MATHEMATICAL PSYCHOLOGY

The Mathematical Psychology of Gratry and Boole

TRANSLATED FROM THE LANGUAGE OF THE HIGHER CALCULUS INTO THAT OF ELEMENTARY GEOMETRY

BY

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TO HENRY MAUDSLEY, M.D.

DEAR DR. MAUDSLEY,—

You have often asked me to explain, for students unaquainted with the Infinitesimal Calculus, certain doctrines expressed in terms of that Calculus by P. Gratry and my late husband. That you permit me to dedicate my attempt to you will, at least, be a guarantee that the main ideas of mathematical psychology are based, not on mystic dreams, but on scientific induction.

I am glad of this opportunity of expressing to you my gratitude, both for your published works, and for much personal kindness to myself.

I am,

Dear Dr. Maudsley,

Yours very truly,

MARY EVEREST BOOLE.

PREFACE

It is usual, in a preface, to express the author's thanks to those who have been helpful in the evolution of the book. Were I to attempt to do this, I should have to mention the names of most of the persons with whom I ever conversed, and most of the authors whose works I have read. The existence of my little volume in its present form is due to Mr. Percy Furnival, F.R.C.S., of St. Bartholomew's; who, on learning the nature of the work which I have long had in hand, insisted that information which may possibly prove to be of some use to members of his profession, ought to be made accessible to them without further delay. I have to thank him for his kind help in the final arrangement of my material.

M. E. B.

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CHAPTER I

INTRODUCTORY

HOW is it possible that questions of general Psychology should be mathematically investigated?

And if such a thing be possible, what class of readers does the matter concern?

In answer to the latter question, it is sufficient to say that I have been invited by members of the medical and educational professions to state, in the ordinary language common to all educated persons, certain conceptions about the nature of the process of original thought which have hitherto been expressed in a terminology intelligible only to mathematical specialists. Many to whom these conceptions have been explained *vivâ voce* seem to feel that it would be well if they were made accessible to all who are concerned either in fostering the development of original faculty, in distinguishing between genius and the mere inspirations of self-conceit, or in preventing the growth of hallucinations:—that is to say, to young persons who either have or suppose themselves to have original ideas; to parents and teachers; and to those within the sphere of whose duties it may come either to minister to the special needs of genius, or to deal with that lurid shadow of genius which we call insanity.

But how can there be such a science as Mathematical Psychology? Mathematics is understood to be the science of number and space. It is true, indeed, that all large masses of data must be reduced statistically before the information which they are capable of affording can be made available as a basis for accurate reasoning; and statistics, of course, belong to the domain of mathematics. In that sense the adjective "mathematical" might conceivably be used to qualify the name of any science; but, in any other sense than that, how can mathematics be concerned in a treatise on human psychology?

This initial question can best be answered by a parable. Suppose that by some means the inhabitants of Mars became possessed of a ladder of human construction—an ordinary ladder such as is used in our orchards; and suppose they were informed that this instrument is one in common use among the intelligent inhabitants of the Earth for the purpose of reaching certain articles of food which would otherwise be unattainable. The first impression of the Martians would probably be that the instrument had some direct relation to the articles desired by men; either that its shape is adapted to catch the fruits, or that the ladder has some property of attracting the fruits and causing them to come of themselves towards the person possessed of it. On that basis they might possibly construct a pseudo-science ot earth-botany; an elaborate scheme of hypothetical assumptions about the nature and growth of earth fruits. It would probably not occur to them at first that the ladder could reveal anything about the nature of man (except, of course, that they would infer a tendency in man to desire fruits growing out of his reach).

But suppose it came to their knowledge that the ladder has no magnetic attraction for fruits, and that its shape has no relation to any property of fruit except the one property of growing out of human reach; their hypothetical botany would then be seen to be chimerical, and they would be driven on to a different line of speculation. If the ladder has no relation to the things to be reached, then it must be constructed in some very close relation to the normal action of the creatures who want to reach the things.

Starting anew on this sounder basis, it might be possible for them to arrive at some knowledge of the anatomy of the human climbing-limbs. Then suppose that beings on the other side of Mars acquired knowledge of our anatomy by some other means,—suppose a book of anatomical plates or the legs of a human body were conveyed to them,—we can easily see how, if the two sets of Martians came together and compared notes, the set who had investigated the ladder might supplement the more direct knowledge acquired by the other set and add to it elements of considerable value.

The science of mathematics is an intellectual ladder. The rather usual assumption that it is "a science of number, size, and form" is as erroneous as would be the assumption that a ladder is an instrument specially adapted to draw cherries from their natural level. Before we can truly grasp the conceptions of Mathematical Psychology, we must begin by realizing a little how far mathematical science is from being harmoniously adjusted to the laws of the subjects to which it is supposed to be necessarily attached. Mathematical processes may almost be said to have no relation to any property of the laws of number or of natural form, except, indeed, the one property of being too difficult for the intellect of man to grasp unaided. When we are clear on this point, when we have come to a true understanding of what mathematics is not, we shall be in a better position for entering on the question :—What it is.

Mankind have invented names for each of the numbers in order, and have agreed on certain written signs to represent those names : 1, 2, 3, etc. Why, on arriving at ten, do we suddenly leave off writing fresh signs and begin to use combinations of the former ones ? No property of numbers is more absolutely certain than that of following in an unbroken and homogeneous series without jolts or breaks. Why do we make a break in representing them ? Because man is a creature who finds it more convenient to use classified combinations of a few signs than an indefinite number of separate signs.

But man easily uses twenty-six letters in his alphabet; why, then, does he cut short his numerical alphabet at less than half that length? Because, though our memory easily retains twenty-six or more separate signs of similar kind, few persons can visualize, or form a direct conception of, such a number as twenty-six. The largest number that an individual can clearly picture to himself or directly conceive varies, according to his degree of culture and his visualizing capacity, from three or four in the very young child to twelve or sixteen in the average adult. Only exceptional persons can conceive of rather larger numbers.

Why, then, have we not chosen as the basis of our numeration the number twelve? It is, by its own nature, an eminently suitable number for a base, because it has several factors, and therefore is divisible in more than one way. Why did we not evolve a duodecimal instead of a decimal notation? Because man has ten fingers to count on, five on each hand; and, therefore, the only two notations which have ever come into ordinary use are that by *fives* (the so-called Roman, in which six is written "five and one," or the fingers of a hand plus one) and that by *tens* (the Arabic, in which eleven is written "ten and one", or the fingers of two hands plus one).

We see, then, that arithmetic corresponds from its very origin to properties which are distinctively human, not numerical. Its processes are, from the first, anthropomorphic, not, as is commonly assumed, purely abstract. Our systems of numeration are permeated through and through with traces of an anatomical truth,—the truth that man has ten fingers; there is, therefore, nothing improbable *a priori* in the suggestion that mathematics may contain indications of psychological truths. Indeed, it has been shown that it does contain such indications; the very existence of any such invention as a system of numeration points to the psychological doctrine that both man's capacity and his desire to register and combine numbers are greatly in excess of his power of directly conceiving them. This particular psychological doctrine happens to be one which we could have discovered without reference to mathematics; therefore the specially mathematical evidence of it attracts little attention. We shall see in the sequel whether mathematics contains evidence of any other truths of psychology—of any truths not easily discovered without its aid.

As we trace arithmetical and algebraic reasoning through successive gradations of increasing complexity, we find at nearly every step the same artificiality. The processes are artificial in relation to the subject-matter, but they are well adapted to meet the limitations of our direct faculties, and to assist us in attaining knowledge unattainable by our unaided powers.

So far as we have now arrived, there is no great difference between arithmetic and the so-called natural sciences. Most of the implements and methods of science betray their distinctively human origin. Optical instruments, for instance, are implements whose primary function is simply to extend the range of human vision; special adaptations are in some cases made for the observing of special classes of objects; but the function of the microscope is essentially to enable man to see more minute objects than he naturally could do; and that of the telescope is to enable him to see more distant ones. The text-books of nearly all the natural sciences contain, mixed up with information about the particular science, instructions about devices for enabling man to supplement his limited faculties, physical or mental, and to extend the scope of their activities. Most scientific books, therefore, contain, implicitly, information about human anatomy or psychology, or both. When a teacher, in order to demonstrate to his class the structure of a leaf, shows a skeleton leaf, the cuticle under a microscope, and the cellular substance, and then appeals to his class to conceive these various portions combined, and acting on each other, in the growing leaf, his method follows, not the nature of vegetation,—it is not the nature of a leaf to grow its cuticle, its ribs, and its parenchyma separately, and then combine them,—but the nature of man, whose faculties are constructed to work under such conditions of alternate analysis and synthesis.

We have arrived at nothing, as yet, to show that mathematics is psychologically more instructive than botany; but we have at least, it may be hoped, cleared out of our way the popular but erroneous conception of mathematics as a science differentiated from most other sciences by being purely abstract, non-human, out of line with vital processes.

If we turn now to another department of elementary mathematics, that which relates, not to number, but to size, form, and motion, we are again confronted with an anomaly similar in kind to that which we encountered in arithmetic. Before we can enter on such a study as the line of a planet-path we have to become familiar with the properties of the plane ellipse. Now, no such thing occurs in Nature, so far as we know, as a plane ellipse. The path of a planet in space is an elliptical spiral. Could anything be more artificial, as regards the planet-path itself, than to pretend that we have cut it by a plane, and then pretend that our paper is that plane, and then pretend that the spiral path has cast a shadow on the paper? Yet that is what we are virtually doing when we draw an ellipse on paper and use it to study or teach the action of natural forces in producing planetary motion. Even supposing there were a real, living ellipse, supposing that any thing in Nature did run in an elliptic line on one plane, what is the meaning of all the straight lines, the axes, tangents, and radii vectores pictured in our text-books? What are the rectilinear co-ordinates of algebraic geometry? All these devices have been invented, not to suit the nature of the motion under investigation,—most of them have no relation to that motion,—but to meet the needs of the human mind in its efforts to understand what is beyond its comprehension. And they answer their purpose perfectly.

Let us turn back now to elementary arithmetic.

Suppose a student desires some knowledge which is not at present in his mind,—say about the mode of fertilization of a particular plant, or the chemical constituents of a particular article of food. He must go outside of himself for the information which he lacks; he must either investigate a specimen of the plant or substance, or he must refer to some book, or be instructed by some one better informed than himself. It may happen, of course, that he is able to recall to memory what he desires to know, but in that case he is only ransacking a library of impressions stored up within him which impressions were produced originally by information from the outside. If he never yet possessed the knowledge he now seeks, he cannot now re-collect it from within; he must seek it outside himself. The mythical philosopher who "evolved a lion out of his moral consciousness," instead of looking at the real lion in the Zoological Gardens, is proverbially ridiculous.

But suppose that you, my reader, have occasion to know how many days there are in two million four hundred and eighty-one thousand seven hundred and forty-nine years,—that is to say, how often the earth will turn on its own axis while it is going round in its orbit a number of times far, far too great for you to conceive or imagine. It is obvious that you cannot acquire the information you need by direct inspection of the facts; you will not live to count the days in seven million odd years. But you will not, in such default of direct source of knowledge, consult any written book or appeal to any human teacher; nor need you distress yourself by laborious ransacking of your memory; the chances are greatly against your ever having known the exact number of days in the precise number of years about which you are now inquiring.

You will take a blank sheet of paper and a pencil,—instruments whose sole function is to register successively your own mental processes as they take place, and so to assist you in keeping order among them,—and within five minutes you will be able to state with absolute certainty the fact of which, five minutes before, you were ignorant. How did you acquire the new knowledge? Not by any process of mere syllogistic logic; there is no similarity between the process of arithmetical multiplication and such ordinary syllogism as "All men are mortal; John is a man; therefore John will die." In any valid syllogism, the conclusion is really contained in the premisses; the syllogistic process only puts into more convenient form knowledge already possessed.

If any fanatic were to propound such an argument as this:—"All men are mortal, therefore the archangel Gabriel will die," how he would be laughed at! What know we of archangels? Nobody ever saw one. Yet it is just as true that nobody ever saw a million years; nobody ever *saw*—truly saw and cognized—a million of anything. Yet we are permitted to infer something about three hundred times two million from the mere datum that three times two are six. This proceeding must certainly be non-syllogistic, extra-logical. We shall see this all the more plainly if we remember that, as was shown above, the names two million and three hundred are purely artificial. Numbers themselves form a homogeneous and unclassified series; there is no more natural connection between the number which we have chosen to call three and the number which we have chosen to call three hundred than there is between three and nine; the whole nomenclature and classification of numbers is an affair of purely human psychology and human anatomy. Yet every school child makes statements about three hundred times two million for which he has no data, except that he has counted what three times two make.

It will be noticed that the knowledge gained by the process of multiplication is, in one sense, relative to ourselves. The answer we get, if large, is not conceivable by us; but it is expressed in terms of the base of our system of numeration; *i.e.*, of some number chosen by us as base because we can conceive it (*e.g.*, ten, the number of our fingers).

The process of multiplication is capable of indefinite expansion. It does not become more intricate as it is pushed further; it is as easy to manipulate billions or trillions as hundreds or thousands; the only extra cost is the labour of writing extra zeros. It is as if, when we reached the top of a cherry-ladder, we found it perfectly simple to add on another length and climb that; and then another, and another, till we reached the stars. In physical sciences, progress is hampered by the constant need for some fresh facts to lean the ladder on; the arithmetical ladder starts, indeed, from a physical basis,—our ten fingers,—but it never needs anything to lean on any more. And there is no limit to its expansibility.

Scientific men of all "Fachs" agree in asserting that the rough guesses of mere average common sense and *a priori* probability need to be corrected by the dicta of science. And the delightful complacency of their consensus on this point has caused them to overlook the fact that the expression "correcting an unscientific guess by scientific investigation" may refer to either one of two processes as unlike each other as possible.

A man widely known for fidelity and

trustworthiness suddenly robs his employer. Three months later he becomes delirious, and soon after dies. Average common sense supposes that he yielded to the pressure of some exceptional temptation, and then went mad of remorse, and that the strain of that remorse caused his death. Medical science declares that common sense has mistaken cause for effect; that the man died of softening of the brain, which must have been coming on before the date of the robbery, and that in that disease decay of the moral instincts frequently precedes intellectual and physical failure.

Or a girl appears depressed in spirits, and talks mysteriously about some one having ill-used her. She dies suddenly. General probability suggests that she made away with herself; the probability seems all the greater when it is reported that she lately bought beetle-poison. Chemical investigation proves, however, that the beetle-poison contained no ingredient fatal to human life; and at the autopsy no trace of poison is discovered, but a clot of blood or a tumour is found on her brain; and medical science declares that this cause accounts for depression, for imaginations about being ill-used, and for sudden death.

In these cases scientific experts can correct unscientific guesswork by reason of having access to facts not generally known.

But now let us take the case of the Eastern potentate who, according to tradition, bade the inventor of the game of chess to name his own reward. The learned man asked that a grain of corn should be given to him for the first square, two for the second, four for the next, and eight for the next; and so on till the end of the series of sixty-four squares. The monarch considered this request a modest one, till the wise man showed him that the year's harvest would not furnish the amount of corn necessary. In this case the scientific expert had access to no facts hidden from his unscientific hearer; there were no facts to be taken into account, except the obvious one that the board was divided into sixty-four squares; the two men differed because the one had gone through a certain set of mental processes which the other had not gone through.

One remarkable difference between mathematical certainty and what is called scientific conviction must here be pointed out.

"The general laws of Nature are not, for the most part, immediate objects of perception.

They are either inductive inferences from a large body of facts, the common truth in which they express, or, in their origin at least, physical hypotheses of a causal nature, serving to explain phenomena and to predict new combinations of them. They are, in all cases, and in the strictest sense of the term, *probable* conclusions; approaching, indeed, ever and ever nearer to certainty, as they receive more and more of the confirmation of experience; but of the character of probability, in the strictest sense of that term, they are never wholly divested. On the other hand, the knowledge of the laws of mind does not require as its basis any extensive collection of observations. The general truth is seen in the particular instance, and it is not confirmed by the repetition of instances; . . . the perception of such general truths is not derived from an induction from many instances, but is involved in the clear apprehension of a single instance. In connection with this truth is seen the not less important one that our knowledge of the laws upon which the science of the intellectual powers rests, whatever may be its extent or its deficiency, is not probable knowledge. For we not only see in the particular example the general truth, but we see it as a certain truth, —a truth our confidence in which will not continue to increase with increasing experience of its verifications."[1]

When a professor illustrates some principle he has been stating by one or two instances only, he uses those instances, not to prove the principle, but simply to make the pupils understand exactly what it is that he asserts.

Now if he is teaching any physical science, the pupils are expected, if not to believe the principle on the word of the teacher, at least to accept it provisionally as a working hypothesis till they gain, by repeated and long-continued experience, the right to hold an opinion one way or other on its truth. But if the principle be a mathematical one, such, *e.g.*, as that $a \times b = b \times a$, the pupils are not expected to suspend judgment till they gain certainty by experience; when once they have grasped what the principle is that is being stated, they see and know for themselves that it is absolutely, certainly, and always true.

One singular characteristic of mathematics is

[1] Boole: *Laws of Thought* Chapter I

the automatic power of self-correction which the mind possesses in relation to it. No such thing is possible as the existence of either a persistent or a widely spread error, or a serious difference of opinion, as to the result of a calculation. And this is not due to any special immunity from error which we have in connection with the subjects called mathematical. Every schoolboy knows that it is as possible to make a mistake in a sum as in any other exercise; and the greatest mathematicians occasionally make mistakes in calculations. But there seems to be some mysterious court of arbitration within man which detects mathematical error. Any two persons who have come to different conclusions as to the result of a calculation can find out which of them was mistaken, without appealing to any external authority. Nor need they examine any outer facts; they need look at nothing outside of themselves except the slates or papers *which contain the register of their own successive mental acts.* They judge between themselves, and judge infallibly. When one child only has done a sum, he may need to look at the "answer in the book"; if his result does not correspond with that answer, he thinks that his processes must have been wrong somewhere. But when two persons have made the same calculation with differing results, there is no need to compare the results with the answer in a book ; they revise and compare their own mental processes as registered by themselves.

This power of absolute and unerring correction and mutual correction, which man possesses in regard to arithmetic, is popularly supposed to be due to something in the nature of the subjects commonly treated mathematically (*i.e.*, number and quantity). "Man can know number and quantity more accurately than he can know anything else," is the ordinary explanation. But when we examine it, we fail to find any warranty for the assumption that man has more natural power of being accurate about number and quantity than about any other kind of truth. The chess-board legend already alluded to, and such well-known puzzles as that about the nails in the shoes of a horse, prove that people make just as wild guesses about number as about any other department of the Unknown when they dispense with systematic investigation and trust to their unaided judgment. And, as was previously shown, we use with ease an alphabet of twenty-six letters ; if our arithmetical alphabet were as large, we should become hopelessly fogged in our calculations. Most of us could probably recognise at sight any one of some hundreds of persons, of words, of flowers, of any sort of objects with which we are familiar ; but when we come to numbers our powers of direct cognition seem much feebler and more limited. If dots are arranged in patterns, each number having its own pattern, as is done in the printing of playing cards, we recognise any one of the first ten numbers easily, as we know our friends apart by their faces ; most of us would know at a glance the look of twelve dots, or perhaps even of any number up to twenty. After twenty we should probably have to begin to spell out the numbers—*i.e.*, to count or calculate. That is as if we could not read at a glance any words except some sixteen or twenty of the shortest and simplest in the language ; or as if we could not recognise by sight more than twenty of our most intimate acquaintance.

This weakness of our power of direct cognition with regard to numbers, as compared to our faculty of cognition in some other directions, may be partly due to lack of exercise. It might be possible to strengthen it by practice. But at least it leaves us no excuse for asserting that numbers, as such, are easier for us to cognise with certainty than facts of other kinds.

What, then, is the true import of our easy and familiar access to the Great Unknown on the side of number? What is the source of the unerring certainty which we can all attain with regard to it?

"Necessity is the mother of invention." Is it not possible that the very limitation of our direct numerical faculties has forced mathematicians to create, unconsciously, the ladder on which man can safely climb to The Infinite?[1] Is it not possible that mathematics is, after all, a science, not of number and quantity, but of the conditions under which man can make his progress towards unknown Truth uniform and safe, and can preserve himself from being seriously misled by the mistakes which he is sure to make on his way?

This question unavoidably presents itself,

[1] "This process is a process of prayer." Gratry: *Logique.* Chapter on *L'Induction Géométrique.*

sooner or later, not, of course, to all mathematicians—that is to say, all users of mathematical formulæ,—but to all serious students of mathematical philosophy. It is in this century that it has taken concrete form and come into open expression in print; but it seems to me that it must always have been haunting the imaginations of the more subtle thinkers, ever since the first Abacus was invented as an aid in reckoning; ever since circles were first drawn by some inspired savage, by means, perhaps, of two thorns linked together by a strip of bark.

But more than I have yet indicated has dawned on the imagination of a few mathematicians (in this century certainly, and I have reason to think earlier).

Is not mathematics emphatically a miniature edition, so to speak, of the science of sane inspiration? Does it not contain the clue to an accurate distinction between the sane inspirations of genius and its aberrations? (Or, to use old Asiatic phraseology, between inspiration from the true God and sorcery or magical possession.) It is easy for flippant critics to say that the whole distinction between inspiration and sorcery was a squabble between the priests of rival deities; but no serious student of the old religious writings can doubt that more was involved than a mere contention as to whether the Source of Inspiration should be called Jehovah, Jove, or Lord. To Pan, the Unknown *X*, the Unity, the cosmic Force, the Great I Am, gave inspiration freely; the question was what internal disposition, what mental attitude, what sequence of mental attitudes, create normal receptivity, putting the human machinery into such a condition that light from the Beyond enables it to see new truth without causing it to mix that truth with delusion. The belief of the mathematicians whose point of view I am trying to give is that mathematical science contains the answer to this great question. What they affirm is, not that studying mathematics as applied to number and quantity tends to promote sanity; that has often been said, and may to some extent be true, but is not the special dictum of Mathematical Psychology. Nor do they say that *all* right and normal action of the human thinking-machinery must be mathematical in its sequence; that is very far, indeed, from being the case, as I hope in the sequel to show. But they affirm that when a human being desires to induce into himself light from an internal and unseen source, he should then set his mental machinery to work on the subject, whatever that may be, as to which he desires such light, in the same order of sequence as that followed the mathematician about number and quantity. When we desire to know how a particular flower is fertilized, how a certain bird builds its nest, how an operation is performed by those skilful in it, that knowledge must come to us from the outside: we must see the facts, or hear lectures, or read books. Or, if we wish to know what the learned suppose to be the interpretation of certain phenomena, we must get the learned to tell us, either by speech or by writing. But if we wish to "think out for ourselves" the meaning of phenomena—*i.e.*, to receive, without human instruction, new light about facts we already know—then we must keep mathematical order in our sequence of mental operations, or a delusion may come upon us, and we shall be likely to believe a lie, and perhaps to fix it on the texture of our thinking-machinery.

Moreover, some mathematicians have believed (though I have not met this statement in print) that such individuals as are, by natural organization, specially liable to receive sudden and involuntary illumination from above or within, should safeguard themselves from hallucinations and fixed delusions by frequently practising the specially mathematical sequence of mental actions, as a hygienic gymnastic for the brain.

It is, therefore, surely desirable that all psychologists should learn to distinguish what is the essentially mathematical order of sequence, so as to be in a position to begin to investigate the two important questions:—whether it is in itself hygienically valuable, and whether it affords the means of arriving at any test of the exact boundary line between the inspired and the insane conditions. The books in which the subject has hitherto been treated are written in the language of the Differential Calculus; but the doctrines of Mathematical Psychology, so far, at least, as they mainly concern the ordinary psychologist, are expressible in terms of ordinary geometry; and it is this task which I have been invited to undertake.

CHAPTER II

GEOMETRIC CO-ORDINATES

WE must now turn our attention again to that branch of mathematics which relates to the measuring of plane or solid figures. Its name, Geometry, indicates that its original function was to measure and compare areas on the earth's surface, both in the interests of land distribution and to assist men in finding their way about on the planet. One important principle of psychology comes out in the very words used to describe direction; we say that a town lies to the "north-west" of us, or that the wind is blowing from "south-east and by east." Just as the signs by which we denote numbers are compounded from the primary signs by which we designate the first ten numbers, so the names of all possible directions are compounded from the names of four directions, taken as primary—north, south, east, and west. Why should four special directions be called by their own simple names, and all others be referred to those? Not for any reason which is re-al—*i.e.*, connected with the nature of the things to be studied—,but for a psychological reason. It happens, indeed, that the particular directions chosen as primary were not in themselves quite arbitrary as to the facts. The north-south direction is parallel to the axis of the earth's rotation, and the east-west direction perpendicular to it. The Equator affords a natural base line from which to measure latitude (*i.e.*, north-south distance); for measuring longitude (*i.e.*, the east-west distance) any meridian can be chosen at pleasure; very often a writer chooses the meridian which passes through the chief observatory of his own country. When we leave navigation and turn to pure geometry, the system of construction lines (as they are called)—*i.e.*, the artificial scaffolding of measurement—seems purely arbitrary; each proposition of Euclid has its own set. But when we come to seriously complex form, the navigator's artifice comes into use again; as a pure artifice this time, without any excuse derived from the earth's axis of rotation. It is a usual feature of mathematical artifice that it fastens itself at first to some natural fact, and by-and-by detaches itself and becomes generalized. In books on algebraic geometry, we are presented with two lines at right angles to each other; these lines are called co-ordinates. One is usually called the axis of *X*, the other the axis of *Y*; the point where they cross is called the origin. (The Equator is, for the navigator, an axis of *X*, and the particular meridian from which he reckons is his axis of *Y*.)

When the geometrician has to deal with solid forms, he uses a third axis, supposed perpendicular to the plane on which *X* and *Y* lie; it is called the axis of *Z*.

When once the artifice of co-ordinates appears in pure mathematics it remains, so to speak in possession of the field. Algebraic geometry and the infinitesimal calculus rest upon it. As the science progresses, it shakes off its bondage to earth and to things; it dissevers itself from all its attachments to outer fact one by one, but without losing its great psychological characteristic. First, as was said, it dislocates itself from the earth's special position in space, and refers itself no longer to a north-south and an east-west axis, but to a *Y* and an *X* axis; any two lines cutting each other at right angles will serve its purpose. Next, the lines need no longer be at right angles. Then it is found that straight lines are no longer necessary; curvilinear co-ordinates are admissible; or one co-ordinate may be the length of a revolving line and the other an angular measure. Or the tangent and radius may be co-ordinates, and in that case the actual curve-direction of one point is one of the co-ordinates for the next. This case is especially interesting psychologically. Later on, time is taken as one co-ordinate, and velocity or force may be the other. But throughout this long and varied chain of devices to facilitate the acquisition of knowledge there runs the same principle:—Man when he withdraws from the observation of outer facts by means of his senses, to seek from beyond or within fresh light upon the meaning of facts already observed, conducts his mental operations by referring them to two or a few selected ideals. These ideals may be partially or wholly fictitious; the question of their objective reality in no way affects their value as psychological factors. The true thinker will use such ideals for his own education without being in bondage to any of them, will reserve to himself always the right to form a judgment about the reality of each in complete independence of its psychological utility. To suppose, for instance, that, because Buddhism has educated thinkers, therefore any such Being as Buddha must necessarily have existed is illogical. But no attitude of mind is more foolish or more superstitious than that of men who harden themselves against religious impressions on the ground that "we have no proof of the actual existence of either gods or devils," and who yet talk of mathematics as "pure truth," and of such ideals as co-ordinates, tangents, and normals as if they were realities. It is natural to man to climb to truth on a ladder of fiction; if mathematical ideals have led to purer truth than theological ones, it is not because mathematicians have abstained from dealing in fictions. Probably the cause rather is that they have been from the first aware of the limits of their own knowledge, honest in avowing ignorance, and careful to distinguish between necessary psychological apparatus and objective truth.

It may be worth noticing that geometric investigation is made easier by choosing for co-ordinates two standard lines or ideas which are essentially independent of each other

The rectangular co-ordinates of algebraic geometry afford a good type of such independence. The two co-ordinates are not opposite to each other, not contradictory: they are not one north and the other south; but one may lie north-south, and the other east-west; so that progress along the one would not *necessarily* involve either progress or retrogression along the other. The law of any particular curve is then ascertained by finding out, in that curve, according to what law progress in one of the polar directions is accompanied by progress or retrogression in the other—*e.g.*, what distance from axis *X* coincides in that curve with a certain distance from the axis *Y*.

The co-ordinates of latitude and longitude, for instance, are so arranged that we can, on a map or globe, follow any meridian from the North to the South Pole without altering the longitude; or a parallel of latitude through all the degrees of longitude without deviating to the north or south. By means of this arrangement, it is made possible to mark exactly the track of any ship by saying that at such a parallel of latitude it was on such a degree of longitude.

When we have properly understood the doctrine of co-ordinates, we begin to see how psychology comes into contact with the law of the parallelogram of forces. If a body of men has to act as a whole, while the individuals composing it are swayed by a variety of motives, empirical common sense and good feeling declare that "each side must yield a little," that "a compromise must necessarily fall short of carrying out to the full the ideal of any one"; but when common sense and good feeling put themselves under the guidance of mathematics, they are introduced to a higher conception of what compromise may mean. In geometry it means something perfectly definite and accurate. It would, of course, be false reasoning to assume that because a ball pushed by two forces follows the diagonal, therefore a diagonal *can always be found* in human affairs; but we have a right to point out that, so far as regards geometry, the human mind has long ago left off guessing empirically at the *best* compromise to come to between two motions, having found itself perfectly capable of knowing exactly what is the *true* compromise; and, moreover, that much inspiration of fresh knowledge has been gained by insisting on arriving at such exact knowledge.

We may observe, too, that the problems presented in life resemble those of the geometric parallelogram in this respect, at least :—Two motives may partially neutralize each other's action, or they may partially assist each other, or they may be so adjusted to each other as to act quite independently. In mathematics, we have, in the first case, the short diagonal of an obtuse-angled parallelogram (with part of the force converted into heat within the body on which the conflicting forces act). In the second case, we have the long diagonal of the parallelogram. In the third case, we have the diagonal of a rectangular parallelogram; that is to say, a complete resultant, which gives full effect to the whole action which each force would exert if acting alone.

CHAPTER III

THE DOCTRINE OF LIMITS

ONE of the mathematical artifices which is most important psychologically is what is called the doctrine of "limits." It may be illustrated by trying to sum up the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ etc. It will be noticed, first, that the series itself has no natural termination; however long we write, we shall never write its last term. Next, that, however many terms we write, we shall never make up a sum quite equal to 2; the sum is always 2 minus a fraction equal to the last term we have written. Thirdly, that the more terms we write, the nearer their sum will approach to 2; so that by writing more terms we can make the sum as near to 2 as we please. The sum of the series approaches the limit 2, as the last term approaches the limit zero. These facts are expressed by the equations:—

Sum of $1 + \frac{1}{2} + \frac{1}{4}$ . . . *ad infn.* = 2.

Last term of series—

$1 + \frac{1}{2} + \frac{1}{4}$ . . . *ad infn.* = 0;

or by the statement that 2 is the limit of the series $1 + \frac{1}{2} + \frac{1}{4}$ as the last term approaches the limit 0. These are fictitious statements; no series is ever written out *ad infn.*

In the same way it is stated that the parabola touches a certain imaginary straight line, the asymptote, "at infinity," or that the asymptote is the "limit" of breadth of the parabola, as it approaches ∞ in length. The parabola is a re-al line, the curve traced by a projectile. In any given case it comes to an abrupt termination, because the projectile is stopped by the earth; but it does not naturally join ends like an ellipse; it is, potentially, of indefinite length. Man, in investigating the parabola, finds it useful to invent the imaginary asymptote, and then to show that, supposing the asymptote existed, the parabola would touch it "at infinity" (*i.e.*, never), but be always approaching nearer and nearer to it the longer it grew. That this device assists man is a psychological fact.

We are all familiar with the device of the *reductio ad absurdum*, as employed by Euclid. He supposes, or guesses, or thinks he sees that a certain thing cannot be; but, instead of arguing: "I should think it cannot be," or "Of course it is not," etc., he boldly faces the supposition:—"Suppose it were," and then traces out the logical consequences of that supposition till he lands himself in some impossible absurdity or sheer logical contradiction. This device has found its way to some extent into ordinary logic. But in mathematical reasoning it allies itself with another, to which I have already alluded: the fact that man is assisted in understanding a curve by the device of inventing the notion of tangents. Take any real curve, say the path of a planet; the pious dogmatist asserts that this curve is traced, as a whole, by the Will of God. The mathematician analyzes the action, at any point, of God-Will, or One-Force, into two imaginary forces: the centrifugal, which is supposed to act along the tangent, and the centripetal, which pulls towards the sun. These two forces are, of course, as imaginary as the conception of Baldur and Loki, or of Ormuzd and Ahriman, or any other pair of antithetical deities. No one ever saw a planet either fly off at a tangent or rush into the sun, nor is the student intended to calculate the probability of any planet doing so. But it is a fact of human psychology that man understands the action of the One-Force better by first splitting it, in thought, into antithetic imaginations, and then combining the conception of them.

We have now, I think, reached a point where we can gain valuable light from mathematics on the solution of certain discussions which disturb mankind. The ordinary good citizen, whether of the unlearned or of the learned world, is fond of saying that you must not push any logical argument to extremes, or it lands you in some absurdity. Wherever mathematics touches, it instructs us to push everything to the extreme (in thought, in words), in order to be landed in absurdities; because only so can our premisses be thoroughly tested. It is only by pushing in thought our tangent to infinity and our radius to the focus that we can properly understand our curve. The object of so pushing to the ultimatum is that we may understand the curve. When we understand it, we will discard altogether those fictitious entities, tangent and radius; *that* is, for the mathematician, a matter of course. But the world imagines that the tangent or the radius—

or both—are meant for lines of direction as to conduct; and is shocked, or argues to prove that such conduct would be unwise. When told that the suggestion in question was never meant to be acted on, the world asks what is the use of discussing unpractical theories that were not meant to be acted on. The reply to that question is to ask another:—What is the use of a tangent? The mathematical step next in order of sequence, after following the tangent into space in its centrifugal flight, and the radius vector to its origin at the focus, is to find out what would be the effect of two forces acting in combination on the same body, one of which forces has been imagined as carrying it along the tangent, and the other as carrying it down the radius vector; in other words, we have to calculate to what kind of curve our two imaginary straight lines are radius and tangent. If any student should stop short of this, should insist that either the tangent or the radius is the planet path, we should remind him that he has not completed his investigation; just as we should do if a child handed in one line only of a sum in long multiplication, mistaking it for the answer to the sum itself. Whenever in the sequel I speak of uncompleted mathematical sequence in thought process, I would wish the reader to understand that I mean something more or less analogous to mistaking the tangent or the radius for the planet-path. I call it completed mathematical sequence when the mind has gone through a succession of processes analogous to those which, in mathematics, lead to a true conception of an orbit.

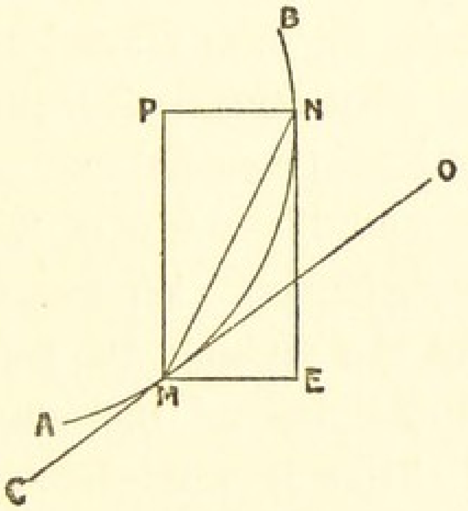
It is interesting to note that biologists in not a few cases find their way, empirically or by instinct, to the exact mode of procedure dictated by Mathematical Psychology. *E.g.:* one such instance has been given in Chapter I.: that of the three fictitious elements of the leaf, by the union of which the student has to construct within his mind a conception of the leaf as it grows. In this case, and I think most others, the analysis, or separation into fictitious elements, is made by projecting the mind outwards, as it were ; by the observation of outer facts. But the synthesis which completes the sequence takes place within.

CHAPTER IV

NEWTON AND SOME OF HIS SUCCESSORS

UNTIL the time of Newton, the Science of Mathematics was comparatively slow in its growth ; mathematicians became accustomed to the use of artificial processes so gradually as hardly to notice their unreality.

Newton and his contemporary, Leibnitz, created an instrument called the Infinitesimal Calculus, or Differential and Integral Calculus, the unreality of which, its lack of conformity to the nature of things, was so glaring that it called forth a momentary protest. I have no right to assume any knowledge of the Calculus on the part of my readers ; and, of course, it would be futile to attempt to analyse it here. It is sufficient to say that the method rests on a certain modification of the theory of the diagonal of forces combined with a peculiar extension of the doctrine of limits. In the case of rectilinear motion along a diagonal, two or more forces are supposed to act for a definite time, each in one straight line. In the case of a curve, no force acts for an instant, or for any fraction of an instant, in one straight line; change of direction is as continuous as the motion itself; we have nowhere to deal with the diagonal of any actual parallelogram. By Newton's method, we, as it were, invent a non-existing parallelogram, *E M P N*, two opposite



angles of which are on the curve. Then we suppose *N* to draw nearer to *M* along the curve. By reason of the curvature, the shape of the parallelogram alters as well as its size; the ratio between the sides changes as well as their actual length; the direction of the diagonal changes as the diagonal diminishes. The Newton method investigates the limiting condition to which the ratio between the sides approaches as the length of the sides approaches zero. This limiting ratio expresses the limiting direction of the diagonal; or, in other words, the actual direction of the tangent at *M*, *M O*.

In speaking of this device, Gratry says:—"We have analysed the finite in order to know the infinitesimal. From what we have learned about the finite we have effaced the character of finiteness; what remains is found to be true for the infinitesimal element, that is to say for the analysis and the study of the indivisible and the infinite (curve). We have analysed the discontinuous, the divisible, the finite, and have thus found the law of the continuous, the indivisible." That is what Gratry writes in this century.

In Newton's time, however, the mathematical world was startled at the roundabout and apparently senseless process invented by him. How can any truth be arrived at by first taking the ratio between two sides of a parallelogram imagined as existing, and then supposing them not to exist? Surely when they vanish to nothing, all reasoning based upon their supposed existence ceases to have any validity.

Reasoning about the relation of black men to white men may be still valid in an island where there remain only two black men and one white one ; but surely no such reasoning can prove anything about a country where all the human inhabitants have been killed off.

The answer is that Newton's method enables men to solve satisfactorily problems hitherto unsolvable. Not only has astronomy made giant strides by means of it, but modern investigations in electricity, mechanics, statistics and actuarial science depend on it. Mathematicians at, or soon after, Newton's time settled down to the use of the artifice invented by him ; accepting it for the most part, without understanding why it should be helpful. And less than half a century ago, it was still possible for a Cambridge graduate to introduce a pupil to the Calculus with the remark that "no one can understand why this method should answer ; it seems contrary to the nature of things ; but if we trust to it the results come out right ; it is like putting corn into a mill-hopper ; you cannot see what happens, but it comes out flour at the end."

What happens is the same as happens when we do a multiplication sum; we put our inspiration-receiving machinery through a process consonant to its nature, though unrelated to the facts we wish to investigate; and we thereby raise it to a position where it can grasp truths previously out of reach.

The lines whose evanishing to zero give us the knowledge which we seek, were construction lines, fictions, not portions of the curve, but made by the investigator as helps in the investigation.

For those who are seeking from the Calculus information about human psychology, it is a fact of overwhelming importance that the special knowledge about the curve which is to be got by means of the varying ratio between two fictitious elements comes only at the moment when those elements vanish; *i.e.*, when the subject-matter which has been under investigation vanishes to No-Thing. When once this principle has been grasped, it becomes easy to see that it is latent in mathematics from the very beginning.

The vivid sense of the sacredness of that psychological moment when the differential elements are reduced to zero and the sides of the parallelogram vanish into No-Thing, has led many mathematicians to feel an impassioned reverence for all forms of worship of The Eternal No-Thing, in Whom all contrasts vanish, Who "is at once the slayer and the slain"; and to conceive an abhorrence of idolatry; by which such men always mean, not the worship of images nor the worship of the *wrong* mental Eidolon, but the act of worship of any concrete entity, any mentally conceivable Eidolon at all. The ordinary pious clamour on this topic, founded on the perfections of Jesus, affect them much as medical men would be affected by a claim that some particular foreign body, being of pure gold, must be fit to introduce into the heart-valves and can do nothing but good. From the point of view of the higher mathematics, the adoration of the concrete and manifested is not a mistake in theology but a brain-vice. The mathematical standpoint is not:—There cannot be an incarnation of good in contrast with a personal devil, but:—If there be an incarnation, the worst use we could put it to is to worship it or mistake it for God. Boulanger, a devout mathematician of the last century and one of the great forces of the intellectual revolution led by the Encyclopedists, wrote a book on the Origin of Eastern

Despotism, with the motto *Monstrum Horrendum Ingens Difforme*, in which he expounds the doctrine that no tyranny could keep itself in existence were not the minds of the peoples weakened by some form or other of the practice of putting a concrete ideal in the place of God. He has defined idolatry in language on which it would be difficult to improve. "Idolatry does not consist, necessarily, in taking a statue, an animal, or a man, as the representative of God; to define it fully we must say that every form of worship or code of law is idolatrous which takes as divine that which is not divine. It is not only idolatrous to treat a stone, a beast, or a mortal as if it were God; we are also guilty of idolatry if we imagine that the words of that man, or the oracles pronounced through that statue, are the very words and decrees of Deity. We are guilty of idolatry when we prefer speculations and mystical chimeras to reason; when we treat any legislative code as if it were dictated by the Almighty; when we endow with a divine character the servants of a theocracy; when we try to regulate the conduct of men here below by laws suited only to celestial beings; when we confuse Heaven with earth; when we mistake our own position and pretend to be more than mortal; and when we forsake our own place as citizens of this world and subjects of the civil government, either to tyrannize over other men in the name of God, or to live as recluses, despising or forgetting our fellow-men." He adds that he considers it essentially idolatrous to regulate our conduct in this world as if we were already in a better.

Boulanger believed in a Revelation from God to man; but he believed that Inspiration comes through the Zeit-Geist of the age; and will surely so come if men are not hindered from receiving that Zeit-Geist in its entirety.

But if we are to take the Zeit-Geist as a whole, we must take into account, amongst other essential psychological factors, the passionate protest of so many earnest men as to the necessity of some concrete human centre for the religious, moral, and emotional life. The desire to understand and do justice to the Christian conception gave, forty years ago, a great impetus to the study of what are called Singular Solutions of Differential Equations, by means of which a whole set, or, as mathematicians say, a "family," of curves is investigated from the point of view of their common relation to some special point or curve. The nature of the method may be indicated by saying that the Singular Solution of a family of Differential Equations is subject to the law which is the characteristic law of the Family, but in a different way :—elements which are constant in one of the others are variable in the Singular Solution ; and elements which are variable in one of them are constant in the Singular Solution. It would be possible, and might for some purposes be instructive, to conceive of an extra planet, moving among the planets and comets of our system, and cutting across their orbits, in a complex curve so arranged as to "touch" each of the existing orbits once. (When two lines coincide at a point without cutting across each other, they are said, in mathematical language, to "touch.") The Equation of this imaginary orbit would be a Singular Solution of the System of Equations of the actual planetary orbits. Now, if certain calculations were found easier to arrange, by referring them to such a fictitious Singular Solution of the general equation of the planetary family, no astronomer would therefore think that there must exist an actual planet running in the orbit he had formulated.

The new science of Singular Solutions turned out to be not only useful in certain kinds of investigation, but intensely fascinating in itself. Its wonderful charm exerted a softening influence on certain mathematicians of whom De Morgan may be taken as a type. It corrected the tendency to conceive Humanity as a collection of individual intellectual powers, each holding isolated communion with The Eternal No-Thing; and brought them, not indeed to actual worship of any concrete eidolon, but to the conception that it may be helpful to prepare for seeking inspiration from the Inconceivable No-Thing by mental contact with an Ideal of Manhood. An eminent mathematician once wrote to me enthusiastically about the inspired utterances of a Christian preacher; and added:—"I have made out what puts the whole subject of Singular Solutions into a state of Unity."

Unscientific persons of pious tendency catch only too willingly at the suggestion that there are Singular Solutions in nature, and that this fact somehow proves the doctrine of Incarnation; on the other hand scientific people accustomed to rigid inference object that the existence of curves in space cannot prove the historic truth of Incarnation. Both parties forget that in the first place Singular Solutions are not objective facts, but psychological entities in the mind of Man; and, in the next place, Mathematical Psychology cannot deal with historic facts; the most it could be supposed to attempt to prove is that a tendency to refer a group of individuals to an Ideal Type, in contact with all the members, yet differing from all, leads in mathematics to a better knowledge of the laws which govern them; and that therefore, when the same tendency is discovered in writings on religion and ethics, it should not be assumed to be abnormal, idolatrous, or unmeaning.

Those who wish to pursue further the subject of the connection between Mathematical Psychology and the higher religious life should read the second volume of Gratry's *Logique*. It is full of most instructive suggestions. We have, however, to bear in mind in reading Gratry that, at the time when he wrote, comparatively little was known about Singular Solutions; and there is no evidence that he was acquainted with even that little. He was groping his way through that part of the subject by sheer instinct, the instinct, moreover, of a Catholic; and the portions of his treatise which refer to "Le Verbe Incarné" are less clear than might be desired.

It has now, I hope, been made sufficiently clear that the modern Science of Mathematical Psychology does not rest on any such false basis as that of assuming analogies between facts of different orders. The analogies on which Mathematical Psychology rest are analogies of logical process, not of facts or things. It would indeed be false reasoning to assume that because a certain order prevails in the star-paths, it must therefore manifest itself among phenomena of other kinds; but it is valid argument to say that a mode of sequence in reasoning which is considered admissible, and which leads to true results, in mathematics, ought not to be treated with contempt as baseless and fanciful when applied to human affairs.

But though modern mathematicians do not offer to the world as proven, statements incapable of proof in the strict scientific sense of the term, those of them who have approached their subject from the standpoint of psychology have received suggestive clues from certain ancient sages who were free from modern conventions and restrictions. The mighty founders of ancient religions did not feel themselves bound to prove their statements; they found out, by instinct and experience, certain things about the conditions on which man can approach Unknown Truth with safety and profit; and they boldly stated what they knew, leaving any one to contradict who dared. They pointed, for instance, to the Rainbow, a temporary explosion of contrasted colours, which soon fades into the Unity of white light, leaving No-Thing behind; and declared that it was the token of God's covenant with man, the ever-young and always fresh messenger from the gods; or, as our sturdy Scandinavian fathers put it, a bridge by which the brave soul reaches the abode of the Divine on its own feet. They instructed disciples to pray that the Will of The Eternal may be done on earth as it is in the Heavens; *i.e.*, by a rhythmic alternation of apparently contrary motions; by incessant, unopposed, and orderly Re-volution. They cut a forked stick from a tree, and held it first in its natural position, and then reversed, putting Unity where there had been separation, and separation where Unity had been; and they called this symbol of analysis and synthesis a "rod of divination," that is to say, of inspired knowledge. They held it as the symbol and instrument of man's conquest over the difficulties which beset his path.

What part mathematicians took in ancient days in the investigation of psychological problems, we have now no means of finding out. Mathematicians were often considered as wizards; and their books were burned by the ignorant and superstitious.

It seems to me certain that ancient sages had arrived, by what process we cannot now ascertain, at a knowledge of the value, for true inspiration, of that psychological element on which the Calculus is founded; the importance of the moment when something which had been elaborated with accurate care vanishes and is reduced to nothing. They tried to teach this psychologic truth to the unthinking masses by object-lessons; declaring that the true Inspirer would descend at the moment when the suppliant had burned to ashes some product of his labour which he considered specially valuable :—(a perfect animal, or the firstfruits of tillage). Their attempts were followed by results often ludicrous, sometimes disastrous. The psychological doctrine became confused with and lost in one pseudo-inspiration after another :—in heathendom with the mere wild animal love of killing, with childish delight in big blazing fires, with triumph over human foes ; in corrupted civilizations, with the diseased love of inflicting torture ; in Palestine, with the desire of hygienists and moralists to insist on the public slaughter and inspection of animals intended for food ; in some cases with that abject lust for indulging the sensation of reckless devotion which is a mere perversion of the sex-instinct ; in other cases with the ascetic disease, which may be described as a form of suicidal mania, bearing the same kind of relation of inversion to man's normal desire for pleasant sensations that actual suicidal mania does to normal Will-to-live. The particular afflatus or pseudo-inspiration which is just at present obscuring and confounding the true mathematical law of *inspiration by extinction* is what is called Altruism ; a doctrine which is partly true, but which sometimes runs into the belief that however foolish an action may be in itself, it is glorified if done at the sacrifice of something valuable to the doer for the sake of somebody else.

Through all these ideas one can detect here and there, in the writings of the more inspired teachers of them, an ever-recurring consciousness of the great truth :—that the true direction for progress is revealed to man at the moment when something which he has been constructing with elaborate care vanishes into No-Thing.

CHAPTER V

THE LAW OF SACRIFICE

THE word sacrifice may seem to strike a keynote which does not belong to any scale in which mathematical conceptions can be expressed; but this should not prevent us from inquiring whether the Laws of Sacrifice find expression in mathematical operations. When we study the question impartially, we are led to see that they do find such expression. Sacrifice forms an integral portion of every true inspirational cycle. In the life of religious inspiration the necessary sacrifices are large enough to involve a serious and often painful moral effort; whereas in mathematics they are so slight and so habitual that we are hardly aware of making any sacrifice at all. But this difference is one of size, not of kind; the acts of sacrifice imposed by the Laws of Mathematics are proportioned to the minute scale on which the sequence is being carried on, and to the comparatively small value of the knowledge sought.

Of course there is a sense in which all study involves sacrifice; the sacrifice of inclination, of laziness, of amusement, and so on. But in mathematics, as in the higher spiritual life, something quite different from this is involved:—the continual giving up, not only of impulses hostile to study, but also of the very results which we have been toiling to attain.

When a child looks out a word in a dictionary, it is one of the words in the passage he will have to read; it is also a real word of the language we wish him to learn. When we tell him a fact, in History or Natural History, we intend him to remember that fact. He may forget it; but, if he does so, he will have to learn it again, for it forms part of the subject he is learning. Every result of his labour, in these cases, is a something which we intend he shall retain as part of the furniture of his memory. The process of learning arithmetic differs in this respect from the acquirement of knowledge from any finite source. Take as an instance the problem, mentioned in a previous chapter, of finding the number of days in 2,481,749 years. The child has to plod conscientiously through the labour (to him no light one) of multiplying the long row of figures by 5; then by 6; then by 3; but none of these laboriously attained results are shown in the final answer. They are, so to speak, immediately effaced, merged in a total which bears no trace of their existence. In a complex calculation the worker goes through a long series of whole sums, the answer to each of which is merged in the next.

Even when a child has finished his sum, he has acquired no tangible knowledge which we wish him to retain in his memory; the result of his toil is, not a stock of information, but skill in drawing information from the Unseen Stores whenever he may need it.

This apparently useless labour, this seeming waste of painstaking effort, is not made necessary by the laws of number; there is no reason in the essential nature of things why the number of days in a year should be expressed by three separate figures, why we should not have a sign to express that number as a whole, and multiply by it directly; the sole reason lies obviously in the nature of man's faculties; the effaced and unrecognised steps in the calculation are rungs of the ladder by means of which our finite intellect reaches up to receive knowledge, direct from The Unseen, about truths beyond our own power to grasp directly.

A touching letter from James Hinton to Ellice Hopkins may be quoted here:—

"A little girl of ten said to me, 'Do tell me about the fluxion.' (I'd been talking in her hearing about Newton's fluxions, which I don't at all understand, a great deal of it.) So what do you think I did? Such a happy thought came to me. I told her plainly, so that she quite understood all I meant.

"I said (but I can only give you the barest notion), 'Multiply 17 by 3. So you know we get 3 times 7 is 21, 1 and carry 2; 3 times 1 is 3 and 2 is 5 = 51. Now,' I said, 'do you see what you have done with that 2? You have put it down, and then rubbed it out; it was necessary to have it, but not to keep it. Now, a fluxion is this: it is a thing we need to have, but are not intended to hold; a thing we rightly make, but in order to unmake.' And indeed that is the whole point. But this simple case shows also perfectly how it comes, how the law of it comes upon our life. For you see our making 21 comes only from our taking the 7 of 17 by itself, isolating it from the 10. In the 3 times 17 there is no 21; we make the 21 simply by separating the 17 into 2 parts and taking one only first. That is, the 21 is an isolation-right, a right or truth that comes by leaving out.

"Now, all isolation-rights are fluxions in this sense; we have to do them, but in order that we may undo them.

"And this is the law of man's life, because Nature is so great and rich that he is compelled to take her piecemeal; and, in all things, he makes for himself, by his inevitable leaving out at first, 'isolation-rights.'

"And the difficulty of his life—the difficulty above all others—is that these isolation-rights have got to be not kept, but used; not held as they come, but 'carried' on into another mode of being, which seems like their being lost.

"This is the pathos of man's life. He makes isolation-rights, and has not known the law of it and recognised its meaning.

"Now, this applies to all things. It is so pretty I can't show you even a fraction of its bearings. But you can think of many of them if you choose. Look at all our sciences. Are they not, clearly enough, every one of them, parts—no more the whole than 7 is, not only of 17, but of a whole page full of figures? Each one of them, therefore, makes for itself an isolation-right or true, a truth which comes only by its being isolated, wants not holding, but to be lost, as the 2 of the 21 is lost in 5, when the one is taken in (*i.e.*, the 1 of the 17).

"And the very relation of the figures in numeration, whereby each term means not one, but many of those before it, comes to have a distinct significance all through our mental life.

"In fact, the world is so beautiful I don't know what to do; only, as you know, the condition of that joy is consenting to bear pain; and one scarcely dares to say one is happy, because it makes the pain confront one, and the words have lost their meaning ere they have passed one's lips.

"Look at this very fluxion; it is such joy to see it, such pain to have to live it. . . . Oh, me! I am happy and sorry; and just now I cannot see a bit whether that gladness I think is coming on the earth is coming or not."[1]

We adults have become so accustomed to sacrifices in arithmetic that we make them without reflection, as mechanically and calmly as the farmer throws down some of his precious crop to rot in the ground; the "sowing in tears" goes on perpetually, only the tears are omitted. When the process goes on on a larger scale, we are tempted to become pessimistic and talk of the "futility of human effort," "the disappointing nature of things," "the unsatisfactoriness of life," and so on. But no one can study much of Mathematical Psychology without discovering that, for creatures constituted as we are, the unsatisfactoriness and futility are conditions of any true inspirational life.

In mathematics, however, much of the painful sense of futility is avoided by our knowing exactly what sacrifice is necessary for obtaining the special knowledge sought. Nearly nineteen centuries ago, a suggestion was made,

[1] *Life and Letters of James Hinton*, Chapter XV.

which, when followed up, imports into the subject of sacrifice for Inspiration much of the exactness which robs mathematical sacrifice of its sting; and changes the sowing in tears recommended by piety from the condition of vague scattering of treasures at random, to that of well-calculated planting of good seed in appropriate soil. The suggestion occurs in a passage in the New Testament, on which much mystical comment has been written by theologians. It was pointed out to me long ago, by Professor Boole, as containing a psychological clue specially well worth following up. It seems to me to sum up and make available for purposes of practical guidance that theoretical knowledge about the nature of our own inspiration-receiving powers which we gain from mathematics generally.

"The wind bloweth where it listeth, and thou hearest the sound thereof, but canst not tell whence it cometh or whither it goeth: so is every one that is born of the Spirit."[1] This evidently cannot refer to wind regarded as travelling in a straight line; we do know of

[1] John iii. 8.

a trade wind whence it comes and whither it goes. The path of the whirlwind was a matter of observation long before the Christian Era, and might have been suggested at any time to an intelligent observer by the motion of particles of dust. If the reader will look at

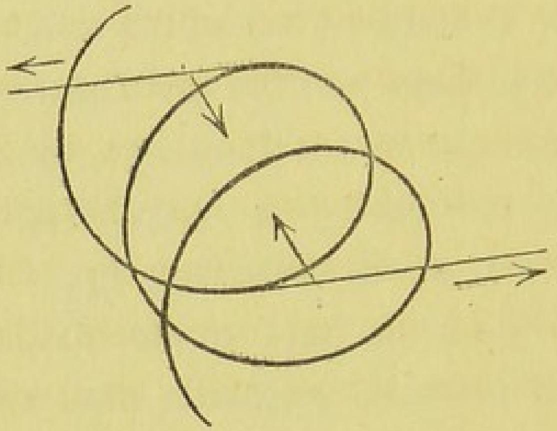


DIAGRAM OF A DUST-WHIRL.

the diagram of a dust-whirl, and suppose himself to be one of a cloud of gnats caught in the gust, or to be steering one of a flotilla of ships caught in a hurricane, he will see that the sentence contains both an admirable picture of a society on which has descended an inspiration to think on some new topic, and a canon of guidance for an individual who desires to find the main line of progress and to preserve his own sanity and safety.

The Zeit-Geist sets men thinking on some common topic, but in all manner of ways; some feel impelled to go north, some south, some east, some west; no one knows how to steer, for no one can judge, merely from the direction in which he feels impelled to go, what is the main direction of progress, or whereabouts safety can be found. If we could trust always to mere momentary inspiration, the matter would be simple enough; each individual would be carried round and round, and advance on the whole with the wind as a whole. But man, when once he has arrived at consciousness, cannot let himself be blown about by every wind; he must organize, formulate, translate momentary direction-impulse into definite tangential direction. It is useless to rail at this tendency; it is a law of the human mind (else, why need mathematicians have invented tangents?). But each man, or each group of men, mistakes *the human tendency to invent a tangent* for *divine inspiration acting in one continuous straight line.* This gives birth to all kinds of conflicting and erroneous opinion; each sect or group believing that the tangential direction it has formulated for itself, by inference from some inspired utterance of its leader, is the true line of inspiration; whereas, in fact, the still centre of calm and uniform progress lies between any two parties whose paths lie in opposite directions; *i.e.*, between any two whose original inspirations were diametrically opposite. Therefore, any individual who sees that another has been blown in the direction exactly opposite to the one in which he himself is drifting, has the clue he requires for finding out where lies the still centre of calm and orderly progress. By steering towards one's absolute opponent, — not in the direction towards which he is drifting, but towards himself, — one would reach the calm centre of the storm. This gives a distinct rule for discovering where lies the still centre.

Of course, as a general rule only the philosopher even desires actually to live in that central position. Most people are content to be swayed to some extent by personal afflatus. But as tangential direction leads at last to drifting away from the centre of force, safety, for every one, depends on keeping not too far from the centre; every one should treat the direction in which he feels impelled to go as one co-ordinate, and the path which leads towards those who seem to him in error as the other co-ordinate, and should frequently correct his own aberrations by acts of conversion towards the centre. Therefore the sacrifice appropriate to any given situation of difficulty or doubt is that which converts the force which is taking the individual along his habitual path into energy of union with opponents. Unfortunately, religious conversion too often takes the form of joining the ranks of former opponents, of adopting their opinions and imitating their customs. A mere glance at the figure is sufficient to show that this is not the kind of conversion which leads to the true centre of progress and of peace; that centre is not on the line which the opponents are following, but on the line which runs from us to our opponents.

The act of conversion from the path of personal afflatus to that along which union with opponents is effected should precede the attempt to learn what is the true path of progress along which the inspiring storm is to carry humanity as a whole.

This interpretation of the geometric figure used by Jesus would seem to be in line with many of His most distinctive utterances, which have been supposed to be canons of some peculiar code of ethics taught by Him. It would seem to me probable that they were hardly intended to supersede, in ordinary cases, the ordinary ethics of the Jewish hearth and home ; but were rather meant as suggestions of methods for bringing about the special state of mind in which pseudo-inspiration or personal afflatus becomes "converted" into energy for reaching the centre of progress. It is interesting to note that ever since the time of Christ, the Christian world has been overrun and perplexed with various theories of "conversion." It seems agreed that true inspiration is to be attained, and the favour of the Inspiring Deity to be secured, when the individual is truly "converted"; but there have been many discussions as to what constitutes valid conversion. About these discussions nothing need here be said, except that the disputants, however they may disagree, are unanimous in neglecting to pay to

Christ the ordinary courtesy expected by every geometrician from his brothers of the craft:—that they will take the trouble to draw the figure which he describes, and study it carefully, before publishing speculations about his probable meaning. It is to be hoped that students of Psychology have been trained in scientific rather than theologic conceptions of the respect due to a Teacher from those who aspire to learn from him.

The Parable of the Wind-storm, when closely analysed, bears out Boulanger's definition of Idolatry quoted above:—We need no higher inspiration (it seems to teach) than that of the Zeit-Geist of our own age; but we must take that inspiration as a whole; we must not let ourselves be tempted too far out of the general movement by mistaking for it *any* tangential direction; we must try to get to the calm centre by unifying contrasted motions. Authorities of one sort or another, religious, political, or educational, priests of this or that Eidolon, are constantly trying to arrest this vital process of conversion, by telling us that this or the other portion of the common afflatus is specially wrong or dangerous. As Boulanger says:—

"Priests were appointed to lead men into truth; but in all ages they have feared lest men should find the truth and walk in it."[1]

From the point which we have now reached, we are able to distinguish pseudo-inspiration from genuine revealed knowledge of the destined path of progress for humanity. Pseudo-inspiration is personal afflatus, formulated into a tangent, and not yet corrected by conversion towards those who are impelled by the opposite afflatus.

[1] Boulanger: *Origine du Despotisme Oriental.*

CHAPTER VI

INSPIRATION VERSUS HABIT

IT will be well to give here a summary of what has been found out mathematically about the contrast between the cultivation of the Inspirational powers, and the training which forms habit and what (in non-mathematical affairs) we call character. The apparent conflict between the two modes of treatment is similar to that between the actions of a gardener who is making soil fertile for plants and that of his assistant who is making paths as unfit as possible to grow weeds ; almost everything that is right for the one to do, is, for the same reason, wrong for the other to do ; but the two workers come into no conflict because the territory is well marked out between them ; whereas, in the treatment of young minds, the psychologist who is endeavouring to foster the growth of original faculty often finds himself in antagonism with the experienced schoolmaster.

He whose mission is practical education has to equip his pupils for life. He has to secure for them, as best he can, such knowledge of facts as is necessary for conquering Nature; such knowledge of prevailing human opinion and sentiment as is necessary for finding a place in society; a set of habits convenient to the individual and not annoying to those with whom he will have to live; and an upright and kindly character.

Now the knowledge of facts is derived, not from inspiration or mathematical induction, but from observation and reading. The knowledge of human opinion and feeling should be based on a solid habit of study. Owing to what is now called suggestion or telepathy, it is easy to form pseudo-inspirational guesses as to what others are thinking or feeling; and this instinctive imbibing of other men's thoughts is among the most dangerous and misleading forms of pseudo-inspiration, unless severely corrected by studying the actual utterances of others. Besides, even for the kind of knowledge which inspiration is an appropriate means of receiving, inspiration, in its true sense of completed synthesis, is a slow and laborious process. He who trusts too many of his actions to the guidance of inspiration will find that he has not time to complete either his analyses or his syntheses; he must act in the meantime; and he will find himself driven to act on pseudo-inspiration, *i.e.*, incomplete sequence; on uncorrected, untested flashes of impulse. In the interests, therefore, of the inspirational life itself, it is essential that a large part of the conduct of the practical life should be guided by habit. Now all formation of habit is destructive of that mental elasticity on the free play of which the possibility of pure inspiration depends. Habit means the arresting of free play in certain directions; it means setting up a preponderating tendency to do this rather than that, so that the organism, unless absolutely controlled to the contrary by the higher centres, will automatically do this and not that. Whereas the cultivation of genius, of original power, of intuitive and inspirational faculties, depends—as a writer in the last century said—on "keeping the mind in a live and sensible (*i.e.*, sensitive) state,"[1] ready to respond freely to the slightest stimulus of fresh suggestion. For we must never forget that true

[1] T. Wedgwood: MS. Treatise on *Genius*.

genius, true inspiration, depends on *correction of error*; and this implies the "right to go wrong." It would seem impossible to combine this with the formation of good habits, except by a careful demarcation, made by each adult for himself, and by parents for each child, of the field in which his genius shall be allowed to play.

The peculiar antithesis between the formation of character and the cultivation of genius (or the exercise of the inspirational faculty) seems to me to throw light on some utterances of certain great ethical teachers, which appear to conflict with family duty and social harmony. For the formation of habit and character, it is of primary importance to cultivate kindly sentiments towards those with whom we come most frequently in contact; *i.e.*, members of our own household, nation, and sect, those with whom we have common interests. But, as was pointed out in a previous chapter, this consolidation of habits leads to the formation of special tangent-paths, along which the individual, or the family, or the group, wanders off from the centre of force. It becomes at times necessary to correct this tendency by a deliberate conversion towards the "enemy," towards those who are following the opposite tangential direction. And as not all members of the group are keen to seek new inspiration thus, there follows a disruption of ties such as has been figuratively described as "hating one's father and mother," etc. When the parable of the circular storm has taken its rightful place in psychology as the basis for the study of the inspirational faculty, much domestic tragedy will, we may hope, lose its tragic character. Here, too, the sowing in tears will still be gone through, but the tears will be fewer, and will have lost some of their bitterness.

The antithesis between the cultivation of habits and that of the intuitional powers has been brought in mathematics to a satisfactory settlement. I do not mean, of course, that all teachers, or the majority of them, teach Arithmetic and Algebra as they should be taught; but that those teachers of mathematics who know their business best are able to make of mathematical lessons a training, perfect so far as the limits of the science extend, for both the formation of habit and the conduct of the intuitional life.

The following principles have been ascertained as to the relation of habit to intuition, in mathematics :—

When an action or sequence of actions has become habitual, it has lost its freshness as a training for the intuitional faculties.

When an action or sequence of actions has been utilized to the full as a training for the intuitional life, the pupil is then in the best possible condition for making of it a settled good habit.

A pupil need never, and should never, be required to act on a principle till he has himself recognised it by the pure exercise of his own faculties, unwarped by any kind of authority from without.

This by no means implies that a pupil is never to go through a process the use or meaning of which he is at the time unable to understand. Very often the best way of leading him to grasp a principle is by causing him to carry it out in practice. But in that case, though the master is guided in the sequence of steps enjoined by reference to the principle which he wishes to illustrate, the pupil acts, for the present, on no principle at all (except one which he already does recognise :—*i.e.*, that he must *do* as he is bidden).

In all ordinary works on education, we may notice constant discussion about the particular mode in which influence should be brought to bear; whether by individual commands, unvarying rules, hopes or fears about consequences, appeals to affection, the contagion of example, or that more subtle form of influence called by the pious "intercessory prayer," and by modern Science "suggestion," or "telepathy." The whole discussion usually turns on the rival merits of the various modes *of bringing influence from without to bear on the pupil.* It seems assumed that it is always legitimate to exert influence. In mathematics, however, the main question kept in view is :—When may the teacher exert influence?

For mathematical purposes, all influence from without, which induces the pupil to admit a principle as valid before his own unbiassed reason recognises its truth, come under the same condemnation.

But when a principle has been admitted as valid, there is usually more than one possible way of carrying it out in any given case. The pupil ought to be able to see that the methods are essentially equivalent; but he is not at first capable of judging which of the several possible alternative methods is the most practically convenient. He will not be able to do so for many years to come; and meantime he ought to be forming the habit of some convenient method. It is here, therefore, that the principle of authority comes in (using the word authority in the sense indicated above, of any force or motive brought to bear on the pupil from the outside to induce him to do this rather than that).

Two examples will illustrate what I have been saying:—

We introduce a child to compound subtraction by asking him what will be left in his purse, if he goes out with so much in it and spends so much. The novice usually begins by reckoning the coins of highest value first. (If asked why he does so, he sometimes says:— "Because if I were really doing it, I should care more how many shillings I had left than how many pennies; so I should count the shillings first.) He should be allowed to use any order he pleases at first. Then he should be asked:—

Would the result be the same if you paid out the coins of smaller value first? He should be allowed to grope his way to the answer at his own pace; for now he is in contact with the Inspirer, and no man should interfere, except by questions calculated to lead him to see for himself whether his process is incomplete or in any particular inaccurate. But when he has satisfied himself that it is immaterial in principle at which end of the line of figures he begins to work, when he is satisfied not only that, in the special case under his notice, the results are the same whichever end he begins at, but that the two methods necessarily lead to identical results in every case, then is the time for authority or influence to be brought to bear on the formation of a mechanical habit of beginning at that end which the experience of mankind has found to be most practically convenient.

Again, a child should not see a multiplication table till he has made one. The squares should be ruled for him, and questions asked:— What are 2×3? 2×4? and so on. Authority should come in only so far as to indicate in what square each answer should be written. If he makes a mistake, he should be led by questions to detect it himself. When the table is complete, he should be encouraged, and assisted by questions, to analyse it in all directions, finding out all that he can about its principles of symmetry. Then, authority should come in to induce him to commit it to memory, and should dictate the precise order in which he repeats it.

Tables of weights and measures belong entirely to the domain of authority; all children should be taught to distinguish between such a question as: What are 6×7? the answer to which comes to him direct from an Unseen Source, and such a question as: How many lbs. are there in a stone? the answer to which must be sought outside of himself.

When once authority has taken possession of a subject and begun to create a habit, it should permit no variation from the selected form or method, at least during school hours. If a child desires to try fresh experiments in working a sum backwards, or in any way altering established order, he should be allowed to do so in times of re-creation, on holy-days; *i.e.*, times of escape from the pressure of allegiance to man, of return to commune with Eternal Unity, in Whose sight, we hope, all honest methods are equivalent.

In some departments of mathematics one method would be abstractly as convenient as another; it is necessary for convenience and mutual understanding that one should be chosen and adhered to. This is the true field for mere *conventionality.* Authority must, in these cases, interpose to create a habit of working in the usual method.

It is important to teach pupils to distinguish between several different kinds of wrongness :—

$5 \times 4 = 19$ is a wrong statement, because untrue in itself.

$\frac{21}{7} = 3$ is wrong when given in answer to the question :—What is $\frac{217}{7}$?

It is wrong to begin a subtraction sum by the coins or number-places of highest value, because it necessitates cumbersome and unnecessary corrections.

It is wrong to write units to the left of tens, because, if you do so, other people will not understand what you have written; and you yourself will become confused in passing from your own figures to those written by others, and will lose the advantage of forming a steady habit for reading and writing figures.

At some points along the chain of mathematical development it is found advisable to cultivate a habit of being able to use with equal facility either of two equivalent processes. For instance, it is for some purposes convenient to be able to calculate with equal facility by the ordinary decimal notation or by some other, such as the duodecimal. On the teacher rests the responsibility of deciding which of the two is to be used in any given case. For if it be desirable to practise different notations, it is for the sake of being able to use, at will, whichever is most convenient on any given occasion. This is not a matter which can be settled by inspiration; questions of convenience must be settled by experience, which the pupil as yet has not. Leaving it to him to decide which notation he will use, is leaving it to be settled by his laziness or his freaks.

But the discipline in mathematics, however strict, is entirely positive. There is in mathematics no "Thou shalt not." Provided that the necessary amount of practice in right methods be secured, during school hours, no one watches to see that wrong ones are avoided at other times. No mathematician worth mentioning has any objection to his pupils trying experiments in bad methods to any extent they please in their spare time. It never occurs to any one that boys are likely to form bad habits, or be the less teachable in class, because of any vagaries they may practise out of it. Mathematical method is a thing about which there exists no *esprit de corps*, no public opinion, nothing resembling either a canon of taste or an ethical code ; and when conventions are imposed on the young by authority, it is with the distinct understanding, from the first, that any one who likes will be at liberty to alter the method in his spare hours ; that he will be even applauded for doing so if he is fortunate enough to hit upon a more convenient one. In no department of study is reverence for precursors more cultivated than in mathematics ; but that reverence is entirely dissevered from the notion that ancient methods should be adhered to : it is connected rather with a feeling of gratitude to those who, by inventing clumsy and imperfect processes, have helped successors to better ones.

Yet mathematics is the science about which there exists no guesses, no disputes, no delusions. And as was pointed out in Chapter I., whatever the reason for its immunity may be, it is certainly not any natural infallibility of individual man about this particular subject.

The only great hiatus, so far as I know, which exists in mathematical teaching between sound theory and the best actual practice, is of this kind:—Mathematical induction is essentially inspirational work; and should be done during periods when discipline is relaxed; periods of leisure, of re-creation from labour; during the Sabbath of the Inspirer, the Unity; ultimately even during sleep. Children are expected to do it to order, under the same conditions as those in which they do ordinary study; *i.e.*, in school, where discipline must not be relaxed; where time must be kept; where examinations are looming ahead. Thus bad habits are formed, which weaken that synthetic faculty on which true inspiration depends, and which cause the pupil to become a victim either to routine or to pseudo-inspiration. I am trying to induce a certain educational society with which I am connected to adopt the custom of introducing the children to each mathematical principle, by some easy examples, in play, several months before it will be used in the work of the schoolroom; so as to utilize each to the full as a growing ground for the inspirational faculty before it is given over to be hardened by the formation of convenient habits.

CHAPTER VII

EXAMPLES OF PRACTICAL APPLICATION OF MATHEMATICAL LAWS OF THOUGHT

THE method of the Calculus was applied by P. Gratry to devise a plan of study adapted to the purpose of training the brain to act in normal sequence. He studied several sciences, each from its own text-books and by the methods specially suited to it; and formed the habit of sitting alone, once a day and so far as possible at the same hour, pen in hand; he suspended thought, making silence within the soul, and then wrote down whatever thoughts came to him about the Unity of Nature. What is so written, Gratry says, is found fruitful of suggestions of syntheses which prove useful for future study. When the habit is fully formed, he adds, the brain will do synthetic work even during sound and refreshing sleep.

It will be necessary for the success of this or any similar method of self-culture that the student should not too much indulge in the reading of books on the analogies of Nature (such, *e.g.*, as those of the late Henry Drummond); such books do for the reader that mental work which it is desirable to induce in the brain the capacity to do for itself.

The precise plan of study adopted to carry out the mathematical method must vary according to circumstances and the nature of the individual's duties. My occupation in life has chiefly been that of assistant or secretary to intellectual men; especially such as are engaged in philosophical or religious reform-movements. I make it a rule to seek and cultivate the personal acquaintance of the chief intellectual opponents of my employer for the time being; especially those of whom he speaks slightingly, or whom he believes to have injured him or been unjust to him. Then, when he comes to a hitch in an investigation, and I find myself unable to help him, I contrive to see one or more of these persons. A conversation with them clears my brain. I do not find it needful or advisable to take them into counsel about the special point on which we need light; usually it happens that one could not do so without breach of confidence. I find that during any serious conversation with them, something is said which helps me to find the clue that we need.

I have been surprised to observe how often, in such a case, a learned man suspects that his assistant is being "led astray" by his opponents, is in danger of becoming "converted" to their opinions or ideas. He seldom seems able to realize, at first, that one is seeking light, not from man, but from that creative Inspirer Who descends into the mind wherein is being effected the union of contrasted polar-opposites; that one is making the sacrifice prescribed by Jesus as a preliminary to receiving inspiration, the sacrifice of prejudice and habit for the sake of effecting union with opponents; that one is being "converted," not to any human opinion, but towards the calm centre of Force.

Gratry's plan of study and my own are intended for the purpose of increasing brain-power and assisting normal brain-action. I will next give a few instances to show how the method of the Diagonal can be applied to the sanitating of brains which are working abnormally.

A death had occurred under circumstances which involved excessive fatigue and strain to the relatives of the deceased. There were, unfortunately, special reasons why the order for the head-stone should be given immediately. The nearest relative of the deceased had religious prejudices against the erection of a cross; the legal owner of the ground had an equally strong objection to any other kind of monument. Either would, in normal health, have given way, but both were ill and feverish; and each imagined it her duty to carry out her convictions. The medical man, James Hinton, said to the owner of the ground:—"If you get your own way, you will have a stone imitation cross on the grave; if you yield, you will have a real one." The whole nervous tension subsided instantly, simply because the matter was presented in a way which did full justice to the patient's own convictions. That is what I mean by a compromise which is—in antithesis to one which is not—a true diagonal. The patient was in a state of tension between two apparently conflicting motives; her conviction that there ought to be a cross on a grave, and her feeling that she ought not to thwart the other sufferer; the doctor's solution enabled her to feel that both convictions were done full justice to. It might conceivably have been arrived at by a happy instinct on his part; but I know that he had just then been studying the parallelogram of forces from the psychological point of view. In the next instance which I will mention, the solution could hardly have been arrived at except by the process of solving a psychological equation to find the exact diagonal.

A girl of seventeen had contracted a habit of falling every night, on lying down, into a condition of semi cataleptic coma; eyes staring, limbs rigid, etc. There were real causes of mental disturbance :—the girl had too early and too often witnessed serious discord among her elders. There was also a certain amount of tension on religious questions, intensified by the influence of one of those electric or magnetic preachers who are responsible for so much hysteria.

Empirical common sense would have suggested, of course, that the disease should be left to *wear itself out*; that the patient should be removed from the influence of the divine; and that religious topics should be avoided.

The mathematical prescription was of precisely opposite kind. The divine had been making himself conspicuous by an angry assault on an agnostic from whom he would have courteously differed had he understood him, but whose meaning he had seriously misrepresented. "Common sense" would have suggested that this painful incident should be kept from the knowledge of the hysterical girl, who was already ill from hearing about misunderstandings and quarrels. Mathematics differed on this point especially from common sense. The patient was therefore told of the blunder made by the preacher, with the remark that it would be a good thing if the curriculum of clerical study included such knowledge as would protect pious men from injuring their own cause by mere linguistic blunders. The hope was gently instilled into her that she might some day do something to aid preachers in understanding the language of antagonists, so as to know how to address them. Under cover of this hope, she was induced to study some beautiful and sympathetic passages from the writings of infidels, especially the one who had called forth the wrath of her pastor. The time chosen for such reading was the last hour before the time at which the comatose fit was due. The reading was followed by normal sleep from the very first. In cases of nervous disturbance aggravated by tension on religious doctrines, I have found it useful to say that I consider it disrespectful to Jesus to know so little as most Christians do of the ritual in which He was brought up. The patient is thus, without discussion or antagonism, induced to attend a synagogue; no sense is aroused of being false to the Christian faith. Going thus, not to criticise, but in the spirit of reverence for the ritual followed by the Founder of Christianity, she is free to absorb the healthful influences of a religious atmosphere where she escapes all allusion to the special doctrines on which there has been tension.

The same principles of treatment along the line of the diagonal may be usefully carried out in dealing with what may be called epidemic or corporate hysterical tension. For instance: I am sometimes asked how I, who know nothing about vivisection and very little about biology, can venture to write, lecture, and take action on committee, on the subject of vivisection. I know at least more of biology than I do of what a million years is; for *that* is, nothing at all. I know as much about vivisection, of my own knowledge, as any astronomer can know, of his own knowledge, about the portion of space in which lies the aphelion of a comet's orbit,—which is also nothing. There is a certain standpoint which I think I am fully justified in taking, by certain psychological facts bearing on the case, which no one disputes.

One party, under the afflatus of desire for knowledge, writes in one way about vivisection. Another party, under the afflatus of a desire to spare suffering to animals, writes in another way about it.

I examine whatever specimens I happen to come across of the two groups of utterances. I observe that the two forms of afflatus are not in any necessary or logical antagonism; there is no essential contradiction between a desire to know biology and a desire to make animals happy. Some compromise ought to be at least theoretically possible which should embody and give full scope for both forms of afflatus. Yet there is a tone of antagonism between the two sets of persons. This (as there is no essential antagonism between their respective aims) is a reason for suspecting that they are not inspired but only pseudo-inspired; that is to say, that each of the two movements is animated by some personal element besides the essential aim which inspires it. To make sure, I test some representatives of each side by confronting them with some scheme[1] (such as Dr. Hickman's scheme for organized morti-section) which promises to open up access to fresh stores of biologic knowledge in ways which do not involve hurting any living creatures; and I watch how they act in presence of it. When a biologist is lukewarm in desiring the success of such a scheme, or when an anti-vivisectionist refuses to aid it because she does not care "to foster in any way the curiosity of those wretches, the biologists," I no longer suspect, I know that the two cases are clear cases of pseudo-inspiration.

There is a third party involved in the

[1] See Appendix.

matter:—the mild, moderate, good doctors, who decide that cruelty for the mere gratification of curiosity is reprehensible, but when it really comes to a question of saving human life it may be necessary to sacrifice animals. Now this may be true, but it fails to give due effect to one main element of the problem, the afflatus of desire for knowledge for its own sake. These moderate people are no doubt safe moral guides in periods of ignorance; but it is not through them that I should expect true inspiration to come. True inspiration comes not from ignoring any pseudo-inspiration, but from converting each to meet its opponent. I have ventured to affirm that light on the vivisection question is to be expected whenever vivisectionists and anti-vivisectionists combine in some positive and active scheme for prosecuting biologic study. Therefore I have, for many years past, been endeavouring to draw off attention from the question of vivisection and fix it on various schemes for the promotion of biologic science which do not involve causing pain. I do this, not because I have an opinion, or the right to form an opinion, as to the merits or feasibility of any such scheme in particular; but because I know that in the direction of such effort lies the hope of a genuine inspiration.

Moderate persons often say to rampant anti-vivisectionists :—" Do not excite yourself so vehemently; do not dwell so constantly on one topic; or you will injure the elasticity of your brain." That sounds to me very like saying to a neurotic patient prone to violent movements :—" Do not toss your arms about so vehemently or so much"; such advice is the resource of well-meaning ignorance. The gymnast, who has studied the laws of sequence, says: "*Do* throw out your arms vigorously; and when you have done so, compensate by an equally sharp movement of return; by practising this normal alternation, you will develop a healthy muscular instinct which will warn you when you have exerted yourself sufficiently, and a healthy condition of the muscles which will enable you to rest." The gymnast does not know, or profess to know, what the patient ought to do with his muscular force when he has it; he only shows how to substitute health for neurotic aberration. And just so, I say to my anti-vivisectionist friends:—"Preach the rights of animals vigorously; but every time you have gone on an anti-vivisectional campaign, take care to compensate, by an equally vigorous movement in the direction of trying to help biologic investigation; then you will get power to receive inspiration as to what to do next."

There are in the world persons who really like causing suffering. This motive is in logical contradiction to that of those who wish to be humane. If the argument were ever used on behalf of vivisection that it affords an opportunity for the exercise of cruelty, we should then have to solve the equation of that case on its own merits, as we do the problems of sport and of war,—by some mental process analogous to that of finding the short diagonal between the obtuse angles of a parallelogram. But I am not aware that any one has yet tried to justify vivisection on the score of its affording an outlet for savage or diseased instincts.

It would be out of place to enter here on any such questions as those of the necessity or utility of vivisection, or of its abstract morality; we are only concerned with its psychology. It appears to me clear that, in the present state of the vivisection controversy, no person can be occupied about the matter in any way, either by writing in favour of or against the practice, by doing it or witnessing it, without grave risk of becoming the victim of some pseudo-inspiration, unless he alternates his action in the matter with vigorous exertions in the direction I have indicated; that of promoting the study of biology in ways which cause no suffering. But there is a further consideration in connection with the subject which ought to be suggested to those who propose to take charge of the health of others.

The physician may not cut open the patient to see from what disease the latter is suffering; the range of direct experiment on patients is also very limited; nearly everything, for diagnosis, depends on being able to make a perfectly accurate synthesis of such data as are readily accessible. Correct diagnosis depends, in fact, on the power of rapidly extracting from any fresh set of data, the truth, the whole truth, and *nothing but the truth* that is in those data. This synthesis must go on within the doctor's mind. All true synthesis is done within a mind; and this is especially important in the case of synthesis for diagnosis. Facts concerning animals other than man may be valuable in assisting to form a judgment; but knowledge of this kind would be dearly purchased, if, in the course of its acquisition, the synthetic faculty itself were either weakened or distorted. If the purely sanitary precaution indicated above were neglected in medical education, the result would probably be pseudo-inspiration of the faculty of diagnosis; or, in other words, a tendency to make brilliant but unreliable diagnoses.

I would wish it to be distinctly understood that I personally have no knowledge whatever as to what amount or kind of vivisection is practised in medical schools in England. I hope, therefore, that what I have said will not be interpreted into an attack on the medical profession on the score of "inhumanity." Medical practitioners, so far as I know of them, compare favourably on the whole, as to kindness and generosity, with any other section of society. But rumours reach the outside world of complaints that "research" is not as free in England as abroad; and that there are those who wish to remove restrictions on its freedom. We hear, moreover, that medical examinations are being made almost yearly more and more stringent, because it is increasingly necessary to make some selection among the too numerous candidates. It concerns the public to know what principle of selection is being acted on in this matter; whether the examinations are being so arranged as to exclude, or to admit, a preponderance of persons in whom the faculty most essential to true synthesis, the faculty of "knowing when they don't know," of "knowing the feel of ignorance" (*i.e.* of knowing instinctively when they have got all they can out of a set of data, when they have touched the exact boundary line between ignorance and knowledge), has been weakened or injured; whether, in fact, medical examinations are being so arranged as to select for success those who can answer a great variety of questions about what goes on, under given conditions, inside of non-human animals, or those who can make an exact synthesis of the information to be got in a given case from the phenomena presented for observation.

CHAPTER VIII

THE SANITY OF TRUE GENIUS

IT seemed to me that before completing my present task I ought to see for myself something of actual insanity. The physician of a large asylum kindly allowed me to spend a week or two occasionally among his patients. This experience cured me of many romantic illusions, especially that of supposing that only a fine line divides genius from insanity. My verdict would rather be that, in one important respect at least, genius and insanity are further apart than either of them is from the ordinary state of man. If they are points on the same ellipse, then one is at the aphelion and the other at the perihelion. Both, indeed, are characterised by periods of strong disinclination, amounting at times to incapacity, for attending to outer facts; by a strong tendency to withdraw at times into some inner world, and there receive fresh light upon facts previously known. But they seem to me to deal with this inner inspiration in opposite ways.

Men of great genius, so far as I know them, however they may differ in some respects, are all marked by one common characteristic:—all develop, at the moment of greatest inspiration, the instinct to protect themselves from aberration by conducting their thinking according to complete mathematical sequence.

When the inspired fit of genius is passing off, the instinct to complete sequences is often the first to fade. If the individual has no conscious knowledge of the Law of Sequence, he is, during that portion of the inspired fit which succeeds the fading of the mathematical instinct, as prone to delusions as any lunatic; and these delusions, owing probably to the extremely sensitive condition of the brain during inspiration, tend to fix themselves on him, and often cause him to mislead his followers; whereas he who consciously knows the Laws of Thought is on his guard, and knows when his good angel is leaving him.

The genius and the hallucinate are alike prone to resent either contradiction or the attempt to argue against their delusions. But I have often tried the experiment of completing the individual's own sequence according to the mathematical formula. Genius ackowledges such addition to its own thought processes, such completion and automatic correction of them, as sound; the lunatic either flies into a rage, or loses interest in the conversation and turns to another topic, or goes away. For instance, in the course of my attempts to sanitate anti-vivisectionist minds by directing their attention to schemes for assisting biologic research on non-pain-causing lines, I meet with more or less of difficulty in persuading common-place enthusiasts to adopt a hygienic mode of activity. But an anti-vivisectionist of true genius, however rampant in his antagonism to biology as it exists, however ignorant and indifferent about science of any sort, recognises by instinct that the path of sanity and wisdom for him lies in the direction of aiding all schemes for prosecuting biology on non-pain-causing lines. The fanatic who is insane on the vivisection question, listens to no argument in favour of assisting biologic study in any manner which does not involve the incessant *denouncing of vivisection.*

I would almost venture to predict that in times to come, the test of lunacy will be, not whether the patient holds this or the other erroneous opinion, not whether he sees ghosts or hears voices or dislikes his near relations, but how he responds when the doctor completes, according to the mathematically ascertained Laws of Thought, the patient's own unfinished thought-sequences.

When I use the word "genius" I apply it only to persons of unmistakable power, who, amid whatever aberrations, have solved some of the world's problems by original and sustained thought. Many pass as inspired in small coteries, who, for any power they possess of making a true thought-sequence or appreciating it when made by some one else, might be as mad as any patient in Bedlam.

What is called Megalo-Mania, I believe to be, in itself, a perfectly normal concomitant of inspiration. It occurs in the highest order of genius as markedly as in some forms of insanity. The "modesty of true genius," about which so much has been written, is the result not of natural humility, of the absence of self-exaltation, but of the completion of a sequence of which self-exaltation is the first half. The inspired person feels sure that some thought which has come into his mind,—or, rather, the fact of its coming there,—is of more consequence to the world than any outer event. The lunatic stops at this conviction; the man of genius completes the sequence, by seeing that thoughts which come to others are also of more importance than outer events; instead of his development being arrested at the stage of Megalo-Mania, it rises to a general conviction that Inspiration is more important than organization; that all possible glory of all visible kingdoms would be dearly purchased at the cost of demoralizing that Kingdom of Heaven which is within. Not only Megalo-Mania, but many other phases of lunacy, are gone through in the minds of men of great genius, and often in their conversation with their intimate friends or trusted assistants; these phases of insane thought are not mere accidental aberrations, they are essential portions of the path to discovery; but the greater the genius the less the individual feels prompted to express his ideas in this immature stage to any persons except those who are actually working with him at his subject. Premature exposure is never a sign of strength.

There is, indeed, one terrible form of brain disease which I think is distinctly traceable to genius; not, indeed, to the patient possessing, or being possessed by, the temperament of genius, but to his not understanding the nature of the gift, to his thinking it his duty to violate its self-protecting instincts.

A man makes a discovery or two in some physical science, or a new departure in some art. He is appreciated, lauded, honoured. He is well aware that his success is due to his having been somehow led to practise some exceptional kind of thought-sequence. It occurs to him that if only mankind knew how to practise that special mode of sequence, most people could do genius-work and make discoveries for themselves in any direction they pleased. He longs to throw open the realm of genius to the world at large. To do this he must initiate the world to the special mode of sequence. If he really understood what that is, he would do as Gratry did,—write a calm and sober treatise about it, which a few thinkers would profit by, and which could do no harm. But he does not understand his own thought processes well enough to write about them abstractly; he tries to give the secret to the world in some concrete form. What concrete subject shall he select to illustrate his method? It would be of no use to write on Physiology, or Mechanics, or Botany, or Musical Composition, or whatever the subject may be by which he made his reputation; if he did that, his new book would only be supposed to be a fresh treatise on that subject; would appeal only to a circle of readers desirous of studying it; and even they would only get out of it technical instruction. He therefore selects some topic of general interest, and proceeds to try, by his method of sequence, to solve some of the problems connected with it, carrying on a process of solution, as it were, *coram populo*. He writes, or lectures, or preaches about,—say,— the relation between rich and poor, education, or the marriage laws; simply thinking aloud, and pouring out to all comers those half-inspired and very erroneous flashy half-truths which are, as I said above, the raw material, out of which appropriate correction would ultimately evolve inspired truth. But long before the poor man has arrived at the ultimate correction of his own absurdities,—which he would have done had he been able to think in peace,—the world has run away with a notion that he means so-and-so, that he is doing so-and-so,—that he is trying to induce his hearers to do so-and-so,—mistaking for seriously-given advice hints which he uttered with a misleading air of inspired earnestness because he feels them to be steps on the road to final illumination, but which he himself would be the last man in the world to dream of acting on. Before he well knows what is happening, he is involved in some tragedy, of which the antagonism and scorn of conservative opponents are the least painful portion ; the grief and anxiety of his true friends is far worse to bear ; but worst of all is the bitter remorse caused by finding that injudicious admirers are attempting to follow advice which he never meant to give. He becomes delirious from sheer moral torture, much greater than what ordinary men could bear ; much greater than anything which ordinary men are usually called on to bear ; and then the world declares that genius is allied to insanity !

The object of the present treatise is not to advocate any special method for either the detection or the cure of lunacy ; but to put within reach of those whom it may concern a clue to the stores of psychological knowledge locked up in mathematical science. It may therefore conclude here with an abstract of some doctrines towards which the Calculus of Logic seems to be tending. It has received sanction from several non-mathematical students of theology, of medical psychology and of mental science, on general grounds of psychology and of religion.

The Creed of Sanity.

Unity is the property of the Infinite, the Absolute, the Eternal. Dividedness is the property of the finite, the phenomenal, the transitory. Every attempt, either to eternalize phenomenal distinctions, or to phenomenalize Eternal Unity is contrary to the true nature of man; and tends towards the destruction of mental health. The great All is a jealous God, and will not suffer His honour to be given to any partial manifestation of good. Every finite or phenomenal good which man invests with attributes belonging only to the Infinite, avenges the majesty of the Unseen Unity by injuring the brain of man.

What force or creative energy is, in its own nature, we do not know; but we know that every mode of it with which we are acquainted works by a pulsation of contrary motions. All forms are evolved by pulsating Force, yet itself is necessarily formless.

Man is the child of this pulsating or alternating Creator; not His mere handiwork, made arbitrarily, unlike Himself; but the outcome of His very thought-processes; and sanity, for us, means thinking as He thinks, so far as we think at all. And He thinks, or, at least, works in an incessant rhythmic pulsation of alternate constructing and sweeping away. Man should imitate this pulsating activity within his own mind; his studies should alternate the formation of defined and contrasted conceptions with the unifying of those conceptions; and his religious exercises should suspend all concrete conceptions in adoration of the inconceivable Unity. If we thus embody the principle of pulsation in our thought-life, it becomes a source of constant power like the movement of our lungs; if we forget it, we waste force at each effort. False religions tend to arrest the natural fading away of things that have served their purpose, whether those things be visible forms or mental conceptions; but the token of the covenant made by Infinite Knowledge with man is the Rainbow, which no man can capture, or embalm, or enshrine; which is made by the breaking up of the one light into many colours, to fade, before long, into the unity of white light again; and which, when it fades, leaves nowhere in the world a trace of its ever having existed, except on man's heart an impression of spiritual beauty, and in his mind a knowledge of the laws of light.

It is vain that we haste to rise early and late take rest, and devour many carefully compiled text-books; to those who love the Invisible, formless, alternate-beating Unity, the knowledge which is power comes even during sleep.

APPENDIX

THE scheme alluded to in the text grew out of the Presidential Address of Dr. Wm. Hickman to the Harveian Society, January, 1883, from which I quote a few sentences :—

" . . . If necropsies were the rule instead of the exception our mortality statistics would be enormously increased in value ; something too might be gained even from the simplest and least doubtful case ; and in private practice much more may often be learned from an individual post-mortem than in public or hospital practice, the patient in the latter case having usually been but a very short time under observation, whereas in private practice the examination may explain and throw light on the history and symptoms of a life-time. The practitioner would also be led to watch, with a greatly increased interest, all the signs and symptoms of every case under his care, and a vigour and certainty would be imparted to his practice which could be obtained under no other circumstances. A large number of cases too occur in private, of which very few are seen in hospital practice, and these present a vast field for future pathological research. What are the obstacles in the way of general mortisection? First, of course, the prejudices of the public. . . . It will be difficult and will take time to overcome these prejudices, but they may and will be overcome. The public must be taught by every means to understand the usefulness and the necessity of post-mortem examinations ; they might be made to see how these would tend to their immediate advantage, in making practitioners more careful and more accurate in their diagnoses ; and if the necropsies were made by independent men, men who made pathology their special study and business, the public would have some test of the knowledge and capabilities of their advisers, by the corroboration or disproof of their opinions afforded by post-mortem examination.

"It has always been a source of wonder to me why the public do not insist rather on post-mortem examinations, and choose advisers who they know have had the greatest experience in them."

In consequence of the publication of the Address in question, a few non-medical persons attempted to found a Mortisection Society. They expressed no opinion as to the amount or kind of benefit to be expected from a more general adoption of the practice of mortisection; they proposed only to give to the medical profession a fair opportunity of testing the truth of such ideas as Dr. Hickman's. He had told a member of the Mortisection Committee that, in his opinion, *certain kinds* of vivisection are scientifically unreliable; and that he believed these particular methods might be superseded by more scientific modes of research if medical practitioners could know with something like certainty, during the lifetime of a patient, that his body would ultimately be available for necropsy. The Mortisection Committee therefore proposed to organize a system of public registration of persons willing to offer their bodies for post-mortem examination. The scheme fell to the ground, owing, not to any one's opposition, but to the indifference of both biologists and anti-vivisectionists towards a movement which did not start by making positive assertions and which contained no element of antagonism to anything or anybody. We are here concerned only with the Psychology of such indifference. From the point of view of the Mathematical Laws of Thought, it seems strange that any one should expect fresh light to illumine the darkness in which the vivisection question is enshrouded, as long as such utterances as those above quoted, emanating from serious students like Dr. Hickman, are lying unheeded, while no one has proved that they are fallacious, and no one cares to carry into effect as much of them as may prove sound. Such light *may* come, but if it does it will be of the nature of an "uncovenanted mercy," and mixed with guesses and errors; it will not be what a mathematical solution is :—the normal response to an orderly mode of appeal "from finite minds to the Infinite for Light on finite concerns."

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