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TO DETERMINE THE DEGREE OF POLARIZATION IN THE CASE OF A RAY OF COMMON LIGHT FALLING OBLIQUELY ON AND BEING REFLECTED OR REFRACTED BY A BUNDLE OF PARALLEL PLATES.

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REGARDING a ray of common light as equivalent to two polarized rays of equal intensity, whose planes of polarization are in and perpendicular to the plane of incidence, then a ray, of intensity 1, is equivalent to two, each equal to $\frac{1}{2}$.

The ray of intensity $\frac{1}{2}$ which is polarized *in the plane of incidence* will give rise to a reflected ray of intensity $\frac{1}{2}v^2$, and a refracted ray of intensity $\frac{1}{2}(1-v^2)$, where v denotes the ratio of the amplitudes of the vibrations of the ether for the reflected and the incident ray.

If ϕ and ϕ_1 be the angles of incidence and refraction on the first surface, then

$$v=\frac{\sin(\phi-\phi_1)}{\sin(\phi+\phi_1)}.$$

The ray of intensity $\frac{1}{2}$, which is polarized in a plane *perpendicular to the plane of incidence,* will give rise to a reflected ray of intensity $\frac{1}{2}w^2$, and a refracted ray of intensity $\frac{1}{2}(1-w^2)$ at the first surface, where

$$w=\frac{\tan(\phi-\phi_1)}{\tan(\phi+\phi_1)}.$$

Now consider first only the light polarized in the plane of incidence.

The ray refracted into the first plate is divided up at its second surface into an internally reflected ray of intensity $\frac{1}{2}(1-v^2)v^2$, and a refracted ray of intensity $\frac{1}{2}(1-v^2)^2$.

A similar division will take place at each surface with every successive internally reflected ray.

The intensity of the light passing out of the plate on the side on which it entered will be

$$\tfrac{1}{2}(1-v^2)^2\{v^2+v^6+v^{10}+\&c.\},$$

and therefore the whole intensity of the reflected beam is

$$\tfrac{1}{2}v^2+\tfrac{1}{2}(1-v^2)^2(v^2+v^6+v^{10}+\&c.)$$

$$=\frac{1}{2}v^2+\frac{1}{2}(1-v^2)^2\frac{v^2}{1-v^4}$$

$$=\frac{1}{2}v^2\left\{1+\frac{1-v^2}{1+v^2}\right\}=\frac{1}{2}\cdot\frac{2v^2}{1+v^2}.$$

Hence the intensity of the beam which is refracted through the plate is

$$\frac{1}{2}\left(1-\frac{2v^2}{1+v^2}\right)=\frac{1}{2}\cdot\frac{1-v^2}{1+v^2},$$

no light being supposed to be absorbed by the plate.

Let the part which is reflected be represented by $\frac{1}{2}k^2$; then the portion which passes through will be represented by $\frac{1}{2}(1-k^2)$.

Now consider the action of the second *plate* on this portion, which has been refracted by the first *plate.* The successive reflections and refractions by the two *plates* will be of the same character as the successive reflections and refractions by the two *surfaces* of the first plate; also the intensities of the portions reflected and refracted by the two *plates* will bear the same relation to k^2, that the intensities of the portions reflected and refracted from the two *surfaces* bear to v^2. Therefore as the intensity of the beam refracted through the two surfaces was changed from $\frac{1}{2}$ to $\frac{1}{2}\cdot\frac{1-v^2}{1+v^2}$, so the intensity of the beam refracted through two plates will be changed from $\frac{1}{2}$ to $\frac{1}{2}\cdot\frac{1-k^2}{1+k^2}$, *i. e.* to $\frac{1}{2}\cdot\frac{1-v^2}{1+3v^2}$; and the intensity of the beam reflected from two *plates* will be

$$\frac{1}{2}\left(1-\frac{1-k^2}{1+k^2}\right)=\frac{1}{2}\cdot\frac{2k^2}{1+k^2}=\frac{1}{2}\cdot\frac{4v^2}{1+3v^2}.$$

Now let

$$\frac{4v^2}{1+3v^2}=k_1^2,$$

then

$$\frac{1-v^2}{1+3v^2}=1-k_1^2;$$

and the intensity of the beam which passes through two plates will be $\frac{1}{2}(1-k_1^2)$.

Now consider the four plates of glass as two *bundles*, each consisting of two *plates*. Then it will be seen that the intensities of the beams reflected and refracted by the two *bundles* will bear the same relation to k_1^2 that the intensities of the beams reflected and refracted by the two plates bear to k^2. Hence the intensity of the light reflected by the two *bundles* is

$$\frac{1}{2}\cdot\frac{2k_1^2}{1+k_1^2}=\frac{1}{2}\cdot\frac{8v^2}{1+7v^2};$$

and the intensity of the light refracted through the two bundles is

$$\frac{1}{2}\cdot\frac{1-k_1^2}{1+k_1^2}=\frac{1}{2}\cdot\frac{1-v^2}{1+7v^2}.$$

Now let us consider the ray of intensity $\frac{1}{2}$ which is polarized in a plane *perpendicular to the plane of incidence.* It is clear that the intensity of the beam arising at any surface from this ray will bear the same relation to w^2 that the intensity of the corresponding beam arising at the same surface from the other ray bears to v^2; hence the intensity of the beam from this ray, which is reflected from four plates, $=\frac{1}{2}\cdot\frac{8w^2}{1+7w^2}$, and the intensity of the refracted beam through four plates $=\frac{1}{2}\cdot\frac{1-w^2}{1+7w^2}$. Hence the whole intensity of the reflected beam is

$$\left(\frac{1}{2}\cdot\frac{8v^2}{1+7v^2}+\frac{1}{2}\cdot\frac{8w^2}{1+7w^2}\right),$$

and the whole intensity of the refracted beam is

$$\frac{1}{2}\cdot\frac{1-v^2}{1+7v^2}+\frac{1}{2}\cdot\frac{1-w^2}{1+7w^2}.$$

The intensity of the polarized light in each beam being

$$\left(\frac{1}{2}\cdot\frac{1-w^2}{1+7w^2}-\frac{1}{2}\cdot\frac{1-v^2}{1+7v^2}\right),$$

the intensity of natural light in the reflected beam is $\frac{8w^2}{1+7w^2}$, and in the refracted beam is $\frac{1-v^2}{1+7v^2}$.

Hence the degree of polarization in the refracted beam is

$$\frac{\frac{1-w^2}{1+7w^2}-\frac{1-v^2}{1+7v^2}}{\frac{1-w^2}{1+7w^2}+\frac{1-v^2}{1+7v^2}}=\frac{8(v^2-w^2)}{(1+7v^2)(1-w^2)+(1+7w^2)(1-v^2)}.$$

Let $\sin(\phi-\phi_1)=y$, and $\sin(\phi+\phi_1)=x$; then $v=\frac{y}{x}$, and

$$1-w^2=1-\frac{\tan^2(\phi-\phi_1)}{\tan^2(\phi+\phi_1)}=1-v^2.\left(\frac{1-x^2}{1-y^2}\right)=\frac{1-v^2}{1-y^2};$$

also

$$v^2-w^2=v^2-v^2\left(\frac{1-x^2}{1-y^2}\right)=\frac{v^2x^2-v^2y^2}{1-y^2}=\frac{y^2(1-v^2)}{1-y^2},$$

and

$$1+7w^2=\frac{1+7v^2-8y^2}{1-y^2}.$$

Dividing out by the common fraction $\frac{1-v^2}{1-y^2}$ in numerator and denominator, and calling p and n the intensities of polarized and natural light, we get

$$\frac{p}{p+n}=\frac{8y^2}{(1+7v^2)+(1+7v^2-8y^2)}=\frac{4y^2}{1+7v^2-4y^2}.$$

From the above relations it appears that

$$1-v^2=(1-w^2).\cos^2(\phi-\phi_1);$$

also

$$v^2-w^2=(1-w^2)\sin^2(\phi-\phi_1)=(1-v^2)\tan^2(\phi-\phi_1),$$

and

$$1+v^2-2y^2=(1+w^2).\cos^2(\phi-\phi_1),$$

$$1+3v^2-4y^2=(1+3w^2).\cos^2(\phi-\phi_1),$$

$$1+7v^2-8y^2=(1+7w^2)\cos^2(\phi-\phi_1).$$

and generally

$$1+(2m-1)v^2-2my^2=\{1+(2m-1)w^2\}\cdot\cos^2(\phi-\phi_1).$$

Hence

$$\frac{p}{n+2p}=\frac{4\sin^2(\phi-\phi_1)}{1+7v^2}=\frac{4}{\frac{1}{\sin^2(\phi-\phi_1)}+\frac{7}{\sin^2(\phi+\phi_1)}},$$

or

$$\frac{p}{n}=\frac{4y^2}{1+7v^2-8y^2}=\frac{4\tan^2(\phi-\phi_1)}{1+7w^2}$$

$$=\frac{4}{\frac{1}{\tan^2(\phi-\phi_1)}+\frac{7}{\tan^2(\phi+\phi_1)}}.$$

It appears, then, for a single refraction at a single surface, that

$$\frac{\frac{1}{2}(v^2-w^2)}{\frac{1}{2}(1-v^2)+\frac{1}{2}(1-w^2)}=\frac{\sin^2(\phi-\phi_1)}{1+\cos^2(\phi-\phi_1)},$$

or the degree of polarized light in the refracted ray depends only on the deviation.

Now

$$\sin\phi=\mu\sin\phi_1;$$

and from this equation we must determine the values of

$$\sin^2(\phi-\phi_1) \text{ and } \frac{\sin^2(\phi-\phi_1)}{\sin^2(\phi+\phi_1)}$$

for different values of ϕ, from which we shall have

$$\frac{p}{p+n}=\frac{4\sin^2(\phi-\phi_1)}{1+7\frac{\sin^2(\phi-\phi_1)}{\sin^2(\phi+\phi_1)}-4\sin^2(\phi-\phi_1)};$$

or

$$\frac{n+p}{p}=\frac{\frac{1}{\sin^2(\phi-\phi_1)}+\frac{7}{\sin^2(\phi+\phi_1)}-4}{4}.$$

Now suppose that before falling on the plate the light is polarized in the plane of incidence, and that p represents the part polarized and n the natural light, then $\frac{p+\frac{n}{2}}{n+p}$ is the intensity of the first ray, and $\frac{n}{2(n+p)}$ the intensity of the second ray; hence the degree of polarization in the refracted beam is

$$\frac{p_1}{p_1+n_1}=\frac{\left(p+\frac{n}{2}\right)\frac{1-v^2}{1+7v^2}-\frac{n}{2}\cdot\frac{1-w^2}{1+7w^2}}{\left(p+\frac{n}{2}\right)\frac{1-v^2}{1+7v^2}+\frac{n}{2}\cdot\frac{1-w^2}{1+7w^2}}$$

$$=\frac{p\frac{1-v^2}{1+7v^2}-\frac{n}{2}\cdot\frac{8(v^2-w^2)}{(1+7v^2)(1+7w^2)}}{p\frac{1-v^2}{1+7v^2}+\frac{n}{2}\left(\frac{1-w^2}{1+7w^2}+\frac{1-v^2}{1+7v^2}\right)}$$

$$=\frac{p(1+7w^2)-\frac{n}{2}\cdot 8\tan^2(\phi-\phi_1)}{p(1+7w^2)+\frac{n}{2}\cdot\frac{1+7v^2+1+7v^2-8\sin^2(\phi-\phi_1)}{\cos^2(\phi-\phi_1)}}$$

$$=\frac{p(1+7w^2)-\frac{n}{2}\cdot 8\tan^2(\phi-\phi_1)}{p(1+7w^2)+\frac{n}{1}\cdot(1+7w^2)+\frac{n}{2}\cdot 8\tan^2(\phi-\phi_1)}.$$

Expressing the result in terms of v^2, we get

$$\frac{p_1}{p_1+n_1}=\frac{p(1+7v^2)-\left(p+\frac{n}{2}\right)8\sin^2(\phi-\phi_1)}{(p+n)(1+7v^2)-\left(p+\frac{n}{2}\right)8\sin^2(\phi-\phi_1)},$$

or

$$\frac{p_1}{n_1}=\frac{p-\left(p+\frac{n}{2}\right)\cdot\frac{8\sin^2(\phi-\phi_1)}{1+7v^2}}{n}.$$

When the light is completely depolarized by the four plates, then

$$\frac{p}{p+\frac{n}{2}}=\frac{8\sin^2(\phi-\phi_1)}{1+7v^2}=\frac{8\sin^2(\phi-\phi_1)}{1+7\frac{\sin^2(\phi-\phi_1)}{\sin^2(\phi+\phi_1)}}$$

$$=\frac{8}{\frac{1}{\sin^2(\phi-\phi_1)}+\frac{7}{\sin^2(\phi+\phi_1)}};$$

hence

$$\frac{p}{n+p}=\frac{4}{\frac{1}{\sin^2(\phi-\phi_1)}+\frac{7}{\sin^2(\phi+\phi_1)}-4},$$

which is the same as the value obtained for the relation of polarized to natural light when a ray of common light is refracted through the plates.

Assuming a value of $\mu=1{\cdot}5$ for crown glass, we may form a Table of the degrees of polarization for different angles of incidence.

Table for $\mu=1{\cdot}5$ (crown-glass). Angle of complete polarization $=56^\circ\ 18'\ 36''$, $\log\mu=\cdot17609$, $\log 7=\cdot84510$.

ϕ.	10°.	15°.	20°.	22° 30′.	25°.	30°.	35°.	40°.
$\frac{p}{n+p}=$	·0107	·02443	·04430	·05683	·07111	·10498	·14745	·19845
For $\mu=1{\cdot}513$ $\frac{p}{n+p}=$	·01103	·02504	·04540	(·05813)	·07267	·1076	·1509	·2029
Probable values for $\mu=1{\cdot}54$. $\frac{p}{n+p}=$	·0116	·0263	·0476	....	·076	·113	·158	·212

ϕ.	45°.	50°.	55°.	56°18′36″	60°.	65°.	70°.	72°.
$\frac{p}{n+p}=$	·25786	·32383	·39231	·40984	·4560	·50530	·5309	·53305
For $\mu=1{\cdot}513$ $\frac{p}{n+p}=$	·2635	·3305	·3999	(For 56° 32′) ·4204	·4642	·5139	·5394	(·5416)
Probable values for $\mu=1{\cdot}54$. $\frac{p}{n+p}=$	·275	·344	·415	....	·481	·531	·556	

From the Table, for the values of $\frac{p}{n+p}$ we see that for four plates the proportion of polarized light in the beam which passes through the plate still goes on increasing when the incidence becomes greater than the angle for complete polarization, and that at about 72° it attains its greatest value, when there is about 53 per cent. of the light polarized.

I have also formed a Table for $\mu=1{\cdot}513$, of which the results are given; and from these we can derive close approximate values for the proportions of polarized light for values of μ not differing much from these values.

The probable values for $\mu=1{\cdot}54$ are given in the Table.

If we employ the four plates as a depolarizer to determine the proportion of polarized light in the incident beam by reducing the light to its ordinary unpolarized state, then (as before explained) the proportion of polarized light in the incident beam will be given, for any angle of complete depolarization, by the value of $\frac{p}{n+p}$ for that angle.

That the reasoning from the analogy between plates and refractive surfaces is correct will readily appear; and the law of intensities may at once be deduced.

Consider only the ray of intensity $\frac{1}{2}$ which is polarized *in the plane* of incidence. $\frac{1}{2}k^2$ is reflected by the first plate, and $\frac{1}{2}(1-k^2)$ is refracted through it. Of this latter portion, which falls on a third refracting surface, $(1-v^2)$ is refracted and v^2 is reflected; so that $\frac{1}{2}(1-k^2)(1-v^2)$ is the intensity of the refracted portion, and $\frac{1}{2}v^2(1-k^2)$ of the reflected portion. In these are included all the rays arising from the successive internal reflections *inside* the plate which take place before reflection by the third surface.

The reflected portion $\frac{1}{2}v^2(1-k^2)$ falls upon the first plate; and the portion of it refracted by the plate is $\frac{1}{2}v^2(1-k^2)^2$, the portion reflected being $\frac{1}{2}v^2k^2(1-k^2)$. These portions include all the rays, however internally reflected, which have only been once reflected by the third surface. The reflected portion $\frac{1}{2}v^2k^2(1-k^2)$ falls on the third surface; and

$\frac{1}{2}v^4k^2(1-k^2)$ is reflected back to the plate,

while

$\frac{1}{2}v^2k^2(1-k^2)(1-v^2)$ is refracted through the surface;

similar reflections and refractions take place; and the whole intensities of the refracted beams will be

$$\tfrac{1}{2}(1-k^2)(1-v^2\{1+v^2k^2+v^4k^4+\&c.\}$$

$$=\tfrac{1}{2}(1-k^2)(1-v^2)\left(\frac{1}{1-v^2k^2}\right);$$

of the reflected beams will be

$$\tfrac{1}{2}v^2+\tfrac{1}{2}(1-k^2)\,.\,\{v^2+v^4k^2+v^6k^4+\&c.\}$$

$$=\frac{1}{2}v^2\left\{1+(1-k^2)^2\left(\frac{1}{1-v^2k^2}\right)\right\}.$$

Of the portion $\frac{1}{2}(1-k^2)$ which falls on a second refracting plate, the intensity of the reflected portion will be

$$\tfrac{1}{2}k^2(1-k^2),$$

and of the refracted portion

$$\tfrac{1}{2}(1-k^2)^2.$$

In these are included all the rays, however internally reflected, which have only been once reflected at the outside of the first surface of the second plate.

The portion again reflected by the first plate is

$$\tfrac{1}{2}k^4(1-k^2),$$

and the portion refracted is

$$\tfrac{1}{2}k^2(1-k^2)^2.$$

This latter portion forms a part of the total beam reflected by the two plates.

Summing up the reflected and refracted portions, we get the intensity of the reflected beam

$$=\tfrac{1}{2}k^2+\tfrac{1}{2}(1-k^2)^2.\{1+k^4+k^8+\&c.\}$$

$$=\frac{1}{2}k^2\left\{1+\frac{1-k^2}{1+k^2}\right\}=\frac{1}{2}\frac{2k^2}{1+k^2}=\frac{2v^2}{1+3v^2}.$$

The intensity of the refracted beam is

$$\frac{1}{2}(1-k^2)^2.\{1+k^4+k^8+\&c.\}=\frac{1}{2}\cdot\frac{1-k^2}{1+k^2}=\frac{1}{2}\cdot\frac{1-v^2}{1+3v^2}.$$

We may in the same way determine the intensities of beams reflected from successive plates; and it will readily be seen that the intensity of the reflected beam is

$$\frac{1}{2}\cdot\frac{2mv^2}{1+(2m-1)v^2},$$

and the intensity of the refracted beam

$$=\frac{1}{2}\cdot\frac{1-v^2}{1+(2m-1)v^2},$$

where m is the number of plates.

If we consider the beam polarized in the plane at right angles to the plane of incidence, the reflected portion will be

$$\frac{1}{2}\cdot\frac{2mw^2}{1+(2m-1)w^2},$$

and the refracted portion

$$=\frac{1}{2}\cdot\frac{1-w^2}{1+(2m-1)w^2}.$$

The intensity of polarized light in each beam is

$$\frac{1}{2}\left\{\frac{1-w^2}{1+(2m-1)w^2}-\frac{1-v^2}{1+(2m-1)v^2}\right\},$$

the intensity of natural light reflected being $\frac{2mw^2}{1+(2m-1)w^2}$, and refracted being $\frac{1-v^2}{1+(2m-1)v^2}$.

Let p and n represent the proportions of polarized and natural light in the refracted beam, then

$$\frac{p}{n+p}=\frac{\dfrac{1-w^2}{1+(2m-1)w^2}-\dfrac{1-v^2}{1+(2m-1)v^2}}{\dfrac{1-w^2}{1+(2m-1)w^2}+\dfrac{1-v^2}{1+(2m-1)v^2}}$$

$$=\frac{2m(v^2-w^2)}{\{1+(2m-1)v^2\}(1-w^2)+(1-v^2)\{1+(2m-1)w^2\}}$$

$$=\frac{2m\sin^2(\phi-\phi_1)}{1+(2m-1)v^2+\cos^2(\phi-\phi_1)\{1+(2m-1)w^2\}}$$

$$=\frac{m\sin^2(\phi-\phi_1)}{1+(2m-1)v^2-m\sin^2(\phi-\phi_1)}$$

or

$$=\frac{m\tan^2(\phi-\phi_1)}{1+(2m-1)w^2+m\tan^2(\phi-\phi_1)}.$$

These expressions may conveniently be put in the form

$$\frac{p}{n+p}=\frac{m}{\dfrac{1}{\sin^2(\phi-\phi_1)}+\dfrac{(2m-1)}{\sin^2(\phi+\phi_1)}-m},$$

or

$$\frac{n+p}{p}=\frac{\dfrac{1}{\sin^2(\phi-\phi_1)}+\dfrac{2m-1}{\sin^2(\phi+\phi_1)}-m}{m}.$$

For a given value of m (as, for instance, when there are four plates) this expression for the degree of polarization may be readily calculated; and, as we have seen, $\frac{p}{n+p}$ expresses the proportion of polarized light in the incident beam when the light is completely depolarized by the plates.