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# NOTES ON OPTICS

## FOR STUDENTS OF OPHTHALMOLOGY

BY

A. S. PERCIVAL, M.A., M.B.
TRINITY COLLEGE, CAMBRIDGE.

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# PREFACE.

THESE "Notes on Optics" are published primarily for those post-graduate students who attend my lectures and demonstrations on Ophthalmology. It will I think be found that they contain all the Optics that are required for an intelligent use of the refraction ophthalmoscope.

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# NOTES ON OPTICS

## FOR STUDENTS OF OPHTHALMOLOGY.

### Conventions used for signs of direction.

In trigonometry we deal with any angle made by a radius vector ($OP$) with an initial line ($OA$). (Fig. 1.) The radius vector may be regarded as the hand of a clock with a pivot at $O$, which can rotate from $OA$ into any position.

(1) Angles measured counter-clockwise are considered positive.
Angles ,, clockwise ,, ,, negative.

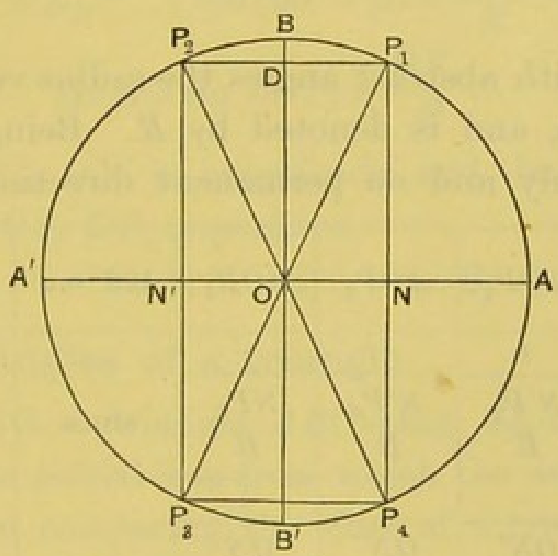


Fig. 1.

The angle between the radius vector $OP_1$ and the initial line $OA$ is read $AOP_1$ if measured in the counter-clockwise direction, but $P_1OA$ if ,, ,, ,, clockwise ,, .

Thus $AOP_1$, $P_1OB$, $AOP_3$, $P_2OB'$ are positive, while $P_1OA$, $BOP_1$, $P_3OA$, $B'OP_2$ are negative.

Note that the angle $AOP_3$ counter-clockwise is positive $= 180° + A'OP_3$, while the angle $AOP_3$ clockwise is negative $= -(P_3OB_1 + 90°)$.

(2) Lines measured from below upwards are considered positive.

Lines measured from above downwards are considered negative.

Thus $OB$, $B'O$ are positive, but $BO$, $OB'$ are negative.

Note that $OB$ is the distance of $B$ from $O$, $BO$ the distance of $O$ from $B$.

Lines measured from left to right are considered positive.

Lines measured from right to left are considered negative.

Thus $OA$, $A'O$ are positive, but $AO$, $OA'$ are negative.

(In order to familiarize the reader with this principle in many of the subsequent figures the direction of the incident light is that from right to left, and the incident direction is then negative.)

**Trigonometrical ratios of abstract angles.**

Erect a perpendicular $NP$ to meet the extremity of the radius vector.

In dealing with abstract angles the radius vector is considered to be of no sign, and is denoted by $R$. Being a rotating rod, it has length only and no permanent direction.

| | $AOP_1$ | $AOP_2$ | $AOP_3$ | $AOP_4$ | $180° - \alpha$ | $180° + \alpha$ | $-\alpha$ |
|---|---|---|---|---|---|---|---|
| sin | + | + | − | − | | | |
| | $\frac{NP_1}{R}$ | $\frac{N'P_2}{R}$ | $\frac{N'P_3}{R}$ | $\frac{NP_4}{R}$ | $\sin\alpha$ | $-\sin\alpha$ | $-\sin\alpha$ |
| cos | + | − | − | + | | | |
| | $\frac{ON}{R}$ | $\frac{ON'}{R}$ | $\frac{ON'}{R}$ | $\frac{ON}{R}$ | $-\cos\alpha$ | $-\cos\alpha$ | $+\cos\alpha$ |
| tan | + | − | + | − | | | |
| | $\frac{NP_1}{ON}$ | $\frac{N'P_2}{ON'}$ | $\frac{N'P_3}{ON'}$ | $\frac{NP_4}{ON}$ | $-\tan\alpha$ | $+\tan\alpha$ | $-\tan\alpha$ |

If $AOP_1$ be denoted by the positive angle $\alpha$, it is easily seen that the last three columns follow from the preceding three columns.

Also $\sin(90° - \beta) = \cos\beta$, and $\sin(90° + \beta) = \cos\beta$.

$\cos(90° - \beta) = \sin\beta$, and $\cos(90° + \beta) = -\sin\beta$.

When comparing the trigonometrical ratios of angles measured from different initial lines, it is necessary to rotate one of them until its initial line coincides with that of the other,

e.g. $$\sin(90° - \beta) = \cos\beta.$$

If $P_1OB$ be denoted by the positive angle $\beta$,

$$AOP_1 = 90° - \beta, \qquad \therefore \sin(90° - \beta) = \frac{NP_1}{R} \text{ (positive).}$$

But $$\cos\beta = \cos(-\beta) = \cos BOP_1 = \frac{OD}{R}.$$

Note that $OD = NP_1$ in magnitude, and that on rotating the initial line $OB$ into the position $OA$, $OD$ is positive.

$$\therefore \sin(90° - \beta) = \cos\beta.$$

Similarly $$\cos(90° + \beta) = -\sin\beta.$$

If $BOP_2$ be denoted by the positive angle $\beta$,

$$AOP_2 = (90° + \beta), \qquad \therefore \cos(90° + \beta) = \frac{ON'}{R} \text{ (negative).}$$

But $$\sin\beta = \sin BOP_2 = \frac{DP_2}{R}.$$

Note that $DP_2 = ON'$ in magnitude, and that on rotating $OB$ into the position $OA$, $DP_2$ is positive.

$$\therefore \cos(90° + \beta) = -\sin\beta.$$

## Sides and Angles of a triangle.

In dealing with a triangle $ABC$ (Fig. 2), each of the sides forms in turn the initial line from which the angles $A$, $B$, $C$, are measured. When comparing the sides of a triangle to the sines

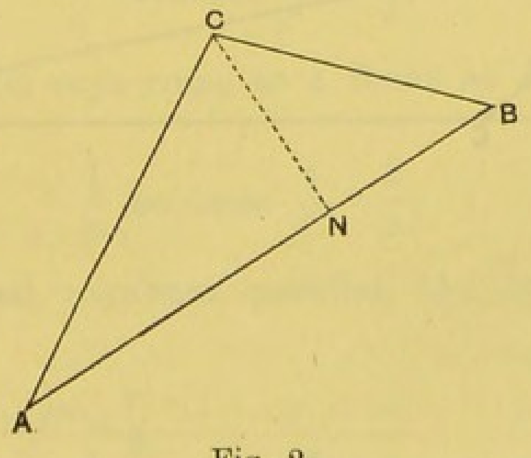


Fig. 2.

of the opposite angles, some convention must be adopted to determine the direction in which the sides are measured.

The simplest convention is that the sides

are considered positive when taken in counter-clockwise order,
,, ,, negative ,, ,, clockwise order.

Thus in Fig. 2, $AB$, $BC$, $CA$ are positive,
while $BA$, $AC$, $CB$ ,, negative.

Now $BAC$ and $CBA$ are concrete positive angles. As the hypothenuse is not a signless radius vector but has a permanent direction or sign,

$$\sin BAC = \frac{NC}{CA} \text{ and } \sin CBA = -\sin ABC = -\left(\frac{NC}{CB}\right) = \frac{NC}{BC},$$

$$\therefore \frac{\sin BAC}{\sin CBA} = \frac{\frac{NC}{CA}}{\frac{NC}{BC}} = \frac{BC}{CA}.$$

The expression $\sin A : \sin B : \sin C = a : b : c$
means $\sin BAC : \sin CBA : \sin ACB = BC : CA : AB$.

(Note that the first and last letters of the expression denoting the angle give the corresponding side and its direction.)

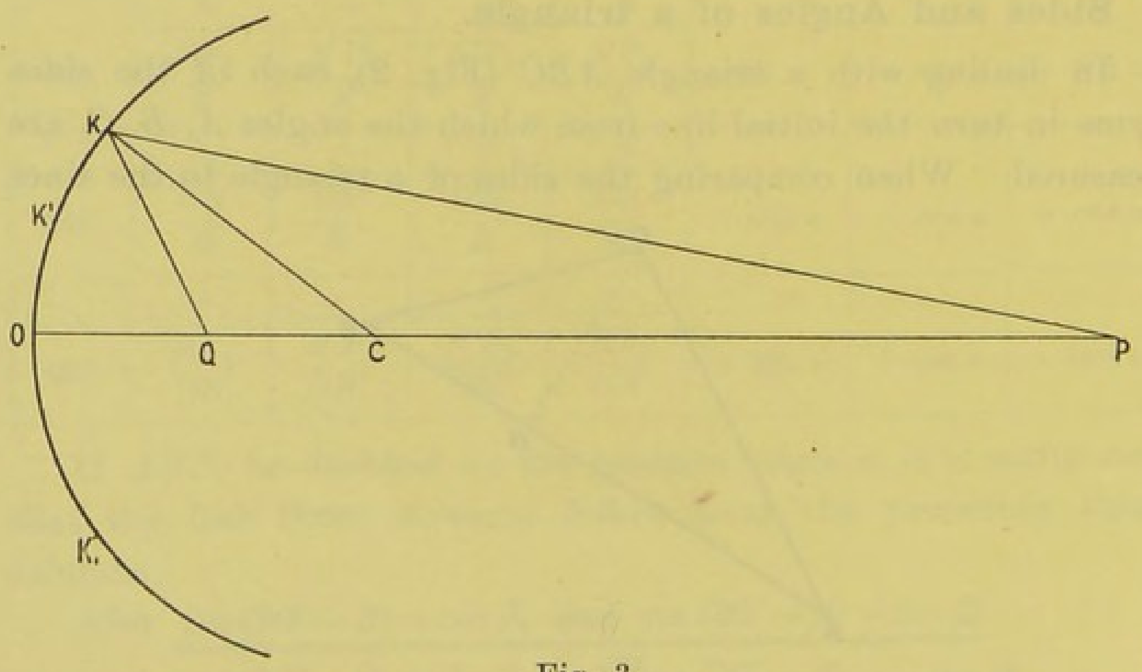


Fig. 3.

**Reflexion at a spherical surface of a small axial pencil.**

(Fig. 3.) Let $P$ be a luminous point on the principal axis of the mirror which is distant $PO$ or $p$ from it.

Let $K$ be a point on the mirror near $O$ and $C$ the centre of the mirror. Join $CK$, then $KC$ is normal to the surface at $K$.

Then light travelling in the direction $PK$ will be reflected at $K$ in the direction $KQ$ such that $CKQ = -CKP$.

Let $QO$ and $CO$ be denoted by $q$ and $r$.

Now in the triangle $PKQ$, by Euc. VI. 3,

$$\frac{PC}{CQ} = \frac{PK}{QK} = \frac{PO}{QO} \text{ in the limit when } K \text{ coincides with } O,$$

$$\therefore \frac{PO - CO}{CO - QO} = \frac{PO}{QO} \text{ or } \frac{p-r}{r-q} = \frac{p}{q},$$

$$\therefore qp - qr = pr - pq \text{ or } qr + pr = 2pq,$$

or on dividing by $pqr$

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{r}.$$

Note that in the figure $p$, $q$ and $r$ are all negative, being measured from right to left; if the incident direction $p$ were positive, $q$ and $r$ would also be positive.

(It is clear that this formula is only true for small centric pencils of which the extreme ray is reflected at a point close to $O$.)

If $p = \infty$, $\frac{1}{q}$ becomes $\frac{1}{f''} = \frac{2}{r}$,

i.e. incident parallel rays come to a focus at $F$, where $2FO$ or $2f'' = r$.

If $q = \infty$, $\frac{1}{p}$ becomes $\frac{1}{f'} = \frac{2}{r}$,

i.e. if the reflected rays are parallel, the luminous point is situated at $F$.

In this case $f' = f'' = \frac{r}{2}$.

The geometrical construction for the image $(ba)$ of an object $(BA)$ is precisely the same as that for an image formed by refraction at a single spherical surface (Figs. 6 and 7), the only difference being that the second principal focus $(F'')$ and the first principal focus $(F')$ in the case of reflexion both coincide at one point $F$ on the incident side of the reflecting surface.

*Size of the image.*

Similarly the same formulæ apply,

$$\frac{i}{o}=\frac{q-r}{p-r}=\frac{f''-q}{f''}=\frac{f'}{f'-p}.$$

In this case however $f'=f''$.

## Refraction.

*Snell's law,* $$\frac{\sin\phi}{\sin\phi'}=\mu.$$

Let $BB'$ represent the bounding surface of a dense medium, glass for instance, and let $SC$ be an incident ray, and $CQ$ the corresponding refracted ray. (Fig. 4.)

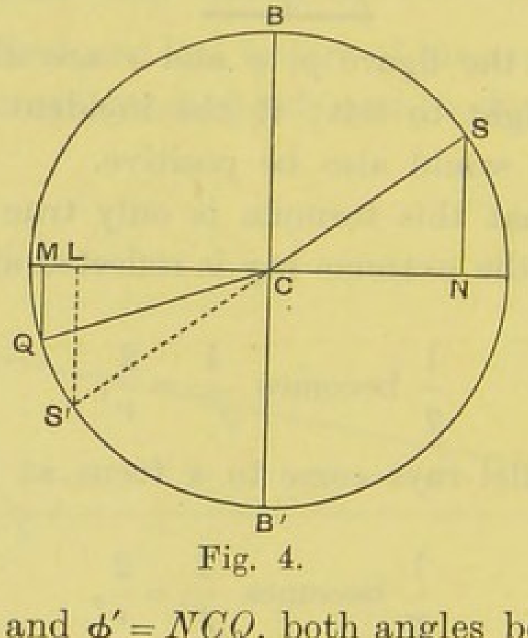


Fig. 4.

Then $\phi=NCS$ and $\phi'=NCQ$, both angles being measured in the same (counter-clockwise) direction.

Now $\sin\phi'=\sin NCQ=\sin(180°+MCQ)$

$$=-\sin MCQ=-\left(\frac{MQ}{R}\right)=\frac{QM}{R}.$$

But $\sin\phi = \sin NCS = \frac{NS}{R}$, $\therefore \frac{\sin\phi}{\sin\phi'} = \frac{NS}{QM} = \frac{S'L}{QM}$.

It will be noted that both the lines $S'L$ and $QM$ are measured in the same direction, and therefore $\mu$ is positive. A study of physical optics shows that $\mu$ is the ratio of the speed of light in the first medium to its speed in the second medium, or

$$\mu = \frac{V_r}{V_a} = \frac{S'L}{QM}.$$

**Refraction at a single spherical surface of a small axial pencil.**

$\phi = NKP$ a negative angle. $\sin NKP = \sin(-180^\circ - PKC)$

$$= -\sin(180^\circ + PKC) = \sin PKC. \quad \text{(Fig. 5.)}$$

$\phi' = NKQ$ a negative angle. $\sin NKQ = \sin(-180^\circ - QKC)$

$$= -\sin(180^\circ + QKC) = \sin QKC.$$

$$\mu = \frac{\sin\phi}{\sin\phi'} = \frac{\sin PKC}{\sin QKC}.$$

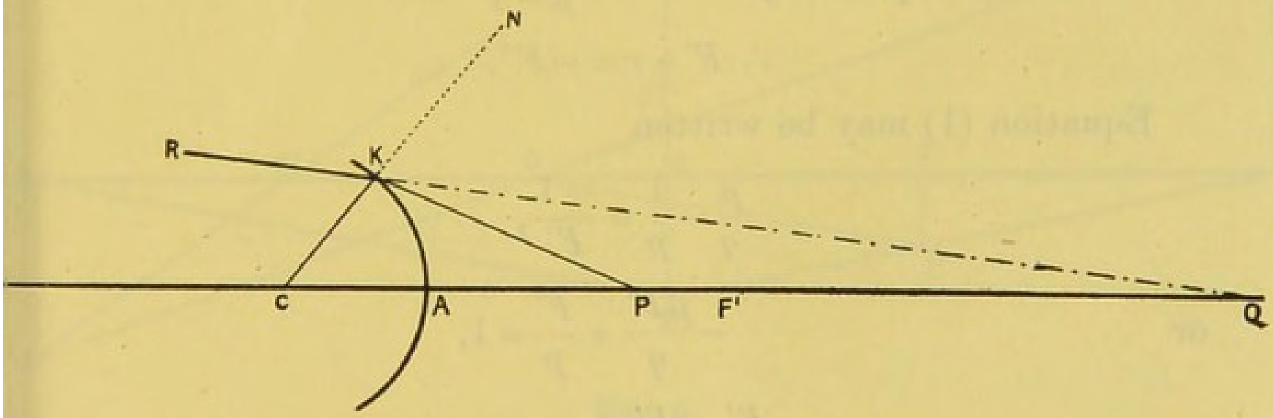


Fig. 5.

Now $\frac{PC}{CK} = \frac{\sin PKC}{\sin CPK}$ and $\frac{QC}{CK} = \frac{\sin QKC}{\sin CQK}$.

$$\therefore \frac{PC}{QC} = \frac{\sin PKC}{\sin QKC} \cdot \frac{\sin CQK}{\sin CPK} = \mu \frac{\sin PQK}{-\sin(180^\circ + CPK)}$$

$$= \mu \frac{\sin PQK}{-\sin QPK} = \mu \frac{\sin PQK}{\sin KPQ} = \mu \frac{PK}{KQ}.$$

Now $PK$ and $KQ$ are of the same sign because they are the sides of a triangle taken in (clockwise) order. But as $K$ approaches $A$, $PK$ and $QK$ differ finally from $PA$ and $QA$ by a negligible quantity.

Therefore for a small axial pencil

$$\frac{PC}{QC} = \mu \frac{PA}{QA} \text{ or } \frac{PA - CA}{QA - CA} = \mu \frac{PA}{QA},$$

i.e. $$\frac{p-r}{q-r} = \mu \frac{p}{q} \text{ or } pq - qr = \mu pq - \mu pr.$$

(1) Therefore on dividing by $pqr$,

$$\underline{\frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r}}.$$

(2) Let $q = \infty$, $p$ then becomes $F'$, or

$$\frac{1}{F'} = -\frac{\mu - 1}{r} \text{ or } F' = \frac{-r}{\mu - 1}.$$

(3) Let $p = \infty$, $q$ then becomes $F''$, or

$$\frac{\mu}{F''} = \frac{\mu - 1}{r} \text{ or } F'' = \frac{\mu r}{\mu - 1} = -\mu F'.$$

$$\therefore F' - r = -F''.$$

Equation (1) may be written

$$\frac{\mu}{q} - \frac{1}{p} = \frac{-1}{F'},$$

or $$-\frac{\mu F'}{q} + \frac{F'}{p} = 1,$$

(4) or $$\frac{F'}{p} + \frac{F''}{q} = 1,$$

or $$pq - pF'' - qF' = 0,$$

and on adding $F'F''$ to both sides of the equation we get

$$pq - pF'' - qF' + F'F'' = F'F'',$$

(5) or $$(p - F')(q - F'') = F'F''.$$

Both (4) and (5) are universally true.

*Donders' eye.* If the incident light is travelling from left to right, $r = -5$ mm. $\mu = \frac{4}{3}$.

$$F' = \frac{-r}{\mu - 1} = \frac{5}{\frac{4}{3} - 1} = +15 \text{ mm.}$$

$$F'' = -\mu F' = -\tfrac{4}{3}(15) = -20 \text{ mm.}$$

*Geometrical construction of image.*

Through the first principal focus $F'$ and through the vertex $(O)$ of the surface draw planes at right angles to the principal axis.

The image $(a)$ of the point $A$ not on the principal axis may be found in two ways.

(1) Draw $AH$ parallel to the principal axis. (Fig. 6.)
,, $HF''$ through the second principal focus.
,, $AC$ through the nodal point (or centre).

The image $a$ is the point of intersection of $HF''$ and $AC$.

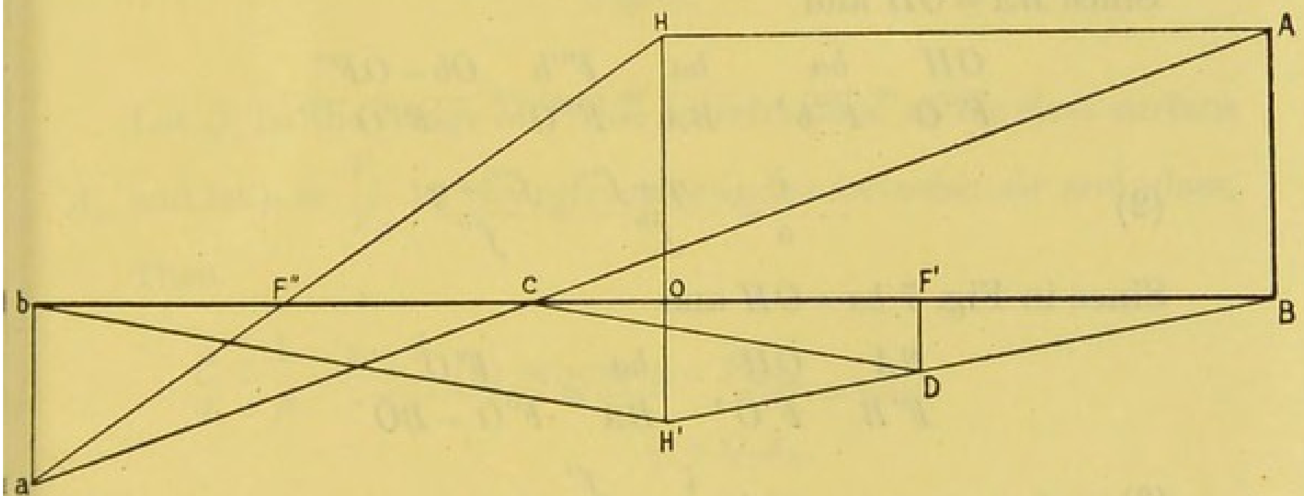


Fig. 6.

(2) Draw $F'AH$ through the first principal focus. (Fig. 7.)
,, $Ha$ parallel to the principal axis.
,, $CAa$ through the nodal point.

The image $a$ is the point of intersection of the two last lines.

The image $(b)$ of the point $B$ on the principal axis may be found by the following device. (Fig. 6.)

Through $B$ draw any ray $BDH'$ cutting the first focal plane in $D$. Join $DC$. Draw $H'b$ parallel to $DC$ cutting the principal axis in $b$.

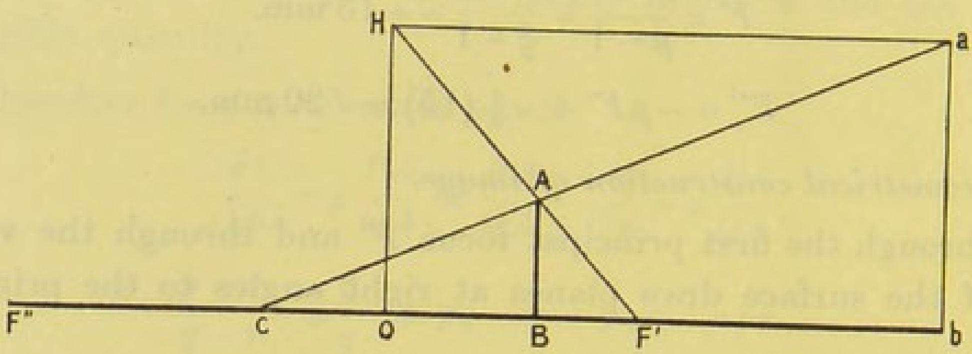


Fig. 7.

*Size of image.*

$$\frac{BA}{CB}=\frac{ba}{Cb}, \therefore \frac{ba}{BA}=\frac{Cb}{CB}=\frac{bC}{BC}=\frac{bO-CO}{BO-CO}. \quad \text{(Fig. 6.)}$$

(1) $$\therefore \frac{i}{o}=\frac{q-r}{p-r}.$$

Since $BA=OH$ and

$$\frac{OH}{F''O}=\frac{ba}{F''b}, \quad \frac{ba}{BA}=\frac{F''b}{F''O}=\frac{Ob-OF''}{F''O}.$$

(2) $$\therefore \frac{i}{o}=\frac{-q+f''}{f''}=\frac{f''-q}{f''}.$$

Since in Fig. 7 $ba=OH$ and

$$\frac{BA}{F'B}=\frac{OH}{F'O}, \quad \frac{ba}{BA}=\frac{F'O}{F'O-BO}.$$

(3) $$\therefore \frac{i}{o}=\frac{f'}{f'-p}.$$

*Note.* Since light after reflexion is travelling at the same velocity in the same medium but in the reverse direction, the formula $\mu=\frac{V_1}{V_2}$ becomes $\frac{V_1}{-V_1}=-1$.

Therefore all the formulæ relating to refraction at a single surface can be converted into those of reflexion by giving $\mu$ the value $-1$.

Thus for

$$\mu = \frac{\sin\phi}{\sin\phi'}, \quad \frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r} \text{ and } F'' = -\mu F',$$

we have

$$-1 = \frac{\sin\phi}{\sin\phi'}, \quad \frac{-1}{q} \frac{-1}{p} = \frac{-2}{r} \text{ and } F'' = -(-1)\, F',$$

or $$\phi = -\phi', \quad \frac{1}{p} + \frac{1}{q} = \frac{2}{r}, \text{ and } F'' = F' \text{ in reflexion.}$$

**Thin Lenses.**

Let $P$ be a luminous point on the principal axis of the lens. (Fig. 8.)

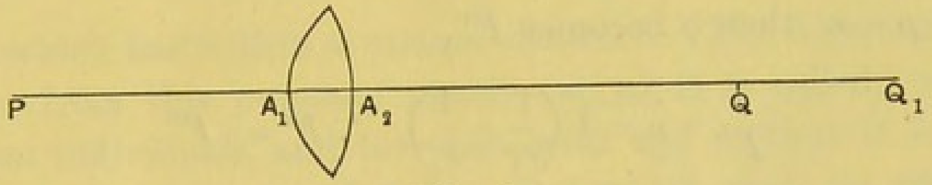


Fig. 8.

Let $Q_1$ be the image of $P$ due to refraction at the first surface $A_1$, and let $\mu$ or $\frac{V_r}{V_d}$ be the refractive index between air and glass.

Then

$$\frac{\mu}{q_1} - \frac{1}{p} = \frac{\mu - 1}{r_1} \quad (\alpha) \text{ where } p = PA_1,$$
$$q_1 = Q_1 A_1,$$
$$r_1 = \text{radius of first surface.}$$

The light from $P$ will traverse the substance of the lens in the direction towards $Q_1$. On emerging at the second surface, refraction takes place from glass to air, the index of refraction is therefore now $\frac{1}{\mu}\left(\text{or } \frac{V_d}{V_r}\right)$, and if the lens is so thin that its thickness is negligible we may regard $Q_1 A_2$ as equal to $Q_1 A_1$ or $q_1$.

To determine the distance $QA_2$ or $q$ we have

$$\frac{1}{\mu q} - \frac{1}{q_1} = \frac{\frac{1}{\mu} - 1}{r_2},$$

or on multiplying by $\mu$, $\quad \frac{1}{q} - \frac{\mu}{q_1} = \frac{1-\mu}{r_2}.$

On adding $(\alpha)$

$$\frac{\mu}{q_1} - \frac{1}{p} = \frac{\mu - 1}{r_1} \text{ we get } \frac{1}{q} - \frac{1}{p} = \overline{\mu - 1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \dots (\beta).$$

Let $q = \infty$ then $p$ becomes $F'$,

$$\frac{1}{F'} = -\overline{\mu - 1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Let $p = \infty$ then $q$ becomes $F''$,

$$\frac{1}{F''} = \overline{\mu - 1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \quad \therefore F' = -F''.$$

Therefore the formula $(\beta)$ may be written

$$\frac{1}{p} - \frac{1}{q} + \frac{1}{F''} = 0,$$

or

$$\frac{F'}{p} + \frac{F''}{q} = 1.$$

*Size of image.*

As before

$$\frac{i}{o} = \frac{F'}{F' - p} \text{ (Fig. 10)} = \frac{F'' - q}{F''}.$$

## Magnification.

Let us first consider the apparent size of an object $BA$ situated on the optic axis $BK$, where $K$ denotes the position of the nodal point of the eye. (Fig. 9.)

Then the apparent size of $BA$ is determined by $\tan\theta$, where $\theta$ denotes the positive visual angle $BKA$ subtended by $BA$ at $K$.

$$\tan\theta = \frac{BA}{KB}.$$

It is evident that $\tan\theta$ could be indefinitely increased by diminishing $KB$ indefinitely. In other words the apparent size of the object could be indefinitely increased by bringing the eye close to it. The eye however is incapable of distinct vision of an

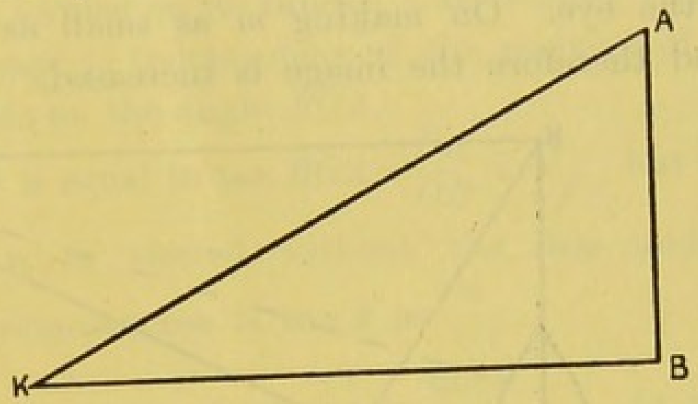

Fig. 9.

object which lies within a certain distance. The distance $BK$ of the eye from this *punctum proximum*, as it is called, varies in different individuals, and increases with age so that it is impossible to assign to it any definite value which shall be applicable to all cases. If $l$ denote the least distance for the individual eye considered, then the greatest value that $\tan\theta$ can actually have is by making $KB$ equal to $-l$ when $\tan\theta = \frac{BA}{-l}$. It is usual to give the value of 10 ins. (or 250 mm.) to $l$. In Fig. 9 $BK$ or $l$ is negative.

If now a convex glass be placed as in Fig. 10 with the object $BA$ within its first focal distance, and the nodal point of the observer's eye be at $K$, a virtual image $ba$ will be formed at a distance $Kb$ from the nodal point. Then if $\theta'$ is the visual angle subtended at $K$ by $ba$,

$$\tan\theta' = \frac{ba}{Kb} = \frac{ba}{-bK} = \frac{ba}{-(bO+OK)} = \frac{i}{-(q+m)},$$

where $m$ is the distance of the nodal point from the principal plane of the lens.

The magnifying power of the lens when placed in this position is

$$\frac{\tan\theta'}{\tan\theta} = \frac{i}{-(m+q)} \cdot \frac{-l}{o} = \frac{f''-q}{f''} \cdot \frac{l}{m+q};$$

when $m+q=l$,

$$\frac{\tan\theta'}{\tan\theta}=\frac{f''-q}{f''}=1-\frac{q}{f''}\text{ or }1+\frac{q}{f'}.$$

In this case the magnification increases the nearer the lens is brought to the eye. On making $m$ as small as possible $q$ is increased and therefore the image is increased.

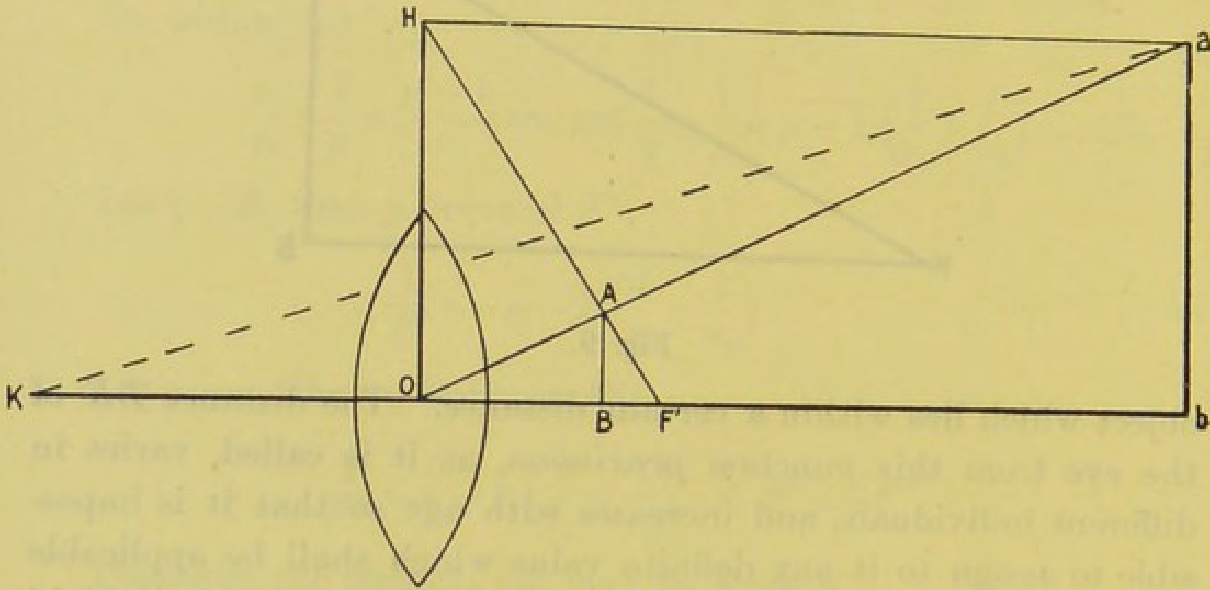


Fig. 10.

If the object $BA$ is placed in the first focal plane of the lens, the principal plane of which is $HOH'$ (Fig. 11), the incident cone of rays $HAO$ diverging from the point $A$ of the object will after traversing the lens proceed as a pencil of parallel rays in the

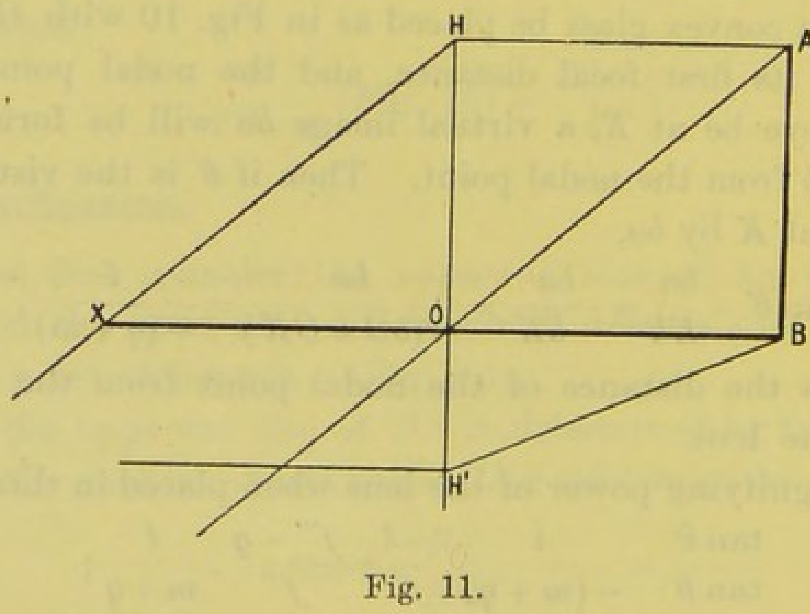


Fig. 11.

direction $AO$. At the same time all the incident rays diverging from $B$ will after refraction proceed as a pencil of parallel rays in the direction of the principal axis $BO$. If now an emmetropic eye be placed behind the lens on the axis $BO$, a complete image of $BA$ will be formed on its fundus. It is clear that the size of the retinal image is independent of the position of the eye, for its size depends on the angle $BOA$.

$\tan\theta'$ now is equal to $\tan BOA = \frac{BA}{OB} = \frac{o}{-f'}$, but the apparent size of $BA$ when viewed without the lens under the most favourable circumstances is $\tan\theta$ or $\frac{o}{-l}$.

Therefore the magnification or $\frac{\tan\theta'}{\tan\theta} = \frac{l}{f'}$ in these circumstances.

On comparing the magnifying power of a convex lens used in these two different ways we see that in the first case it is $1 + \frac{q}{f'}$, and in the second case it is $\frac{l}{f'}$ or $\frac{m+q}{f'}$. Therefore if $m$ is less than $f'$ the first method gives the higher magnification and *vice versâ*.

## Illustrations of the use of Donders' eye.

(1) If an eye be 2 mm. longer than normal, where will its *punctum remotum* be?

$$\frac{F'}{p} + \frac{F''}{q} = 1.$$

In this case $q = -22$ mm., $F' = +15$ mm., $F'' = -20$ mm.

$$\therefore \frac{15}{p} = 1 - \frac{-20}{-22} = \frac{1}{11}, \text{ so } p = 165 \text{ mm.}$$

(2) If the height of an object be 5 mm. situated at 165 mm. from the cornea, what will be the height of its retinal image in the above myopic eye?

$$\frac{i}{o} = \frac{F'}{F' - p} = \frac{15}{15 - 165} = \frac{15}{-150} = \frac{1}{-10},$$

$$\therefore i = \frac{5}{-10} = -\cdot 5 \text{ mm.}$$

The negative sign shows that the image is inverted.

As the transverse diameter of a foveal cone is ·002 mm. the height of the retinal image would cover about $\frac{\cdot 5}{\cdot 002}$ or 250 foveal cones.

(3) In a normal eye accommodating for the same object at the same distance (165 mm.), what will be the height of the retinal image?

$$\frac{F'}{p}+\frac{F''}{q}=1 \text{ or } \frac{-F''}{\mu p}+\frac{F''}{q}=1.$$

Here $q=-20$, but owing to the act of accommodation $F''$ has a new value,

$$F''\left(\frac{1}{-20}-\frac{3}{4\times 165}\right)=1, \quad \therefore F''=-\frac{220}{12}=-\frac{55}{3}\text{ mm.}$$

$$\frac{i}{o}=\frac{F''-q}{F''}=\frac{-\frac{55}{3}+20}{-\frac{55}{3}}=\frac{1}{-11},$$

$$\therefore i=-\frac{o}{11}=-\frac{5}{11}=-\cdot 45\text{ mm.}$$

Therefore for short distances myopic eyes see better than emmetropic eyes.

(4) In a normal eye what will be the minimum visual angle? The extreme minimum will be the angle ($a$) subtended by a

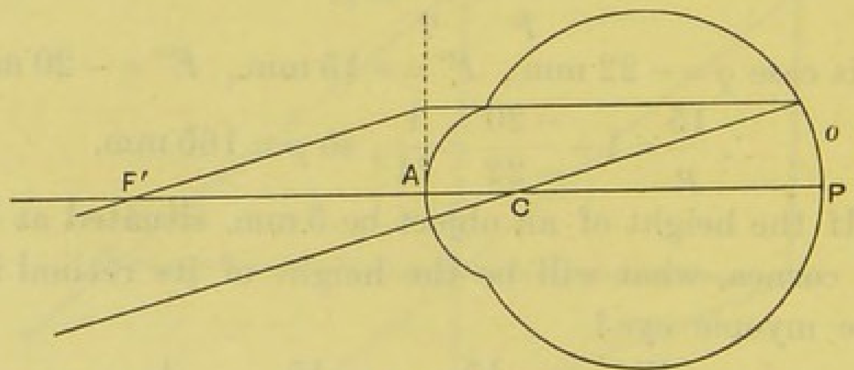


Fig. 12.

foveal cone at the nodal point ($C$) of the eye. (Fig. 12.) The distance $CP$ of the back of the eye from the nodal point is

$20-5$ or 15 mm. or $CP=F'$. If then $c$ denote the transverse diameter of a foveal cone, i.e. ·002 mm.,

$$\tan\frac{a}{2}=\frac{\frac{c}{2}}{CP}=\frac{\cdot 001}{15}=\cdot 0000\dot{6}.$$

$$\therefore a=27''\cdot 5.$$

However in order that two points may be distinguished as such by the eye, their retinal images must be separated by at least one unexcited retinal cone. Practically the minimum visual angle is $\theta=2a=55''$.

(On using the accurate value for $CP=K''F''$ (p. 28), $\theta$ is found to be $53''\cdot 235$.)

## Correction of Ametropia.

There is an advantage gained by placing the correcting lens in the first focal plane of the eye (i.e. about half an inch from the cornea), as when so placed the retinal image of a distant object formed by the combined refractive system is of precisely the same size as that which would be formed by an eye the retina of which is in its second focal plane. Hence the tests of visual acuity in ametropia are strictly comparable with those in emmetropia.

Let $p$ denote the distance of the first principal plane of the eye from the *punctum remotum*.

Ametropia is corrected, when such a lens is placed in the first focal plane of the eye, that incident parallel rays after traversing it proceed as if from the *punctum proximum*. If $f''$ denote the distance of the lens from its second principal focus, $f''=p-F'$. If $l$ denote the distance of the retina from the second principal focus of the eye, $l=-(q-F'')$.

But as $$(p-F')(q-F'')=F'F'',$$

$$f''(-l)=F'F''=(15)(-20)=-300,$$

$$\therefore f''=\frac{300}{l}=\frac{1000}{-x}$$ where $x$ is the strength of the glass in dioptres.

Thus if an eye require $+3D$ convex to correct its refraction and $l = -\cdot 3x = -\cdot 9$ mm. the eye is ·9 mm. too short. If an eye be 2 mm. too long it will require a lens of $\frac{-l}{\cdot 3}$ or $\frac{-2}{\cdot 3}$ i.e. $-6\cdot\dot{6}D$ to correct it.

(On using the accurate values of $F'$ and $F''$ (p. 28), $l = -\cdot 321x$.)

## Ophthalmoscope.

*Direct Method.* If the eye is emmetropic, no image of the fundus is formed except by the eye of the observer. Let $o$ be the height of the fundus seen (Fig. 12), and let $C$ be the nodal point of the eye. Then $\frac{o}{CP} = \tan\theta'$ where $\theta'$ is the positive visual angle which the fundus subtends at the eye of the observer. But $\tan\theta = \frac{o}{-l}$. Therefore the magnification or

$$\frac{\tan\theta'}{\tan\theta} = \frac{-l}{CP} = \frac{250}{15} = 16\tfrac{2}{3}$$

as

$$CP = F' = 15 \text{ mm.}$$

One can make a rough estimate of a patient's refraction by noting the highest glass with which his fundus can be distinctly seen. The number of this glass *minus* the observer's error of refraction gives the patient's correction.

*Indirect Method.* (1) If the eye is emmetropic and a lens of $2\frac{1}{2}$ ins. (or 63 mm.) focal length is used, a real inverted image

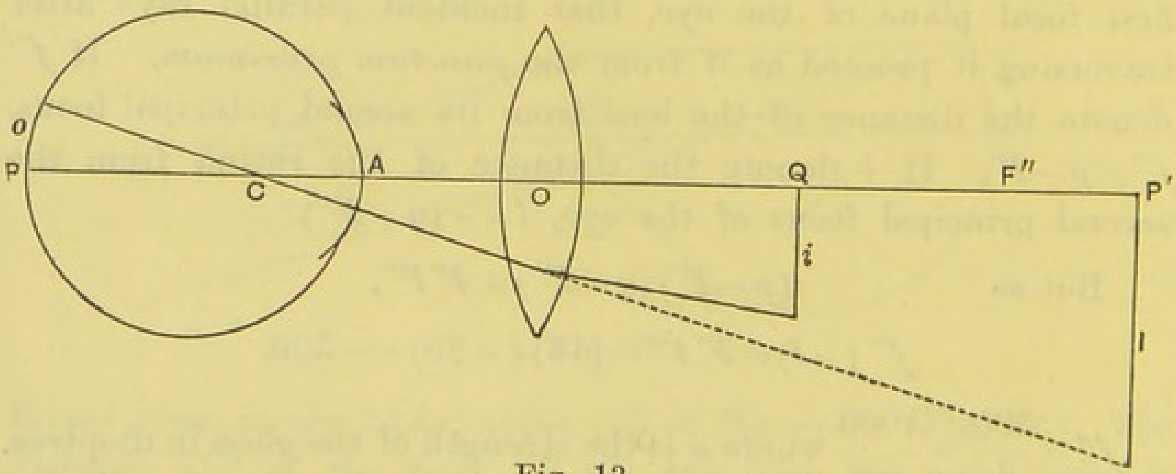


Fig. 13.

of the fundus is formed in the second focal plane of the lens, as the incident light consists of parallel rays. A figure like Fig. 13 can easily be drawn in which the image $i$ is formed in the plane at $F''$,

$$\therefore \frac{o}{CP} = \frac{i}{OF''}, \quad \therefore \frac{i}{o} = \frac{F''O}{PC} = \frac{-63}{15} = -4\tfrac{1}{5}.$$

Note that the negative sign shows that the image is inverted, and arises from the fact that $F''O$ is measured in the reverse direction to $PC$. If the inverted image is viewed at the *punctum proximum* of the observer, it is $4\frac{1}{5}$ times larger than the fundus would be when seen at this distance.

(2) If the eye is myopic, for instance 2 mm. longer than normal, an inverted image ($I$) of the fundus will be formed at $P'$ where $AP' = 165$ mm. (Fig. 13). On interposing the lens at $O$, a smaller image ($i$) will be formed at $Q$.

Then $$\frac{i}{I} = \frac{f'}{f'-p} = \frac{f'}{f'+OP'} = \frac{f'}{f'+AP'-AO} = \frac{63}{228-AO}.$$

Now as $p$ is negative the size of the image ($i$) increases with the distance $AO$, i.e. with the distance of the lens from the eye.

(3) If the eye is hypermetropic, the fundus being nearer the cornea than the focus, a virtual erect image ($I$) will be formed behind the eye at $P'$. Let $A$ and $O$ denote the situations of the cornea and lens. Then $p = P'A + AO$ is the distance of the lens from the virtual image. Of $I$ the lens will form a real inverted image ($i$) which is viewed by the observer. The reader can easily draw a diagram for himself,

$$\frac{i}{I} = \frac{f'}{f'-p} = \frac{63}{63-(PA+AO)}.$$

Now as $p$ is positive and greater than $f'$, the negative distance $f'-p$ increases as $AO$ increases, i.e. the size of the image diminishes as the distance of the lens from the eye increases.

### Retinoscopy.

The light $A$ (Fig. 14) is placed behind the patient's head, 150 cm. distant from the concave mirror of focal length 25 cm.

held by the observer. A real inverted image of the light will be formed at *a*, 30 cm. from the mirror, for

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}, \quad \therefore \frac{1}{q}=\frac{1}{25}-\frac{1}{150}=\frac{1}{30}.$$

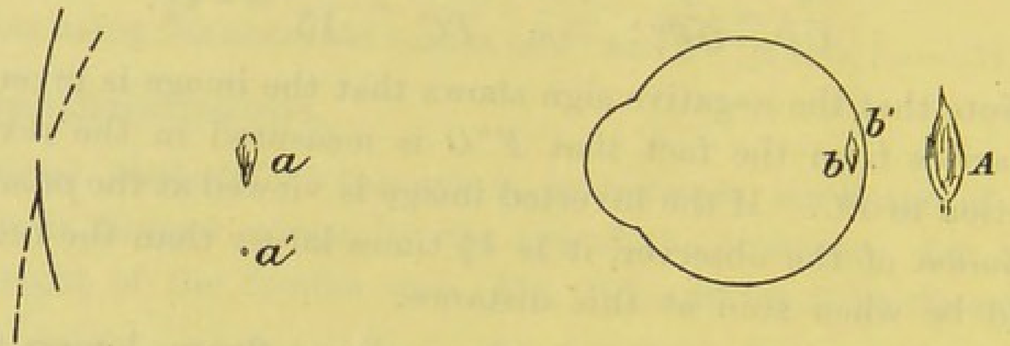


Fig. 14.

The light from this image (*a*) will illumine a small area of the patient's retina at *b*; in fact an inverted image of *a* will tend to be formed at *b*. Now on turning the mirror slightly downwards the image at *a* will move downwards to *a'*. Consequently the illuminated area on the patient's retina will move upwards to *b'*, whether the patient be hypermetropic or myopic. If the patient's distance from the observer be less than that of his far point, the observer will see a magnified erect image of the patch of light through the patient's pupil. Suppose the patient be 1 metre from the observer; then if the patient be hypermetropic, or even myopic but to a less extent than 1*D*, on turning the mirror, the observer will see the light move in the reverse direction across the patient's pupil.

If the myopia is greater than 1*D*, an inverted image of *b* will be formed at the patient's far point, which is situated somewhere between him and the observer. This inverted image will move in the reverse direction to *b*, and therefore in the same direction as that in which the mirror is turned.

Suppose a patient, whose eyes are fully dilated with atropine, keep his eyes fixed on the mirror, and, on turning the mirror from side to side, the red reflex is seen to move across his pupil in the same direction. Concave glasses of increasing strength are now placed in the trial frames before the patient's eyes until one is found which just reverses the movement of the light. Suppose with $-3{\cdot}5D$ the light moves with the mirror, but with $-4D$ it moves in the reverse direction from side to side. The patient's far point in the horizontal meridian when wearing $-3{\cdot}5D$ is situated just on his side of the observer. If the patient be a little more than 1 metre from the observer, we may expect that $-3{\cdot}5D-1D$ or $-4{\cdot}5D$ will correct his horizontal meridian for distance.

If the same result is observed on turning the mirror from above downwards we conclude that there is no astigmatism.

If however the light moves in the same direction as the mirror in the vertical meridian with $-5{\cdot}5D$, and is only reversed by a $-6D$ lens, we conclude that his vertical meridian requires $-6{\cdot}5D$ to correct it. The glasses for such a patient will be $-4{\cdot}5D$ sph. $-2D$ cyl. *ax—o*, and if confirmed by the subjective examination with test types, these will be ordered.

There are a few other optical formulæ which are occasionally required in ophthalmic practice, the proof of which demands more than the elementary knowledge of mathematics hitherto employed. In this place it will be sufficient to state without proof those connected with thin eccentric pencils, and with the circle of least confusion[1].

### Eccentric Pencils.

If $U$, $V_1$ and $V_2$ represent the distances of the eccentric portion of the spherical surface from the source of light, the first focal line and the second focal line, and if $\phi$ and $\phi'$ be the angles of incidence and refraction at the point of incidence,

$$(1) \qquad \frac{\mu\cos^2\phi'}{V_1}-\frac{\cos^2\phi}{U}=\frac{\mu}{V_2}-\frac{1}{U}=\frac{\sin(\phi-\phi')}{r\sin\phi'}.$$

This formula gives the relations between $U$, $V_1$ and $V_2$ for refraction at a single spherical surface.

1 The reader is referred to the author's manual of 'Optics' for the proof of the following formulæ, and for further information. Different conventions for signs were adopted there.

The corresponding formula for reflexion can be obtained from this by giving $\mu$ the value $-1$ and $\phi'$ the value $-\phi$ (p. 14).

(2) $$\frac{1}{U}+\frac{1}{V_1}=\frac{2}{r\cos\phi} \text{ and } \frac{1}{U}+\frac{1}{V_2}=\frac{2\cos\phi}{r}.$$

*Oblique Centric Pencils through a Lens.*

$$\frac{\cos^2\phi'}{V_1}-\frac{\cos^2\phi}{U}=\frac{1}{V_2}-\frac{1}{U}=\frac{\sin(\phi-\phi')}{\sin\phi'}\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Thus if a $+10D$ lens ($R_2=-R_1=108$ mm.) be inclined at 20° with the incident wave front, as $\mu=1{\cdot}54$, $V_1=-84{\cdot}88$ mm., $V_2=-96{\cdot}13$ mm., or the lens in this position before the eye acts in much the same way as

$$+10{\cdot}4D \text{ sph. } +1{\cdot}38 \text{ cyl. } ax\text{—}o.$$

Tilting strong spherical spectacles downwards before the eyes in this way is sometimes sufficient to correct the astigmatism in aphakic or highly myopic patients.

## Circle of least confusion.

When the cross-section of a reflected or refracted pencil is circular in shape, as is always the case when it is received by the circular pupil of the eye, the formulæ for the size and position of the circle of confusion formed by an optical system such as a lens become considerably simplified.

*Position.* Let $l$ denote the distance of the lens from the eye, and $x$ the distance of the lens from the circle of least confusion.

Then $$x=\frac{2V_1V_2-l(V_1+V_2)}{V_1+V_2-2l}.$$

In the case of refraction $l$ is negative, but in that of reflexion $l$ is positive, if the incident light is travelling from left to right.

*Size.* If $y$ is the radius of the equivalent pupil in the first principal plane of the eye, and if $R$ is the radius of the circle of least confusion,

$$R=\frac{y(V_2-V_1)}{V_1+V_2-2l}.$$

*Retinal confusion circle.* If $r$ is the radius of the retinal confusion circle,

$$\frac{r}{R} = \frac{F'}{F' - p},$$

where $F'$ is the first focal distance of the eye and $p$ is the distance of the eye from the original confusion circle. Therefore, when the lens (as in the case of spectacles) is placed in the first focal plane of the eye, $F' - p$ is the distance of the confusion circle from the lens, i.e. $-x$.

$$\therefore r = \frac{RF'}{-x} = -\frac{y(V_2 - V_1)F'}{2V_1V_2 - l(V_1 + V_2)}.$$

## The Eye.

The following are the mean values of the optical constants of the normal emmetropic eye (Fig. 15). The affixed signs are on

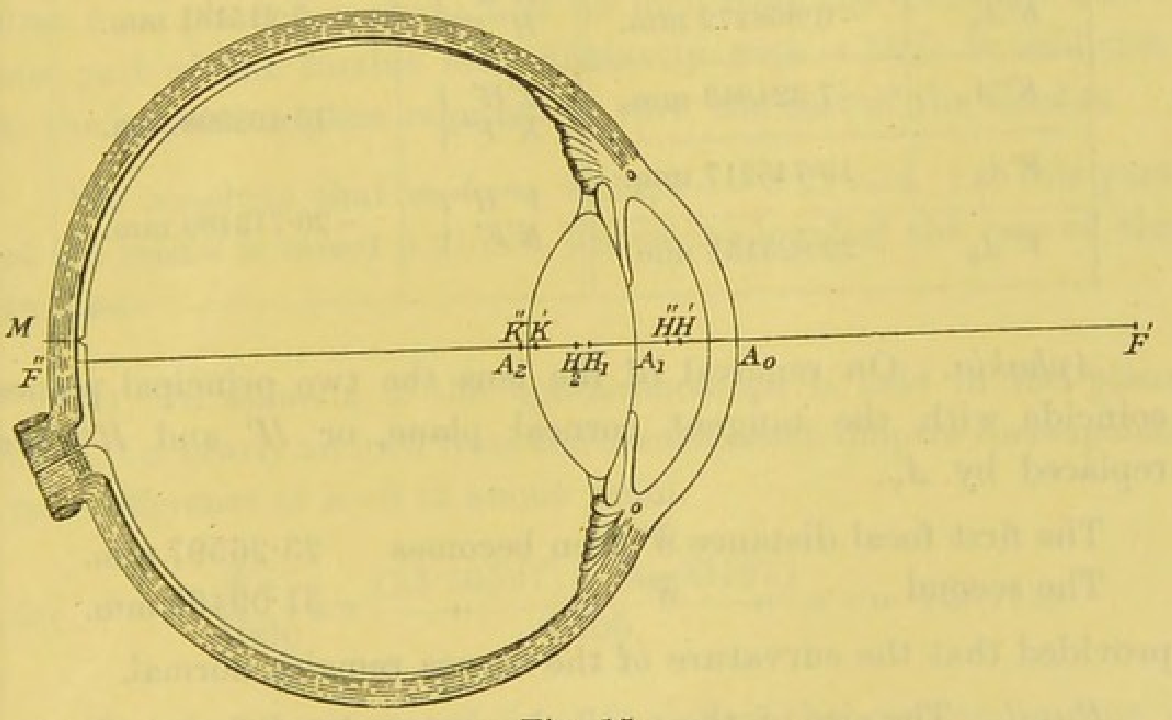


Fig. 15.

the assumption that the incident light is travelling from left to right. The figure should therefore be reversed.

The radius of curvature of

| | | |
|---|---|---|
| (1) | The anterior surface of the cornea | $r_0 = -7{\cdot}829$ mm. |
| (2) | The ,, ,, ,, lens | $r_1 = -10{\cdot}000$ mm. |
| (3) | The posterior ,, ,, lens | $r_2 = 6{\cdot}000$ mm. |

The distance of the lens from the cornea $A_0A_1 = 3{\cdot}6$ mm.
The thickness of the lens $A_1A_2 = 3{\cdot}6$ mm.

The indices of refraction are

For the aqueous and vitreous $\mu = 1{\cdot}3365$,
For the lens (total) $\mu' = 1{\cdot}4371$.

*Cardinal points of the Eye,*

where $H'H''$, $K'K''$, $F'F''$ are the first and second principal, nodal, and focal points respectively.

| | | | |
|---|---|---|---|
| $H'A_0$ | $-1{\cdot}753091$ mm. | $H'H''$, $K'K''$ | $\cdot356571$ mm. |
| $H''A_0$ | $-2{\cdot}109662$ mm. | | |
| $K'A_0$ | $-6{\cdot}968272$ mm. | $H'K'$, $H''K''$ | $5{\cdot}215181$ mm. |
| $K''A_0$ | $-7{\cdot}324843$ mm. | $F'H'$, $K''F''$ | $15{\cdot}498308$ mm. |
| $F'A_0$ | $13{\cdot}745217$ mm. | $F''H''$, $K'F'$ | $-20{\cdot}713489$ mm. |
| $F''A_0$ | $-22{\cdot}823151$ mm. | | |

*Aphakia.* On removal of the lens the two principal planes coincide with the tangent corneal plane, or $H'$ and $H''$ are replaced by $A_0$.

The first focal distance $\mathfrak{F}'$ then becomes $23{\cdot}26597$ mm.
The second ,, ,, $\mathfrak{F}''$ ,, ,, $-31{\cdot}09497$ mm.

provided that the curvature of the cornea remains normal.

*Pupil.* The size of the pupil changes under different circumstances, its radius in health varying from about $1{\cdot}2$ mm. to $2{\cdot}8$ mm. Perhaps its average value may be taken as $1{\cdot}6$ mm.

*Centre of Motility.* Slight movements of the eyeball may be considered as movements of rotation about a fixed centre, which may usually be taken to be $13{\cdot}4$ mm. behind the cornea. Its position varies in different eyes.

### Illustrations of the use of these constants.

Note that $F' = F'H'$ and $p = PH'$, i.e. the distance of the first principal point from $F'$ and $P$ respectively. Similarly $F'' = F''H''$ and $q = QH''$, i.e. the distance of the second principal point from $F''$ and $Q$ respectively.

(1) We have seen (p. 21) that $\frac{1000}{x} = \frac{F'F''}{l}$.

On giving the correct values to $F'$ and $F''$ we find that

$$l = \frac{F'F''}{1000}x = \frac{(15{\cdot}498\ldots) \times (-20{\cdot}713\ldots)}{1000}x = -{\cdot}321\ldots x.$$

On examining an eye with the ophthalmoscope held in the first focal plane (i.e. about half an inch from the cornea) we find one part of the fundus seen distinctly with $+10D$ in addition to the correcting glass required to view the rest of the fundus.

We conclude that as $l = -{\cdot}321x$, $l = -3{\cdot}21$ mm., i.e. this part of the retina is raised 3·21 mm. above the level of the rest of the fundus.

(2) In aphakia if the ophthalmoscope is held in the plane of $\mathfrak{F}'$ (i.e. nearly an inch from the cornea), each dioptre corresponds to a difference of level of about $\frac{3}{4}$ mm.

for $$l = \frac{\mathfrak{F}'\mathfrak{F}''}{1000}x = \frac{(23{\cdot}26597)(-31{\cdot}09497)}{1000}x = -{\cdot}723\ldots x.$$

(3) As above stated (p. 21), when the correcting lens is placed in the first focal plane of the eye, the retinal image of a distant object is precisely the same size as that of an eye the retina of which is in its second focal plane. In other words the position of the second nodal point is altered so that the distance of the retina from it is equal to the first focal distance.

Now in a normal eye (Fig. 12) the tangent of the visual angle

$\theta = \frac{i}{CP} = \frac{i}{F'}$, in an aphakic eye with its correcting lens placed in its first focal plane the tangent of the visual angle $= \frac{i'}{\mathfrak{F}'}$,

$$\therefore \frac{i'}{i} = \frac{\mathfrak{F}'}{F'} = \frac{23{\cdot}6597}{15{\cdot}4983} = 1{\cdot}526\ldots$$

If the correcting lens is placed in the usual position about half an inch in front of the cornea, $\frac{i'}{i} = 1{\cdot}337\ldots$

This accounts for the extreme discomfort that patients suffer from, if one eye is aphakic and corrected, while the other eye is normal, for the aphakic eye sees an object about a third larger than the other eye.

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