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DeForest's Formula for "An Unsymmetrical Probability Curve"

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DeFOREST'S FORMULA FOR "AN UNSYMMETRICAL PROBABILITY CURVE"

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In presenting a long-forgotten investigation by E. L. DeForest ('82–'83) on "an unsymmetrical probability curve," the writer wishes to call attention to the fact that the first systematic analysis of the subject was attempted by DeForest and as a result he obtained a formula which is identical with that for Professor Pearson's ('95) generalized probability curve. DeForest suggests further that by retaining the higher derivatives a more general formula, of which the formula already found will be a particular case, may be obtained from his original differential equation. Thus DeForest's investigation is not only interesting from an historical standpoint, but still more from the fact that the same formula, though in different terms, has been derived from entirely different methods of analysis by Professor Pearson. This fact furnishes good evidence as to the validity of Professor Pearson's theoretical assumption.

As the investigation was published a number of years ago, the original paper by DeForest is difficult to obtain, and so, for the reader who is anxious to see the method of mathematical analysis adopted by him, I venture to present in the following pages some of the important points which directly concern the derivation of his final formula. I shall also add a mathematical process of transformation of Professor Pearson's formula to that of DeForest. For numerous other important and interesting points, the reader must refer to the original memoirs.

DeForest employed this reasoning:
Let the following be a given polynomial

$$\lambda_{-m} Z^{-m} + \ldots\ldots + \lambda_{-1} Z^{-1} + \lambda_0 + \lambda_1 Z^1 + \ldots\ldots + \lambda_m Z^m. \quad (1)$$

Its expansion to the κ power may be written

$$l_{-\kappa m} Z^{-\kappa m} + \ldots\ldots + l_{-1} Z^{-1} + l_0 + l_1 Z^1 + \ldots\ldots + l_{\kappa m} Z^{\kappa m}. \quad (2)$$

From the relations

$$(\lambda_{-m} Z^{-m} + \ldots\ldots + \lambda_m Z^m)^\kappa = l_{-\kappa m} Z^{-\kappa m} + \ldots\ldots + l_{\kappa m} Z^{\kappa m}$$

we have

$$\kappa \log (\lambda_{-m} Z^{-m} + \ldots\ldots + \lambda_m Z^m) = \log (l_{-\kappa m} Z^{-\kappa m} + \ldots\ldots + l_{\kappa m} Z^{\kappa m})$$

which holds good for all values of Z. By differentiation with respect to Z and then clearing of fractions it becomes

$$\begin{aligned} &\kappa(-m\lambda_{-m} Z^{-m-1} \ldots\ldots + m\lambda_m Z^{m-1})\,(l_{-\kappa m} Z^{-\kappa m} + \ldots\ldots + l_{\kappa m} Z^{\kappa m}) = \\ &(\lambda_{-m} Z^{-m} + \ldots\ldots + \lambda_m Z^m)\,(-\kappa m l_{-\kappa m} Z^{-\kappa m-1} \ldots\ldots + \kappa m l_{\kappa m} Z^{\kappa m-1}.) \end{aligned} \quad (3)$$

Forming the coefficient of Z^{i-1} in the polynomial product, and remembering also that the rank of the middle l of this group reckoned from l_0 is i, we get, by equating the two to each other by the principle of undetermined coefficients,

$$\kappa(-m\lambda_{-m} l_{i+m} \ldots\ldots + m\lambda_m l_{i-m}) = (i+m)\lambda_{-m} l_{i+m} + \ldots\ldots + (i-m)\lambda_m l_{i-m}.$$

In the second member, let that part which does not have the coefficient i be transferred to the first member, then

$$-m\lambda_{-m} l_{i+m} - \ldots\ldots + m\lambda_m l_{i-m} = \frac{i}{\kappa+1}(\lambda_{-m} l_{i+m} + \ldots + \lambda_m l_{i-m}). \quad (4)$$

Clearly then any coefficient l_i in the expansion, and the $2m$ coefficients nearest to it, will be connected by the relation

$$\frac{(\lambda_1 l_{i-1} - \lambda_{-1} l_{i+1}) + 2(\lambda_2 l_{i-2} - \lambda_{-2} l_{i+2}) + \ldots\ldots + m(\lambda_m l_{i-m} - \lambda_{-m} l_{i+m})}{\lambda_0 l_i + (\lambda_1 l_{i-1} + \lambda_{-1} l_{i+1}) + (\lambda_2 l_{i-2} + \lambda_{-2} l_{i+2}) + \ldots + (\lambda_m l_{i-m} + \lambda_{-m} l_{i+m})} = \frac{i}{\kappa+1}. \quad (5)$$

This is the fundamental principle of DeForest's analysis in his numerous interesting studies on the theory of probability. Let

l_{i+1}, l_{i-1}, etc., in (5) be expressed in terms of l_i and their differences. For this DeForest refers to a convenient formula given by Lacroix (Cal. diff. et intég., Paris, 1819) as follows:

$$l_{i+n}=l_i+\frac{n}{1}\Delta_1+\frac{n^2}{2!}\Delta_2+\frac{n(n^2-1^2)}{3!}\Delta_3+\frac{n^2(n^2-1^2)}{4!}\Delta_4+\frac{n(n^2-1^2)(n^2-2^2)}{5!}\Delta_5+\text{etc.} \quad (6)$$

For brevity let us write also

$$\begin{aligned} b_0 &= \lambda_0+(\lambda_1+\lambda_{-1})+(\lambda_2+\lambda_{-2})+\ldots\ldots+(\lambda_m+\lambda_{-m}) \\ b_1 &= 1(\lambda_1-\lambda_{-1})+2(\lambda_2-\lambda_{-2})+\ldots\ldots+m(\lambda_m-\lambda_{-m}) \\ b_2 &= 1^2(\lambda_1+\lambda_{-1})+2^2(\lambda_2+\lambda_{-2})+\ldots\ldots+m^2(\lambda_m-\lambda_{-m}) \\ b_3 &= 1^3(\lambda_1-\lambda_{-1})+2^3(\lambda_2-\lambda_{-2})+\ldots\ldots+m^3(\lambda_m-\lambda_{-m}) \qquad (7) \\ &\qquad \text{etc.,} \qquad\qquad \text{etc.} \end{aligned}$$

Denoting the numerator and denominator in the first member of (5) by N and D respectively, we get

$$\begin{aligned} N &= b_1 l_i - b_2\Delta_1+\frac{1}{2}b_3\Delta_2-\frac{1}{3!}(b_4-b_2)\Delta_3+\frac{1}{4!}(b_5-b_3)\Delta_4 \\ &\quad -\frac{1}{5!}(b_6-5b_4+4b_2)\Delta_5+\frac{1}{6!}(b_7-5b_5+4b_3)\Delta_6 \\ &\quad -\frac{1}{7!}(b_8-8b_6+19b_4-12b_2)\Delta_7+\ldots\ldots\ldots \\ D &= b_0 l_i - b_1\Delta_1+\frac{1}{2}b_2\Delta_2-\frac{1}{3!}(b_3-b_1)\Delta_3+\frac{1}{4!}(b_4-b_2)\Delta_4 \\ &\quad -\frac{1}{5!}(b_5-5b_3+4b_1)\Delta_5+\frac{1}{6!}(b_6-5b_4+4b_2)\Delta_6 \\ &\quad -\frac{1}{7!}(b_7-8b_5+19b_3-12b_1)\Delta_7+\ldots\ldots\ldots \qquad (8) \end{aligned}$$

or $$\frac{N}{D}=\frac{i}{\kappa+1}$$

When κ becomes infinite, and the successive values of l are regarded as consecutive ordinates to a limiting curve, we have

$$l_i=y \qquad \Delta_1=dy \qquad \Delta_2=d^2y \qquad \Delta_3=d^3y, \text{ etc.,}$$

and at the same time when the ordinates are set close together, the abscissa x corresponding to any y is $x = idx$. Thus (8) becomes the differential equation of the curve, and b_0, b_1, b_2, etc., are constants, and in fact are the successive moments of the area bounded by the curve and the axis of abscissas, these moments being taken about a vertical axis. Since any given polynomial may be reduced to one in which $\Sigma(\lambda) = 1$, by dividing it throughout by the sum of its coefficients, we therefore consider $b_0 = 1$. If a constant number is added to or subtracted from all the exponents of z in (1), it will not alter the value of l in (2). Hence by making Z^0 the abscissa of the center of gravity, b_1 becomes zero. Then any constant b_n in (7) will denote the sum of the products formed by multiplying each λ into the nth power of its abscissa reckoned from the new origin, if the common interval Δx between the abscissa is regarded as unity. With the above transformations, we may now write (8) in the following forms:

$$\frac{b_2dy - \frac{1}{2}b_3d^2y + \frac{1}{6}(b_4 - b_2)d^3y - \text{etc.}}{y + \frac{1}{2}b_2d^2y - \frac{1}{6}b_3d^3y + \text{etc.}} = \frac{-x}{(\kappa+1)dx} \tag{9}$$

In the denominator of the first member let d^2y, d^3y, etc., be neglected in comparison with y and in the numerator let d^3y, d^4y, etc., be neglected in comparison with dy. Since κ is infinitely large, we may write κ instead of $\kappa+1$.
Therefore

$$\frac{dy - \frac{1}{2}(b_3 \div b_2)d^2y}{y} = \frac{-x}{\kappa b_2 dx}.$$

Invert both members of this equation, subtract $\frac{1}{2}(b_3 \div b_2)$ from each and invert them both back again. This gives

$$\frac{dy - \frac{1}{2}(b_3 \div b_2)d^2y}{y - \frac{1}{2}(b_3 \div d_2)dy + \frac{1}{4}(b_3 \div d_2)^2d^2y} = \frac{-x}{\kappa b_2dx + \frac{1}{2}(b_3 \div b_2)x}. \tag{10}$$

Thus far we have carried on our treatment on the assumption that the origin of Z^0 in the expansion is located at the center of gravity for the coefficient l in (2), which became the ordinate y

to the limiting curve. Now in (10) let the origin be transferred from the center of gravity to another convenient point by putting

$$x - \frac{2\kappa b_2^2 dx}{b^3} \tag{11}$$

in place of x. This gives

$$\frac{dy - \frac{1}{2}(b_3 \div b_2)d^2y}{y - \frac{1}{2}(b_3 \div b_2)dy} = \frac{4\kappa b_2 dx - 2(b_3 \div b_2)x}{(b_3 \div b_2)^2 x}. \tag{12}$$

In the first member, the numerator is the differential of the denominator. Without any further change of origin, we can write approximately as follows:

$$y = y + \tfrac{1}{2}(b_3 \div b_2)dy, \qquad x = x + \tfrac{1}{2}(b_3 \div b_2)dx$$

Neglecting d^3y in the numerator and d^2y in the denominator, we get

$$\frac{dy}{y} = \frac{4\kappa b_2 dx - (b_3 \div b_2)^2 dx - 2(b_3 \div b_2)x}{(b_3 \div b_2)^2[x + \frac{1}{2}(b_3 \div b_2)dx]}.$$

Since the denominator y in the first member is supposed to be infinitely greater than the numerator dy, the denominator in the second member must be infinitely greater than its numerator, so that in the denominator we may neglect dx in comparison with x. Further let the constants be expressed by means of the two new constants

$$a = \frac{2b_2(dx)^2}{b_3(dx)^3}, \qquad b = \kappa b_2(dx)^2. \tag{13}$$

Since κ is supposed to be an infinity of the second order, b represents a finite area. The equation will now stand

$$\frac{dy}{y} = \frac{dx}{x}(a^2b - 1) - adx, \tag{14}$$

and integration gives

$$\begin{aligned} \log y &= (a^2b - 1)\log x - ax + \log C \\ \therefore\ y &= Cx^{a^2b-1}\, e^{-ax} \end{aligned} \tag{15}$$

It now remains to determine the constant C in (15). Since

$\Sigma(\lambda) = 1$ in the given polynomial and $\Sigma(l) = 1$ in its expansion, we shall have $\Sigma(y) = 1$ in the formula (15). The y which DeForest uses, represents an elementary area, so that it should be understood to mean ydx in modern notation. Thus equation (18), omitting dx, gives the equation of the curve. Thus we have in DeForest's notation:

$$\frac{1}{dx}\int_0^\infty ydx = 1 \therefore \frac{C}{a^{a^2b}dx}\int_0^\infty (ax)^{a^2b-1}e^{-ax}d(ax) = 1,$$

which gives at once the value of C and we have

$$y = \frac{adx}{\Gamma(a^2b)}(ax)^{a^2b-1}e^{-ax} \tag{16}$$

the complete equation of the curve sought.

If we now transfer the origin of coördinates to the center of gravity by putting $x + \frac{2\kappa b_2^2 dx}{b^3}$ in (11) or $x + ab$ in place of x in (16), we have

$$y = \frac{dx}{ab\Gamma(a^2b)}\left(\frac{a^2b}{e}\right)^{a^2b}\left(1+\frac{x}{ab}\right)^{a^2b-1}e^{-ax} \tag{17}$$

Applying a known formula for $\Gamma(n)$

$$\Gamma(n) = \left(\frac{n}{e}\right)^n\sqrt{\frac{2\pi}{n}}\left(1+\frac{1}{12n}+\frac{1}{288\,n^2}+\text{etc.}\right)$$

(17) is reduced to

$$y = \frac{dx}{\kappa\sqrt{2\pi b}}\left(1+\frac{x}{ab}\right)^{a^2b-1}e^{-ax} \tag{18}$$

$$\text{where } \kappa = 1 + \frac{1}{12\,a^2b} + \frac{1}{288(a^2b)^2} + \text{etc.}$$

Returning to the meaning of the constants, a in (13) may be written

$$a = 2\left(\frac{b_2\,(dx)^2}{b^3\,(dx)^3}\right) = 2\left(\frac{\kappa b_2(dx)^2}{\kappa b_3(dx)^3}\right) \tag{19}$$

This shows that the part within the parenthesis may be regarded as the square of the quadratic radius divided by the cube of the cubic radius, either in the first power of the polynomials or in its expansion to the κ power.

The value of a and b may thus be expressed by means of the coefficients λ in the given polynomial, or by means of the ordinates y to the limiting curve. When the λ's and y's are all positive $\kappa b_2(dx)^2$ is the square of the quadratic mean error "ϵ" and $\kappa b_3(dx)^3$ is the cube of what DeForest calls the cubic mean inequality "ζ."

The constants in (13) will then be

$$a = 2\epsilon^2 \div \zeta^3 \qquad\qquad b = \epsilon^2$$

It will be seen then that the constants ϵ^2 and ζ^3 are respectively the second and third moments of Pearson and therefore can be advantageously determined by his method. The above sketch should enable the reader to get an idea of the method of DeForest's analysis, and this was my object in presenting it. The properties of the formula as well as the method of transformation of the present formula to the normal probability form are adequately treated in the original paper of DeForest. However, regarding these points, the reader will get still better information from Pearson's discussion on his curve of Type III.

Although I have not given the process of transformation of the formula to the normal form, DeForest's statement in this connection will be worth noting. He states that he would have obtained the normal form directly from the equation (9) if he had neglected d^2y. If instead of retaining only dy and d^2y he should also retain d^3y, the resulting equation, provided such is integrable, would doubtless give a limiting curve of a still more general form, of which the curve derived from (18) is but a particular case. Thus he thought that the probability curve and his curve (18) are only the first and second approximations to the actual form of an expansion to a high power.

From the foregoing discussion the reader will notice a close similarity between DeForest's formula, and Pearson's formula for the curve of Type III. For convenience, I shall enumerate some of the similar properties in these two curves.

(1) Both are the skew binomial curves.

(2) The curve is limited on one side the mean.

(3) The analytical constants are determined from the first three moments.

(4) Both can be reduced to the normal form.

(5) Each is a particular case of a more general formula.

It will be demonstrated in the following pages that although these two formulas show no more apparent similarity yet the formulas are identical:

From the differential equation $\frac{1}{y}\ \frac{dy}{dx} = \dfrac{x + \dfrac{\mu_3}{2\mu_2}}{\mu_2 + \dfrac{\mu_3}{2\mu_2}\,x}$

Professor Pearson obtained his formula for the curve of Type III which is usually written in the following form:

$$y = \frac{\alpha}{a} \cdot \frac{p^{p+1}}{e^p\,\Gamma(p+1)} \left(1 + \frac{x}{a}\right)^{\nu a} e^{-\nu x} \tag{20}$$

The following relations are also given

$$\mu_2 = \frac{p+1}{\nu^2}\ ;\ \nu = \frac{2\mu_2}{\mu_3}\ ;\ p = \frac{4\,\mu_2^3}{\mu_3^2} - 1\ ;\ a = \frac{p}{\nu}.$$

Since the distance of the centroid vertical from the axis of y or maximum ordinate is $\frac{1}{2}\,\frac{\mu_3}{\mu_2}$, by changing the value of x, that is, putting

$$x = x + \tfrac{1}{2}\,\frac{\mu_3}{\mu_2}$$

(20) is reduced into the following

$$y = \frac{\alpha}{a} \cdot \frac{p^{p+1}}{e^p\Gamma(p+1)} \left(1 + \frac{\frac{1}{2}\frac{\mu_3}{\mu_2} + x}{\frac{2\mu_2^2}{\mu_3} - \frac{\mu_3}{2\mu_2}}\right)^{\frac{4\,\mu_2^3}{\mu_3^2} - 1} e^{-\frac{2\mu_2}{\mu_3}\left(x + \frac{\mu_3}{2\mu_2}\right)}$$

$$= \frac{\alpha}{a}\ \frac{p^{p+1}\,e^{-(p+1)}}{\Gamma(p+1)\left(1 - \frac{\mu_3^2}{4\,\mu_2^3}\right)^p} \left(1 + \frac{x}{2\,\mu_2^2 \div \mu_3}\right)^{\frac{4\mu_2^3}{\mu_3^2} - 1} e^{-\frac{2\mu_2}{\mu_3}x}$$

$$= \frac{\alpha}{a\sqrt{2\pi\mu_2}} \frac{\sqrt{2\pi\mu_2}\, p^{p+1} e^{-(p+1)}}{\Gamma(p+1)\left(\dfrac{p}{p+1}\right)^p}\left(1+\frac{x}{2\mu_2^2 \div \mu_3}\right)^{\frac{4\mu_2^3}{\mu_3^2}-1} e^{-\frac{2\mu_2}{\mu_3}x}$$

$$= \frac{\alpha}{a\sqrt{2\pi\mu_2}} \frac{\sqrt{2\pi(p+1)}\, a\, p^p e^{-(p+1)}}{\Gamma(p+1)\dfrac{p^p}{(p+1)^p}}\left(1+\frac{x}{2\mu_2^2 \div \mu_3}\right)^{\frac{4\mu_2^3}{\mu_3^2}-1} e^{-\frac{2\mu_2}{\mu_3}x}$$

and finally, as the result of transferring the origin to the centroid vertical, we obtain

$$y = \frac{\alpha}{\sqrt{2\pi\mu_2}} \frac{\sqrt{2\pi(p+1)}\, e^{-(p+1)}(p+1)^p}{\Gamma(p+1)}\left(1+\frac{x}{2\mu_2^2 \div \mu_3}\right)^{\frac{4\mu_2^3}{\mu_3^2}-1} e^{-\frac{2\mu_2}{\mu_3}x} \quad (21)$$

If we now apply to the above (21) DeForest's notation, that is,

$$\mu_2 = b \text{ and } 2\mu_2 \div \mu_3 = a$$

we obtain at once

$$y = y_1\left(1+\frac{x}{ab}\right)^{a^2b-1} e^{-ax}$$

where

$$y_1 = \frac{\alpha}{\sqrt{2\pi\mu_2}} \frac{\sqrt{2\pi(p+1)}\, e^{-(p+1)}\,(p+1)^p}{\Gamma(p+1)}.$$

It only remains to see whether or not y_1 in Pearson's formula is identical with DeForest's C.

We have

$$y_1 = \frac{\alpha}{\sqrt{2\pi b}} \frac{\sqrt{2\pi(a^2b)}\; e^{-a^2b} a^2b^{a^2b-1}}{\Gamma(a^2b)}$$

$$= \frac{\alpha}{\Gamma(a^2b)}\; ae^{-a^2b}\; a^2b^{a^2b-1}$$

Using the approximation formula for $\Gamma(n)$ which DeForest uses (18) we have

$$y_1 = \frac{\alpha e^{-a^2b} a^3 b^{a^2b-1}}{\left(\dfrac{a^2b}{e}\right)^{a^2b}\sqrt{\left(\dfrac{2\pi}{a^2b}\right)}\kappa}$$

$$= \frac{\alpha}{\kappa\sqrt{2\pi}} \cdot \frac{a\sqrt{a^2b}}{a^2b} = \frac{\alpha}{\kappa\sqrt{2\pi b}}$$

$$\text{where } \kappa = 1 + \frac{1}{12a^2b} + \frac{1}{288(a^2b)^2} + \text{etc.}$$

Since α is unity in DeForest's formula, thus Pearson's formula for the curve of Type III immediately reduces to DeForest's. That is

$$y = \frac{1}{\kappa\sqrt{2\pi b}}\left(1 + \frac{x}{ab}\right)^{a^2b-1} e^{-ax}.$$

Thus DeForest's formula presents several interesting points which I herewith enumerate as the conclusion of the present report.

(1) DeForest's investigation gives an additional proof for the theoretical basis of Pearson's generalized probability curve.

(2) DeForest's investigation is interesting from an historical standpoint since he actually obtained one of Pearson's curves many years ago, and his work suggests a more generalized curve.

(3) Since DeForest's formula (see (18), p. 286) retains an elementary character, the curve fitting can be accomplished with comparatively small labor, and it can advantageously be used in place of the formula of Pearson for the curve of Type III.

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