$$

The data from which I have calculated the values of ϵ corresponding to each planet and satellite are contained in the following Tables.

TABLE I.*

SATELLITES.		n	a : δ	E : M	ε
Moon,		1·00	$\frac{2153}{7926 \times 59·964}$	87·73	$\frac{1}{24524}$
Satellites of Jupiter,	1st,	1·00	$\frac{2508}{87 \times 6048}$	$\frac{1000000000}{17328}$	$\frac{1}{32·00}$
	2nd,	1·00	$\frac{2068}{87 \times 9623}$	$\frac{1000000000}{23235}$	$\frac{1}{308·32}$
	3rd,	1·00	$\frac{3377}{87 \times 15350}$	$\frac{1000000000}{88497}$	$\frac{1}{1094·5}$
	4th,	1·00	$\frac{2890}{87 \times 26998}$	$\frac{1000000000}{42659}$	$\frac{1}{4580·3}$

TABLE II.*

PLANETS.	n	a : δ	E : M	ε
Mercury,	87·6	$\frac{3140}{190 \times 387098}$	4865751	$\frac{1}{36082}$
Venus,	230·9	$\frac{7800}{190 \times 723331}$	401839	$\frac{1}{103970}$
Earth,	365·25	$\frac{7926}{190000000}$	389551	$\frac{1}{205121}$
Mars,	669·7	$\frac{4100}{190 \times 1523692}$	2680337	$\frac{1}{529573}$
Jupiter,	10468	$\frac{87000}{190 \times 5202776}$	1047·871	$\frac{1}{1084480}$
Saturn,	24631	$\frac{79160}{190 \times 9538786}$	3501·600	$\frac{1}{1589015}$
Uranus,	77524	$\frac{34500}{190 \times 19182390}$	24905	$\frac{1}{11029750}$

* The figures contained in the first three columns of Table I. are taken from the third edition of Sir JOHN F. W. HERSCHEL's Astronomy, pp. 331, 649, 650. The corresponding figures of Table II. are calculated from the Tables of the same book, pp. 647, 648.

From the foregoing Tables, it would appear that the effect of the planets in elongating the figures of their satellites was greater than the effect of the Sun upon the planets; and so far the conclusion to be drawn from the calculation accords with the idea of LAPLACE. But a slight consideration will show that the amount of contraction assigned to the planets is much too small. In fact, we are entitled by the nebular hypothesis to assume that each planet, at the time of its separation, extended at least as far as the orbit of its most distant satellite; this consideration supplies us with another and safer measure of the contraction of those planets which have satellites.

The following Table contains the values of $\sqrt[5]{n}$, which express the amount of contraction used in Tables I. and II., and also the value of the ellipticity of each planet, supposed homogeneous and extending to the orbit of its outermost satellite.

TABLE III.

PLANETS.	$\sqrt[5]{n}$	$a:\delta$	ϵ
Mercury, ...	2·4462	—	—
Venus, ...	2·9695	—	—
Earth, ...	3·2547	$\frac{7926 \times 59 \cdot 964}{190000000}$	$\frac{1}{32 \cdot 801}$
Mars, ...	3·6743	—	—
Jupiter, ...	6·3675	$\frac{87000 \times 26 \cdot 998}{190 \times 5202776}$	$\frac{1}{14228}$
Saturn, ...	7·5560	$\frac{79160 \times 64 \cdot 359}{190 \times 9538786}$	$\frac{1}{2571 \cdot 3}$
Uranus, ...	9·5035	$\frac{34500 \times 22 \cdot 8^{*}}{190 \times 19182390}$	$\frac{1}{798591}$

From the first column of this Table, it appears that the original radius of the planets used in Tables I. and II. in no case exceeded ten times the present radius, which is too small for the planets with satellites, especially the Earth and Saturn, and probably too small for all the other planets. From a comparison

* These figures refer to the fourth satellite.

of the ellipticities in Tables I., II., III., we are led to infer that the action of the Sun in elongating Jupiter, and so by internal friction causing his movements of rotation and revolution to become equal, was much less powerful than the corresponding action of Jupiter upon his satellites; hence the physical cause assigned by LAPLACE for this equality may be admitted in the case of Jupiter's satellites. But this conclusion will not apply to the Earth. From Table I. it appears, that the elongating action of the Earth upon the Moon is represented by the fraction $\frac{1}{24524}$; while Table III. shows that the similar action of the Sun upon the Earth is represented by the fraction $\frac{1}{32 \cdot 801}$.

Before quitting this subject it may be useful to consider the various explanations which might be offered to explain the difficulty which undoubtedly exists in the case of the Earth and Moon.

We are not at liberty to assume that the planets separated from the central mass as annuli, and the satellites as nodules, which would give to the planets a quicker rotation than to the satellites. In this case $\varpi > \phi$, and therefore $\epsilon < 4\epsilon'$; hence the couche, on the friction of which the effect in question depends, would be less for the planets, *ceteris paribus*, than for the satellites. But this assumption is not admissible, since the only annuli with which we are acquainted in the solar system occur among the satellites. Neither are we at liberty to assume greater friction among the particles of the satellites than of the planets, for, according to the nebular hypothesis, they are probably composed of the same materials. It is possible to explain the difficulty by assuming a sufficient amount of contraction in the Moon. It is, in fact, easy to prove that the effect of the Earth upon the Moon would be equal to that of the Sun upon the Earth and Moon, supposed to extend as far as the orbit of the Moon, provided the Moon extended to a distance represented by the equation

$$\frac{\delta}{a} = 24 \cdot 322, \quad \text{or,} \quad \frac{a}{\mathbf{a}} = 9 \cdot 076 ;$$

and this amount of contraction is physically possible, since it is less than the distance from the Moon at which a particle would be equally attracted by the Moon and Earth. But how are we to reconcile this amount of contraction with the observed facts, without tacitly assuming that the internal friction of the Moon, supposed fluid, was greater than that of the Earth ; an assumption which is purely arbitrary, and made to explain the difficulty.

There remains one real difference between the case of the planets and satellites, which, so far as it operates, is a *vera causa*, and acts in the direction required. The effect of the internal friction in destroying the increment of angular velocity must be greater in proportion as the mass of the planet or satellite is less; as we observe small rivers more retarded by the friction of their bed than large rivers. But it may be doubted whether this cause is sufficient to account for the remarkable difference which exists between the planets and satellites.

The conclusion which the foregoing calculations appear to warrant us in drawing is the following: that the nebular hypothesis does not explain the equality of the mean movements of revolution and rotation of the satellites, although it cannot be said to be absolutely inconsistent with it.

II.—*Figure of the Earth and Planets.*

It is well known that on the hypothesis of the original fluidity of the planets, it is necessary that the ellipticity of each planet should lie between two limits, which are, respectively, five-fourths and one-half of the fraction which expresses the ratio of centrifugal force to gravity at the surface of each planet;* the first or major limit corresponding to the case of homogeneity, and the second or minor limit corresponding to the case of infinite density at the centre. It is possible to compare this theory with observation in the case of five planets and the Moon. In the following Table, m denotes the ratio of centrifugal force to gravity at the surface of each planet, gravity being expressed in feet, and calculated from the formula

$$G = g\frac{P}{E}\frac{R^2}{r^2}, \qquad (6)$$

in which G, g, denote gravity on the surface of the planet and Earth respectively; P, E, the masses of the planet and Earth; R, r, the radii of the Earth and planet. The centrifugal force at the equator of each planet is calculated from the ordinary formula

$$f = 4\pi^2\frac{r}{T^2},$$

in which r is expressed in feet, and T, the time of rotation, in seconds.

* Clairaut, Figure de la Terre, p. 294.

TABLE IV.

PLANETS.	Gravity.	Centrifugal Force.	Minor Limit, $\frac{1}{2}m$	Major Limit, $\frac{5}{4}m$	Observed Ellipticity.	
					e	Observer.
Earth, . . .	32·088	0·111	$\frac{1}{578}$	$\frac{1}{231}$	$\frac{1}{299\cdot152}$	Bessel.
Mars, ...	17·428	0·054	$\frac{1}{644}$	$\frac{1}{258}$	$\frac{1}{16\cdot3}$	W. Herschel.*
					$\frac{1}{38\cdot8}$	Arago.†
Jupiter, . .	99·007	7·090	$\frac{1}{27\cdot92}$	$\frac{1}{11\cdot17}$	$\frac{1}{17\cdot7}$	Arago.‡
Saturn, . . .	35·787	5·792	$\frac{1}{12\cdot35}$	$\frac{1}{4\cdot94}$	$\frac{1}{10\cdot37}$	W. Herschel. §
Uranus, . .	26·490	3·074	$\frac{1}{17\cdot22}$	$\frac{1}{6\cdot88}$	$\frac{1}{9\cdot92}$	Mädler.

On comparing the observed ellipticities with the limits calculated in the preceding Table, it appears that the ellipticity of Mars exceeds the major limit admissible on the fluid hypothesis; the inference from which fact is, either that gravity is not perpendicular to the surface of Mars, or that his interior structure is not that which would be assumed by a fluid body. The first of these suppositions appears inadmissible from the fact, that there is reason to believe, that there are degrading and disintegrating forces at work on the surface of that planet, similar to those now in operation on the Earth, and which would render the surface perpendicular to gravity, if not so originally. The second supposition would appear to be inconsistent with the idea that Mars derived his present figure from having been originally fluid; at least, we are scarcely justified

* Transactions of Royal Society of London, for the year 1784. The ratio of the axes of the planet Mars, deduced from observation, is 1355 : 1272.

† Exposition du Systeme du Monde, p. 37. The ratio of axes deduced from observation by ARAGO is 194 : 189.

‡ Exposition du Systeme du Monde, p. 39. The ratio of axes is 177 : 167.

§ Transactions of Royal Society of London, for the year 1790. The ratio of axes is 2281 : 2061.

in assuming the original fluidity of all the planets, when there exists so remarkable an exception in the case of the planet Mars.*

III.—*On the Structure of the Earth, supposed partly Fluid and partly Solid.*

In the following investigation I shall suppose the Earth composed of elliptical couches of small ellipticity, the density of each couche being constant and a function of its distance from the centre. The surfaces bounding the couches must be perpendicular to the resultant of the forces acting upon the particles composing them, in the parts of the Earth which are supposed fluid, and also at the boundary between the solid and fluid parts, since the friction of the fluid would render the bounding surface perpendicular to the resultant, if not so originally. The only external forces supposed to act upon the particles are the centrifugal forces arising from the earth's rotation.

The condition that any surface bounding one of the couches of equal density should be perpendicular to gravity is contained in the following equation:

$$\text{const} = V + N, \tag{7}$$

in which V is the potential of the earth, and

$$N = \tfrac{1}{3}\omega^2 r^2 - \tfrac{1}{2}\omega^2 r^2 \varkappa; \tag{8}$$

r denoting the radius of the surface, ω the angular velocity, and $\varkappa = \cos^2\theta - \frac{1}{3}$, θ being the angle contained between the radius vector and the axis of rotation.† The potential contained in (7) is composed of two parts, one relating to the couches inside the surface considered, and the other to the couches outside the same surface. The value of the potential of a body constituted as we have supposed the earth, on an external point, is,

* It has been remarked by LAPLACE (Mec. Cel. Tom. II. p. 370, and Tom. V. p. 287), that the ellipticities of the principal sections of the Moon, deduced from the moments of inertia obtained by the observations of TOBIAS MAYER and NICOLLET, are nearly $\frac{1}{1675}$ and $\frac{1}{1773}$, and that both these ellipticities are greater than those of the figure of the Moon, if supposed fluid and homogeneous, which would give the maximum ellipticity. We have, therefore, in the Moon a case similar to Mars, viz., the actual ellipticity is greater than the major limit of the fluid hypothesis; but it is easier to admit that gravity is not perpendicular to the surface in the case of the Moon.

† Mec. Celeste, Tom. II. p. 66.

$$V = \frac{4\pi \int \rho a^2}{r} - \frac{4\pi\varpi}{5r^3} \int \rho \frac{d \,.\, a^5 e}{da}; \tag{9}$$

in which ρ is the density of any couche, a the radius of its equi-capacious sphere, and e its ellipticity.

The potential of a shell composed of couches arranged in the manner supposed, on an internal point, is,

$$V = 4\pi \int \rho a - \frac{4\pi r^2 \varpi}{5} \int \rho \frac{de}{da}. \tag{10}$$

The radius vector of the surface of each couche is given by the following equation,

$$r = a\,(1 - e\varpi)\,; \tag{11}$$

from which may be deduced the values of the equatorial and polar axes, viz., $a\,(1 + \frac{1}{3}e)$, and $a\,(1 - \frac{2}{3}e)$. Substituting from the foregoing equations in (7), we find

$$\begin{aligned} \text{const} = &\frac{4\pi}{a}\,(1 + e\varpi) \int_0^a \rho a^2 - \frac{4\pi}{5a^3}\,\varpi \int_0^a \rho \frac{d \,.\, a^5 e}{da} \\ &+ 4\pi \int_a^{\mathbf{a}} \rho a - \frac{4\pi a^2}{5}\,\varpi \int_a^{\mathbf{a}_1} \rho \frac{de}{da} - \frac{4\pi a^2}{5}\,\varpi \int_{\mathbf{a}_1}^{\mathbf{a}} \rho \frac{de}{da} \\ &+ \frac{4\pi a^2 m}{3\mathbf{a}^3} \int_0^{\mathbf{a}} \rho a^2 - \frac{4\pi a^2 m}{2\mathbf{a}^3}\,\varpi \int_0^{\mathbf{a}} \rho a^2; \end{aligned}$$

$\mathbf{a}$ denoting the mean radius of the external surface, $\mathbf{a}_1$ the mean radius of the internal surface of the shell supposed solid, and m the ratio of centrifugal force to gravity at the equator. This equation consists of two parts, one independent of ϖ, which is satisfied by means of the constant; the second, which is the coefficient of ϖ, gives the condition,

$$\frac{e}{a} \int_0^a \rho a^2 - \frac{1}{5a^3} \int_0^a \rho \frac{d \,.\, a^5 e}{da} - \frac{a^2}{5} \int_a^{\mathbf{a}} \rho \frac{de}{da} - \frac{m a^2}{2\mathbf{a}^3} \int_0^{\mathbf{a}} \rho a^2 = 0. \tag{12}$$

This equation expresses the fact, that each fluid surface is perpendicular to the resultant of all the forces acting upon the particles composing it.

Differentiating this equation, so as to banish the integrals, we obtain,

$$\frac{d^2 e}{da^2} + \frac{2\rho a^2}{\int_0^a \rho a^2} \frac{de}{da} - \frac{6e}{a^2}\left(1 - \frac{\rho a^3}{3\int_0^a \rho a^2}\right) = 0. \tag{13}$$

This equation is identical with that derived from the supposition that the Earth is completely fluid, and is therefore independent of the law of density and ellipticity of the solid parts of the Earth; it determines the relation which necessarily exists between the law of density and ellipticity of the fluid portions of the Earth. If the law of density of the fluid parts be given, the integral of this differential equation will give the law of ellipticity, involving two constants, one of which is determined by the condition that the density does not become infinite at the centre, and the other constant may be expressed in terms of the ellipticity of the surface which bounds the fluid. If we suppose that there is a fluid nucleus inside the Earth, whose radius is $\mathbf{a}_1$, and ellipticity ϵ_1, equation (12) will give for the bounding surface of the nucleus the following,

$$\frac{\epsilon_1}{\mathbf{a}_1}\int_0^{\mathbf{a}_1}\rho a^2 - \frac{1}{5\mathbf{a}_1^3}\int_0^{\mathbf{a}_1}\rho\,\frac{d.a^5e}{da} - \frac{\mathbf{a}_1^2}{5}\int_{\mathbf{a}_1}^{\mathbf{a}}\rho\,\frac{de}{da} = \frac{m\mathbf{a}_1^2}{2\mathbf{a}^3}\int_0^{\mathbf{a}}\rho a^2. \qquad (14)$$

If, also, we assume, as we may in the case of the Earth, that the external surface is perpendicular to gravity, equation (12) may be applied to this surface, although not fluid. Hence we obtain,

$$\frac{2}{5}\int_0^{\mathbf{a}}\rho\frac{d.a^5e}{da} = (2\epsilon - m)\,\mathbf{a}^2\int_0^{\mathbf{a}}\rho a^2. \qquad (15)$$

Equations (14) and (15) assert, respectively, that the inner and outer surfaces of the solid shell are perpendicular to gravity.

In the case of the Earth, the integral at the right-hand side of these equations is known, because the mean density of the Earth is known. The integral at the left-hand side of equation (15) is also known; since it may be expressed in terms of the difference of the moments of inertia with respect to the polar and equatorial axes, which is given by the inequalities of the Moon's motion produced by the structure of the Earth, or by the phenomena of precession and nutation, which are produced by the same cause. In fact, if C, A denote the moments of inertia with respect to the polar and equatorial axis respectively,

$$C - A = \frac{8\pi}{15}\int_0^{\mathbf{a}}\rho\frac{d.a^5e}{da}. \qquad (16)$$

Also the first and second integrals, on the left-hand side of equation (14) are known from the differential equation (13), if we assume the law of density of the fluid parts to be known.

There remains, however, the third integral on the left-hand side of (14), which cannot be known without assuming a law of density and also of ellipticity for the solid portion of the Earth.

We are thus led to the conclusion, that it is necessary to assume *three* hypotheses with respect to the internal structure of the Earth, before we can be in a position to assert how far it is solid and how far fluid. The three necessary hypotheses are:—1st. The law of density of the fluid parts. 2nd. The law of density of the solid parts. 3rd. The law of ellipticity of the solid parts.

If we suppose that these are given, then equations (14), (15) will become,

$$\begin{aligned} F(\mathbf{a}, \mathbf{a}_1, \epsilon, \epsilon_1, m) &= 0; \\ f(\mathbf{a}, \mathbf{a}_1, \epsilon, \epsilon_1, m) &= 0; \end{aligned} \qquad (17)$$

in which F, f denote known functions. In these equations $\mathbf{a}$, ϵ, m are known, and $\mathbf{a}_1$, ϵ_1, are determined by the equations themselves.

If we suppose that the fluid parts of the earth are bounded on both surfaces by solids, we should then have three equations, analogous to (14) and (15), belonging to the two surfaces of the fluid, and to the external surface respectively. From these, assuming the law of density of the fluid, and of density and ellipticity of the solid parts, we should obtain

$$\begin{aligned} \Phi(\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2, \epsilon, \epsilon_1, \epsilon_2, m) &= 0; \\ X(\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2, \epsilon, \epsilon_1, \epsilon_2, m) &= 0; \\ \Psi(\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2, \epsilon, \epsilon_1, \epsilon_2, m) &= 0; \end{aligned} \qquad (18)$$

$\mathbf{a}_2$, ϵ_2 being the radius and ellipticity of the second surface of the fluid. In equations (18), as before, $\mathbf{a}$, ϵ, m are known; but the number of unknown quantities is greater than the number of equations, the unknown quantities being four, viz., $\mathbf{a}_1$, $\mathbf{a}_2$, ϵ_1, ϵ_2, while there are only three equations. The problem is therefore not so definite as the last, and requires an additional hypothesis.

Confining our attention to the simplest case (17), we see that before a single step can be made towards using equations (14) and (15), we must assume three laws, respecting facts of which we have no certain knowledge, and probably never shall. The subject would thus appear to be excluded from the domain of positive science, and to possess an interest for the mathematician alone.

I shall conclude this investigation by examining the structure of the Earth on the simple but improbable hypothesis of homogeneity, and by determining how far the density belonging to the rocks of the surface may extend to the materials composing the interior of the Earth.

If the Earth be supposed to be composed of a solid shell, having the density of the rocks at its surface, and of a fluid homogeneous nucleus, equations (14) and (15) will become

$$\frac{2}{5}\rho\epsilon_1 - \frac{3}{5}\rho_0(\epsilon - \epsilon_1) = \Delta\frac{m}{2}, \tag{19}$$

and

$$\frac{6}{5}\{\rho_0\epsilon \mathbf{a}^5 + (\rho - \rho_0)\epsilon_1\mathbf{a}_1^5\} = (2\epsilon - m)\Delta\mathbf{a}^5; \tag{20}$$

in which ρ_0 signifies the density of the rocks of the shell, ρ the density of the nucleus, and Δ the mean density of the whole Earth. To equations (19) and (20) must be added the following, which expresses that the mass of the Earth is equal to the sum of the masses of its shell and nucleus.

$$\rho - \rho_0 = (\Delta - \rho_0)\frac{\mathbf{a}^3}{\mathbf{a}_1^3}. \tag{21}$$

Eliminating ρ from (19) and (20) by means of (21), they become respectively

$$\frac{2}{5}\left\{\rho_0 + (\Delta - \rho_0)\frac{\mathbf{a}^3}{\mathbf{a}_1^3}\right\}\epsilon_1 - \frac{3}{5}\rho_0(\epsilon - \epsilon_1) = \Delta\frac{m}{2}, \tag{22}$$

and

$$\frac{6}{5}\left\{\rho_0\epsilon\frac{\mathbf{a}^2}{\mathbf{a}_1^2} + (\Delta - \rho_0)\epsilon_1\right\} = (2\epsilon - m)\Delta\frac{\mathbf{a}^2}{\mathbf{a}_1^2}. \tag{23}$$

In the case of the Earth $\Delta = 2\rho_0$; substituting this value of the mean density, and solving equations (22) and (23) with respect to ϵ_1, we find

$$\epsilon_1 = \frac{5m + 3\epsilon}{5 + 2\phi^3}; \tag{24}$$

$$\epsilon_1 = \frac{7\epsilon - 5m}{3}\phi^2; \tag{25}$$

ϕ being used to denote the fraction $\frac{\mathbf{a}}{\mathbf{a}_1}$.

These are the equations which correspond, on the supposition of homogeneity, to the equations (17). Equating the values of ϵ_1, we obtain the following equation to determine ϕ:

$$2\phi^5 + 5\phi^2 = 3\,\frac{5m + 3\epsilon}{7\epsilon - 5m}. \qquad (26)$$

Substituting in this equation for m and ϵ their values in the case of the Earth, viz., $\frac{1}{289}$ and $\frac{1}{300}$, we find,

$$2\phi^5 + 5\phi^2 = 13{\cdot}57743. \qquad (27)$$

Applying STURM's theorem to this equation, it is easy to prove that it has only one real root, which lies between $\phi = 1$ and $\phi = 2$. The numerical value of this root is

$$\frac{\mathbf{a}}{\mathbf{a}_1} = \phi = 1{\cdot}2407.$$

Hence, since $\mathbf{a} = 3958$ miles; $\mathbf{a}_1 = 3190$ miles, and

$$\mathbf{a} - \mathbf{a}_1 = 768 \text{ miles}. \qquad (28)$$

This is the thickness of the earth's crust, on the hypothesis that both the crust and nucleus are homogeneous, and the surfaces of both perpendicular to gravity.

I shall now prove that this thickness of crust is a major limit to the depth to which the density of the rocks at the surface can extend into the interior; the density being supposed heterogeneous.

The difference of the moments of inertia of the nucleus with respect to its polar and equatorial axis may be expressed as follows:

$$C - A = \frac{8\pi}{15}\int_0^{\mathbf{a}_1} \rho\,\frac{d.a^5 e}{da} = \frac{8\pi}{15}\,\rho\,\frac{\epsilon_1}{\sigma}\,\mathbf{a}_1^5, \qquad (29)$$

σ denoting an unknown number, depending on the structure of the nucleus, and which, if the nucleus be supposed fluid, is greater than unity.

Substituting from (29) in equations (14) and (15) we find

$$\epsilon_1\left(\frac{1}{3} - \frac{1}{5\sigma}\right)\{\rho_0 + (\Delta - \rho_0)\,\phi^3\} - \frac{1}{5}\rho_0\,(\epsilon - \epsilon_1) = \frac{1}{6}\,\Delta m\,; \qquad (30)$$

and,

$$\frac{2}{5}\left[\rho_0\,(\epsilon\phi^5 - \epsilon_1) + \frac{\epsilon_1}{\sigma}\,\{\rho_0 + (\Delta - \rho_0)\,\phi^3\}\right] = \frac{1}{3}\,\Delta\,(2\epsilon - m)\,\phi^5. \qquad (31)$$

Solving these equations with respect to ϵ_1, and making $\Delta = 2\rho_0$ we find

$$\epsilon_1 = \frac{m + \frac{3}{5}\epsilon}{\frac{3}{5} + \left(1 - \frac{3}{5\sigma}\right)(\phi^3 + 1)} ; \qquad (32)$$

$$\epsilon_1 = \frac{(7\epsilon - 5m)\,\phi^5}{\frac{3\,(\phi^3 + 1)}{\sigma} - 3} . \qquad (33)$$

Eliminating ϵ_1 and solving for σ, we find,

$$\frac{1}{\sigma} = \frac{(3\phi^5 + A) + 5\phi^5\,(\phi^3 + 1)}{(3\phi^5 + A)\,(\phi^3 + 1)} , \qquad (34)$$

in which $A = 3\,\frac{5m + 3\epsilon}{7\epsilon - 5m}$.

But the nucleus being supposed fluid, the denominator of the right-hand member of (34) is greater than its numerator; consequently we have the inequality

$$2\phi^5 + 5\phi^2 < 3\,\frac{5m + 3\epsilon}{7\epsilon - 5m} . \qquad (35)$$

The value $\phi = 1{\cdot}2407$ renders the left-hand member of (35) equal to the right, and therefore ϕ must be less than 1·2407, and, consequently, the depth to which the density of the surface extends is less than 768 miles.

The results which have just been obtained are to be regarded merely as examples of the manner in which equations (14) and (15) should be used, if we were acquainted with the laws of density and ellipticity of the fluid and solid parts of the Earth. So long as we are ignorant of these laws, we cannot calculate numerical values, and indeed the chief use of the investigation I have just given appears to be, to enable us to estimate at their just value speculations relating to the interior of the Earth, of whose real structure we are, and must remain, hopelessly ignorant.

NOTES.

No. I., *referred to in page* 252.—" Un des phénomènes les plus singuliers du système solaire, est l'égalité rigoureuse que l'on observe entre les mouvemens angulaires de rotation et de révolution de chaque satellite. Il y a l'infini contre un à parier qu'il n'est point l'effet du hasard. La théorie de la pesanteur universelle fait disparaître l'infini, de cette invraisemblance, en nous montrant qu'il suffit pour l'existence du phénomène, qu'à l'origine, ces mouvemens aient été très peu différens. Alors l'attraction de la planète a établi entre eux, une parfaite égalité; mais en même temps, elle a donné naissance à une oscillation périodique dans l'axe du satellite, dirigé vers la planète, oscillation dont l'étendue dépend de la différence primitive des deux mouvemens. Les observations de MAYER sur la libration de la lune, et celles que MM. BOUVARD et NICOLLET viennent de faire sur le même objet, à ma prière, n'ayant point fait reconnaître cette oscillation, la différence dont elle dépend, doit être très petite; ce qui indique avec une extrême vraisemblance, une cause spéciale qui d'abord a renfermé cette différence dans les limites fort resserrées où l'attraction de la planète a pu établir entre les mouvemens moyens de rotation et de révolution, une égalité rigoureuse, et qui ensuite a fini par détruire l'oscillation que cette égalité a fait naître. L'un et l'autre de ces effets résultent de notre hypothèse; car on conçoit que la lune à l'état de vapeurs, formait par l'attraction puissante de la terre, un sphéroïde allongé dont le grand axe devait être dirigé sans cesse vers cette planète, par la facilité avec laquelle les vapeurs cèdent aux plus petites forces qui les animent. L'attraction terrestre continuant d'agir de la même manière, tant que la lune a été dans un état fluide, a dû à la longue, en rapprochant sans cesse les deux mouvemens de ce satellite, faire tomber leur différence, dans les limites où commence à s'établir leur égalité rigoureuse. Ensuite, cette attraction a dû anéantir peu à peu l'oscillation que cette égalité a produite dans le grand axe du sphéroïde, dirigé vers la terre. C'est ainsi que les fluides qui recouvrent cette planète, ont détruit par leur frottement et par leur résistance, les oscillations primitives de son axe de rotation, qui maintenant n'est plus assujetti qu'à la nutation résultante des actions du soleil et de la lune. Il est facile de se convaincre que l'égalité des mouvemens de rotation et de révolution des satellites a dû mettre obstacle à la formation d'anneaux et de satellites secondaires, par les atmosphères de ces corps. Aussi l'observation n'a-t-elle jusqu'à présent, rien indiqué de semblable."—LAPLACE, *Exposition du Systeme du Monde*, pp. 472, 473.

No. II., *added March* 25, 1852.—Since the foregoing communication was offered to the Academy, I have become acquainted with Mr. HENNESSEY's Researches in Terrestrial Physics, published by the Royal Society of London in the Philosophical Transactions, Part II., for 1851. In these Researches, pp. 544, 545, Mr. HENNESSEY obtains numerical values for the major and minor limit of the thickness of the Earth's crust, the interior being supposed fluid. These limits are 600 miles and 18 miles respectively. The first limit is obtained by assuming LAPLACE's law of density for the fluid nucleus of the Earth, and the same law for the solid shell, with an alteration of the constants to correspond with the supposed alteration of density of the shell in passing from the fluid to the solid condition. As the hypotheses used to obtain this limit are arbitrary, the limit itself must be considered only as of the same value as the limit in equation (28), deduced from the improbable hypothesis of homogeneity in the shell and nucleus. The other limit is more interesting, being assumed to be a minor limit to the thickness of the Earth's crust, and independent of the law of density of the interior.

On a careful examination of the hypotheses on which the determination of this limit depends, I believe that it will be found, that one of them is inadmissible, and others arbitrary. If I understand Mr. HENNESSEY aright, the following are the statements from which he deduces his minor limit of the thickness of the Earth's crust:

1st. The shell is homogeneous and of the density of the rocks at the surface.

2nd. The shell is bounded by similar surfaces, whose ellipticity is $\frac{1}{300}$.

3rd. The internal surface of the shell is perpendicular to gravity.

4th. The external surface of the shell is not perpendicular to gravity, and its ellipticity, if it were so, would be $\frac{1}{294}$.

The fourth of these statements appears to me to be inadmissible for the following reasons: the ellipticity of the surface perpendicular to gravity is assumed by Mr. HENNESSEY to be $\frac{1}{294}$, which is a mean between the ellipticities $\frac{1}{288}$ and $\frac{1}{300}$, deduced from the pendulum, and lunar inequalities,* but the ellipticity deduced from the lunar observations, $\frac{1}{300}$, is identical with that deduced from the measurement of meridian arcs, and although there may be some chance in this agreement, yet it is sufficient to suggest the idea, that the surface of the Earth is rigorously perpendicular to gravity, and that the pendulum experiments are influenced by variations of local attraction, arising from variable density in the rocks, or from the position of land and water. Such are the usual explanations of the difference between the ellipticity obtained from the pendulum and that deduced from lunar observations; and unless some explanation be offered of the agreement between the ellipticity of the actual surface obtained from meridian arcs, and the ellipticity of the surface perpendicular to gravity deduced from the lunar inequalities, it is not allowable to assume, that the mean of the results of the pendulum and lunar observations gives the surface perpendicular to gravity.

In fact, the observations of the pendulum and of the Moon should give exactly the same ellipticity, and would do so, were it not that the pendulum is liable to local variations, from which the other method is exempt; the result of the latter is, therefore, more trustworthy, and this result is almost identical with the ellipticity of the actual surface. It is certainly unphilosophic to take the mean of observations which differ more from each other than they differ from the quantity with

* The figures here given are those adopted by Mr. HENNESSEY, and are probably as near the truth as any others which have been deduced. The ellipticity deducible from SABINE's pendulum experiments is $\frac{1}{288 \cdot 7}$; and from BOUVARD, BURCKHARDT, and BURG's lunar observations, is $\frac{1}{304 \cdot 1}$. (Mec. Cel., Tom. v. p. 45.)

which we wish to compare them, and then to assume that the difference between the mean so found and that quantity is a real difference.

Adopting the four hypotheses above mentioned, Mr. HENNESSEY has deduced from his formulæ the following value for the ratio of the radius of the nucleus to the radius of the exterior surface, p. 545;

$$a_1^5 = \frac{2}{3} + \frac{1}{3}\,\frac{\frac{5}{4}m - (e)}{\frac{5}{4}m - e}. \qquad (1)$$

In this equation a_1 denotes the ratio of the radius of the nucleus to the radius of the external surface, which is assumed equal to unity; $m = \frac{1}{289}$ is the ratio of centrifugal force to gravity at equator; $e = \frac{1}{300}$ is the ellipticity of the actual surface of the Earth; and $(e) = \frac{1}{294}$ (the mean of the fractions $\frac{1}{288}$ and $\frac{1}{300}$, obtained from the pendulum and lunar observations), is the ellipticity of the surface, if perpendicular to gravity. Substituting these values in equation (1), Mr. HENNESSEY obtains $a_1^5 = 0{\cdot}97714$, and $a_1 = 0{\cdot}99539$, $1 - a_1 = 0{\cdot}00461$, from which he infers, that "consistently with observation, the least thickness of the Earth's crust cannot be less than 18 miles." It is very easy to prove, that if the shell be bounded by similar surfaces, both of which are perpendicular to gravity, that its thickness is zero; this I believe to be the true minor limit of the thickness of the crust.

But even admitting Mr. HENNESSEY's assumption, that the outer surface of the Earth is not perpendicular to gravity, I am unable to agree with him as to the formula from which its thickness should be calculated. In equation (1), which is deduced from the previous equations, a_1 is the reciprocal of the quantity I have called ϕ. This equation contains only the fifth power of a_1 or ϕ, whereas, the equation deducible from the investigation which I have given contains both the fifth and third powers of ϕ, and gives a numerical result which differs materially from Mr. HENNESSEY's. The investigation is as follows. Assuming $\epsilon_1 = \epsilon = e$ in equation (30), which asserts that gravity is perpendicular to the inner surface of the crust and is deduced from (14), and solving for σ, we find, making $\Delta = 2\rho_0$.

$$\frac{3}{5\sigma} = \frac{(\phi^3 + 1)\,e - m}{(\phi^3 + 1)\,e}. \qquad (2)$$

In equation (15), the external surface is supposed perpendicular to gravity, and, therefore, the ellipticity ϵ of its right-hand member must be replaced by (e); the integral at the left-hand side of this equation is proportional to the difference of the moments of inertia of the Earth with respect to its polar and equatorial axes (16), and does not require the surface to be perpendicular to gravity; in fact, the left-hand side of this equation may be supposed to belong to any body having the same difference of moments of inertia as that belonging to the Earth. Separating the integral into two parts, belonging respectively to the shell and nucleus of the Earth, the external surface being supposed similar to the inner, and not perpendicular to gravity, we find,

$$3\,(\phi^5 - 1)\,e + 3\,(\phi^3 + 1)\,\frac{e}{\sigma} = 5\,\{2(e) - m\}\,\phi^5;$$

which might have been deduced directly from (31), by making $\epsilon = \epsilon_1 = e$ on the left-hand side, $\epsilon = (e)$ on the right, and $\Delta = 2\rho_0$. Solving this equation for σ, we find,

$$\frac{3}{5\sigma} = \frac{\{10(e) - 3e - 5m\}\,\phi^5 + 3e}{5e(\phi^3 + 1)}. \tag{3}$$

Eliminating σ from equations (2) and (3), we obtain finally,

$$\phi^5 + \frac{5m - 2e}{10(e) - 3e - 5m} = \frac{5e}{10(e) - 3e - 5m}\,\phi^3. \tag{4}$$

In this equation ϕ is the reciprocal of a_1, and the other letters are the same as the corresponding letters used in equation (1). Equation (4) differs widely from the equation (1) obtained by Mr. HENNESSEY; the hypotheses used in obtaining it are the four hypotheses used by him; and yet I am unable to perceive any error in the process by which (4) is found.

Substituting for $m, e, (e)$; their values $\frac{1}{289}$, $\frac{1}{300}$, and $\frac{1}{294}$, we find,

$$\phi^5 + 1{\cdot}58425 = 2{\cdot}48290\,\phi^3. \tag{5}$$

Applying STURM's theorem to this equation, I find that it has three real roots, one negative and two positive; the latter lying between $\phi = 1$ and $\phi = 2$. These roots are

$$\phi = 1{\cdot}0436; \qquad \phi = 1{\cdot}3626.$$

Rejecting the negative root, as being not applicable to the question in hand, it would appear at first sight as if there were two solutions, corresponding to the two real positive roots just found; but it is evident, by referring to equation (28), that the second value of ϕ, being greater than 1·2407, is to be rejected as well as the negative root; in fact, the second value of ϕ would give a thickness to the crust of the Earth greater than the depth to which the density of the rocks at the surface can extend; and such a thickness, as has been already shown, is inconsistent with the supposition of a fluid nucleus. Calculating the thickness of crust corresponding to the least positive root of equation (5), we find,

$$\mathbf{a} - \mathbf{a}_1 = 166 \text{ miles}. \tag{6}$$

This result differs materially from that obtained from the same data by Mr. HENNESSEY, but as the hypothesis on which it is founded is untenable, the result itself is of little value, except so far as it illustrates the use of the equations already given. As I have before stated, the thickness of the crust would be zero, if we were to admit the first three statements, and combine with them an assertion that the surface of the Earth is perpendicular to gravity. This I believe to be the true minor limit of the thickness of the Earth's crust; and the major limit appears to me to require for its numerical calculation a knowledge of facts, respecting which we must be content to remain in ignorance.