# GENERAL AND PRACTICAL OPTICS

LIONEL LAURANCE

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BY

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# CONTENTS

## OPTICAL SYMBOLS AND ABBREVIATIONS

| | |
|---|---|
| *S. or Sph.* | Spherical. |
| *C. or Cyl.* | Cylindrical. |
| *Pr.* | Prism. |
| *Ax.* | Axis. |
| *Pc. or Peris.* | Periscopic. |
| *Pcx. or Pcvx.* | Periscopic convex. |
| *Pcc. or Pcvc.* | Periscopic concave. |
| *Dcx. or Dcvx.* | Double convex. |
| *Dcc. or Dcvc.* | Double concave. |
| *F. or P.F.* | Principal focal distance or focus. |
| $F_1$ *and* $F_2$ | Anterior and posterior focal distances or foci. |
| $f_1$ *and* $f_2$ | Conjugate focal distances or foci. |
| *O.* | Object. |
| *I.* | Image. |
| *Hor. or H.* | Horizontal. |
| *Ver. or V.* | Vertical. |
| *Mer.* | Meridian. |
| *D.* | Diopter. |
| +, *Cx., or Cvx.* | Plus, Convex. |
| −, *Cc., or Cvc.* | Minus, Concave. |
| △ *PD.* | Prism diopter. |
| ▽ | Centrad. |
| Λ | Metran. |
| ∞ | Infinity, a distance infinitely great. |
| ⊃ | Combined with. |
| $\mu$, *m., or n.* | The index of refraction. |
| *M., n., or* $\nu$ | A medium. |
| △ *or* $\delta$ | The difference between (applied to lines of the spectrum). |
| $w$ *or* $\omega$ | The ratio between the dispersion and refraction of a medium. |
| $\nu$ | The ratio between the refraction and dispersion of a medium. |

## MATHEMATICAL SYMBOLS AND ABBREVIATIONS

| | |
|---|---|
| *M.* | Metres. |
| *cm.* | Centimetres. |
| *mm.* | Millimetres. |
| $\mu$ | Microns. |
| $\mu\mu$ | Micromillimetres. |
| *Ft. or* ′ | Foot. |
| *In. or* ″ | Inch. |
| ‴ | Line. |
| ° | Degree. |
| °*d.* | Degree of deviation. |
| ′ | Minute. |
| ″ | Second. |
| ∞ *or* 1/0 | Infinity, a number infinitely great. |
| 0 *or* 1/∞ | Zero, a number infinitely small. |
| ∠ | Angle. |
| : | Is to, the ratio between. |
| : : | So is, as or equals (used with ratios). |
| ∴ | Therefore. |
| ∵ | Because. |
| ∝ | Varies as, proportional to. |
| ⊥ | Perpendicular to. |
| ‖ | Parallel to. |
| ∟ | Right angles to. |
| $\pi$ | (Pi) Ratio of circumference to diameter. |
| $r$ *or* $\rho$ | Radius. |
| $r^\circ$ *or* $\rho^\circ$ | Radius in degrees. |
| $r'$ *or* $\rho'$ | Radius in minutes. |
| $r''$ *or* $\rho''$ | Radius in seconds. |
| $\theta$ $\phi$ | Any angles. |
| + | Plus, addition. |
| − | Minus, subtraction. |
| ± | Either + or −. |
| × | Multiplied by. |
| ÷ *or* : | Divided by. |
| ∼ | The difference between. |
| $\sqrt{\ }$ | The square root of. |
| $\sqrt[3]{\ }$ | The cube root of. |
| $\sqrt[n]{\ }$ | The $n$th root of. |
| $x^2$ | $x$ squared ($xx$). |
| $x^3$ | $x$ cubed ($xxx$). |
| $x^n$ | $x$ raised to the power of a number equal to $n$. |
| $\overline{a+b}$ | Bond or vinculum, showing that the numbers are to be taken together. Is the same as $(a+b)$. |
| = | Equal to. |
| > | Greater than. |
| < | Less than. |

*For ophthalmic abbreviations and symbols, see "Visual Optics and Sight-Testing."*

# PREFACE

No one recognises more fully than I the errors of omission and commission to be found in the first edition of "General and Practical Optics." It has, however, apparently served the purpose for which it was designed, and I trust that some, at least, of its faults will be found remedied in this second edition, in which the subject-matter has been entirely rewritten, there being but few paragraphs of the original left untouched. In addition, the arrangement has been materially changed, some seemingly unnecessary matter having been omitted, while a large amount of new matter has been introduced, and most of the diagrams are new.

Although primarily intended as a textbook for candidates for the examination of the Worshipful Company of Spectacle Makers, it is written also for other students of optics as a reference book for those engaged in spectacle work, and as an introduction to the study of more pretentious volumes and those dealing with special branches of optical science.

In the preface to the first edition I acknowledged my indebtedness to Dr. George Lindsay Johnson. In this I have to acknowledge the valuable aid, in writing, compiling, revising, and correcting the work, of Mr. H. Oscar Wood, who has also made all the new diagrams.

As I have said in the preface to "Visual Optics and Sight-Testing," I have endeavoured to cover in the two works all that is essential for the sight-testing optician.

LIONEL LAURANCE.

# GENERAL AND PRACTICAL OPTICS

## CHAPTER I

### LIGHT

**Light.**—Everything we see around us is rendered visible by means of a form of radiant energy which is termed light. With the exception of certain manifestations such as fluorescence, phosphorescence, etc., all light has its source in bodies which are in a condition of white heat or incandescence. The source of light itself may not be visible, but the reflected light by which objects—the sky, moon, trees, houses, etc.—are seen can invariably be traced to the sun, or to some artificial source of incandescence.

It was once supposed that light was something which radiated from the eye to the objects seen, and later it was thought to be due to minute corpuscles which proceeded from a visible object to the eye at great speed, but it has now been proved that light is due to vibrations set up in the luminiferous ether by the molecular agitations of an incandescent body.

**Ether.**—This is a medium believed to occupy all space throughout the universe, penetrating between the molecules and atoms of which bodies are composed, so that every body is saturated with ether, nor can any vacuum, however perfect, remove the slightest fraction of it. Exceedingly little is known about its nature, its properties being chiefly negative, since it cannot be appreciated by any of the senses. It has been concluded, however, that it possesses density, rigidity and elasticity, properties enabling it to propagate transverse undulations or waves, which are generated by vibrations in incandescent material bodies ; these waves travel to an infinite distance without appreciable loss of energy. Ether is the connecting medium of the universe, and it is due to its presence that material bodies are capable of acting on one another at a distance, and by which such forms of energy as light, heat, magnetism, electricity, etc., are made manifest.

**Light Waves and Rays.**—Since every point of a source of light generates an oscillation which travels in every direction, let one of these parts *L* (Fig. 1) be considered a point of vibrating incandescent matter. This forms the centre of a tiny sphere whose diameter equals a wave length, and according to the accepted theory of Huyghen, every point on the circumference of this sphere forms a new centre of disturbance which generates a fresh sphere, and each of these spheres again forms fresh ones, and so on. Now, as these tiny spheres may be supposed to lie side by side overlapping each other, tangents to points on their combined circumference (which points are ends of radii from the primary centre of disturbance) will, if taken collectively, form a wave-front (*a b c d e*). As each wave-front forms a centre for the formation of a fresh row of spheres, the diameter of each sphere is equal to a wave-length. Each successive wave-front may therefore be considered as the crest (*W* or 2*W*), and the space between it and the next wave-front as the trough of a wave ($\frac{1}{2}W$ or $1\frac{1}{2}W$). But although we may consider light as advancing in the form of a simple wave-front which forms part of an ever-enlarging sphere, yet in reality the process is exceedingly complex, and cannot be entered into here.

The wave motion of the ether is always *transverse*, i.e. at right-angles to the direction of propagation of the light. The ether particles themselves

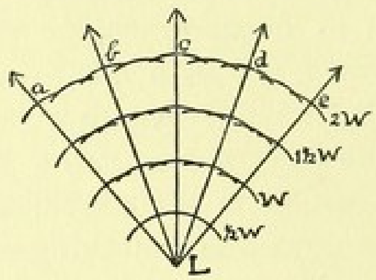



FIG. 1.

do not travel, but merely oscillate, much in the same way as a cork bobs up and down in the water as a wave passes by; or, to employ another illustration, as the vibrations of a rope, fixed at one end, travel along it when it is shaken at the other extremity.

Although light is propagated from a luminous point in a series of wave fronts, it is more convenient to consider the direction of propagation of any particular point on the main wave, which can be shown as a straight line. From the luminous point *L* (Fig. 1) the light radiates in every direction, and any line of propagation such as *La*, *Lb*, etc., is termed a *ray* of light. Thus "rays" are really the imaginary radii of the wave fronts, and as such have no material existence. For diagrammatic purposes, however, their assumption is most convenient, as they indicate the directions in which portions of the real wave-front are travelling.

**Wave Length and Frequency.**—The frequency of a wave motion is the time taken by it to perform one undulation, or to travel over a distance of one wave length. If, therefore, V be the velocity of light in mm. per second, L the wave-length in mm., and T the number of vibrations executed in one second, V = LT. In free ether (i.e. space) all waves travel with the same velocity and, therefore, it follows that the short waves of blue and violet must have a higher frequency than the longer red waves in order that their velocities may be equal. It is only when light passes into material bodies, like glass or water, that the velocities of the various waves become unequal, the natural result being the phenomenon of *refraction* (q.v.).

**Radiant Energy.**—When the temperature of a body is raised, the increased molecular activity causes a generation of ether waves of certain length and frequency, which constitutes what is termed *radiant heat.* If the temperature is raised still more, the activity is proportionally increased, so that the waves become shorter and the vibrations more rapid. Thus, when the temperature of a body reaches about 500° centigrade, it not only emits the relatively long waves of *heat*, but also the shorter waves of *light*; the difference between the two forms of radiant energy—heat and light—existing solely in the difference in length of the waves. The undulations must be of a certain shortness and rapidity in order to become "light" as distinct from "heat."

Some bodies transmit light and not heat rays, and others the reverse. Bodies which transmit the invisible heat rays without becoming quickly warmed themselves are termed *diathermanous*; those which do not transmit radiant heat without themselves becoming rapidly heated, are termed *athermanous* or adiathermanous.

The longest light waves, i.e. those of least frequency, give rise to the visual sensation of red when the temperature of a body is raised to about 500° C. On further raising the temperature of the body, shorter waves are also produced which, being of different lengths and frequencies, cause the sensation of various colours, varying from red, the longest, to violet, the shortest visible waves. White is a sensation caused by the combined action of all waves ranging between red and violet, and is produced when the temperature reaches about 1000° C.

The existence of what is known as the *infra red* waves, or those beyond the visible red of the spectrum which are too long, or too slow, to cause vision, may be shown in various ways. Thus a blackened thermometer bulb placed just beyond where the red in the spectrum ceases will show a rise of temperature, proving the existence of heat rays. Again, by employing a lens made of rocksalt, which readily transmits the long heat waves, the latter can be demonstrated when the visible spectrum is cut off.

Similarly the spectrum extends beyond the visible violet end, this portion, called the *ultra-violet*, consisting of waves whose vibrations are too rapid, or whose length is too short, to cause the sensation of sight. The existence of the ultra-violet waves can be proved by placing beyond the visible violet a screen painted with a solution of a fluorescent liquid such as quinine, which fluoresces brightly under the influence of the ultra-violet light. A quartz prism, which is very transparent to the short vibrations, must be used to produce the spectrum.

In addition to the effect on the eye, and the sensation of heat, it is obvious that light waves possess many other properties, especially the chemical actions which occur in photography, bleaching, the generation of carbonic acid, and the formation of chlorophyll necessary for vegetable life, although, for the latter, the heat rays may be equally active or may be more so than the short waves.

Thus it may be said that, in general, the spectrum within certain limits consists of the long infra-red (heat) waves, the luminous or visible portion, and the short ultra-violet actinic (chemical) waves. In addition there are the long Hertzian (electrical) waves beyond the infra-red, and what are supposed to be the X rays beyond the ultra-violet, as shown in the table on page 9.

The incandescence of the sun is, of course, the principal source from which light on the earth is derived. Impact, friction, electricity, chemical combination, combustion, in fact anything which causes increased molecular motion also may give rise to light.

**Density of Media.**—The speed with which light travels within a certain medium depends on the nature of the latter or, more exactly, on the elasticity of the ether within it; thus light travels more slowly in a dense medium, i.e. one in which its component particles are crowded together like glass, than in a rare one, such as air.

**Velocity of Light.**—Light travels in air at about 186,000 miles or 300,000 kilometres per second; the velocity is lessened in denser media, the decrease being roughly proportional to the density, although this is not invariably the case. Thus, in glass, the rate of progression is about one third less, and in water one fourth less, than it is in air. In air the speed is slightly less than in space or a vacuum. 186,000 miles is a distance equal to about eight times the circumference of the earth at the equator, a journey travelled by light in one second. From the sun it takes about eight minutes for light to reach the earth, some 93 million miles distant. At this rate light travels six million million miles in a year, and the distance of a fixed star, being so enormous, is measured in light years, thus expressing the number of years the light from the star takes to reach the earth.

**Measurement of Light-Speed.**—There are at least four methods by which the velocity of light has been measured. The earliest methods, by reason of the imperfection of optical instruments, were of necessity astronomical ones.

**Römer's Method.**—It was known that one of Jupiter's moons $M$ (Fig. 2) became eclipsed by the planet $J$ every $48\frac{1}{2}$ hours. At a certain period of the earth's annual revolution round the sun it is in opposition to Jupiter. If light were to travel instantaneously, the eclipse, and its observation by an observer on the earth, would occur simultaneously. The light, however, has to travel from Jupiter to the earth before the eclipse can be seen. Let R and r be respectively the radii of the orbits of Jupiter and the earth round the sun. Then $J\ E$ (i.e., R − r) is the distance the light has to travel at a velocity V. This time therefore will be (R − r)/V seconds after the eclipse has taken place. After six months the earth and Jupiter will again be in opposition, the earth now being at $E'$ on the other side of the sun. The eclipse will therefore be observed (R + r)/V seconds after the occurrence, the difference between the two observations being equal to 2r = 186 million miles.

Römer observed that, as the earth moved from $E$ to $E'$, the observed time steadily exceeded the calculated time. Thus he found that an eclipse observed when the earth was at $E'$ occurred 995 seconds later than when it was observed at $E$. Since the diameter of the earth's orbit is 186 million miles, V = 186,000,000/995 = 186,000 miles per second (approx.).

**Bradley's Method.**—The apparent direction of light from a star, owing to the earth's motion, makes an angle with its true direction. As the earth pursues its elliptical orbit round the sun it must move in an opposite direc-

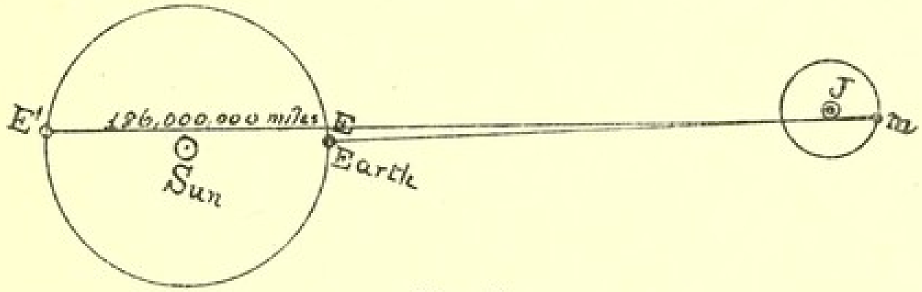



Fig. 2.

tion to that which it took six months before, so that a telescope directed to a star somewhere along a line at right angles to the earth's motion must be pointed slightly in front of the mean calculated position at the first period of observation, and a similar distance behind at the second observation. The angle which the telescope makes between the calculated and the observed position is called the aberration of the star.

Bradley knew the velocity of the earth's motion, he measured the angle of aberration, and from these data he proved the velocity of light to be,

$$V = \frac{\text{velocity of earth}}{\text{tan of angle}} = \frac{18 \text{ miles}}{\tan 20''} = \frac{18}{\cdot 0001} = 180{,}000 \text{ miles per sec.}$$

Bradley's method may be illustrated as follows; if a shot from a cannon $C$ (Fig. 3) be fired at a ship, moving at right angles to the direction of the shot, the latter will not pass through the ship at right angles to its line of travel, but obliquely as if the shot came in the direction of the dotted line $C'$.

**Fizeau's Method.**—Fizeau's method depends on the interruption of a beam of light by the teeth of a revolving wheel. The light from a source $S$

(Fig. 4)—rendered convergent by a lens $L$—falls on a plane unsilvered mirror $m$ which is inclined at 45° and situated between the lens and its focus $F$, the latter being at the teeth of the wheel. Another lens $L'$, placed at its principal focal length on the other side of the wheel and in a line with the mirror, renders the light from $F$ parallel. The beam of light is collected by a third lens $L''$, situated at a distance (say four miles), and is brought to a focus on a spherical mirror $M$, from which it is reflected, so as to return along the same path, finally forming a real image at $F$ which is viewed by the observer at $E$ through an eyepiece.

Suppose the light escapes through the first gap while the wheel is turning

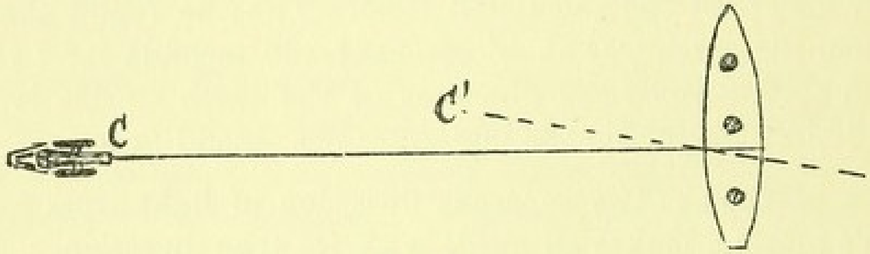



Fig. 3.

slowly, then it will, after travelling eight miles, pass through the same opening and a flickering image is seen. If the speed is greater the second tooth blocks out the light, but if still greater the light passes through the second gap, the wheel having revolved one tooth while the light travelled eight miles, and so reappears to an observer at $E$. The result is checked

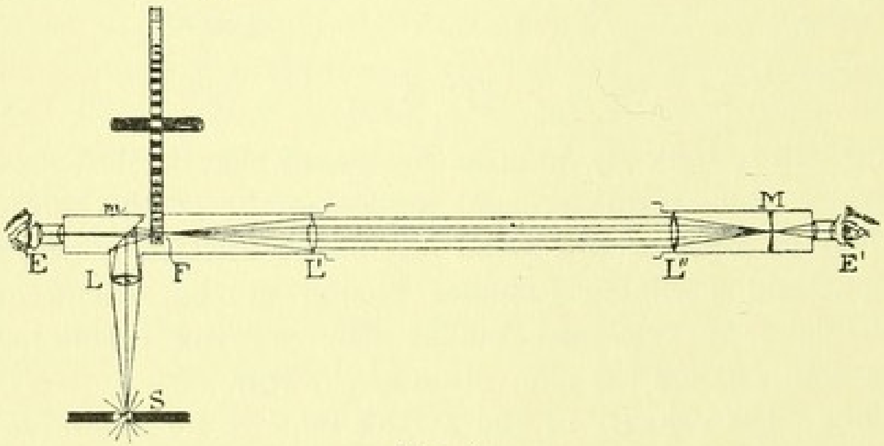



Fig. 4.

by another observer at $E'$ who sees the light through an opening in $M$. The speed of the wheel being further increased the light appears and disappears as an additional tooth or gap passes by before the light returns. The speed of the toothed wheel, the size of the teeth, and the distance between $m$ and $M$ being known, Fizeau, and later Cornu, who improved on the apparatus, found the velocity of light in air to be about 300,000 km. per second.

**Foucault's Method.**—Light (Fig. 5) is passed through a slit $S$ and a lens $L$ on to a plane mirror $M_1$, whence the light passes to a concave mirror $M_2$ placed at a distance equal to its radius. From $M_2$ the light is again reflected back to $M_1$ and retracing its path is partly reflected by the glass plate $M_3$ to the eye at $T$. If $M_1$ is then rapidly rotated it will have had time to turn through an appreciable angle during the time that the light has travelled from $M_1$ to $M_2$ and back again, so that it will not be reflected back to the same spot on the mirror $M_3$. Thus the image seen by the observer through the telescope will not be formed on the cross wires at $a$, but will be found shifted to some point $b$. If the speed be known at which the mirror $M_1$ is rotated, and the distance which the light has to travel from $M_1$ to $M_2$ and back (which in this case is equal to eight yards) the velocity of light can be calculated by the displacement of the image from $a$ to $b$ as seen through the telescope $T$.

**Solar Light**, which is white, is a combination of seven distinct colours—namely, red, orange, yellow, green, blue, indigo, and violet. Some authorities omit indigo and consider the spectrum to consist of six main colours, and some even omit the yellow, which colour, indeed, occupies but a small space

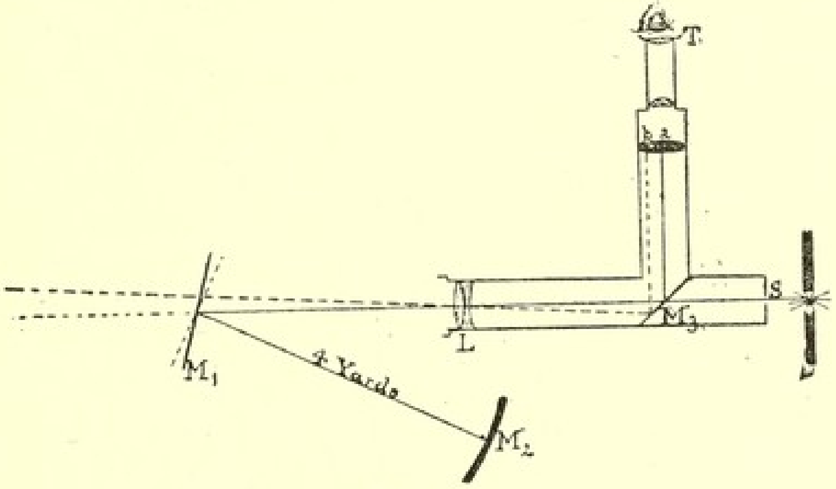



Fig. 5.

in the spectrum. The combination of these colours in correct proportion produces white light.

Sunlight is said to consist of about 50 parts red, 30 parts green, and 20 parts violet in 100, and has about 30 per cent. of luminous rays. Artificial light has a higher proportion of heat or red rays, and the proportion of luminous rays is much smaller, varying from 20 per cent. for electricity (arc), 10 per cent. for oils and coal-gas, to one per cent. for alcohol. With the exception of the electric arc and similar sources, artificial light is very deficient in actinic (violet and ultra-violet) light.

**Cause of Colour.**—Ethereal waves of certain length and certain frequency always produce a mental sensation of a definite colour, in a person of normal colour perception. Whether the length of the wave or its frequency, or both, give rise to the definite sensation, and whether the retina or the mind differentiates between the various waves, are points which are not yet precisely settled. The sensation of red is produced by comparatively long waves of low frequency, the sensation of violet by short waves of high frequency, while the remaining colours are produced by wave-lengths and frequencies between these two.

**The Spectrum.**—When sunlight passes through a dense medium, the shorter violet waves are more retarded and, if refracted, are bent to a greater extent than the longer red waves, so that the component colours become separated. The dispersed colours, caused by refraction of white light by a prism, can be seen on a screen as a bright-coloured band, called the spectrum, which contains red, orange, yellow, green, blue, indigo and violet. The various colours are not sharply separated, but merge so imperceptibly into one another that it is almost impossible to locate where one colour ends and another commences. The space in the spectrum, formed by a prism, occupied by the different colours varies with the refracting medium used for its production. If a spectrum of solar rays, refracted by a given prism of flint

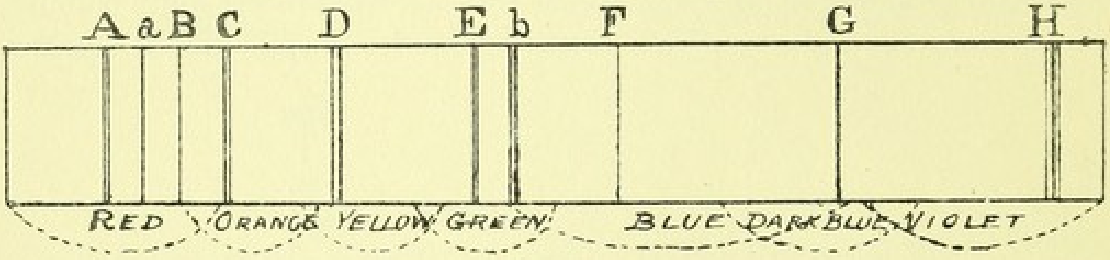



FIG. 6.

| Line. | Position in Spectrum. | Metal or Gas producing the Line. | Wave-Lengths. |
|---|---|---|---|
| | | | $\mu\mu$ |
| A | Red .. .. .. | Oxygen (O) .. .. .. | 759 |
| a | Red .. .. .. | Water Vapour .. .. | 733 |
| B | Red .. .. .. | Oxygen .. .. .. | 686 |
| C | Orange-red .. .. | Hydrogen (H) .. .. | 656 |
| D | Yellow .. .. .. | Sodium (Na) .. .. .. | 589 |
| E | Green .. .. .. | Iron (Fe) Calcium (Ca) .. | 527 |
| b | Blue-green .. .. | Magnesium (Mg) .. .. | 518 |
| F | Blue .. .. .. | Hydrogen .. .. .. | 486 |
| G | Dark Blue .. .. | Hydrogen Iron .. .. | 430 |
| H | Violet .. .. .. | Calcium (Bright Line) .. | 397 |

glass, the red being somewhat crowded and the violet drawn out, be divided into 360 parts the proportional space occupied by each colour will be approximately as follows—red 50, orange 35, yellow 15, green 50, blue 60, indigo 50, violet 100; total, 360.

**Fraunhöfer's Lines.**—When a gas is rendered incandescent the spectrum of the light, emitted by it, consists of one or more isolated *bright* lines on a dark ground which are characteristic of the gas in question; this is known as a *line* spectrum.

The solar spectrum is of the *continuous* variety caused by the intensely incandescent nucleus, crossed by dark bands or lines on the bright ground. These lines, which are very numerous and of varying widths, are called the *Fraunhöfer lines.* The experiments of Kirchhoff, Bunsen and Fraunhöfer have proved that the flame of each element radiates characteristic wave lengths which produce the bright lines of its spectrum, and that the vapour of this same element at a lower temperature transmits freely all wave-lengths except those which it would itself give out if it were incandescent, and these waves it absorbs. Thus salt, if burnt in a Bunsen flame, emits monochromatic yellow light, and white light from a *hotter* source would be robbed of precisely the same colour, i.e. yellow, on its passage through a sodium flame. The dark absorption bands, of the solar spectrum, correspond to the bright lines of specific substances, and are the result of the absorption of certain wave lengths from the hot nucleus of the sun by the relatively cooler layers of incandescent gases continually being ejected to form the outer envelope. Some of the Fraunhöfer lines are due to certain unknown substances, while some are said to be due to absorption by the terrestrial atmosphere. Absorption spectra can be produced experimentally. The chief Fraunhöfer lines are indicated by letters of the alphabet, and as they always correspond to rays of a definite wave-length, they form a convenient means of identifying any particular part of the spectrum. Fig. 6 shows their approximate positions.

TABLE OF WAVE-LENGTHS AND FREQUENCIES.

| Wave-Lengths in $\mu\mu$. | Number of Vibrations in Billions Per Second. | | Character. |
|---|---|---|---|
| 100,000,000 (100 mm.) | | | Electrical vibrations (Hertzian waves). |
| 3,000,000 (3 mm.) | | | Shortest are about 3 mm. Longest 1 meter to several miles. |
| 61,000 | 4·8 | Infra-red spectrum | Longest heat waves measured by Langley by his bolometer. |
| 8,000 | 37 | | Longest heat waves measured by Ruebens and Snow by fluor-spar prism and bolometer. |
| 812 | 370 | | Longest waves capable of being seen by the spectroscope, according to Helmholtz. |
| 750 | 400 | Ordinary visible spectrum | Red. |
| 650 | 460 | | Orange. |
| 590 | 508 | | Yellow. |
| 530 | 566 | | Green. |
| 460 | 652 | | Blue. |
| 420 | 710 | | Indigo. |
| 375 | 800 | | Violet. |
| 330 | 909 | Ultra-violet spectrum | Shortest waves visible according to Soret. |
| 210 | 1,430 | | Shortest waves visible according to Mascart. |
| 185 | 1,620 | | Shortest waves photographed through fluor-spar prism alone. |
| 100 | 3,000 | | Shortest waves photographed by means of fluor-spar prism, vacuum camera and bromide of silver plate without gelatine. |
| — | — | | X and Röntgen Rays (?). |

NOTE.—A billion is a million times a million. A micromillimetre $\mu\mu$ = one-millionth part of a millimetre or the billionth part of a kilometre. A micron $\mu$ = one thousandth of a millimetre.

**Speed and Frequency of Light.**—The visible spectrum consists of those light waves whose lengths vary approximately between 750 and 400 $\mu\mu$, and whose vibrations respectively vary between 400 and 750 billions per second. The speed of light in air is 300,000 kilometres per second, and if we express the length of the waves in billionths of a kilometre, that is, in $\mu\mu$, and the frequencies in billions per second, then by dividing 300,000 by the wave-length in $\mu\mu$ the number of billions of frequencies per second for any kind of light is obtained. The wave-length multiplied by the frequency of any part of the spectrum is a constant, i.e. $LT = V = 300{,}000$.

In the yellow, which is the most luminous part of the solar spectrum, the number of billionths of a kilometre of the wave-length is equal to the billions of frequencies per second, namely, about 548. The mean refractive index of glass, or any other substance, is expressed by that of yellow light (the D line).

**Luminous Bodies.**—Waves of light are termed *incident* when they fall on a body. A body is said to be luminous when it is, in itself, an original source of light. Every visible body, which is not in itself a source of light, is illuminated by the light it receives from a luminous source, but it may be convenient to consider that every visible body is luminous, since light is emitted or radiated from every point of it. The rays diverging from these points travel without change so long as they are in the same medium.

**Transparency and Translucency.**—A body is said to be transparent when light passes freely through it, with a minimum of absorption or reflection. It is translucent when it transmits only a portion of the light, as frosted glass and tortoise-shell. Much of the light incident on such a body is reflected, scattered or absorbed, so that objects cannot be seen clearly through it.

**Opacity.**—A substance is said to be opaque when all the rays of light, incident on it, are either absorbed or reflected, so that none traverse it.

**Reflection.**—Reflection is the rebound of light waves from the surface, on which they are incident, into the original medium. The reflection is *regular* from a polished surface and *irregular* from a roughened surface. Irregularly reflected light causes the reflecting surface to become visible ; regularly reflected light causes the image of the original source of light to be seen, the reflecting surface being practically invisible.

The rougher the surface, the greater is the proportion of irregularly reflected light ; the smoother the surface, the greater that of regularly reflected light. The proportion of light regularly reflected from a partially roughened surface is increased as the angle of incidence of the light becomes greater, so that a reflected and fairly distinct image may be obtained with very oblique incidence of the light from a body which ordinarily gives no definite reflected image.

Total regular reflection never occurs, for even a silvered mirror or highly polished surface of metal fails to reflect all the light falling on it, but the proportion reflected by metallic surfaces does not vary so much with the incidence of the light as it does with glass. Polished silver reflects some 90 per cent., polished steel some 60 per cent., and mirrors reflect about 70 to 85 per cent. of the incident light. Nor is there ever total irregular reflection; even fresh snow absorbs some of the light it receives.

**Opacity, Transparency, Absorption and Reflection.**—No substance is absolutely transparent, the clearest glass or water absorbing some of the incident light. It is estimated that below 50 fathoms the sea is pitch dark, at least to the human eye, and even glass of sufficient thickness is opaque. Again any ordinary opaque object such as stone, metal, etc., may be ground or hammered into a sheet so thin as to permit the passage of some light through it. Thus gold leaf of sufficient thinness is translucent and transmits greenish rays. It follows, therefore, that transparency and opacity are relative, and depend not only on the nature of the medium, but also on its thickness.

A body which is usually opaque may be rendered translucent by making it less capable of reflection. This fact is very often made use of in practice. For instance, if a drop of Canada Balsam be dropped on to a camera focussing-screen, and a cover glass pressed over it, the screen becomes immediately transparent at that spot, so that the aerial image may be readily focussed with a magnifying-glass, and very minute details observed. The liquid occupies the spaces between the particles of the surface and, being of the same index of refraction, converts the whole into a homogeneous refracting body which transmits nearly all the light. Moistening a piece of paper with oil or water makes it much more translucent for the same reason. The fibres of which the paper is made are of a higher index of refraction than the air, so that, when the latter is replaced by oil or water, the two indices are then more nearly alike; and being homogeneous, less light is scattered. The glass tube of a soda-water siphon is plainly visible in the water, but if the latter were replaced by oil of the same refractive index as that of the tube, the tube would be rendered invisible.

Some of the incident light is reflected from the polished surface of a transparent body, and the proportion reflected varies with the nature of the body and with the angle of incidence, it being greater as such angle increases. The proportion reflected is very small (about eight per cent.) when the light is incident perpendicularly, and it is almost totally reflected if the angle of incidence is nearly 90°.

If with perpendicular incidence practically all the light is transmitted and none reflected, and if with an extremely oblique incidence (nearly 90°) practically none is transmitted and all reflected, there must be some angle of incidence at which half the light is reflected and half transmitted and refracted. This occurs when the light is incident at about 70° with the normal to the point of incidence. Also the proportion reflected increases as the index of refraction of the medium is greater and *vice versa*. If glass is dusty, the irregularly reflected light is increased and the glass becomes more visible. Scratches on a piece of glass roughen the surface and so tend to destroy its transparency by irregularly reflecting the light. If the scratches be multiplied indefinitely, the glass ceases to be transparent and becomes translucent. Thus, in the case of every transparent body, some of the incident light is always transmitted, some absorbed and some reflected. Of the light falling from all sides on to a piece of well-polished transparent glass, about 75 per cent. is refracted and transmitted, 15 per cent. is regularly reflected and gives an image of the source from which the light proceeds, about five per cent. is irregularly reflected, and so makes the glass itself visible, while the remainder is lost, being absorbed and changed into heat, etc.

**Linear Propagation of Light.**—The propagation of light is rectilinear, and the familiar instance of sunlight, admitted through a hole in the shutter into a darkened room, illustrates this fact by the illumination of the dust particles in the air along its path. The illuminated dust renders the course of the light visible, for, were the air to be deprived of it, by filtration, the space over which the light passes would be invisible.

A circle may be regarded as the common terminal of a multitude of straight lines diverging from a point. A wave front as it advances is an arc of a circle of which the luminous point is the centre; the multitude of straight lines contained in the arc are termed rays of light. Thus the rays of light diverging from a luminous point form a cone, of which the point itself is the vertex, and such a collection of rays is called a *pencil* of light.

**Divergence of Light.**—In nature, rays of light always diverge from luminous points, but if the luminous point be very distant the angle of divergence becomes so small that the rays may be considered parallel to each other, and the luminous point is then said to be at infinity. A collection of parallel rays is called a *beam* of light.

As light radiates from luminous points which have no real magnitude, any body on which it falls must be larger than such points, the pencil from any given point constituting a cone of which the point of origin is the apex (or vertex) and the illuminated body the base. From a luminant of sensible size an innumerable number of such cones of light proceed, all having as their common base the illuminated object itself.

The angle of divergence is that angle included between the rays, proceeding from the luminous point, which fall on the outermost edges of the object; consequently the angle of divergence varies inversely with the distance between the source of light and the illuminated body, and directly with the size of the latter. Rays of light which diverge from a very distant point are always regarded as parallel, and those from a near point as divergent. This being so, there must be some distance at which divergence can be assumed, for practical purposes, to merge into parallelism.

**Parallel Light.**—In visual optics 20 feet or 6 metres marks the shortest distance from which light is regarded as parallel, and this distance, or any beyond it, is regarded as infinity, which is written thus: $\infty$. For some branches of optics a much greater distance is taken as the divergence limit. Thus in photographic optics it may amount to 100 yards or more, while in astronomy the nearest $\infty$ point may be taken as several miles. If $d$ is the angle of divergence, $a$ the aperture of the lens, and $S$ the distance of the source, the angular divergence of light is, with sufficient exactitude, found from $\tan d = a/S$. For example, suppose the source of light is at 6 M., and the pupil of the eye to be 3·5 mm. in diameter, then the visual angle of divergence will be 2′, since

$$\tan d = \frac{3{\cdot}5}{6000} = {\cdot}0006 = \tan 2'.$$

Since a divergence of 2′ is so small as to be negligible, it explains why 6 M. is considered the same as $\infty$ in this connection. At 20 cm., with the same pupil, the divergence of the light is one degree.

Similarly, therefore, if light is *converging* to a focus a great distance off, it

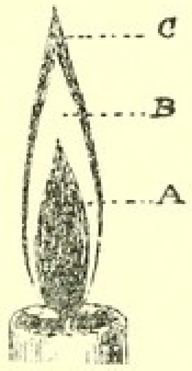



Fig. 7.

may be considered parallel—for visual purposes—at any distance greater than 6 M. from the focus. Light is never naturally convergent, but can be rendered so by means of a lens or reflector. A collection of convergent rays is also called a *pencil* of light; the apex of the pencil, towards which they are convergent, is the focus.

**The Flame.**—A flame (Fig. 7) consists of three cone-shaped portions, viz.:—

(A) The dark central portion surrounding the wick is called the cone of generation or obscure cone. It is of low temperature and is composed of gaseous products holding in suspension fine carbon particles which have not yet become incandescent.

(B) The luminous part surrounding A, called the cone of decomposition or luminous cone, in which the carbon is in a state of intense incandescence, and in which luminosity is greatest.

(C) The thin external envelope which is light yellow towards the summit and light blue at the base. It is the cone of complete combustion giving but little light, and is the main source of heat. Here the temperature is high and combustion complete on account of the free access of the oxygen of the air.

The flame in general is brighter at the top where the light predominates, and darker towards the base where heat is in excess. The outer envelope, being mixed with oxygen, is called the oxydising element, while the inner cone, consisting mainly of unconsumed gas, is called the reducing element of the flame, since at that spot metals may be reduced from their compounds.

A flame is produced by the incandescence of carbon particles which have been brought to a high temperature, the combustion, when once started, being continued owing to the heat produced by the chemical action itself. In a lamp or candle flame the material consumed is drawn up by capillarity through the wick.

Heat being produced by combustion, and luminosity being the result of the incandescence of unconsumed particles of carbon, the luminosity of a flame is low when combustion is complete, as is the case with the flame of some gases and of alcohol. It is high in a coal-gas flame, or in that produced by the combustion of oils and fats, where a considerable quantity of incandescent carbon is present. If the combustion be intensified by the introduction and intimate mixture of a sufficient supply of oxygen, as is done in the ordinary blow-pipe or Bunsen burner in which coal-gas is consumed, luminosity is decreased and heat is increased; the flame produced is then of a faint blue instead of the usual yellowish colour. The oxyhydrogen flame also gives very great heat, and yet is of a pale bluish colour and almost invisible; but when made to impinge on a lime cylinder, it renders it white hot at the point of contact, giving rise to an intensely brilliant spot of light, so that the temperature of a flame is neither indicated by the luminosity nor by the colour alone. To obtain maximum luminosity the supply of air must be neither too large nor too small. If too large the carbon is consumed too quickly, and if too small the carbon passes off unconsumed as soot.

On the other hand, although the *temperature* of the Bunsen flame, or any other source of complete combustion, is very much higher than that of luminous or incandescent sources, yet its power of *radiation* is considerably less. This can be illustrated by means of an experiment with a Bunsen burner and a thermopile, the latter being an apparatus exceedingly sensitive to radiant heat and its detection when placed some distance from a source.

With the complete combustion flame practically no rise in temperature is indicated by the thermopile, but when the oxygen is cut off and the flame becomes luminous, the index of the pile immediately swings over to a higher reading. Thus it will be seen that, for the production of radiant heat, the source must consist of rapidly vibrating incandescent particles capable of transferring their energy to the surrounding ether. For local heat, from conduction and convection air currents, the greatest temperatures are secured by complete combustion, where practically no energy is wasted in agitating the surrounding ether.

# CHAPTER II

## SHADOWS AND PHOTOMETRY

**Shadows.**—Since light travels in straight lines with the waves vibrating at right angles to their line of travel, any opaque obstacle in their path will arrest their march and produce a negative image of the object, called a *shadow*.

**Umbra and Penumbra.**—When the source of light is very small and in line with the centre of the obstacle, and the ground on which the shadow is

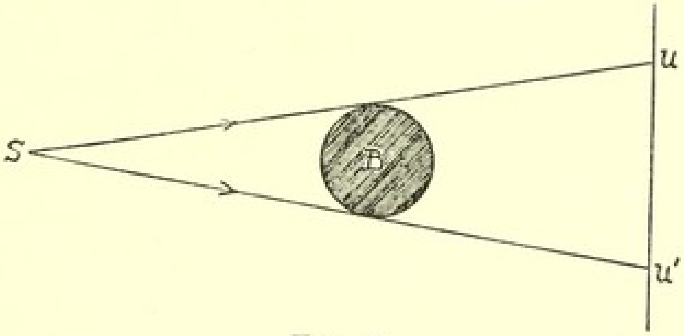



Fig. 8.

cast is at right angles to the central ray of the pencil of light, the shadow has an outline exactly corresponding to that of the body, because then, as in Fig. 8, the periphery of *B* cuts off the light equally in every direction. The shape of the shadow otherwise depends on the inclination of the screen to the

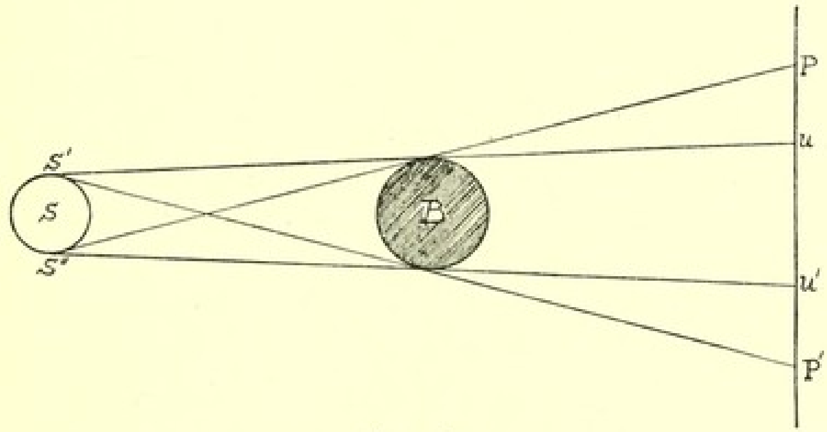



Fig. 9.

opaque body and the source of light. If the light *S* is, or approximates to, a point, the shadow is uniformly dark and its edges clearly defined as at *u u′* on the screen.

If, however, the source of light *S* is of definite size relative to the intercepting body (Fig. 9), the edges of the shadow are not sharp and the shadow exhibits two parts, viz., a very dark centre *u u′* called the umbra, from which the light is entirely cut off, and a less black outer portion *Pu*, *P′u′*, called the penumbra, which receives a certain amount of illumination. The space *Pu* receives light from *S′*, but none from *S‴*, while *P′u′* receives light from *S‴*, but none from *S′*. The area *u u′* receives light from neither *S′* nor *S‴*.

Fig. 9 shows the umbra and penumbra when the luminant *S* is smaller than the intercepting body *B*. Both become larger as the shadow is further from the intercepting body, since the umbral and penumbral cones are divergent.

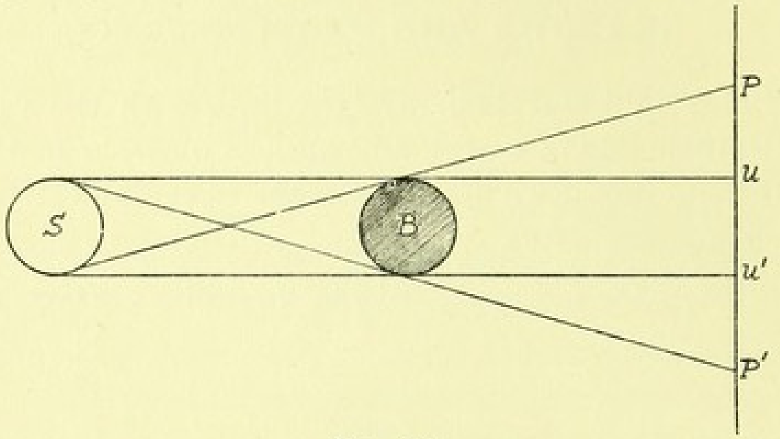



Fig. 10.

When the luminant *S* and the obstructing body *B* are of equal size (Fig. 10), the umbra is cylindrical in section and does not vary in size with its distance from the body or screen, but the penumbra increases as the screen is further away.

Fig. 11 shows the source larger than the intercepting body. In this case, as the distance between *B* and the screen increases, the umbra decreases, since the umbral cone is convergent, while the penumbra increases owing to

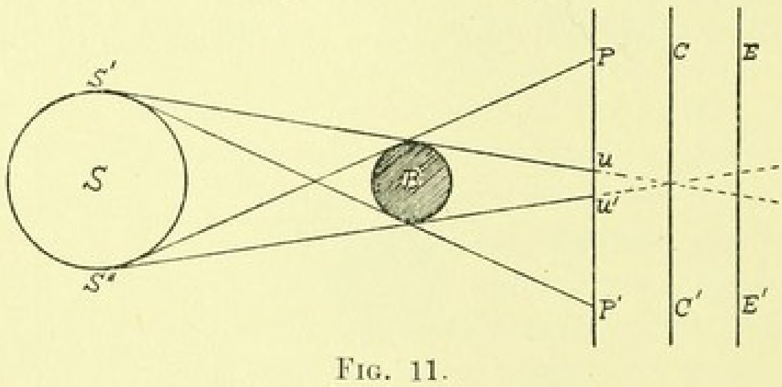



Fig. 11.

the penumbral cone being divergent; beyond a certain point there is no umbra, as when the screen is at *C C′* or beyond it at *E E′*. When the hand is held close to a wall, in a well-illuminated room, the projected shadow is almost entirely umbra; as the hand is moved away the umbra decreases and the penumbra increases until, at a certain distance, the whole shadow becomes penumbral. The larger the size of the luminant as compared with that of the intercepting body, the smaller is the umbra, and the larger the penumbra, and *vice versa*.

The umbral and penumbral portions of a shadow are not sharply separated, but merge imperceptibly into each other. Generally the brighter the light, the deeper is the shadow cast, for then the contrast between the illuminated ground and the part from which the light is totally or partially obstructed is greater than in a dull light, when shadows are barely perceptible.

**Calculations of Umbræ and Penumbræ.**—The calculations for determining the size of the umbra and penumbra are somewhat complicated and vary with the conditions under which the shadow is cast, so that every case must be worked out on its own merits, and from general principles. But if we assume that the size of the luminant is small compared with its distance from the intercepting body (and this is practically what occurs in the great majority of cases), most of the complications disappear, enabling the necessary calculations to be much simplified. Here, the angle subtended by the luminant at the intercepting body being small, either the edge or centre of the luminant may be assumed to be in line with either edge of the body, so that the edge of the geometrical shadow may always be regarded as *exactly bisecting the penumbral cone on either side.* By the geometrical shadow is meant an imaginary space on the screen equal in size to the intercepting body, so that an umbra is formed when *P*, the calculated size of a penumbral cone, is smaller than *B*, but there is no umbra if *P* is greater than *B*. In this latter case the encroachment of the penumbral disc on either side of the geometrical shadow is more than half the size of the geometrical shadow itself.

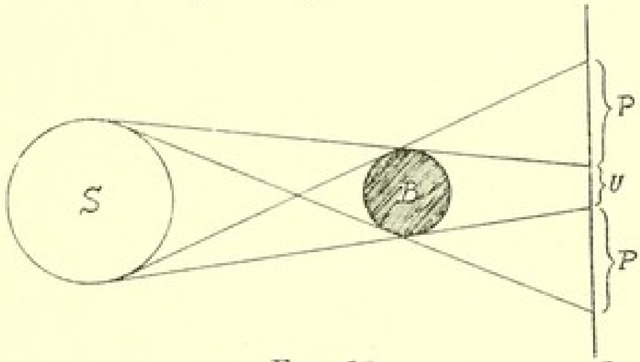


FIG. 12.

In Fig. 12 let *U* be the size of the umbra, *P* that of either penumbral cone, *S* that of the source of light, *B* that of the intercepting body, and *C* the screen. The central line of the whole shadow may be considered as coinciding with the central line connecting *S*, *B* and *C*. Now the angle subtended by *S* at the edge of *B* equals the angle of the penumbral cone, so that the penumbra on each side of *U* can be calculated from the simple proportion $P/S = d_1/d_2$, where $d_2$ is the distance of *S* to *B*, and $d_1$ that of *B* to screen. Thus

$$P = \frac{Sd_1}{d_2}$$

and from what has been said above it may be taken that (Fig. 12)—

$$U = B - P$$

and the total penumbra $= 2P + U$

If $P$ is greater than $B$, then $U$ is negative, and must be reckoned as such in finding the size of the total penumbra.

As an example, if $S$ be a square window 2 ft. in diameter, the size of $U$ and $P$ on a wall 20 ft. distant, cast by a coin 1 inch in diameter held 1 ft. from the wall, would be calculated as follows.

$$P = \frac{24 \times 1}{19} = 1{\cdot}26 \text{ inches}$$

$$U = 1 - 1{\cdot}26 = -{\cdot}26 \text{ inches.}$$

Total penumbra $= 2{\cdot}52 - {\cdot}26 = 2{\cdot}26$ inches.

Thus there is no umbra, it being a negative quantity, as on $EE'$ Fig. 11. If the coin were 2 in. in diameter, the other conditions being similar, we should have

$$P = \frac{24 \times 1}{19} = 1{\cdot}26 \text{ inches}$$

$$U = 2 - 1{\cdot}26 = {\cdot}74 \text{ inches.}$$

Total penumbra $= 2{\cdot}52 + {\cdot}74 = 3{\cdot}26$ inches.

Here the umbra is real or positive, as in Fig. 12.

When the size of the luminant is unknown, or is inaccessible, the size can be calculated from

$$P = d_1 \tan a$$

when $a$ is the angle subtended at the edge of $B$ by the luminant. The values of $U$ and of the total penumbra are found as before, after $P$ has been calculated.

Thus if $B$ is 3 inches in diameter, and 100 inches from a wall, $S$ being the sun subtending an angle of 30′,

$$P = 100 \tan 30' = 100 \times {\cdot}0087 = {\cdot}87 \text{ inch.}$$

$$U = 3 - {\cdot}87 = 2{\cdot}13''$$

The total penumbra $= {\cdot}87 + 3 = 3{\cdot}87$ inches.

**Shadows cast on the Ground.**—In the case of shadows cast by vertical objects on to an horizontal plane, generally a simple proportion will suffice. For example, what is the length of the shadow cast by a stick 3 ft. long, 20 ft. from a small lamp 10 ft. from the ground? Then, if the length of the shadow be x, the distance of the lamp to the end of the shadow is 20 + x. and 20 + x : x as 10 : 3; therefore

$$(20 + x)/x = 10/3 \text{ or } 10x = 60 + 3x.$$

$$7x = 60, \text{ and } x = 8\ 4/7 \text{ ft. which is the length of the shadow.}$$

**Shadows cast by Lenses.**—A concave lens, when placed between a small source of light and a screen, casts a shadow like a semi-opaque body. The transmitted rays being divergent, only very few impinge on the screen immediately behind the central portion of the lens, and the diverged rays fall on the screen away from the axial line, on a space which receives increased illumination, so that the shadow is surrounded by a luminous zone. The luminous zone becomes larger and fainter as the distance between the screen and the lens is increased. A convex lens throws a very bright image on a screen if placed near the focus, because it condenses to a small area all the light passing through it. The bright area is surrounded by a shadow, this being the area from which all light is excluded. If bright light be passed through a prism the space on a screen immediately behind it exhibits a shadow, the light deviated by the prism falling on another part of the screen, which, being also illuminated directly, exhibits there a bright area.

**Intensity of Illumination.**—In order to illustrate how the intensity of illumination varies with the distance between a source of light and an illuminated area, let the source of light, say a candle flame, be supposed to be at the centre of a sphere of one foot radius, and let the intensity of the light at the surface be considered as unity. The area of a sphere is equal to $4\pi r^2$, r being the radius. Now if the radius of the spherical envelope be increased from one foot to two feet, its area will then be quadrupled, since the superficial area of a sphere varies as the square of its radius, and there-

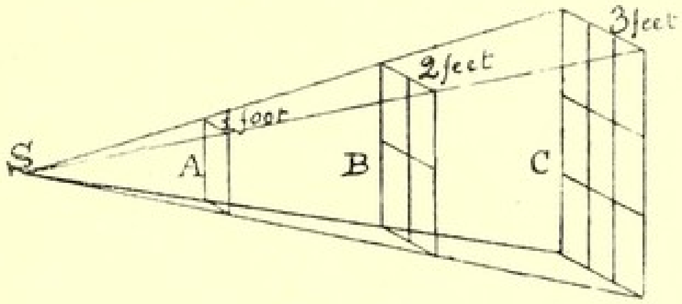



FIG. 13.

fore the amount of light received on each point of the sphere is one-fourth of what it was when the radius was one foot. If the sphere be three feet in radius its area will be increased nine times. In this latter case, the available light is distributed over nine times the area of the one foot sphere, so that the intensity of illumination over a given area is but one-ninth that of the first sphere, and in this way the intensity may be calculated for a sphere of any size.

**The Law of Inverse Squares.**—Since any flat surface virtually forms a portion of a sphere having the source of light for its centre, it may be stated, without much error, *that the illumination of a flat surface also varies inversely with the square of its distance from the source of light.* This distribution of the illuminating agent is illustrated in Fig. 13.

Let *S* (Fig. 13) be the source of light and *A*, *B* and *C* screens subtending equal angles, placed vertically at distances of 1, 2 and 3 feet respectively. The same amount of light from *S* is received by all, but *C*, being at a distance from *S* which is three times greater than that of *A*, is superficially nine times as large; and it follows, therefore, that each unit of area of *C* receives only 1/9th of the quantity of light received by each similar unit of

*A*, while *B* at 2 feet receives 1/4th. If at a given distance, say 1 foot, a certain intensity of illumination L is obtained from a lamp, and the lamp be moved to a greater distance, say 9 feet, then the intensity becomes $1/9^2 = 1/81$ of the illumination received at a distance of one foot. If it be increased to 10 feet it will require $10^2 = 100$ luminants to obtain an equal intensity as at 1 foot.

**Obliquity of Illuminated Surface.**—The intensity of illumination depends also on the inclination of the surface to the light, with which it varies as the cosine of the angle which the surface makes with the normal.

Suppose for example parallel light impinges on a vertical screen *A B*. If *A B* be inclined to the position *B D*, so that the angle of inclination $A B D = 60°$, then only those rays between *C* and *B* will fall on *B D*, as in Fig. 14. Now $\cos C B D = B C / B D = 1/2$. Also from inspection it is clear that *B C* is half of *B A*, which equals *B D*. Therefore, if the screen be nclined 60° from the vertical it will receive half the light that it does when it is vertical.

Suppose L be the amount of light falling on a unit of A, the area of the

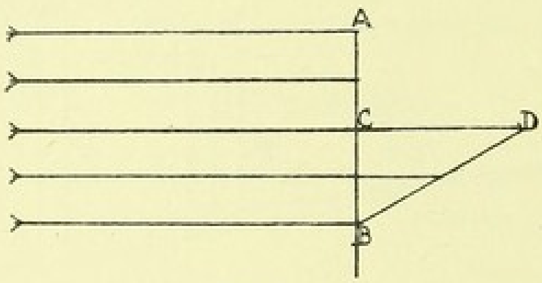



Fig. 14.

screen, 1 metre from the light; then if the screen is inclined at 45° to the normal, and removed to five times the distance, the intensity of illumination per unit area is $\cos 45°/d^2 = \cdot 7071/25 = \cdot 028$, or about 1/36 of the light received on the screen at 1 M distance. The total amount of light received each instant is = L A, and the amount of light received on the screen inclined at 45° is therefore L A/36. This holds good also for the light reflected from a surface, as can be seen from Fig. 14, where the oblique surface *B D* = *A B*.

**Apparent Exceptions.**—An object or source of light appears equally bright at all distances from the eye. The brightness of an object varies inversely as the square of the distance, so that an object at one yard is four times as bright as one at two yards, but at the same time the image on the retina occupies four times the area, so that it is only a fourth as bright as it would be if the object were twice as far away. Thus the light gained by bringing the object nearer is exactly neutralized by spreading it over a proportionately larger area. It may therefore be said that the law of inverse squares holds good only for light received directly on a screen, and that if it passes through a lens system so as to form an image, as in a camera or the eye, the brightness of the image is the same whatever the distance of the object may be, provided the distance between lens and screen is not altered.

A luminous or illuminated surface appears equally bright at whatever angle it is seen. This apparently contradicts the law of cosines, but although an inclined surface receives less light, the area perceived is correspondingly diminished. Therefore its brightness as perceived by the eye is the same in both cases since the foreshortening which the tilted reflecting surface undergoes is, like the amount of light it receives, proportional to the cosine of the angle of inclination.

The sun and moon appear as flat discs and not as hemispheres, since their surfaces are apparently equally illuminated, and in the same way a cannon ball or cylinder of metal, heated white hot, appears quite flat.

When light is condensed by a lens or mirror, the illumination of a screen varies directly as the square of the distance up to the focus ; beyond the focus it varies inversely as the square of the distance.

**Photometry.**—The measurement of the luminosity of a light source, or of the *illumination* of a surface, is termed photometry, and the instrument or apparatus employed is called a photometer.

A luminous source, unless it be a minute point such as a star, has a definite surface which is seldom of equal luminosity throughout. The quantity of light emitted varies at different points of the surface, but the sum of the light emitted from every point is the total luminosity, and it is this which is measured by the photometer. It is necessary to differentiate between luminosity, or the illuminating power of the source light, and illumination or the amount of light received from the source of a body. The intrinsic intensity of *luminosity* I is the mean quantity of light emitted normally from a unit of surface. It is expressed by $I = \phi/S$, where $\phi$ is the total amount of light emitted, and S is the area of the luminous source. The intensity of *illumination* is the total amount of light which falls on a unit of the illuminated surface.

The power of a light source is expressed in "standard candles" as described in the next article ; the term "candle-feet" expresses the luminosity of so many standard candles at 1 foot distance.

**Photometric Standards.**—The usual standard of illumination in Great Britain is that given by a sperm candle $\frac{7}{8}$ inch in diameter, $\frac{1}{6}$ of a pound in weight, and burning 120 grains per hour. It has a variation of about 20%. The luminosity of gas, with an ordinary burner, is equal to that of from 12 to 16 British candles (B.C.).

There are various other photometric units, among them the following:—

In Germany the standard is the Hefner-Alteneck lamp, called a "Hefner-lamp" (H), having a cylindrical wick 8 mm. in diameter burning amylacetate, the flame being 40 mm. high. It is correct to about 2%.

The "Pentane" standard consists of a mixture of pentane gas and air, which is burnt at the rate of ½ cubic foot per hour; the flame is circular, 2½ inches high and ¼ inch in diameter. There is neither wick nor chimney to the flame. Pentane is a volatile liquid, like naptha, prepared from petroleum. The form designed by Vernon Harcourt is a 10 candle-power standard, and is largely used in this country. It is said to vary less than 1%.

The French "Carcel" is a lamp of special construction burning 42 grammes of colza oil per hour.

The "Violle" or absolute unit was the standard invented by M. Violle, and adopted at the International Congress at Paris in 1884. It consists of the light emitted from a square cm. of platinum heated to its melting point. Of all the standards it is the most exact and reliable, but it is expensive and difficult to apply.

The International Congress of 1890 adopted as the standard the "Bougie-decimale" or decimal candle, the unit illumination of a surface being that produced by one bougie-decimal at one metre.

The British candle and the bougie-decimal have about the same intensities.

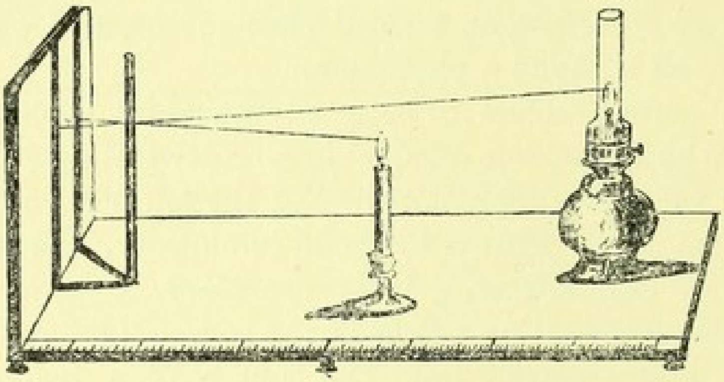


Fig. 15.

The "Carcel" equals about 9½ candles, and the "Violle" unit about 20 candles. Thus 20 bougie-decimals = 19·75 B.C. = 22·8 Hefner = 2·08 Carcel = 1 Violle.

**Measurement of Light Sources.**—Photometry consists in making a comparison of the unknown illuminating power of any source of light with that of a standard unit. Direct comparison would be difficult, but the stronger light can be placed at a greater distance, where it produces an intensity of illumination equal to that of the standard light at some shorter distance. The illuminating powers of the two sources of light are respectively as the squares of the distances at which, on a given surface, they produce equal intensities of illumination. If we represent the respective luminosities of the source to be measured and that of the standard candle by L and C, and the distances of the two when they are equal in intensity by a and b, then

$$\frac{L}{a^2} = \frac{C}{b^2} \qquad \text{or} \qquad L = \frac{Ca^2}{b^2}$$

If a standard candle at 1 ft. and light at 4 ft. give equal intensity of illumination at some common point, then the greater luminant is $1 \times 4^2 =$ 16 c.p.

Four candles 4 feet from a screen have the same effect as one candle at 2 feet, for $2^2/4^2 = 4/16 = 1/4$.

**The Rumford Photometer.**—The *shadow* or *Rumford* photometer consists of a vertical white screen before which is placed a rod. The standard candle is placed (preferably at one foot) in front of the screen and the rod casts a shadow. The lamp or other luminant (Fig. 15) to be measured is placed so far away that the shadow cast by the rod from its light is of equal intensity to that of the other. The space on the screen, occupied by the candle's shadow, is illuminated only by the light from the lamp, while that occupied by the lamp's shadow is illuminated only by the candle. It is these intensities of illuminations that are actually compared, although apparently it is the shadows themselves. The lights should be placed so that the two shadows lie near to each other without overlapping. The luminant measured is of so many candle power according to the distance at which the shadow pertaining to it equals in depth that pertaining to the standard candle, for then $L/a^2 = C/b^2$.

**The Bunsen Photometer.**—The *grease spot* or *Bunsen* photometer consists of a sheet of white paper, suitably mounted in a frame, on which there is

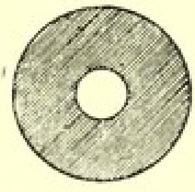


FIG. 16.

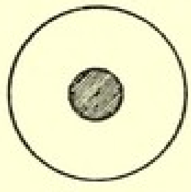


FIG. 17.

a spot rendered semi-transparent by grease or oil. If the paper be viewed on the side remote from the candle the grease spot looks lighter than the balance of the paper, because more light penetrates (Fig. 16). Viewed from the other side, the greased spot looks darker, because less light is reflected from it than from the rest of the paper (Fig. 17). Used as a photometer, the paper is placed one foot from the standard candle, the light from which is totally reflected by the ungreased part of the paper and transmitted to a great extent by the grease spot. The luminant to be tested is placed on the other side of the screen at such a distance that the amount of light from it, transmitted by the grease spot, equals that passing the other way; then the paper appears of uniform brightness all over. If we take one foot as unity, then the candle power of the light to be tested will be equal to the square of its distance in feet from the grease spot.

**The Slab Photometer.**—The *paraffin slab* photometer consists of two thick slabs of solid paraffin separated by an opaque layer of tin foil. The two lights to be compared are placed one on either side, and their intensities are compared by viewing the sides of the two slabs simultaneously.

**The Lummer-Brodhun Photometer.**—This photometer is largely used in scientific laboratories, being accurate to about 1%. Its superiority over the Bunsen and some other photometers is due to the fact that, with these, the two images to be compared cannot be seen simultaneously. With the Lummer-Brodhun instrument only one combined image is seen by one eye.

The instrument (Fig. 18) consists of a rail on which the two luminants $L_1$ and $L_2$ can be made to travel at right angles to the opaque screen *A B*, which is whitened on both sides. From *A B* the light is reflected to the two mirrors $M_1$ and $M_2$ and thence through the cube of glass *C D* made of two right angle prisms cemented together, the hypothenuse of one of which is partly cut away.

The observer looks through a short telescope placed in front of *D*. The light from $L_1$ which reaches the telescope passes through the central cemented portion of *C* and *D*, while that from $L_2$ is reflected from the peripheral portion of *D*. The two lights therefore enter the eye simultaneously in two

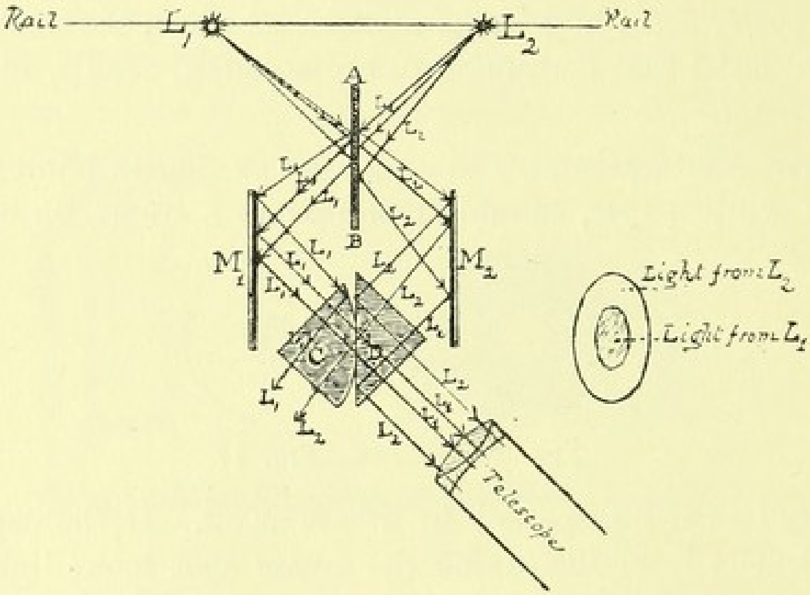



FIG. 18.

concentric rings, as shown in the figure, the lights being moved to and fro along the rail until the two circles appear equally bright.

**The Simmance-Abady "Flicker" Photometer.**—This consists essentially of a white circular disc or wheel, the edge of which is peculiarly bevelled by being "chucked" eccentrically at two positions with the turning tool set obliquely at 45°. Thus the periphery of the wheel, when revolved, presents a bevel of 45° on the one side, say the right, and no bevel on the left, then graduates to a knife edge, and finally to a bevel of 45° on the left and no bevel on the right.

This wheel is so fixed in a box that part only of it projects, and immediately in front of it, but leaving its projecting portions unobscured, there is a sighting tube carrying a Cx. lens for magnifying purposes. The box contains a clockwork arrangement by means of which the wheel is made to revolve at a rapid speed. The box itself is fixed on a bar 60 inches long, scaled in terms of a standard candle, and along which the apparatus can be freely moved.

The two luminants which are to be compared are placed one at each end of the bar, and the light from them falls on that part of the revolving disc which projects from the box. When the light falls on the bevelled edge at 45° it is reflected, and passing through the sighting tube, is seen by the observer. When incident on the unbevelled part of the disc, the light does not pass through the sighting tube, so that each luminant is alternately *light* and *dark* to the observer's eye, and both are light at the same time when the knife edge is immediately in front of the sighting tube. Then when the intensities are equal the light is absolutely steady, while it flickers when they are not. If there is flickering the apparatus is moved until this disappears, and the position is found where $L/a^2 = C/b^2$. The smallest alteration of the position of the apparatus towards either light causes flicker. The test is made more sensitive, and the point of *balanced intensities* more exactly located, when the speed of revolution of the wheel is lessened. The apparatus can be set obliquely for measuring lights at any angle.

**Photometry of Coloured Lights.**—One of the great difficulties of photometry is the difference in the nature and colour of various lights ; and the comparison or measurement of actually coloured or monochromatic lights is still more difficult, or rather impossible, by ordinary photometry.

The eye, although fairly accurate in judging the difference of hue of two sources, is very deficient in the comparison of the relative intensities of two differently coloured lights. These difficulties seem, however, to be obviated by the Simmance-Abady photometer. Here the rapidly alternating light from the sources does not afford the eye sufficient time to appreciate the difference of colour but only their difference of intensity, since the flicker depends on intensity of illumination on the two sides of the bevelled disc, and is independent of the colour of these illuminations.

Therefore by the flicker photometer coloured lights, and therefore also the transmissive qualities of coloured and smoked glasses, can be compared and measured. By it also the illuminating power of the effect of daylight can be measured as well as that of different sources of artificial light. Coloured lights may, however, be compared by occlusion, using for the purpose a serie of properly graduated smoked glasses.

**Calculations in Photometry.**—Having by means of a photometer made the intensities of illumination equal, the candle power of the luminant is calculated from the formula $L = Ca^2/b^2$. When b is unity (say 1 foot) of course no division is necessary, as the square of 1 is 1. Thus if the luminant at 5 feet is equal to the standard candle at 1 foot, the former is of $5^2 = 25$ c.p. If the candle is at 2 feet and the luminant at 8 feet the latter is $8^2/2^2 = 64/4 = 16$ c.p.

To compare the intensity of illumination of two sources of light L and

C of different powers, if L be 30 c.p. placed at 20 ft., while C is 200 c.p. at 70 feet, their relative intensities are

$$L = \frac{30}{400} \qquad \text{and} \qquad C = \frac{200}{4900}$$

$$\text{so that the intensity } C = \frac{200}{4900} \times \frac{400}{30} = \frac{5}{9} \text{ L (approx.)}$$

The relative distances for equality of illumination of two sources of 9 c.p. and 36 c.p. are as $\sqrt{9} : \sqrt{36} = 3 : 6$ or as $1 : 2$.

What candle power in a lamp at 100 feet would give the same illumination as one of 1,000 candles at 30 feet?

Now since $$\frac{L}{a^2} = \frac{C}{b^2} \therefore \frac{L}{100^2} = \frac{1000}{30^2},$$

or $900\text{ L} = 10{,}000{,}000$, so that $L = 11{,}111$.

At what distance should an arc lamp of 1,200 c.p. be placed so as to give an illumination three times as great as that of an incandescent light of 70 c.p. at 15 feet?

$$\frac{70}{15^2} \times 3 = \frac{1200}{b^2}$$

therefore $210b^2 = 1200 \times 15^2$; that is $b = 36$ feet (approx.)

The c.p. needed at 13 feet to give on a wall an illumination of 5 candle feet is $L/13^2 = 5$, or $L = 845$ c.p.

**Neutral Glasses.**—The absorption of light by smoke glass can be calculated fairly closely by means of a simple photometer such as the Bunsen. Take any two sources of light, A and B, balance them photometrically in the usual way, and measure the distance d in feet or inches of one of them, say B, from the screen. Then interpose the smoke glass to be tested between A and the screen, when it will be found necessary to withdraw B in order to secure a second balance; let this distance be b. Then the relative intensities of illumination of A, with and without the smoke glass, are as $1/b^2 : 1/d^2$. If the first distance d is unity, the fraction of light transmitted by the glass is $1/b^2$, or $100/b^2$ per cent., and the amount blocked out is

$$\frac{b^2 - 1}{b^2} \qquad \text{or} \qquad \frac{100\,(b^2 - 1)}{b^2}\,\%.$$

It will be noticed that only B is moved, A remaining fixed; nor does the actual distance of the latter from the screen affect the result in any way. The smoke glass should, of course, be sufficiently large to cover the light completely, but its position between A and screen is immaterial.

Thus if B, when at 2 ft., balanced A, and had to be moved to 3 ft. when a smoke glass was interposed, the light transmitted is $2^2/3^2 = 4/9$ or 45%, and the amount blocked out is 5/9 or 55%.

**Coloured Glasses.**—To measure the absorptive or transmissive power of a coloured glass the method described above can be employed, but, for the reason given previously, ordinary artificial lights, which are generally white or yellowish, cannot be employed alone. To overcome this difficulty the following procedure may be followed. Suppose the glass to be measured is green. Place over the two sources A and B a green glass lighter in tint than the one to be measured ; this renders the light uniform, though duller, in tint, and the necessary measurements can then be carried out exactly as for neutral glasses.

**Small Apertures.**—Since light travels in straight lines, if that from a candle be allowed to pass through a small aperture on to a white screen, an inverted image of the flame is formed on the latter. The relative sizes of image and object are as their respective distances from the aperture; thus they are equal in size when the two are equi-distant from the aperture. The image is smaller if the screen be brought nearer to the aperture, or if the candle be moved further away, and vice versa. Generally the smaller the aperture, the sharper but less bright is the image. The shape of the small aperture does not materially affect the distinctness of the image, nor does it have any appreciable effect on its shape, because it is immaterial whether the image be made up of innumerable circles, squares, triangles, etc., of confusion so long as they are sufficiently small to lose their identity to the eye in the slight overlapping which takes place between the images of adjacent points of the object. This is seen when the sun shines through the gaps in the foliage of a tree. Each of these gaps varies in size and shape, but the luminous images of the sun form bright discs on the ground, all identical in shape unless the gaps are large.

In order that a distinct image of a flame may be seen on a screen, it is necessary that the rays from each point of the luminous body should have a separate focus on a screen. This may be said to occur when the light passes through a minute aperture, because then only a very narrow pencil of light —the cross section of which is similar in shape to that of the aperture—from each point can reach the screen, and for the same reason the image thus formed is faint. If twenty apertures be made near one another, twenty images of the flame will be seen on the screen, and the number of images will increase with the number of holes, until the images will so overlap one another that it will be found impossible to distinguish them separately, in which case there will be a general illumination of the screen.

Although the smaller the pin-hole the better is the image defined, yet if the aperture be too small the image is blurred by diffraction. Hence the aperture must be theoretically that diameter which is too small for diffusion and too large for diffraction to blur the image. The aperture is found from $\sqrt{4f_2\lambda}$, where $f_2$ is the distance of the screen from the aperture, and $\lambda$ is the wave-length, this being ·0004 for photographic and ·0006 for visual effect. A (the aperture), $f_2$ and $\lambda$ are expressed in mm. If $\lambda$ be a constant ·0004, and $f_2$ be in inches, we can simplify the above to $A = \cdot 2\sqrt{f_2}$, the value of A being in mm. The intensity of the light is $A/f_2$. The respective sizes of object and image $O/I = f_1/f_2$, where $f_1$ is the distance of the object.

## CHAPTER III

## REFLECTION AND MIRRORS

A *normal* is a straight line perpendicular to a given point, as *P C* in Fig. 19, and the *angle of incidence* is that which an incident ray makes with the normal at the point of incidence.

**Irregular Reflection.**—When light falls on an unpolished surface such as ground glass it is, owing to the irregular nature of the surface, incident at all conceivable angles, at each point of the surface. The incident light is broken up so that each point of the surface, giving rise to a fresh series of waves, becomes a source of light. No image is therefore formed either of the original source, or of any external object, but the diffused light diverging in every direction renders the surface visible, no matter from what

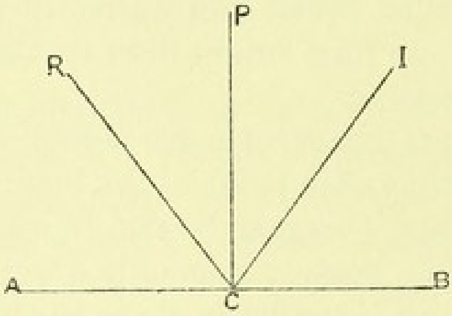


Fig. 19.

direction it is viewed, and it is either coloured or white according as some wave-lengths are, or are not, absorbed.

**Regular Reflection.**—When light falls on a smooth polished surface it is regularly reflected in definite directions according to the following laws:—

(1) The angle of reflection is equal to the angle of incidence.

(2) The incident and reflected rays are in the same plane as the normal to the point of incidence, and lie on opposite sides of it.

**Oblique Incidence.**—In Fig. 19, *A B* is a reflecting surface at which the ray *I C* is incident at the point *C*, and reflected in the direction *C R*. *P C* is the normal to *A B* at *C*, and the angle of reflection *R C P* is equal to the angle of incidence *I C P*. The perpendicular divides equally the angle *I C R* between the incident and reflected rays, and all three lines are in the plane of the paper.

**Perpendicular Incidence.**—If the ray be incident in a direction $PC$ normal to the surface the angle of incidence is zero, and therefore the angle of reflection is also zero; the ray is thus reflected back along its original path.

**Images.**—An image of a point is formed when the light, diverging from it, is caused, by reflection or refraction, to converge to, or to appear to diverge from, some other point. An image is said to be *real* or *positive* when the reflected or refracted rays from the original object point are made to *converge* and *actually meet* in the image point. If the original rays, after reflection or refraction, are *divergent*, they are referred back by the eye to an imaginary image point, and the latter is then said to be *virtual* or *negative* (see page 86). Similarly the real or virtual image of an object is made up of the real or virtual images of its innumerable points.

A real image can be received and seen on a screen, or it can be seen in the air, where it actually exists. A virtual image cannot be formed on a screen; it is only mentally conceived where it appears to be.

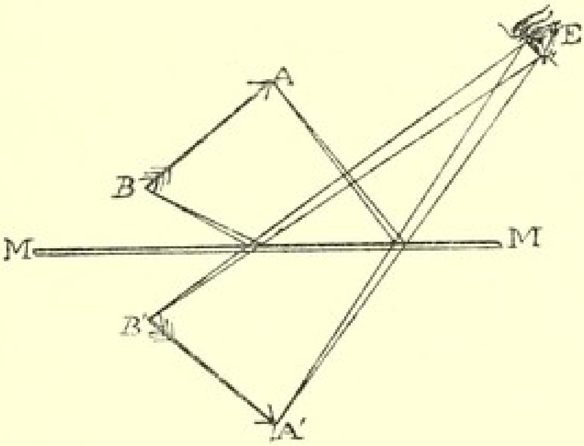



Fig. 20.

**Mirror.**—A mirror is an opaque body with a highly polished surface. It is usually made of glass backed by a film of mercurial amalgam, or coated with an extremely thin layer of silver.

**Reflection by Plane Mirror.**—If a beam of parallel light falls on a plane mirror all the rays, having similar angles of incidence, are reflected under equal angles, and are therefore reflected as parallel light. If a pencil of divergent rays be thus incident, after reflection they are equally divergent, and appear to come from a point as far behind the mirror as the original luminous point is situated in front of it. Accordingly, if an object stands in front of a plane mirror the rays, diverging from each point on it, are reflected from the surface of the mirror and enter the eye of an observer as so many cones of light diverging from so many points behind the mirror, and these points, from which the light appears to diverge, constitute the virtual image of the original object.

If the object is parallel to the surface of the mirror the image is also parallel; if the object is oblique to the surface the image forms a similar angle with it.

**Construction of Image.**—The image can be graphically constructed by drawing straight lines from the extremities of the object, perpendicular to the mirror or plane of the mirror, and continuing such lines as far behind the mirror as the object points are in front of it. Thus, in Fig. 20, if a line be drawn from *B* to *B′*, and another from *A* to *A′*, and *B′ A′* be connected, the image *B′ A′* is obtained. Rays diverging from *A*, after reflection, enter the eye *E*, and are projected to a virtual focus at *A′*, from which point they appear to diverge. Those from *B* are projected to *B′*, so that *A′ B′* is the virtual image of *A B*. *A′* is apparently as far behind *M M* as *A* is in front of it; so also *B* and *B′* are equally distant from *M M*. The complete image is erect and corresponds exactly as regards shape, distance, and size to the object itself, the relative directions of the rays from each point on the object being unchanged by the reflection.

**Lateral Inversion by Reflection.**—The image is, however, laterally inverted, the right hand of a person becoming the left of his image in the

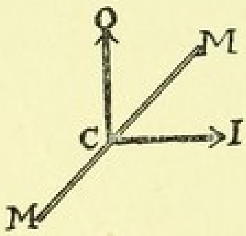


Fig. 21.

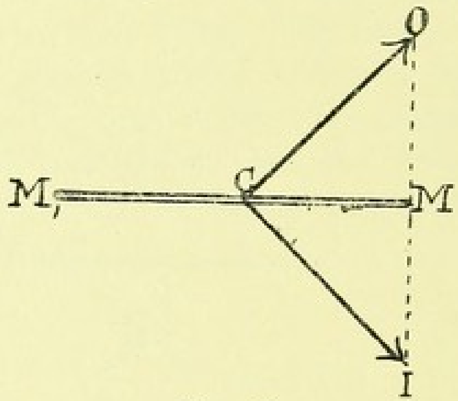


Fig. 22.

mirror, and vice versa. If the eye regards *A B* (Fig. 20) directly, *A* is to the right of *A B*, but looking into the mirror *A′* is seen to the left of *A′ B′*. If the top of an inverted page of printed matter be held obliquely downwards against a mirror the letters will be in their true order from left to right, and at the same angle to the mirror, as the page, but they will be upside down. Engravers sometimes use a mirror in front of the letters or objects they wish to draw on a wood-block and copy the image they see in the mirror. On taking an impression of the block the letters or objects are in their right position.

**Distance of Image.**—If a person stands in front of a plane mirror, say at 2 ft., and looks into it he sees an image of himself at a distance of 4 feet. If an object is placed in contact with a glass mirror its image appears behind the silvered surface, and only twice the thickness of the glass itself separates object and image, although the image appears rather nearer owing to vertical displacement by refraction. If the mirror is of polished metal the two are in contact.

**Position of Image.**—Since the angle *O C M*, between the mirror *M M* and the object *O C* (Fig. 21) and the angle *I C M*, between the mirror and the image $C\,I$, are equal, it follows that the angle $O\,C\,I$ between the object and the image is twice as large as either ; therefore if the mirror be placed at an angle of 45° with the object, the object and image are at right angles to each other, as is shown in Fig. 22.

**Angular Displacement of Image.**—If a mirror be turned through any angle the image will move through twice that angle. This is easily proved from the first law of regular reflection, for since the angles of incidence and reflection are equal, it follows that the total angular displacement between the incident and reflected rays is twice the angle of incidence. But the angle of incidence of a ray is the same as the angle of inclination of the mirror. Therefore any reflected ray or image must turn through twice the angle of inclination of the mirror, and must travel at twice the angular speed. This fact must be allowed for in the construction of the sextant. In the reflecting galvanometer it is an advantage in that it doubles the delicacy of the readings.

**The Sextant** (Fig. 23) is used to measure the angle subtended at the eye by the sun and the horizon, from which the angular elevation of the sun

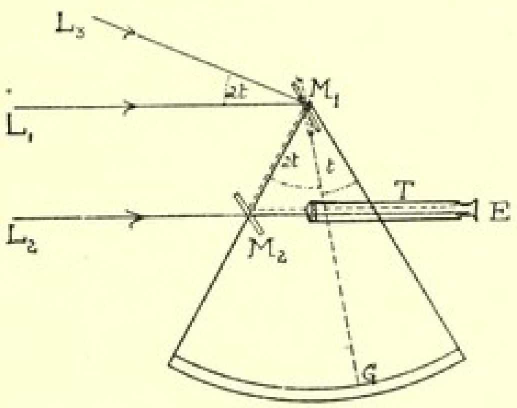



Fig. 23.

can be calculated. It also serves to measure the angle between two inaccessible objects.

A small mirror $M_1$ revolves about a horizontal axis to which is attached a pointer $G$ moving over a scale of degrees. $M_2$ is a small fixed mirror of which one half is silvered and the other half is clear, and is so inclined that when $M_1$ and $M_2$ are parallel the pointer indicates zero on the scale. T is a small telescope so directed forwards that it receives at the same time light from the horizon by direct transmission through the clear part of $M_2$, and by reflection, from the silvered part, the light which has been reflected to $M_2$ from $M_1$.

Let $L_3$ be a ray emanating from the sun, and $L_2$ a ray from the horizon. Then to an eye $E$ the image of the sun along the path $L_3$ will apparently coincide with the image of the horizon seen directly along $L_2$. The angle which $L_3$ makes with $L_1$, which is parallel to $L_2$, is the angular distance between the sun and the horizon, but $G$, the pointer, only moves through $t$, which is half this angle; therefore the scale over which $G$ moves is divided into half degree spaces, which, however, are numbered as whole degrees in order that direct readings may be taken from the scale, to which also a vernier (q.v.) is attached for greater accuracy.

**Size of Mirror.**—The smallest plane mirror which will enable a person to see the whole of himself reflected is one which is about half his height, the top of the mirror being on a level with a point midway between the eyes and the top of the head, also it must be half the breadth, one eye being closed, and rather less if both are open. To see in a mirror the whole of a test chart placed over one's head, the size of the mirror should be one half that of the chart in both diameters; for other distances of object and observer see page 349.

**Multiple Images.**—When there is but one reflecting surface, as in a metal mirror, there is but one image, but in a glass mirror having two

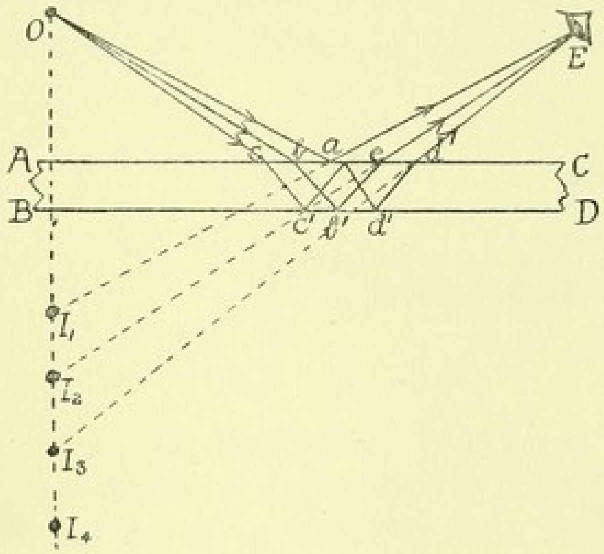



Fig. 24.

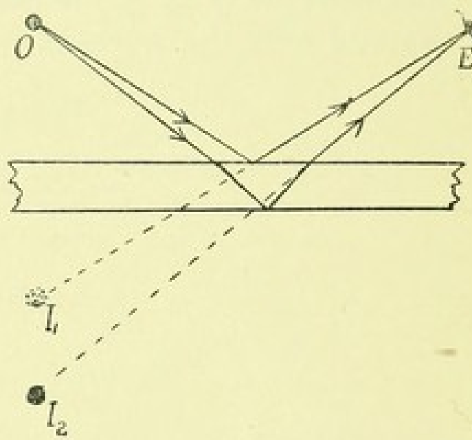



Fig. 25.

reflecting surfaces, namely, the front surface of the glass $A\ C$ (Fig. 24) and its silvered back surface $B\ D$, there are multiple images of an object. Let a candle flame $O$ be held near to a glass mirror and a series of images will be seen by an eye $E$; the first image $I_1$, that nearest to the candle, is formed by direct reflection from the front surface of the glass along $a\ E$; the second image $I_2$, which is the brightest, is directly reflected from the silvered surface along $e\ E$.

The other images $I_3$, $I_4$, etc., all equally distant from each other, are formed by repeated internal reflection between the silvered surface and the front of the glass, but some of the light escaping by refraction at $e\ d$ - - - after each reflection, the images become progressively fainter. Ordinarily on looking into a mirror only two images are noticeable, the faint one reflected from the front and the bright one from the back surface (Fig. 25), but the more oblique the line of view and consequently the greater the angle of incidence of the light to the mirror, the greater is the separation and number of images seen. The total number visible also depends, of course, on the luminosity of the flame.

**Parallel Mirrors.**—If two plane mirrors *M* and *M′* (Fig. 26) are parallel to each other, and an object *O* is placed between them, a series of images (the first of which are *I* and *I′*), infinite in number, is produced by reflection of the light backwards and forwards between the two mirrors. As with the

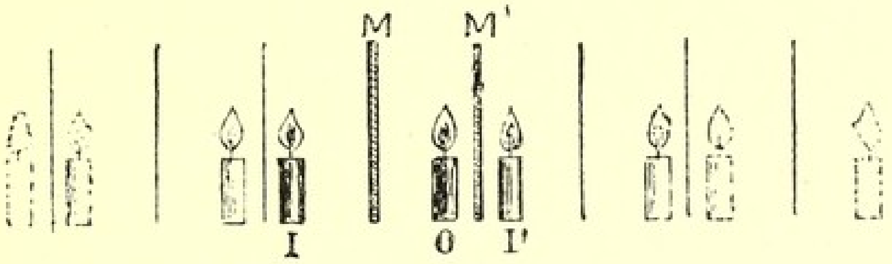



Fig. 26.

single mirror, the repeatedly reflected light soon becomes too feeble for the images to remain visible. The number depends, therefore, on the brightness of O.

**Inclined Mirror.**—When two mirrors *A M*, *B M* are mutually inclined (Fig. 27), the multiple images formed are situated on an imaginary circle passing through the object, and whose radius is equal to the distance of the object from the junction of the mirrors. There being 360° in the complete circle the number of images produced, including the object itself, is found

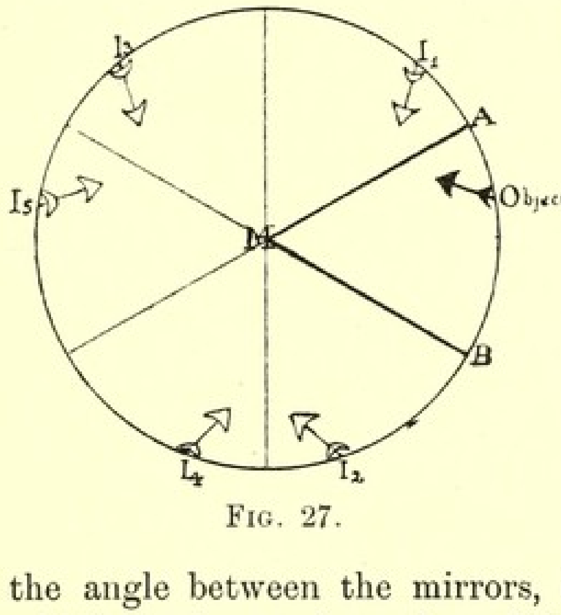



Fig. 27.

by dividing 360° by the angle between the mirrors, or the angle may be calculated by dividing 360° by the total number of images seen, including the object. Thus, if the angle is 90° there are four, if 60° there are six, and if 45° there are eight images. A single mirror may be regarded as two inclined to each other at an angle of 180°; there are then two images, or rather the object itself and its single image in the mirrors. If two mirrors are parallel the angle between them is zero and the images are therefore

360/0 = an infinitely great number, although, as stated above, comparatively few are visible to the eye.

When the number of degrees between the mirrors is an exact even divisor of 360, as 45° or 60°, the complete figure is symmetrical; if the number is an exact odd divisor of 360, such as 120° and 72°, the figure is not symmetrical from every point of view, as when the angle is an exact even divisor of 360°. If the number is not an exact divisor of 360, the figure is asymmetrical, as some of the images are either incomplete or overlapping.

**Construction of Multiple Images.**—To find by construction the images formed by inclined mirrors, let $M\ A$ and $M\ B$ (Fig. 27) be the mirrors at any angle, and $O$ the object between them. With $M$ as centre and $M\ O$ as radius, describe a circle; measure off $A\ I_1$ equal to $O\ A$, and $B\ I_2$ equal to $O\ B$; measure off $A\ I_3$ equal to $A\ I_2$, and similarly $B\ I_4$ equal to $B\ I_1$. Then take $A\ I_5$ equal to $A\ I_4$, and so on until two images coincide or overlap.

**Kaleidoscope.**—The principle of the kaleidoscope depends on the multiple reflection caused by two inclined mirrors. The mirrors are placed lengthways in a tube, which is closed at one end by a disc of transparent glass, beyond which is one of frosted glass. Between these two glass discs there are a number of small coloured objects, or fragments of coloured glass. Looking through the open end of the tube an image is seen consisting of a certain number of images, the whole forming a more or less symmetrical figure. The usual form of kaleidoscope has three mirrors inclined to each other at 60°, and the figure is symmetrically hexagonal, or rather it looks triangular, as shown in Fig. 27. The whole central figure, as seen in a kaleidoscope, is surrounded by others formed by repeated reflections of the light.

## Curved Mirrors.

**Spherical Mirrors.**—A spherical mirror is a portion of a sphere, the cross section of which is an arc of a circle; its centre of curvature is the centre of the sphere of which it forms a part. It may be either concave or convex, and can be considered as made up of an infinite number of minute plane mirrors, each at right angles to one of the radii of the sphere.

**Concave Mirror.**—Let $A\ B$ be a concave mirror (Fig. 28) and $C$ its centre of curvature. Then all straight lines drawn from $C$ to any part of $A\ B$ are radii. They are therefore all of equal length and perpendicular or normal to the surface of the mirror. All rays therefore starting from $C$, on reaching the surface of the mirror, will be reflected back along the same paths and form an image at the same point $C$.

The point $D$ is the *vertex* or *pole*, and the surface $A\ B$ between the extremities of the reflecting surface is the *aperture*. The line passing through $C$ and $D$ is the *principal axis*; all other lines passing through $C$ to the surface are *secondary axes*.

If a luminous point be situated infinitely far away, on the principal axis the angle of divergence being very small, the rays are considered parallel to each other and to the principal axis. Let *A′A*, *B′B*, *D′D*, etc., be such rays, and let *CA*, *C B*, and *C D* be joined; then, since these latter are radii, they each form a right angle at *A*, *B* and *D* respectively with the surface of the mirror. Therefore *A C* is a normal to the surface at *A*, and the ray *A′A* will be reflected to *F*, making the angle of reflection *FAC* equal to the angle of incidence *A′A C*. All the other rays, in the same way, are, provided the

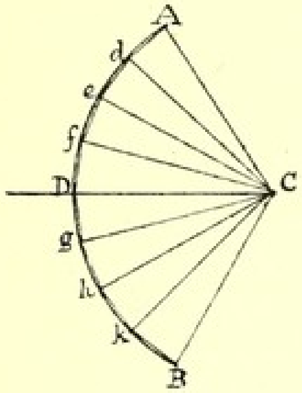



Fig. 28.

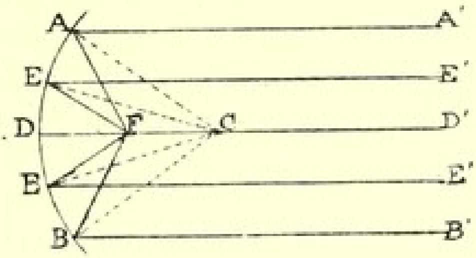



Fig. 29.

aperture is not too great, reflected to *F*, which is the common image of a luminous point situated at ∞. *F* is the *principal focus* of the mirror, and the distance *D F* is the *principal focal distance* or *focal length*. *D F* is equal to half the radius *D C*.

A Cc. mirror therefore renders parallel light convergent, and since the image can be received on a screen, or seen in the air in front of the mirror,

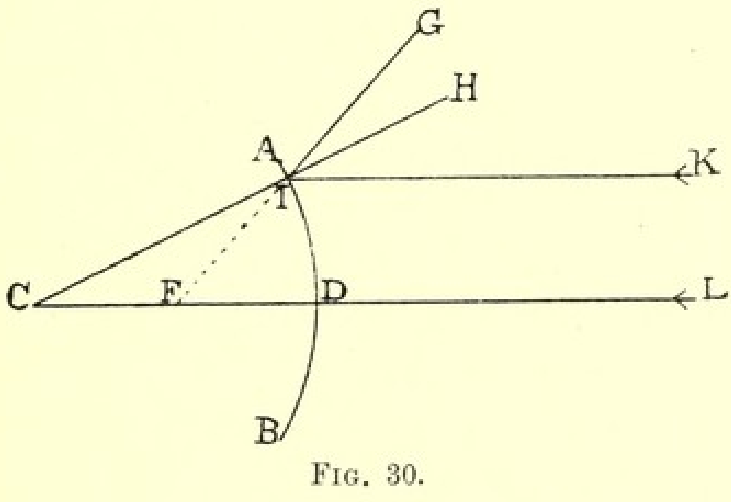



Fig. 30.

the focus of a Cc. mirror is *real* or positive. If light is divergent it is made, by a Cc. mirror, convergent, parallel, or less divergent as the case may be.

The course of a ray can be traced backwards along the same path as that by which it arrived ; so that if *F* be the object-point, the rays *F E*, *F A*, etc., will be reflected parallel to the axis along the lines *E E′*, *A A′*, etc. Thus, image and object are interchangeable.

**Convex Mirror.**—Let *A B* (Fig. 30) be a convex mirror, *C* the centre of curvature, *D* the pole, and *C D L* the principal axis. Then if the object point is at $\infty$ on the principal axis the rays proceeding from it to the mirror are parallel. Let $K I$ be one of these rays meeting the mirror at $I$, and let $C H$ be a normal to the surface. The ray $K I$ will be reflected at $I$ to $I G$, so that the angle of reflection $H I G$ is equal to the angle of incidence $H I K$, and the reflected ray $I G$, produced backwards, cuts the axis at $F$, which is the principal focus of the mirror.

A Cx. mirror therefore renders parallel light divergent, and divergent light still more divergent. The image thus formed is imaginary, so that the focus of a convex mirror is *virtual* or negative.

**Conjugate Foci.**—Conjugate foci may be defined as the positions of object and image. If the image is *real* these positions are such that they may be reversed and the object placed where the image previously was and vice-versa, the distance of the two conjugate points from the mirror remaining the same. In other words the direction of the light can be reversed without altering the positions of the two *conjugate foci*. Conjugate focal distances are the distances of the conjugates from the mirror.

**Conjugate Focal Distances. Cc. Mirror.**—If the object-point (Fig. 31) be on the principal axis between $C$ and $\infty$, say at $f$, the image must be at $f_1$

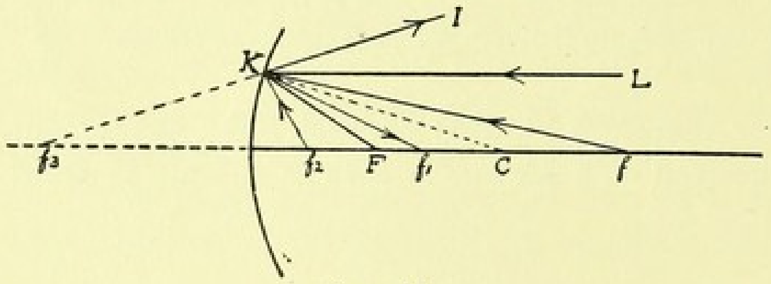



Fig. 31.

somewhere between $F$ and $C$. An object at $\infty$ will have its image at $F$, and it is obvious that the angle of incidence $f K C$ is smaller than the angle $L K C$; therefore the angle of reflection $f_1 K C$, which equals the angle of incidence $f K C$, must in the same way be smaller than the angle $F K C$, and therefore $f_1$, the image of $f$, will lie nearer to $C$ than $F$.

As the object-point approaches $C$, its real image also approaches $C$; when the object-point arrives at $C$ the image will also be at $C$, the ray $C K$ being reflected back along its own path. When the object-point arrives at $f_1$ the real image is obviously at $f$, and when it reaches $F$ its image is at $\infty$.

When the object-point, as $f_2$, passes $F$ towards the vertex, the reflected ray $K I$ lies outside $K L$. Then the focus will no longer be on the same side of the mirror as the object, but will be found by prolonging the ray $K I$ backwards to $f_3$ on the other side of the mirror. In this case the image is not actually formed, but is virtual or negative, existing only in the brain of an observer whose eye is looking into the mirror. As the object-point travels on towards the vertex the image $f_3$ also approaches until the two meet at, and touch, the mirror.

**Conjugate Focal Distances, Cx. Mirror.**—As the object point $f$ (Fig. 32) approaches the mirror the image $f_1$ also approaches the mirror from $F$ to $D$, because the angle of incidence $LKI$ increases with the nearness of the object, until at $D$ object and image coincide, so that, no matter where the object is, the image is always formed behind the mirror either at $F$, or between it and $D$, by prolongation backwards of the divergent rays, and is always imaginary or virtual.

**Images on Secondary Axes.**—In the preceding cases the object is supposed to be on the principal axis, so that the image is also on the principal

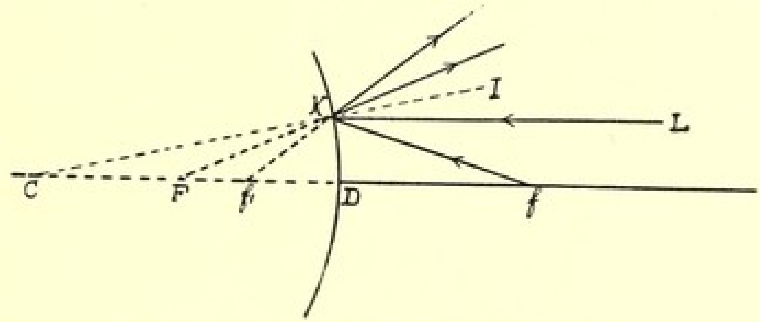



Fig. 32.

axis. If the object be situated on some secondary axis the image is on that same secondary axis. Also the object hitherto has been considered as a point; it can now be supposed to have a definite size.

**Construction of Images—Cc. Mirror.**—It is known that (1) a ray parallel to the principal axis passes, after reflection, through the principal focus; (2) a ray passing through $F$, after reflection, is parallel to the principal axis; (3) a ray proceeding through $C$, the centre of curvature, is reflected along its original path. It is possible to make a graphical construction of the image of an object placed in front of a spherical mirror by tracing any

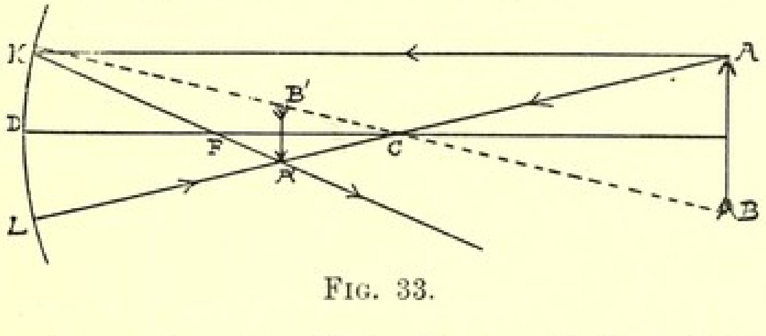



Fig. 33.

two of such rays from the extremities of the object, and their course after reflection. The point where these rays meet is the point where all the rays, which diverge from the object-point, also meet, and is therefore the image of that point.

The graphical construction when the object is beyond $C$ is as follows.

Let $A\,B$ (Fig. 33) be the object, $C$ the centre of curvature, and let $F$ be the principal focus. Draw $A\,K$ parallel to the axis, connect $K\,F$, and produce it onward; draw $A\,L$ through $C$. These two lines cut each other in $A'$, which is therefore the image of $A$, situated on the secondary axis $A\,C\,L$.

In the same way, rays drawn from $B$ meet at $B'$, and both $B$ and $B'$ are on the secondary axis $B\,C\,K$. By connecting $B'$ and $A'$ the image of $A\,B$ is obtained, and it is real, inverted, and smaller than the object. If the object were at $B'\,A'$ within the centre of curvature and beyond $F$, the image would be $A\,B$, real and inverted, but larger.

The course of any ray other than those mentioned can be constructed by drawing the normal to the point of incidence and making the angle of reflection equal to the angle of incidence.

Graphical construction when the object is within $F$.

Let $A\,B$ (Fig. 34) be the object. Draw $A\,K$, connect $F$ and $K$, and pro-

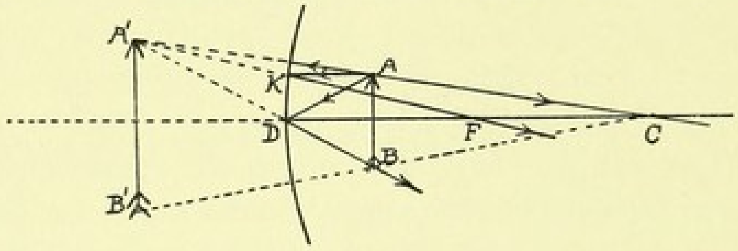



FIG. 34.

duce towards $A'$; draw $C\,A$, producing it similarly. These lines meet on the secondary axis $C\,A\,A'$ in the point $A'$, which is therefore the image point of $A$. Any ray $A\,D$ is reflected as if proceeding from $A'$. In the same way $B'$ can be shown to be the image of $B$. By connecting $B'$ and $A'$ the image $B'A'$ is obtained. It is virtual, erect and enlarged.

The graphical construction of an image formed by an object at $F$ resolves itself into lines parallel to the secondary axes, so that the image is at infinity (Fig. 35).

If the object is at $C$, object and real image coincide, but the image is

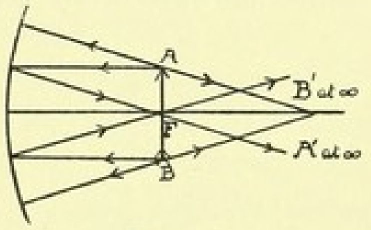



FIG. 35.

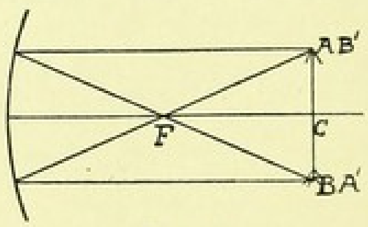



FIG. 36.

inverted (Fig. 36). If the object is at $D$ (Fig. 34), no rays can be drawn, since object and virtual image are in contact with the mirror and coincide.

**Construction of Images—Cx. Mirror.**—Draw $A\,K$ (Fig. 37) and connect $K$ with $F$; join $A\,C$. Where these cut each other at $A'$ is the image of $A$; it is on the secondary axis $A\,A'\,C$. Any other ray from $A$ can be shown to be reflected as if proceeding from $A'$.

By similar construction the position of $B'$, the image-point of $B$, is determined, and connecting $A'B'$ the complete image of the object $A\,B$ is obtained, $B'$ being on the secondary axis $B\,C$.

In the case of a convex mirror, wherever the object may be placed, the image $A'$ $B'$ is always virtual (imaginary) erect and smaller than the object, but if $A$ $B$ is in contact with the mirror, the image $A'$ $B'$ coincides with it.

**Relative Sizes of Image and Object.**—Since, as will be seen from the foregoing figures, both object and image subtend the same angle at the vertex of the mirror, or at the centre of curvature, the relative magnitude of object and image are proportional to their respective distances from the mirror, or from its centre of curvature, and this rule holds good for all images, both virtual and real, and for convex and concave mirrors.

### Conjugate Foci of Spherical Mirrors.

For the convention of optical signs see page 86. List of symbols faces preface.

**Conjugate Focal Distances.**—If F be the principal focal distance, then 1/F is the reflecting power of the mirror, the two being reciprocals of each other; thus, if F be 10, then 1/F = 1/10. If $f_1$ be the distance of the object from which light diverges to the mirror, we can represent the divergence of the light by $1/f_1$ and this quantity is considered negative.

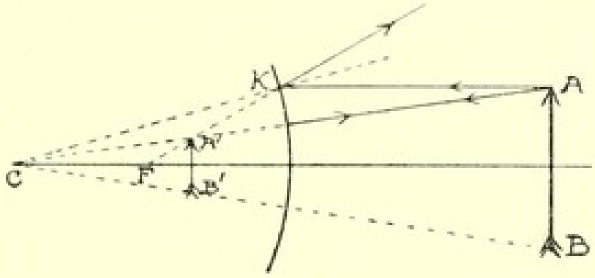



FIG. 37.

Now a Cc. mirror converges the incident light, but if the latter proceeds from a very distant object the divergence is negligible, and therefore the focus of the light is at F as a result of the converging power 1/F of the mirror, which is reckoned positive. If, however, the light proceeds from a near object the divergence is appreciable, and a focus is obtained at some other distance, $f_2$, which distance is determined by the addition of the divergence of the light to the converging power of the mirror, i.e. $1/\mathrm{F} - 1/f_1 = 1/f_2$, where $1/f_2$ is the reciprocal of the distance $f_2$.

A Cx. mirror diverges incident parallel light; its power is negative and representative by $-1/\mathrm{F}$. When light is parallel $f_2$ is at F, but when it is divergent $f_2$ is determined by adding the divergence of the light to the diverging power of the mirror, that is, $-1/\mathrm{F} - 1/f_1 = 1/f_2$. In the case of a Cc. mirror $f_2$ is positive or negative according as $1/f_1$ is respectively a smaller or greater quantity than 1/F. With a Cx. mirror $f_2$ is always negative.

Since convergence is considered positive and divergence negative, if $-1/f_1$ represents the divergence of light from a distance $f_1$, then $1/f_1$ represents convergence to the distance $f_1$, while $1/f_2$ is that power which causes parallel rays to converge to $f_2$, and $-1/f_2$ that power causing them to diverge from that distance. Thus the total power of a mirror 1/F is equal to the sum of the powers which represent the distances of the object and image. In other words the reciprocal of the principal focal distance is equal to the sum of the reciprocals of any pair of conjugate foci. Then we can write the formula

$$\frac{1}{F}-\frac{1}{f_1}=\frac{1}{f_2} \quad \text{or} \quad \frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}$$

This formula is one of the most important in optics. It enables us to find the focal length of a mirror if $f_1$ and $f_2$ are known; or if F and $f_1$ are known we can find $f_2$ (the image). It is universal and holds true for both concave and convex mirrors and, as will be seen, for lenses as well. Since the two fractions $1/f_1 + 1/f_2$ added together always produce the same sum, it follows that however much the one is increased the other is decreased in the same proportion. Thus if a Cc. mirror be of 20 in. radius or 10 in. focal length the sum $1/f_1 + 1/f_2$ is always 1/10. If a Cx. mirror has $F = -10$ in., the sum $1/f_1 + 1/f_2$ is always $-1/10$; here $f_2$ is always negative, as it may be also when 1/F is positive. The formula may also be written $F/f_1 + F/f_2 = 1$.

**Geometrical Proof of the Formula—Cc. Mirror.**—Let *O* (Fig. 38) be any object point situated on the axis of the mirror *D M*. Let *C M* be any radius

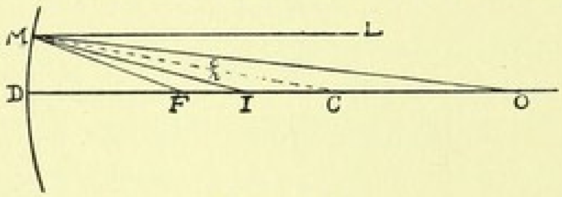



Fig. 38.

and *O M* a ray incident on *M* and reflected to *I*, the image point, such that the angles *O M C* and *I M C* are equal.

If, in any triangle, one of the angles be bisected by a line passing through the opposite side, the latter is so divided that the ratio of the one segment to the adjacent side is equal to that between the other segment and its adjacent side. Thus in the triangle *O M I*,

$$IC : IM :: OC : OM$$

or

$$\frac{IC}{IM}=\frac{OC}{OM}$$

But since the semi-aperture *M D* is considered small, *I M* may be taken as equal to *I D*, and similarly *O M* to *O D*

$$\therefore \quad \frac{IC}{ID}=\frac{OC}{OD}$$

But $ID=f_2$, $OD=f_1$, and $DC=r$

Then

$$\frac{r-f_2}{f_2}=\frac{f_1-r}{f_1} \quad \text{or} \quad \frac{f_1+f_2}{f_1 f_2}=\frac{2}{r}$$

Whence $1/f_1 + 1/f_2 = 2/r = 1/F$

because the radius of curvature is twice the principal focal distance. This is easily proved, for let $LM$ be a ray parallel to the axis and incident at $M$ so that, after reflection, it passes through $F$. Then the angles $LMC$ and $FMC$ are equal, as are the angles $LMC$ and $MCD$ between the parallels $LM$ and $OD$. Now $MCD$ is equal to $FMC$; therefore the angles $FMC$ and $FCM$ are also equal, so that $FM = FC$.

But, *the aperture being small*, $DF$ may be considered equal to $MF = FC$. Therefore $DC$ is equal to twice $DF$—in other words, the radius of curvature is twice the principal focal length.

**Another Proof.**—In Fig. 39 $AB$ is an object whose image is $B'A'$. The

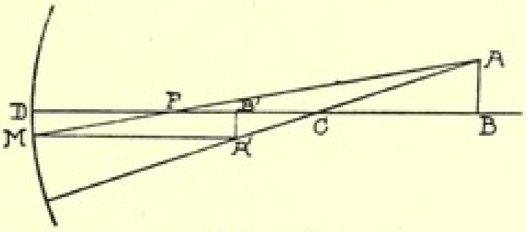



FIG. 39.

aperture of the mirror is considered small, so that $DM$ may be considered a straight line. In the similar triangles $AFC$ and $AMA'$

$$\frac{FC}{MA'} = \frac{AF}{AM} = \frac{BF}{BD}$$

But $FC = F$, $MA' = DB' = f_2$, $BF = f_1 - F$, and $BD = f_1$

Then

$$\frac{F}{f_2} = \frac{f_1 - F}{f_1}$$

That is $F/f_2 + F/f_1 = 1$ or $1/f_1 + 1/f_2 = 1/F$

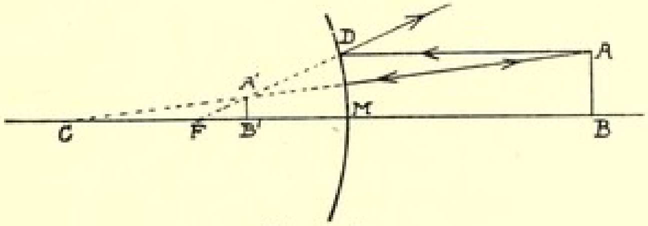



FIG. 40.

**Proof for Cx. Mirror.**—The aperture being small, $DM$ (Fig. 40), as before, may be considered a straight line. Then

$$\frac{CB'}{CB} = \frac{A'B'}{AB} = \frac{A'B'}{DM} = \frac{FB'}{FM}$$

$CB' = 2F - f_2$, $CB = 2F + f_1$, $FB' = F - f_2$, and $FM = F$

Then

$$\frac{2F - f_2}{2F + f_1} = \frac{F - f_2}{F}, \qquad \text{whence} \qquad F(f_2 - f_1) = f_1 f_2$$

That is $$-1/F = 1/f_1 - 1/f_2$$

In all the above proofs it will be noticed that an approximation is introduced by considering the portion of the reflecting surface utilized as flat. If this were not done the formulæ would be very complicated, and a different value would be got out for each size of aperture—in other words the *spherical aberration* of the mirror is ignored in our simple formula, which is indeed permissible seeing that the portion of the mirror chiefly responsible for the production of the image is that immediately surrounding the vertex.

**Calculations on Conjugate Foci.**—A positive result when working these problems denotes either a real image, or a concave mirror. If the result is negative the image is virtual, or if the principal focus is being found, the mirror is convex.

**Examples—Cc. Mirror.**—Let the mirror be of 10 in. F, and let the object be at $\infty$; then we have

$$1/f_2 = 1/F - 1/f_1 = 1/10 - 1/\infty = 1/10 - 0 = 1/10$$

The image is real and at the principal focal distance, since $f_2 = F$.

It must be remembered that we regard, in these calculations, any *considerable* distance as $\infty$. In visual objects any distance beyond 6 M. or 20 feet is so taken, but there we are dealing with short focal length systems.

If the object be at F the calculation is

$$1/f_2 = 1/F - 1/F = 0/F \qquad \therefore \qquad f_2 = F/0 = \infty$$

so that the image is at $\infty$, and F and $\infty$ are conjugate distances.

If the object is at 30 in., we get

$$1/f_2 = 1/10 - 1/30 = 1/15$$

Therefore the image is 15 in. and real.

If the object were at 15 in. we find

$$1/f_2 = 1/10 - 1/15 = 1/30$$

15 in. and 30 in. are conjugate foci with respect to a 10 in. concave mirror; if the object be at 15 in. its image is at 30 in.; if the former is at 30 in. the latter is at 15 in.

If the object be at twice F, that is, at the centre of curvature, say 20 in., in front of a 10 in. concave mirror, the image is at the same distance, since

$$1/f_2 = 1/10 - 1/20 = 1/20 \text{ or } f_2 = 20 \text{ ins.}$$

When the object is within the principal focal distance, a higher number than 1/F being deducted from it, the result is a negative quantity. Thus if the object be placed 6 inches in front of a 10 in. concave mirror, then

$$1/f_2 = 1/10 - 1/6 = -1/15$$

so that the virtual image is 15 inches behind the mirror.

Here $-15$ in. is the conjugate of 6 in. in respect to a 10 in. concave mirror, and 6 in. is the conjugate of $-15$ in., *but not of* 15 *in.* That is to say, if the rays of light incident on the mirror are *convergent* to a point 15 in. behind it, they will be reflected so as to come to a focus 6 in. in front of it.

If light is incident on a 10″ mirror convergent to 15″ behind it we get a real image at 6″ for

$$1/f_2 = 1/10 + 1/15 = 1/6$$

From these calculations it will be seen that a real or positive image is obtained with a concave mirror so long as the object is beyond F, and that the image becomes virtual or negative when the object is nearer than F. Also that in all cases $1/F = 1/f_1 + 1/f_2$. Thus when the light diverges from 30″ and is converged to 15″ we find $1/10 = 1/30 + 1/15$. When it diverges from 6″ before reflection, and from 15″ after reflection we get $1/10 = 1/6 + (-1/15)$.

**Relative Distances of O and I.**—The nearer the object is to F, the more distant is the real image ; as the object recedes from F, the image approaches it, but no positive image of an object can be nearer than F since no object can be more distant than $\infty$. If, however, the rays are convergent before reflection, then $f_2$ passes to the mirror side of F.

The planes of unit magnification for real images lie at the point where the object coincides with the centre of curvature of the Cc. mirror, for then the image is equal in size to the object and at the same distance.

The nearer the object is to F, the more distant also is the negative image. As the object recedes from F and approaches the mirror, the image also approaches the mirror, but the image is always more distant than the object. When the object touches the reflecting surface, the image does so likewise, this being the plane of unit magnification for virtual images formed by a Cc. mirror.

**Examples—Cx. Mirror.**—Let the mirror be of 10 in. F and the object at $\infty$. Then

$$1/f_2 = -1/F - 1/\infty = -1/10 - 0 = -1/10$$

The image is virtual or negative and at F.

If the object be in front of the mirror at a distance equal to F of a Cx. mirror, the image is at half F. Thus with a 10″ mirror

$$1/f_2 = -1/10 - 1/10 = -1/5$$

If an object is situated 30 in. in front of the convex mirror

$$1/f_2 = -1/10 - 1/30 = -1/7\tfrac{1}{2} \text{ in.}$$

The image is virtual and $7\frac{1}{2}$ in. behind the mirror.

$-7\frac{1}{2}$ in. is conjugate to 30 in. with respect to a 10 in. convex mirror, and 30 in. is the conjugate of $-7\frac{1}{2}$ in. but not of $7\frac{1}{2}$ in. If light were convergent to a point $7\frac{1}{2}$ in. behind the surface, the convergence would be, by reflection, so much reduced that an image would be formed 30 in. in front of a 10 in. convex mirror.

Thus if light converges to a point within F the image is real, if convergent to F the light is parallel after reflection, since the convergence of the light and divergence of the mirror neutralise each other. If the light is convergent to a point beyond F the virtual image formed is also beyond F. In all cases, however, $1/F = 1/f_1 + 1/f_2$.

**Relative Distances of O and I.**—The image of a real object formed by a convex mirror is therefore always virtual, and cannot be at a greater distance from it than F, the object being then at $\infty$. When the object is nearer than $\infty$ the image recedes from F towards the mirror, and when the object touches the surface the image does likewise. This is the plane of unit magnification of a Cx. mirror.

**Another Expression for Conjugate Foci.**—If the distance of the object from $F = A$, and that of the image from $F = B$, then $AB = F^2$. That is, $B = F^2/A$. This is generally known as Newton's formula. Following are some examples :

Let $F = 10$ and $f_1 = 30$, th n $A = 30 - 10 = 20$

and $B = 100/20 = 5$, or $f_2 = 5 + 10 = 15$ in.

Let $F = 10$ and $f_1 = 6$, then $A = 6 - 10 = -4$

and $B = 100/-4 = -25$, o $f_2 = -25 + 10 = -15$ in.

Let $F = -10$ and $f_1 = 30$, then $A = 30 - (-10) = 40$

and $B = 100/40 = 2{\cdot}5$, or $f_2 = 2{\cdot}5 + (-10) = -7{\cdot}5$ in.

These examples should be compared with those worked by the ordinary formula.

**Size of the Image formed by a Spherical Mirror.**—In the case of both Cc. and Cx. mirrors the size of the image bears the same relation to the size of the object, as the distance of the image does to the distance of the object from the mirror, or from the centre of curvature. That is

$$h_2/h_1 = f_2/f_1 \qquad \text{or} \qquad h_2 = h_1 f_2/f_1$$

where $h_1$ is the size of the object, $h_2$ is the size of the image, $f_2$ is the distance of the image, and $f_1$ is the distance of the object. In the calculation $f_1$ and $f_2$ must be in the same terms, then $h_2$ will be in the same terms as $h_1$. This rule holds good in all cases, whether the image be real or virtual, *since both object and image subtend the same angle at the centre of curvature or at the vertex, so that their respective sizes depend solely on their respective distances from C or from the vertex.*

Suppose the object is 30″ from a 10 in. Cc. mirror and 2″ in height, while the image is at 15″, then

$$h_2 = h_1 f_2/f_1 = 2 \times 15/30 = 1''$$

If the object were at 15 in. and the image therefore at 30 in, the image would be 4″ high if the object were 2″.

When the object is 30″ from a 10 in. Cx. mirror its image is virtual at $7\frac{1}{2}$ in. and if $h_1 = 2''$

$$h_2 = \frac{2 \times 7{\cdot}5}{30} = {\cdot}5''$$

In the above examples the distances from the mirror have been taken, but the same ratios would exist were they taken from *C*. Thus 30″ and 15″ from a 10″ Cc. mirror are $30 - 20 = 10''$ and $20 - 15 = 5''$ from *C*, and obviously $30/15 = 10/5$. Similarly with the Cx. mirror, 30″ and $7\frac{1}{2}$ in. from it are $30 + 20 = 50$ and $20 - 7\frac{1}{2} = 12\frac{1}{2}$ from the centre of curvature, and $30/7{\cdot}5 = 50/12{\cdot}5$. All formulæ for magnification with lenses apply also to mirrors.

It must not be forgotten that although a distance may be taken as $\infty$ for the calculation of $f_2$, its *definite* distance is needed for calculating $h_2$. Thus suppose an object 3 yards high is an eighth of a mile from a 30″ mirror. What is the distance and size of the image? The image is at $F = 30''$, and

$$h_2 = \frac{3 \times 30}{220 \times 36} = \frac{1}{88} \text{ yard or } {\cdot}4 \text{ inch.}$$

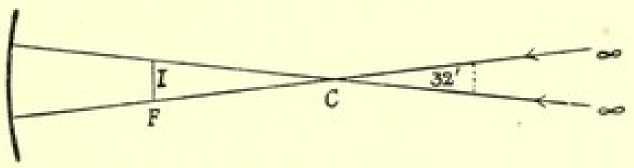



Fig. 41.

When the object is inaccessible so that the actual size cannot be determined, the size of image can readily be found from the angle which the object subtends at the centre of curvature. For example, what will be the size of the image of the moon formed by a concave mirror of 16 in. focus?

The object being at $\infty$ (Fig. 41) the rays from each point of it are parallel, and the image *I* will be formed at the principal focus and $\therefore f_2 = 16$ ins. But although the object is at $\infty$ it has a definite size subtending an angle $a$ of 32′, and we utilise the size of the angle subtended by the object instead of the object itself. Now object and image subtend the same angle at *C*, and since $\tan 32' = {\cdot}0093$ and $f_2 = 16$ ins., we have, as the size of the image,

$$16 \times {\cdot}0093 = 0{\cdot}1488 \text{ in., or about } \tfrac{1}{7} \text{ in.}$$

**Recapitulation of Conjugates.—Cc. Mirror.**

When *O* is at $\infty$, *I* is real, inverted, diminished and at ***F***.

When *O* is between $\infty$ and $2F$, *I* is real, inverted, diminished and between $2F$ and ***F***.

When *O* is at $2F$, *I* is real, inverted, equal to *O* and at $2F$.

When *O* is between $2F$ and *F*, *I* is real, inverted, enlarged and between $2F$ and $\infty$.

When $O$ is at $F$, $I$ is infinitely great and at $\infty$.

When $O$ is within $F$, $I$ is virtual, erect, enlarged and on the other side of the mirror.

When $O$ is at the mirror, $I$ is virtual, erect, equal to $O$ and at the mirror.

**Cx. Mirror.**

When $O$ is $\infty$, $I$ is virtual, erect, diminished and at $F$.

When $O$ is within $\infty$, $I$ is virtual, erect, diminished and within $F$.

When $O$ is at the mirror, $I$ is virtual, erect, equal to $O$ and at the mirror.

The virtual image of a Cx. or Cc. mirror is laterally inverted as in a plane mirror. The real image of a Cc. mirror is entirely reversed, and therefore not laterally inverted in this sense.

**Aperture of a Mirror.**—In order that a true image of a point may be obtained with a spherical mirror, it is essential that the aperture should be

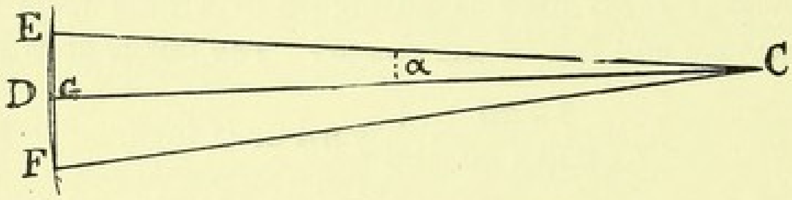



Fig. 42.

small compared with its radius, subtending an angle of, say, not more than 20° at $C$, so that the arc of the aperture may be approximately a straight line. Suppose $E\ F$ (Fig. 42) be the aperture of the mirror, $C\ D$ the principal axis, and $C$ the centre of curvature. Join $E\ F$. Then if the angle $E\ C\ D$ be small (under 10°) the distance $D\ G$ will also be small, so that $C\ G$ may, without much error, be taken as equal to $C\ E$; also

$$\text{E D C} = \text{G E C} = \text{E G C} = \text{a right angle.}$$

$$\text{Now } \tan a = \frac{\text{E G}}{\text{G C}},\ \sin a = \frac{\text{E G}}{\text{E C}},\ \text{and } \cos a = \frac{\text{G C}}{\text{E C}}.$$

Since $E\ C$ is taken as equal $G\ C$, $\sin a = \tan a =$ the arc $E\ D$, and $\cos a = 1/1 = 1$.

Thus all calculations involving mirrors—and, as will be seen later, lenses also—are greatly simplified, since the sine and tangent may be considered equal for small angles and can be replaced by the arc, and the cosine by unity, whenever the angular aperture is s ı all.

## CHAPTER IV

## REFRACTION AND THE REFRACTIVE INDEX

**Normal Incidence of Light.**—The fact that the velocity of light is lessened in a dense medium is the cause of refraction. When a beam of light, traversing the air, is incident normally on a refracting medium such as a sheet of glass, the whole of the wave-front is retarded simultaneously and equally. The plane of the wave after entering the glass is unchanged in direction and continues so during its progression through the denser medium. On reaching the second surface, the whole of the wave-front is again incident at the same time, and each part of it is equally increased in speed as it

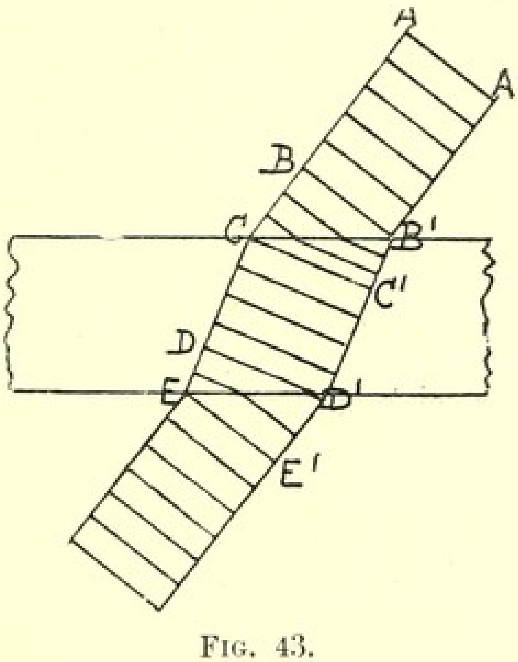



Fig. 43.

passes again into the rarer medium, so that its line of progression remains unchanged.

**Oblique Incidence of Light.**—But if the plane wave-front $A\ A'$ (Fig. 43) be incident on the first surface obliquely, one part $B'$ meets the denser medium sooner than the rest and this is retarded, while the others are still in the rarer medium advancing at an undiminished rate of speed. Each wavelet on reaching the glass becomes retarded, one by one, until the whole of the wave-front has passed into the denser medium, and in consequence the wave-front is changed in direction. The angular change of direction depends on the distance that the more rapidly advancing parts of the wave-front travel before their speed is also checked, that is, on the obliquity of the incidence of the light, and on the retardation itself, that is, on the refracting power of the medium. When the whole of the wave-front $C C'$ has entered the denser medium, it travels without further deviation but at a diminished rate of speed. On reaching the second surface of the glass the wave-front $D D'$ is again incident sooner at one point $D'$ than at others. The wavelet at that point increases its speed, while the remainder is still moving less rapidly in the denser medium; then the other wavelets emerge and increase their speed until, having passed into a rarer medium, the entire wave-front $E E'$ travels with its original velocity and in a direction parallel to its original direction.

**The Laws of Refraction.**—When a ray of *monochromatic* light (i.e. light of a single wave-length) is incident obliquely on the boundary between two media of different optical densities:—

(1) The incident and refracted rays are in the same plane as the normal to the point of incidence, and lie on opposite sides of it.

(2) A constant ratio exists between the sines of the angles of incidence and refraction; this ratio is governed solely by the relative density of the two media, and is known as the *index of refraction* with respect to the two media.

It will be seen later that the value of the index also depends upon the colour of the light, but, for the present, we shall consider all light as monochromatic.

From the second law we can deduce the following:

A ray passing obliquely from a rarer into a denser medium is refracted towards the normal at the boundary plane between the two media.

A ray passing obliquely from a denser into a rarer medium is refracted away from the normal at the boundary plane between the two media.

A ray suffers no deviation if, at the point of incidence, it is normal to the surface of the medium which it enters.

**Index of Refraction.**—The index of refraction between two media is *the ratio of the velocities of the light in these media.* For example, supposing light to travel at a speed of $x$ in the first medium and at a speed of $y$ in the second, then the index of refraction from the first to the second medium is $x/y$, but if the direction of the light is reversed, then the index is said to be $y/x$. If the light travels three miles in the first, while, in the same time, it is travelling two miles in the second, then the index is $3/2 = 1{\cdot}5$; from the second to the first the index would be $2/3$.

**Snell's Law of Sines.**—The second law of refraction states that the constant ratio between the sines of the angles of incidence and refraction is the index of refraction; it can be shown that the ratio of the velocities of the light in the two media are as the sines of the angles of incidence and refraction. In Fig. 44, let $S S$ be the bounding surface between the media—the second being denser than the first—in which the velocities of the light are respectively $V_1$ and $V_2$. Let $DC$ be a plane wave-front incident on the surface at the angle $DCA = i$ which, after refraction, passes into the second medium at the angle of refraction $CAE = r$.

Then $AD$ and $CE$ are distances travelled by the extremities of the wave in equal times, the one portion at $D$ being in the rarer and that at $C$ in the denser medium; it follows, therefore, that the ratio $AD/CE$ is the refractive index for light travelling in that direction between the two media. Now since the "rays" $AD$ and $CE$ are perpendicular to the wave-front, the angles $ADC$ and $AEC$ are right angles. Therefore, the hypotenuse $AC$ being common to the triangles $ADC$ and $AEC$, $AD$ and $CE$ are numerically the sines of the angles of incidence and refraction respectively, so that

$$\frac{V_1}{V_2} = \frac{AD}{CE} = \frac{\sin i}{\sin r}$$

*That is, the index of refraction, or the relative velocities of light in the two media, is given by the ratio of the sines of the angles of incidence and refraction.*

This proof holds equally when the incident wave is curved, since we may

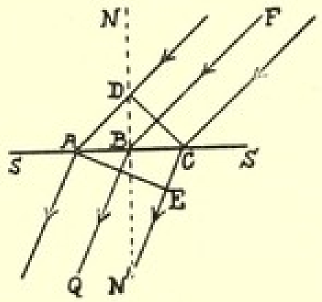



FIG. 44.

assume the portion $DC$ under consideration to be so small as to be sensibly straight—in other words, we may work from the "ray" $FB$, which is really the path taken by a very small portion of the wave-front itself. Similarly if the medium $SS$ is curved, the portion $AC$ may also be considered plane. Further the ratio $\sin i/\sin r$ is quite independent of the angle of incidence since, whatever course a wave may take through the same pair of media, its velocity in each, and therefore the ratio of the velocities, must remain constant.

**Absolute Refractive Index.**—With the exception of certain metals, the velocity of light is a maximum in free ether, i.e. a vacuum, through which all waves, of whatever length, travel with equal speed. Its progression through air is very slightly slower, but for all practical purposes no distinction need be, nor is, made between free ether and the atmosphere. The optical density of air is therefore taken as unity or 1, and the density of any other medium, such as water or glass, is expressed in terms of this unit and is called the *absolute index of refraction*, generally denoted by the Greek letter $\mu$ (mu). Thus if the $\mu$ of a certain kind of glass is 1·5, it implies that light travels one and a half times as fast in air as in the glass; or to put it in another way, the velocity in the glass is only $1/1{\cdot}5 = 2/3$ that in air. Therefore $\mu$, or the *absolute index of refraction*, expresses the optical density of a medium, and if $\mu = 1{\cdot}5$, the medium to which it pertains has an optical density 1·5 times greater than air. To a certain extent the optical density varies directly with the true density, but there are notable exceptions, as for instance with some of the metals, but these are opaque; the departure from the rule is more noticeable in comparing such transparent media as oil and water; the former has the greater optical density but the lesser specific gravity than the latter.

When reference is made to the $\mu$ of a substance it is invariably understood to mean the *absolute index as compared with air*. In addition $\mu$, unless otherwise stated, refers to *yellow light*; the reason for this restriction will appear later when chromatism and chromatic aberration are discussed.

It should be noted that the angle of incidence of a *wave* is that which it makes with the *surface*, as $A\,C\,D$ in Fig. 44, the corresponding angle of refraction being $E\,A\,C$. The angle of incidence of a *ray* is that which it makes with the *normal* as $F\,B\,N$, the angle of refraction being $Q\,B\,N'$. It is immaterial, however, as to whether the ray or the wave is taken since obviously the angles of incidence $A\,C\,D$ and $F\,B\,N$ are equal, as are the angles of refraction $E\,A\,C$ and $Q\,B\,N'$.

It is usual, when several media are involved in a calculation, to refer to their indices as $\mu_1$, $\mu_2$, $\mu_3$, etc., but when there are only two media, one of which is air, the index of the denser is denoted simply by $\mu$ without any suffix, that of air being, as before stated, always taken as 1, although actually it is about 1·000294.

**Relative Refractive Index.**—The relative index of refraction (usually written $\mu_r$) is the expression of the refractivity when light passes from one dense medium into another, say, from water into glass or vice versa. It is found by dividing the absolute index of the medium into which the ray passes, by the absolute index of the medium from which it proceeds; thus when light passes from water $\mu = 1{\cdot}333$ into glass $\mu = 1{\cdot}545$ the relative index is

$$\mu_r = \frac{\sin i}{\sin r} = \frac{1{\cdot}545}{1{\cdot}333} = 1{\cdot}16$$

Again, the sines of the angles of incidence and refraction, as light passes through two such media, are to each other as the velocities of the light in those two media.

**Reciprocal $\mu$'s.**—In the case of any two such media A and B the index of refraction for light passing from A into B is the reciprocal of the index for light passing from B into A. Thus, when light passes from air into glass, the sines of the angles of incidence and refraction are, say, as 3 : 2, and the index is 3/2. If it passes from glass into air, the sines of the two angles are as 2 : 3 and the index is 2/3. Taking the example of the last paragraph, the relative $\mu$ is 1·16, but if the light passed from the glass to the water $\mu = 1{\cdot}333/1{\cdot}545 = 1/1{\cdot}16$.

**The Course of a Ray.**—As an example of the application of the sine law, let a ray be incident from air ($\mu = 1$) at 30° with the normal, to glass of $\mu = 1{\cdot}5$, and it is required to find the course of the ray after refraction. We have $\sin i/\sin r = \mu$, from which $\sin r = \sin i/\mu$.

$$\therefore \qquad \sin r = \frac{\sin 30^\circ}{1{\cdot}5} = \frac{{\cdot}5000}{1{\cdot}5} = {\cdot}3333$$

Now $\cdot 3333 = \sin 19^\circ\ 30'$ (approx.), so that $r = 19^\circ\ 30'$. Therefore the ray has been deviated towards the normal by an amount equal to $30^\circ - 19^\circ\ 30' = 10{\cdot}5^\circ$.

It should be observed that, apart from the angle of incidence, the actual deviation which light undergoes, when passing from one medium into another of different density, depends *on the ratio between the $\mu$'s of the two media*, and not on the high value of the $\mu$ of the second medium. Thus

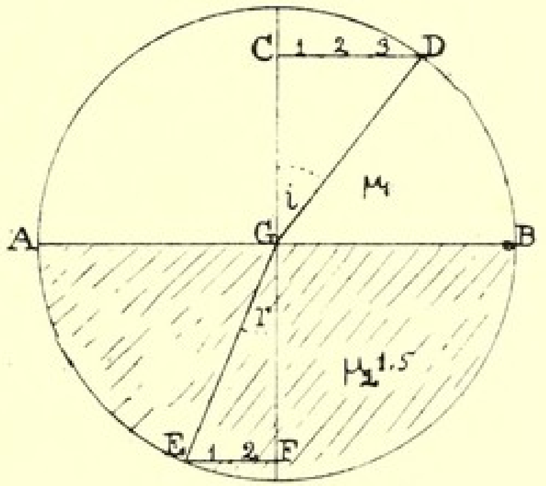



FIG. 45.

the refraction is greater when light passes from air into glass of $\mu = 1{\cdot}5$ than when it passes from water into glass of $\mu = 1{\cdot}6$.

**Graphical Constructions.**—The course of the light can also be graphically constructed in the following manner. Let $DG$ (Fig. 45) be the ray incident at the angle $i$ to the surface $AB$ of $\mu = 1{\cdot}5$. At the point of incidence $G$ drop the normal $CF$ and with $G$ as centre describe a circle of any radius—the longer the latter the more accurate will be the construction. From the point $D$ where the circle cuts the incident ray drop the perpendicular $DC$ and divide it into three equal parts. Then drop a similar perpendicular $EF$ from $E$ on the other side of the surface, such that the length of $EF$ is equal to two of the parts into which $DC$ was divided. Then $GE$ is the course of the refracted ray. This construction is merely a graphical representation of the sine law because if, in the right angled triangles $CDG$ and $FEG$, the hypotenuse $DG$ is equal to $GE$, $CD$ and $EF$ are numerically the sines of $i$ and $r$ respectively, and as these have been divided in the ratio of 3 and 2, $GE$ must be the direction of the refracted ray.

This construction is universal and can be applied to any pair of media. Thus suppose, in the above example, that the first medium was water of $\mu = 1{\cdot}33$ instead of air, $CD$ and $FE$ would have to be divided into 15 and 13·3 parts respectively. To calculate the deviation, if $i$ be 30°

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1} = \frac{1{\cdot}5}{1{\cdot}33}$$

or

$$\sin r = \frac{1{\cdot}33 \sin i}{1{\cdot}5} = \frac{1{\cdot}33 \times {\cdot}5}{1{\cdot}5} = {\cdot}4444$$

whence

$$r = 26° \text{ (approx.)}$$

The deviation, therefore, in this case is only $30 - 26 = 4°$.

Another and perhaps more simple construction is shown in Fig. 46. Let $DG$ be any ray incident at $G$ on the surface $SS$ of a medium whose $\mu = 1{\cdot}5$ or 3/2. From $D$ drop the perpendicular $DB$ and divide $BG$ into three equal spaces. Then from $G$ mark off $GA$ equal to two such spaces. From $A$ drop a perpendicular and from $G$ draw a line $GE$, *equal in length to* $GD$, cutting the perpendicular from $A$ in $E$. Then $GE$ is the direction of the refracted ray. In this construction $BG$ and $AG$ takes the place respectively of $DC$ and $EF$ in Fig. 45.

In order to trace the course of a ray of light through any refracting body, with plane or curved surfaces, the procedure is the same, but in the case of a curved surface the tangent to the curve, at the point on which the ray is incident, is considered to contain the plane of incidence and of refraction.

TABLE OF REFRACTIVE INDICES (FOR THE *D* LINE).

(*For other Media see Appendix.*)

| | | |
|---|---|---|
| Air | .. | 1·000 |
| Water | .. | 1·336 |
| Alcohol | .. | 1·366 |
| Pebble | .. | 1·544 |
| Canada Balsam | .. | 1·535 |
| Tourmaline | .. | 1·636 |
| Crown glass | say | 1·500 to 1·600 |
| Flint ,, | ,, | 1·530 to 1·800 |
| Diamond | ,, | 2·47 |

The index of glass varies with the materials used in its manufacture, and as a rule the higher the $\mu$ the softer is the glass.

**Refractivity.**—It should be particularly noted that the *refraction* or deviation caused by any medium is not proportional to its $\mu$ but to the difference between its $\mu$ and that of air $= 1$. We say that the *mean refractivity of a substance is* $(\mu - 1)$. The refracting values of any two media—lenses, prisms, etc., having similar forms—would vary not as their $\mu$'s, but as their $(\mu - 1)$'s.

**Dispersion.**—The shorter waves, with rare exceptions, are retarded by a medium, more than the longer waves, so that when white light undergoes refraction its components are refracted to different extents and the various colours become separated, producing what is known as dispersion or chromatism, which subject is treated in a later chapter.

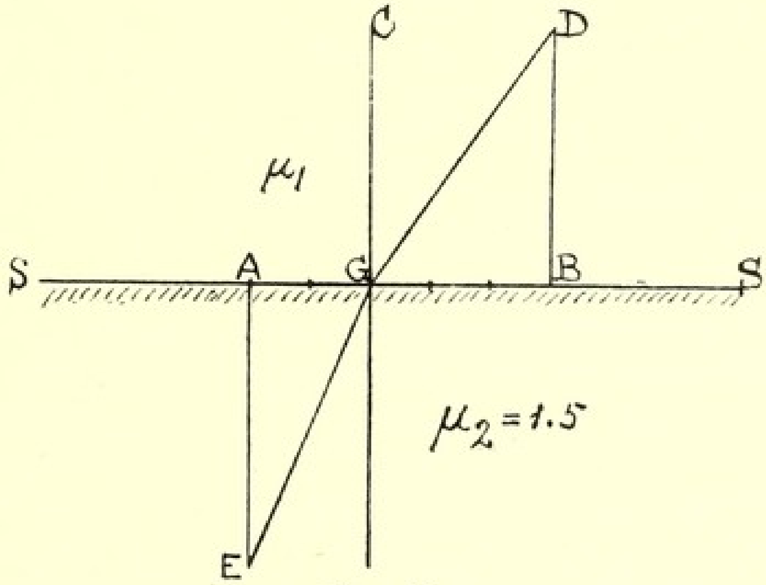



FIG. 46.

**Critical Angle and Total Reflection.**—When a ray of light passes from a dense into a rare medium, it is bent away from the normal, with which it makes a larger angle than before refraction. In Fig. 47 let $\mu_2$ be the index of the dense and $\mu_1$ that of the rare medium, and let *A B* be the incident and *B C* the refracted ray. As *A B* makes a larger angle with the normal the corresponding angle of refraction becomes still larger. Hence if the

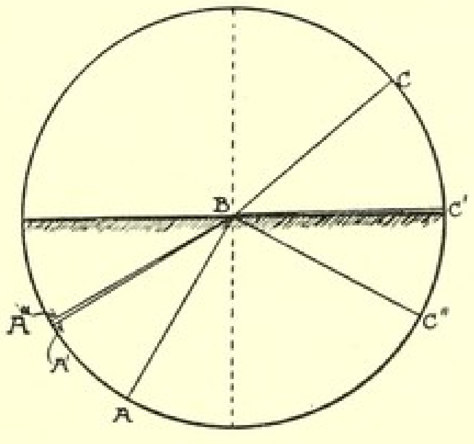



FIG. 47.

ray *A′ B* be incident at an angle sufficiently large, the angle of refraction becomes a right angle, and the refracted ray *B C′* will skim along the bounding surface. The angle of incidence in the denser medium which produces this result is termed the *critical angle*, because the slightest further increase of it prevents the ray from passing out of the denser medium. If the incident ray be *A″ B* it is reflected as *B C″*, and *total internal reflection* takes place. Internal reflection is termed *total* to distinguish it from ordinary reflection, which is always accompanied by a certain amount of absorption or transmission.

Let $C$ be the critical angle, which is equal to $i$ in the dense medium. Then

$$\frac{\sin i}{\mu_1} = \frac{\sin r}{\mu_2}$$

But $r = 90°$, and $\sin 90° = 1$, the greatest possible value of any sine. Therefore

$$\frac{\sin C}{\mu_1} = \frac{1}{\mu_2} \qquad \text{or} \qquad \sin C = \frac{\mu_1}{\mu_2}$$

Thus *the sine of the critical angle is equal to the relative index from the denser to the rarer medium.* If the rarer medium be air, $\mu = 1$, so that, for a denser medium bounded by air,

$$\sin C = \frac{1}{\mu}$$

Suppose the ray to pass from glass $\mu = 1{\cdot}5$ to air; then $1/\mu = 1/1{\cdot}5 = 0{\cdot}666$, so that the sine of the critical angle is ·666, which is sin 41° 46′. This is the greatest angle at which a ray can be incident in order to emerge from glass of $\mu = 1{\cdot}5$ into air, and the emergent ray is then parallel to the surface.

For light passing from one dense medium into another

$$\sin C = \frac{\mu_1}{\mu_2} = \frac{1}{\mu_r}$$

where $\mu_r$ is the *relative* index of refraction for the two media. Thus for glass and water where $\mu_r = 1{\cdot}52/1{\cdot}33 = 1{\cdot}14$, $\sin C = 1/1{\cdot}14 = {\cdot}877 = \sin 61{\cdot}18'$.

This principle affords a method of determining the refractive index of a medium. If the angle in the denser medium, at which the incident ray just ceases to emerge into the other, be measured, 1 divided by the sine of that angle is equal to the $\mu$ of the medium, or the relative $\mu$ of the two media.

The critical angle when light passes through several media is the same as that which obtains directly between the first and the last.

TABLE OF CRITICAL OR LIMITING ANGLES.

| Medium | Index of Refraction. | Critical Angle. |
|---|---|---|
| Chromate of lead .. .. .. .. | 2·92 | 20° |
| Diamond .. .. .. .. .. | 2·47 | 24° |
| Various precious stones .. .. .. | — | 25° to 30° |
| Flint glass .. .. .. .. .. | — | 38° to 40° |
| Crown glass .. .. .. .. .. | — | 40° to 43° |
| Pebble .. .. .. .. .. | 1·54 | 40° |
| Water .. .. .. .. .. | 1·33 | 48° 30′ |

It will be seen, from the above, that the critical angle varies inversely with $\mu$. That of glass *in general* is about 40°.

**Some Effects of Total Reflection.**—On looking upwards through the side of an aquarium tank the surface above one's head glistens like quicksilver, owing to the light being reflected downwards. A metal ball, blackened by a smoky flame, immersed in water appears brilliantly polished, because the thin film of air surrounding it totally reflects the light.

If a tank half full of water has some benzine on the top, the two liquids, owing to their different specific gravities, do not mix. As the benzine has the higher index, a beam of light from above may be totally reflected at the surface of the water and emerges upwards, the surface common to the two liquids, seen obliquely from above, glistening like polished silver. If a tank, containing carbon-disulphide be filled up with water, the lower liquid has the higher refractive index, so that but little light is reflected and the boundary surface will appear a dull matt grey.

A tank filled with water has a glass window with a collimating lens behind which is a light, and an aperture opposite to it which can be opened by a tap. On opening the aperture the light, in a parallel beam, emerges with

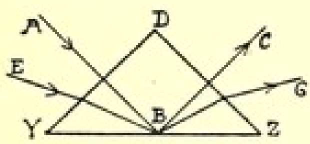


FIG. 48.

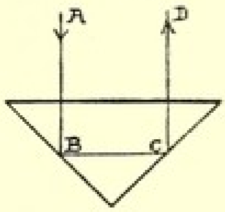


FIG. 49.

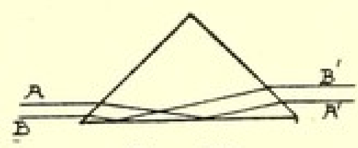


FIG. 50.

the stream of water, which it follows on account of internal reflection. The appearance of the jet is such that it is called the *cascade of silver*; were the jet of water perfectly smooth it would appear dark. Similarly, if a solid bent tube of glass has a strong source of light brought close to one end, and the other is placed against the opening of the stage of a microscope, the light traversing the tube by internal reflection forms a powerful evenly-illuminated disc under the stage, and a slide will be uniformly illuminated from below. Here no light is scattered since the tube is smooth.

**Reflecting Prisms.**—If the principal angle of a prism exceeds twice the critical angle of the medium of which it is made, total reflection ensues for incident light. All glass has a critical angle of less than 45°. If, therefore (Fig. 48), a ray $A\,B$ enters a right-angled prism normally making an angle of 45° with the normal to the surface $Y\,Z$, the ray will be totally reflected in the direction $B\,C$. The light is not refracted at the surfaces $D\,Y$ and $D\,Z$ of the prism because it is incident normally to each. Thus a right-angled prism serves as a total reflector when the light is incident perpendicularly to the one face, the direction of the emergent light being at right angles to the original course. There is reflection even if the light does not enter at right angles to $D\,Y$, provided it makes, after refraction, an angle greater than 42°

with the normal to the hypotenuse *Y Z*. Thus the ray *E B G* will be also totally reflected. The dispersion which takes place as the ray enters is reversed as it leaves the prism, so that the emergent ray consists of white light similar to that which entered.

If the light falls normally on the hypotenuse side of a right-angled prism it causes total reflection twice at *B* and *C*, as in Fig. 49, so that the final direction *C D* of the light is parallel to its original course *A B*. These forms of prisms are, with variations of shape, extensively employed in prism binoculars, range finders, etc.

By means of a right-angled prism, as indicated in Fig. 50, vertical without lateral inversion may be obtained. This prism is largely used in process photography.

Use is made of the property of total reflection in order to learn whether a prism is ground to a right angle. If the ray *l m* (Fig. 51) enters the

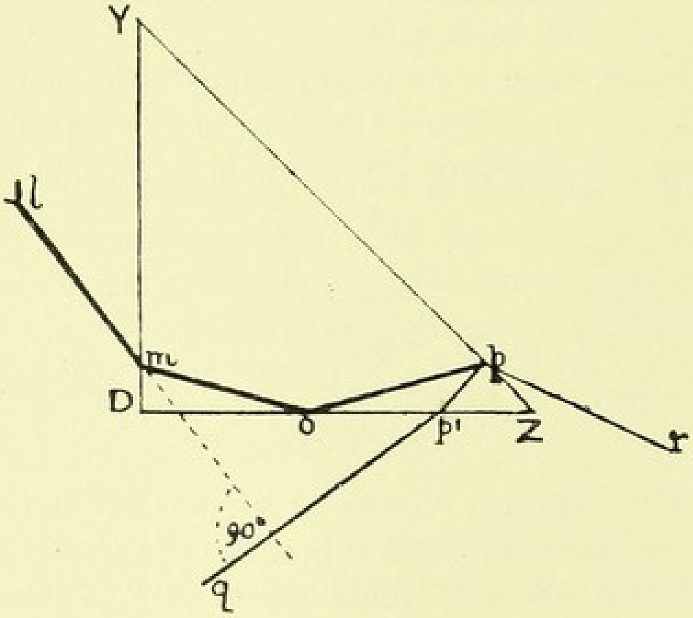



Fig. 51.

prism at *m* it is reflected at *O* in the direction *p*, and is then partly reflected and partly refracted. The reflected ray emerges at *p′* in the direction *p′q*, which, if the prism be truly worked, makes an angle of 90° with *l m* produced, no matter what the direction of *l m* may be; but if there is any error in the angles of the prism, *p′q* will not meet *l m* at right angles.

**Displacement due to Refraction.**—In Fig. 52 let *C* be a luminous point in a dense medium from which, after refraction, rays diverge away from the normal at the surface of the medium, and enter an observer's eye. These rays being projected backwards intersect at *C′*, the virtual image of *C*, which is situated nearer to the refracting surface, at a point dependent on the obliquity of the emergent rays and the index of refraction of the dense medium. This explains why a stick *A B C* partly immersed in water, in an oblique direction, appears bent towards the surface, the bend commencing at the level of the water.

The apparent position of an object in a dense medium depends upon the position of the observing eye with respect to the surface; the nearer the eye to the latter, the greater must be the obliquity of the emergent light and the greater also the apparent raising of the object. If the eye be practically on the surface of the medium, the object is also apparently raised to the surface,

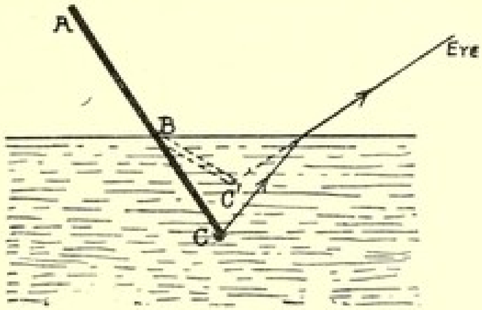



FIG. 52.

but is very distorted and indistinct. This will be seen from Fig. 53, which represents the surface of a dense medium and $O$ the object. A ray $O\,A\,N$, normal to the surface, passes out unrefracted, but other rays from $O$ which are oblique to the surface, are bent away from the normal and when referred

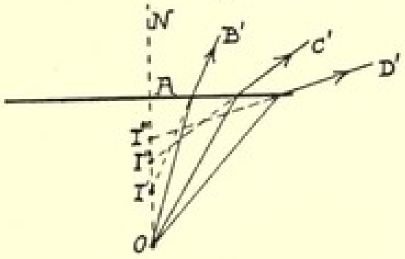



FIG. 53.

back by the eye appear to come from points $I'$, $I''$, $I'''$ on $O\,N$, these being the images of $O$ when the eye is at $B'$, $C'$, and $D'$ respectively. Actually, however, the images are formed nearer than $O\,N$ on a curve, this being known as a *caustic by refraction*, but it is sufficient for our purpose to consider the

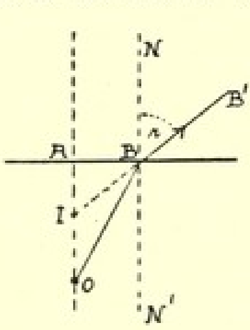



FIG. 54.

images as lying on the normal $O\,N$. The exact position of any particular image depends upon the obliquity of the line of vision, the $\mu$ of the medium, and the depth of the object, and can be calculated as follows.

In Fig. 54, let $O$ be the object and $O\,B$ any ray making an angle with the normal $N\,N'$. After refraction it will take the course $B\,B'$, so that to an eye at $B'$ the object is apparently raised to $I$. It is required to find the apparent depth of the image, i.e. the distance $AI$ in terms of the real depth $AO$, $\mu$ and $r$, the angle of view with the normal.

Now the angle $AOB = OBN' = i$ and $AIB = NBB' = r$

Also $AO = AB/\tan i$, and $AI = AB/\tan r$

Therefore $AI/AO = \tan i/\tan r$

and

$$AI = AO\frac{\tan i}{\tan r} = AO\frac{\cos r}{\mu \cos i}$$

Let the real depth $AO$ be $t$ and the apparent depth $t'$. Then

$$t' = \frac{t \cos r}{\mu \cos i}.$$

Thus, knowing $\mu$ and $r$ we can calculate the value of $i$, and after that $t'$ from the known depth $t$ for any angle of view.

**Vertical Displacement.**—If, however, the eye be on, or near to, the normal $OA$, the above expression can be greatly simplified, because then the angles $r$ and $i$ are very small, only a very narrow pencil being able to enter the pupil of the eye. In these circumstances both $\cos r$ and $\cos i$ are practically unity, since the cosine of $0°$ is 1. Therefore without any appreciable error we may say that

$$t' = \frac{t}{\mu}$$

that is, *the apparent depth of a medium viewed vertically from above is equal to the real depth divided by the $\mu$ of the medium.* If the medium be water whose index is 4/3, then the apparent depth is 3/4 that of the real depth; with glass $\mu = 1{\cdot}5$, $t' = 2/3\ t$.

On the other hand, if the eye were supposed to be in the dense medium and viewing an object in air, the apparent position of $O$ would be greater than the real distance such that $t' = t\mu$.

An object in a dense medium is apparently raised a distance $d$ of

$$d = t - t/\mu = t\,\frac{(\mu - 1)}{\mu}$$

$(\mu - 1)/\mu$ is about 1/4 in the case of water, and 1/3 for glass.

The foregoing explains why a fish appears nearer the surface than it really is, and also why, when the eye is near the surface, it appears distorted, being thinner if length-ways (parallel) to the surface, and stunted if viewed with its head towards the surface. Light from its under portions suffers relatively more deviation than that from the upper, thus giving the idea of vertical compression. Supposing the course of a bullet to be unaltered by the water, one would have to aim, with a rifle, well beneath a fish in order to hit it. To reach a coin at the bottom of a bath one would have to dive towards a point apparently nearer. Again if a coin were hidden from view by the rim of a basin, it may come into view if water be poured into the basin.

**Refraction through a Parallel Plate.**—If a ray *A B* (Fig. 55) be incident on a medium with parallel surfaces such as a plate of glass in air, it will be refracted towards the normal at the first surface in the direction *B C*, and will emerge at the second as *C D* parallel to its original course *A B*. This is due to the refraction at the second surface being exactly reversed to that at the first, so that the angular deviation is zero.

**Lateral Displacement.**—The ray, however, as a whole is laterally displaced over the distance *H D*, the extent depending upon the angle of incidence $i$, the $\mu$ of the medium, and its thickness $t$. Let $d$ be the displacement and $r$ the angle of refraction in the plate.

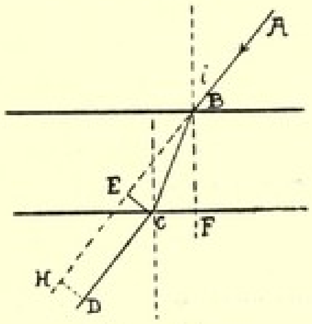



FIG. 55.

Now $HD = EC$, the angle $EBF = i$, $CBF = r$, and $EBC = i - r$, while $FB = t$ the thickness.

Then $$EC = BC \sin EBC = BC \sin (i - r)$$

but $$BC = BF/\cos CBF = t/\cos r$$

Therefore $$EC = d = t\,\frac{\sin (i - r)}{\cos r}$$

The value of $r$ must be first found from $i$ and $\mu$, and then the displacement $d$ calculated.

If the angle of incidence be small, and the displacement taken as if it were on the flat surface, we can get an approximate value for $d$ as follows:

$d = t(\tan i - \tan r)$, and substituting $\tan i/\mu$ for $\tan r$ we have $d = t$ $(\tan i - \tan i/\mu)$

Whence $$d = \frac{t \tan i\,(\mu - 1)}{\mu}$$

Lateral displacement causes slight distortion of a near object when viewed through a plate, but if the thickness is small, the effect is unappreciable. If the object be at such a distance that the incident light may be considered parallel, there can be no distortion whatever.

**Multiple Parallel Media.**—If any number of parallel plates of different indices be superposed their combined action is similar to that of a single plate of uniform index. That is, light incident on the first surface emerges parallel from the last *provided always that the first and the last media have the same index and that the various component layers have themselves parallel surfaces.* The refraction that occurs on the passage of light through various parallel media would be such that sin $i \times \mu$ of 1st medium = sin $r \times \mu$ of last medium, and if the 1st and last media are of equal optical density sin $i$ = sin $r$. When the first and last media have not the same index the deviation suffered by the light, on emergence, is the same as if the light entered from the first directly into the last medium.

**Lateral and Vertical Displacements.**—In Fig. 56 the point $L$ viewed obliquely by an eye at $A\ A$ through a transparent medium $N$, whose two refracting surfaces are plane and parallel, is seen as $L'$ laterally displaced and nearer. If $L$ is viewed from $B$ situated vertically above, it appears to be

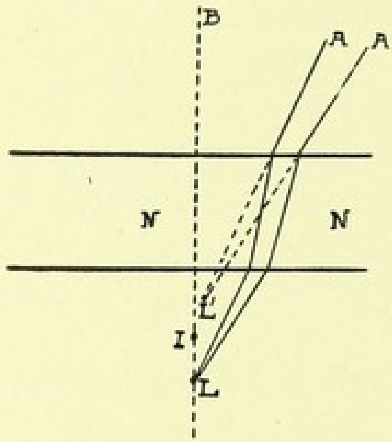



Fig. 56.

nearer, at $I$, but not laterally displaced. It is easily proved that the distance $L\ I$ is given by

$$\frac{t\ (\mu - 1)}{\mu}$$

as for an object situated in a dense medium as previously described. Thus for glass of index 1·5, any object viewed through a parallel plate would always appear nearer by about 1/3 the thickness of the plate.

**Eye in Dense Media.**—There is, of course, no critical angle for light passing from a rare into a dense medium, so that to an eye under, say, water, a field of 180°, including all external objects down to the surface, is visible. Owing, however, to refraction, the light from external objects is crowded into a cone whose angle is twice the critical angle of—in this case—water. This can be seen from Fig. 57, where $E$ is an eye looking upwards from the dense medium.

Rays $A\ B$ and $D\ C$ from objects level with the surface are practically parallel to the latter and therefore are refracted into the water at the critical angle $E\ B\ N'$ and $E\ C\ N'$ and are referred back in the direction $G$ and $F$. The cone $F\ E\ G$ contains a distorted view of all external objects, and its angle $CEB$ is equal to the sum of the angles $EBN'$ and $ECN'$, that is, to twice the critical angle of water—about 96°. Also, as previously mentioned, objects directly above appear more distant by an amount equal to about 1/3

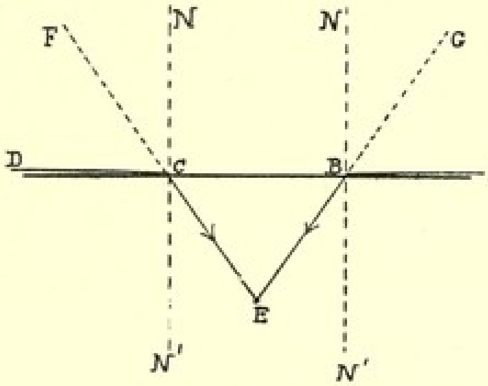



Fig. 57.

their real distance, but those closer to the surface are displaced to a rather greater extent. The distortion and indistinctness are greatest for objects near the surface, and disappear for those directly above.

## CHAPTER V

### REFRACTION BY PRISMS

If the two surfaces of a refracting medium are not parallel to each other, all incident light must suffer refraction, since no ray can be perpendicular to both surfaces.

**Prism.**—An optical prism is a transparent body, usually made of glass, but it may for special reasons consist of quartz, rocksalt, fluorspar, etc. It has two plane refracting surfaces *A B*, *A C* (Fig. 58), which meet in a line at *A*, termed the apex or edge of the prism. The third side *B C*, opposite this edge and joining the two refracting surfaces, is called the base. The latter may slope in any direction, as it does not affect the course of the light.

If a ray be incident in a direction perpendicular to the first surface, it

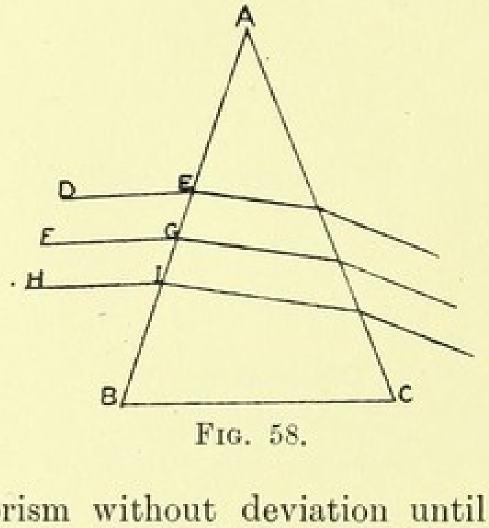



Fig. 58.

passes through the prism without deviation until it reaches the second surface, when it is refracted away from the normal. If a ray be incident otherwise than normally on the first surface, as it passes from the rarer into the denser medium, it is refracted towards the normal to the first surface, and on emergence is again refracted at the second surface as it passes from the denser into the rarer medium.

Provided that the angle of incidence be the same, the rays are refracted to the same extent, no matter on what part of the first surface of a prism they are incident. If the rays (Fig. 58), incident on the prism, are parallel before refraction, they are similarly situated in relation to each other after refraction, and emerge from the prism parallel. If they are divergent before refraction (Fig. 59) they emerge from the prism divergent; if they are convergent, they are convergent on passing out. Nevertheless, as will be seen later, the degree of divergence or convergence is not quite the same after refraction as before.

**The Principal Angle.**—In Fig. 58 the angle formed at $A$, by the two refracting surfaces, is called the *principal angle,* sometimes known as the

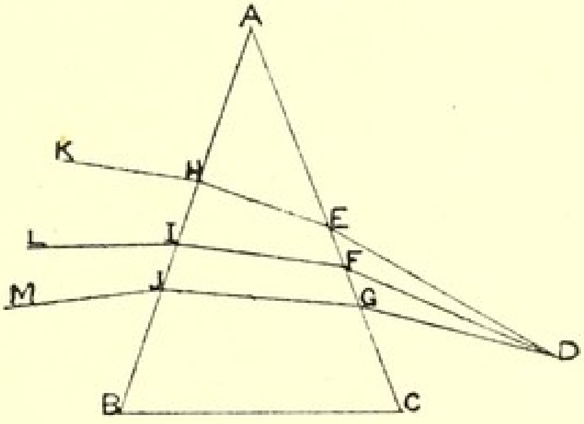



FIG. 59.

*refracting angle*, or *angle of inclination.* While the principal angle merely indicates the shape of the prism, yet the refracting power of the latter is governed chiefly by it.

**The Degree.**—A prism is usually indicated by the number of degrees included between its two inclined sides. A prism of, say, 10° is one in which the two sides enclose an angle of that amount.

**The Angle of Deviation.**—Let the incident ray $D\,E$ (Fig. 60) be directed towards a point $H$ in the centre of the prism. Being refracted at $E$ it takes

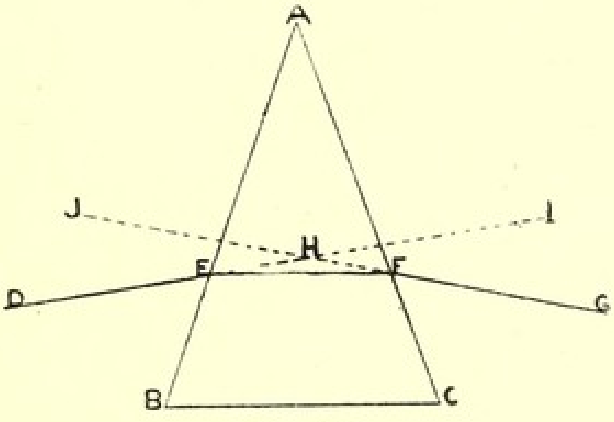



FIG. 60.

the direction $E\,F$ and passes out in the direction $F\,G$ as if proceeding from $J$. The angle of deviation of the prism is, in this case, $I\,H\,G$, because $D\,E$, instead of following the path $H\,I$, appears after refraction to follow the path $J\,F\,G$. An object at $D$, when viewed through the prism from $G$, appears as if it were situated at $J$. The deviating angle constitutes the important optical property of a prism and expresses its power or refracting effect.

**Defining Terms.**—In the prism (Fig. 61) the line of junction *A B* of the two refracting surfaces is termed the *edge*. *F C D E* is the *base*, *A B D C* and *A B E F* are the two *refracting surfaces*. The plane *A B K I* containing the edge of the prism and situated symmetrically with respect to the two surfaces is the *base apex plane ;* generally it bisects the base. Any line, as *L M*, at right angles to the edge of the prism and lying in the base-apex plane, is a *base apex line*. The line *G H*, parallel to the edge and lying in the

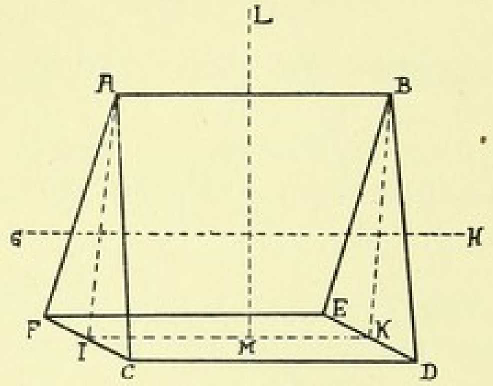



Fig. 61.

base-apex plane, midway between the edge and base, is the *axis* of the prism. A *principal section* of a prism is any section, as *A F C*, cutting it from edge to base perpendicularly to the axis.

**Shape of Prism.**—A prism, as regards the outer margins of its refracting surfaces, may be of any shape—square, circular, or oval ; neither the shape nor size of its refracting surfaces influences the course of the light passing through it.

In a circular or oval prism the thinnest part *L* (Fig. 62) is considered to

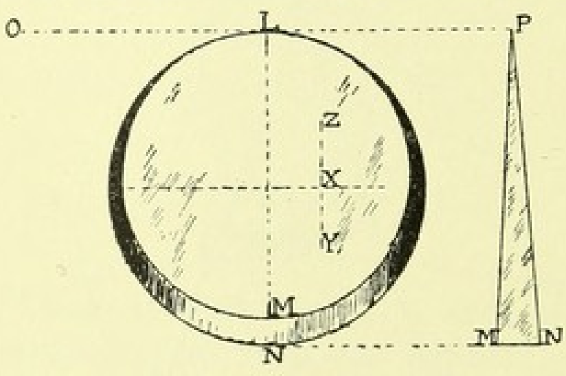



Fig. 62.

be the *apex*. *M N* is the base opposite to the apex. The *central* line *L M* of the plane (*A B K I* of Fig. 60) connecting the thinnest and thickest parts of a round or oval prism is called its *base apex line:* It is usually marked on the circular prisms of the trial case by two small scratches, one at the apex and the other at the base. *O P*, tangent to *L* and perpendicular to *L M*, is the imaginary edge. *P M N* shows a section of such a prism along the base-apex line.

**Deviation by a Prism.**—The apparent deviation of an object caused by a prism is the combined result of the refraction suffered by the rays at the two surfaces, and although commonly said to be towards the apex it is actually towards the edge of a square prism, or the imaginary edge of a circular prism, in a direction parallel to the base-apex plane. A ray incident at $X$ (Fig. 62), from an object beyond the prism, is refracted towards $Y$ and is referred back towards $Z$, the line of deviation $Z\,X\,Y$ being parallel to the base apex-line $L\,M$. The effect of a prism is to bend the light *towards the base* and an object seen *through the prism* appears deviated *towards the edge.*

**Deviating Power of Prism.**—The deviation of a ray passing through a prism depends on (1) the angle of the prism, (2) the index of refraction of the medium, and (3) the angle of incidence of the ray. The larger the angle formed by the two refracting surfaces the greater is the angle formed by the incident ray and the normal, and, therefore, the greater is the deviating effect of the prism. The deviating effect also depends on the index of refraction of the medium of which the prism is made, since the higher the index the more is a ray, incident at a given angle, refracted.

As will be seen in the next paragraph the deviation of a ray passing through a prism is a minimum when the incident ray makes a certain angle with the first surface, and since, unless otherwise stated, minimum deviation is implied, (3) above is usually ignored when the power of a prism is considered.

**Minimum Deviation.**—For every prism there is one position in which an incident ray will be less deflected than in any other. From this position, if the prism be rotated round its axis so that either the edge or base is advanced towards the source of light, an object viewed through the prism appears still more deviated towards the edge of the prism.

*Minimum deviation obtains when the incident and emergent rays are equidistant from the edge, and, as shown in Fig.* 63, *the angles of incidence and emergence* ($i$ *and* $e$) *are also equal.* In this position the course of the ray, as it traverses the prism, forms, with its sides, the base of an isosceles triangle, and a perpendicular let fall on it from the prism apex will bisect it.

For any other incidence of the ray, as $i$ increases, $e$ decreases less rapidly; while if $i$ decreases, $e$ increases more rapidly, so that, in any case, the total deviation is greater.

**To Calculate $\mu$ of a Prism.**—In the prism $B\,A\,C$ (Fig. 63) $P$ is the principal angle and $d$ is the deviating angle. Let $i$ be the angle of incidence, $r$ the angle of refraction at the first service, $u$ the angle of incidence at the second surface, $e$ the angle of emergence which the refracted ray makes with the normal $F\,N$, and $\mu$ the index of the prism.

Draw $A\,M$ to bisect the principal angle $P$. Produce the incident ray $D\,E$ to $K$, and produce the emergent ray $G\,F$ backward to meet $D\,K$ in $Q$;

then the total deviation of the incident ray is equal to $d$. Produce the normals $L\ E$ and $N\ F$ to meet at $M$.

As the prism is in the position of minimum deviation

$$i = e, \text{ and } i = p + r, \text{ and } e = q + u.$$

In the triangle $Q\ E\ F$ the external angle $d$ equals the two equal internal and opposite angles $p$ and $q$.

In the right-angled triangle $A\ E\ M$ the angle at $A = 90° - A\ M\ E$. In the right-angled triangle $O\ E\ M$, the angle $r = 90° - A\ M\ E$. Therefore $O\ E\ M = E\ A\ M$. That is $r = \mathrm{P}/2$.

Now the angle $p = q = d/2$

Then $$i = r + p = \mathrm{P}/2 + d/2 = (\mathrm{P} + d)/2$$

Therefore

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin\left(\frac{\mathrm{P}+d}{2}\right)}{\sin\left(\frac{\mathrm{P}}{2}\right)}$$

This formula enables us to find the index of refraction of a prism when $P$ the principal angle and $d$ the angle of minimum deviation are known, $P$ and $d$ being measured by the spectrometer (q.v.).

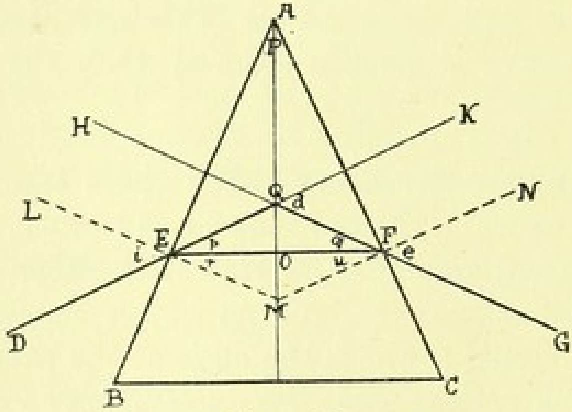



FIG. 63.

**Example.**—What is the index of a prism whose angle of minimum deviation is 28° and principal angle 45°? We have

$$\mu = \frac{\sin\left(\frac{\mathrm{P}+d}{2}\right)}{\sin\left(\frac{\mathrm{P}}{2}\right)} = \frac{\sin\left(\frac{45°+28°}{2}\right)}{\sin\left(\frac{45°}{2}\right)} = \frac{\sin\ 36°\ 30'}{\sin\ 22°\ 30'} = \frac{\cdot 5948}{\cdot 3827} = 1 \cdot 554$$

**To Calculate the Angle of Deviation.**—If $\mu$ and $P$ are known $d$ can be found thus:—

Since $$\mu = \frac{\sin\left(\frac{\mathrm{P}+d}{2}\right)}{\sin\left(\frac{\mathrm{P}}{2}\right)} \qquad \mu \sin \mathrm{P}/2 = \mathrm{Sin}\ (\mathrm{P} + d)/2$$

Let $(\mathrm{P} + d)/2$ be called $a$.

Then $2a = \mathrm{P} + d$ and $2a - \mathrm{P} = d$.

To find the value of $d$ we require two steps, thus

(1) Find $a$ from $\sin a = \mu \sin P/2$ ; (2) then $d = 2a - P$.

Expressed as a formula this becomes

$$d = 2\left[\sin^{-1}(\mu \sin P/2)\right] - P$$

that is, $d$ equals twice that angle whose sine is $\mu \sin P/2$ less P.

**Example.**—What is the angle of deviation of a prism whose principal angle is 60° and index 1·62 ?

Here $\mu \sin P/2 = 1{\cdot}62 \sin 30° = 1{\cdot}62 \times {\cdot}5 = {\cdot}81$

and ${\cdot}81 = \sin 54°$ (nearly)

Therefore $d = (2 \times 54) - 60 = 48°$

**To Calculate the Principal Angle.**—The angle $P$ at which a prism of known index must be ground, so that a certain angle of deviation be obtained is found as follows :—

$$\mu = \frac{\sin\left(\frac{P+d}{2}\right)}{\sin\left(\frac{P}{2}\right)} = \frac{\sin\left(\frac{P}{2}+\frac{d}{2}\right)}{\sin\left(\frac{P}{2}\right)}$$

$$= \frac{\sin P/2 \cos d/2 + \cos P/2 \sin d/2}{\sin P/2} = \cos d/2 + \frac{\sin d/2}{\tan P/2}$$

that is

$$\mu = \cos d/2 + \frac{\sin d/2}{\tan P/2}$$

whence

$$\tan P/2 = \frac{\sin d/2}{\mu - \cos d/2}$$

**Example.**—What angle must be given to a prism of 36° minimum deviation when $\mu = 1{\cdot}586$ ?

$$\tan\frac{P}{2} = \frac{\sin 18°}{1{\cdot}586 - \cos 18°} = \frac{{\cdot}3090}{1{\cdot}586 - {\cdot}9511} = \frac{{\cdot}3090}{{\cdot}6349} = {\cdot}4882$$

whence $P/2 = 26°$ (nearly), and $P = 52°$

**Simplified Formulæ.**—When the angle of incidence or emergence is zero, i.e. when the incident ray is normal to the first surface, or the emergent ray is normal to the second, the formulæ for finding $\mu$, $d$ or $P$ when the other two values are known, become simplified to

$$\mu = \frac{\sin(d+P)}{\sin P}$$

$$d = \left[\sin^{-1}(\mu \sin P)\right] - P$$

and

$$\tan P = \frac{\sin d}{\mu - \cos d}$$

**Further Simplified Formulæ.**—By substituting angles for their sines, which can be done without serious error when the angle of the prism is small, as in ophthalmic prisms, the formulæ may be greatly simplified. The original formula can be written

$$\mu = \frac{d + P}{P} = \frac{d}{P} + 1$$

whence $$d = P\,(\mu - 1)$$

and $$P = \frac{d}{\mu - 1}$$

If the refractive index $= 1 \cdot 5$ then $\mu - 1 = 1/2$ and

$$d = \frac{P}{2}.$$

Thus for a prism of 5° the angle of deviation would be taken to be 2° 30′. It is thus usual to consider that the deviation of ophthalmic prisms is half the principal angle, although the glass used has an index slightly greater than 1·5. In addition, when a prism is thin, any moderate departure from the position of minimum deviation does not result in any appreciable increase of deviation, so that this factor may also be ignored.

**Examples.**—If the refracting angle of a prism is 10° and the deviating angle 5·25, then

$$\mu = \frac{5 \cdot 25}{10} + 1 = 1 \cdot 525.$$

A prism of 10° principal angle, whose index is 1·54, has an angle of deviation of

$$d = 10 \times \cdot 54 = 5 \cdot 4° = 5° \; 24'.$$

If a prism of 6·25° deviation is required, the index of refraction being 1·56, the prism angle is

$$P = \frac{6 \cdot 25}{\cdot 56} = 11 \cdot 166 \text{ or } 11° \; 10'.$$

**The Angle of Incidence.**—In Fig. 63, we have seen that, when deviation is a minimum, the angle of refraction $r$ at the first surface is equal to half the principal angle $P$, so that

$$\sin i = \mu \sin r = \mu \sin \frac{P}{2}$$

When the incidence is normal at the first surface, the principal angle of the prism is equal to the angle of incidence at the second surface; when the emergent light is normal at the second surface, the principal angle is equal to the angle of refraction at the first surface. Hence for normal emergence $\sin i = \mu \sin P/2$.

For thin prisms, i.e. those having $P$ of not more than about 10° or so, we may omit the sines and then

$$i = \frac{\mu P}{2}$$

so that if $\mu$ be taken as 1·5, which is about that found in ophthalmic prisms, $i = 3P/4$.

**Neutralising Prisms.**—Two prisms of similar angle $d$ will, when placed in opposition, i.e. base to apex, neutralise each other. If they are also of similar $P$ and $\mu$ they act as a plate, having parallel surfaces, on light passing obliquely through them. If the $\mu$'s are unequal, so also must be the angles $P$, that is, $P_1 (\mu_1 - 1) = P_2 (\mu_2 - 1)$.

Therefore to calculate the thin prism $P_1$, made of glass of a certain index of refraction, which will neutralise the deviation of another $P_2$, whose index is different, we have only to put

$$P_1 = \frac{P_2 (\mu_2 - 1)}{(\mu_1 - 1)}$$

Then the prisms, being placed base over apex, act as a single plate

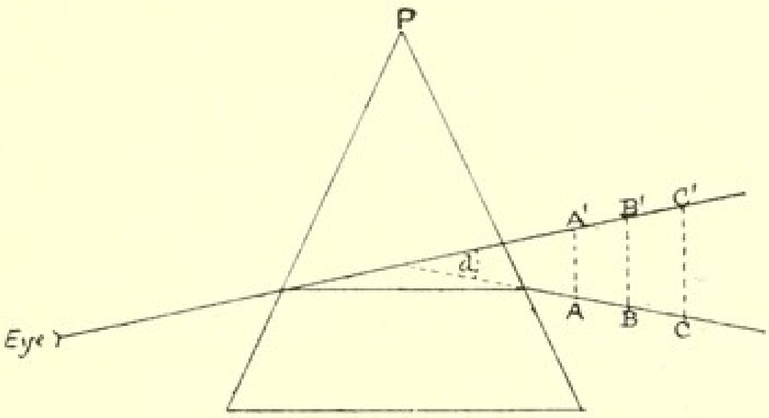



FIG. 64.

(although the surfaces are not parallel) since their respective deviations are equal and opposite.

Thus if a crown glass prism of 15°, whose $\mu = 1·54$, has to be neutralised by a flint prism whose index = 1·62, then from the above formula

$$P_2 = \frac{15 \times ·54}{·62} = 13°.$$

Of course, should the prisms be thick, $P_1$ and $P_2$ are not then directly proportional to $d$ or to $(\mu - 1)$, so that here the value of the deviation of P must first be calculated from the exact formula, and from this the corresponding value of $P_2$ for an index of $\mu_2$ may be obtained.

**Displacement by a Prism.**—In Fig. 64 the object $A$ is seen, through the prism, at $A'$; if the object is at $B$ or at $C$, it is seen at $B'$ or $C'$ respectively. The angular displacement of the object by a given prism depends entirely on the magnitude of $d$, and no matter how near or how distant the object (as may be seen from the figure) the angle $d$ is invariable; but the actual displacement $A\ A'$, $B\ B'$, $C\ C'$ is proportional to the distance of the object, and is represented by the tangent of the angle of deviation at the distance.

**Construction.**—To trace the course of a ray $D\ E$ refracted by the prism $A\ B\ C$ of $\mu$ 1·5 it is only necessary to use a double construction like that of Fig. 46. Draw $D\ F$ (Fig. 65) normal to $A\ C$ and divide $E\ F$ into three equal parts; from $E$ on $E\ A$ mark off $E\ G$ equal to two such parts. Draw $G\ I$ normal to $E\ A$ and connect $E$ with $G\ I$ by a line $E\ H$ (cutting $A\ B$ at the

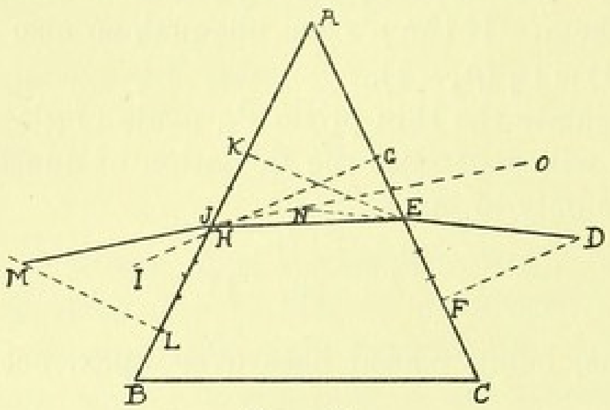



FIG. 65.

point $J$) such that $E\ H = E\ D$. Then $E\ J$ will be the direction of the refracted ray in the prism. Draw $E\ K$ normal to $A\ B$; divide $J\ K$ into two equal parts; on $J\ B$, from $J$, mark off $J\ L$ equal to three such parts; draw $M\ L$ normal to $A\ B$; connect $J$ with $M\ L$ by a line $J\ M$, such that $J\ M = J\ E$. Then $J\ M$ is the direction of the ray of emergence. The angle of deviation $D\ N\ O$ is found by prolonging $M\ J$ backwards and $D\ E$ forwards so that they meet at $N$.

Should the $\mu$ of the prism have any other value than 1·5 then $E\ F$ and $J\ L$ must be to $E\ G$ and $J\ K$ respectively as $\mu : 1$. Thus if $\mu = 1{\cdot}6$ the proportional parts would be 16 and 10, or 8 and 5.

## CHAPTER VI

## REFRACTION BY CURVED SURFACES

**Curved Surface.**—A refracting surface is one which separates two media of different densities, so that, when light passes from the one to the other, refraction takes place. Only one refraction occurs and in this respect a surface differs essentially from a lens, where there are at least two surfaces and two refractions of the light which traverses it.

Since every line drawn from the centre to the circumference of a sphere

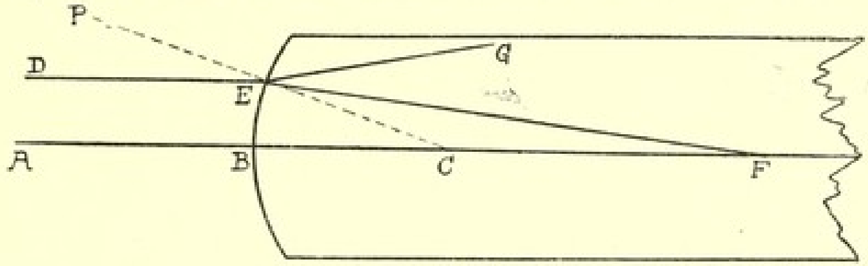



FIG. 66.

is a radius of curvature, every point on the circumference may be regarded as a minute plane at right angles to a radius. Thus $CE$ (Fig. 66) is a normal to the surface at $E$, and also when prolonged beyond the circumference.

**Course of Light.**—Let Fig. 66 represent a transparent body having a curved surface with its centre at $C$. Any ray of light $AB$ or $PE$ directed towards $C$, is normal at the point of incidence, and passes into the medium

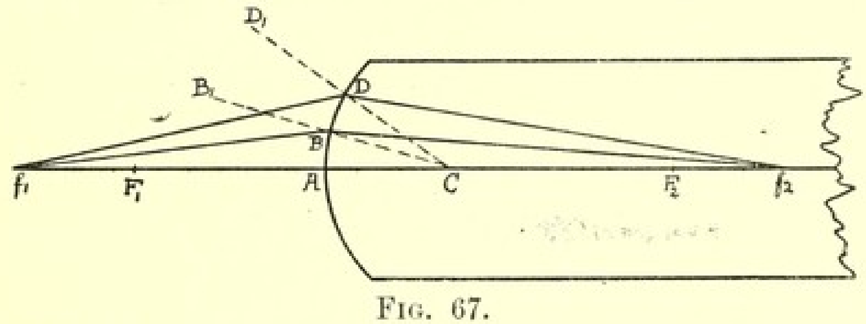



FIG. 67.

without deviation. A ray $DE$, which is not normal to the surface, is bent towards the perpendicular $PEC$ in the direction E F, if the medium is of a higher index of refraction, or it is bent away from the perpendicular in the direction $EG$, if of a lower index than that of the medium from which the light proceeds. Both media are supposed to be of indefinite extent.

**Cx. and Cc. Surfaces.**—In Fig. 67 let a mass of glass have a convex surface, and the outer medium be air. The ray $f_1 A$ directed towards $A$ is normal to the refracting surface and passes onward without deviation. The rays $f_1 B$ and $f_1 D$ form certain angles with the normals to the surface, and each, on passing into the denser medium, is bent towards the perpendicular to an amount governed by the ratio between the sines of the angles of incidence and refraction. Thus $f_1 D$ is bent more than $f_1 B$, and the two meet the line $f_1 f_2$ at the point $f_2$. Similarly, all rays diverging from $f_1$ are refracted to $f_2$; $f_2$ is, therefore, called the focus or the image of the source of light $f_1$, and the points $f_1$ and $f_2$ are conjugate foci. If the object were at $f_2$, the image would be at $f_1$.

The focus thus formed by a *convex* surface of a medium of higher index

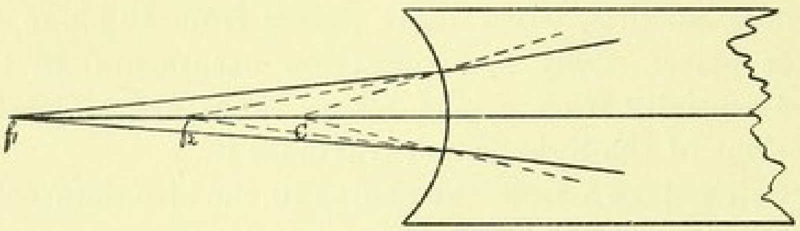


FIG. 68.

of refraction is *positive or real*. If the medium is of lower index, light entering it is rendered divergent, and the focus is *negative or virtual*. If the surface is concave, as in Fig. 68, the reverse is the case, and $f_2$ is virtual and on the same side of the surface as $f_1$. The boundary plane between the two media may be regarded either as the convex surface of the one or the concave surface of the other, but it is more convenient to express it as convex or concave to the medium of lower index, which usually is air. Thus for a dense medium having a convex surface in contact with air we could calculate the position of $f_2$ from the refractivity and curvature of the dense medium, or from those of the rare medium, and the result is the same in the two cases,

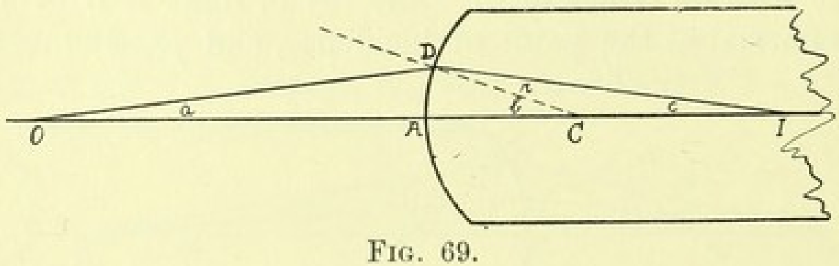



FIG. 69.

for if the curvature of the latter is taken as negative while that of the former is positive the refractivity of the latter is also negative.

**Defining Terms.**—The line $f_1 A f_2$ (Fig. 67), which is perpendicular to the refracting surface and passes through the centre of curvature $C$ and the principal foci, is the *principal axis;* all other lines passing through $C$ are *secondary axes.* $A C = r$ is the *radius of curvature*, and $C$ is the *centre of curvature.* $f_1$ and $f_2$ are the positions occupied by object and image. $F_1$ is the *anterior principal focus* formed by the light proceeding from a distant source on the principal axis in the denser medium, and $F_2$ is the *posterior principal focus* formed by light proceeding from a distant source in the rarer medium.

**Formulæ connecting $f_1$ and $f_2$.**—In Fig. 69, let $O$ be any object on the axis of a single surface, and $I$ its image formed by direct refraction of the ray $O\,D$ incident at $D$. From $C$ draw the radius $C\,D$ and let the angle $A\,O\,D = a$, $A\,C\,D = b$ and $A\,I\,D = c$. Suppose the indices of refraction of the first and second medium be $\mu_1$ and $\mu_2$.

Then $$\mu_1 \sin i = \mu_2 \sin r$$

But $$i = a + b \text{ and } r = b - c$$

Therefore $$\mu_1 \sin (a + b) = \mu_2 \sin (b - c).$$

If the incident pencil be considered small and axial, the angles $a$, $b$ and $c$ are also small, so that we can write

$$\mu_1(\sin a + \sin b) = \mu_2(\sin b - \sin c).$$

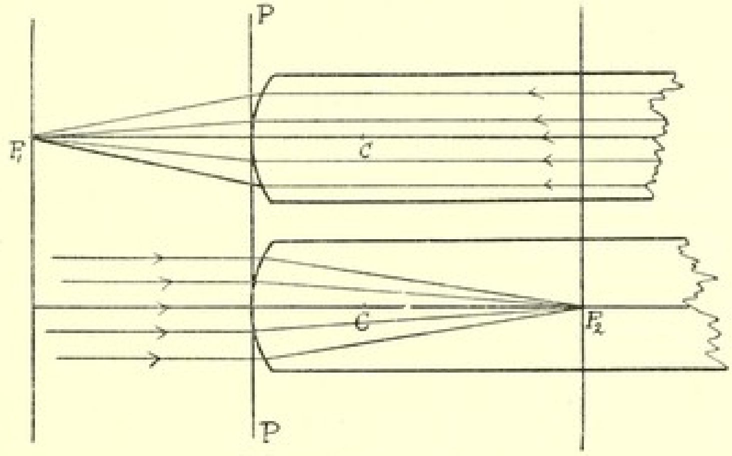
Let $O\,A = f_1$, $A\,I = f_2$ and $A\,C = r$ the radius of curvature.

Then replacing sines by circular measure

$$\mu_1\left(\frac{1}{f_1} + \frac{1}{r}\right) = \mu_2\left(\frac{1}{r} - \frac{1}{f_2}\right) \qquad \text{or} \qquad \frac{\mu_1}{f_1} + \frac{\mu_1}{r} = \frac{\mu_2}{r} - \frac{\mu_2}{f_2}$$

Therefore $$\frac{\mu_1}{f_1} + \frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r}$$

**The Focal Lengths and Power of a Curved Surface.**—The refractive power of a curved surface depends on its curvature and the refractive




FIGS. 70 AND 71.

index of the medium, so that an increase in either is accompanied by increase of power. The focal length depends on the refractive power, the one being inversely proportional to the other, i.e. the greater the power, the shorter is the focal length.

In Figs. 70 and 71 $P$ is the principal or refracting plane of the surface at which all refraction is presumed to take place. $C$ is the optical centre (or nodal point) because all rays passing through it are unrefracted. In Fig. 70 the light is parallel to the principal axis in the dense medium, and on emergence into the rare medium is refracted so as to meet at the point $F_1$ situated on the principal axis. $P\,F_1$ is the *anterior focal distance*, and $F_1$ the *anterior principal focus*.

In Fig. 71 the light is parallel to the principal axis in the rare medium, and after refraction meets at $F_2$ in the denser medium. $P F_2$ is the *posterior principal focal distance* and $F_2$ the *posterior principal focus.* In the formula given in the preceding article, if $f_2$ is at $\infty$ so that the light in the denser medium may be regarded as parallel, the term $\mu_2/f_2$ becomes $\mu_2/\infty = O$. Thus

$$\frac{\mu_1}{f_1} = \frac{\mu_2 - \mu_1}{r}$$

But $f_1$ has now become the anterior principal focal length, and may be written $F_1$. Therefore

$$F_1 = \frac{\mu_1 r}{\mu_2 - \mu_1}$$

If $f_1$ is at $\infty$ so that the light in the rarer medium (air) is parallel, the term $\mu_1/f_1$ becomes $\mu_1/\infty = O$. Therefore

$$\frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r}$$

Here $f_2$ has become the posterior principal focal length $F_2$. Therefore

$$F_2 = \frac{\mu_2 r}{\mu_2 - \mu_1}$$

The planes passing through $F_1$ and $F_2$ perpendicular to the principal axis are respectively the anterior and posterior focal planes.

If the one medium is air, $\mu = 1$, so that it can be omitted from the formulæ ; the index of the dense medium we can then call $\mu$, and therefore the formulæ become simplified to

$$F_1 = \frac{r}{\mu - 1} \qquad \text{and} \qquad F_2 = \frac{\mu r}{\mu - 1}$$

These formulæ hold good only when the focus lies within the medium to which the radius $r$ pertains.

**Examples.**—If $\mu$ of the dense medium is 1·5 and the other medium is air, for a radius of curvature of 8 in. we have

$$F_1 = \frac{8}{1{\cdot}5 - 1} = 16 \text{ in. from P, or } 16 + 8 = 24 \text{ in. from C.}$$

$$F_2 = \frac{1{\cdot}5 \times 8}{(1{\cdot}5 - 1)} = 24 \text{ in. from P, or } 24 - 8 = 16 \text{ in. from C.}$$

Thus, ordinary glass having an index of about 1·5, the anterior and posterior focal distances of a curved surface are approximately twice and three times the radius respectively.

If the surface is concave towards the light the radius is negative and would be prefixed in the formulæ by a − sign, so that $F_1$ and $F_2$ become negative quantities, and are situated on the same side of the surface as the source of light, i.e., $F_2$ is in air and $F_1$ is in the dense medium.

Thus let $r = -8''$ and $\mu = 1{\cdot}5$, then

$$F_1 = \frac{-8}{\cdot 5} = -16 \text{ in. from P, or } -16 + (-8) = 24 \text{ in. from C.}$$

$$F_2 = \frac{1{\cdot}5 \times (-8)}{\cdot 5} = -24 \text{ in. from P, or } -24 - (-8) = 16 \text{ in. from C.}$$

Suppose parallel light passes from water $\mu = 1{\cdot}33$ into glass $\mu = 1{\cdot}5$ and let the radius be eight inches; then

$$F_2 = \frac{1{\cdot}5 \times 8}{1{\cdot}5 - 1{\cdot}33} = \frac{12}{\cdot 17} = 70{\cdot}6 \text{ in. from P, or } 70{\cdot}6 - 8 = 62{\cdot}6 \text{ inches from C.}$$

If the light passes from glass into water,

$$F_1 = \frac{1{\cdot}33 \times 8}{1{\cdot}5 - 1{\cdot}33} = \frac{10{\cdot}64}{\cdot 17} = 62{\cdot}6 \text{ in. from P, or } 62{\cdot}6 + 8 = 70{\cdot}6 \text{ inches from C.}$$

In these formulæ the relative $\mu$, which equals $\mu_2/\mu_1$, can be found and the calculation then made as if the lower $\mu$ were 1.

**Relationship of $F_1$ and $F_2$.**—The anterior and posterior focal distances measured from the refracting surface are proportional to the indices of refraction of the two media.

Thus in the examples given we find respectively

$$\frac{F_2}{F_1} = \frac{\mu_2}{\mu_1} = \frac{24}{16} = \frac{1{\cdot}5}{1} \qquad \text{and} \qquad \frac{F_2}{F_1} = \frac{70{\cdot}6}{62{\cdot}6} = \frac{1{\cdot}5}{1{\cdot}33}$$

In a refracting body with a single curved surface $r = F_2 - F_1$; this holds good whatever the refractive indices may be. Thus $F_1 = F_2 - r$ and $F_2 = F_1 + r$. That $F_1$ is shorter than $F_2$ follows from the law of sines. If the two media are respectively of $\mu_1 = 1$ and $\mu_2 = 1{\cdot}5$, when light passes into the denser medium sin $r$ is 2/3 sin $i$, whereas when light passes into the rarer medium sin $r$ is 3/2 sin $i$; hence for parallel light the angular deviation is greater when the focus is formed in the air than when it is formed in the dense medium, it being in the first case about 1/3 $i$, and in the second case 1/2 of $i$, the incidence being the same in both cases.

In addition to what is stated in the first paragraph of this chapter, a surface differs from a lens in that, with the former, the first and last media being different, $F_1$ differs from $F_2$, whereas with a lens $F_1 = F_2$. Also as shown in Fig. 66, the optical centre or nodal point $C$ does not coincide with the principal point which marks the refracting plane at $B$, the apex of the surface.

**To find r or $\mu$.**—The radius or the refractive index can be found by substituting known values for the symbols given in the above formulæ, and then equating. Thus if $F_2$ be 30 in. and the indices of refraction be respectively 1·5 and 1, we have

$$30 = \frac{1{\cdot}5\,r}{{\cdot}5}, \text{ that is } r = \frac{15}{1{\cdot}5} = 10 \text{ in.}$$

If $\mu_2 = 1{\cdot}5$, r = 8, and $F_2 = 70{\cdot}6$, we find $\mu_1$ as follows :—

$$70{\cdot}6 = \frac{8 \times 1{\cdot}5}{1{\cdot}5 - \mu_1} \text{ and } 105{\cdot}9 - 70{\cdot}6\,\mu_1 = 12$$

$$\text{or } 70{\cdot}6\,\mu_1 = 93{\cdot}9. \quad \text{Therefore } \mu_1 = 1{\cdot}33.$$

All the formulæ apply when the denser medium has a concave surface, but care must be taken that the – sign be given to *F* and to *r* in all calculations.

**Secondary Axes.**—The principal axis of a refracting body passes through *C* the centre of curvature and the principal foci (Fig. 72). All other lines as *B C*, *D C* are secondary axes; they are radii of curvature of the surface and therefore normals thereto. An object point situated on the *principal axis* has its image on the *principal axis.* An object point situated on a

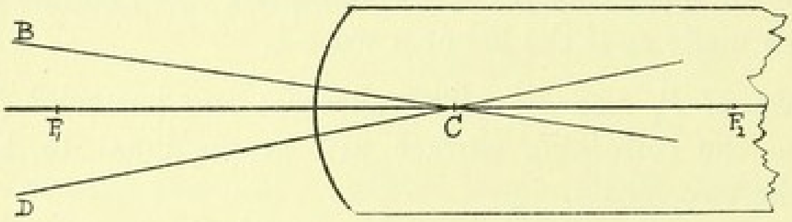



Fig. 72.

*secondary axis* has its images on *that same axis* and the focus is a *secondary focus.* An object can have only one point on it situated on the principal axis; every other point on its surface is situated on a different secondary axis, and similarly of course with the image of the object.

**Position of an Image Point.**—The image of a luminous point being on a line drawn from that point through *C*, its position on that line can be determined by calculation or construction. It is on the opposite side of the refracting surface if the rays converge after refraction; and on the same side if, after refraction, they diverge from the axis on which the point is situated. The greater the convergence or divergence the sooner do the rays meet or appear to meet and form the image of the object point from which they originally diverged.

**Construction of Image—Cx. Surface.**—In Fig. 73 *A B* is an object situated in front of the refracting surface *D P H*. Rays diverging from *A* and *B* have their images respectively at *A′* and *B′*, so that *B′ A′* is the image of the object *A B* and can be constructed in the following way.

There are three rays emanating from any point of the object, say the point *A*, the course of which can be easily traced, viz.,

(1) *A* ray *A C* directed towards the centre of curvature *C*. This being normal to the refracting surface passes into the second medium without deviation.

(2) A ray *A D* parallel to the principal axis. This, after refraction passes through the posterior principal focus $F_2$.

(3) A ray *A G* passing through the anterior principal $F_1$. This, after refraction, is parallel to the principal axis in the second medium.

The point where these rays meet at *A′* is common to all the other rays diverging from *A* and constitutes the image of that point. Similar rays from *B* form an image at *B′*. Any two of the rays mentioned suffice for the con-

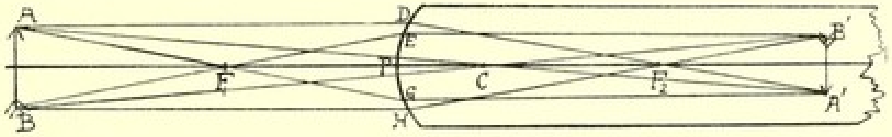



FIG. 73.

struction of the image points *A′* and *B′*, and the latter define the position and size of the entire image of the object *A B*.

The image here is *real* and *inverted*; it is *smaller* or *larger* than the object according as the image is *nearer to*, or *further from*, the centre of curvature of the refracting surface than the object itself.

If the object is nearer to the surface than $F_1$, as *A B* in Fig. 74, the light after refraction is still divergent, although less so than before refraction, and as the rays cannot meet no real image is formed. The rays can, however, be referred back so as to meet in front of the refracting surface as *A′ B′*.

This is shown by the construction employed. From *A* draw *A C*; since

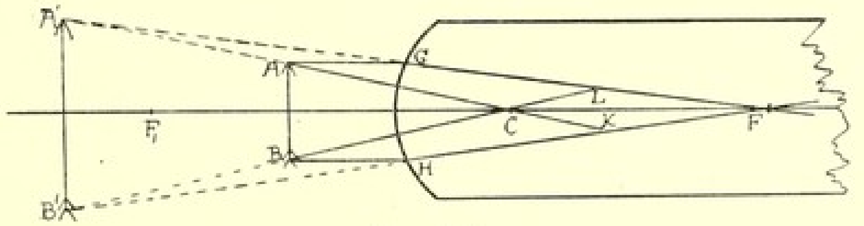



FIG. 74.

this passes through *C* it undergoes no refraction; draw *A G* parallel to the axis; this is refracted so as to pass through $F_2$. Since the lines *A C K* and *G* $F_2$, after refraction, diverge, they can meet only by being prolonged back to *A′*. Similarly, *B C* and *B H* may be drawn, and produced backwards to meet at *B′*. Thus *A′B′* is the virtual image of *A B*, is further away than the object, and is *virtual* or *negative*, *erect* and *magnified*.

When the object is situated in the anterior focal plane the rays, diverging from any point on it are, after refraction, parallel to each other and to a secondary axis in the denser medium so that the image, in theory, is formed at ∞. Similarly if the object lies in the posterior focal plane the light is parallel in the rarer medium after refraction.

**Course of Any Ray—Cx. Surface.**—From the foregoing we are able to construct the course of any ray refracted by a Cx. surface. If an object point were at $D$ in the anterior focal plane (Fig. 75) the light diverging from it is, after refraction, parallel to the secondary axis $D\,C$. Therefore any ray $S\,D\,Q$ incident on the refracting surface, passes through the first focal plane at $D$ and through the principal plane at $Q$, and its course, after refraction at $Q$, will be parallel to $D\,C$ drawn from $D$ through $C$; it therefore takes the direction $Q\,f_2$. This construction is useful if it be required to locate the image of a luminous point situated on the principal axis; if $S$ is thus situated, $f_2$ is its image.

The distance $a$, between the ray and the principal axis in the refracting

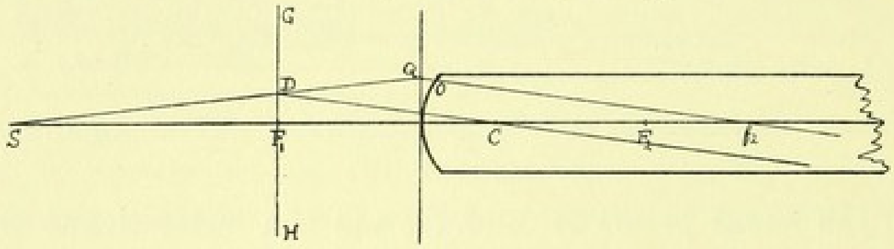



FIG. 75.

plane, is equal to the sum of $b$ and $c$, the distances between the ray and the axis in, respectively, the first and second focal planes. This fact gives an alternative construction, because the point $f_2$ can be located by measuring off on the second focal plane $c = a - b$, and then connecting $Q$ through that point to $f_2$.

**Construction of Image — Cc. Surface.**—$A\,B$ is the object (Fig. 76). Draw $A\,G$; this, after refraction, is diverged as if proceeding from $F_2$. Draw

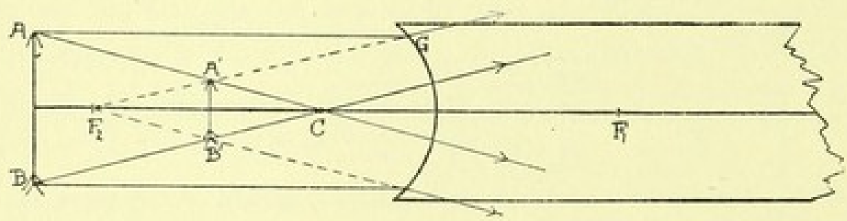



FIG. 76.

$A\,C$ through the centre of curvature; this is unchanged in direction by refraction. Now $A\,C$ and $A\,G$ are more divergent in the denser medium than they originally were in the rarer medium, and when projected backwards meet at $A'$, which is the virtual image of $A$. Similar rays from $B$ locate its image at $B'$. Consequently $A'\,B'$ is the image of the object $A\,B$, and is *virtual or negative*, *erect* and *diminished*.

**Course of any Ray—Cc. Surface.**—To trace the course of any ray as $S\,G$ (Fig. 77) incident on a Cc. surface, we know that if rays diverge from the focal plane they have their image in a plane $D\,E$ midway between $F_2$ and the surface. Now $S\,G$ cuts the focal plane in $H$, and if from $H$ we draw the secondary axis $H\,C$ we determine the point $K$, where it cuts $D\,E$. The ray is refracted as if it came in the direction $f_2 K G$, so that if $S$ is on the principal axis, $f_2$ is its image.

If the object point $S$ (Fig. 78) is within $F_2$, draw $H S$ connecting $S$ with the focal plane and the surface. Draw $H C$ cutting $D E$ in $K$; connect

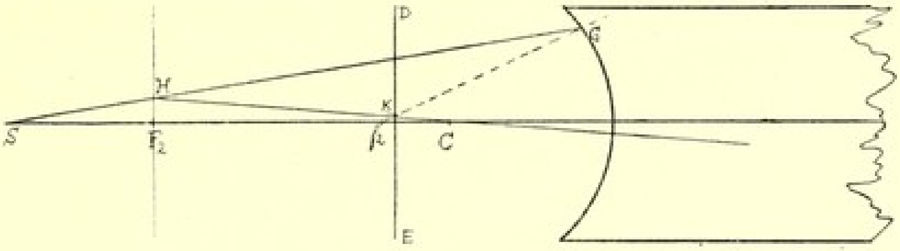



FIG. 77.

$K$ with the surface to meet there $S H$, crossing the axis in $f_2$, which is the image of $S$ if the latter is on the principal axis.

**Construction for the Course of a Ray.**—When $F_1$ and $F_2$ are not known the construction as is illustrated in Fig. 79 can be employed. Let $A B$ be the incident ray on a surface of the medium whose centre is $C$ and $\mu = 1{\cdot}6$.

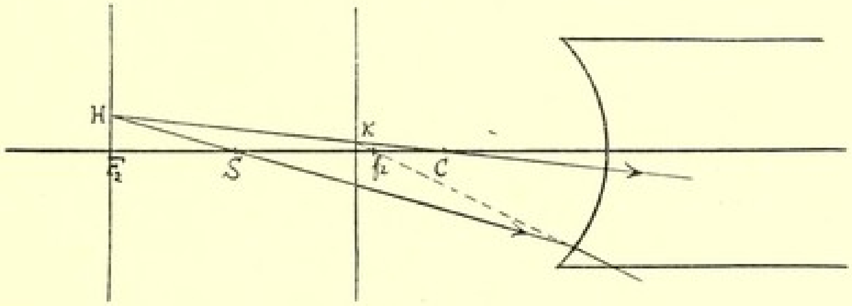



FIG. 78.

Draw $D B C$ normally at the point of incidence, and draw a tangent to the surface at $B$, and at right angles to $D B C$; then $H K$ is the refracting plane. From any point $G$ drop $G H$ normally to $E F$. Measure $H B$ and mark off $B K$ equal to 10/16 of $H B$. Drop the normal $K L$ and mark off the line $B M$ *whose length equals that of* $B G$. Then $B M N$ is the course of the refracted ray.

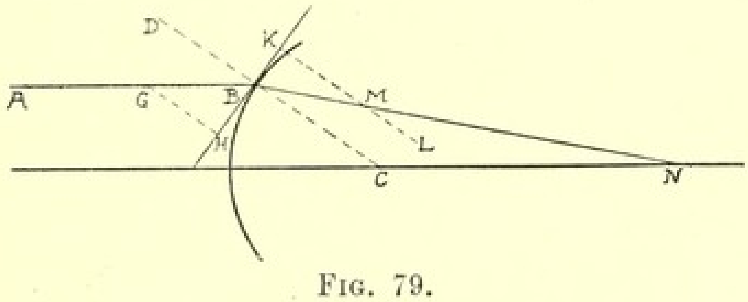



FIG. 79.

This method serves also for spheres and lenses by making a second construction for the second surface.

**Formula for Conjugate Foci.**—The formula, previously given, for calculating conjugate foci of a single surface is

$$\frac{\mu_1}{f_1} + \frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r} \qquad \text{or} \qquad \frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r} - \frac{\mu_1}{f_1}$$

where $f_1$ and $f_2$ are the two conjugates, and $\mu_2$ is the refractive index of the denser medium. $\mu_1$ *always pertains to the medium in which the object is situated*, the other being that of the medium towards which the light proceeds, but which may or may not be that in which the image is actually situated, since this may be either real or virtual. The radius $r$ is positive or + when the surface is convex towards the object, and negative or − when concave towards it.

**Size of Image.**—*Whatever may be the distance of the object, its size and that of its image are to each other as their respective distances from the centre of curvature, where the axial rays cross each other.* This is shown in Fig. 80, and whether the image be real or virtual it can be seen that the object and image always subtend the same angle at $C$.

Let the distance of the image from the surface be $f_2$ and that of the object $f_1$, and let $r$ be the distance from the surface to the centre of curvature. Let the size of the image be $h_2$ and that of the object $h_1$ and their distances

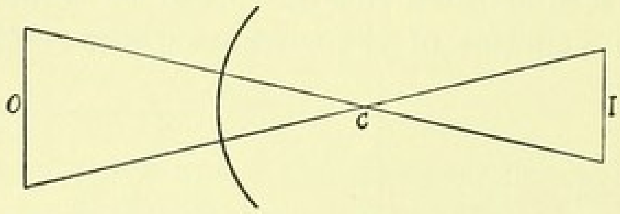



FIG. 80.

from $C$ respectively $IC$ and $OC$; then the magnification, in the case of a real image, is

$$M = \frac{h_2}{h_1} = \frac{IC}{OC} = \frac{f_2 - r}{f_1 + r}$$

Should the distance of $f_2$ or $f_1$ not be known, $M$ can be found from $F_2/(f_1 - F_1)$ or $(f_2 - F_2)/F_1$ respectively.

The linear size of $I$ or $O$ is found, when the size of the other is known, from

$$h_2 = \frac{h_1(f_2 - r)}{f_1 + r} \qquad \text{and} \qquad h_1 = \frac{h_2(f_1 + r)}{f_2 - r}$$

If $h_1$ and $f_1$ are in the same terms, i.e., inches, cm., etc., then $h_2$ is expressed in the same terms as $f_2$.

A more useful expression, however, can be found for the magnification as follows. From the original formula

$$\frac{\mu_1}{f_1} + \frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r} = \frac{\mu_2}{r} - \frac{\mu_1}{r} \qquad \text{or} \qquad \frac{\mu_1}{f_1} + \frac{\mu_1}{r} = \frac{\mu_2}{r} - \frac{\mu_2}{f_2}$$

that is,

$$\frac{\mu_1(f_1 + r)}{f_1 r} = \frac{\mu_2(f_2 - r)}{f_2 r} \qquad \text{or} \qquad \frac{f_2\mu_1}{f_1\mu_2} = \frac{f_2 - r}{f_1 + r}$$

so that we can write

$$M = \frac{h_2}{h_1} = \frac{f_2\mu_1}{f_1\mu_2} \qquad \text{in place of} \qquad \frac{f_2 - r}{f_1 + r}$$

Thus
$$h_2 = \frac{h_1 f_2 \mu_1}{f_1 \mu_2}$$

which is a much more convenient formula for calculating the size of the image.

**Unit Magnification.**—When the object and its real image are situated at equal distances from $C$, and on opposite sides of the refracting surface, they are equal in size and situated in the planes of unit magnification. In this case $O$ is at twice the anterior focal distance and $I$ is at twice the posterior focal distance from the surface, or $O$ is at $2F_1 + r$ and $I$ is at $2F_2 - r$ from $C$. That is, $M = 1$ when $f_1 + r = f_2 - r$, or when $\mu_1 f_2 = \mu_2 f_1$. The two conjugates are consequently at $2F_1$ and $2F_2$ respectively from the surface. If the image is virtual, with a concave or convex surface, unit magnification can only occur when $O$ and $I$ are both at the refracting surface and, of course, therefore equi-distant from $C$. Since the axial rays cross at $C$, the image formed at $F_2$ of a surface is of a size equal to that formed by a lens whose $F = F_1$, and that formed at $F_1$ is the same as that formed by a lens whose $F = F_2$.

**Examples.**—Let $r = 10$ mm., $\mu_2 = 1{\cdot}5$, $\mu_1 = 1$, and $f_1$ be in the air at 100 mm. from the surface; then

$$\frac{1{\cdot}5}{f_2} = \frac{1{\cdot}5 - 1}{10} - \frac{1}{100} = \frac{4}{100} \qquad \text{that is, } f_2 = \frac{150}{4} = 37{\cdot}5 \text{ mm.}$$

The image is real and inverted and its size relative to the object is

$$\text{M} = \frac{h_2}{h_1} = \frac{37{\cdot}5}{100 \times 1{\cdot}5} = \frac{37{\cdot}5}{150}$$

If $h_1$ be, say, 10 mm. then

$$h_2 = \frac{10 \times 37{\cdot}5}{150} = 2{\cdot}5 \text{ mm.}$$

Let $r = 8$ mm., $\mu_1 = 1{\cdot}333$, $\mu_2 = 1$, and the object be at 3·6 mm. behind the surface and 2 mm. in size; then

$$\frac{1}{f_2} = \frac{1 - 1{\cdot}333}{-8} - \frac{1{\cdot}333}{3{\cdot}6} = \frac{-1}{3{\cdot}05}$$

Here $\mu_1$ is the denser medium containing the object, towards which the surface is concave. The image is virtual at 3·05 mm. behind the surface, and its size is

$$h_2 = \frac{2 \times 3{\cdot}05 \times 1{\cdot}33}{3{\cdot}6} = 2{\cdot}25 \text{ mm.}$$

That is to say, the pupil of the eye, if 2 mm. in diameter, and 3·6 mm. from the cornea, appears to be 2·25 mm. in diameter and about 3 mm. behind the cornea.

Suppose $r = -3''$, $\mu_2 = 1{\cdot}5$, $\mu_1 = 1$, $f_1 = 20''$ and $h_1 = 2''$. Then

$$\frac{1{\cdot}5}{f_2} = \frac{1{\cdot}5 - 1}{-3} - \frac{1}{20} = -\frac{13}{60} \qquad \text{therefore } f_2 = \frac{-90}{13} = -6\tfrac{12}{13}''$$

and

$$h_2 = \frac{2 \times 6\frac{12}{13}}{20 \times 1{\cdot}5} = \frac{6}{13} \text{ in.}$$

**Another Expression for Conjugate Foci.**

Since

$$\frac{\mu_1}{f_1} + \frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r} = \frac{1}{F_1} \text{ when } f_2 = \infty$$

we could write

$$\frac{\mu_1 F_1}{f_1} + \frac{\mu_2 F_1}{f_2} = 1$$

and if $\mu_1 = 1$ and since $\mu_2 F_1 = F_2$ we have the most useful formulæ for conjugate foci in

$$\frac{F_1}{f_1} + \frac{F_2}{f_2} = 1 \qquad \text{or} \qquad f_2 = \frac{f_1 F_2}{f_1 - F_1} \qquad \text{and} \qquad f_1 = \frac{f_2 F_1}{f_2 - F_2}$$

**Examples.**—Suppose the object be situated 20 inches in front of a convex surface where $F_1 = 6$ in. and $F_2 = 9$ in. Then the image is real and

$$f_2 = \frac{20 \times 9}{20 - 6} = \frac{180}{14} = 12\tfrac{6}{7} \text{ in.}$$

Let an object be situated 5 inches in front of a surface where $F_1 = 6$ in. and $F_2 = 9$ in. Then

$$f_2 = \frac{5 \times 9}{5 - 6} = \frac{45}{-1} = -45 \text{ in.}$$

The image $f_2$ is negative or virtual and on the same side of the surface as $f_1$.

If $f_1$ is situated at $F_1$ the divisor of the fraction is 0, so that $f_2$ is at $\infty$; also if $f_1$ is at $\infty$ then $f_2$ corresponds to $F_2$.

Let an object be $12\frac{6}{7}$ in. in the dense medium having a convex surface whose $F_2 = 9$ in., and $F_1 = 6$ in., then

$$f_2 = \frac{12\frac{6}{7} \times 6}{12\frac{6}{7} - 9} = \frac{77\frac{1}{7}}{3\frac{6}{7}} = 20 \text{ in.}$$

This example should be compared with the one previously given, where the object is in front of the refracting surface, 20 inches and $12\frac{6}{7}$ inches being conjugate distances for the given refracting medium.

An object 20 inches from a Cc. surface, whose $F_1$ and $F_2$ are respectively $-6$ and $-9$ in. has its virtual image

$$f_2 = \frac{20 \times (-9)}{20 - (-6)} = \frac{-180}{26} = -6\tfrac{12}{13} \text{ in.}$$

**Conjugate Focal Distances—Cx. Surfaces.**—If the object is at ∞ represented by $A$ (Fig. 81), the light is parallel and, after refraction, meets at $F_2$. This is the nearest point to the refracting surface at which a real focus can be obtained.

If the light diverges from an object $f_1$ at a finite distance from the refracting surface, some of the converging power of the medium is required to neutralise the divergence of the light and there is less residual convergence; the light therefore is convergent to a greater distance behind the refracting surface than if the light had been previously parallel; the image in the denser medium is at some point $f_2$ situated between $F_2$ and ∞.

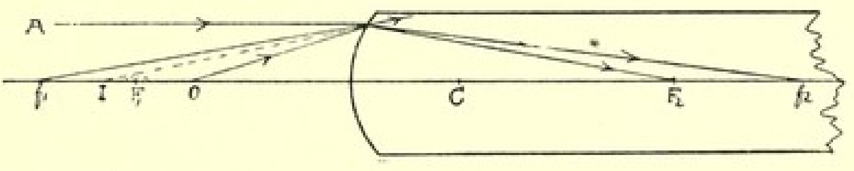


FIG. 81.

As the object approaches from ∞ the image recedes from $F_2$ and vice versa, until when the object is at $F_1$ the image is at ∞.

If the object is nearer than $F_1$ as at $O$ the image is at $I$ on the same side of the surface. As $O$ then further approaches the surface so also does $I$, and when $O$ touches the surface $I$ does so also.

When $O$ is within the dense medium and the light is parallel $I$ is at $F_1$; as $O$ approaches $F_2$ so $I$ recedes from $F_1$, and when $O$ is at $F_2$ the image is at ∞. When $O$ lies nearer to the surface than $F_2$ the image is virtual and on the same side of the surface as $O$.

Thus in Fig. 82 if the object is at $O'$ the image is further away at $I'$; if

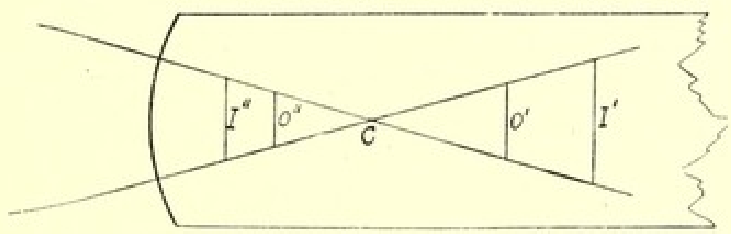


FIG. 82.

the object is at $C$ then the image is at $C$, and if the object is at $O''$ then the image is nearer the surface at $I''$; when $O$ touches the surface $I$ does so also. If light diverges from a point beyond $C$ it becomes less divergent by refraction at the surface, and if from a point nearer than $C$ it becomes more divergent. It should be particularly noticed, however, that as the object moves away from the surface in a convex dense medium, the image increases in size until, when $O$ is at $C$, $I$ is there also, but with a magnification equal to $\mu$ (the dense medium being assumed to be bounded by air). As the object moves beyond $C$ towards $F_2$ the image continues to increase in size, until, when $O$ is at $F_2$, $I$ is at ∞ and infinitely magnified.

**Conjugate Focal Distances—Cc. Surfaces.**—When the surface of the dense medium is concave and the object is at $\infty$ the image is at $F_2$. This is the most distant point from the surface at which an image can be formed. If the object is within $\infty$, the original rays being divergent are rendered still more divergent after refraction than if they had been orginally parallel; hence the image is formed nearer to the surface, that is, as $O$ approaches the surface so also does $I$.

The virtual image is nearer to the surface than the object so long as $O$ is beyond $C$; when $O$ is at $C$ so also is $I$, but the latter is diminished by $\mu$ times; when $O$ is within $C$ then $I$ is beyond $O$, and when $O$ touches the surface $I$ does so also.

When the object is in a concave dense medium, unit magnification occurs when $O$ touches the surface; as $O$ moves away towards $F_1$ the image becomes progressively smaller until when $O$ is at $\infty$, $I$ is at $F_1$ and infinitely diminished.

**Virtual Conjugates.**—Virtual conjugate foci, formed by Cx. or Cc. surfaces, are not interchangeable as are real conjugates, but if the light were directed converging towards $f_2$ the image formed would be at $f_1$.

**Other Formulæ for Conjugate Foci and M.**—If $A$ and $B$ be respectively the distance of $O$ from $F_1$ and of $I$ from $F_2$, then

$$A\,B = F_1\,F_2 \qquad \text{and} \qquad M = \frac{I}{O} = \frac{F_1}{A} = \frac{B}{F_2}$$

**Dioptral Formulæ for a Single Refracting Surface.**—The diopter $D$ is an expression for the refracting power of a surface and has a value of $D = 100/F$, $F$ being in cm. We get

$$D_A = \frac{100\ (\mu_2 - \mu_1)}{r\ \mu_1} \qquad\qquad D_P = \frac{100\ (\mu_2 - \mu_1)}{r\ \mu_2}$$

Where $\mu_2$ is refractive index of the dense medium,
,, $\mu_1$ ,, ,, ,, ,, rare medium,
,, $r$ ,, the radius of curvature of the surface in cm.
,, $D_A$ ,, the dioptric power corresponding to $F_1$.
,, $D_P$ ,, ,, ,, ,, to $F_2$.

$$D_A : D_P \text{ as } \mu_2 : \mu_1.$$

**Example.**—Find the power of a surface of radius 8 mm. and $\mu = 1{\cdot}333$ in air.

$$D_A = \frac{100 \times (1{\cdot}333 - 1)}{{\cdot}8 \times 1} = 41{\cdot}66$$

$$D_P = \frac{100 \times (1{\cdot}333 - 1)}{{\cdot}8 \times 1{\cdot}33} = 31{\cdot}25$$

$$31{\cdot}25 : 41{\cdot}66 \text{ as } 1 : 1{\cdot}333.$$

**Conjugate Foci.**

Since $\frac{F_1}{f_1}+\frac{F_2}{f_2}=1$ and $F_1\mu=F_2$ when the 1st medium is air, we have

$$\frac{F_2}{f_1\mu}+\frac{F_2}{f_2}=1 \qquad \text{or} \qquad \frac{F_1}{f_1}+\frac{F_1\mu}{f_2}=1$$

hence

$$\frac{1}{f_1\mu}+\frac{1}{f_2}=\frac{1}{F_2} \qquad \text{or} \qquad \frac{1}{f_1}+\frac{\mu}{f_2}=\frac{1}{F_1}$$

which expressed in diopters becomes

$$d_1/\mu+d_2=D_P \qquad \text{or} \qquad d_1+d_2\mu=D_A$$

where $d_1$ and $d_2$ are the conjugate focal distances expressed dioptrally.

# CHAPTER VII

## THIN LENSES

**Images** have been defined in Chapter III. When light from an object enters the eye a real image is formed at the retina, the stimulation produced by the point foci being conveyed to the brain where the sense of sight exists. The retinal image is not seen, nor is the original object; what is seen is the mental conception of the light stimulation which is projected out into space, and usually this coincides in size and distance with the object itself. The mental image is virtual, and when it coincides with the object we say that we see the latter, but if it does not thus coincide we say that we see a virtual image of the object. This occurs whenever, by reflection or refraction, the course of the incident light is changed as by mirrors, prisms or lenses.

Light diverges from a real image, formed on a screen or in the air, and it is seen in the same way as an object; the mental image of the real image is projected so as to coincide with it; but it can be viewed through a lens or prism, and a virtual image formed of this real image, as occurs when a microscope or telescope is employed. A *real* image is an actual thing which exists; a *virtual* image, as the term is commonly employed, is imaginary, it merely appearing to exist.

**Position of Object.**—It is always taken that an object is in *front* of a lens or mirror, and the image is in front or behind according as it is, respectively, on the same side as, or on the opposite side to, the object.

**Optical Signs.**—In this work the following convention is followed. Since light always diverges from luminous points *divergence* is considered *negative*, and therefore *convergence* is considered *positive*.

Surfaces, mirrors or lenses that cause, or tend to cause, convergence of light, are similarly positive, as also are their focal lengths and powers, and the real images and foci produced by them; to all these the + sign is assigned.

Surfaces, mirrors or lenses that produce, or tend to produce, divergence, together with their focal lengths and powers, and virtual images and foci, are negative and given the − sign.

Thus when a convex spherical surface of glass is in contact with air, refraction occurs, and this may be taken as due either to the Cx. glass surface or to the Cc. air surface; both are + since both cause convergence of parallel light. If a double Cc. air lens be in water we could consider the converging effect which results by calculation of the two Cc. air surfaces, or of the two Cx. water surfaces. A Cx. surface is not necessarily positive, nor a Cc. negative; when they are reflecting surfaces they are the reverse, as they are, also, when refracting if they are of lower $\mu$ than the adjacent medium. Usually, however, in optics, a Cx. refracting surface is positive and a Cc. is negative because it has a higher $\mu$ than the adjacent medium, the latter being air, but this may not be the case when light passes successively through various media.

**Important Consideration.**—It is most essential to differentiate between the direction of axial rays and that of the rays from the various points on an object with reference to their axes.

From each point of the object a pencil of rays diverges and each pencil has an axis, which is the axial ray of that pencil. Axial rays *always converge* to the optical centre of the lens, and their convergence governs the *size of the angle subtended by the object and the image at the lens.*

The rays themselves *always diverge* from the luminous point to the lens, and their divergence governs the *position or distance of the image*, the rays after refraction being more or less divergent or convergent, according to the

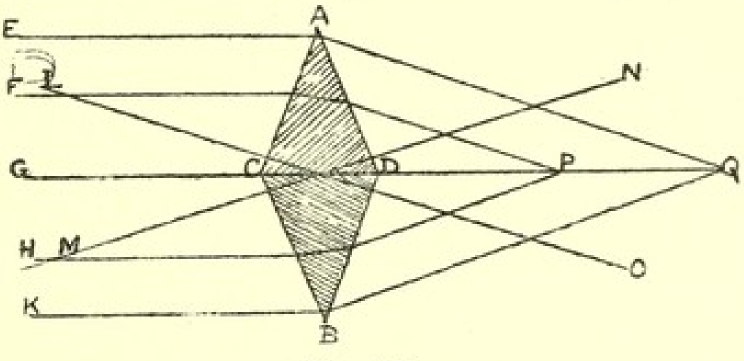



FIG. 83.

degree of original divergence and the diverging or converging power of the lens.

Parallel light is merely light having a negligible degree of divergence.

These *most important considerations*, for students are apt to confuse the conditions, should be carefully noted. Thus, in a diagram which shows light parallel to the axis, and incident on various parts of the lens surface, the rays are presumed to originate, not in various points, but in one single point on the axis.

These considerations apply not only to all lenses, but to surfaces and curved mirrors as well, and all positions of the object.

**Definition of Lens.**—A lens is a transparent body bounded by one curved and one plane surface, or by any two curved surfaces, and is usually surrounded by air. This definition, therefore, covers all forms of convex and concave sphericals as well as cylindrical and other special forms of lenses.

**Prismatic Formation.**—If two similar prisms *A C D* and *B C D* be placed base to base as in Fig. 83 incident rays *E* and *F* are bent towards the base of the prism *A C D*, and rays *H* and *K* are bent towards the base of the prism *B C D*, so that those refracted by the one prism meet those refracted by the other. One ray, viz., *G C D P* suffers no deviation since it coincides with the base of both prisms; also *L* and *M* may be considered incident perpendicular to the two refracting surfaces, and are therefore also not deviated.

If two prisms *C A D*, *E A F*, as in Fig. 84, be joined edge to edge all rays incident on them, being refracted towards the bases, are therefore diverging from the common edge, except the central ray incident at the junction of the two edges.

What is true of two is also true of any number of prisms, and a convex

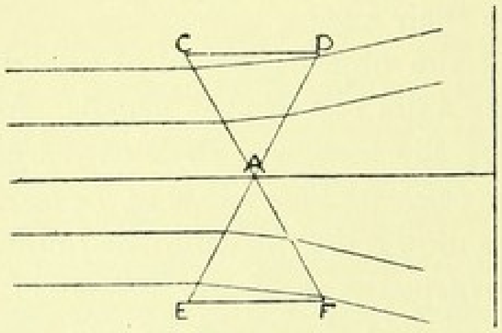



FIG. 84.

or concave lens may be considered as formed of prisms whose bases or apices respectively have a common centre; also every meridian must be considered as if formed of a series of truncated prisms of different angles of inclination, but having a common base apex line.

Any two point areas *A* and *B* (Fig. 85) opposite each other constitute a portion of a prism whose base, in the Cx., and whose apex, in the Cc., is turned towards the principal axis of the lens. The areas *A* and *B*, near the

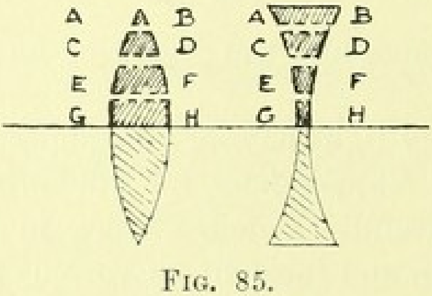



FIG. 85.

periphery of the lens, are more inclined towards each other than *C* and *D*, situated nearer to the axis, and the inclination between the surfaces decreases gradually until at *G H* on the principal axis they are parallel. Since the angle formed by *A* and *B* is greater than that formed by *C* and *D*, a ray passing through *A B* is bent to a greater extent than one passing through *C D*, while the ray which passes along the axis is not deviated at all.

Each zone of a lens, therefore, whether concave or convex, has a refractive power which becomes greater as its distance from the axis is increased, and it is due to this fact that rays diverging from a point, and incident on the lens, are brought to a common focus practically as a point.

**Forms of Lenses.**—There are (Fig. 86) four forms of thin convex and four of concave spherical lenses :—

1. Equi-convex. Two convex surfaces of equal curvature.
1′. Equi-concave. ,, concave ,, ,, ,,
2. Bi-convex. Two convex surfaces of unequal curvature.
2′. Bi-concave. ,, concave ,, ,, ,,
3. Plano-convex. One side convex, the other plane.
3′. Plano-concave. ,, concave, ,, ,,
4. Positive meniscus or periscopic convex, Cx. on one side and Cc. on the other, the Cc. being the weaker power.
4′. Negative meniscus or periscopic concave, Cc. on one side and Cx. on the other, the Cx. being the weaker power.

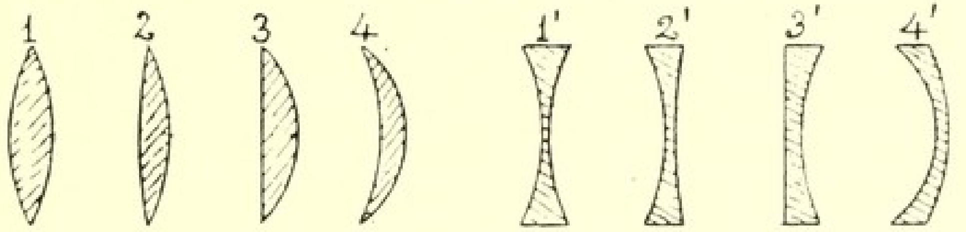



Fig. 86.

Variations of the above are made by increasing the interval between the two surfaces ; these are treated in the chapter on thick lenses.

In any lens of whatever nature or shape, there are innumerable pairs of points on the two surfaces such that tangents drawn to the surfaces at these points are parallel. If, therefore, a ray incident at one of these points

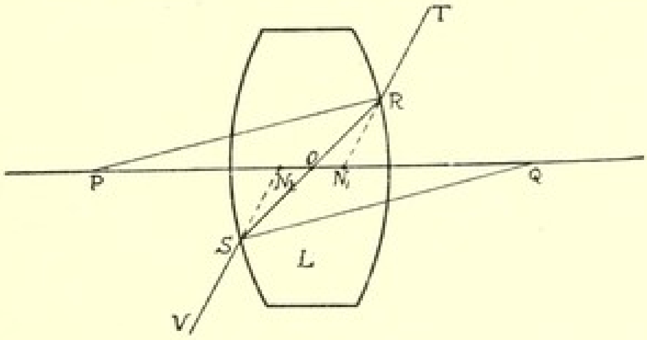



Fig. 87.

emerges from the other, its final direction is parallel to its initial course just as though it had been incident on a parallel plate. The ray, therefore, is not deviated but merely laterally displaced by an amount depending upon the thickness and $\mu$ of the lens. In Fig. 87 let $R$ and $S$ be two parallel points on the surfaces of the bi-convex lens $L$. They are located by drawing from the centres of curvature $P$ and $Q$ any two mutually parallel radii $PR$ and $QS$, and therefore, if a ray $TR$ is so incident at $R$ as to emerge at $S$, the final direction $SV$ of the ray will be parallel to the original course $TR$. While in the lens the ray will take the direction $RS$, cutting the principal axis $PQ$ in $O$, which is a fixed point no matter what the position of $RS$ may be. Thus any number of pairs of parallel points may be located by drawing a corresponding number of pairs of parallel radii, and if lines be drawn, as $R S$, they will all be found to cut the axis in $O$. This fixed point $O$ is the *optical centre*, and any line $R S$ passing through it is termed a *secondary axis*, since it suffers no angular deviation, but only a lateral displacement depending upon the thickness and $\mu$ of the lens.

The imaginary point $N_1$ on the principal axis, towards which a secondary axis is directed, is called the *first nodal point*, and the corresponding point $N_2$ from which it apparently emerges is the *second nodal point*. It is from these points that the principal and all conjugate focal distances are measured, since, as will be shown later in the chapter on thick lenses, it is on planes drawn perpendicular to the axis through $N_1$ and $N_2$ that the refraction of the surfaces of the lens is presumed to take place. The nodal points are also frequently referred to as *principal* or *equivalent* points, but as we shall not be dealing with them until later it is not necessary to explain here the exact difference between these terms.

Fig. 89 shows the positions of the optical centre for every type of lens, as well as the general formation of the latter by the intersection or non-intersection of spheres and planes.

For our purpose in this chapter we shall, however, regard the thickness of all lenses as negligible in comparison with the focal length of such lenses. All lenses employed in visual optics are treated thus, and are said to be *thin*, as distinct from others whose thickness cannot be disregarded without introducing considerable error in calculating the power and focal length.

This being so, the action of a lens, considered as thin, is greatly simplified since we may assume the interval between the nodal points and optical centre as so small that all three fuse into a single point to which we apply the single term optical centre. Similarly the equivalent planes passing through the nodal points are also considered to unite into a single refracting plane passing through the optical centre, from which all distances and foci are measured. The position of the optical centre depends only upon the curvature and thickness of the lens, and is distant from each surface by an amount proportional to the relative radii of curvature.

**Terms of a Lens.**—Let Fig. 88 represent a thin Cx. lens; $C C$ are the centres of curvature, and $O$ the optical centre. The line $A O B$ passing through the two centres of curvature, and the optical centre, is the principal axis; it is perpendicular to both surfaces of the lens. The plane $L O L$ passing through $O$, perpendicular to $A B$, is the refracting plane, on which all the refraction effected by both surfaces of a thin lens is presumed to take place. Any lines as $D D$, $E E$ directed to $O$, are secondary axes, which pass, obliquely to the principal axis, through the lens, and the latter being thin, they are *presumed* to suffer no deviation at all.

**Formation of Lenses.**—In each of the diagrams in Fig. 89, which shows the actual formation of lenses by the intersection or non-intersection of spheres and planes, the radius of curvature is a line drawn from the centre of each sphere to the corresponding surface of the lens. The optical centre in each case is marked $O$.

In the equi-Cx. and bi-Cx. (1 and 2), and the equi-Cc. and bi-Cc. (5 and 6), the centres $C$ and $C'$ are on opposite sides of the lens.

In the plano-Cx. (3) and plano-Cc. (7) the curvature of the plano surface

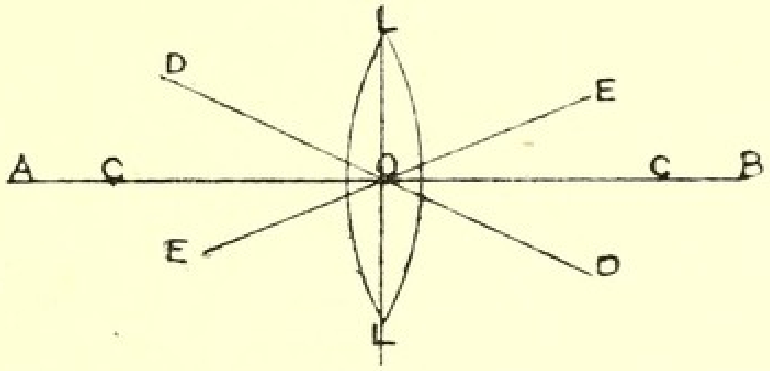



FIG. 88.

may be considered to be of infinite radius; the centre then being at infinity can be considered to be on either side.

A Cx. lens consisting of a complete sphere has the centres of its opposite surfaces coincident.

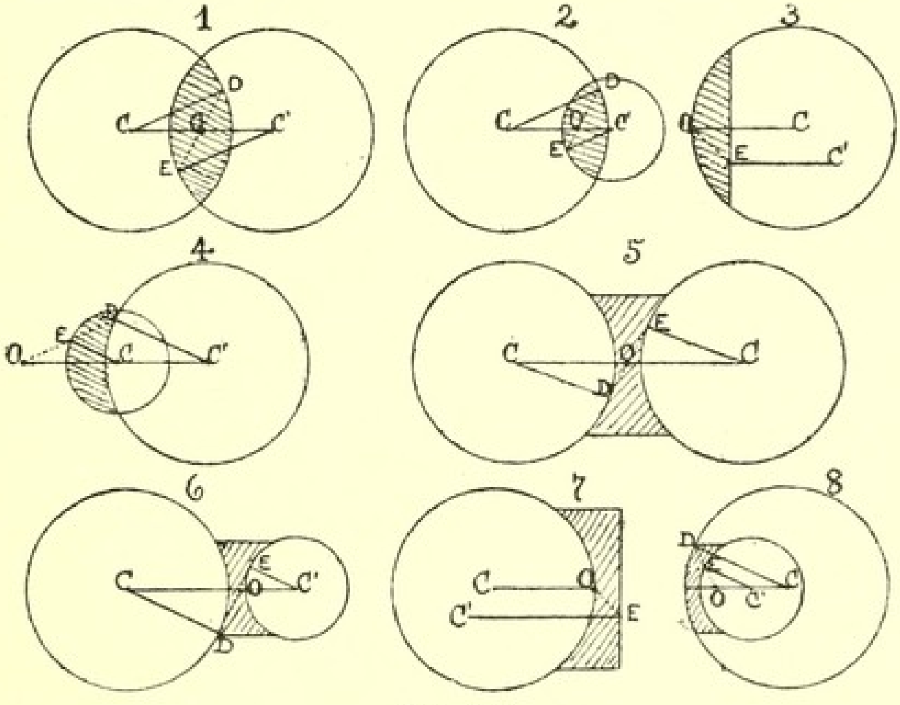



FIG. 89.

In the periscopic Cx. (4) and periscopic Cc. (8) the centres are on the same side.

**Position of Optical Centre.**—By calculation, the position of the optical centre of any lens is found by dividing the thickness of the glass, on the principal axis, in the ratio of the two radii of curvature; so that, if the two surfaces are equal, $O$ is equally distant from each, but it is nearer to the more curved surface if the two are unequal. If $r_1$ and $r_2$ be the radii of the two surfaces, $t$ the thickness of the lens, and $O$ the optical centre, then the distance of the latter from the surfaces whose radii are $r_1$ and $r_2$ is respectively

$$O = \frac{tr_1}{r_1 + r_2} \text{ and } \frac{tr_2}{r_1 + r_2}$$

Thus in a bi-convex lens where $t$ = ·2 inch, and $r_1$ and $r_2$ are respectively 6 and 10 inches,

$$O = \frac{\cdot 2 \times 6}{6 + 10} = \cdot 075 \text{ in. from } r_1, \qquad \text{and} \qquad \frac{\cdot 2 + 10}{6 + 10} = \cdot 125 \text{ in. from } r_2$$

The thickness is divided into 6 + 10 = 16 parts, and $O$ lies on the axis six of these parts from the pole of the shorter curve, or ten parts from the pole of the longer curve. In lenses whose surfaces are both convex or both concave $O$ lies within the lens, but in periscopic lenses $O$ lies outside the lens on the side of the surface of greater power.

Suppose a periscopic convex in which $t$ = ·2 in., $r_1$ of the Cx. surface being 9 in., and $r_2$ of the Cc. − 12 in. Then $r_1 + r_2 = 9 - 12 = -3$ and $O = 1{\cdot}8/-3 = -{\cdot}6$ in. from $r_1$ and $-2{\cdot}4/-3 = +{\cdot}8$ in. from $r_2$.

The distance from the convex surface being negative must be reckoned *away from* it, and the two distances coincide ·6 in. from the convex surface. If the lens were periscopic concave $O$ would be on the Cc. side.

With a plano lens the one surface having $r_1 = 9$ in. the other $r_2 = \infty$, if $t = \cdot 2$ in., then

$$O = \frac{\cdot 2 \times 9}{9 + \infty} = 0$$

since any number divided by $\infty = 0$. $O$ therefore lies on the curved surface in plano Cx. and Cc. lenses.

**Construction of Optical Centre.**—The method of finding the optical centre of any form of lens is shown in Fig. 89. From the centre of curvature $C$, in any of the diagrams, draw a radius $C\,D$ to the curved surface, of which $C$ is the centre. From $C'$ draw a radius $C'\,E$ to its corresponding surface, and parallel to $C\,D$. Connect the extremities of the two radii by the line $D\,E$ and where it cuts the principal axis at $O$, is the optical centre of the lens.

In (3) and (7) $C'$ being at $\infty$, the only radius that can be drawn from $C$, parallel to $C'\,E$, corresponds to the principal axis itself.

In (4) and (8) the line connecting $D$ and $E$ has to be produced in order to cut the principal axis.

**Properties of Lenses.**—A convex lens has positive refracting power and, therefore, can form a real focus and a real image; it renders parallel rays convergent and divergent rays less divergent, parallel or convergent as the case may be.

A concave lens has negative refracting power and, therefore, can only form a virtual or negative focus or image; it renders parallel rays divergent and divergent rays more divergent.

The general effect of every spherical (and cylindrical) lens is, as with a prism, to bend every incident ray of light towards the thickest part. This property and the foregoing ones apply if the surrounding medium is of lower density than that of the lens; otherwise the reverse occurs. When discussing lenses we take them, unless otherwise stated, to be in air.

**The Focus.**—A real focus, formed by a lens, is that point at which rays diverging from a point meet after refraction.

A virtual focus is that point where rays diverging from a point meet when produced backwards, or whence they appear to diverge, they being still divergent after refraction.

**Principal Focus and Focal Distance.**—A principal focus is one formed on the principal axis by the convergence or divergence of originally parallel rays. A secondary focus is one formed on a secondary axis.

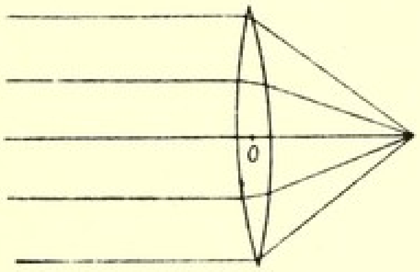



FIG. 90.

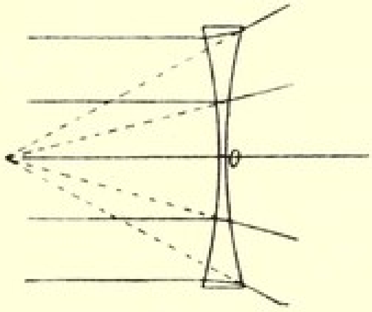



FIG. 91.

The principal focus of a convex lens is positive and is situated on the principal axis on the opposite side of the lens from the source of light.

Natural rays signify those rays which proceed from a source of light and whose course is not altered by a lens or mirror; they may be parallel or divergent, but never convergent.

The distance between the optical centre and the principal focus is the principal focal distance of a thin convex lens (Fig. 90), the focus being that point at which, after refraction, parallel rays meet. It is the *nearest* point to a convex lens at which a focus of natural rays can be obtained. The parallel rays in the figure are presumed to diverge from a single point on the principal axis at $\infty$.

The principal focus of a concave lens is negative, and is situated on the principal axis on the same side of the lens as the source of light. Thus in Fig. 91 the distance between $O$ and the principal focus is the principal focal distance of a concave lens. The principal focus being the point from which, after refraction, parallel rays appear to diverge, is the *furthest* point from a concave lens at which a focus can be obtained for natural rays.

*The value of a lens* is expressed either by its principal focal length $F$ or by its *refractive power* $D$ or $1/F$, the latter expression being sometimes termed the *focal power*; both properties depend solely on the curvature and the refractive index, thickness being neglected. $F$ and $D$ vary inversely with each other, they being reciprocals; as the one is increased the other is proportionately diminished. Thus $F = 1/D$, and $D = 1/F$. The meaning of $D$ and its relationship to $F$ will be found in the next chapter.

**Distance of Principal Focus.**—Whether the one side or the other of a thin equi-convex or equi-concave lens is exposed to the light, $F$ is at the same distance from the back surface of the lens since $O$ is situated equally

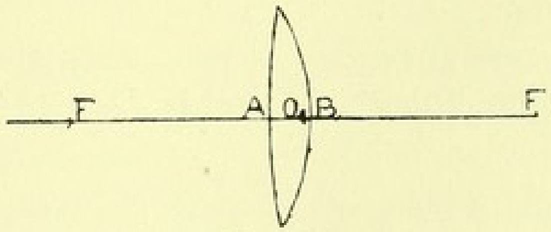



FIG. 92.

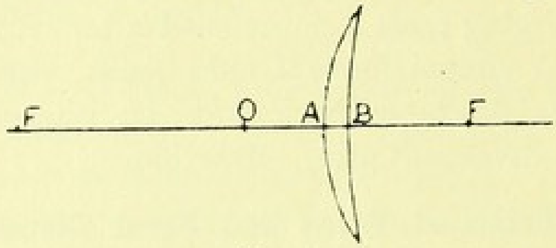



FIG. 93.

distant from each surface; but this is not the case with other forms of spherical lenses. In Fig. 92 the principal focal distance $O F$ of a bi-convex lens being measured from $O$, it follows that the distance of $F$ behind the posterior surface of the lens depends on whether the less curved surface $A$, or the more curved surface $B$, is exposed to the light. If $A$ is thus exposed, $F$ lies further from $B$ than it does from $A$ when $B$ faces the light. The same applies to the bi-concave. With the periscopic convex, as shown in Fig. 93, and the periscopic concave, the difference in the distance of $F$ as measured to the right from $B$ or to the left from $A$ is very marked. Similarly with

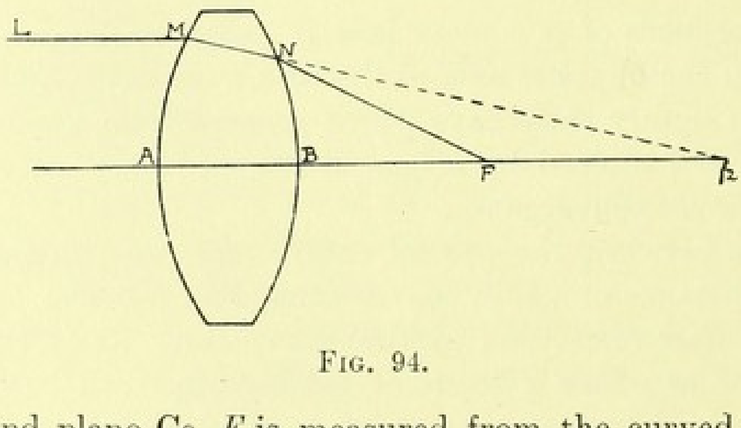



FIG. 94.

the plano Cx. and plano Cc. $F$ is measured from the curved surface since $O$ is situated thereon. In all cases $O F$ is the same either way, i.e., $F_1 = F_2$.

To express the focal length of a lens in terms of the radii of curvature, the refraction at each surface of the lens must be considered, and the two combined into one expression.

Let $A B$ (Fig. 94) be a bi-convex lens of radii $r_1$ and $r_2$ and index $\mu_2$, that of the surrounding medium being $\mu_1$. Then if any ray $L M$ parallel to the principal axis be incident at $M$ it will be refracted and tend to focus at $f_2$ the posterior focal distance of the first surface.

Thus

$$f_2 = \frac{\mu_2 r_1}{\mu_2 - \mu_1}$$

Therefore $f_2$ is virtually an object with respect to the second surface, and the final image is formed at $F$, which is the principal focus of the whole lens. Since the thickness of the lens is disregarded we may take $A\ F$ as being equal to $B\ F$. For the second surface, $f_1'$ and $f_2'$ being the conjugates

$$\frac{\mu_2}{f_1'} + \frac{\mu_1}{f_2'} = \frac{\mu_2 - \mu_1}{r_2}$$

But the image distance $f_2$ of the first surface becomes the virtual object distance $f_1'$ of the second surface, so that we must substitute the former for the latter, using the negative sign. Then we get

$$\frac{\mu_1}{f_2'} - \frac{\mu_2}{\mu_2 r_1/(\mu_2 - \mu_1)} = \frac{\mu_2 - \mu_1}{r_2}$$

or

$$\frac{\mu_1}{f_2'} = \frac{\mu_2 - \mu_1}{r_1} + \frac{\mu_2 - \mu_1}{r_2}$$

Now the final image distance $f_2'$ is the principal focal distance $F$;

$$\therefore \qquad \frac{\mu_1}{\mathrm{F}} = (\mu_2 - \mu_1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad \text{or} \qquad \mathrm{F} = \frac{\mu_1 r_1 r_2}{(r_1 + r_2)(\mu_2 - \mu_1)}$$

These are the general formulæ for a thin lens in any medium, but if $\mu_1$ is air, which is usually the case, and taking $\mu$ as the index of the lens, the above simplify to

$$\frac{1}{\mathrm{F}} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad \text{or} \qquad \mathrm{F} = \frac{r_1 r_2}{(r_1 + r_2)(\mu - 1)}$$

Since $1/r_1$ and $1/r_2$ represent the curvatures of the two surfaces, *the power of a lens is equal to the sum of its curvatures multiplied by the refractivity of the medium of which it is made.*

A convex surface with regard to a lens is considered positive and a concave surface negative, as already stated, and this facilitates calculation where we have only two surfaces to deal with.

We have to consider the three following conditions:

(*a*) If both surfaces are of the same nature.

**Example.**—A Cx. lens of $\mu = 1{\cdot}54$ and having surfaces of radii of 8 in. and 5 in. The focus is here positive, thus

$$\mathrm{F} = \frac{8 \times 5}{(8 + 5)(1{\cdot}54 - 1)} = \frac{40}{7{\cdot}02} = 5{\cdot}7 \text{ in.}$$

If the surfaces are concave the negative sign must be prefixed to each; the focus also is negative.

$$F = \frac{-8 \times (-5)}{(-8 - 5) \times (1{\cdot}54 - 1)} = \frac{40}{-7{\cdot}02} = -5{\cdot}7 \text{ in.}$$

If both surfaces have the same radius, i.e. $r_1 = r_2$, as in an equi-Cx. or equi-Cc. lens, the formula becomes simplified, for

$$F = \frac{r_1 r_2}{(r_1 + r_2)(\mu - 1)} = \frac{r^2}{2r(\mu - 1)} = \frac{r}{2(\mu - 1}$$

Thus if $r_1$ and $r_2 = 5$ and $\mu = 1{\cdot}54$

$$F = \frac{5}{\cdot 54 \times 2} = \frac{5}{1{\cdot}08} = 4{\cdot}63 \text{ in.}$$

If $\mu = 1{\cdot}5$ it can be seen that in equi-Cx. or equi-Cc. lenses the focal length is equal to the radius.

(*b*) If one surface is plane, then $r_1 = \infty$ and $1/r_1 = 1/\infty = 0$, so that it may be ignored and only the curved surface considered, and the original formula simplifies to

$$F = \frac{r}{\mu - 1}$$

If $\mu = 1{\cdot}5$, it can be also seen that in a plano Cx. or Cc. lens the focus is twice the radius.

(*c*) If one surface is positive and the other negative, the focus will be positive or negative as the one or other predominates.

**Example.**—In a periscopic Cx. let the two surfaces be respectively $-8$ in. and $+4$ in. and $\mu = 1{\cdot}6$. Here

$$F = \frac{-8 \times 4}{(-8 + 4) \times \cdot 6} = \frac{-32}{-2{\cdot}4} = +13{\cdot}3 \text{ in.}$$

In a periscopic Cc., if the surfaces are $+8$ in. and $-4$ in. respectively.

$$F = \frac{8 \times (-4)}{(8 - 4) \times \cdot 6} = \frac{-32}{2{\cdot}4} = -13{\cdot}3 \text{ in.}$$

**Relative Powers.**—It can be seen from the above that, the radii being constant, the power of a lens in air is proportional, not to $\mu$, but to $(\mu - 1)$, the latter being termed the refractivity of the medium. Thus if two lenses $A$ and $B$ be ground to the same radii but on glasses of different $\mu$'s, the ratio of their powers is as $(\mu_A - 1) : (\mu_B - 1)$, their focal lengths being as $(\mu_B - 1) : (\mu_A - 1)$.

**To find** $r$.—To calculate the curvature of one of the surfaces $r_1$ or $r_2$ when that of the other, as well as $\mu$ and F, are known, it is necessary to substitute the values of the known quantities and then equate as in the following examples.

What radius should be given to the second surface of a lens so that F = 6 in. $r_1 = 8$ in. and $\mu = 1{\cdot}5$?

$$F = \frac{r_1 r_2}{(r_1 + r_2)\ (\mu - 1)} \qquad 6 = \frac{8r_1}{(8 + r_1) \times {\cdot}5}$$

then

$$6 = \frac{8r_1}{4 + {\cdot}5r_1}; \qquad \text{or} \qquad 24 + 3r_1 = 8r_1$$

$$\therefore \qquad 5r_1 = 24 \qquad \text{or} \qquad r_1 = +4{\cdot}8$$

What should be the radius of the Cc. surface of a meniscus when that of the Cx. is 5 in., F being 12 in. and $\mu = 1{\cdot}6$?

Then

$$12 = \frac{5r_1}{(5 + r_1)\ {\cdot}6}; \qquad \text{or} \qquad 5r_1 = 12 \times (3 + {\cdot}6r_1)$$

and

$$r_1 = \frac{36}{-2{\cdot}2} = -16{\cdot}36 \text{ in.}$$

**To find** $\mu$.—Similarly by substitution $\mu$ can be calculated. For example, F = 24 cm. and the radii are +6 and − 12 cm., then

$$24 = \frac{6 \times -12}{(6 - 12)\ (\mu - 1)} = \frac{-72}{-6\,\mu + 6}$$

$$\therefore \qquad -72 = -144\,\mu + 144 \qquad \text{and} \qquad \mu = 1{\cdot}5$$

**Calculations when $\mu_1$ is not Air.**—When the first and last media are not air—that is to say, if the lens is situated in a dense medium—the original formula is required :—

$$F = \frac{r_1 r_2 \mu_1}{(r_1 + r_2)\ (\mu_2 - \mu_1)}$$

Thus, suppose a double Cx. lens of $\mu = 1{\cdot}54$ and 8 cm. radius placed in water ; in that medium

$$F = \frac{8 \times 8 \times 1{\cdot}33}{(8 + 8)\ (1{\cdot}54 - 1{\cdot}33)} = \frac{85{\cdot}12}{3{\cdot}36} = 25{\cdot}33 \text{ cm.}$$

Or the relative index $\mu_r$ may be found by dividing $\mu_2$ by $\mu_1$, and the formula then becomes as with thin lenses in air. Here $\mu_r = 1{\cdot}54/1{\cdot}33 = 1{\cdot}158$, and

$$F = \frac{8 \times 8}{(8 + 8) \times {\cdot}158} = 25{\cdot}33 \text{ cm.}$$

Let a similar lens, but of $\mu = 1{\cdot}33$, be placed in cedar oil of $\mu = 1{\cdot}54$, then

$$F = \frac{8 \times 8 \times 1{\cdot}54}{(8+8)\,(1{\cdot}33 - 1{\cdot}54)} = \frac{98{\cdot}66}{-3{\cdot}36} = -29{\cdot}33 \text{ cm.}$$

Here the lens acts with a negative effect, and it shows us that an air lens in water must have a concave curvature in order that it may have a positive refracting power. Dr. Dudgeon constructed such a lens to enable divers, without helmets, to see under water. It consisted of two small watch-glasses of very deep curvature cemented into each end of a vulcanite ring, the convex surfaces facing each other inside the ring. The lens had no magnifying power out of water, as it only contained air. In water, however, the concavity of the lens produced a convexity of the water in contact with it on each side, and this convexity gave the required refractive power.

Let a Cc. air lens be of 10 inch radius on both surfaces. What will its focus be in water?

$$F = \frac{-10 \times -10 \times 1{\cdot}33}{(-10 - 10)\,(1 - 1{\cdot}33)} = \frac{100 \times 1{\cdot}33}{-20 \times -{\cdot}33} = \frac{133}{6{\cdot}66} = +20.$$

Since a Cx. water lens of the same radius in air has F = 15 in., it will be noticed that the effect is not the same when the conditions are reversed. This arises from a similar cause to that which produces a difference in the anterior and posterior foci of a single refracting surface. If light passes finally into a rare medium the focal distance is shorter than when it thus passes finally into a dense medium.

**Change of F in Dense Media.**—The change undergone by the power and focal length of a lens when transferred from air to some denser medium is greater than we might perhaps expect at first sight. It has been previously shown that F is inversely proportional to $(\mu - 1)$, so that when a lens of index $\mu_1$ is immersed in a medium of index $\mu_2$, we have $F : F' : : (\mu_r - 1) : (\mu_1 - 1)$, where $\mu_r$ is the relative index $\mu_1/\mu_2$. In other words the lens has a focal length $F'$ in the medium as if it were made of a substance whose index is $\mu_r$, and surrounded by air. Thus

$$F' = \frac{F(\mu_1 - 1)}{(\mu_r - 1)} \qquad \text{or} \qquad D' = \frac{D(\mu_r - 1)}{(\mu_1 - 1)}$$

For instance a Cx. lens of radius 8″ and $\mu = 1{\cdot}5$ has in air a focal length of 8″ and a power of 5 D. If placed in water of $\mu = 4/3$, $F'$ is 32 in. and D is 1·25. Thus a glass lens in water has its focal length increased about four times, and its power correspondingly reduced to a quarter.

The crystalline lens of the eye suffers an even greater change of power. *In situ* its power is about 22 D, but in air it becomes about 125 D. In this case the relative index between the lens medium and the surrounding aqueous and vitreous is only some 1·09.

**Cases of Various Media.**—When a thin lens of $\mu_2$ separates two media of $\mu_1$ and $\mu_3$—that is, when there are three different media separated by two curved surfaces—the following formula serves for finding the focal length :—

$$\frac{\mu_3}{F} = \frac{\mu_2 - \mu_1}{r_1} + \frac{\mu_3 - \mu_2}{r_2}$$

If there are four media we have

$$\frac{\mu_4}{F} = \frac{\mu_2 - \mu_1}{r_1} + \frac{\mu_3 - \mu_2}{r_2} + \frac{\mu_4 - \mu_3}{r_3}$$

The power of any number of surfaces separated by negligible distances can be found by taking the sum of their anterior focal powers and multiplying it by the reciprocal of the last $\mu$—i.e., by $1/\mu_3$ or $1/\mu_4$ as the case may be. If the last medium be air, like the first, we have $1/F$ equal to the sum of the anterior focal powers of all the media.

It should be particularly noted that in the numerator of each fraction the preceding $\mu$ is always deducted from the $\mu$ following—e.g. $\mu_3 - \mu_2$, and that $r$ is positive or negative according as it is respectively Cx. or Cc. towards the direction of the light. In this way all calculations involved in fused or inset bifocals are rendered comparatively easy.

We have a case of four media when light passes from air to a surface of $\mu_2$, then to another surface of $\mu_3$, and finally, by a third surface again into air. Such a combination exists if a bi-focal be made by the insertion of a deeply curved convex segment of high $\mu$ into a space made for it in a larger lens of low $\mu$. Such a combination is also formed by the contact of a double Cc. lens of, say, $\mu = 1{\cdot}5$ with a double Cx. lens of, say, $\mu = 1{\cdot}6$, the two being of equal curvature. The focal power can be found by calculating for each lens separately and then adding them together, or by calculating for each surface separately, as indicated above.

**Recapitulation of Formulæ.**—The following is a recapitulation of the formulæ for finding the focal length of the various spherical refracting bodies when the light passes from air. Where numerical examples are appended they are in each case for $r = 6$ in. and $\mu = 1{\cdot}5$.

| | | Approx. Value in $r$. |
|---|---|---|
| Posterior F of a single surface | $= \frac{\mu r}{\mu - 1}$ | $= 18$ in. $= 3r$ |
| Anterior F of a single surface | $= \frac{r}{\mu - 1}$ | $= 12$ in. $= 2r$ |
| F of all forms of thin lens | $= \frac{r_1 r_2}{(r_1 + r_2)(\mu - 1)}$ | |
| F of a thin equi-lens | $= \frac{r}{2(\mu - 1)}$ | $= 6$ in. $= r$ |
| F of a thin plano-lens | $= \frac{r}{\mu - 1}$ | $= 12$ in. $= 2r$ |

### Dioptral Formulæ.

**Lens in Air.**—To find the dioptral power D of a Cx. or Cc. lens, the radii being in cm.:—

$$D = \frac{100\,(\mu - 1)\,(r_1 + r_2)}{r_1 r_2} \qquad \text{or} \qquad \left(\frac{100}{r_1} + \frac{100}{r_2}\right)(\mu - 1)$$

which formulæ simplify to

$$D = \frac{2 \times 100\,(\mu - 1)}{r} \text{ for an equi Cx. or Cc.}$$

$$D = \frac{100\,(\mu - 1)}{r} \text{ for a plano Cx. or Cc.}$$

**Lens in a Medium Denser than Air.**—$\mu_2$ pertains to the lens and $\mu_1$ to the medium in which it is placed.

$$D = \left(\frac{100}{r_1} + \frac{100}{r_2}\right)\left(\frac{\mu_2 - \mu_1}{\mu_1}\right)$$

### The Construction of Images formed by Thin Lenses.

**Course of Light—Cx. Lens.**—If a beam of rays shown by the thick lines in Fig. 95 be incident on the surface of a Cx. lens in a direction parallel to the principal axis $F_1 F_2$ they are refracted to meet at the point $F_2$, the principal focus or second focal point, situated on the axis. A line $C D$ drawn through this point perpendicular to the axis is the *second focal plane.* The distance from $O$, the optical centre of the lens, to $F_1$ or $F_2$, is the focal length of the lens. In the same way parallel rays which are incident on the other surface of the lens (shown by the dotted lines) meet in a point at $F_1$, the first focal point. A line $A B$ drawn through it perpendicular to the axis is the *first focal plane.* The distance $O F_1$ is equal to $O F_2$, and $L O L$ is the refracting plane of the lens.

Whatever course a ray takes in passing through a lens (or any number of lenses), if the light retraces its course, it follows the same path. It is clear, therefore, that if the source of light be at $F_1$ or $F_2$ the rays, after refraction, pass out of the lens parallel to the principal axis. All rays which diverge from a luminous point on the principal axis are refracted on passing through a lens, with the exception of the principal axial ray which passes through the optical centre and undergoes no refraction.

If, instead of the object point being on the principal axis, it is situated on a secondary axis $E F_2'$, as in Fig. 95, the rays are similarly bent to meet in a focus at $F_2'$, and any ray passing through $O$ obliquely to $F_1 F_2$ is presumed to be undeviated by the lens.

**I of Point on the Axis—Cx. Lens.**—The object point $A$ being beyond $F_1$, draw the axis $A B$ (Fig. 96) and through $F_1$ draw the focal plane $G H$. From

$A$ draw any line $A\,K\,D$, cutting the first focal plane at $K$ and the refracting plane of the lens at $D$. From $K$ draw a line through the optical centre $O$ and from $D$ draw $D\,B$ parallel to $K\,O$. This refracted ray $D\,B$ cuts the

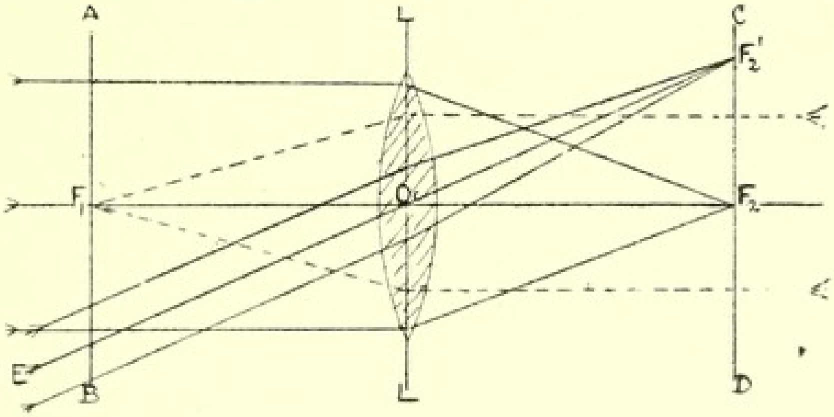



Fig. 95.

principal axis at $B$, which is the image of the point $A$. This construction holds good because rays diverging from any point in the focal plane are parallel to each other after refraction.

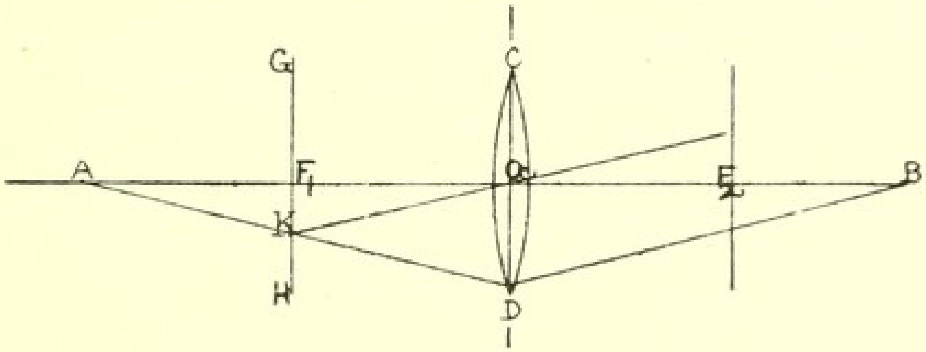



Fig. 96.

The object point $A$ being on the axis nearer than $F_1$ (Fig. 97) from $A$ draw any line $A\,E$, and from $F_1$ draw $F_1\,C$ parallel to $A\,E$. Draw $C\,D$ parallel to the principal axis cutting the second focal plane in $D$. Connect

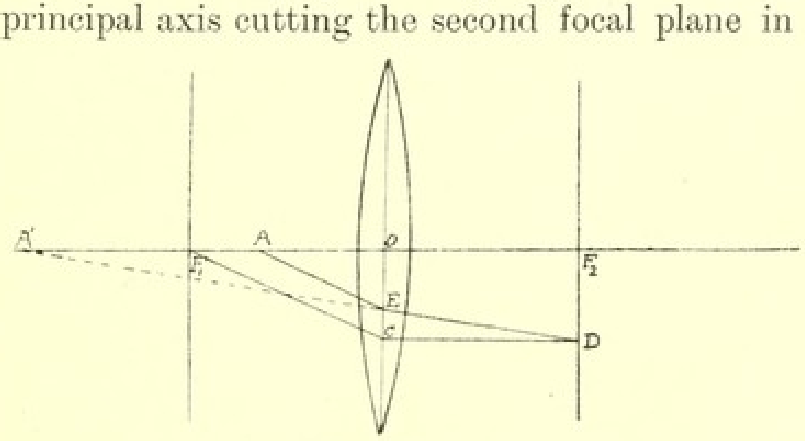



Fig. 97.

$D$ and $E$ and produce to $A'$ on the principal axis; then $A'$ is the virtual image of $A$. This construction holds good because $A\,E$ and $F_1\,C$ are parallel and therefore meet in the second focal plane; also $F_1\,C$ is, after refraction, parallel to the principal axis.

**Construction of I for Cx. Lens.**—In order to construct the image of an object formed by a Cx. lens we have three rays diverging from any point whose course, after refraction, it is easy to follow, viz. :—

(*a*) The ray parallel to the principal axis, passing, after refraction, through $F_2$.

(*b*) The ray which passes through $F_1$ and, after refraction, is parallel to the principal axis.

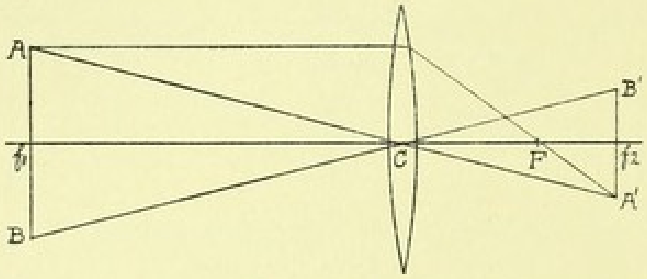



Fig. 98.

(*c*) The ray which passes through the optical centre, and whose course is not altered by refraction.

It is necessary to draw only two of these rays in order to locate the I of a point, since where any two rays diverging from a point meet, all other rays diverging from that same point also meet.

**Real I.**—In order to construct the complete I of an O, the images $A'$ and $B'$ of the two extreme points $A$ and $B$ should be found, and these suffice to

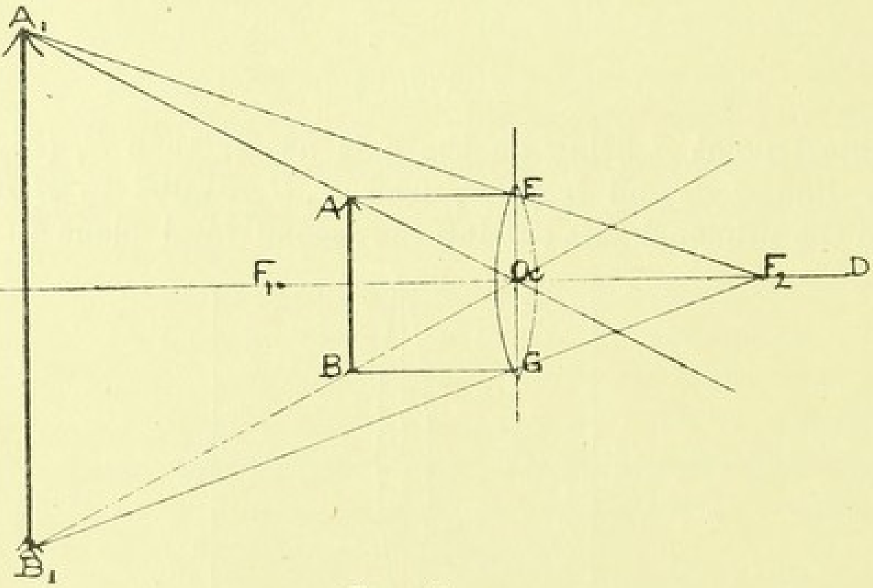



Fig. 99.

show the location and size of the image (Fig. 98). Draw from $A$ the ray parallel to the axis; this ray, when refracted, passes through $F$. Draw the secondary axis $A\ A'$ passing straight through $C$. These lines meet at $A'$, the image of $A$. In the same way $B'$, the image of $B$, can be constructed. The images of all intermediate points between $A$ and $B$ could be constructed, but are not necessary, for *B′ A′* shows the position and size of the real inverted image of the object *A B*.

**Virtual I.**—When the object is nearer the lens than $F_1$ (Fig. 99), from *A* draw *A E* parallel to *C D*; *E* $F_2$ is the course of the ray after refraction. Draw *A Oc* passing through the optical centre.

Since these rays are divergent after refraction, no real image can be obtained, but by producing them backwards they are made to meet at *A′*,

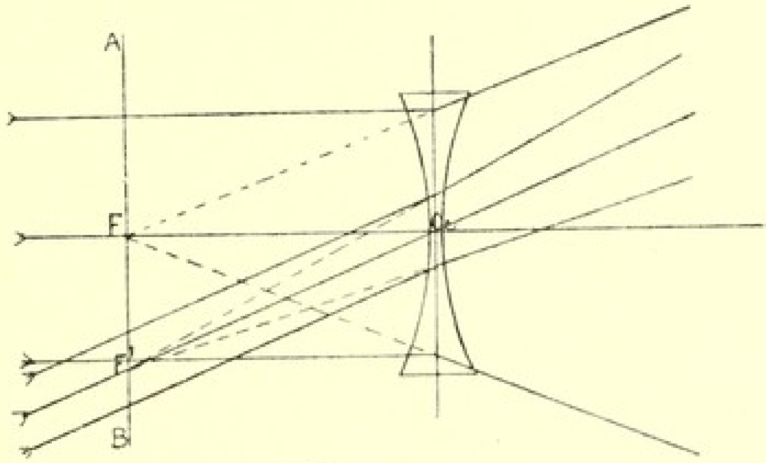



FIG. 100.

which is the virtual image of *A*. Similar rays drawn from *B* locate its image as *B′*, and *A′ B′* is the complete virtual erect image of the object *A B*.

**I at ∞.**—When the object is at $F_1$, the rays, after refraction, are parallel to their axes, and, therefore, no image can be constructed, since it lies at infinity.

**Course of Light—Cc. Lens.**—If a beam of parallel rays (Fig. 100) is incident on the surface of a Cc. lens they apparently diverge, after refraction,

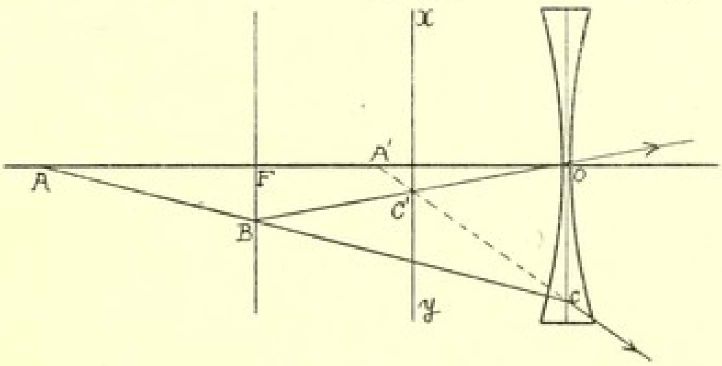



FIG. 101.

from *F*, and a plane *A B* perpendicular to the axis passing through *F* is the focal plane. *Oc F* is the principal focal distance. Every ray passing through the lens is refracted, except that passing along the principal axis or a secondary axis. A point on any axis as *F′ Oc* has its image on that same axis.

**I of Point on the Axis—Cc. Lens.**—To construct the image produced by a concave lens, the object being a point *A* on the axis, draw any ray *A B C* cutting the focal plane in *B* and the refracting plane in *C* (Fig. 101).

From *B* draw *B O* through *O* the optical centre. Now *B C* and *B O* diverging from the focal plane apparently come from *C′* on *X Y*, the latter being a plane midway between the focal and refracting planes. Prolong *C C′* to *A′* on the principal axis; *A′* is then the image of the point *A*. This construction holds good because rays diverging from the focal plans of a Cc. lens after refraction apparently come from the plane midway between the focal and refracting planes.

**Construction of I for Cc. Lens.**—The construction of the image formed by a concave lens is the same wherever the object is situated, since the image is always formed on the same side as the object, and between the principal focus and the lens. In Fig. 102, let *A B* be an object placed in front of a concave lens of which *F* is the principal focus. From *A* trace *A O*, the axial ray. Draw *A E* parallel to the axis before refraction, and diverging as if from *F* after refraction. These rays, being divergent, can only unite by being prolonged backwards, when they meet at *A′*. Similar rays from *B* meet at *B′*, its image. The complete image of *A B* is *A′ B′*.

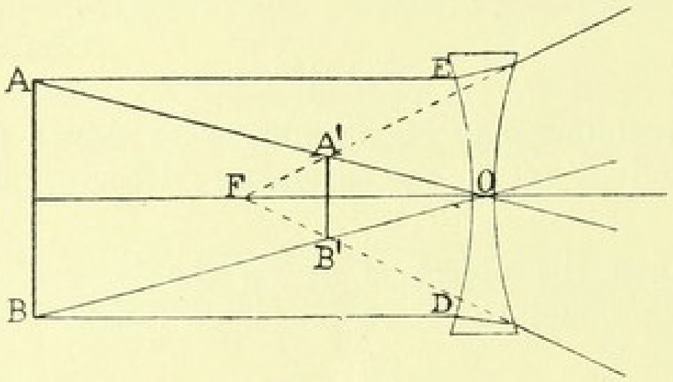



FIG. 102.

**Construction for the Course of a Ray.**—This can be done on the principle shown in Fig. 79, as it is given for a single surface. In the case of a lens or sphere, after the course has been determined from the first surface, a second construction is needed for the second surface.

**Characteristics of a Convex or Positive Lens.**

(*a*) It is thicker at the centre than at the edge.

(*b*) It forms a magnified image of an object held within the focus.

(*c*) At the proper distance, it forms on a screen an inverted real image of a luminous object, as a flame or window.

(*d*) It causes the image of an object, viewed through it, to move in the *contrary* direction as the lens is moved.

**Characteristics of a Concave or Negative Lens.**

(*a*) It is thinner at the centre than at the edge.

(*b*) It diminishes the apparent size of an object seen through it.

(*c*) No image can be projected by it on a screen.

(*d*) When moved, an object seen through it appears to move in the *same* direction.

**Refraction and Reflection Compared.**—A curved mirror may be regarded as a dioptric system in which $\mu_1 = \mu_2$ and therefore $F_1 = F_2$. The principal point is at the vertex of the curve, and the optical centre is at the centre of curvature, they being similar to those of a single refracting surface. When F of a refracting system lies in a medium similar to that from which the light proceeds $F_1 = F_2$, and both are equally distant from the principal and nodal points, which are then combined. In a reflecting system the source of light and F lie in air. F is midway between, and therefore equally distant from the principal and nodal points.

For comparison, the following figures show the difference between the focal lengths when an incident beam of light is reflected from, or refracted by, the surface of a thin plano Cx. or Cc. glass lens of $\mu = 1{\cdot}5$.

In Figs. 103 and 104, *C* is the centre of curvature. Rays of light

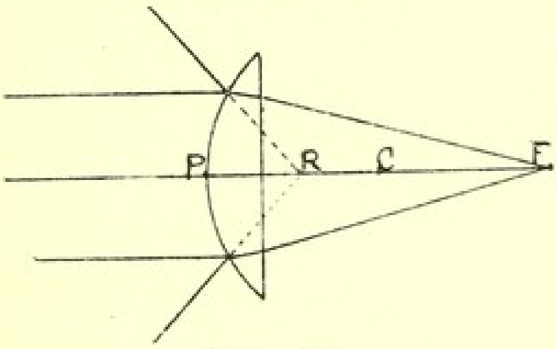



FIG. 103.

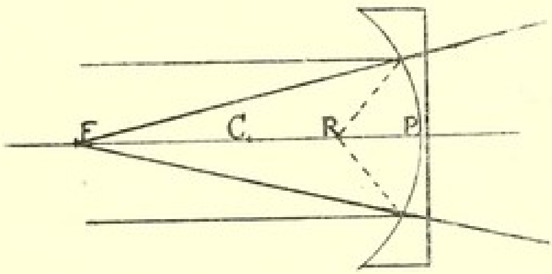



FIG. 104.

parallel to the axis, if reflected, meet at *R*, which is half the distance of *C* from the pole *P*; if refracted they meet at *F*, which is twice the distance of *C* from the surface. The thick lines represent the course of the refracted rays and the dotted lines that of the reflected rays.

If we use the surface of a Cc. lens as a reflector and find F, then $\mu$ being taken as 1·5, the refracting F of that surface is four times as long; or if the lens be double Cc. the F of the lens is twice as long as that shown by reflection. Thus with a plano Cc. if the reflection F is 10 cm., the dioptric F is 40 cm.

If we measure the curvature of a mirror by a lens measure scaled in diopters, the measurement shown is about 1/4 that of reflection, i.e. F of the mirror is 1/4 that shown by the scale. Thus if a mirror shows 2·5 D its F is $100/2{\cdot}5 \times 4 = 10$ cm.

## CHAPTER VIII

### THIN LENS CALCULATIONS

Lenses are numbered by two principal systems, namely, the inch and the dioptric.

**The Inch System of Numeration** is based on the measurement of the focal length of a lens, and the unit is a lens of one inch focus. Since F varies inversely with the power, a lens which brings parallel light to a focus at 10 ins. or at 20 ins., has respectively 1/10 or 1/20 the power of the unit; while one whose F = 1/2 in. has twice the power. The abbreviations Cx. for convex and Cc. for concave are commonly employed in conjunction with the focal notation of lenses.

**The Disadvantages of the F System** are that the inch in various countries differs in value, so that a lens of given focal length in one country may not be the same as one of similar number in another. There are 37 French inches, while there are 39·37 English inches, in the metre, so that a lens of 18 French (Paris) inches focal length is about equivalent to one of 20 English or American inches.

Again, the intervals between the lenses, although regular as to their focal lengths, are irregular as to their refractive powers; thus there is a far greater difference between the powers of a 5 and a 6 inch, than between a 15 and a 16 inch lens. Further, the unit being a very strong lens, and the lenses mostly required being weak ones, calculations involve the use of vulgar fractions.

**Dioptric System.**—The dioptric system is based on the refractive power of lenses, and the unit is the diopter, which is that power which causes parallel light to focus at 1 metre. The diopter of refraction is a measure of converging or diverging power, and is not, strictly speaking, synonymous with the metre, which is a unit of linear measurement; nevertheless, it is often convenient to express distances in dioptric measure. The symbols + and − are always used with this system.

The dioptric system is much more simple than the inch, and is now universally recognised. The unit being weak, the power of most other lenses is expressed by whole numbers, while if fractions are involved they are expressed as decimals. The intervals between the lenses are uniform as regards their refracting powers. The power of a combination of two or more lenses is obtained from simple algebraical addition of their numbers.

If a 1 D lens has F = 1 M, a 4 D lens, having four times as much power, has F = 1/4 M. But since the M can be sub-divided into 100 cm. or 1000 mm. the focal length of a 4 D is more conveniently expressed as 100/4 = 25 cm. A 10 D lens has ten times the power of the unit; therefore its F = 100/10 = 10 cm., or 1/10 that of the unit. A 0·50 D has half the power of the unit; consequently its F = 100/·5 = 200 cm., or twice that of the standard lens.

**Conversion.**—Since the +1 D lens has F = 1 M, or 40 inches, it is equal to No. 40 of the inch system, and a 40 D lens is the same as a 1 inch lens. Now the metre (or 100 centimetres) = 39·37 English inches, and for all practical purposes may be regarded as equivalent to either 40 or 39 inches. Therefore for conversion from either scale into the other, it is only necessary to divide 40 or 39 (whichever is the most convenient) by the known number. For instance,

| | |
|---|---|
| 2·5 D = 40/2·5 = 16 in., | 13 D = 39/13 = 3 in., |
| 2 in. = 40/2 = 20 D | 13 in. = 39/13 = 3 D |

Since many numbers will not divide evenly into 40 or 39, there is frequently a small remainder which need not be considered beyond the 1/4, 1/2 and 3/4 in the lower inch numbers, and ·25, ·50, and ·75 in the dioptral numbers. Some numbers of both scales have no exact equivalent in spectacle lenses, numbered according to the other, and the nearest must be taken as the equivalent power. For instance, it is considered that 3·50 D = No. 11″; 3·25 D = No. 12″; 4·50 D = No. 9″, etc.

**To Find F or D.**—Dividing 40 or 100 or 1000 by the dioptral number gives F in inches, in cm., or in mm. respectively. Thus, a 5 D lens has F = 40/5 = 8 in., 100/5 = 20 cm., or 1000/5 = 200 mm.

If F is known in cm., mm., or inches, the dioptral number is found by dividing respectively into 100 or 1000 or 40; thus, if F = 200 mm., then D = 1000/200 = 5; if F = 40 cm., D = 100/40 = 2·5; if F = 160 in., D = 40/160 = ·25.

**Old Curvature System.**—Originally the inch system of numeration was based on the radius of curvature. No. 10 implied a double Cx. or Cc. lens having a radius of curvature of 10″ on each surface.

**Old Cc. System.**—In England concave sphericals were formerly numbered by an arbitrary system commencing at 0000—the weakest—and terminating with No. 20—the strongest. The values of these numbers in the inch and dioptric scales are to be found in the appendix, but the system is now obsolete.

**Cyls.**—The numeration of cyls. is the same as that of sph's.

**Addition of Lenses.**—The combined strength 1/F of the two thin lenses in contact, whose values are indicated by their focal lengths $F_1$ and $F_2$ respectively, is obtained by the addition of their refractive powers, thus

$$1/F = 1/F_1 + 1/F_2$$

If the two lenses be, say, 24 inch Cx. and 10 inch Cx. their powers are respectively 1/24 and 1/10; the combined power is

$$1/24 + 1/10 = 34/240 = 1/7 \text{ approx.}$$

The two are equivalent to a 1/7 Cx. or a lens of 7 in. F. It is evident that F of the combination must be shorter than that of either lens alone.

If the two lenses are concave, say 5 and 8, they equal

$$-1/5 + (-1/8) = -13/40 = -1/3 \text{ approx.}$$

When the one lens is convex and the other concave 1/F is positive or negative according as $F_1$ or $F_2$ is the shorter. The two neutralise each other more or less, and the residual power of the stronger is the power of the combination. Thus, a 15 Cx. and a 12 Cc. when combined make a lens of 60 inch negative F, thus

$$1/15 + (-1/12) = 12/180 - 15/180 = -3/180 = -1/60$$

A 20 Cc. and a 10 Cx. together give $1/10 + (-1/20) = +1/20$ i.e., a 20 Cx.

The summing up of three or four lenses is achieved in a similar manner; thus 10 Cx., 16 Cx., 7 Cx., and 5 Cc. make together $1/10 + 1/16 + 1/7 - 1/5 = 59/560$, that is, $9\frac{1}{2}$ Cx. approx.

The strength D of combined dioptral lenses in contact is obtained by adding them together algebraically, thus

$$D = D_1 + D_2$$

$D_1$ being the power of the one, $D_2$ that of the other lens, and D that of the two combined. For example:

$$+2 \text{ D and } +4 \text{ D} = +6 \text{ D}; \quad +4 \text{ D and } -3 \text{ D} = +1 \text{ D}$$

$-5{\cdot}25$ D and $-2{\cdot}50$ D $= -7{\cdot}75$ D; $+3$ D and $-3$ D $= 0$, i.e. they neutralise each other.

$$+7 \text{ D} + 4{\cdot}50 \text{ D} + 1{\cdot}75 \text{ D and } -6{\cdot}50 \text{ D} = +6{\cdot}75 \text{ D}$$

## Conjugate Foci with Thin Lenses.

**Conjugate Foci.**—In the following, let O and I represent object and image respectively.

The focal distance of a thin Cx. lens is the distance from the optical centre, which marks the refracting plane, to the plane in which originally parallel light meets after refraction, and it is that distance from which light must diverge in order to be parallel after refraction. In the case of a thin

Cc. lens, it is that distance from the refracting plane from which originally parallel light appears to diverge after refraction. If F is the focal length, its reciprocal 1/F is the focal power of the lens. If $f_1$ be the distance from the optical centre from which light from the object diverges, then $1/f_1$ represents that divergence; if $f_2$ is the distance of the image from the optical centre, then $1/f_2$ is the convergence or divergence of the light which produces the image. 1/F is positive or negative according as it pertains to a converging or diverging lens respectively, while $1/f_1$ is always negative. The value of $1/f_2$ is found by adding the divergence of the light $1/f_1$ to the converging or diverging power of the lens, that is

$$1/f_2 = 1/F - 1/f_1 \qquad \text{whence } 1/F = 1/f_1 + 1/f_2,$$

that is, the power of the lens is always equal to the sum of the reciprocals of any pair of conjugate foci, or to the sum of its actions on the light.

With a Cx. lens $f_2$ is positive or negative according as the convergence of the lens 1/F is greater or less than the divergence of the light

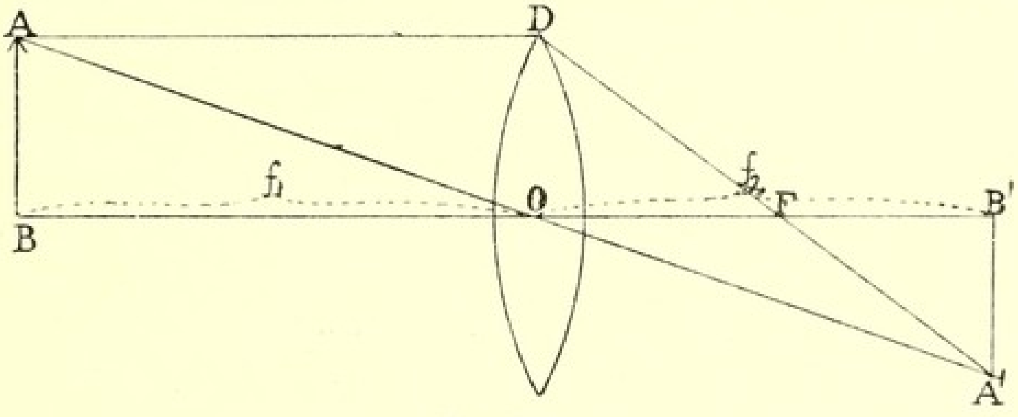



Fig. 105.

$1/f_1$. With a Cc. lens $1/f_2$ is always negative, since the divergence of the light is added to the divergence of the lens.

By inverting the formula we get a variation which is sometimes more convenient to use.

$$F = \frac{f_1 f_2}{f_1 + f_2}$$

therefore

$$f_2 = \frac{f_1 F}{f_1 - F} \qquad \text{and} \qquad f_1 = \frac{f_2 F}{f_2 - F}.$$

It can also be wri ten

$$F/f_1 + F/f_2 = 1.$$

**Geometrical Proofs.**—In Fig. 105 *A B* is the object and *B′A′* is the image. *O* is the optical centre of a Cx. lens, *O F* is its focal length, and *F* is the principal focus. Since the triangles *A A′ D* and *O A′ F* are similar

$$\frac{A\,D}{O\,F} = \frac{A\,A'}{O\,A'} = \frac{B\,B'}{O\,B'}$$

Now

$$A\,D = O\,B = f_1,\ O\,B' = f_2,\ \text{and}\ O\,F = F$$

Therefore

$$\frac{f_1}{F} = \frac{f_1 + f_2}{f_2} \quad \text{or} \quad \frac{1}{F} = \frac{f_1 + f_2}{f_1 f_2}$$

That is

$$1/F = 1/f_1 + 1/f_2$$

For a Cc. lens (Fig. 106) in the pairs of similar triangles A O B and A′ O B′, D F O and A′ F B′,

$$\frac{O\,B'}{O\,B} = \frac{A'\,B'}{A\,B} = \frac{A'\,B'}{D\,O} = \frac{B'\,F}{O\,F}$$

Now

$$O\,B' = f_2,\ O\,B = f_1,\ \text{and}\ O\,F = F$$

Therefore

$$\frac{f_2}{f_1} = \frac{F - f_2}{F}, \quad \text{or} \quad F\,f_2 = F\,f_1 - f_1 f_2$$

so that

$$F(f_1 - f_2) = f_1 f_1 \quad \text{and} \quad 1/F = -1/f_1 + 1/f_2$$

that is

$$-1/F = 1/f_1 - 1/f_2$$

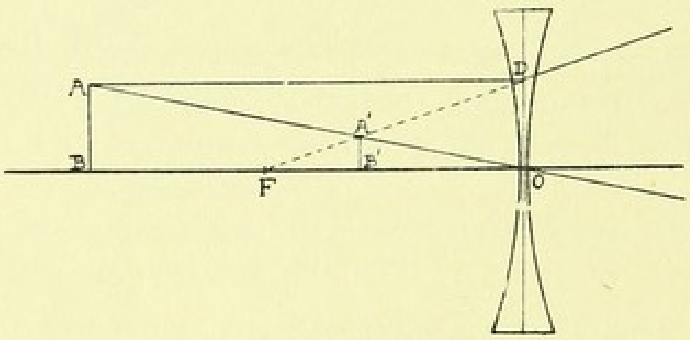



Fig. 106.

**Conjugate Distances of O and I and Examples—Cx. Lens.**—A convex lens renders rays convergent, parallel or less divergent, according as the point of divergence is respectively beyond, at, or within F; the converging property of the lens is decreased, neutralised, or exceeded by the divergence of the light due to the nearness of the object, and since any approach of the object to a Cx. lens causes the light to be less convergent after refraction, it follows that any real conjugate focus is more distant than F, so that F is *the nearest point* to the lens at which a real image can be formed by natural rays.

If a lens has $F = 8$ inches and $f_1$ is at 40 inches, then $f_2$ will be at 10 inches and real, for $1/f_2 = 1/8 - 1/40 = 4/40 = 1/10$.

This is proved by $1/10 + 1/40 = 1/8$, and $8/40 + 8/10 = 1$.

A real image is 16 inches behind a 7 inch Cx., at what distance is the object in front of the lens?

$1/f_1 = 1/7 - 1/16 = 9/112$. The object is at $12\frac{4}{9}$ in.

If, however, the incident light were *converging*, I would be nearer than F. Thus if light converges to 15″ behind a 6″ Cx. lens we have

$$1/f_2 = 1/6 + 1/15 = 21/90$$

The image is at $4\frac{2}{7}$″, which is nearer to the lens than F.

When O is at ∞, then I is at F, since 1/F − 1/∞ = 1/F − 0 = 1/F.

As O approaches the lens I recedes from F; when O is at 2 F then I is at an equal distance. This is the plane of *unit magnification* for real images.

When O is at F the power of the lens is just sufficient to render the incident rays parallel and I is at ∞, for then $1/f_2 = 1/F - 1/F = 0$ and $f_2 = \infty$.

Therefore ∞ and F are conjugate focal distances.

When O is situated nearer than F, the power of the lens is insufficient to render the light parallel, and it emerges divergent after refraction, although less so than before. No real focus is obtained, but if projected backwards the light meets in front of the lens (on the same side as the object) and forms a negative focus and virtual image at $f_2$. Whereas the light diverged originally from $f_1$ it appears after refraction to diverge from $f_2$. Since the divergence $1/f_1$ is greater than the convergence 1/F, on deducting the former from the latter the result is a negative quantity. Thus let the object be 6 in. from an 8 in. Cx. lens, then

$$1/f_2 = 1/8 - 1/6 = -1/24.$$

The I is virtual or negative at 24 inches on the same side of the lens as O.

As O approaches the lens from $F_1$ its virtual I also approaches, and when O touches the lens so also does I, this being the *plane of unit magnification for virtual images.*

**Conjugate Distances of O and I—Cc. Lens.**—A concave lens renders parallel light divergent, and increases the divergence of divergent light; therefore any distance of O nearer than ∞, causes I to be nearer than F, so that the *most distant conjugate focus of a Cc. lens is F.* Thus let the lens be −1/10 and $f_1$ at 40 in.; then $1/f_2 = -1/10 - 1/40 = -5/40 = -1/8$ the image being at 8 in. negative or virtual. This is proved by the power of the lens 1/F being equal to −1/8 + 1/40 = −1/10.

When O is at ∞, then I is at F, and as O approaches the lens so also does I, until when O touches the surface, I does also, *this being the plane of unit virtual magnification.* If, however, the incident light were convergent, I would be beyond F—for instance, should light be converging towards a point 15″ behind a 6″ Cc. lens, then $1/f_2 = -1/6 + 1/15 = -9/90$; the image is virtual at 10″. If he light converged to 6″ it would be rendered parallel by the Cc. lens, and if the convergence were to 5″ we should have −1/6 + 1/5 = +1/30, or a convergence to 30″.

**Reciprocity of Conjugates.**—Real conjugate foci are interchangeable distances in the sense that if O is at either of them, I is at the other. Thus when O is at 40″ in front of an 8″ Cx. lens, I is at 10″, and if I were at 10″, O would be at 40″. Virtual conjugates are not interchangeable in this sense. If O is at 6″ from an 8″ Cx. lens I is at 24″ virtual. O could not be at – 24″, which is a negative distance, and if it were at 24″ actually, I would not be at 6″. These conjugates are interchangeable merely in the sense that if light *converged towards* the virtual focus, in this case 24″, then I would be at the real distance 6″.

The same occurs with the virtual conjugate of a Cc. lens. If O is at 40″ and the lens is – 1/10 the I is at 8″ virtual; light would need to converge to 8″ behind the lens in order that a real image be formed at 40″.

## Dioptral Formulæ.

**Conjugates of a Cx. Lens.**—*The reciprocal of the focal distance in terms of a metre, or its value expressed in diopters, indicates the power of the lens.* A + 5 D lens has a focal length of 20 cm., and consequently light diverging from 20 cm. is rendered parallel by it, the converging power of the lens just

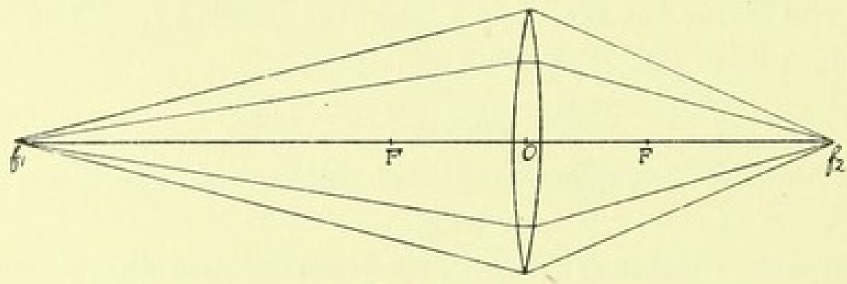



Fig. 107.

neutralising the divergence of the light from 20 cm. Similarly light from ∞ is brought to a focus at 20 cm. by a + 5 D lens.

If the light diverges from some point within ∞ it has then a divergence equal to that of a Cc. lens whose F is equal to the distance; the resulting image $d_2$ is the dioptral result of the addition of the dioptral divergence of the light $d_1$ and the dioptral power of the lens D. That is

$$D - d_1 = d_2 \qquad \text{or} \qquad D = d_1 + d_2$$

The power of a lens is equal to the sum of the two conjugates $f_1$ and $f_2$ expressed in diopters as $d_1$ and $d_2$.

Suppose a + 5 D (Fig. 107) and let $f_1$ be 100 cm. distant. The lens has a converging power of 5 D, and the light has a divergence, expressed in diopters, of 1 D. Consequently after refraction the light has a convergence of 5 – 1 = 4 D, the I being at 25 cm.

In Fig. 108 the + 5 D is shown as if split into two lenses, the + 1 D rendering parallel the light diverging from $f_1$, while the + 4 D brings the parallel rays to a focus at 25 cm.

The two distances 100 cm. and 25 cm. expressed in diopters are +1 and +4 respectively. We may therefore write :—

$1+4=5$ D = the power of the lens.
$5-1=4$ D = the dioptral distance of I.
$5-4=1$ D = the dioptral distance of O.

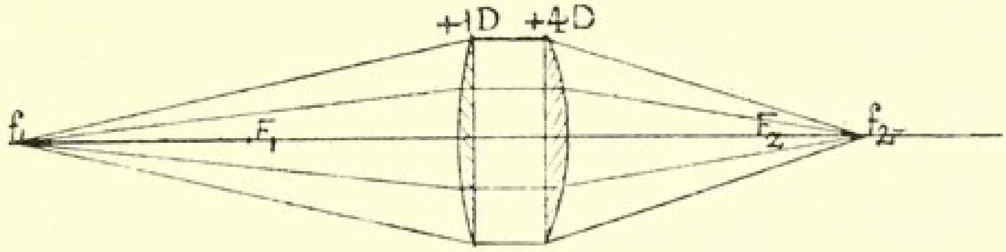



Fig. 108.

**Examples.**—Suppose the object be placed 50 cm. in front of a lens having its image 12·5 cm. behind it, then to find the power of the lens

$$d_1 = 100/50 = 2, \qquad d_2 = 100/12{\cdot}5 = 8\,;$$

therefore $$D = 2 + 8 = 10.$$

Suppose an object is 200 cm. in front of a 7 D lens, where will the image be ?

Here $$d_1 = 100/200 = {\cdot}5, \qquad d_2 = 7 - {\cdot}5 = 6{\cdot}5\,;$$

therefore $$f_2 = 100/6{\cdot}5 = 15 \text{ cm.}$$

An image is 22 cm. behind an 8 D lens, where is the object ?

We have $$d_2 = 100/22 = 4{\cdot}5, \qquad d_1 = 8 - 4{\cdot}5 = 3{\cdot}5\,;$$

therefore $$f_1 = 100/3{\cdot}5 = 30 \text{ cm.}$$

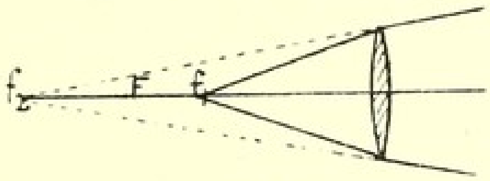



Fig. 109.

If O is at ∞, then $d_1 = 100/\infty = 0$ ;

so that $$D - 0 = D \quad \text{and} \quad 100/D = F,$$

consequently I is at F.

If O is at F, then $d_1 = 100/F = D$,

and $$D - D = 0 \quad \text{and} \quad 100/0 = \infty,$$

consequently I is at ∞.

If light converges to 50 cm. behind a +5 D lens we have $5+2=+7$ D, or 14 cm. as the distance of the I, which is nearer than F.

Let the lens be +5 D and O be at 14 cm. (Fig. 109) then $d_1 = 100/14 = 7$ ;

$d_2 = 5 - 7 = -2$, and $100/-2 = -50$, so that $f_2$ is at 50 cm. virtual in front of the lens. While the lens has a converging power of 5 D, the light has a divergence of 7 D; therefore, after refraction, there is a residual divergence of 2 D. We have $d_1 + d_2 = \text{D}$, that is $7 + (-2) = +5$ D.

**Conjugates of a Cc. Lens.**—A concave lens refracts light divergently so that parallel rays, after refraction, appear to diverge from $F$ (Fig. 110). If the power of the lens is $-5$ D, the virtual $F$ will be at $100/5 = 20$ cm. or 8 ins.

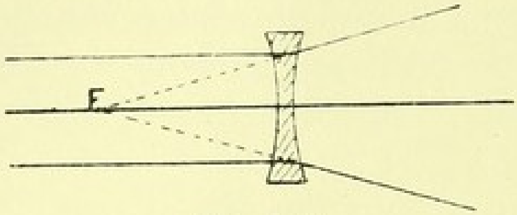



Fig. 110.

When $f_1$ is nearer than $\infty$ (Fig. 111) the light incident on the lens being divergent before refraction is rendered still more divergent; the divergence of the light is augmented by that of the lens, consequently the conjugate focus is nearer than $F$. Here again $f_1$ and $-f_2$ are conjugates just as in the

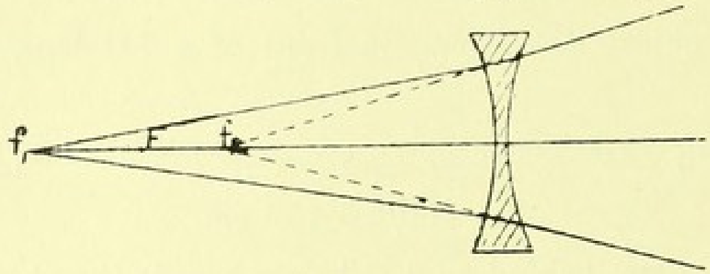



Fig. 111.

case of the virtual focus obtained with a convex lens, because the sum of their powers $d_1 + (-d_2) = \text{D}$.

Let the lens be $-5$ D and $f_1$ at 100 cm.; then $d_2 = -5\text{ D} - 1\text{ D} = -6$ D, and $100/-6 = -16{\cdot}66$ cm.; $f_2$ is therefore virtual and 16·66 cm. in front of

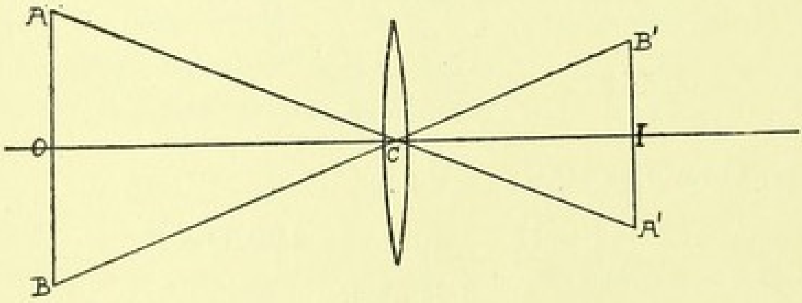



Fig. 112.

the lens. If light diverges from 100 cm. to a $-5$ D lens, after refraction it is divergent as if from 16·66 cm. If convergent to a point 16·66 cm. behind a $-5$ D lens it is, after refraction, convergent to 100 cm.

**Magnification or Relative Sizes of O and I.**—In Fig. 112, the object $O$ and the image $I$ *subtend equal angles at* $C$, the optical centre of the lens, since both are always contained between the extreme secondary axes $A\ A'$ and $B\ B'$. It is obvious that the triangles $A\ C\ B$ and $A'\ C\ B'$ are similar.

Therefore $$\frac{IC}{OC}=\frac{B'A'}{AB}$$

Thus the *relative sizes of O and I are proportional to their respective distances from the optical centre of the lens, and this holds equally true for virtual images of both Cx. and Cc. lenses.*

The ratio $B'A'/AB$ is the *magnification*, and denotes the linear *increase* or *decrease* in the size of the image with respect to the object. Superficial magnification applies to area, and is the linear magnification squared.

So long as O is beyond 2 F the I must be smaller than O, since it is nearer to the lens. When O is at 2 F the size of I is the same as that of O, because both are at the same distance in what are termed the *symmetrical planes*. When O is within 2 F, I is larger, because it is further from the lens than O.

To calculate the size of I or of O the following formulæ are applicable to all cases, whether the lens be Cx. or Cc. or the image real or virtual.

$$M=\frac{h_2}{h_1}=\frac{f_2}{f_1}, \text{ that is, } h_2=\frac{h_1 f_2}{f_1} \text{ and } h_1=\frac{h_2 f_1}{f_2}$$

where $f_1$ and $f_2$ are the distances of O and I respectively from the lens, $h_1$ is the linear size of O, and $h_2$ that of I. In the first formula $h_1$ and $f_1$ must be in similar terms, but not necessarily that of $f_2$; $h_2$ will then be in the same terms as $f_2$, whether inches, cm., etc. In the second formula $h_2$ and $f_2$ must be in the same terms; and $h_1$ will be in that of $f_1$.

For example let O be at 2 M, and I ·625 cm. long at 25 cm. distance from the lens; then

$$h_1=\cdot625\times2\times100/25=5 \text{ cm.}$$

O is eight times the size of I. If O were at 25 cm. and I at 2 M, then I would be eight times the size of O.

Let O, 4 yards long, be $\frac{1}{4}$ mile distant from a +5 D lens; then the object being at $\infty$, $f_2=20$ cm. and

$$h_2=4\times20/440=\cdot18 \text{ cm.}$$

The answer here is in cm., showing that O and I need not be in the same terms, so long as $h_1$ and $f_1$ are.

When the I formed by a Cx. lens is virtual, it is always larger than O, since it is always more distant from the lens. With a Cc. lens the virtual I formed is always smaller than O, since it is always nearer to the lens.

The relative size of the object to the real and the virtual image formed by a given Cx. lens is the same when O is as far beyond F in the first case as it is within F in the second case. Thus, suppose O situated at 14 in. and at 6 in. respectively in front of a 10 in. Cx. lens, it being in either position 4 in. from F, then the size of the image in each case is $2\frac{1}{2}$ times that of the object.

**Planes of Unit Magnification.**—In order that O and I be equal in size they must be equally distant from the lens, i.e., they must be situated in the planes of unit magnification which, for real images, are the symmetrical planes, which cut the axis at twice the principal focal distance. It can be seen that then $h_1 = h_2$. For a virtual I to be equal in size to O, it must be in contact with the lens. This is true for both Cx. and Cc. lenses, so that the planes of unit magnification for virtual images is zero. It may be remarked that both planes of unit magnification are distant from F a distance equal to F.

**Recapitulation of Conjugate Foci.—Cx. Lens.**

| | |
|---|---|
| When O is at $\infty$ | I is real, inverted, infinitely diminished compared with size of O, and at F. |
| When O is between $\infty$ and 2 F | I s real, inverted, diminished and between F and 2 F. |
| When O is at 2 F | I is real, inverted, equal to O and at 2 F. |
| When O is between 2 F and F | I is real, inverted, enlarged and between 2 F and $\infty$. |
| When O is at F | I is infinitely great and at $\infty$. |
| When O is within F | I is virtual, erect, enlarged and on same side as O. |
| When O is at the lens | I is virtual, erect, equal to O and at the lens. |

**Cc. Lens.**

| | |
|---|---|
| When O is at $\infty$ | I is virtual, erect, infinitely diminished compared with size of O, and at F. |
| When O is within $\infty$ | I is virtual, erect, diminished and within F. |
| When O is at the lens | I is virtual, erect, equal to O and at the lens. |

**Reciprocity of Conjugate Distances from F.**—If the distance of the two conjugates $f_1$ and $f_2$ of a Cx. lens be measured respectively from $F_1$ and $F_2$ they are reciprocals of each other in terms of F. If $f_1$ is at a distance $n$F beyond $F_1$, then $f_2$ is $(1/n)$F or F/$n$ beyond $F_2$. Thus, for instance, if the distance of O to $F_1$ is 2 F, then the distance of I to $F_2$ is F/2.

For M (magnification) = 1, the one conjugate must be at F + F, the other being at F + F also. For M = 2 the one must be at F + 2 F, the other being at F + F/2. For M = 3 the one must be at F + 3 F, the other being at F + F/3, and so on. Then we find the rather curious relationship of the two conjugates, that if the object is distant $n$ F, the image, with a Cx. lens, is distant $n\,F/(n-1)$, and with a Cc. the latter is at $n\,F/(n+1)$. Thus if the distance from a 5″ Cx. lens is 5 × 4 = 20″, the image is at 5 × 4/3 = 6·66″; in the case of a 5″ Cc. if the object is at 5 × 4 = 20″, the image is at 5 × 4/5 = 4″.

Let the distance of O to $F_1$ be called A, and $f_2$ to $F_2$ be called B; then since $n \times 1/n = 1$, it follows that $n\,F \times F/n = F^2$, and $A\,B = F^2$.

**Newton's Formula for Conjugate Foci.**—Let the distances A and B be as defined in the last article. Now the ordinary formula for conjugate foci is

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{F-A}+\frac{1}{F-B}$$

that is
$$AB=F^2$$

This last gives us an alternative formula for calculating conjugate foci.

The ratio between the sizes of image and object $h_2$ and $h_1$ is

$$M=\frac{h_2}{h_1}=\frac{F}{A}=\frac{B}{F}$$

Since, with a given lens, $F^2$ is a constant, the value of A B, the multiple of the distances beyond F of any pair of conjugates, is also a constant.

When employing these formulæ it is essential to remember that positive quantities are measured forwards from $F_1$ and backwards from $F_2$; also that in Cc. lenses $F_1$ is on the remote side of the lens, and $F_2$ on the object side. A is always reckoned from $F_1$ and B from $F_2$. These points make this otherwise valuable formula difficult of application. To obtain $f_1$ or $f_2$ the value of F must be added to A or B respectively.

**Examples.**—Thus, suppose $f_1$ to be 50 cm. in front of a Cx. lens of 10 cm. focus, we get $40\ B=10^2=100$, so that $B=100/40=2{\cdot}5$, and $f_2=2{\cdot}5+10=12{\cdot}5$ cm.

If O is 5 cm. high, we have $h_2/5=10/40$, so that $40\ h_2=50$, or $h_2=1{\cdot}25$ cm.

If an object 5 cm. high be placed 8 cm. in front of a lens of 10 cm. F, then $A=8-10=-2$, and $-2\ B=10^2=100$, so that $B=100/-2=-50$, and $f_2=-50+10=-40$ cm.

$h_2/5=10/2$, so that $2h_2=50$, or $h_2=25$ cm. The image is negative at 40 cm. and is 25 cm. high.

If an object 5 cm. high be placed 50 cm. in front of a Cc. lens, whose $F=10$ cm., then $A=50-(-10)=60$, and $60\ B=10^2=100$. $B=100/60=1{\cdot}66$ and $f_2=1{\cdot}66+(-10)=-8{\cdot}33$ cm.

$h_2/5=10/60$, so that $60h_2=50$, or $h_2={\cdot}833$ cm. The image is negative at 8·33 cm. and is ·833 cm. high.

**Geometrical Proof.**—In Fig. 113, showing the object $A\,B$ and the image $B'A'$, it can be seen that the triangles $A\,B\,O$ and $A'B'O$ are similar. Therefore

$$\frac{O\,B}{O\,B'}=\frac{A\,B}{A'\,B'}=\frac{M\,O}{A'\,B'}=\frac{O\,F_2}{B'\,F_2}$$

the triangles $M\,O\,F_2$ and $A'B'F_2$ being also similar.

But $O\,B=F+A$, $O\,B'=F+B$, $O\,F_2=F$, and $B'F_2=B$. Therefore

$$\frac{F+A}{F+B}=\frac{F}{B} \qquad \text{or} \qquad AB=F^2$$

**Power of a Lens.**—It has been proved that the power of a Cx. or Cc. lens L is equal to the sum of the powers of any pair of its conjugate foci, whether the image be real or virtual; the following examples illustrate this law.

| | | |
|---|---|---|
| If the conjugates are | 20 and 50 cm. | L is $5+2=+7$ D |
| ,, ,, ,, | 20 and $-50$ cm. | ,, $5-2=+3$ D |
| ,, ,, ,, | $-20$ and 50 cm. | ,, $-5+2=-3$ D |
| ,, ,, ,, | 5″ and 10″ | ,, $1/5+1/10=3/10$ |
| ,, ,, ,, | 5″ and $-10$″ | ,, $1/5-1/10=1/10$ |
| ,, ,, ,, | $-5$″ and 10″ | ,, $-1/5+1/10=-1/10$ |

**Light Divergent.**—Whether light actually diverges from some point nearer than ∞, say 50 cm., or whether parallel light is rendered divergent by an added $-2$ D lens, the converging effect of, say, a $+5$ D lens is equally reduced, and in both cases $f_2$ is at $+5-2=3$ D $=33$ cm. behind the lens. If $f_1$ were 14 cm. (7 D) in front of a $+5$ D, or if $-7$ D were added to a $+5$ D, the effect in both cases would be that the light, after refraction, diverged as if proceeding from 50 cm.

Similarly whether light diverges from 50 cm. (2 D) in front of a $-5$ D lens, or whether a $-2$ D be added to the $-5$ D, and the two combined act on

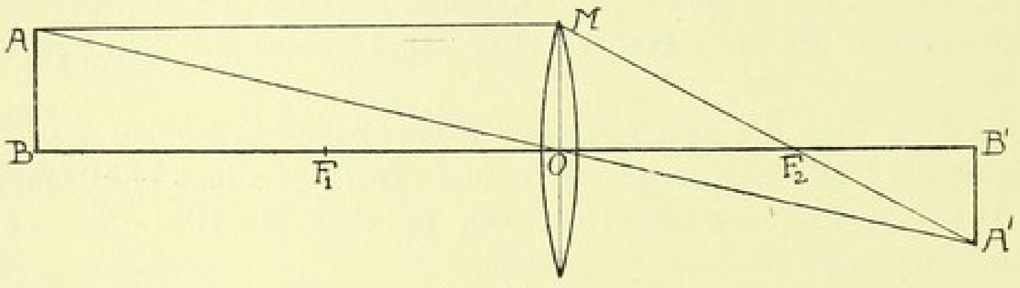



Fig. 113.

parallel light, the focus $f_2$ in either cases is at $-5-2=-7$ D or 14 cm. negative.

**Removal of I.**—To move the image from $f_2$ to some other position $f_2'$ more distant or nearer, there must be added to the lens another Cc. or Cx. respectively whose power is the difference between $1/f_2'$ and $1/f_2$.

Thus, supposing $f_2$ to be at 20 cm. and $f_2'$ to be 25 cm., the required lens is Cc. because $f_2'$ is more distant than $f_2$. The power necessary is $4-5=-1$ D. If it is required to place the image at 16 in. behind the lens instead of at $f_2$, which is 20 in., then the added lens must be positive of $1/16-1/20$ or 80 inches F.

**Position of O for given M.**—Supposing it be required to find where O should be placed from a given lens so that the image be a certain number of times larger or smaller. For example, the lens is a 6 in. Cx., the object 2 inches long, and it is required that the real image should measure 18 inches.

In this case if $x$ is the object conjugate it follows that $18x/2$ or $9x$ must be the image conjugate, so that

$$1/6 = 1/x + 1/9x = 10/9x$$

whence $9x = 60$ in. and $x = 6\frac{2}{3}$ in.

O, therefore, must be placed $6\frac{2}{3}$ in. from the lens, and the image will be at $6\frac{2}{3} \times 9 = 60$ in. from the lens.

If a virtual image is required to measure 18 in., then $1/9x$ is negative, and the calculation becomes $1/6 = 1/x - 1/9x = 8/9x$, whence $x = 5\frac{1}{3}$ in.

When either the object distance $f_1$ or the image distance $f_2$ is not known the magnification M of the image is found respectively from

$$M = \frac{f_2 - F}{F} \qquad \text{and} \qquad M = \frac{F}{f_1 - F}$$

These are deduced from the ordinary conjugate foci expression. M is positive for a real, but negative for a virtual, image, and is expressed as a fraction when there is diminution. By transposing the above and substituting $f_2$ for $Mf_1$ a further variation of the original expression is obtained for finding the position of O and I when the one has to be magnified a certain number of times, such calculations being readily solved by the formulæ

$$f_2 = F(M + 1) \qquad \text{and} \qquad f_1 = f_2/M$$

As before, M is negative when I is virtual, with either a Cx. or a Cc. lens, and it is expressed as a fraction when diminution is required.

**Position of Lens for given Distance between O and I.**—The calculation of the position of a given lens between two given points, so that O be at the one and I at the other, necessitates finding two conjugate distances such that the sum of their reciprocals is equal to the power of the lens. Let $d$ be the distance between object and image, and $x$ represent the one conjugate; then

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{d - x} \qquad \text{or} \qquad x^2 - dx = -dF$$

the solution of which involves a quadratic equation.

With a Cx. lens when $d$ is not less than 4 F, the image is real and may be at either conjugate, and there are two positions for the convex lens, between object and image, which will fulfil the given conditions. In every case $d$ is the sum of the two solutions. When $d$ is less than 4F, the shorter conjugate is positive and is the distance of the object; the greater is negative and is that of the virtual image, $d$ then being a negative quantity.

When the lens is concave, $d$ is positive but F is negative. The greater conjugate is positive and is the distance of the object, while the smaller is negative and is that of the virtual image.

Let F = 7 in. and the distance between O and I be 36 in. Then

$$x^2 - 36x = -252$$

To find $x$ we must add to each side of the equation the square of half the coefficient of $x$, viz. 36, that is $18^2 = 324$. This turns the whole of the left hand side into a perfect square, so that it only remains to extract the square root of each side, and solve the resultant simple equation.

Thus $$x^2 - 36x + 324 = -252 + 324 = 72$$

$$\sqrt{x^2 - 36x + 324} = \sqrt{72}.$$

Therefore $$x - 18 = \pm 8{\cdot}5$$

and $$x = +8{\cdot}5 + 18 = 26{\cdot}5 \quad \text{or} \quad -8{\cdot}5 + 18 = 9{\cdot}5$$

The lens may be either 9·5 in. or 26·5 from O.

Let F = 5 in. and the distance between O and I be 16 in.; then $d$ is negative, so that

$$x^2 + 16x = +80$$

$$x^2 + 16x + 64 = 80 + 64 = 144$$

extracting the square roots, we get $x + 8 = \pm 12$

and $$x = +12 - 8 = +4 \quad \text{or} \quad -12 - 8 = -20$$

The lens is 4 in. from the O and 20 in. from the virtual I.

Let F be 5 in. Cc. and $d$, as before, 16 in.

$$x^2 - 16x = 80$$
$$x^2 - 16x + 64 = 80 + 64 = 144$$
$$x - 8 = \pm 12$$
$$x = +12 + 8 = +20, \text{ or } -12 + 8 = -4.$$

Therefore the lens is 20 in. from O and 4 in. from the virtual I.

If the strength of the lens is expressed in diopters it is better to convert it into focal length for this calculation, but the two distances A and B can also be calculated by the following method, in which two numbers, whose sum and multiple are known, have to be found. Thus

$$A + B = d, \quad \text{and} \quad A\,B = 100d/D$$

## CHAPTER IX

# MAGNIFYING POWER OF LENSES

**Apparent Magnification.**—Hitherto those chapters on lenses have only dealt with the ratio between the actual sizes of object and image (real and virtual), which ratio may vary to an indefinite extent depending upon the position of the object with respect to the lens. In the present article, however, we shall deal with what is known as the *apparent* magnification of the object—or rather, its image—when viewed through a Cx. lens used as a reader or loupe. In contra-distinction to the real magnification mentioned above, the apparent magnification is not subject to such great variations.

Magnification, as before mentioned, is expressed by the linear increase of the image with respect to the object, the superficial magnification being the square of the linear. Thus $\times 3$ implies an increase of three diameters, while $\times 1/3$ expresses a corresponding reduction, the image being one third the size of the original object.

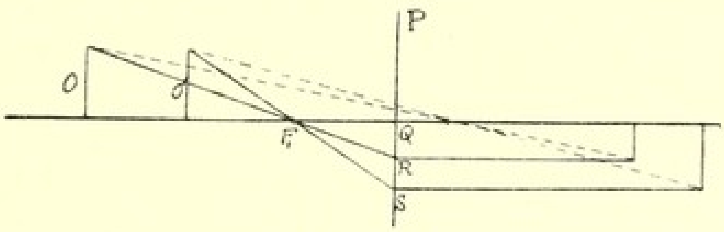



Fig. 114.

**Apparent Size of Object.**—The apparent size of any object depends solely upon the angle it subtends at either the nodal or anterior focal point of the eye. Generally, *the visual angle*, as it is known, is taken at the nodal point, but for our purpose in the present chapter it will be more convenient to work from the angle subtended by the object at the anterior focus.

In Fig. 114 let $P$ be the refracting plane of the eye, $F_1$ the anterior focus, $x$ the corresponding anterior focal length, and $O$ any object distant $d$ from $F_1$. Then any ray drawn from the extremities of $O$, through $F_1$, must, after refraction at $P$, be parallel to the principal axis within the eye. Therefore $Q\ R$ on the refracting plane, will represent the size of the retinal image of $O$, while $Q\ S$ will denote the corresponding image when $O$ is moved to $O'$ at a distance $d'$ from $F_1$. Therefore the size of the retinal image is proportional to the distance of the object from $F_1$, since $O$ and $x$ remain constant, although $O$ changes position. Actually the projection of $QR$ and $QS$ on to the retina would give the images formed were the retina capable of movement to and fro like an ordinary screen to receive the real image as in the camera.

Now, in the ordinary way, when a person views a near object so as to get the best possible general view of it, he unconsciously holds it, not at his near point, but at the most convenient distance called the distance of most distinct vision. This distance of most distinct vision varies, of course, considerably in different individuals, depending upon age, length of eyeball, etc., but is taken, on the average, to be 10″. Further, in order to see the object clearly at this distance the emmetropic person (whom we shall take in illustration) must exert a certain amount of accommodation, namely, 4 D for 10″, and this, by increasing the refraction of the eye and shortening the anterior focus, slightly reduces the size of the retinal image as compared with what he would obtain if no accommodation were used. In the latter case, however, the image would be blurred; but by means of a pinhole in the anterior focus it could be sharpened up, and its size would then be seen to be larger than that obtained in the usual way by accommodation. In other words, if an emmetrope views an object at 10″ through a + 4 D lens, the object is at the principal focus, the light emerges parallel, and all accommodation is suppressed. The result is an increase in the retinal image which is simply the difference between what he obtained *without* the diminishing effect of accommodation, and the reduced image seen *with* the necessary accommodation, the lens being removed. But, provided the lens is so placed that its optical centre—or in the case of a combination of lenses, its second nodal point—coincides with the anterior focus of the eye, there is no magnification due to the lens itself because, as will be seen from Fig. 114, it cannot, in this position, alter the direction of the extreme rays $F_1R$, $F_1S$ governing the size of the images $QR$ and $QS$. Thus if the lens be so placed (and such is generally the case, $F_1$ being some 15 mm. from the cornea) no matter what its strength, *it cannot alter the size of the retinal image*, although, by throwing it out of focus, there may be some appearance of magnification. The only effect a lens, when used as a simple microscope, can have *is to enable the object to be seen under a larger angle* by overcoming the extreme divergence of the rays from a very near distance. Thus if a watchmaker fixes a 2″ lens in front of his eye at the anterior focus, he sees an object 6 times as large as he would without the lens at 10″. If he were able to see distinctly at 2″ with the same accommodation, the object would appear the same size as with the lens. It is necessary to assume that the same degree of accommodation is used in all such cases in order to estimate the true apparent magnification due to the lens. Light from an object at 2″ is, however, so divergent that it cannot be focussed on the retina by the unaided eye. If, however, no accommodation be used at 10″ or 2″ the magnification would be simply the ratio $d_2/d_1$ (Fig. 114) in this case 5. But if accommodation be

that is to say, the quantity $1 + d/F$ is the same as $d/f_1$, where $f_1$ is the conjugate focus of $d$.

This is the usually accepted formula to express the magnifying power when the lens is placed at $F_1$, or near to it. Since $d$ is taken as 10″ we have

$$M = 1 + 10/F$$

Thus with a +2″ lens $M = 1 + 10/2 = 6$.

For lenses expressed in diopters,

$$M = 1 + \frac{10}{F} = 1 + \frac{10}{40/D} = 1 + \frac{10\,D}{40} = 1 + D/4$$

Thus with a +20 D the $M = 1 + 20/4 = 6$.

When the lens is very strong the formula may be simplified to

$$M = 10/F \qquad \text{or} \qquad D/4$$

Thus with a 1/4″ lens $M = 10/\frac{1}{4} = 40$ instead of $1 + 40 = 41$.

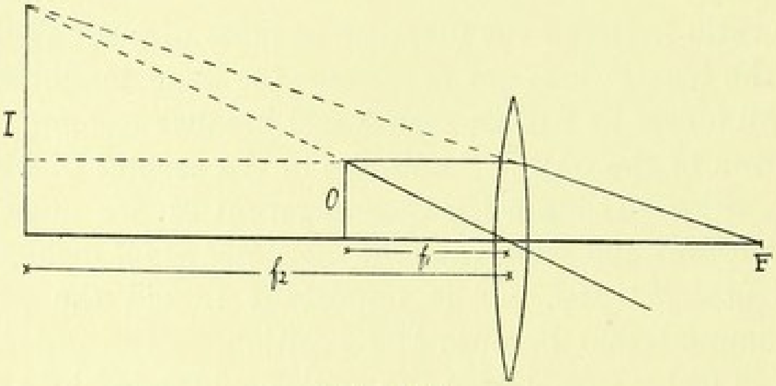



Fig. 115.

Magnification being practically the ratio between the distance of most distinct vision and the focal length of the lens, it follows that the magnifying power of any lens is smaller for a myopic eye whose distance of distinct vision is shorter than 10″. On the other hand, the magnification is greater for the hypermetrope whose position of most acute vision is greater than 10″.

Thus for an emmetrope and a 2″ lens where $d = 10$, $M = 1 + 10/2 = 6$.

For a hypermetrope, where $d = 16$, $M = 1 + 16/2 = 9$.

For a myope, where $d = 6$, $M = 1 + 6/2 = 4$.

It may be noticed in passing that the M is the size of the image formed at $I$ (Fig. 115) compared with the projection of $O$—indicated by the dotted line—on to the plane of $I$ at the distance $d$.

If the object be within the focus of the lens, and the eye be withdrawn from the latter, the retinal image becomes smaller, but when the O is beyond F, i.e., adapted for a hyperope, the retinal image increases in size as the eye is drawn back. Should, however, the object be exactly in the focal plane, the retinal image undergoes no change, since the emergent light is parallel. In all three cases, however, the field of view is reduced, less being seen of the object than when the eye is close to the lens.

For an object to be seen at its best through a loupe or hand glass, it should be placed slightly within the focus. Firstly, because, owing to the curvature of the field, especially in strong lenses, only the central portions are clearly defined, the object having to be moved nearer to bring the peripheral parts into focus, whereas, if the edges are rendered clear by bringing the object within the focus, a slight effort of accommodation will render also the centre sharp. Secondly, it is almost impossible to view a near object without involuntary accommodation, and therefore its exertion to a slight extent renders the observation more comfortable.

Apart from the actual magnification obtained with a lens there are one or two mental conceptions which influence the effect. Thus, since a Cx. lens suppresses accommodation, the object is conceived to be more distant, and, therefore, for a given retinal image, to be larger in size. Also a Cx. lens reduces the divergence of the light, which is referred back by the mind to an image whose distance enhances its apparent size in excess of the actual calculated magnification.

Needless to say the whole of the foregoing remarks and formulæ apply equally to a combination of lenses like that found in an ordinary eyepiece, provided the equivalent focal length and position of the nodal points be known.

In the same way that magnification results from vision of a near object through a Cx. lens, because the angle under which the image is seen is then larger, diminution is obtained with a Cc. lens because the angle under which the image is seen is then correspondingly smaller.

Finally, for any position of the object, and for any Cx. lens, withdrawal of the lens towards the object at first increases the magnification, which reaches a maximum when half way between the anterior focus and the object; M then decreases until, when the lens touches the object, the magnification is the same as what it was when the lens was coincident with the anterior focus, this being zero. This maximum, when a Cx. lens is about mid-way between eye and object, holds good in *all* cases, but is quite independent of the *clearness* of the image, which may either be blurred or sharp, depending upon the strength of the lens. Similarly the greatest diminution occurs when any concave lens is mid-way between eye and object, but in this case, provided there is sufficient accommodative power, the image is clear. These facts explain some of the phenomena in connection with spectacle lenses.

## CHAPTER X

### CYLINDRICALS

**The Cylinder.**—A cylinder is a body (Fig. 116) generated by the revolution of a rectangle about one of its sides as an axis. Such a body consists of two flat circular ends and an intermediate convex surface.

The cylinder possesses no curvature in any line parallel to the axis *A B*. At right angles to the axis, in any plane parallel to the direction *C D*, the curvature is spherical and has its maximum value. In any other direction, as *E F*, the curvature is that of an ellipse of which *E′F′* is an example. The curvature is always less than that of the circle *C D*, diminishing as the direction departs from *C D* and approaches that of *A B*.

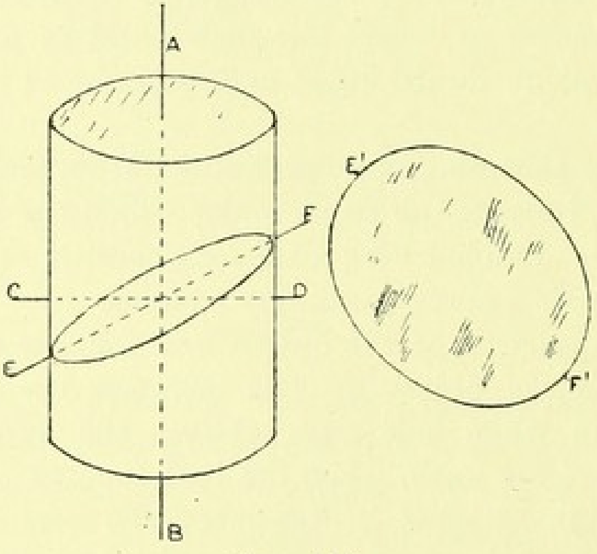



FIG. 116.

Therefore any section of the cylinder taken at right angles to its axis is a circle whose centre lies on the axis of the cylinder; a section in the plane of the axis is a parallelogram, and one anywhere between these two is an ellipse.

Fig. 117 represents a Cx. cylindrical lens. It is a segment of a cylinder on the one side and has a plane surface on the other; it is formed by a cylinder and a plane which intersect each other. The Cc. cylindrical lens (Fig 118) has a hollowed surface on one side; it is formed by a cylinder and a plane which do not intersect each other.

A Cx. cyl. lens may be conceived as formed of a series of prisms whose bases are directed towards a central line and whose apices are outwards. In the same way a Cc. cyl. may be considered to be formed of prisms whose apices are directed towards a central line and whose bases are outwards.

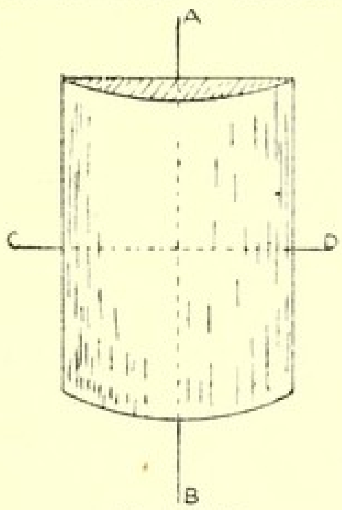



FIG. 117.

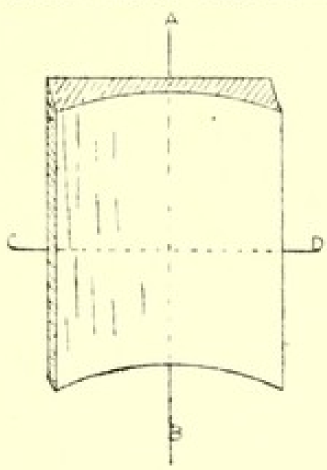



FIG. 118.

**Meridian.**—The term meridian in connection with lenses signifies a plane passing through the geometrical centre of a lens, as shown in Fig. 119.

**The Principal Meridians.**—Since in the direction of its axis (Fig. 120) a cyl. lens has no curvature, it has in that direction no refractive power; the

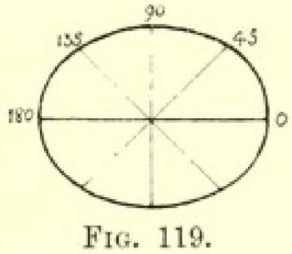



FIG. 119.

directions of maximum curvature *A B C*, *D E F*, *G H K*, are at right angles to the axis. The meridian of no refraction—i.e. the axis—and the meridian of greatest refraction, at right angles to the axis, are termed the *two principal*

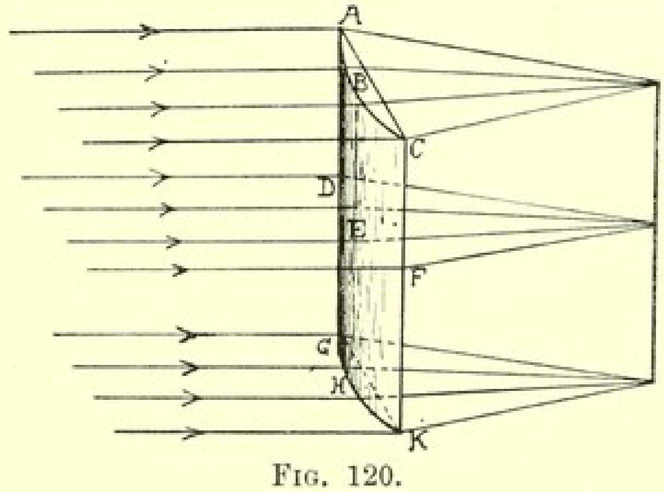



FIG. 120.

*meridians*, and these alone need be considered when treating of cyl. lenses. The position of a cylindrical is indicated by the direction in which its axis is placed, while its power is expressed by the maximum refractivity. By means of these two meridians the path of all rays passing through the lens may be traced.

**The Refraction of a Cylindrical.**—A sph. lens has equal curvature and therefore similar refractivity in every meridian, so that a point image of a point object is obtained. In a cyl. it is only the meridian at right angles to the axis that can form a focus, because a ray proceeding from a point and meeting the surface in an intermediate meridian cannot meet the other rays which pass through the same meridian. All the light from an object point refracted by a cyl. passes through two focal lines, one of which is at the focal distance of the meridian of greatest refraction, and the other at $\infty$. The cylindrical lens has therefore two focal distances, and the image of a point is not a point, as with the spherical, but two lines. But since the one focus is at $\infty$, it need not be considered, so that we can say that the image of a point formed by a cyl. is a line, as shown in Fig. 120.

Using for illustration a + 5 D cyl. axis vertical, if the object be a point of light the image will be a row of focal points along a Ver. line, which fuse into a thin streak of light at 20 cm. *parallel to the axis of the cyl.*, and this is called the *focal line*. If the cyl. be rotated around its centre *E* the streak also will be rotated with it.

In any intermediate mer. the refraction is such that a ray from a distant point is deviated so as to meet all the other rays in the focal line, and the deviation is less than in the maximum, and more than in the axial, meridian. Thus although intermediate meridians of a cyl. are elliptical in curvature and have no true foci, we can say that the power of a cyl. varies from 0 at the axis to its maximum power D in the meridian at right angles thereto. These intermediate powers can be expressed by a formula, as will be shown later.

The focal line is situated at the focal distance of the meridian of greatest power, which, in this case, is the horizontal meridian, and the lens being + 5 D cyl., it is at 20 cm. At any other distance the streak broadens out into a band of light, and a section of the emergent light, at any distance from the lens, is rectangular in outline.

Since the image of an object consists of the images of its various points and each object point has its own line image, the complete image consists of an infinite number of streaks, parallel to the axis, and narrow as the focal length is short and vice-versa. The length of a streak is that of the aperture of the lens in the meridian of its axis. We commonly refer to this streak image as the focal line.

The refraction of a Cc. cylindrical is similar to that of a Cx., but, of course, the focus and the focal line are virtual, and are formed in front of the lens. The shape of the refracted pencil, however, is not so apparent, because the pupil of the eye acts as a very small stop, so that the focal line, on looking through the lens, is so short as to appear little different from the point focus of a concave spherical, unless the cyl. be very strong. A single cylindrical power is called a plano- or simple cyl.

A square seen through a cyl. lens appears to be a rectangle of natural size in the meridian of the axis, but magnified by a Cx., and diminished by a Cc., across the axis.

If the cyl. is oblique the square takes the form of a parallelogram, the obliquity of the sides being due to the fact that the light from each point diverges from, or tends towards, a line parallel to the axis. The skew thus given causes the light from each point on the object to appear to come from points in space other than what they actually do. The series of oblique parallel lines (or ellipses) which constitute the virtual object of which the retinal image is formed results in vertical and horizontal lines appearing oblique. This explains also the *dipping* of cross lines as a cyl. is rotated (vide neutralisation). The apparent obliquity is lessened if the lens is near to the object or very near to the eye.

Viewing a circular object, say a shilling, through a Cx. cyl. axis Ver., the image is an oblate ellipse in form, having its minor axis vertical and equal to the diameter of the shilling. With a similar Cc. cyl. the image is a prolate ellipse.

**Combined Cylindricals.**—If two Cx. cyls. of similar power be placed in contact with their axes corresponding in, say, the vertical meridian, the cyl. power is doubled. If the second cyl. be at right angles to the first, the result will be *equivalent to a sph. lens* of the same power as the single cyl. In this case the greatest power of the one corresponds with the axis of the other, and in all intermediate meridians any deficiency of power in the one is supplied by the other, so that if the second cyl. be rotated from axis vertical to axis horizontal the vertical image streak will be seen to shrink until, when the two axes are at right angles, it will have become a point of light, or a complete image as the case may be. When the axes are oblique to one another, the effect is that of some sph.-cyl., whose two principal powers vary with the angle between the axes. Two unlike cyls. are always equivalent to some sph.-cyl. combination no matter what may be the inclination of their axes, except in the case of the axes being parallel, when they constitute a plano-cyl. The effect is the same whether the two cyl. powers be ground on opposite sides of a piece of glass, or whether two plano-cyls. be placed in contact. A lens having such powers is termed a cross-cyl.

**The Refraction of a Sphero-Cylindrical.**—When a sph. is combined with a cyl., the curvature of the former is ground on the one side of the lens, and that of the latter on the other. Such a combination is called a sphero-cylindrical or compound cylindrical, in contradistinction to a plano or simple cyl. Since there is no curvature and consequently no refractive power along the axis of the cyl., only the power of the sph. exists in that Mer., whereas at right angles to the axis there is the united power of the sph. and the cyl. As with the plano-cyl., these are the two principal Mers. of the combination, which alone need be considered in practice.

Let us consider a + 4 D Sph. + 4 D Cyl. Axis Ver. and let the object be a point at ∞. All the light incident on the lens is so refracted as to pass through a vertical line at 12·5 cm. to which it is converged; thence, expanding horizontally and converging vertically, it again meets in a horizontal line at 25 cm. The vertical meridian acts as a plano- Cx. lens, the horizontal as a double Cx. lens, and these principal meridians have true foci. A ray incident in any intermediate meridian is converged to a certain extent in the vertical plane and to a still greater extent in the horizontal, the resultant deviation being intermediate as to direction and extent. Hence, although an intermediate meridian of a sphero-cyl. has no true focus, since a ray, passing through it, does not meet the other rays passing through it, we can assign to it a definite dioptral power as in the case of a plano-cyl. Such power depends on the angular distance of the meridian in question from the axis, and it is, in all cases, somewhere between the highest and lowest powers of the combination.

The action of a concave sph.-cyl. is similar to that of the corresponding convex sph.-cyl., the only difference being that the foci and images are virtual.

As with the simple cyl., the images at the two focal planes of an object of definite size, consist of bands of light, whose width and length depend on the powers of the lens and its aperture or diameter.

A section of the cone of light emergent from the lens is elliptical except where the Hor. and Ver. lines are formed; also at some position between the lines where the cone of light has equal diameter in both Mers. producing what is termed *the circle of least confusion*.

**The Principal Meridians and Powers.**—The power of the sph. alone is that of the Mer. in which the axis lies. The power of the sph. plus that of the cyl. constitutes that of the Mer. at right angles to the axis. For examples see *Transposing*. The positions of the principal Mers. are easily recognised when Ver. and Hor.; when they are oblique their angular positions can be estimated or determined (vide *Neutralisation*).

**The Interval of Sturm.**—Let a screen be held close behind a Cx. sph.-cyl., say + 4 Sph. ⊂⊃ + 4 Cyl. Axis Ver.; then the light from a small bright source, some distance in front of the lens, is cast as a light patch on the screen. If now the latter be gradually drawn away from the lens (Fig. 121) at the distance 12·5 cm., which is equal to F of the combined sph. and cyl. powers, a Ver. line is formed at $F_1$; as the screen is still slowly receded the line developes gradually into a Ver. (prolate) oval at $C$, an almost perfect circle at $B$, a Hor. (oblate) oval at $A$, and finally into a Hor. line at $F_2$. The screen is then at 25 cm., which is F of the sph. The space between the two principal focal distances $F_1$ and $F_2$, represented by the two sharp lines, is termed the *interval of Sturm*. As the screen is still further removed from the lens, the patch of light takes the form of an ever enlarging Hor. ellipse.

**Calculations of the Interval of Sturm.**—The two focal lines are at the focal distances of the two principal meridians, and their lengths are proportional to the diameter of aperture of the lens and to their distances from the latter.

Let the lengths be $L_1$ and $L_2$, the focal distances $F_1$ and $F_2$, and the dioptric powers $D_1$ and $D_2$. The formulæ are given in terms of F and D.

$$L_1F_2 = L_2F_1 \qquad L_1D_1 = L_2D_2$$

Let $d$ be the effective aperture of the lens, and $S$ the length of the interval of Sturm, i.e. the distance between $F_1$ and $F_2$ (Fig. 121), or the dioptric difference $D_1 - D_2$.

$$L_1 = d\,S/F_2 = d\,S/D_1 \qquad L_2 = d\,S/F_1 = d\,S/D_2$$

The circular disc of confusion $B$ divides $S$ into two parts $b$ and $a$ distant, respectively, from $L_1$ and $L_2$ proportional to their distances from the lens.

$$S = a + b \quad \text{and} \quad b/a = L_1/L_2 = F_1/F_2 = D_2/D_1$$

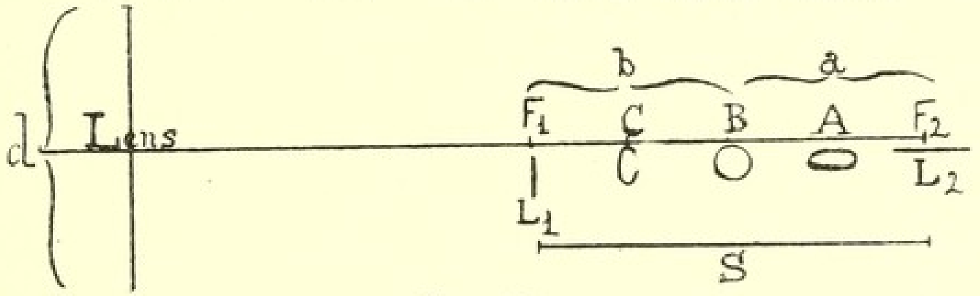



Fig. 121.

Then $B$ is distant from $L_1$ and $L_2$ respectively

$$b = \frac{S\,F_1}{F_1 + F_2} = \frac{S\,D_2}{D_1 + D_2} \quad \text{and} \quad a = \frac{S\,F_2}{F_1 + F_2} = \frac{S\,D_1}{D_1 + D_2}$$

$B$ cannot be midway between $L_1$ and $L_2$, but is always nearer to $L_1$. Its size is

$$B = bL_2/S = aL_1/S$$

Its distance from the lens is

$$C = \frac{2F_1F_2}{F_1 + F_2} = \frac{200}{D_1 + D_2}$$

Example with + 4 D Sph. ⊂ + 2 D Cyl. having d = 5 cm.

$$F_1 = 16{\cdot}66 \text{ cm.} \qquad F_2 = 25 \text{ cm.} \qquad S = 25 - 16{\cdot}66 = 8{\cdot}33 \text{ cm.}$$

$$L_1 = \frac{5 \times 8{\cdot}33}{25} = 1{\cdot}66 \text{ cm.} \qquad L_2 = \frac{5 \times 8{\cdot}33}{16{\cdot}66} = 2{\cdot}5 \text{ cm.}$$

$$b = \frac{8{\cdot}33 \times 16{\cdot}66}{25 + 16{\cdot}66} = 3{\cdot}33 \text{ cm.} \qquad a = \frac{8{\cdot}33 \times 25}{25 + 16{\cdot}66} = 5 \text{ cm.}$$

$$B = \frac{3{\cdot}33 \times 2{\cdot}5}{8{\cdot}33} = \frac{5 \times 1{\cdot}66}{8{\cdot}33} = 1 \text{ cm.} \qquad C = \frac{2 \times 16{\cdot}66 \times 25}{16{\cdot}66 + 25} = 20 \text{ cm.}$$

When the combination is negative the interval of Sturm is also negative; when the combination is *mixed*, it is peculiar; thus when $F_1 = 10''$ and $F_2 = -20''$, and $d = 1''$ we find $B$ behind $L_1$ and negative. $S = -30$, $L_1 = +1{\cdot}5$, $L_2 = -3$, $b = +30$, $a = -60$, $B = -3$, $C = +40$.

If the power of shorter F is concave, $B$ lies in front of $L_1$ and is again negative. If the + and − powers are numerically equal, $B$ is at $\infty$.

## CHAPTER XI

### TRANSPOSING AND TORICS

#### Angle Notation.

**Standard Notation.**—The Standard angle notation for the location of the various meridians of a lens (Fig. 119), refers to both the right and left eyes. The numeration commences on the right-hand of the imaginary horizontal line drawn through the lens when looked at from the front, the front of the lens being that face of it remote from the eye of the wearer.

This notation corresponds with the trigonometrical division of the circle into 360 degrees. The upper right quadrant contains the angles between 0° and 90°, and the upper left those between 90° and 180°. The notation need not be carried beyond 180° (the half-circle), since a meridian corresponds to a diameter, i.e., to two continuous radii—for instance, 45° is the same meridian as 225°; 10° the same as 190°, etc. The vertical meridian is 90°, and the horizontal is 0° or 180°, but is preferably indicated as 180°.

**Other Notations.**—Some trial frames and prescription forms are marked differently from that shown in Fig. 119, and it frequently occurs that the optician has to transfer from one notation to another. The most commonly met with are the bi-nasal and the bi-temporal methods, in which the zero is placed at, respectively, the two nasal and the two temporal extremities of the horizontal line of the eye, the numeration running upwards or conversely running downwards. Sometimes the zero is placed in the vertical meridian, the numeration proceeding to the right and left. Indeed, there are many different methods of *notating* the *two eyes*, but it is hardly necessary to attempt to detail them all here. Fig. 122 shows a notation reverse to the *standard*.

Suppose a prescription be written with the indicated cylindrical axis at 125° according to the notation of Fig. 122. To translate this to standard notation, it must be considered how many degrees the required position is from the horizontal or the vertical. In this case 125°, in Fig. 122, is 35° from the vertical on the right and, therefore, corresponds to 55° of Fig. 119. If the location of the axis is 40° from the horizontal on the right, it would be 40° in Fig. 119 and 140° in Fig. 122. The same mode of calculating applies if the cylindrical axis is indicated as so many degrees with a stroke to show the direction of inclination. This last-mentioned method of axis indication is unfortunately used by many medical men, thus making the reading of their prescriptions difficult to the optician. Many oculists also do not use the ⊂ sign, but write the combination with a dividing line, thus :—

$$\frac{+4\ \text{S.}}{+2{\cdot}50\ \text{C. Axis } 70}$$

In all these methods it is, however, understood that the direction indicated refers to the front of the lens, or the surface away from the wearer's eye.

### Transposing: The Powers of Cylindrical and Toric Lenses.

**Transposition of Sph. Lenses.**—A Cx. sph., say, + 6 D, can be made in the form of a plano-Cx., in which all the power is on the one side ; as an equi-Cx., in which the power is equally divided between the two surfaces ; as a bi-Cx., in which the powers are unequally divided between the two surfaces ; or as a periscopic-Cx., in which the Cx. power on the one side is more than 6 D, but the total is reduced to + 6 D by the necessary Cc. curvature of the other surface. Similarly, a concave spherical can be made in the various

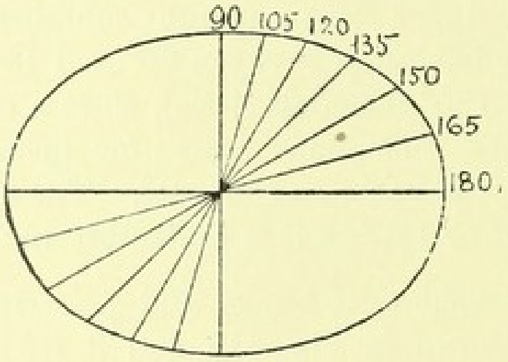



Fig. 122.

forms as indicated above. The change from one form to another, without altering the refractive power of a lens, is called a *transposition*. The power of the one surface increases proportionately as that of the other decreases, so that the number of possible forms for a given sph. power is endless ; but the position of the optical centre varies with the different forms of a lens.

**Transposition of Cyl. Lenses.**—Lenses which contain a cyl. element are susceptible of only two or three changes of form, and it is to such a change, which does not alter the refractive powers of the two principal meridians, that the term "transposition" is generally applied.

When the two powers have the same sign, they are said to be of *like* nature, or congeneric ; when they are of opposite signs (the one + and the other − ) they are of *unlike* nature, or contrageneric. A *plano* (or simple) cyl. possesses no sph. element, but may be regarded as a sph.-cyl., whose sph. element is of infinitely great radius, and it will be so treated in this article.

A *sph.-cyl.* has sph. and cyl. elements, and may be a *compound* cyl., having + or − powers in both principal meridians, or a *mixed* cyl., having a + power in the one and − power in the other.

A *cross* cyl. is one formed of two similar or two dissimilar cyls. crossed at right angles.

**Powers and Principal Meridians.**—The one principal Mer. of a sph.-cyl. corresponds to the axis of the cyl., and its power is that of the sph. alone; the other is at right angles to the axis of the cyl., and its power is the algebraical sum of the sph. and cyl. Thus

The powers of + 3 S. ⊂⊃ + 2 C. Ax. 70° are + 3 at 70° and + 5 at 160°.

Those of + 3 S. ⊂⊃ − 1 C. Ax. 110° are + 3 at 110° and + 2 at 20°.

Those of + 3 S. ⊂⊃ − 3 C. Ax. 5° are + 3 at 5° and 0 at 95°.

Those of + 3 S. ⊂⊃ − 5 C. Ax. 120° are + 3 at 120° and − 2 at 30°.

In the cross-cyl. the two principal powers are those of the cyls. themselves, each being in the Mer. corresponding to the axis of the other. Thus the powers of + 2 C. Ax. 40° ⊂⊃ + 5 C. Ax. 130° are + 2 at 130° and + 5 at 40°. Those of + 2 C. Ax. 70° ⊂⊃ − 4 C. Ax. 160° are + 2 at 160° and − 4 at 70°.

**Possible Combinations.**—A cyl. combination may consist of two different powers of similar nature, as + 2 and + 5, or − 3 and − 7, or of two powers

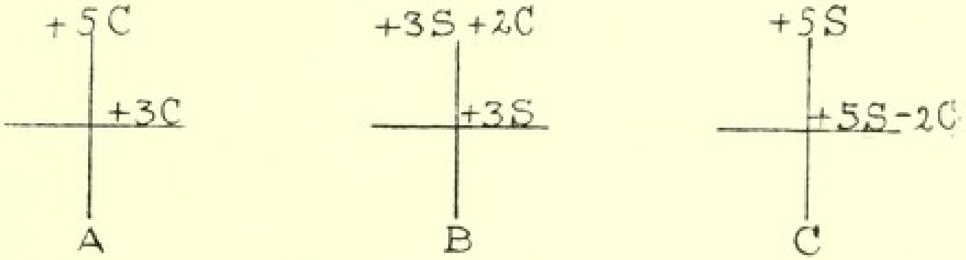



FIG. 123.

of dissimilar nature as + 2 and − 2, or + 3 and − 4. They can be made in three forms, viz., a cross-cyl. and two forms of sph.-cyl., but if the one power is 0 it can be made only as a plano-cyl., and in one form of sph.-cyl. If there are two similar equal powers the possible forms are only those of a cross-cyl. and of a sph.

**The Various Forms of a Lens with Cyl. Element.**—Where two unequal powers in the two principal meridians are required, as + 3 at 180° and + 5 at 90°.

(*a*) The + 3 needed at 180° (Fig. 123) can be obtained from + 3 C. Ax. 90°, and the + 5 at 90° from + 5 C. Ax. 180°, the axis of each cyl. being at right angles to the direction in which the power is required, as in *A*.

(*b*) The 3 needed at 180° can be obtained from + 3 sph., which also supplies 3 of the + 5 D needed at 90°, the balance of the latter being obtained from + 2 C. Ax. 180°, which gives + 2 at 90° and 0 at 180°.

(*c*) The + 5 needed at 90° can be obtained from + 5 sph., but this not only supplies the + 3 needed for 180°, but is 2 D too strong. To reduce the latter to + 3 D a − 2 C. Ax. 90° is required, this giving − 2 at 180° and 0 at 90°, as in *C*.

If the two parts of any of the forms (*a*) (*b*) (*c*) be placed over one another, the total combination is, in each case, $+5$ at 90° and $+3$ at 180°. The three forms are thus made up by—

(*a*) A cyl. of each of the two powers, the axis of each being at right angles to the meridian where the power is needed.

(*b*) A sph. of the lower power and a cyl. of the difference between the two powers, the axis corresponding to the meridian of least power. If the lower power is 0, the sph. is also 0.

(*c*) A sph. of the higher power and a cyl. of the opposite sign and of the difference between the two, the axis being in the meridian of greater power.

Whether the two powers are of like or unlike nature, the number of the cyl. is obtained by the *algebraical subtraction* of the power taken as the sph. from that of the other principal power. Thus in the example the powers are $+3$ and $+5$, so that if the sph. is $+3$, the cyl. is $+2$; if the sph. is $+5$ the cyl. is $-2$. If the two powers are $+2$ and $-3$, then, if the sph. is $+2$, the cyl. is $-5$; if the sph. is $-3$, the cyl. is $+5$.

For $-4$ at 60° and $-7$ at 150°, the three forms are:

(*a*) $-4$ C. Ax. 150° ⊂ $-7$ C. Ax. 60°.
(*b*) $-4$ S. ⊂ $-3$ C. Ax. 60°.
(*c*) $-7$ S. ⊂ $+3$ C. Ax. 150°.

For $-1$ at 45° and $+5$ at 135° they are:

(*a*) $-1$ C. Ax. 135° ⊂ $+5$ C. Ax. 45°.
(*b*) $-1$ S. ⊂ $+6$ C. Ax. 45°.
(*c*) $+5$ S. ⊂ $-6$ C. Ax. 135°.

For $+3$ at 120° and 0 at 30° they are:

(*a*) 0 S. ⊂ $+3$ C. Ax. 30°.
(*b*) $+3$ S. ⊂ $-3$ C. Ax. 120°.

**Rules.**—(1) *To transpose a sph.-cyl. or plano-cyl. into another form of sph.-cyl. or plano-cyl.*

The following apply to all cases, but when the original or the transposed form is a plano-cyl., the one power being 0, the sph. may also be 0.

(*a*) The new sph. is found by adding *algebraically* the power of the sph. to that of the cyl.

(*b*) The new cyl. has the same power as the original cyl., but its sign is changed and its axis is at right angles.

(2) *To transpose a sph.-cyl. into a cross-cyl.*

(*a*) The one cyl. of the new form has the same number and sign as that of the original sph. and its axis is at right angles to that of the original cyl.

(*b*) The other cyl. has its axis in the same meridian as that of the original cyl., and a sign and number which results from the *algebraical addition* of the powers of the original sph. and the original cyl.

(3) *To transpose a cross-cyl. into a sph.-cyl.*

(*a*) The sph. of the new form has the number and sign of the first original cyl.

(*b*) The new cyl. has its axis corresponding to that of the second original cyl. and a sign and number which results from the *algebraical subtraction* of the first from that of the second original cyl.

Since either original cyl. may be taken as the first, there are two forms of sph.-cyls. into which a cross-cyl. can be transposed.

**Examples.**—The above rules can be better appreciated by studying the example at the same time. In the following examples, which illustrate all possible combinations, the first is the original, and those following are the forms into which it can be transposed.

(1) +4 S. ⊂⊃ +2 C. Ax. 20° =
+6 S. ⊂⊃ − 2 C. Ax. 110°
+4 C. Ax. 110° ⊂⊃ +6 C. Ax. 20

(2) − 2·50 S. ⊂⊃ − 1·50 C. Ax. 175° =
− 4·00 S. ⊂⊃ + 1·50 C. Ax. 85°
− 2·50 C. Ax. 85° ⊂⊃ − 4·00 C. Ax. 175°

(3) +3·50 S. ⊂⊃ − 2·50 C. Ax. 45° =
+1·00 S. ⊂⊃ + 2·50 C. Ax. 135°
+1·00 C. Ax. 45° ⊂⊃ + 3·50 C. Ax. 135°

(4) +3 S. ⊂⊃ − 3 C. Ax. 105° =
+3 C. Ax. 15°

(5) +2·50 S. ⊂⊃ − 4·50 C. Ax. 115° =
− 2·00 S. ⊂⊃ + 4·50 C. Ax. 25°
+2·50 C. Ax. 25° ⊂⊃ − 2·00 C. Ax. 115°

(6) − 1·25 S. ⊂⊃ + 1·75 C. Ax. 160° =
+0·50 S. ⊂⊃ − 1·75 C. Ax. 70°
− 1·25 C. Ax. 70° ⊂⊃ + 0·50 C. Ax. 160°

(7) +2·75 C. Ax. 95° =
+2·75 S. ⊂⊃ − 2·75 C. Ax. 5°

(8) +2 C. Ax. 80° ⊂⊃ + 3 C. Ax. 170° =
+2 S. ⊂⊃ + 1 C. Ax. 170°
+3 S. ⊂⊃ − 1 C. Ax. 80°

(9) − 5·50 C. Ax. 155° ⊂⊃ − 2·50 C. Ax. 65° =
− 2·50 S. ⊂⊃ − 3 C. Ax. 155°
− 5·50 S. ⊂⊃ + 3 C. Ax. 65°

(10) +2·25 C. Ax. 75° ⊂⊃ − 2·25 C. Ax. 165° =
+2·25 S. ⊂⊃ − 4·50 C. Ax. 165°
− 2·25 S. ⊂⊃ + 4·50 C. Ax. 75°

(11) + 3·50 C. Ax. 120° ⊂ – 0·75 C. Ax. 30° =
+ 3·50 S. ⊂ – 4·25 C. Ax. 30°
– 0·75 S. ⊂ + 4·25 C. Ax. 120°

(12) – 10·00 C. Ax. 180° ⊂ + 2 C. Ax. 90° =
+ 2 S. ⊂ – 12 C. Ax. 180°
– 10 S. ⊂ + 12 C. Ax. 90

(13) + 3·50 C. Ax. 90 ⊂ + 3·50 C. Ax. 180°
+ 3·50 S.

(14) – 4 S. =
– 4 C. ⊂ – 4 C. with axes at right angles.

**Comparison of Original and Transposed Forms.**—The two principal powers and Mers. of the original form of a combination can be extracted and compared with those of the transposed form, and they must be alike if the transposition is correct. Thus, suppose – 3 S. ⊂ + 4 C. Ax. 90°. The two principal powers are – 3 at 90° and + 1 at 180°. The power of the – 3 Sph. is in both principal meridians, while that of the + 4 C. Ax. 90° is only at 180°; its axis, being at 90°, contributes no refractive power to that meridian.

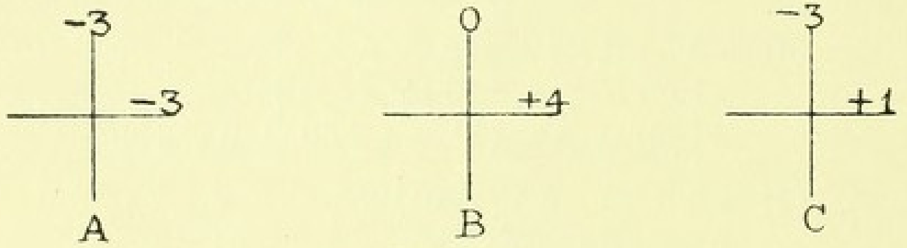



Fig. 124.

The two components separated are represented by *A* and *B* of Fig. 124. When combined they are represented by *C*. The two forms into which they can be transposed are

(*a*) + 1 S. ⊂ – 4 C. Ax. 180°
(*b*) + 1 C. Ax. 90° ⊂ – 3 C. Ax 180°

**Proof by Neutralisation.**—Since a transposition simply assigns the needed powers in a different way, as regards the two surfaces of a lens, and does not change the refractive power of the combination, that combination which will neutralise the original form will also neutralise the transposed forms. Thus—

(*a*) + 1 S. ⊂ – 4 C. Ax. 180° transposes into
(*b*) – 3 S. ⊂ + 4 C. Ax. 90°

(*a*) is neutralised by – 1 S. ⊂ + 4 C. Ax. 180°, and these also neutralise (*b*) as can be seen by adding them together thus—

( – 3 S. ⊂ + 4 C. Ax. 90°) + ( – 1 S. ⊂ + 4 C. Ax. 180°)

The 2 sphs. = – 4 S., the 2 cyls. = + 4 S.; – 4 S. + 4 S. = 0.

It is never required in practice to give crossed cyls. for any combination, since the effect can be equally well obtained from a sph.-cyl., and at much less cost. The best form to employ is usually a + Sph. ⊂ − Cyl., or a − Sph. ⊂ + Cyl., since from these we obtain a certain periscopic effect without additional expense.

### Toric or Toroidal Lenses.

A toric lens is one having two principal powers worked on the same surface with their axes at right angles to each other, as shown in Fig. 125. The curvature of the lens along *A B* is, say, + 3 D, while along *C D* it is, say, + 5 D. It is, therefore, equal to + 3 S. ⊂ + 2 C. and has the same optical effects. The name is derived from the tore or arched moulding used at the base of pillars. It can be illustrated by a bent tube or rod ; the side of an egg or the bowl of a spoon resembles a toric surface.

The curvature of a toroidal surface is spherical in the two principal meridians, and elliptical in the intermediate ones. It can be either convex or

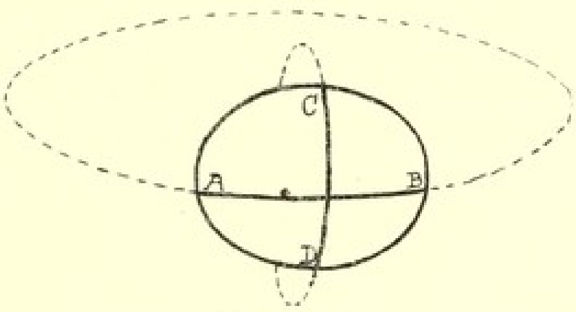



Fig. 125.

concave. Astigmatism of the cornea is due to the toroidal shape of the latter.

Since the possible toric forms of a combination is exceedingly great, it is usual to employ tools of a given base curve. Often an assortment is kept of toric lenses having the one surface unworked, and on which any spherical curve can be ground. The *base curve* indicates the standard or *fixed* power of the toric surface. It is usually the lower of the two powers, but may be, and occasionally is, the higher. In the following it will be taken as if it were always the lower toric power.

**Advantages of the Toric Lens.**—The utility of the toric form is that by its means the refracting power of a lens can more nearly be divided between the two surfaces. Thus if + 10 S. ⊂ + 1 C. be required, instead of there being + 10 S. on the one surface and + 1 C. on the other, it can be made with + 4 S. on the one surface and + 6 C. ⊂ + 7 C. on the other. Or it can be made with any other convex spherical power, the virtual cylindricals of the toric surface being accordingly stronger or weaker, but always having 1 D difference between them. Thus, a strong lens as needed in aphakia or high myopia can be made less thick and unsightly, and more nearly resembling a Dcx. or Dcc. Another, and perhaps greater, advantage of the toric surface is that, with it, a sph.-cyl. can be made periscopic, if so needed, to any extent, as shown in the following example. The advantages of highly periscopic lenses are mentioned under *Menisci.*

Required + ·50 S. ⊃ + ·25 C. Ax. 90°. As a toric periscopic lens on a Cx. base of 6 D, the combination would be

$$\frac{-5{\cdot}50\ \text{S.}}{+6{\cdot}0\ \text{C. Ax. } 180^\circ \supset +6{\cdot}25\ \text{C. Ax. } 90^\circ}$$

**Conversion to Toric Form.**—Toric tools or plano-toric lenses are usually made on a base curve of either 3 D, or 6 D, or sometimes 9 D, the other curve always being the *stronger* by an amount *equal to the cylindrical effect* required in the finished lens. The number of the toric tool or blank is therefore simply the difference in the principal powers, since we always know the weaker power to be either 3, 6, or 9 as the case may be ; thus a 2 D tool is one having a 6 D curve in one direction and 8 D at right angles ; or if the series is on a 3 D base, the curvatures would be 3 and 5.

Therefore if a + 1 S. ⊃ + 1·5 C. were required in toric form with + 6 base curve, a 1·5 tool would be used, giving on the Cx. surface powers of + 6 and + 7·5. On the other surface − 5 D sph. would be necessary in order to reduce the principal powers to + 1 D and + 2·5 D, as required in the original lens. The result, therefore, is the same as that of an ordinary sph.-cyl., but has the advantage of the periscopic form.

The series of tools being in pairs, the toric surface can be made concave if the powers of the original lens are not suitable for a convex toric. For example, the above sph.-cyl. could be made with − 6 D and − 7·5 D powers on the one side, the adjusting spherical on the other being + 8·5 D. Here a greater periscopic effect is secured with the same toric powers than when they were convex.

To convert a sph.-cyl. combination into a toric form, it is merely necessary to select the base curve and required tool, and then add to the other surfaee that Cx. or Cc. sph. necessary to bring the powers to the requisite strength. Due regard must, of course, be paid to the powers of the original lens when selecting the base curve in order to get the best result. If this is not done the result may be a periscopic little greater than what could be obtained from the ordinary sph.-cyl., and on the other hand it may be so deep as to render the lens clumsy or unsuitable for mounting in a frame. Thus if − 5 S. ⊃ − 1 C. Ax. 45° be required in toric form, on a − 6 base,

$$\frac{+1\ \text{S.}}{-6\ \text{C. Ax. } 135^\circ \supset -7\ \text{C. Ax. } 45^\circ}$$

would differ but little from the sph.-cyl. form of − 6 S. ⊃ + 1 C. Ax. 135° ; and on a + 6 base, the sph. being − 12 D, the lens would be thicker and heavier with no great increase in advantage over the ordinary sph.-cyl. It is in Cx. sph.-cyls. where the toric is most useful, because generally only a very small periscopic effect can be obtained by ordinary transposition. A toric surface may be expressed as *two powers* in certain meridians, or as the *two virtual cylindricals*, which are contained therein, with their axes at right angles and in certain meridians. The latter method is followed in this article. It may be added that the usual base power is 6 D.

## Rules for Conversion to Toric Form.

(1) Convert the combination into cross-cyls. with their axes.

(2) Find the difference between the two powers.

(3) The lower power of the toric surface is the base curve.

(4) The higher power is the base *plus* the difference found in (2).

(5) Find the spherical to be worked on the other surface to reduce the powers to the originals as found in (1).

(6) Assign the axes of the toric to conform with those of (1).

*For example* +1 Sph. ⊂ +2 Cyl. Ax. 45° to a base curve +6 D.

(1) The combination = +1 C. Ax. 135° ⊂ +3 C. Ax. 45°.

(2) The difference is 2.

(3) The base curve is + 6.

(4) The other toric curve is + 8.

(5) The spherical is − 5.

(6) The axis of (3) is 135°; that of (4) is 45°, i.e.

$$\frac{-5\text{ S.}}{+6\text{ C. Ax. }135^\circ \subset +8\text{ C. Ax. }45^\circ}$$

*The same combination* on a base curve of − 6 D.

(1) and (2) are the same.

(3) The base is − 6.

(4) The other toric curve is − 8.

(5) The adjusting sph. is + 9.

(6) The axis of (3) is 45°; that of (4) is 135°, i.e.

$$\frac{+9\text{ S.}}{-6\text{ C. Ax. }45^\circ \subset -8\text{ C. Ax. }135^\circ}$$

The above rules seem somewhat complicated, but after a little practice only (5) and (6) require any consideration. It is useful to remember that

(*a*) With the base of same sign as the two powers required, the sph. is the base *less* the *lower* power, and the base axis is reverse to the original cyl. axis.

(*b*) With the base of opposite sign to the two powers required, the sph. is the base *plus* the *higher* power, and the base axis is the same as the original cyl. axis.

When the combination is mixed these last rules cannot apply.

**Advantages of Menisci.**—Of course the advantages derived from the use of periscopic sph. lenses apply also to toric lenses, which are merely deep menisci possessing a cyl. element.

A periscopic Cx. or Cc. sph. is preferable to a Dcx. or Dcc. A + sph. ⊂⊃ − cyl. is better than a + sph. ⊂⊃ + cyl. A concave surface near to the eye prevents side reflections of light, allows of the frame being fitted closer, and, what is most important of all, the field of clear view is widened by the elimination of the oblique aberrations, coma and radial astigmatism.

There is one defect noticeable sometimes in toric and deep meniscus lenses, especially if the powers be weak. The wearer complains of "ghost" or secondary image of any bright object such as a window. This is caused in the same way as the multiple images in plane mirrors, since a weak lens made deeply periscopic has its surfaces nearly parallel. Also the elimination of oblique point aberration tends to render more prominent any residual distortion in the peripheral portions of the field, or when using the edges of the lens.

Again, a deep Cc. rear surface is apt to act as a powerful Cc. mirror and to produce magnified virtual images of the wearer's own corneal reflections, these being most marked when looking in the neighbourhood of bright lights.

In a meniscus lens the optical centre lies on the remote side of a Cx., and on the near side of a Cc., with respect to the eye, so that the effect of the lens is somewhat increased in both cases as compared with an ordinary lens of similar dioptric number. But since the distance of the optical centre depends largely on the thickness of the lens, any difference of effect resulting from the toric or meniscus form is negligible if the lens be thin, as is usually the case with spectacle lenses. It is, however, sometimes noticeable with strong lenses. The term toric is often misapplied to deep meniscus spherical lenses.

**Wide-angle or Periscopic Lenses.**—The form of lens which allows of best vision over a fair range, of say 50° or 60°, i.e., 25° or 30° on each side of the axis, is one which eliminates radial astigmatism and produces a flat field; the two do not necessarily accompany each other, and the former is the more important.

The subject has been treated by Ostwalt and Wollaston, and more recently by Dr. Percival in his "Prescribing of Spectacles" and by Mr. A. Whitwell in the *Optician*. The calculations, which are of a complicated nature, are based on motion of the eye about the centre of rotation some 27 mm. behind the plane of the lens. The actual best forms as to the curvature of the two surfaces vary with the power of the lens, with the $\mu$ and with the distance of vision; it is, however, nearly always a deep meniscus. With refractive indices between 1·5 and 1·54, for Cx. lenses up to, say, + 8 D, the one surface is about − 7 D or about + 20 D; for concaves the one surface is either − 7 D or − 20 D added to half the power of the required lens. These are only very approximate figures for distant vision; the true figures differ for every $\mu$, every power of lens, every distance of lens from eye, and, moreover, for every distance of vision.

## CHAPTER XII

### ANALYSIS AND NEUTRALISATION OF THIN LENSES AND PRISMS

**Neutralisation.**—Neutralisation consists of finding that lens (or lenses) of opposite refraction and of known power (from the test case) which stops the movement caused by the lens to be analysed.

A Cx. and a Cc. lens (Fig. 126) of the same power, when placed in contact, have no converging or diverging effect, the convergence of the Cx. being counteracted by the divergence of the Cc., and incident parallel light emerges parallel. Two such lenses, when moved in front of the eye, cause no movement of the image of the object viewed through them, as occurs with a plane glass.

**Cx. and Cc. Lenses.**—If an object is viewed through a lens and the lens be then moved, the virtual image seen moves in the *opposite direction with a Cx. lens*, and in the *same direction with a Cc. lens*. If the lens is dis-

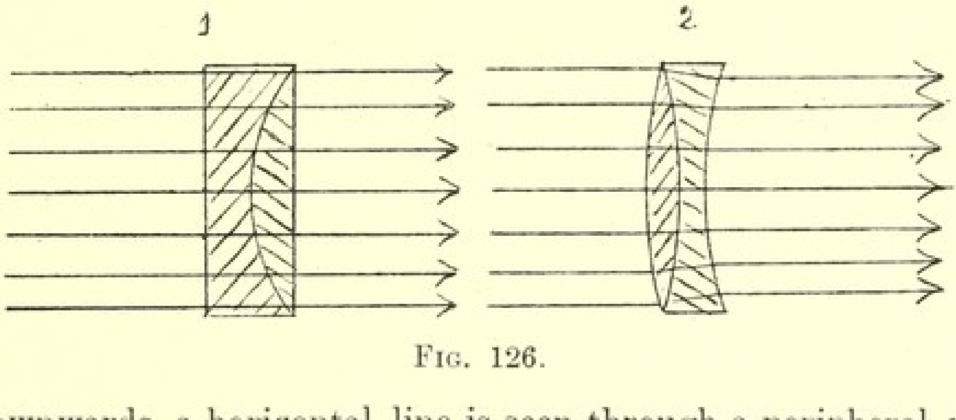



FIG. 126.

placed downwards, a horizontal line is seen through a peripheral portion of the lens, which is of greater deviating power than the centre, and the line appears deviated in the direction of the apices of the virtual prisms of which the lens is formed, that is, towards the edge of a Cx., and towards the centre of a Cc. lens. The degree of deviation and the rapidity of movement of the line is proportional to the strength of the lens; also the deviation is greater, as the part of the lens looked through is near the periphery. The apparent motion of the object viewed, as the lens is moved, *is due to the fact that the lens increases gradually in prismatic or deviating power from centre to periphery.* If the line be first viewed through, say, the bottom of the lens, and this then moved downwards, the motion of the image is continuously *with* or *against* throughout the journey. If, instead of the lens, the head is moved, an image seen goes with the head if the lens is Cx., and in the opposite direction if it is Cc.; for if the head is moved, say, to the right it produces the same effect as if the lens had been moved to the left.

**Analysing Card.**—Analysis and neutralisation are facilitated by the use of an analysing card, as shown in Fig. 127, although, in its absence, any clearly-defined straight vertical, or horizontal line, as the sash of a window, serves the purpose. The card should be 18 or 20 inches square, with two crossed black lines about $\frac{1}{4}$ inch in width, running vertically and horizontally, and for most work should be distant not less than 3 or 4 feet.

A square card viewed through a sph. is slightly increased or decreased in size equally in every direction and (disregarding distortion) remains a true square. If viewed through a cyl., axis vertical, the square is apparently increased in size across the axis, by a Cx., and diminished by a Cc.; the size is not altered in the direction of the axis, so that the square appears a rectangle in both cases. Diminution caused by a Cc., and magnification caused by a Cx., disappear when the two of equal power are placed together.

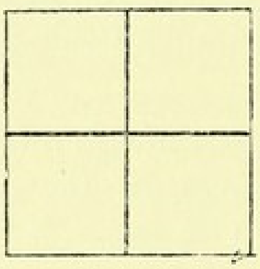


Fig. 127.

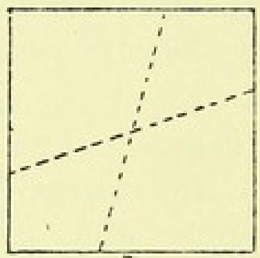


Fig. 128.

**Determination of Cylindrical Element.**—The first step in the analysis of a spectacle lens is to learn whether or not it contains a cyl. element. A lens having a sph. power only, on being rotated around its geometrical centre in a plane parallel to the card, does not cause any change in the appearance of the lines of the analysing chart, because its refractive power is alike in all meridians. If the lens has a cyl. element the lines become oblique, as shown in Fig. 128, where the dotted lines represent the black lines of the chart as seen when the lens is rotated. This obliquity occurs because the power of the lens is not the same in all meridians.

**Determination of Nature of Sphericals.**—If the lens has only sph. power, the next necessary step is to learn whether it is Cx. or Cc. by moving the lens horizontally while observing the vertical line, or vertically while observing the horizontal line.

When the vertical line is first viewed through the centre of the lens the part *A B* seen through the glass is continuous with the parts *C* and *D* seen beyond its edges (Fig. 129). Then if the lens is moved, say, to the *right*, *A B* becomes broken away from *C* and *D* to the *left* if the lens is Cx. (Fig. 130), and to the *right* if it is Cc. (Fig. 131). When making this test the lens should be moved slowly in a certain direction, and not rapidly from side to side or up and down. If the lens is held too close to the eyes the line *C* and *D* beyond the edges cannot be seen, so that the best distance is about 8 or 10 inches. If, however, the lens is a strong Cx. it must be held nearer the observer's eyes, or nothing can be seen through it owing to the strong convergence of the light, but the nature of such a lens can be at once recognised both from its form and from the fact that the lines, seen through it, are indistinct. Again, if held at a distance somewhat greater than its focal length, for instance, if a 4 in. Cx. be held 10″ in front of the eye, the light will have crossed in the air, to enter the eye divergently, and the apparent movement of the object when the lens is moved is the same as with a Cc. lens. What is really observed is an inverted aerial image of the object, but the inversion of the chart may not be noticeable. Only very strong Cx. lenses can, when held a few inches from the eye, form an aerial image sufficiently far away to be distinctly seen. The central thickness, however, of a strong Cx. lens sufficiently indicates its character.

If the glass is *plano*, that part *A B* of the vertical line seen through the glass remains *continuous* with the parts *C* and *D* on either side; no displace-

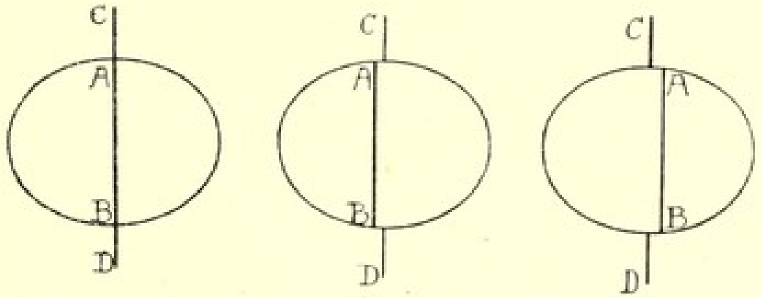



FIG. 129. FIG. 130. FIG. 131.

ment occurs on moving the glass or on rotating it. Also there is no movement of either line on moving a prism in front of the eye, but both lines are not continuous, nor do they remain stationary on rotation of the prism.

**Neutralisation of Sphericals.**—If the unknown lens is Cx., a Cc. is selected from the trial case, as near the power as can be judged, and then the two held together are again moved. If the movement is still that of a Cx. the power of the neutralising Cc. is insufficient, and a stronger one must be tried. If with the first neutralising lens the movement of the two combined is that of a Cc., the neutralising lens is too strong, and a weaker one must be taken. A few trials will enable one to find a lens which, when placed in contact with the unknown lens and moved, causes no displacement of the line; then the number of the neutralised Cx. equals that of the neutralising Cc. To find the power of an unknown Cc. lens, a neutralising Cx. must, of course, be used. Practice will soon enable one to judge, by the degree or rapidity of movement, the approximate neutralising power needed, as well as to appreciate such slight movements as occur when neutralisation is nearly, but not quite, effected. When neutralising, the lenses must be in actual contact, because if separated the Cx. acts with increased effect.

**The Principal Meridians of a Cylindrical.**—If the lens contains a cyl. element the cross lines of the analysing card are seen continuous within and beyond the edges of the lens, as in Fig. 132, only when the axis of the lens is horizontal or vertical. The two principal Mers. then correspond in direction to the lines of the chart. Such a position for a cyl. must be found in order (*a*) to learn whether it is a plano- or a sph.-cyl., (*b*) to determine whether it is Cx. or Cc., and (*c*) to neutralise it. This position being found, the lens is first moved vertically and then horizontally. If no movement is observed in the one direction it is a plano-cyl.; if there is movement in both directions it is a sph.-cyl., or its equivalent, a cross-cyl.

Movement *against* indicates Cx. power and movement *with* indicates Cc. power *in that meridian*. If there is movement in both Mers. they may be both *against*, both *with*, or the one *against* and the other *with*. The movement in the one Mer. differs in degree or nature from that in the other if there is a cyl. element.

The axis of a plano-cyl. lies in the meridian in which there is *no* movement. The axis of the cyl., in a sph.-cyl., which has two positive or two

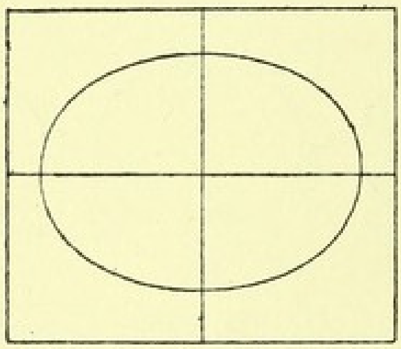


Fig. 132.

negative powers, is in that principal meridian in which there is the *lesser* movement. When there are + and − powers the axis of the cyl. is also *presumed* to be in the principal meridian of *lesser* movement. In all cases the axis of the actual cyl. might be in the meridian of greater movement, because the same principal powers can be obtained in lenses of various forms. (See *Transposing*.)

The angular *position* of the axis is *the same as that of the neutralising cyl.* This can be determined after a little practice, with a fair degree of accuracy, when the lens is held as when in use. With more accuracy its numerical position can be determined by holding the lens against the neutralising lenses when the latter are in a trial frame, with the long diameter of the neutralised lens horizontal. The axis of the trial lens, being marked by a scratch, can be read off from the notation of the frame.

There are several forms of inclinometers or axis-finders—that of Dr. Maddox, for instance, is a most excellent one—designed for the purpose of aiding in the location of the axis of an unknown cylindrical spectacle lens. A quick and fairly accurate method of locating the axis is by means of the protractor on the "Orthops" rule. For real accuracy the procedure is as follows:

Holding the lens and the neutralisers in position, the axis of the cyl. is marked with a grease pencil on the lens by a line coinciding with the axis of the neutralising cyl.; also the optical centre is marked by a dot. The lens is then placed on a protractor with the dot at the centre, the long diameter of the lens being exactly horizontal; the angular position of the axis is then indicated on the protractor. Care must be taken that the same meridian is covered by the marked grease line both above and below the central horizontal line. When the axis is oblique and the lens is not in a frame, consideration must be given as to which of the two faces of the lens is supposed to be directed outwards, since the location of the axis varies accordingly. The rule is that the less convex, or the more concave, surface of a lens is placed next to the eye.

**Neutralisation of Cylindricals.**—To neutralise a plano-cyl. the procedure is the same as with a sph., only that cyls. of opposite nature are employed. Care must be taken that the cyl. axis is always exactly vertical or horizontal, and that the axis of the neutraliser precisely corresponds to it. In order that this may be the case, *continuity of the crossed lines at the edges of the lens must be looked for, and constantly maintained during the process of neutralisation.*

In a sph.-cyl. the lesser movement is that due to the sph. alone, while the greater movement is caused by the united powers of the sph. and the cyl. The lens being held with its axis, say, *vertical*, that sph. of opposite refraction is found which neutralises, in the Ver. meridian, the movement of the Hor. line. This having been achieved, the lens and the neutralising sph. are held together, and the cyl. element is then neutralised with a cyl. *axis vertical*, of opposite refraction, in the same manner as if the lens were a plano-cyl. The rapidity and exactitude of the neutralisation depends, as with a plano-cyl., on the care exercised *in keeping the principal meridians exactly parallel to the two lines of the card, and the axes of the two cyls. exactly corresponding.*

Neutralisation of a sph.-cyl. can also be effected by neutralising each principal meridian separately with a sph. or with a cyl. whose axis is placed at right angles to the meridian that is being neutralised, the two powers thus found being transposed into a sph.-cyl. combination. These methods are, however, not so exact, especially for beginners.

Cross cyls., torics and obliquely crossed cyls. are all merely special forms of sph.-cyls. and so are analysed and neutralised in a similar manner to these latter.

**Expressing Sphero-Cylindricals.**—Since any lens which has two principal meridians can be put up in various forms, the neutralising combination, while correctly indicating the refracting powers of the lens, may not represent the exact form in which it is made. It is always correct to express a combination as a sph.-cyl. *with a sph. of the lower power.*

**True Form of a Lens.**—This can be learnt by (*a*) ordinary inspection, (*b*) reflection from the surfaces, (*c*) the lens measure or spherometer, (*d*) by a straight-edge which, when in contact, easily shows the difference between Cx. and Cc. curvature.

**The Scissors Movement.**—On rotating a cyl. in a plane parallel to the analysing chart the lines on the latter appear to make a scissors-like movement, and if the rotation be continued, appear to move back again, the amount of *dipping* being dependent on the strength of the cyl. Each line appears to bend towards the meridian of greatest positive, or least negative, refraction, so that they both rotate towards the axis of a Cc. or away from the axis of a Cx. cyl., and since they incline towards each other, they are never at right angles except when the principal meridians of the lens correspond to them in direction.

The inclining of the cross lines is due to the prismatic formation of the lens, *the apparent displacement being towards the edges of the virtual prisms contained in the lens.* Thus rotation is useless for determining the nature of a

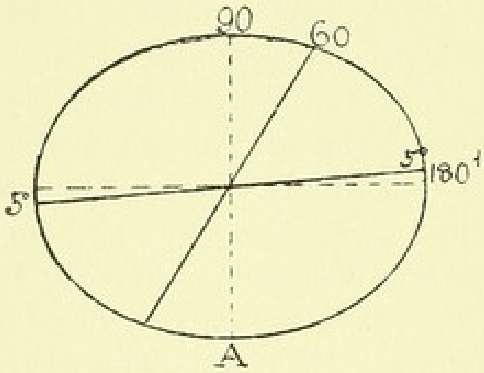



FIG. 133.

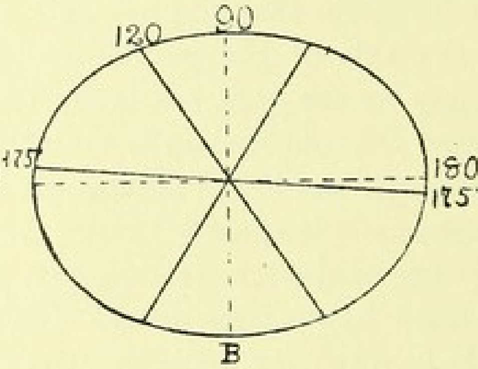



FIG. 134.

cyl. since the scissors movement is the same for both Cx. and Cc., the one end of the horizontal line moving up and the other down, one end of the vertical line moving to the right and the other to the left. For instance, a Cx. cyl. axis Ver. and a Cc. cyl. axis Hor., both rotated, say, clock-wise, cause similar movements of the cross lines. An attempt to neutralise by "stopping" the apparent inclinations might result in selecting for that purpose another cyl. of similar power and nature, the two together making a sph. lens.

**Reversion of a Cyl.**—If a cylindrical (Fig. 133), having its axis at, say, 60° when the one face is to the front, is turned over so that the other face becomes the front, the axis is then at 120° (Fig. 134). If the one position were 5°, the other would be 175°. It is only when the axis is vertical or horizontal that no change occurs on turning the lens over. When the one inclination is 45° or at 135°, turning the lens over brings the axis to a position at right angles to the former one. The change in the numerical position of the axis, caused by turning an oblique cylindrical, is calculated as so many degrees above or below the horizontal, or to the right or to the left of the vertical, and assigning its position accordingly; or it is done by simply deducting the numerical position of the axis from 180°. Thus, suppose the axis is at 60°, this is 30° to the right of the vertical; on turning the lens the axis is at $90° + 30° = 120°$, i.e., 30° to the left of the vertical, or more simply by $180° - 60° = 120°$. The corresponding positions of the axis as the one face or the other is in front are shown in the following table:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1st Position | .. | 0° | 10° | 20° | 30° | 40° | 50° | 60° | 70° | 80° | 90° |
| When reversed | .. | 180° | 170° | 160° | 150° | 140° | 130° | 120° | 110° | 100° | 90° |

**Prisms.—The Base-Apex Plane.**—As a prism, a sphero-prism, or a decentered sph. is rotated in front of the analyser, it is found that, in a certain position, there is *a continuity of one of the lines within and beyond the edges of the glass*, as in Fig. 135, where the vertical line *A B* is continuous. The direction of this line indicates that of the base-apex plane of the prism, or of the prismatic element of the lens. If the Hor. line *C D* is deflected upwards, as to *E F*, the apex is then pointing upwards towards *A*, and the

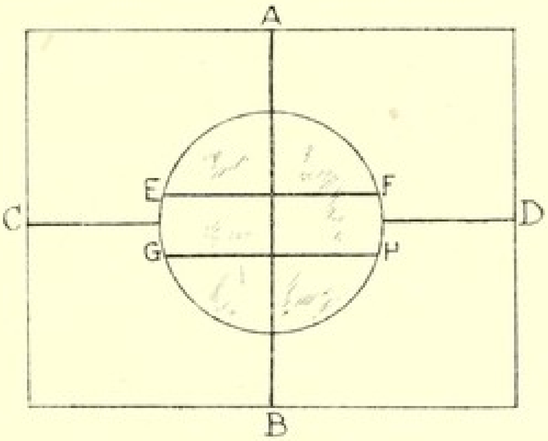


Fig. 135.

base is down towards *B*. If the deflection of *C D* is downwards towards *G H* the edge of the prism is pointing downwards, and the base is up. If properly marked, the indicating scratches of circular trial prisms lie over *A B* when that line appears unbroken by the prism.

**Neutralisation of Prisms.**—The strength of a prism can be learnt by neutralisation. The base apex line being located, the displacement of a bar of the analyser can be neutralised by trying one prism after another from the test case and placing it in opposition to the unknown prism; that is, placing the base of the former over the edge of the latter, until that test prism is found which causes the bar to be seen continuous beyond and through the two prisms. The number of the test prism, which neutralises the unknown prism, indicates the value of the latter. By this method the *deviating angle is really neutralised*, although the neutraliser may be numbered according to its principal angle.

If the prism is combined with a sph. (or a sph.-cyl.) this latter must be first neutralised. With the lens and the neutraliser held together, the two being *geometrically centred*, the prismatic element is located and neutralised. It must be remembered that the smallest decentration, with respect to each other, of neutralising Cx. and Cc. lenses, introduces considerable prismatic effect not actually existing in the lens which is under analysis. Therefore *the geometrical centres of all the lenses should exactly coincide* when any prismatic element is suspected, or is being measured.

If the angular inclination of an oblique prism is needed the base apex line, when located, should be marked with a grease pencil and the angular position determined on a protractor, as with the axis of a cyl.

**Points on Neutralising.**—It is essential that the front of the lens, i e., the Cx. surface of a periscopic, or the more Cx. or less Cc. surface of any lens, be held towards the observer.

A lens possessing sph., cyl. and prismatic elements should be neutralised in that order. Practice is necessary to neutralise rapidly and correctly, and it is well to commence with simple sphs., and then proceed to plano-cyls. and finally to sph.-cyls. and other compound lenses.

If the sph. is strong compared with the cyl. it is difficult to appreciate the latter until the sph. is partly neutralised. Similarly it is difficult to appreciate a weak sph., when combined with a strong cyl., until the latter is wholly or partly neutralised. When the two powers of a sph.-cyl. are nearly equal it is not always easy to determine in which Mer. the movement is the lesser, but this becomes easy enough when the lens is partly neutralised.

A simple prism may be mistaken for a plano, since neither causes *movement;* rotation is needed to distinguish between them.

It is necessary to guard against supposing a prismatic element to exist, when it may be produced by holding the neutralising lens out of centre with the lens which is being tested.

Holding several lenses together is difficult, but is rendered easier if the surfaces are fitted together, i.e. Cx. to Cc.

To determine very weak powers, or to determine whether neutralisation is obtained, hold the lens or combination at arm's length, and move in the ordinary way; by this means the apparent movement of the object is increased and enables very weak powers to be detected.

**Strong Opposite Lenses.**—It is difficult to get absolute neutralisation with strong lenses, say over 10 D, there being always some slight movement in the peripheral portion of the lenses, although near the centre there may be practically none. This is due to the thickness of the Cx., or rather to the interval between the optical centres of the two lenses. As shown in Fig. 136 by the dotted lines, the two lenses actually constitute a Cx. meniscus, for, with the same radius of curvature, the total lens is one formed of two intersecting circles.

The thickness of a Cc. lens in the centre, no matter how strong it be, can be ignored, but this is not the case with a strong Cx. If the focal length of the Cx. is equal to that of the Cc., it is clear that $F_V$ of the Cc. is further from the centre of the combination than is $F_X$ of the Cx. Parallel light incident on $B$ is rendered divergent as if proceeding from $F_V$, a point outside $F_X$, and is therefore slightly convergent after refraction by $A$. If parallel light is incident on $A$, it is converged to $F_X$, a point nearer than $F_V$, so that it is still slightly convergent after refraction. Thus a strong Cx. and Cc. lens of similar $\mu$ and radius do not actually neutralise each other.

This can also be explained in another way. If the light be incident on the Cx. it is converged, and the convergence is increased as it traverses the thickness of the two lenses to an extent that the final Cc. surface is unable to neutralise. On the other hand, if the light is incident first on the Cc. surface it is diverged, but in passing through the Cx. some divergence is lost, with the result that the Cx surface over-neutralises it and produces a slight positive effect. Thus with either lens to the front, the result is the *same* Cx. power, but when the Cx. lens is in advance of the Cc. the *effectivity* of the resultant Cx. power is enhanced. Again, if a ray of light originally parallel to the axis traverses first a Cx. and then a Cc. of equal dioptric power, or vice-versa, its passage *in both cases* is, in the Cx., at a part of the lens more distant

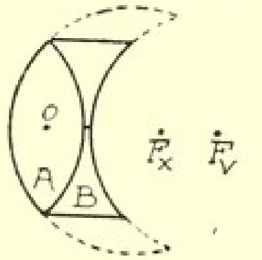



FIG. 136.

from the axis than that of the Cc. and, therefore, where the prismatic element is greater in the former.

The Cc. being thin at its axis, its required radius for a given focal length would be calculated by the formula where thickness is neglected, while that of the Cx. would need to be calculated with its thickness considered. The +20 D from a trial case, being of a large diameter, is about ·75 cm. thick in the centre, and its radius would need to be shorter than that of the −20 D to have equal *equivalent* power. Giving the same radius to each, the *true* or *equivalent* power of the Cx. is weaker than that of the Cc. In order that two strong opposite lenses should neutralise, the Cc. must be the more powerful, the focal length of the Cx. being approximately one-third its thickness longer than that of the Cc., which, however, is not the case when the radii of curvature of the two are equal. In short, although a thick Cx. has a longer equivalent focal length than a Cc. of similar radius and $\mu$, it is not sufficiently so for the Cx. to be neutralised by the Cc. For a −20 D whose F = −5 cm. to neutralise a Cx. having a thickness of ·75 cm., the latter would need have F = 5·25 cm., or D = +19, and if $\mu = 1·5$ would require a radius of curvature of 5·125 cm. In other words the *back foci* of the lenses must be equal. If, therefore, a Cx. and a Cc. do neutralise, the latter is stronger, but the difference is quite inappreciable in weak lenses, and not of importance in spectacle lenses, even if strong.

In modern cases of test lenses the concaves are of their indicated strength, but the convexes are made to neutralise the concaves of similar numerical value. Up to 10 D the difference is inappreciable, but the + 10 D is only 9·8 D approximately, and the + 20 is + 18·75 D approx., the intermediate

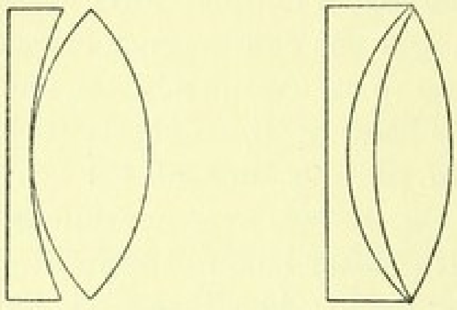


FIG. 137.

numbers being of proportional nominal value. Whether there is good reason for this arrangement is very doubtful indeed.

Strong Cx. and Cc. lenses may neutralise each other at the centre and not at the periphery or vice-versa, with excess of either Cx. or Cc. effect at the one or other; or there may be Cc. effect at the centre and Cx. at the periphery, or the reverse. In such cases the lenses are not in contact either at the centre or at the periphery (Fig. 137), and the phenomenon is due to the increased effect of a Cx. owing to separation. It is this separation that renders the neutralisation of torics and deep menisci so difficult.

## CHAPTER XIII

## OBLIQUE CYLINDRICALS AND OBLIQUE SPHERICALS

**Powers of a Single Cyl.**—If a lens measure be placed in contact with the maximum meridian $M$ of a cyl. (Fig. 138) we obtain the highest possible curvature from that cylindrical. Along the axis the instrument would indicate 0, and between these two the recorded power would vary. Suppose the two fixed legs touch at $d$ $d$, then the sag. of the central leg indicates the power which is based on the formula $r = d^2/2$ S (vide The Spherometer), and the curvature c = 2 S/$d^2$, where $d$ is half the distance $d$ $d$. If the sag were a fixed quantity c varies inversely with $d^2$ Let the instrument be turned so that the legs lie on the meridian $M_1$ at an angle $b$ with $M$; then $d/\cos b = d_1$. If now the sag were the same as before it is because the distance between the legs is greater, curvature $c_1$ at $M_1$ bearing to the curvature c at $M$ the relationship $c_1/c = d^2/d_1^2$, where $d_1$ is the new distance $d_1$ between the

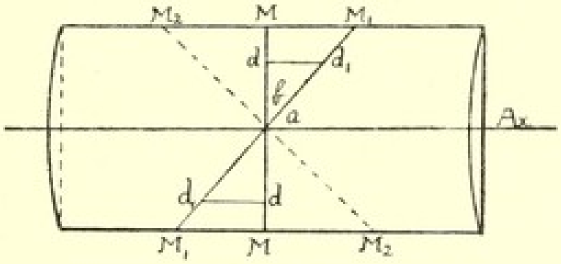



Fig. 138.

central and one of the fixed legs in the meridian $M_1$. But $d_1 = d/\cos b$, so that

$$\frac{c_1}{c} = \frac{d^2}{d_1^2} = \frac{d^2 \cos^2 b}{d^2} = \cos^2 b$$

or

$$c_1 = c \cos^2 b$$

Now the dioptric powers D at $M$, and $D_1$ at $M_1$ are directly proportional to the curvatures c and $c_1$ respectively, so that in the meridian $M_1$ the power of the lens $D_1 = D \cos^2 b$, or what is the same, $D_1 = D \sin^2 a$ where $a$ is the angle between $M_1$ and the axis. Similarly it can be shown that in the meridian $M_2$ at right angles to $M_1$ the power $D_2 = D \sin^2 b$, or $D \cos^2 a$. If we consider the angle $a$ between a given meridian and the axis of a cyl., the power varies as $\sin^2 a$; if we consider the angle $b$ between it and the maximum meridian, the power varies as $\cos^2 b$.

Although we thus refer to the refractive power of a cyl. in any oblique meridian, yet this latter does not cause a point focus. A cyl. brings incident light to a line focus, parallel to the axis, and if a meridian of maximum power be isolated by means of a stenopæic slit, so that the oblique meridians are cut off, the line is reduced to a point, since the effective curvature in that meridian is spherical, and the result is similar to that of an ordinary spherical lens. If the slit be slowly rotated the meridians successively uncovered are elliptical in curvature, and the point focus, first obtained, gradually widens into a line parallel to the axis, showing that, although the effective part of the lens is oblique, the effective *curvature* is always that of the maximum meridian. If the rotation be continued until the slit is parallel to the axis, the line reaches its maximum length just as though the whole lens were uncovered. Thus the only meridian capable of producing a true focus is the maximum principal meridian, which has a spherical curvature. It is, however, as mentioned in Chap. XII., useful to assume that the oblique meridians of a cyl. have certain powers relative to the maximum, and the following is a brief summary of the necessary calculations; a distinction

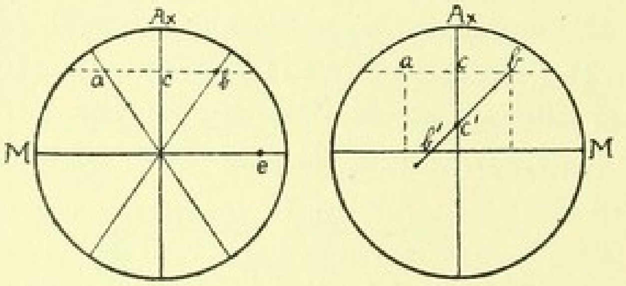



Fig. 139. Fig. 140.

must, however, be drawn between the incomplete line foci of such powers and the point foci produced by spherical curvatures.

**Oblique Refractions of a Cyl.**—Fig. 139 represents a Cx. cyl. lens whose axis *Ax.* is vertical, and whose maximum power *M* is horizontal. Let this lens be a + 5 D, and the object be a point at ∞. Any ray of light incident in the meridian *Ax*, central to the meridian *M*, suffers no deviation, it being normal to the lens at both surfaces. Any ray incident in Mer. *M* is refracted to an extent governed by its distance from the central point of Ax., such that it meets all other rays, incident in that meridian, in a point in line with, and 20 cm. behind it. Any ray, as *b*, incident in an intermediate meridian, say that of 70°, is refracted so as to meet all other rays, incident in the plane *b c a*, in a point in line with *c*, and also 20 cm. distant. The deviation suffered by the ray *b*, refracted in an intermediate meridian, is less than that which occurs when refracted as *e* in meridian *M*, both being equidistant from the central point of Ax. The total image is a Ver. line.

In the case of a sph.-cyl. (Fig. 140) a ray incident at *b* in an oblique meridian is refracted by the sph. to a point on the principal axis, and by the cyl. to a point in line with *c*, with the resultant deflection in the direction $b'$, so that it meets rays incident in a plane $b\ c\ a$ parallel to $M$ in $b'$, and those incident in a plane parallel to $Ax$ in $c'$; or if the lens be regarded as consisting of crossed cyls., the deviation is towards both axes, resulting in an oblique deviation towards $c'$ in the first, and $b'$ in the second focal line.

Let D be the maximum power of a cyl., $D_1$ the power in a given Mer., and $D_2$ that at right angles to $D_1$; let $a$ be the angle between the axis and the Mer. of $D_1$. Then the powers of a cyl. in any pair of given meridians are, as shown on page 153, found by

$$D_1 = D \sin^2 a \qquad \text{and} \qquad D_2 = D \cos^2 a$$

The power along the axis is 0, and at right angles it is D, so that the total power of this pair of opposite Mers. is $D + 0 = D$. Likewise the sum of the powers of any pair of opposite Mers. is equal to D, for $\sin^2 a + \cos^2 a = 1$, so that $D \sin^2 a + D \cos^2 a = D_1 + D_2 = D$.

Thus the powers of a + 3 D. Cyl. Ax. 180°, at 20° and 110°, are:—

$$D_1 = 3 \times {\cdot}11696 = {\cdot}35 \text{ D. at } 20°; \; D_2 = 3 \times {\cdot}88303 = 2{\cdot}65 \text{ D. at } 110°.$$

Let $A\ Y$ and $A\ Z$ (Fig. 141) represent the forces exerted, respectively, in

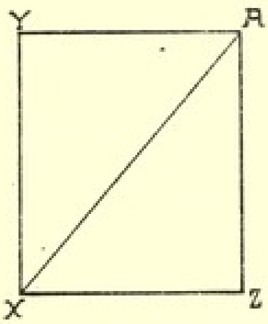



Fig. 141.

the Hor. and Ver. mers. by, say, a 3·5 D. Cyl. Ax. 60°. Let H be the horizontal and V the vertical effect. Now $X\ Y = A\ Z = \sin 60°$, and $X\ Z = A\ Y = \cos 60°$, whence $H = D \sin^2 a = 3{\cdot}5 \times {\cdot}75 = 2{\cdot}625$, $V = D \cos^2 a = 3{\cdot}5 \times {\cdot}25 = {\cdot}875$ and $2{\cdot}625 + {\cdot}875 = 3{\cdot}5 = D$.

In these calculations it is merely necessary to find either $D_1$ or $D_2$ since the other can be obtained by subtraction from D. Thus if $V = {\cdot}875$, $H = 3{\cdot}5 - {\cdot}875 = 2{\cdot}625$ and vice-versa.

Following are the approximate powers of unit cyl. in different Mers. calculated as mentioned.

| Degrees from Ax. | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Proportional Power .. | 0 | ·01 | ·03 | ·07 | ·12 | ·18 | ·25 | ·33 | ·42 | ·50 | ·58 | ·67 | ·75 | ·82 | ·88 | ·93 | ·97 | ·99 | 1·0 |

To find the powers in any Mer. of a given cyl., multiply the decimal corresponding to the angle between the axis and the Mer., by D of the lens. Thus 4·5 D cyl. at 25° from the axis = ·18 × 4·5 = ·81 D.

**Obliquely crossed Cylindricals.**—If two cyls. D and D′ are placed with their axes corresponding in the Ver. Mer. their combined Ver. power $=0$, and the Hor. $=D+D'$. If one or both cyls. be rotated, they are equivalent to a combination of some two other principal powers. When the two axes are at right angles the combination is equal to a sph. if $D=D'$, and to an ordinary cross-cyl. if D and D′ are unequal. It should be particularly noted that, *with any obliquity of the axes, two (or more) cyls. are always equivalent to some other cross-cyl. whose axes are at right angles, and are, therefore, also equivalent to some sph.-cyl.* The sum of the two principal powers $D_1$ and $D_2$ is always equal to the sum of the individual maximum powers D and D′, that is

$$D+D'=D_1+D_2$$

Not only the powers of the principal mers., but also *the sum of the powers of any pair of mers. at right angles to each other* $=D+D'$. Rotation of the axis of one or both cyls., merely locates the refraction in varying quantities as

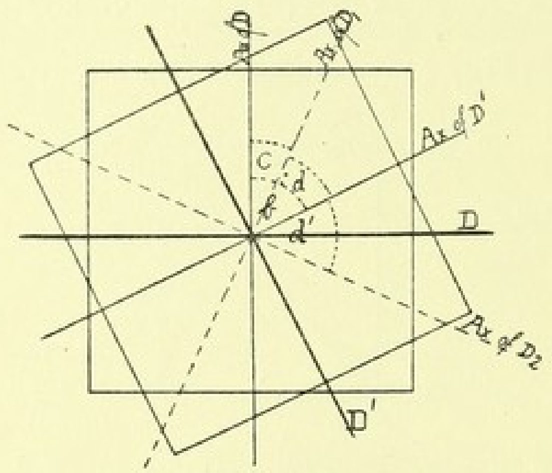



FIG. 142.

regards each of any pair of opposite meridians, and does not alter the total power.

Let $b$ (Fig. 142) be the angle between the axes of two cylindricals $D$ and $D'$, of which $D$ is the higher of the two. Let $D_1$ and $D_2$ be the two resulting powers, $D_1$ being the higher. Let $c$ be the angle which the axis of $D_1$ makes with that of $D$, the stronger original lens, and let $d$ be the angle which the axis of $D_1$ makes with that of D′. Then angle $b=c+d$. Now $D_1$ corresponds with the axis of $D_2$, and $D_2$ with the axis of $D_1$.

From the foregoing we have $D+D'=D_1+D_2$

and $D \sin^2 c + D' \sin^2 d = D_2$, also $D \cos^2 c + D' \cos^2 d = D_1$

Multiplying these together we get

$$D_1D_2 = D^2 \sin^2 c \cos^2 c + D'^2 \sin^2 d \cos^2 d + D\,D' \sin^2 c \cos^2 d + D\,D' \sin^2 d \cos^2 c$$

but $D^2 \sin^2 c \cos^2 c + D'^2 \sin^2 d \cos^2 d = 2D\,D' \sin c \cos c \sin d \cos d$

so that

$$D_1D_2 = D\,D' (\sin^2 c \cos^2 d + \sin^2 d \cos^2 c) + 2D\,D' \sin c \cos c \sin d \cos d$$
$$= D\,D' (\sin c \cos d + \sin d \cos c)^2$$

but $\sin c \cos d + \sin d \cos c = \sin (c + d) = \sin b$

therefore $D_1D_2 = D\,D' \sin^2 b$

and since $D_1 + D_2 = D + D'$

we can, knowing the *multiple* and the *sum* of the two numbers, arrive at their difference N, thus

$$N = D_1 - D_2 = \sqrt{(D + D')^2 - 4D\,D' \sin^2 b}$$

Then we get in the resultant combination

The higher power $D_1 = \dfrac{D + D' + N}{2}$

and the lower power $D_2 = \dfrac{D + D' - N}{2}$

$D_1$ is the spherical + the cylindrical; $D_2$ is the spherical; N is the cylindrical.

The following relationships exist.

$$D_1 - D = D' - D_2 = \frac{N + D' - D}{2}$$

and
$$\frac{D^2}{\sin^2 d \cos^2 d} = \frac{D'^2}{\sin^2 c \cos^2 c} = \frac{N^2}{\sin^2 b \cos^2 b}$$

also $N \sin c \cos c = D' \sin b \cos b$

Now from above $N^2 \sin^2 c \cos^2 c = D'^2 \sin^2 b \cos^2 b$
$$= D' \sin^2 b\,(D' - D' \sin^2 b)$$

but $D' = D_1 + D_2 - D$

so that

$$N^2 \sin^2 c \cos^2 c = D' \sin^2 b\,(D_1 + D_2 - D - D' \sin^2 b)$$
$$= D_1D' \sin^2 b + D_2D' \sin^2 b - D\,D' \sin^2 b - (D' \sin^2 b)^2$$

substituting $D_1D_2$ for $D\,D' \sin^2 b$ we get

$$N^2 \sin^2 c \cos^2 c = D_1D' \sin^2 b + D_2D' \sin^2 b - D_1D_2 - (D' \sin^2 b)^2$$
$$= (D_1 - D' \sin^2 b)\,(D' \sin^2 b - D_2)$$

Since $\sin^2 c + \cos^2 c = 1, \qquad N = N \sin^2 c + N \cos^2 c$

But $N = D_1 + D_2 = (D_1 - D' \sin^2 b) + (D' \sin^2 b - D_2)$

That is $N \sin^2 c + N \cos^2 c = (D_1 - D \sin^2 b) + (D' \sin^2 b - D_2$

And from above

$$N \sin^2 c \times N \cos^2 c = (D_1 - D \sin^2 b)\,(D' \sin^2 b - D_2)$$

Then we deduce that

$$N \sin^2 c = D' \sin^2 b - D_2 \qquad \text{and} \qquad N \cos^2 c = D_1 - D' \sin^2 b$$

Now $$\mathrm{N} \sin c \cos c = \mathrm{D}' \sin b \cos b$$

Therefore $$\tan c = \frac{\mathrm{N} \sin c \cos c}{\mathrm{N} \cos^2 c} = \frac{\mathrm{D}' \sin b \cos b}{\mathrm{D}_1 - \mathrm{D}' \sin^2 b}$$

or $$\tan c = \frac{\mathrm{N} \sin^2 c}{\mathrm{N} \sin c \cos c} = \frac{\mathrm{D}' \sin^2 b - \mathrm{D}_2}{\mathrm{D}' \sin b \cos b}$$

But $$\frac{\mathrm{D}' \sin^2 b - \mathrm{D}_2}{\mathrm{D}' \sin b \cos b} = \frac{\mathrm{D}' \sin^2 b}{\mathrm{D}' \sin b \cos b} - \frac{\mathrm{D}\,\mathrm{D}' \sin^2 b}{\mathrm{D}_1 \mathrm{D}' \sin b \cos b} = \tan b - \frac{\mathrm{D}}{\mathrm{D}_1} \tan b$$

So that $$\tan c = \frac{(\mathrm{D}_1 - \mathrm{D}) \tan b}{\mathrm{D}_1}$$

The value of $c$ is the angular distance of $\mathrm{D}_1$, the stronger resultant cyl., from that of D, the stronger original. We could find a formula for $d$, but it is unnecessary, and of course $c + d = b$. The distance $d'$ of the axis of $\mathrm{D}_2$, the weaker resultant, from that of D′, the weaker original cyl., is found from

$$\tan d' = \frac{(\mathrm{D}_2 - \mathrm{D}') \tan b}{\mathrm{D}_2}$$

Now since $(\mathrm{D}_2 - \mathrm{D}') \tan b = (\mathrm{D}_1 - \mathrm{D}) \tan b$, it is an easy matter to confirm the calculations, but care must be taken with the − signs. When D and D′ are of similar signs $d'$ is negative; also both $c$ and $d'$ are taken as negative when D and D′ are of opposite signs. The two resultant axes must be 90° apart, i.e. $b - (c - d') = 90°$. A positive measurement is towards the other axis, and a negative one is away from it.

**Example.**— + 3 C. Ax. 70° ⊂ + 2 C. Ax. 20°, $\mathrm{D} + \mathrm{D}' = +5$, $b = 50°$, $\mathrm{D}_1 = 4{\cdot}15$, $\mathrm{D}_2 = {\cdot}85$, $c = 18°\ 18'$, $c$ being measured towards D′ from D. The combination is + ·85 S. ⊂ + 3·30 C. Ax. 51° 42′.

It will be seen that $\mathrm{D}\,\mathrm{D}' \sin^2 b = \mathrm{D}_1 \mathrm{D}_2$, i.e. $3 \times 2 \times {\cdot}5868 = 4{\cdot}15 \times {\cdot}85 = 3{\cdot}52$.

The sum of the maximum powers of the two original cyls., in this example + 5 D, is not changed by altering the position of the two axes with respect to each other, for the sum of the two principal meridians of the resultant cyls. is similarly + 5 D. That is, $\mathrm{D}_1 + \mathrm{D}_2 = 4{\cdot}15 + {\cdot}85 = 5$ D.

**Example.**— + 4 C. Ax. 20° ⊂ − 2·75 C. Ax. 65°, $\mathrm{D} + \mathrm{D}' = +1{\cdot}25$, $b = 45°$, $\mathrm{D}_1 = 3{\cdot}05$, $\mathrm{D}_2 = -1{\cdot}80$, $c = 17°\ 15'$. The combination is − 1·80 S. ⊂ + 4·85 C. Ax. 2° 45′, or + 3·05 S. ⊂ − 4·85 C. Ax. 92° 45′.

$\mathrm{D}_1 + \mathrm{D}_2 = +3{\cdot}05 - 1{\cdot}80 = +1{\cdot}25$.

Here by calculation tan $c$ is a minus quantity, and the angle is measured from the axis of D away from the axis of D′ instead of towards it.

**Two Equal Like Cyls.**—Here the calculation is simplified, for when $\mathrm{D} = \mathrm{D}'$, $c = d$, so that it is unnecessary to calculate N or $c$. Thus

$$2\ \mathrm{D} \sin^2 b/2 = \mathrm{D}_1 \qquad 2\ \mathrm{D} \cos^2 b/2 = \mathrm{D}_2$$

and, as stated, $$c = b/2$$

Thus $+4$ D C. Ax. 10° ⊂ $+4$ D C. Ax. 60°
$= +1{\cdot}4288$ S. ⊂ $+5{\cdot}1424$ C. Ax. 35°.

**Two Equal Unlike Cyls.**—Here also the calculation is simplified, for $D = -D'$, $D + D' = 0$, $D_1 + D_2 = 0$, and N needs no calculation.

$D\,D' \sin^2 b = D_1 D_2 = -D^2 \sin^2 b = D_1^2$ or $D_2^2$; therefore $D \sin b = D_1$, and $-D' \sin b = -D_2$.

$\tan c = \dfrac{\cos b}{1 + \sin b}$ measured negatively from the Cx., for the resultant Cx., or from the Cc. for the resultant Cc.

or $\tan c = \dfrac{1 + \sin b}{\cos b}$ measured from the Cx. positively for the resultant Cc., or from the Cc. for the resultant Cx.

The two measurements = 90°.

Thus $+4$ D C. Ax. 60° ⊂ $-4$ D C. Ax. 120°
$= +3{\cdot}464$ C. Ax. 45° ⊂ $-3{\cdot}464$ C. Ax. 135°.

### Graphical Illustration of Formulæ.

Draw $A\,D$ (Fig. 143) in units of length $= D$, and $A\,D' = D'$, making the angle $D'A\,D = b$. On $A\,D$ mark off $A\,H = D_1 - D$, and prolong $A\,D$ a

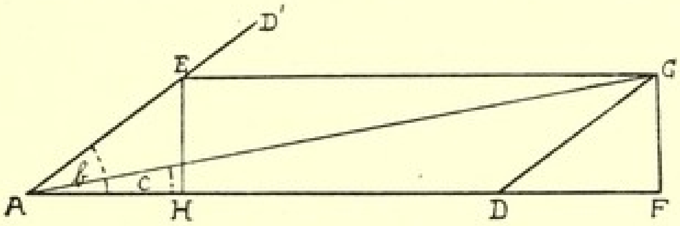



FIG. 143.

distance $D\,F = A\,H = D_1 - D$ so that $A\,F = D_1$. From $E$ drop $E\,H$ normal to $H$, and $E\,H = (D_1 - D) \tan b$. From $D$ draw $D\,G$ equal and parallel to $A\,E$; connect $E\,G$ and from $G$ drop the normal $G\,F$ to $F$ so that $G\,F = E\,H$. Connect $A\,G$. Then $G\,A\,F = c$, and $G\,F = \tan c = \dfrac{(D_1 - D) \tan b}{D_1}$.

**To find b** the angle between two cyls. D and D′ in order to produce any two effects $D_1$ and $D_2$, we have $\sin^2 b = D_1 D_2 / D\,D'$, but of course it is possible only when $D_1 + D_2 = D + D'$.

### The Cylindrical Effect of Oblique Sphericals.

Let Fig. 144 represent the face of a Cx. lens placed normally to the light. Let the effect of the refraction in the vertical plane be ignored and that of the horizontal considered by itself. Rays of light parallel to the axis passing through $c\,c'$ would meet in a point behind, and in line with, O.; similar rays incident at $d\,d'$ and $e\,e'$ would meet in corresponding points behind the lens, forming a radial line focus parallel to $B\,B'$. Now if a beam of light be incident to the lens obliquely in the vertical plane, it is so refracted that the focal line is nearer to the lens and inclined to the principal axis in proportion as the incidence of the light is oblique.

Again, considering the vertical plane by itself, the refracting effect is to produce a tangential line focus parallel to *A A′*. The tangential line meeting its corresponding radial line combine to form point foci for rays parallel to the principal axis, but when the incidence is oblique the two do not combine, the tangential focal line lies nearer to the lens than the radial.

Thus with normal incidence of the ray there is a point focus of a point source in the focal plane ; with oblique incidence a point source gives rise to two focal lines, the tangential at a shorter distance than the radial and both within F. This, as an aberration, is called radial astigmatism (q.v.).

Now since a spherical lens acts with an astigmatic effect on an oblique pencil of light, a spherical held obliquely to the incident light acts as if it were a sphero-cylindrical lens. When a spherical lens is held upright, parallel to a screen, and at its focal distance, a luminous point on the axis will have a point image on the screen. If now the lens be rotated around, say, a horizontal axis, the image becomes confused and drawn out as if a

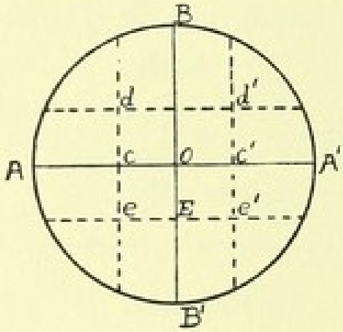



FIG. 144.

cylindrical had been added to the spherical. Two bright focal lines are formed on the screen when the lens is held at the proper distance for each. The second focal line is, in this case, vertical, and slightly within the focus of the lens, the first line being horizontal and still nearer to the lens. *Thus the effect produced by obliquity of a spherical is that of a slightly stronger spherical combined with a cylindrical whose axis corresponds to the axis of rotation.* The refraction is therefore increased in both meridians, but mostly in that at right angles to the axis of rotation.

The increase of power in the meridian of the rotation is owing to the fact that the light has to pass through a rather greater thickness of lens when the latter is oblique than when it is placed normally. The increase in power in the meridian at right angles to the axis of rotation is due partly to the same cause, but is much enhanced by the increased obliquity of the light to the lens surface. It is this increase of power which enables some people who are astigmatic or under corrected to see better by looking obliquely through their glasses.

In Fig. 145 let *a* represent the angle of rotation of the lens, *F* the focal length, and $F_1$ and $F_2$ the effective focal lengths of, respectively, the meridians of greatest and least power. $AB$ is the principal axis of the lens, and $CD$ is the secondary axial ray on which the focal lines are formed. The pencil of incident light is presumed to be parallel to $C\,D$, so that rays as $d$ and $e$, or $d'$ and $e'$, incident in planes parallel to the axis of rotation, meet each other to form the radial focal line $F_2$. Rays as $d$ and $d'$, or $e$ and $e'$, incident in planes at right angles to the first, meet each other to form the tangential focal line $F_1$. The angle of rotation $a$ is that between $C\,D$ and the principal axis, and $b$ is the angle of refraction at the first surface. The distances of $F_1$ and $F_2$ are found from the following formulæ,

$$F_2 = F \times \frac{\sin a - \sin b}{\sin (a-b)} = F \times \frac{\mu \sin b - \sin b}{\sin a \cos b - \sin b \cos a}$$

$$F_2 = \frac{F\,(\mu - 1)}{\mu \cos b - \cos a} \quad \text{and} \quad F_1 = \frac{F\,(\mu-1)\cos^2 a}{\mu \cos b - \cos a} = F_2 \cos^2 a$$

or

$$D_2 = \frac{D\,(\mu \cos b - \cos a)}{\mu - 1} \quad \text{and} \quad D_1 = \frac{D\,(\mu \cos b - \cos a)}{(\mu - 1)\cos^2 a} = \frac{D_2}{\cos^2 a}$$

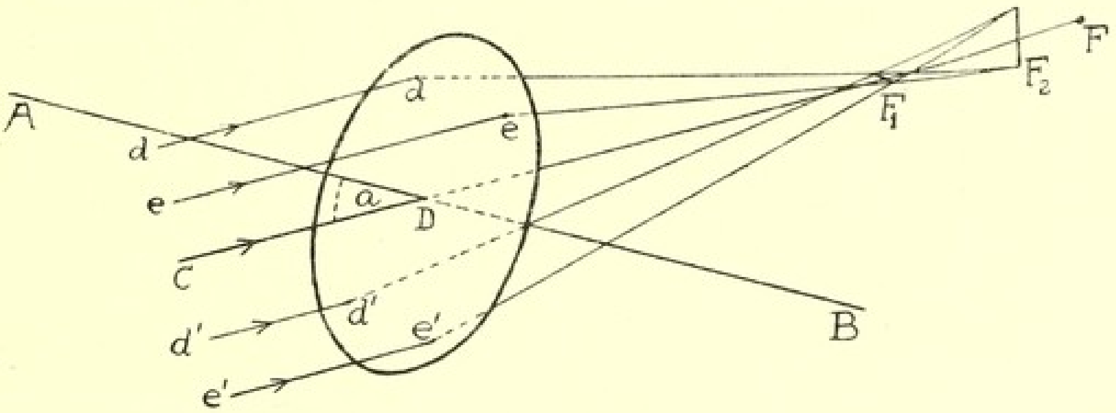



FIG. 145.

Therefore with a distant source of light, if the two focal distances be measured, the angle of rotation of the lens can be found from the equations

$$F_1/F_2 = D_2/D_1 = \cos^2 a$$

When $\mu = 1{\cdot}5$ simplified approximate formulæ are obtained by substitution, in those given, of $\frac{2\mu - \sin^2 a}{2\mu}$ for $\frac{\mu - 1}{\mu \cos b - \cos a}$, which hold good for small angles. They can be written

$$F_2 = F\,(3 - \sin^2 a)/3 \qquad F_1 = F_2 \cos^2 a$$
$$D_2 = 3D/(3 - \sin^2 a) \qquad D_1 = D_2/\cos^2 a$$

**Examples.**—A 10″ Cx. lens is rotated 20° ;

$$F_2 = 10(3 - {\cdot}117)/3 = 9{\cdot}61 \quad \text{and} \quad F_1 = 9{\cdot}6 \times {\cdot}883 = 8{\cdot}48''$$

If a +4 D be rotated 20°

$$D_2 = 4 \times 3/(3 - \cdot 117) = 5 \cdot 3 \text{ and } D_1 = 4 \cdot 13/\cdot 883 = 4.68$$

Since $D_2$ does not vary greatly from D, the increased or cylindrical effect produced by obliquity of a spherical lens is

$$C = D_1 - D_2 = D/\cos^2 a - D = D \tan^2 a.$$

**Table of Cylindrical Effect of Oblique Sphericals.**

The following table gives the approximate effects obtained by rotating a 1 D lens; the effect on other lenses is proportional. The rotation is supposed to be around a horizontal axis.

| Angle of Rotation. | $F_1$ Ver. Mer. | $F_2$ Hor. Mer. | Sph.-Cyl. Combination. $D_2$ $D_1 - D_2$ | Cyl. rotated or $D_1$. |
|---|---|---|---|---|
| 5° | 99 | 100 | 1·00 ⊂⊃ 0·01 | 1·01 |
| 10° | 96 | 99 | 1·01 ⊂⊃ 0·03 | 1·04 |
| 15° | 91 | 98 | 1·02 ⊂⊃ 0·07 | 1·09 |
| 20° | 84 | 96 | 1·04 ⊂⊃ 0·16 | 1·20 |
| 25° | 77 | 94 | 1·06 ⊂⊃ 0·24 | 1·30 |
| 30° | 70 | 91 | 1·09 ⊂⊃ 0·34 | 1·43 |
| 35° | 59 | 88 | 1·13 ⊂⊃ 0·57 | 1·70 |
| 40° | 50 | 86 | 1·16 ⊂⊃ 0·84 | 2·00 |
| 45° | 42 | 83 | 1·20 ⊂⊃ 1·20 | 2·40 |

The effect increases rapidly with a greater obliquity.

The effective power of a cylindrical rotated around its axis is found by the same formulæ as for $D_1$ and $F_1$. It is, in effect, a stronger cyl. If it is rotated across its axis its effect also is that of a slightly stronger cyl. If a sph.-cyl., both powers being of similar nature, be rotated round the axis of the cyl., the cyl. effect is increased. If rotated round its meridian of greatest power, the sph. effect is increased, and the cyl. decreased. A rotation oblique to the principal Mers. results in a new combination altogether.

## CHAPTER XIV

### OPHTHALMIC PRISMS

**The Deviation caused by a Prism.**—When a glass, possessing a prismatic element, is rotated around its geometrical centre, the base apex plane and the edge of the prism are similarly rotated. If the cross lines of the chart *A B C D* (Fig. 146) be observed they, being deviated towards the edge of the prism, move around with the latter, the junction *Z* of the cross lines being deflected towards the edge of the prism. As the glass is rotated the vertical line moves horizontally and the horizontal line moves vertically, but the two always remain at right angles to each other, and do not become distorted as when a cylindrical is rotated. The movement is the same whether the prismatic element be derived from a prism or from decentration.

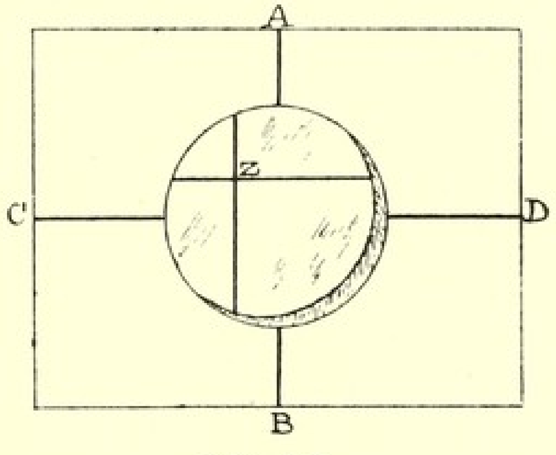



Fig. 146.

Ophthalmic prisms are presumed to be *thin,* i.e., not exceeding, say, 20° principal angle.

### The Notation of Prisms.

**Principal Angle.**—The numeration of prisms according to the principal angle (i.e. its form) is similar to the numeration of lenses according to their curvature. Slight differences cannot be easily recognised, and the true optical effect is not indicated. Two prisms of, say, 3°, the one of $\mu = 1{\cdot}5$, and the other $\mu$ 1·54, are both prisms of 3°, but their optical properties are not the same.

**Angle of Deviation.**—The angle of deviation indicates the true optical value of the prism, it being the result of the angle of inclination of the two refracting surfaces and of the refracting power of the medium, with both of which it varies directly. This system has the drawback that the angle itself is inconvenient to measure in practice. The unit is a prism of 1° deviation (1°d).

**Relationship of the ° and the °d.**—If $\mu = 1{\cdot}5$, $1° = 1/2°$d, but the °d increases with respect to the ° as the $\mu$ is higher. The number of degrees in the deviating angle of a prism being about half that of its principal angle, the °d is practically double the value of the °. Therefore, if two prisms of the same strength be numbered respectively in the two systems, its number in °d would be about half that in °. In the following paragraphs, however, $\mu$ is taken as 1·52, and, therefore, the relative values are slightly less than two to one.

**Prism Diopter.**—This prism notation, introduced by Mr. Charles Prentice of New York, being based on the linear deviation itself, presents many advantages. The unit is the 1Δ, which is the strength of a prism that causes a deviation of 1 cm. (on a tangent) at a distance of 1 metre. The deviation is, therefore, 1 in 100, and $N\Delta/100 = \tan d$. Separate prisms, numbered in prism diopters, when placed together are not, however, exactly equal to the sum of their powers. This is due to the fact that, the deviation being measured on a tangent surface, equal increase in the linear deviation does not result in a corresponding increase of angular deviation. Thus 1Δ is equal to 34′ 22½′ and 10Δ = 5° 45′, but if 10 single 1Δ prisms were combined the total angular deviation, caused by them, would be 34′ 22½″ × 10 = 5° 43′ 45″. The difference is, however, so very inconsiderable, especially in the weak prisms needed in spectacle work, as to be of no practical importance.

The Δ is nearly equal to the ° when the glass has $\mu = 1{\cdot}54$. The 1° has then ·54° deviation or 32′ 12″, and the tangent included by such an angle at 1 M. is ·94 cm. That of the Δ being 1 cm., there is a difference of about 6%. When $\mu = 1{\cdot}54$, the principal angle required to produce 1Δ is 1° 3′. If $\mu = 1{\cdot}575$, the ° = Δ, for ·575° = 34′ 30″, the tangent of which is ·01. When $\mu = 1{\cdot}52$, the ° = ·9Δ, and this is the refractive index of the glass usually employed. It must, however, be remembered that these values can only be considered true for small angles such as occur in the optics of spectacle work.

**Relative Values.**—The relative values of the three units mentioned, in terms of the deviation they cause at 1 M., are, the ° = ·9; the Δ = 1; the °d = 1·745, or say 1·75. Their equivalences are as follows:—

1° = ·52°d = ·9Δ
1Δ = ·57°d = 1·1°
1°d = 1·745Δ = 1·9°

**Calculations in Prism Measurement.**—Calculations with prisms can be made as follows, but for degrees and degrees of deviation, while sufficiently accurate for practical purposes, they are not exact, since tangents of angles are used in place of angles themselves.

Let P represent the power of the prism, M its distance in metres from the object viewed, C the deviation in centimetres, and K a constant for each system of prism notation. Then

$$C = P\,M\,K.$$

| | |
|---|---|
| For the ° | $C = P \times M \times \cdot 9$ |
| For the °d | $C = P \times M \times 1{\cdot}75$ |
| For the Δ | $C = P \times M.$ |

Thus at 3 metres, the deviation caused by a 4°, a 4°d, and a 4Δ respectively is

$$4^\circ \times 3 \times \cdot 9 = 10{\cdot}8 \text{ cm.}$$
$$4^\circ d \times 3 \times 1{\cdot}75 = 21 \text{ cm.}$$
$$4\Delta \times 3 = 12 \text{ cm.}$$

If the deviation caused by a prism at four metres is 5 cm., the prism is

$$\frac{5}{4 \times \cdot 9} = 1{\cdot}4^\circ, \quad \text{or} \quad \frac{5}{4 \times 1{\cdot}75} = \cdot 7^\circ\text{d}, \quad \text{or} \quad \frac{5}{4} = 1{\cdot}25\Delta.$$

The distance at which a prism of 5°, one of 5°d, and one of 5Δ respectively causes a deviation of 15 cm., is

$$\frac{15}{5^\circ \times \cdot 9} = 3{\cdot}33 \text{ M.} \qquad \frac{15}{5^\circ\text{d} \times 1{\cdot}75} = 1{\cdot}75 \text{ M.} \qquad \frac{15}{5\Delta} = 3 \text{ M.}$$

**Prism Nomenclature.**—A prism placed in a spectacle frame with its base towards the nose is termed + or *base in*, while a prism placed with its base towards the temple is termed − or *base out*. A prism is called *horizontal* or *vertical* according as the base-apex line is horizontal or vertical respectively.

**Conversion of Prismatic Values.**—For conversion from one system of prism notation to another it is only necessary to remember the relative linear deviation that each unit produces at 1 M., $\mu$ being 1·52. Thus

$$4^\circ = 4 \times \cdot 9 = 3{\cdot}6\Delta, \text{ or } 4 \times \cdot 9/1{\cdot}75 = 2{\cdot}06^\circ\text{d}$$
$$4\Delta = 4/\cdot 9 = 4{\cdot}44^\circ, \text{ or } 4/1{\cdot}75 = 2{\cdot}28^\circ\text{d}$$
$$4^\circ\text{d} = 4 \times 1{\cdot}75/\cdot 9 = 7{\cdot}77^\circ, \text{ or } 4 \times 1{\cdot}75 = 7\Delta.$$

**Centrad.**—Another prism unit is the centrad, which causes at 1 M. a deviation of 1 cm. on the arc of the circle. The deviation is again 1 in 100, and the difference between the arc and the tangent of small angles being negligible, the centrad and Δ may be considered equal. A given prism numbered in Δ would be of fractionally higher number than if numbered in centrads. The centrad more nearly agrees with the metre angle (which is measured by the sine of the angle) than the prism diopter, because there is less difference in value between the sine and the arc than between the sine and the tangent. It is, however, very much more inconvenient to measure on a curved than on a flat surface, and the centrad has never come into general use. $N\Delta/100 = \text{arc d}$. Calculations for centrads can be taken as the same as for $\Delta$'s.

**The Metran.**—Another unit prism suggested by L. Laurance is the metran. This is a prism which causes a deviation of 3 cm. when placed in front of the eye at one metre from the scale. It has, therefore, about 1·75° (or 1° 45′) deviation, and is the same as the metre angle for the average inter-pupillary distance of 2⅜ in. or 60 mm. The symbol is thus $4 \wedge$.

**False Images of a Prism.**—On looking at a candle flame through a prism a second fainter image can be seen, which is often a source of annoyance to the wearer. This image is formed by internal reflection of some of the light incident on the prism from the flame, and is projected parallel to the base-apex line under an angle about five or six times the deviating angle of the prism, so that in a strong prism it lies too far away to be observed unless specially sought for. In his work "The Clinical Use of Prisms," Dr. E. E. Maddox indicates that it can be utilised for the exact horizontal or vertical adjustment of the base-apex line of weak prisms by noting that the direct and the reflected image are in the same horizontal or vertical plane.

**Measurement of Prisms.**—The measurement of the principal angle of a prism is *goniometry*, that of its deviating angle is *prismetry*.

**Determining the Principal Angle.**—The *principal* angle of a prism can be roughly measured by enclosing it between the legs of a pair of compasses and measuring the angle so obtained on a protractor or by any instrument made for the purpose. A goniometer, consisting of a pivotted arm, at one end of which there are two legs which rest on the face of the prism, serves the same purpose, the other end indicating the angle on a scale. It can also be determined by the *pin* method described further on in Chap. XXVIII., and most accurately of all by the *spectrometer* described in Chap. XXIII. Also, without much error, for weak ophthalmic prisms, the *tangent scale* can be, and is, generally employed.

**Determining the Deviating Angle.**—For this the *spectrometer* method (Chap. XXIII.) is the true one; an approximate *pin* method is described in Chap. XXVIII. The *tangent scale* and *neutralisation* (Chap. XII.) methods are the practical ones for thin prisms.

**The Tangent Scale.**—A tangent scale, shown in Fig. 148, constitutes the most convenient method of measuring ophthalmic prisms. It consists of a card, say, 12 inches wide and 30 inches long, scaled so that the intervals between the divisions represent the tangents of the angles of deviation, and was originally designed by Dr. Maddox. The intervals vary in size with the distance at which the card is used.

The line $A\ C$ (Fig. 147) is looked at through the prism, which is held sufficiently low for the figures on the card to be seen over it, the base being directed towards $A$ while the edge points to $B$. If the line $A\ B$ is displaced upwards or downwards the prism must be rotated in a plane parallel to the card until $A\ B$ is *continuous and seen unbroken through the prism.* The base-apex line being horizontal, the horizontal deviation is greater than with any other position of the prism in a plane parallel to the card, and therefore the number towards which the deviated part of $A\ C$ points indicates the prismatic power of the prism in degrees of deviation. Generally, however, the scale is so arranged as to read prism diopters or degrees. The deviation caused by the prism varies if its position departs from that of the minimum deviation; consequently, when $A\ B$ is unbroken, the prism must be rotated on its axis in order to secure minimum deviation, this being the numerical strength of the prism. Thus, in Fig. 147, the prism is presumed to be in the position of minimum deviation, and the indicated number is 3, but if the edge of the prism were turned either towards or away from the scale, the indicated deviation would be greater than 3.

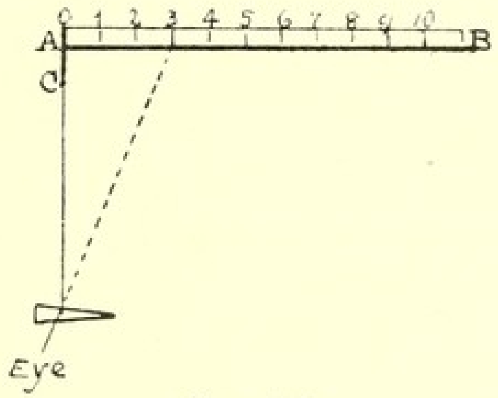



FIG. 147.

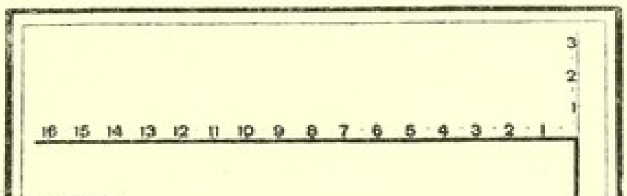



FIG. 148.

If the prism is combined with sph. or cyl. powers, these must be neutralised before the prismatic power can be measured on a tangent scale, *care being taken that the geometrical centres of neutralising and neutralised lenses exactly coincide;* otherwise a false measure of the prismatic power is obtained.

A tangent scale arranged for one system could be utilised for others by holding the prism at the proper distance. Thus, the intervals of the "Orthops" scale (Fig. 148) are 3·5 cm., so that used at 2 M. the numbers indicate degrees of deviation, and at 3·5 M. prism diopters. If used at 4 M. it serves for ordinary degrees.

**Another Tangent Measurement.**—The deviation of a prism can be measured by the following modification of the ordinary tangent scale. Parallel light is passed through a suitable Cx. cyl. and brought to a sharp focus as a vertical line at the zero of a tangent scale. The prism is then introduced, quite close to the cylindrical, with its edge towards the zero and at right angles to the horizontal line; the sharply focussed line of light is then deviated to some number on the scale, which indicates the value of the prism. This method is suggested by Dr. Maddox.

## Oblique Prisms.

**Direction of Deviation.**—A prism so changes the direction of light that an object viewed through it appears in a different position from that which

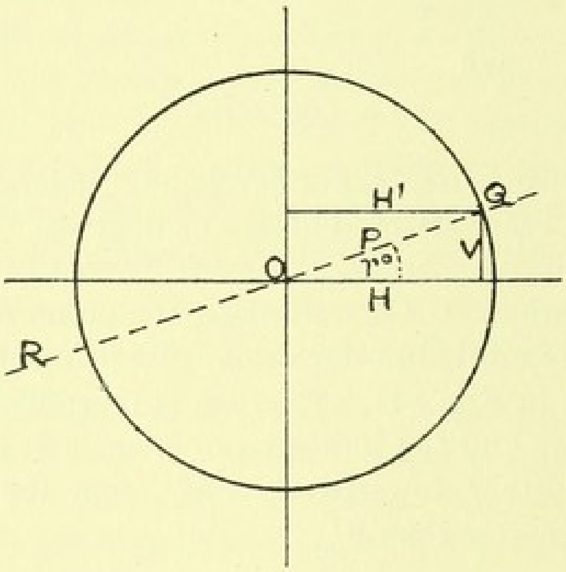



Fig. 149.

it really occupies. The deviation is parallel to the base-apex line and towards the edge of the prism.

If a cross bar be viewed through a prism held with base-apex line horizontal, the vertical bar is displaced horizontally to an extent dependent on the strength of the prism, and there is no vertical displacement of the horizontal bar. If, now, the prism be rotated a few degrees in a plane parallel to the card, so that the base-apex line is oblique to both bars, the horizontal deviation becomes less, and a vertical deviation is introduced (Fig. 149). If the rotation be continued, the horizontal deviation continues to decrease and the vertical to increase, until when the base-apex line is vertical all the deviation is vertical, and there is none in the horizontal plane. The maximum effect $d$ of the prism is always in the base-apex plane, and when the latter is oblique, its effect can be divided into V, a vertical, and H, a horizontal component, which are equal when the base-apex line is at 45°.

**Indirect Effects.**—Suppose V to represent the vertical and H = H′ the horizontal forces of a rotated prism. Let P = O Q represent the power of the prism, and $r$ the angular rotation of its base-apex line from the horizontal. Then, since $\sin r = V/P$ and $\cos r = H/P$, $V = P \sin r$ and $H = P \cos r$. Thus let the base-apex line of a 5°d prism be at 20° from the horizontal; then $V = 5 \times {\cdot}3420 = 1{\cdot}71°$, and $H = 5 \times {\cdot}9397 = 4{\cdot}698°$. If the base-apex line is at 45°, a 6Δ has $V = 6 \times {\cdot}7071 = 4{\cdot}24\Delta$ and $H = 6 \times {\cdot}7071 = 4{\cdot}24\Delta$.

Given a 4°d prism, the position of the base-apex line so that the vertical effect be 1°d is $\sin r = 1/4 = {\cdot}25 = \sin 14° \; 29'$ from the horizontal. Then $V = 4 \times {\cdot}25 = 1°$d and $H = 4 \times {\cdot}9681 = 3{\cdot}872°$d. If with a 6Δ a horizontal effect of 3Δ is needed, $\cos r = 3/6 = {\cdot}5 = \cos 60°$, so that the base-apex line must be at 60°, V being ${\cdot}6 \times {\cdot}866 = 5{\cdot}2\Delta$.

If, instead of the angular distance of the base-apex line from the horizontal, its distance from the vertical is considered, the sine would apply to the horizontal, and the cosine to the vertical meridian in these calculations.

If P′ represent the effect of a prism in a given meridian, P the power of the prism, and $r$ the angle between the given meridian and the base-apex line, the effect in the given meridian is $P' = P \cos r$. Thus to find the effect at 40° of a 4° prism whose base-apex line is vertical, $P' = 4 \times {\cdot}6427 = 2{\cdot}57°$, $r$ being 50°, the cosine of which is ·6427.

**Neutralisation.**—The vertical or horizontal effects of an oblique prism, or the effect in any oblique meridian of a vertical or horizontal prism, can also be obtained by direct neutralisation in the meridian whose power has to be learnt.

Following are the approximate powers of unit prism at different Mers. calculated as shown above.

| Degrees from base-apex line | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Proportional power .. | 1 | ·99 | ·98 | ·97 | ·94 | ·91 | ·87 | ·82 | ·77 | ·71 | ·64 | ·57 | ·5 | ·44 | ·34 | ·26 | ·17 | ·09 | 0 |

## Resultant Prisms.

A resultant prism is the combined effect of two prisms whose base-apex lines are oblique, or at right angles to, each other; the term can also be applied to any number of prisms with base apex-lines in various directions, since the combined effect is always that of some single prism.

In Fig. 150 let *A B* and *A C* represent the deviations caused by two prisms $P_1$ and $P_2$ whose base-apex lines are crossed at the angle $a$. To construct graphically the resultant deviation we have only to complete the parallelogram *A B C D* by drawing *C D* equal and parallel to *A B*, and *B D* equal and parallel to *A C*. Then *A D* is the resultant deviation, and $r$ is the angle it makes with the horizontal. If a third prism $P_3$ were now introduced, a similar construction between *A D* and $P_3$ would give the single resultant of the three prisms $P_1$, $P_2$ and $P_3$, and so on for any further number.

**Calculation of Resultant Prism.**—By means of the cosine formula in trigonometry, it can be proved that $AD^2 = AB^2 + AC^2 + 2AB \cdot AC \cos a$. But AD is the resultant prism P, and AB and AC the original prisms $P_1$ and $P_2$ respectively. Therefore

$$P = \sqrt{P_1^2 + P_2^2 + 2P_1P_2 \cos a}$$

and

$$\tan r = \frac{P_2 \sin a}{P_1 + P_2 \cos a}.$$

Suppose two prisms of 6° and 8° respectively whose base-apex lines are 30° apart, we find

$$P = \sqrt{6^2 + 8^2 + 2 \times 6 \times 8 \times \cdot 866}$$

$$= \sqrt{36 + 64 + 83 \cdot 136} = \sqrt{183 \cdot 136} = 13 \cdot 53°.$$

and

$$\tan r = \frac{8 \times \cdot 5}{6 + (8 \times \cdot 866)} = \cdot 3091 = \tan 17° \; 11'$$

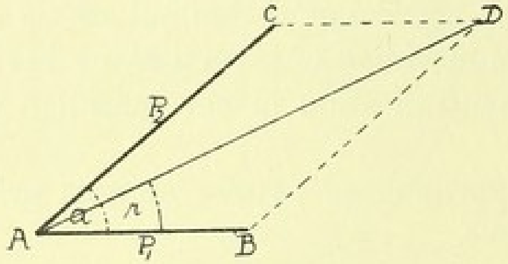



FIG. 150.

The resultant prism is 13·53°, and its base-apex line is 17° 11′ from that of the 6° prism.

When $P_1 = P_2$ the formulæ simplify to $P = (P_1 + P_2) \cos a/2$, and $r = a/2$.

**Prisms at Right Angles.**—It is, however, rare that mutually oblique prisms are required; in the great majority of cases the components are vertical and horizontal. When such is the case $a = 90°$, so that, since $\sin 90° = 1$, and $\cos 90° = 0$, the formulæ simplify to

$$P = \sqrt{V^2 + H^2}$$

and

$$\tan r = V/H$$

Or, with a reasonable degree of accuracy, the resultant base-apex line may be found by dividing 90° by the sum of V and H, and multiplying the result by the weaker of the two original figures. This gives the angular distance from the stronger of the original prisms.

Thus, suppose a 3°d base-apex line horizontal, and 2°d base-apex line vertical be required, then

$$P = \sqrt{3^2 + 2^2} = \sqrt{13} = 3{\cdot}6°d$$

$$\tan r = V/H = 2/3 = {\cdot}666 = \tan 33° 40' \text{ so that } r = 33° 40'$$

The resultant prism needed is 3·6°d (3° 36′) with its base-apex line inclined 33° 40′ from the horizontal, or 56° 20′ from the vertical; that is, approximately, 3·5°d base-apex line at 35°. Or by the simplified method the resultant base-apex line would have been found $90/(3+2) = 18$, and $18 \times 2 = 36°$ from the 3° prism, or $18 \times 3 = 54°$ from the 2° prism.

**Construction.**—To construct graphically the resultant of two prisms at right angles draw a straight vertical line $V$ (Fig. 151) as many inches (or cm.) long as there are units (degrees, etc.) in the vertical prism, and a horizontal line $H$ as many inches (or cm.) long as there are degrees in the horizontal prism. The ends of these two lines being connected by a third line, $P$, the number of inches (or cm.) in $P$ represents the number of degrees

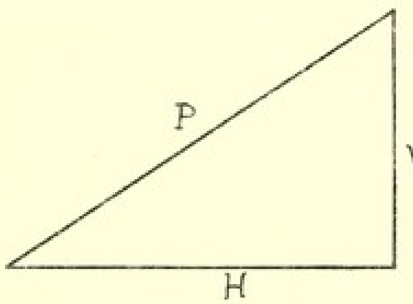



FIG. 151.

in the resultant prism; and the inclination of $P$ with respect to $H$ and $V$, measured by a protractor, is the inclination of the base-apex line of the resultant prism in its relation to the horizontal and vertical.

**Practical Measurement.**—A resultant prism can be found by holding the horizontal and vertical prisms together and finding on a tangent scale the maximum oblique deviation, that is, the value of the resultant prism. Or the two original prisms can be put into a trial frame and neutralised by a single prism from the trial case; the power of the neutraliser is that of the resultant prism, and its inclination is at once indicated.

**Rotary Prism.**—A rotary prism consists of two vertical prisms of equal power conveniently mounted. In the primary position the base of the one coincides with the edge of the other, so that the effect is 0. From this position they are rotated towards the horizontal, so that their bases approach each other; thus a gradually increasing horizontal effect is obtained while the vertical effect always remains 0. The maximum effect is obtained when the two bases coincide in the horizontal meridian. If the primary position is horizontal a similar vertical effect is obtained by rotation.

# CHAPTER XV

## DECENTRATION

### Prismatic Effect of Lenses.

**Optical Centre.**—The optical centre of a sph. lens lies, as mentioned previously, on the principal axis at a distance from each surface proportional to its radius of curvature. It is situated, therefore, on the line passing through the thickest part of a convex and the thinnest part of a concave lens, and is that point through which the secondary axes pass.

**Geometrical Centre.**—The geometrical centre is that point of the lens which is equi-distant from the opposite edges. It can be located by inspection, or, more exactly, by drawing a horizontal line across the lens, connecting the two extremities of the long diameter, and a vertical line connecting

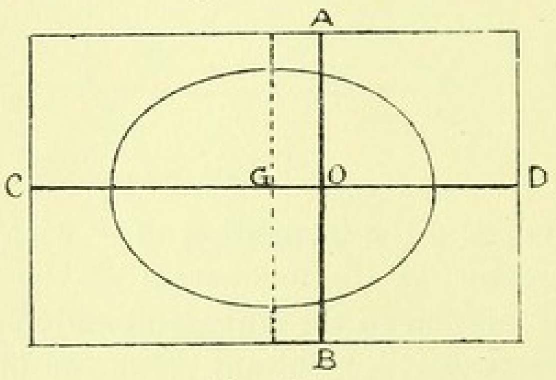



Fig. 152.

the highest and lowest points; where the two lines cut each other is the geometrical centre.

**Locating the O. C.**—To locate the optical centre the lens must be moved about until cross lines seen through it are continuous with the parts of the lines seen beyond the edges, as in Fig. 152, where *G* is the geometrical centre of the lens. The optical centre *O* coincides with that point of the lens opposite to the intersection of the cross lines, and can be, if necessary, marked by a dot with a grease pencil or pen and ink. The test should be made with fine cross lines drawn on a small card placed on the table, the lens being held steadily a short distance above the card, and in a plane parallel to it. This method is preferable for strong lenses, but the analysing card at a reasonable distance is better for a very weak lens. Accuracy is enhanced by employing a pinhole, through which the observation is made, when the near test card is used.

The same procedure is employed for a sph.-cyl., but the principal meridians *must be parallel with the lines of the card.* With a plano-cyl., there being only a line of no prismatic effect, and, therefore, no optical centre, the central point of the axis may be regarded as such, when the cross lines are seen unbroken.

**Centered and Decentered Lenses.**—A lens is said to be centered when its optical and geometrical centres coincide, and is said to be decentered when they do not. When an object is viewed through the geometrical centre of a decentered lens the effect is precisely the same as if the lens were combined with a prism. Similarly, if a centered lens is looked through at a point which is not in line with the optical and geometrical centres the effect is the same as if a sphero-prism were substituted.

To learn whether a spherical lens is truly centered it must be held parallel to the analysing card and viewed through its geometrical centre. If centered (Fig. 153) the junction of the two lines of the card is seen in line

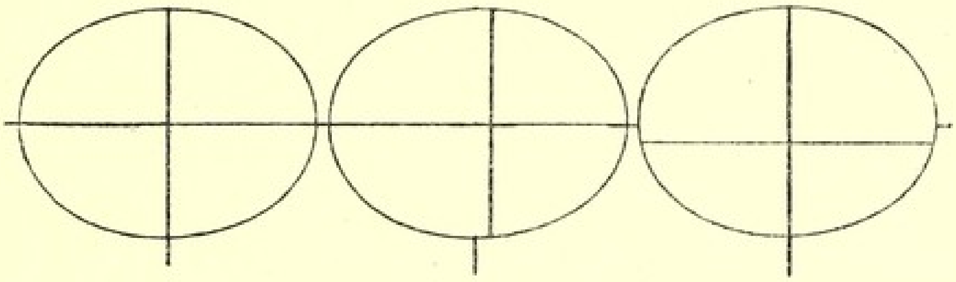


FIG. 153. FIG. 154. FIG. 155.

with the exact centre of the lens, the lines being continuous beyond the edges as in Fig. 153. If decentered the junction of the two lines is seen not to coincide with the exact centre of the lens and the vertical line, as in Fig. 154, or the horizontal line as in Fig. 155, is broken at the edges of the lens, or both are broken.

In a sph. lens there is only one *point*, i.e. the optical centre in the refracting plane of the lens where there is no prismatic effect. In a plano-cylindrical there is a *line* without prismatic effect along the axis.

In Fig. 156 let the lens be a + cyl., whose axis *A X* is at 45°, *B C* being a vertical, and *D E* a horizontal line. On looking through the lens the points *F G H* on the vertical line *B C* are seen deflected by the prismatic action of the cyl. to *F′G′H′*, upwards and to the left, the virtual prisms being base down and to the right. The points *K L M* on the horizontal line *DE* are seen deflected to *K′L′M′*, also upwards and to the left, the virtual prisms being base down and to the right. On the other side of the axis the virtual prisms are base up and to the left, and the deflections are downwards and to the right. Thus a convex cylindrical axis, say, 45°, causes a vertical line

*BC* to appear as *B′C′*, and a horizontal line *DE* to appear as *D′E′*, both being deviated away from the axis.

If another equal + cyl. be placed axis at right angles to the first, the horizontal deviation of the vertical line, and the vertical deviation of the horizontal, are neutralised, but the vertical effect in the vertical meridian

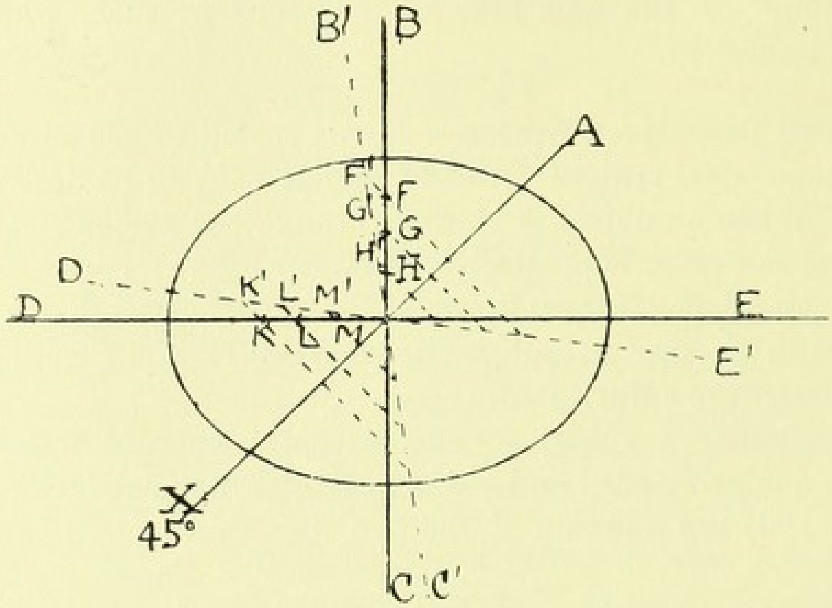



Fig. 156.

and the horizontal effect in the horizontal are doubled, the combination being equivalent to a sph. lens in which the prismatic effects are equal in every meridian.

With a concave cyl. the edges of the virtual prisms are towards the axis,

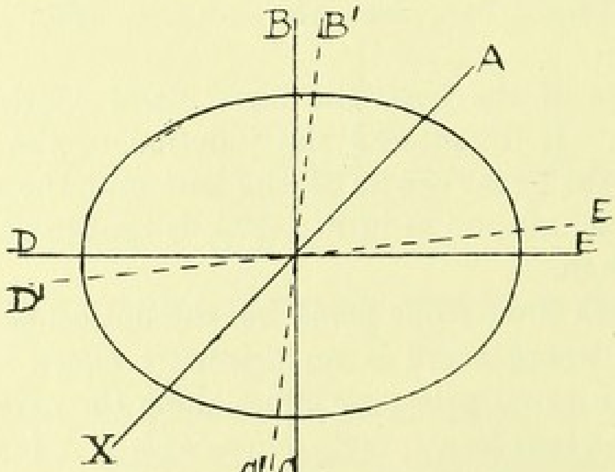



Fig. 157.

and if a − cyl. Ax. 45° be looked through (Fig. 157) a vertical line *B C* appears as *B′ C′*, and a horizontal line *D E* appears as *D′ E′*, the deviation of these lines being towards the axis of the lens, or towards the apices of the virtual prisms.

In a sph.-cyl. lens there is (as in the case of a sph.) a point of no prismatic effect. This is where the axis of the cylindrical cuts that of the spherical, and it is therefore at the geometrical centre of a centered lens.

In Fig. 158 let the lens be a + sph.-cyl., whose axis *A X* is at 45°. Let *B* be a point situated between the vertical and the axis. There is, at this point, the effect *O B* of a prism base down to the left derived from the sph. The cyl. contributes a prismatic effect *P B*, the base of the virtual prism being down to the right. Thus there are two vertical effects both

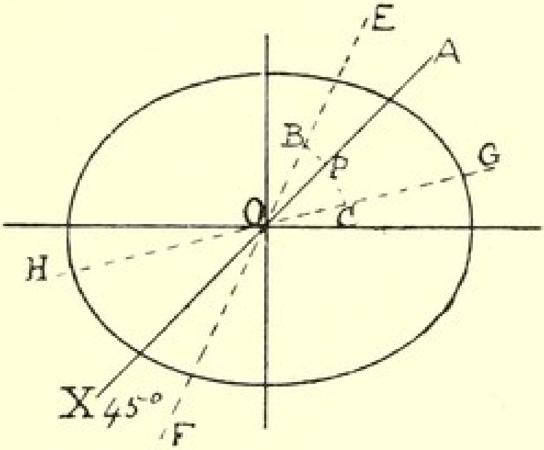



Fig. 158.

directed upwards, and two horizontal, the one directed to the left and the other to the right. These latter neutralise each other at some point *B*, and similarly at every point on the line *E F*.

Between the axis and the horizontal, at some point *C*, there is the effect *O C* of a prism base down and to the left derived from the sph., and from the cyl. there is the effect *P C* of a prism base up and to the left. There are

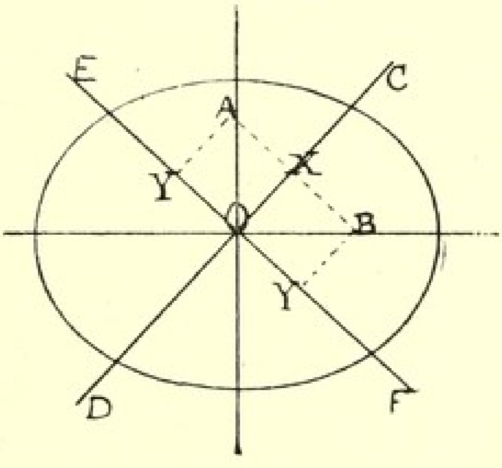



Fig. 159.

thus two horizontal effects both directed to the left, and two vertical, the one up and the other down. At some point *C* the opposing vertical effects neutralise each other, and similarly we have a neutralising effect all along the line *G H*.

In a Cc. sph.-cyl. there are similar prismatic effects, but in the opposite directions.

Let Fig. 159 be a combination of + Cyl. Ax. 45° ⊂⊃ − Cyl. Ax. 135°, the two being of equal power. *C D* is the axis of the convex cyl., and *E F* that of the concave. At some point *A* the convex cyl. has an effect *XA* of a prism base down and to the right, the concave has an effect *YA* of one base up and to the right. The up and down vertical effects neutralise each other, but there is a combined lateral effect. At the point *B* the convex acts with an effect *X B* base up and to the left, and the concave with an effect *Y B* base up and to the right. The right and left horizontal effects neutralising each other, the combined deviation being vertical. Thus the point *A* is deviated to the left, and *B* is deviated downwards. A vertical line is seen inclined to the left above, and to the right below ; a horizontal line is inclined downwards on the right, and upwards on the left.

**Locating the Lines of No Prism Effect.**—If an oblique sph.-cyl. be moved horizontally until the oblique image of a vertical line is seen in contact at *B* (Fig. 160), at the upper edge of the lens, with the line itself seen above the lens, and similar contact is then obtained at the lower edge of the lens, say at *C*, the line connecting these two contact points indicates the line of no

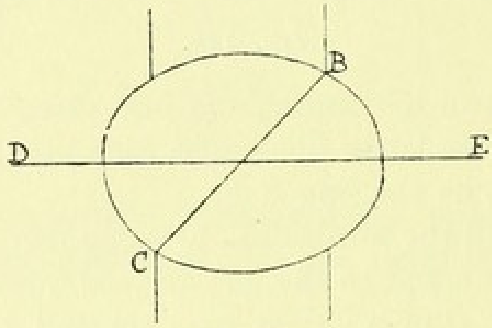



FIG. 160.

horizontal prismatic effect. Similarly the points can be found where, by moving the lens vertically upwards and downwards, a horizontal bar is, at each side, in contact with its image ; the line connecting them indicates the line of no vertical prismatic effect.

## The Decentering of Spherical Lenses.

Prismatic effect can be obtained by decentering a lens as well as by combining a prism with it. The prismatic effect thus obtained by decentration is called a *virtual prism*.

**How to Decenter.**—Decentering is achieved by so cutting the unedged glass disc that the optical centre is nearer than the geometrical centre to one part of the edge of the finished lens. Thus in Fig. 161 *G* is the geometrical centre of the finished lens, and *O*, the optical centre, lies nearer the right edge. In a centered lens *O* and *G* coincide. To decenter a lens, the optical centre *O* is located, as previously described, by means of a card having two fine cross lines. *O* is marked by a dot, the amount of decentering is measuring off, and the point which is to be the geometrical centre of the edged lens is marked as *G*. The lens is then cut out so that *G* is the geometrical centre, the distance *O G* being the decentration of the lens.

It should be observed that the decentration is indicated by the *position of O, the fixed point of a lens, with reference to G.* To achieve this the distance is marked off *contrary to the decentration required*, so that *G* is actually altered. If *O* has to be *in*, the distance measured off, in order to mark where *G* is to be, is *out* from *O*.

Another method is to place in contact with the lens a prism, equal to the effect required, with its base in the opposite direction; mark the optical centre as then found, and this point is the geometrical centre needed. By

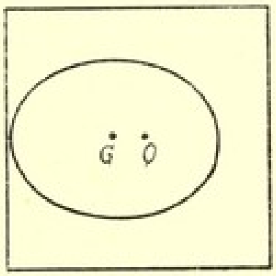



Fig. 161.

this method one can at once see whether the required effect can be obtained by decentering, it not being possible if the marked point be too near the periphery, much less if, as may be the case, it is beyond the lens. It is specially suitable for oblique decentrations and for oblique sph.-cyls. Thus suppose 1△ *base in* effect is needed on a +4 D lens; a 1△ *base out* is placed with the lens, the O. C. is then displaced outwards, the lens must be shifted inwards to find the point of no prism effect, and this being marked indicates where the geometrical centre of the finished lens must be.

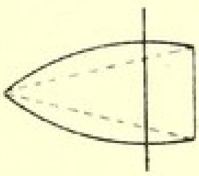


Fig. 162.

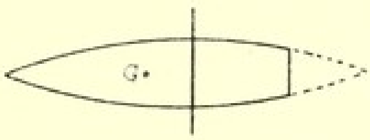



Fig. 163.

**To Measure Decentration.**—To measure the decentration of a lens, the geometrical centre must be marked with a fine dot, and the optical centre found and similarly marked; the distance between them is the decentration. This distance can be measured by placing the lens on a metric rule.

**Sphero-Prism and Decentered Lenses.**—When a prism is combined with a sph., the curved surfaces are inclined towards each other at an angle (Fig. 162), just as if the lens had been split and a prism inserted. If from a large lens (Fig. 163) one part be cut away, the effect is the same; in both figures the principal axis of the lens is shown by the thick vertical line.

The two surfaces of a lens are inclined towards each at an angle which increases from the centre to the periphery, although the curvature remains the same; therefore the effect produced on a ray of light by the outer zones of a lens is as if a prism of given power were used, *the prismatic power being greater as the part of the lens, through which the ray passes, is more distant from the axis.*

A ray of light, *A B* (Fig. 164), passes through the optical centre *O* of a lens *L*, and through the prism *P*; it is undeviated by the lens, but is bent by the prism towards its base in a direction to the right in the diagram. If the prism be removed (Fig. 165) and the convex lens decentered to the

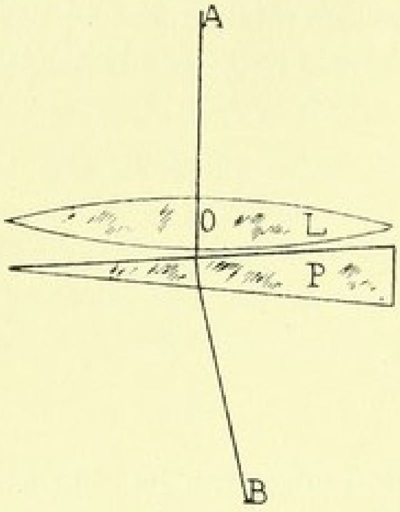



Fig. 164.

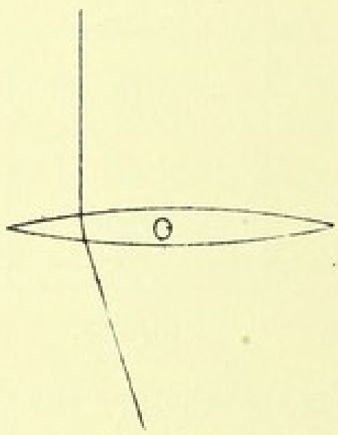



Fig. 165.

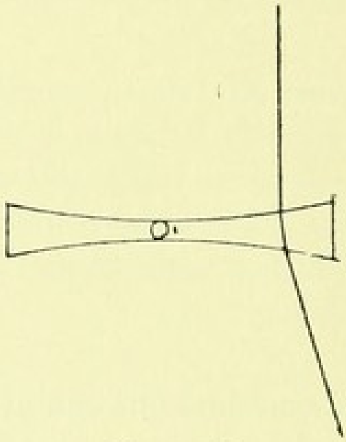



Fig. 166.

right, the ray *A B* is bent in the same way as if it had passed through *O* and a prism. If a concave lens (Fig. 166) be decentered to the left the same effect is obtained. Similarly all the rays contained in a beam of light, parallel to the axis, and refracted by a spherical lens, are bent towards, or away from, the axis to an extent dependent on the distance from the axis of that part of the lens through which *each* ray passes. Thus all the rays parallel to the axis before refraction meet, after refraction (disregarding aberration) on the axis at a single point.

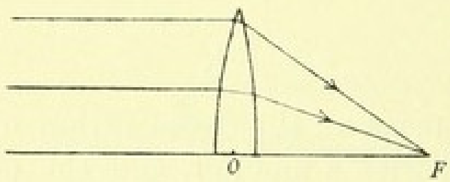



Fig. 167.

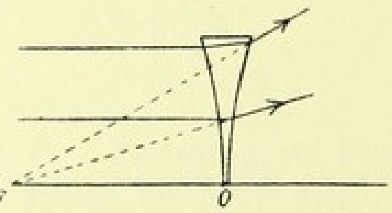



Fig. 168.

If the principal axis passes through the *geometrical* centre of the lens, the rays, after refraction, converge towards, or diverge from, a point (F) on a straight line drawn from the luminous point through the geometrical centre of the lens.

But if *O*, the optical centre of the lens, is displaced, the rays are not only rendered convergent or divergent, but are also bent towards or away from the displaced axis (Figs. 167, 168) in the same manner as they would be if a prism had been added to the lens.

**Direction of Decentration.**—To produce the effect of a prism with its base in a certain direction, a convex lens must be decentered in *that same direction*, and a concave in the *opposite direction.*

**Effect Produced by Decentering.**—The prismatic power obtained by decentration is directly proportional to the amount of decentration and to the strength of the lens, so that the decentration necessary to obtain a desired prismatic effect is directly proportional to the effect required, and inversely proportional to the power of the lens. The calculation may be made in any of the systems of prism notation, but the prismatic effect of decentering lenses can best be illustrated in connection with prism diopters.

In Fig. 169 let *S S* be a screen situated at the focal distance of a + 1 D lens ; the distance *O F* is therefore one metre. All rays of light as *A B*, *C D*, parallel to the axis are bent so as to meet at *F*. The ray *A B* incident at *B* situated, say, 1 cm. from the axis, instead of falling on the screen at *B′*, as it would if it were unrefracted, cuts the axis at *F*. Consequently, the ray is deviated the distance $B'F = B\,O = 1$ cm. at 1 M. The ray *C D* incident, say, 2 cm. from the axis, is deviated the distance $D'F = D\,O = 2$ cm. at 1 M since

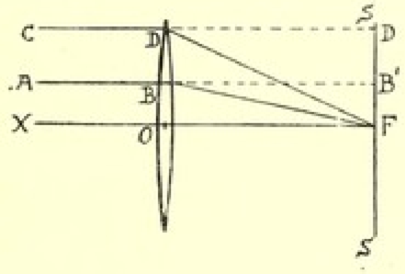



FIG. 169.

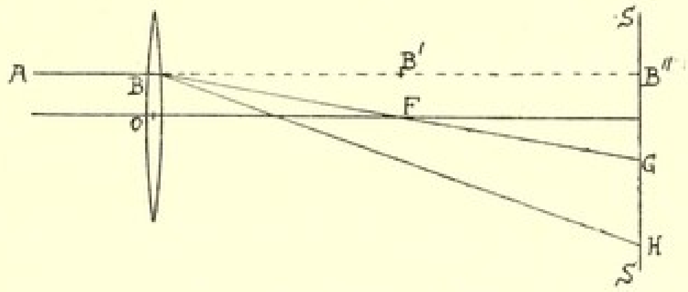



FIG. 170.

it also meets the axis at *F*. The effect of placing the geometrical centre of the lens at *B* or at *D* would, therefore, be the same as having the lens normally centered and combined with a prism of 1△ or 2△ respectively, since these prisms also have the effect of deviating a ray 1 cm. or 2 cm. respectively at a distance of one metre.

In Fig. 170 the lens is a + 2 D and *S S*, the screen, is 1 M from it ; a ray *A B*, parallel to the axis, and incident at *B*, 1 cm. from the axis, meets the latter at *F*, 50 cm. from the lens and the screen at *G* instead of at *B″*. The ray is deviated a distance $B'F = 1$ cm. at 50 cm., and $B''G = 2$ cm. at 1 M, so that the prismatic effect is the same as that of a 2△ acting on a ray unrefracted by the lens. If the lens were + 4 D, *A B* would meet the axis at 25 cm. from the lens and would be there deviated 1 cm., while at 1 M it would be deviated $B''H = 4$ cm. and have the effect of 4△. The + 2 D at a point 2 cm. from the axis has the effect of 4△, while the 4 D at the same point has the effect of an added 8△.

**Limitations to Decentering.**—The smaller the size of the lens required and the larger the disc from which it is cut, the greater is the extent of decentration possible with the ordinary disc used in the trade. If the edged lens were nearly as large as the unedged disc no decentering would be possible. Since the usual finished lens is longer in the horizontal than in the vertical diameter, a greater vertical than horizontal decentration is possible. Thus a lens which must be of No. 1 eye size when edged can be decentered about 4 mm. horizontally and about 7 mm. vertically; for 2 or 3 eye lenses the extents are greater, while for 0 and 00 eyes they are smaller. The average size of the uncut disc is 40 mm. square.

**Formulæ for Decentration.**—Since a +1 D lens decentered 1 cm. has the effect of 1△, the formulæ for calculating decentrations and their effect are extremely simple when the prismatic power is expressed in prism diopters. Let P represent the prismatic effect needed, D the dioptral number of the lens, and C the decentration *in centimetres,* then

$$P = D\,C \qquad \text{and} \qquad C = P/D.$$

Thus if, on a 4·5 D lens, the effect of 2△ is required, the lens must be decentered

$$C = 2/4{\cdot}5 = {\cdot}444 \text{ cm.}$$

If a 4 D lens is decentered ·75 cm. the prismatic effect in △ is

$$P = {\cdot}75 \times 4 = 3^{\triangle}.$$

If the optical centre of the lens is found to be ·75 cm. from the geometrical centre, and the prismatic effect, as measured on a tangent scale, is 3△, the lens is

$$D = 3/{\cdot}75 = 4 \text{ D.}$$

Two similar lenses decentered with respect to each other, have no prismatic effect; the one is base out, the other is base in. Two lenses which neutralise, if slid one over the other, have a prismatic effect introduced thereby, the Cx. being moved the one way, and the Cc. the other. If C be the amount of the slide, D C = △, or △/C = D, where D is the power of the one lens, the base of the virtual prism being towards the Cx. lens. In this case the effect is doubled.

By introducing the necessary constant K, the formulæ for prism diopters apply also for degrees, whose constant is ·94 when $\mu = 1{\cdot}54$, is ·9 when $\mu = 1{\cdot}52$ and ·87 when $\mu = 1{\cdot}5$. For degrees of deviation the constant is 1·745; or with a sufficient degree of accuracy 1·75, so that

$$P = D\,C/K$$

Suppose the effect of 2°d on an 8 D lens is required, the decentration will be

$$C = 2 \times 1{\cdot}75/8 = {\cdot}44 \text{ cm.}$$

If on a 5 D lens the effect of 3·5° is required, it must be decentered

$$C = 3{\cdot}5 \times {\cdot}9/5 = {\cdot}63 \text{ cm.}$$

A 4 D is decentered ·75 cm., the prismatic effect will then be

$$P = {\cdot}75 \times 4/1{\cdot}75 = 1{\cdot}75^\circ d, \qquad \text{or} \qquad {\cdot}75 \times 4/{\cdot}9 = 3{\cdot}3^\circ.$$

In using these formulæ, parts of degrees should be expressed as decimals, and not as minutes and seconds, and the decentration in cm. and decimals thereof. These rules, while sufficiently accurate for practical spectacle work, especially as no lens can be decentered to a very great extent, are not exact since the variation of angles have been taken as equivalent to that of their tangents.

**Formulæ Involving F.**—Where F or 1/F is given, it is easier, for calculating decentrations, to convert F into diopters, but the calculations can be made by the following formula, where both F and the decentration are expressed in inches or cm., K being the constant—

$$\text{Decentration} = PFK/100$$

Thus, how much should a 4 in. lens be decentered for 1°?

$$C = 1 \times 4 \times {\cdot}9/100 = {\cdot}036 \text{ in.}$$

**More Exact Formulæ.**—More accurate formulæ for decentration for degrees of deviation are as follows, where F and D have the usual significations, and P is the degree of deviation—

$$\tan P = C/F \qquad \text{or} \qquad C = F \tan P;$$

and

$$\tan P = C\,D/100 \qquad \text{or} \qquad C = 100 \tan P/D.$$

These formulæ are illustrated in Fig. 169, where *D O* is the tangent of the angle of deviation of the ray *C D*.

### Resultant Decentrations.

The value of a prism in any meridian is P cos $r$, where $r$ is the angle between the base-apex line and the meridian in question. Similarly the prismatic value of a decentration is at its maximum along the line of decentration, and its value at any other meridian is D C cos $r$. Therefore, if a lens had to be decentered for both vertical and horizontal prismatic effects, each may be made separately or the two obtained by a single oblique decentration; put in another way, a lens decentered obliquely causes a vertical and horizontal prismatic effect equal to D C cos $r$, where $r$ is the angular distance between the direction of decentration and the horizontal or vertical.

In Fig. 171 let $O$ be the optical centre. If the geometrical centre is moved horizontally from $O$ to $h$ and vertically to $v$, the true displacement is along $O\,v$. A resultant decentration is calculated by finding, by the formulæ previously given, the oblique prismatic effect required and its angle $r$, and then decentering accordingly.

Thus suppose a + 5 D lens has to be decentered for a horizontal effect of $2^{\triangle}$, and a vertical effect of $1{\cdot}5^{\triangle}$; then

$$P = \sqrt{2^2 + 1{\cdot}5^2} = 2{\cdot}5^{\triangle}$$

$$\tan r = 1{\cdot}5/2 = {\cdot}75 = \tan 36^\circ\ 52'.$$

$$\text{and } C = 2{\cdot}5/5 = {\cdot}5 \text{ cm.}$$

The two needed prismatic effects are obtained by decentering the lens ·5 cm. along meridian, say, 37°.

The Ver. and Hor. decentration V and H could be found separately and a resultant decentration then calculated, but the above method is more

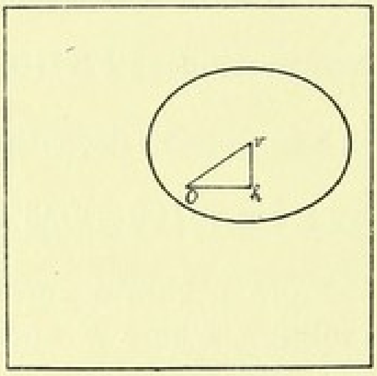



Fig. 171.

simple. Thus, in the above example, H = ·4 cm., V = ·3 cm., and $\sqrt{{\cdot}4^2 + {\cdot}3^2} = {\cdot}5$ cm., the angle $r$ being found as shown above.

Instead of finding tan $r$, the direction can be obtained without serious error, for small values, by dividing 90° proportionally to the two needed decentrations, in this case 4 and 3. We should find $90/(4+3) = 12{\cdot}8$ and $12{\cdot}8 \times 3 = 38{\cdot}5^\circ$ from the stronger of the two original decentrations—that is, about 38° from the horizontal.

**The Effects of Oblique Decentering.**—When a sph. is decentered obliquely we have, in the meridian of decentration, P = D C, and since $O\,h = \cos r$, and $V\,h = \sin r$, the Hor. effect H, and the Ver. effect V, of an oblique decentration P, are found by the equations

$$H = P \cos r \qquad \text{and} \qquad V = P \sin r.$$

But P = D C, where D is the dioptral number of the lens, and C the decentration in cm.; therefore

$$H = D\,C \cos r \qquad \text{and} \qquad V = D\,C \sin r.$$

Thus, if a $+7$ D sph. be decentered ·6 cm. at 30°, $P = 7 \times \cdot 6 = 4\cdot 2^{\triangle}$, $H = 7 \times \cdot 6 \times \cdot 866 = 3\cdot 637^{\triangle}$, and $V = 7 \times \cdot 6 \times \cdot 5 = 2\cdot 1^{\triangle}$.

The necessary constants can be introduced into the formulæ when the prismatic effects are required in degrees or degrees of deviation, or the effects in $^{\triangle}$ can be converted by the usual methods.

### The Decentration of Cylindricals.

In the following article, for the sake of brevity, we will term *upright* those cyls. and sph.-cyls. whose principal Mers. are Ver. and Hor., in contradistinction to those which are *oblique*.

A lens which possesses a cyl. element should not be decentered except in its principal meridians, that is to say, upright cyls. ought never to be decentered obliquely, nor should oblique cyls. be decentered horizontally or vertically. However, as will be shown later on, such decentrations can be made, but the results are difficult to calculate owing to the fact that the virtual prisms in a cyl. have their base-apex lines at right angles to the axis. The reason why sphs. are so easy to decenter is because the base-apex lines of all the virtual prisms radiate from the optical centre, so that any possible decentration must always lie in a virtual base-apex plane.

**Upright Cyls.**—The effect of decentering a cyl. *across* its axis is the same as decentering a sph. in that direction; *along* the axis there is, of course, no effect, since there is no refractive power. Thus a cyl. axis Ver. can be decentered horizontally, but not vertically; a cyl. axis Hor. can only be decentered vertically.

If $+4$ C. Ax. 90° requires decentration for a horizontal prismatic effect of $2^{\triangle}$, $C = 2/4 = \cdot 5$ cm.

**Oblique Cyls.**—The decentration of an oblique cyl. *along* the axis has no effect, while *across* the axis the effect is, in the principal meridians, the same as with the Ver. and Hor. decentration of upright cyls. Thus if a 4 cyl. Ax. 60° be decentered 4 mm. at 150°, the principal effect P along 150° = D C, that is, $P = 4 \times \cdot 4 = 1\cdot 6^{\triangle}$. This, however, produces Hor. and Ver. effects, because any single oblique displacement can always be resolved into two components at right angles.

If a plano-cyl. axis oblique be moved horizontally, vertically or in any oblique direction in front of the eye, *any object viewed through it will appear to move in a direction across the axis*, thus showing that the resultant effect is always at right angles to the axis, or in the meridian of maximum power. Indeed this result is only to be expected, since the virtual prisms in a cyl. lie only in one direction with their base-apex lines across the axis. Therefore no matter what oblique decentration be made to a cyl., the resultant effect is as though a smaller decentration had been made in the principal power at right angles to the axis.

In Fig. 172 let $x\,y$ be the axis of the cyl. at an angle $a$ with the horizontal, and let $x'y'$ be its position when the lens is decentered from $o$ to $r$ in the meridian of maximum refraction. We can resolve the single displacement $o\,r$ into two components $o\,d$ and $o\,e = d\,r$ lying in the Hor. and Ver. planes respectively, and these expressed in terms of the single decentration $o\,r$ will enable us to find the resulting Hor. and Ver. prismatic effects H and V.

Now $o\,r\,d = a$. $\therefore\ o\,d = o\,r \sin a$, and $d\,r = o\,r \cos a$.

Let C represent the distance $o\,r$ and D the maximum power of the cyl. Then

$$P = D\,C,\ H = D\,C \sin a,\ V = D\,C \cos a.$$

Precisely similar prismatic effects are obtained if the lens is decentered horizontally to $h$ or vertically to $v$, but the maximum effect remains in the dir3ction $o\,r$, and can then be considered the resultant effect of the Hor. and

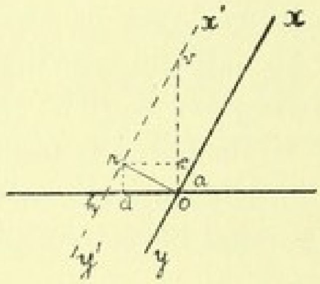



Fig. 172.

Vert. components $o\,d$ and $o\,e$ with the base-apex line of the virtual prism parallel to $o\,r$.

Now $o\,r = o\,h \sin a = o\,v \cos a$, so that if the decentration of an oblique cyl. is Hor. we have to write C sin $a$ instead of C, and then

$$P = D\,C \sin a,\ H = D\,C \sin^2 a,\ V = D\,C \cos a \sin a.$$

If the decentering is Ver., C cos $a$ replaces C, so that

$$P = D\,C \cos a,\ V = D\,C \cos^2 a,\ H = D\,C \sin a \cos a.$$

As an example, let the lens be +4 D. axis 60° decentered along $o\,r = C = \cdot 4$ cm. at 150°, as in the example already given; then

$$P = 4 \times \cdot 4 = 1 \cdot 6^{\Delta}$$

$$H = 4 \times \cdot 4 \times \cdot 866 = 1 \cdot 386^{\Delta},\ V = 4 \times \cdot 4 \times \cdot 5 = \cdot 8^{\Delta}$$

$$o\,h = o\,r/\sin 60° = 4/\cdot 866 = \cdot 462 \text{ cm. } o\,v = o\,r/\cos 60° = 4/\cdot 5 = \cdot 8 \text{ cm.}$$

Then if a +4 C. Ax. 60° is decentered ·462 cm. horizontally,

$$P = 4 \times \cdot 462 \times \cdot 866 = 1 \cdot 6^{\Delta}$$

$$H = 4 \times \cdot 462 \times \cdot 75 = 1 \cdot 386^{\Delta},\ V = 4 \times \cdot 462 \times \cdot 5 \times \cdot 866 = \cdot 8^{\Delta}$$

Or if a +4 D Cyl. Ax. 60° be decentered ·8 cm. vertically,

$$P = 4 \times \cdot8 \times \cdot5 = 1\cdot6^{\triangle}$$

$$H = 4 \times \cdot8 \times \cdot866 \times \cdot5 = 1\cdot386^{\triangle},\ V = 4 \times \cdot8 \times \cdot25 = \cdot8^{\triangle}$$

It should be noted that *a horizontal or vertical effect alone can never be obtained by decentering an oblique cyl.*, and that is why it is inadvisable to decenter such lenses.

The maximum Ver. effect of a Hor. displacement and vice versa results when the axis is at 45°. Also since sin 45° = cos 45°, the effect is equal in both directions, no matter how decentered.

Only if the maximum meridian nearly corresponds to the line of decentering can the effects in other meridians be ignored. Indeed it may occur that the Hor. decentering of an oblique cyl. results in a greater Ver., effect and vice versa.

The Ver. effect of a Hor. decentration of a cyl. whose axis is at, say, 30° is the came as when the axis is at 60°. This occurs because although the distance *o r* (Fig. 172) is less in the first case, the Ver. power of the lens is greater.

To illustrate these rather peculiar effects referred to let a +4 C. be decentered horizontally ·33 cm., the axis being respectively at 45°, 30° and 60°. Then

| | | | |
|---|---|---|---|
| With axis at 45° | P = ·93△ | H = ·66△ | V = ·66△ |
| ,, ,, 30° | P = ·66△ | H = ·33△ | V = ·57△ |
| ,, ,, 60° | P = 1·15△ | H = 1△ | V = ·57△ |

**Upright Sph.-Cyls.**—Decentering a sph.-cyl. across the axis of the cyl. has the same effect as decentering a sph. whose power is that of the two powers combined ; while in the direction of the axis it is the same as decentering the sph. alone.

If + 3 S. ⊂⊃ + 2 C. Ax. 90° is to be decentered for 2△ horizontally, the power in the Hor. Mer. is 3 + 2 = 5 D ; therefore the amount of decentration is

$$C = 2/5 = \cdot4 \text{ cm.}$$

If +3 S. ⊂⊃ +2 C. Ax. 90° needs to be decentered for 2△ vertically, the power in the Ver. Mer. is 3 D, so that

$$C = 2/3 = \cdot66 \text{ cm.}$$

**Oblique Sph.-Cyls.**—The effect of decentering an oblique sph.-cyl. in the principal meridians is the same as with Hor. and Ver. decentration when the axis is Hor. and Ver. respectively.

Suppose a + 3 S. ⊂⊃ + 2 C. Ax. 30° is decentered ·4 cm. at 30° (Fig. 173), then

$$P = 3 \times \cdot4 = 1\cdot2^{\triangle}$$

Here the decentration being along the axis of the cyl. only the sph. is decentered, but there are, besides the effect P in the principal meridian at 30°, certain Hor. and Ver. effects introduced due to the oblique decentering of the sph. Now $a = 30°$, and as in Fig. 173, when $o$ is moved to $r$, there is a Hor. decentration $o\,h = r\,v = o\,r \cos a$, and a Ver. one $o\,v = r\,h = o\,r \sin a$, therefore

$$H = D\,C \cos a, \qquad V = D\,C \sin a$$

and in the above example

$$H = 3 \times {\cdot}4 \times {\cdot}866 = 1{\cdot}04^{\Delta}, \qquad V = 3 \times {\cdot}4 \times {\cdot}5 = {\cdot}6^{\Delta}$$

When the decentration is across the axis of the cyl. (Fig. 174) we have in the Mer. of decentration, and in the Hor. and Ver. Mer. the effects of both the sph. and cyl., so that if D is the power of the sph. and D′ that of the cyl. we have

$$P = (D + D')\,C, \qquad H = (D + D')\,C \sin a, \qquad V = (D + D')\,C \cos a.$$

Thus suppose + 3 Sph. ⊃ + 2 Cyl. Ax. 30° be decentered ·4 cm. at 120°

$$P = (3 + 2) \times {\cdot}4 = 2^{\Delta}.$$

$$H = (3 + 2) \times {\cdot}4 \times {\cdot}5 = 1^{\Delta}, \qquad V = (3 + 2) \times {\cdot}4 \times {\cdot}866 = 1{\cdot}732^{\Delta}$$

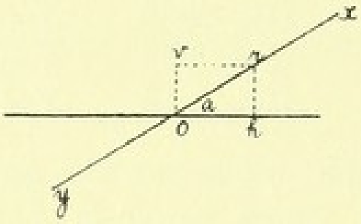



FIG. 173.

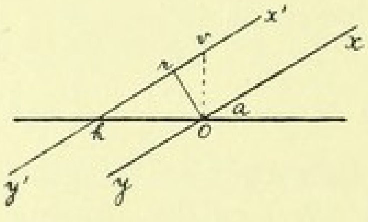



FIG. 174.

Now if the lens were decentered horizontally to $h$ or vertically to $v$, while the effects from the cyl. are the same, those from the sph. are different.

If the lens be decentered horizontally, the sph. causes no Ver. effect, but the cyl. acts as does the plano-cyl. Let D be the power of the sph., D′ that of the cyl. and P′ the effect across the axis of the latter. Then

$$P' = (D + D')\,C \sin a, \qquad H = (D + D' \sin^2 a)\,C, \qquad V = D'\,C \cos a \sin a$$

If the decentration is Ver. the sph. causes no Hor. effect, but the cyl. acts as when not combined with a sph., and

$$P' = (D + D')\,C \cos a, \qquad H = D'\,C \sin a \cos a, \qquad V = (D + D' \cos^2 a)\,C$$

Let the axis of the cyl. be 30°, as in the example given; then $oh = or/\sin\ 30° = 4/{\cdot}5 = {\cdot}8$ cm. and $o\,v = or/\cos\ 30° = 4/{\cdot}866 = {\cdot}462$ cm.

Thus if a $+3$ Sph. ◯ $+2$ Cyl. axis 30° be decentered ·8 cm. horizontally we get

$$P' = (3+2) \times \cdot 8 \times \cdot 5 = 2^{\Delta}$$

$$H = (3 + 2 \times \cdot 25) \times \cdot 8 = 2 \cdot 8^{\Delta}, \qquad V = 2 \times \cdot 8 \times \cdot 866 \times \cdot 5 = \cdot 7^{\Delta}$$

If the same lens be decentered ·462 cm. vertically

$$P' = (3+2) \times \cdot 462 \times \cdot 866 = 2^{\Delta}$$

$$H = 2 \times \cdot 462 \times \cdot 5 \times \cdot 866 = \cdot 4^{\Delta}, \qquad V = (3 + 2 \times \cdot 75) \times \cdot 462 = 2 \cdot 08^{\Delta}$$

**Actual Resultant Prismatic Effects.**—While an oblique cyl. decentered horizontally or vertically always has its greatest prismatic effect across the axis, with an oblique sph.-cyl. the effect across the axis might, or might not, be greater than in that meridian in which the displacement is made, this depending on the power of the spherical. In the last examples P′ is less than H in the one case and less than V in the other, and the *actual resultant prismatic power lies between P′ and the meridian of decentration.*

The Hor. and Ver. effects of decentering an oblique sph.-cyl. having been calculated the actual resultant effect P can be obtained from the formulæ given. Suppose H = $2 \cdot 8^{\Delta}$ and V = $\cdot 7^{\Delta}$; then

$$P = \sqrt{H^2 + V^2} = \sqrt{2 \cdot 8^2 + \cdot 7^2} = \sqrt{8 \cdot 33} = 2 \cdot 88^{\Delta}$$

$$\tan r = V/H = \cdot 7/2 \cdot 8 = \cdot 25 = \tan 14°$$

These effects may be met with in the case of a frame that is incorrect as to width or height when the lenses are oblique cyls. or sph.-cyls.

It should be noted that while a Hor. or Ver. prismatic effect can never be obtained with an oblique plano-cyl., this is often possible with a sphero-cyl. by adjustment of the decentering so as to neutralise the unneeded effects introduced. Practically this is best achieved, if it be possible, by employing a prism for the marking as described in "How to Decenter." It is always possible if the sph. is strong compared with the cyl.

**The Formulæ and Deductions.**—Although definite formulæ have been given in the case of obliquely decentered cyls. and sph.-cyls., and for finding the Hor. and Ver. components and the main effects, they can be worked, in each case, from first principles as indicated in the following. This may be necessary if the decentering is neither Hor. nor Ver. nor in the principal meridians. When a sph.-cyl. is decentered in a direction obliquely to the principal meridians, there are the displacements which would take place if the sph. and the cyl. were separately decentered. The prismatic effect due to the sph. alone lies in the meridian of decentration, while that due to the cyl., as we have already shown, lies in the meridian at right angles to the axis. The total effect, therefore, is that of a prism whose base-apex line corresponds neither to that of the decentration, nor to that of the maximum power of the cyl., but to some meridian between them if the powers of the sph. and cyl. are of the same sign, and outside them if they are of opposite sign. Thus suppose a + 6 S. ⊂⊃ + 2 C. Ax. 30° be decentered 2 mm. upwards in meridian 70°. The prismatic effect due to the sph. is 6 × ·2 = 1·2△ base up at 70°, while that due to the cyl. is 2 × ·2 × ·6428 = ·26△ where ·6428 = sin 40° since the direction of decentration 70°, is 40° from the axis of the cyl. Therefore the cyl. produces ·26△ whose base-apex line is at 120°, and whose base is up. Therefore there are two prismatic effects crossed at 50° (120° − 70°) and the resultant of these can be found from

$$P = \sqrt{1{\cdot}2^2 + {\cdot}26^2 + 2 \times 1{\cdot}2 \times {\cdot}26 \times {\cdot}6428} = \sqrt{1{\cdot}9087} = 1{\cdot}38$$

$$\tan r = \frac{{\cdot}26 \times {\cdot}766}{1{\cdot}2 + {\cdot}26 \times {\cdot}6428} = {\cdot}14 = \tan 8°$$

So that the effect of decentering + 6 S. ⊂⊃ + 2 C. Ax. 30° 2 mm. up in meridian 70°, is 1·38△ base up at 78°. Still further, this oblique prismatic effect may be resolved into its Hor. and Ver., components from the formulæ given previously.

Any possible case of decentration can be worked from general principles, as in the example just given, provided, of course, that proper attention be paid to signs, etc., but, as can be seen, the procedure is complicated.

## CHAPTER XVI

## EFFECTIVITY AND BACK FOCAL DISTANCE

**Effect of Altered Position of a Cx. Lens.**—The power of a lens 1/F or D is a fixed quantity ; nevertheless the effect of a lens, in relation to a given plane behind it, varies with its distance from that plane.

Thus a 10 in. Cx. lens $L$ (Fig. 175) placed 6 in. in front of the plane $P\ P'$, has $F$ 4 in. behind it. If now we place a 40 in. Cx. lens in contact with $L$, the two combined will have $1/F = 1/10 + 1/40 = 1/8$ or $F' = 8''$ so that the combined focus will be 2 in. behind $PP'$. The same effect would be produced if we moved the 10 in. lens 2″ forward to $L'$. Therefore a Cx. lens moved away from a plane acts with increased effectivity, i.e. it acts like a lens of shorter focus. If we place a 60 in. Cc. lens in contact with the 10 in. Cx. the two combined will have $1/F = 1/10 - 1/60 = 1/12$, i.e. F = 12 in. and will be at $F''$ 6 in. behind $PP'$. The same effect is produced if

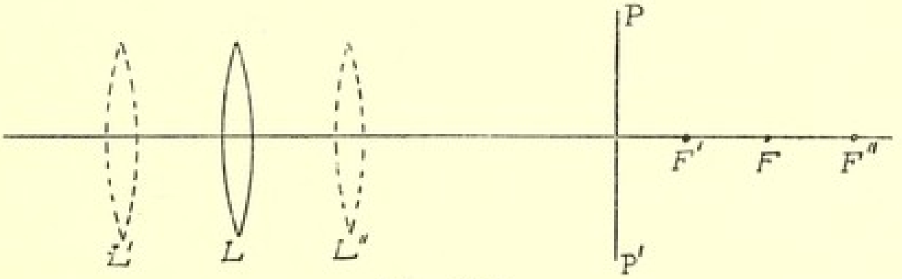



FIG. 175.

the 10 in. Cx. were moved to $L''$ 2″ nearer to $PP'$. Therefore, a Cx. lens acts with a lessened effect as regards the plane when brought nearer to it.

The effectivity $1/F_V$ of a Cx. lens when moved through a given distance $d$ in the direction of the object is, for parallel light,

$$1/F_V = 1/(F - d) \qquad \text{or} \qquad F_V = F - d$$

If $d$ is equal to F, then $1/(F - d) = 1/(F - F) = 1/0 = \infty$, in other words the converging effect will be infinite at the plane, when the lens is placed at its focal distance in front of it.

If the lens be moved beyond its focal length, since $d$ is then greater than F, the effectivity will be negative. Thus, if a + 10 in. lens is 12 in. from a screen, its effect there is $1/F = 1/(10 - 12) = -1/2$, or that of a 2 in. Cc., since the light diverges from 2 in. in front of the plane.

**Effect of Altered Position of a Cc. Lens.**—The effects produced by similarly moving a Cc. lens are opposite in character. Let $PP'$ be the plane, and $L$ a 10 in. Cc. lens placed 3 in. in front of it (Fig. 176); $F$ will then be 13 in. in front of $PP'$. If a + 60 in. lens be placed in contact with the first lens the two combined $1/F' = -1/10 + 1/60 = -1/12$ or $F' = -12''$, so that $F$ lies $12 + 3 = 15''$ in front of $PP'$, and the same effect is produced if the lens be carried to $L'$ 2 in. further from $PP'$. Thus, the effectivity of a Cc. is decreased by increasing the distance between it and a given plane behind it, the lens acting as one of longer focus. If a 40 in. Cc. is added $1/F = -1/10 - 1/40 = -1/8$, the same as if the 10 in. Cc. were carried to $L''$ 2 in. nearer to $PP'$, so that a Cc. lens acts with an increased effect when brought nearer to a given plane, F″ being then 2 in. nearer.

The effect of a Cc. lens when moved in the direction of the object through a given distance $d$, for parallel light, is

$$1/F_V = 1/(-F-d) = 1/-(F+d), \text{ or } F_V = -F-d$$

**Change of Effect.**—The altered effect of a lens when moved from one position to another in front of a plane, or in front of another lens, is the

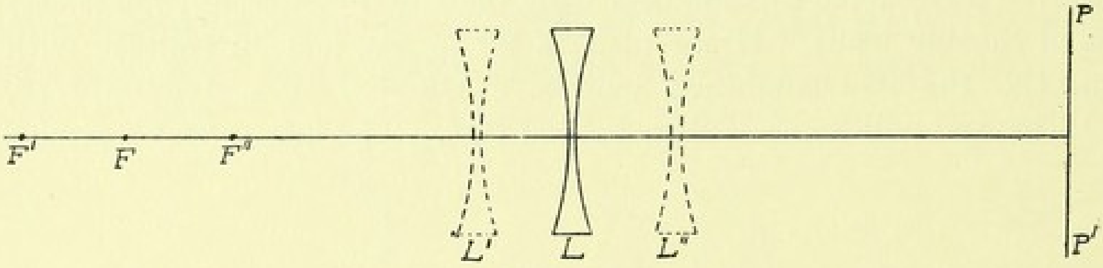



FIG. 176.

difference between its effectivity in its original, and in its new, position; thus, if a 5 in. Cx. lens be moved from 1 in. to 2 in. away from a plane, the change is

$$\frac{1}{5-2} - \frac{1}{5-1} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

or an increase of effect equal to that of an added 1/12 Cx. A 5 in. Cc. similarly moved causes a decrease of effect just as if a 1/42 convex had been added to the concave, as shown by,

$$\frac{1}{-5-2} - \frac{1}{-5-1} = -\frac{1}{7} - \left(-\frac{1}{6}\right) = +\frac{1}{42}$$

**Variation of Effectivity for Near Objects.**—Let a 5 in. Cx. $L$ (Fig. 177) be placed 5 in. in front of $PP'$, and if the light diverges from $f_1$ at 12 in. $f_2$ is at $8\frac{4}{7}$ in. behind the lens, or $8\frac{4}{7} - 5 = 3\frac{4}{7}$ in. behind $PP'$.

If the lens be now carried outwards 2 in. from its original position, to $L'$, it is distant 10 in. from $f_1$ and 7 in. from $PP'$; $f_2$ is now $f_2'$ at 10 in. behind the lens, or 3 in. behind $P\,P'$, the focus being shortened from $3\frac{4}{7}$ to 3 in. with reference to $P\,P'$. *Removal, therefore, of a Cx. lens towards the source of light causes increased effectivity so long as the distance between the Cx. lens and the object* $f_1$ *is not less than* 2F. At this distance the lens has the highest possible effectivity for the *given position of the object* with respect to $P\,P'$, *which is reduced by any further withdrawal of the lens outwards.* Thus, if the lens is at $L''$ 9 in. from $f_1$, then $f_2''$ will be at $11\frac{1}{4} - 8 = 3\frac{1}{4}''$ behind $P\,P'$, the focus being lengthened $\frac{1}{4}$ in. as compared with the position when the lens is 10 in. from $f_1$ and 7 in. from $P\,P'$; there is a lessened effect of 1/39 at the screen.

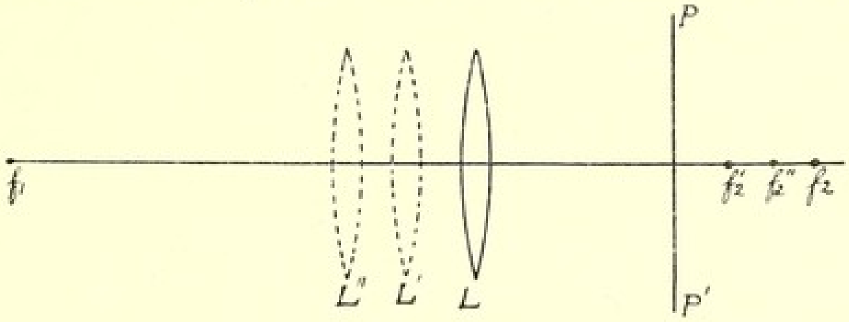



FIG. 177.

Further withdrawal of the lens results in a rapid increase in the distance of $f_2$ behind $P\,P'$ and a corresponding decrease of effectivity; thus if the lens is 6 in. from the source of light, $f_2$ is 19 in. behind $P\,P'$, and when the lens is 5 in. from $f_1$ the light, after refraction, is parallel, and $f_2$ is at infinity; still further removal of the lens from $P\,P'$ towards $f_1$ renders the light divergent after refraction. When the + lens is in contact with $f_1$ all effect vanishes.

In Fig. 178 a 5 in. Cc. lens $L$ is placed at $P\,P'$, and if the light proceeds

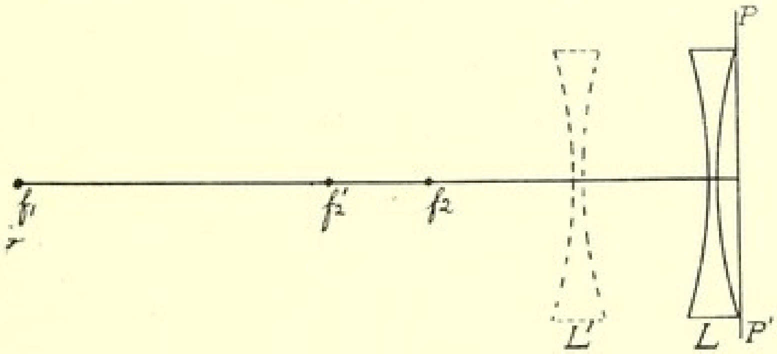



FIG. 178.

from $f_1$ 12 in. distant, $f_2$ is at $3\frac{9}{17}$ in. in front of $P\,P'$. If the lens be withdrawn to $L'$ 4 in. from $P\,P'$ and 8 in. from $f_1$, then $f'_2$ lies $3\frac{1}{13}$ in. in front of the lens, and $7\frac{1}{13}$ in. in front of $P\,P'$. If the lens is 3 in. from $f_1$ then $f_2$ is about 11 in. from $P\,P'$. When the Cc. lens reaches $f_1$ its effect is zero as with a Cx.

Therefore if incident light be divergent an increased effect may be obtained by increasing the distance between a Cx. lens and a plane behind it, but the increase for a given movement is less than if the light were parallel; there will be a decreased effect if $f_1$ is less than 2 F. With a Cc. lens the resultant effect is always decreased, but the change is smaller as the distance between the object and lens is less. Thus the change varies not only with the strength of the lens, but also with the increased or decreased divergence of the light. When either a Cx. or Cc. lens is in contact with the object, the light diverges as if the lens were not there at all.

Quite apart from changes of effect due to change of position of the lens, increased divergence of the incident light reduces the effect of a Cx. lens and increases that of a Cc. When light is parallel there is only the effect due to change of position of the lens to consider, but when the light is divergent, there is also the *increased divergence of the light* from the object to be reckoned for, and the latter tends always to decrease the effect, due to movement, that would have resulted had the light been parallel and, indeed, with Cx. lenses may more than neutralise it.

If any lens is in contact with a given plane, then the effectivity at that plane is represented by the power of the lens itself, e.g. the effectivity of an 8″ lens in contact with the cornea is 1/8 or − 1/8 as the case may be.

**Dioptral Expression.**—If the power of the lens be expressed in diopters, its effective power $D_v$ in a new position becomes

$$D_v = 1000/(F - d)$$

F and $d$ being expressed in mm., or by

$$D_v = D/(1 - D\,d)$$

$d$ being expressed in terms of a metre.

Thus, suppose a + 8 D lens is moved from contact with a given plane to a position 10 mm. further forward, i.e. towards the source of light, which is at $\infty$, then since F is 1000/8 = 125 mm.

$$D_v = 1000/(125 - 10) = 8{\cdot}7$$

The effectivity of the lens is increased + ·7 D.

If a + 10 D lens be moved from 15 to 20 mm. in front of a given plane, the altered values for parallel light, since F = 100 mm., are

$$\text{at 15 mm.}\quad D_v = 1000/(100 - 15) = 11{\cdot}77$$
$$\text{at 20 mm.}\quad D_v = 1000/(100 - 20) = 12{\cdot}5$$

so that the effectivity is increased by 12·5 − 11·77 = ·73 D.

Similarly, moving the lens back from 20 to 15 mm. decreases the effectivity to a like extent.

The distance $d$ (in cms.), which a lens must be in advance of a given plane in order that it may have a given effectivity at that plane, is found by

$$d = 100/D - 100/D_v$$

where D is the power of the lens and $D_v$ is that of its required effect.

**Effectivity of Two Cx. Lenses.**—It has been shown that the combined power of two thin lenses, placed together, is equal to the sum of their individual powers, thus

$$\frac{1}{F} = \frac{1}{F_1} + \frac{1}{F_2} = \frac{F_1 + F_2}{F_1 F_2} \qquad \text{or} \qquad F = \frac{F_1 F_2}{F_1 + F_2}$$

But if the two thin Cx. lenses are separated by an interval $d$ the resultant effect is not the same as if the two were in contact. The distance of $F$ behind the back lens, that is, the back surface or effective focal distance $F_B$, is shorter than when they are in contact, since the effectivity of the front lens has now become $1/(F_1 - d)$ in the plane of the second lens $F_2$. Therefore

$$\frac{1}{F_B} = \frac{1}{F_1 - d} + \frac{1}{F_2} = \frac{F_1 + F_2 - d}{(F_1 - d) F_2} \qquad \text{or} \qquad F_B = \frac{(F_1 - d) F_2}{F_1 + F_2 - d}$$

where $F_1$ is the front, $F_2$ the back lens, and $d$ the distance between them.

In Fig. 179 let $L_1$ and $L_2$ be two thin lenses of 10″ and 7″ focal length respectively, separated by 2″, then

$$F_B = \frac{(10 - 2) \times 7}{10 + 7 - 2} = \frac{56}{15} = 3\tfrac{11}{15} \text{ in.}$$

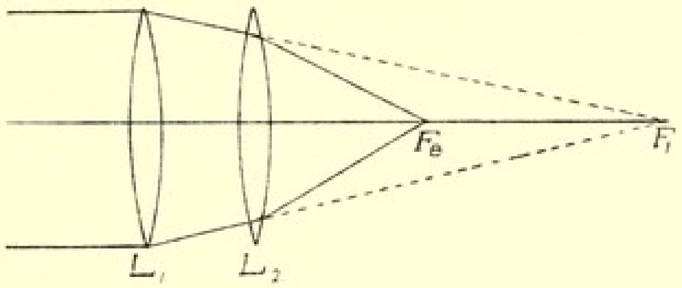



FIG. 179.

Parallel light incident on $L_1$ is converged towards a point $F_1$, 10 in. behind it, but on its way the light meets, at 2 in. from $L_1$, the 7 in. Cx. lens $L_2$, and converges towards a point 10 − 2 = 8 in. behind the latter. The effectivity of $L_1$ in the plane of $L_2$ is that of 1/8, or the effect is the same as if an 8 in. lens were in contact with $L_2$, and the common focus $F_B$ is at $3\frac{11}{15}$ in. instead of $4\frac{2}{17}$ in., where it would be if $L_1$ were touching $L_2$. The separation of the lenses by carrying $L_1$ out from $L_2$ is to increase the effectivity of the combination with respect to a plane behind it.

The distance of $F_B$ differs considerably, when the two lenses are of different powers, according as the one or the other lens faces the light. Thus, if the combination were reversed so that the 7 in. Cx. faced the light, and the 10 in. Cx. were 2 in. behind it, $F_B = 3\frac{1}{3}$ in. instead of $3\frac{11}{15}$ in. $F_B$ is equal on both sides only when the lenses are equal and of same nature.

When $d$ is greater than $F_1$ the $F_B$ is negative.

**When d = F₁.**—If a Cx. lens $L_1$ (Fig. 180) is placed at its principal focal distance in front of another Cx. lens $L_2$ the latter has no effect whatever. Thus when $d = F_1$, the $F_B = 0$ or $1/F_B = \infty$.

**When $d = F_1 + F_2$.**—If the interval between two similar Cx. lenses (Fig. 181), is equal to the sum of their focal lengths, parallel light refracted by $L_1$, converges to $F$, which is also the anterior principal focal distance of $L_2$. Therefore, as the light from $F$ is incident on $L_2$ diverging from its principal focal distance, after refraction by $L_2$ it emerges parallel as before

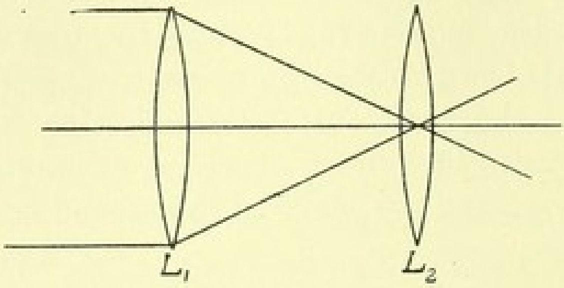



FIG. 180.

refraction, but the light and, therefore, the virtual image seen through $L_2$ are inverted. Such an arrangement is found in terrestrial telescopes for reinverting the inverted image formed by the objective, and is known as *the erecting eye-piece*. In like manner if two unequal Cx. lenses (Fig. 182) are separated by a distance equal to the sum of their focal lengths, so that

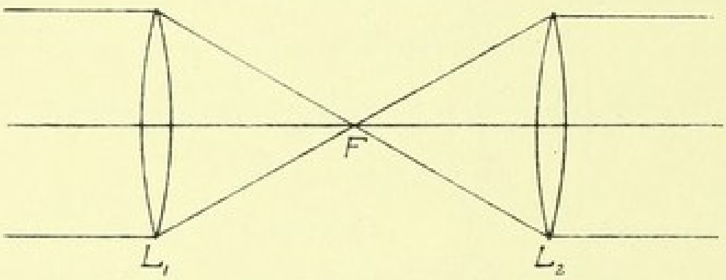



FIG. 181.

$d = F_1 + F_2$, parallel light, after refraction, emerges parallel and reversed. If the lens of greater focal length $L_1$ is to the front, the combination represents the principle of the *astronomical telescope*.

**The Telescope** is used for obtaining an enlarged view of distant objects, and consists (Fig. 183) of an objective $L_1$ of long, and an eye-piece $L_2$ of short

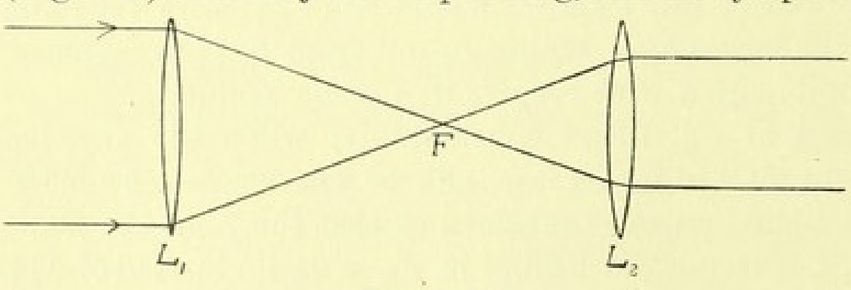



FIG. 182.

$F$, both corrected for spherical and chromatic aberration. The objective forms a real inverted image $S\,T$ of a distant object $P\,R$, subtending an angle $a$, and this image is viewed through the eye-piece $L_2$. For an emmetropic eye the distance between $L_1$ and $L_2$ is equal to the sum of their focal lengths, so that the light, after refraction by both lenses, may enter the eye in parallel beams. The magnification depends on the ratio between the angle $a$, which the object subtends, and $b$, which the final image $R' P'$ subtends.

Let $Q$ be the optical centre of the objective, and $P R$ the extreme axial rays of the object at $\infty$, the one extremity $P$ being assumed to be on the principal axis $Q O$ of the telescope. Then $P Q R = a$ the angle subtended by the object at $Q$, and $S T$ is the real image formed in the focal plane of the ocular, of which $O$ is the optical centre. The angle under which this image is seen is $b$. The magnification therefore is the ratio between the angle $a$, under which the object would be seen by the naked eye, and the angle $b$, under which it is apparently seen when the telescope is in use. Thus

$$M = b/a = b/a'$$

But as these angles involved are very small, we may replace them by their tangents. Now $\tan a' = S T/T Q$ and $\tan b = S T/T O$. Then

$$M = ST/TO \div ST/TQ = TQ/TO$$

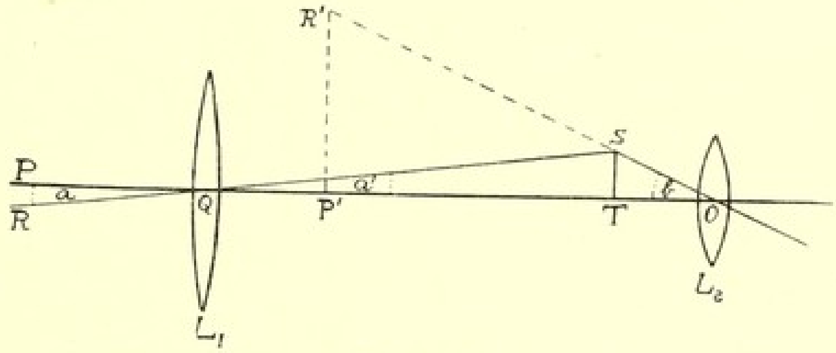



FIG. 183.

But $T Q = F_1$ the focal length of the objective, and $T O = F_2$, that of the eye-piece. Therefore

$$M = F_1/F_2$$

A hypermetrope would adjust the telescope so that the distance between the lenses is greater than $F_1 + F_2$, the light then entering the eye convergently, while the myope, in order to obtain divergent light, would make the interval less than $F_1 + F_2$.

The final image is inverted with respect to the object, and for terrestrial purposes this difficulty is overcome by means of an erecting eye-piece which, when suitably placed between the objective and eye-piece, causes a reinversion of the image. For astronomical purposes an erector is not needed, since inversion of a heavenly body is of no importance, while, on the other hand, loss of light owing to increase in the number of the refracting surfaces is avoided.

**The Compound Microscope** consists of a similar combination, the lens of shorter focal length being to the front, but in this case, the object viewed lies just in front of $F$ of the first lens, and the separation of the lenses is much greater than $F_1 + F_2$. The compound microscope is used to obtain a magnified view of a small near object, the distance between the lenses being dependent on the available length of the instrument, usually from 6 to 10 in., the distance of most distinct vision of the observer being generally 10 in. also. The first lens $L_1$, called the objective, is a short focus combination, highly corrected for aberrations, and the second, called the eye-piece, or ocular, $L_2$, is also a strong combination, but less so than the other (Fig. 184).

A small object $A B$ is placed just beyond $F$ of the objective $L_1$, so that the latter forms a real, inverted, magnified image, $B'A'$, of the object. This image is formed practically in the focal plane of the eye-piece $L_2$, and an eye placed behind the latter sees an enlarged virtual image $B''A''$ of $B'A'$ at the distance of most distinct vision. Hence there is magnification due both to

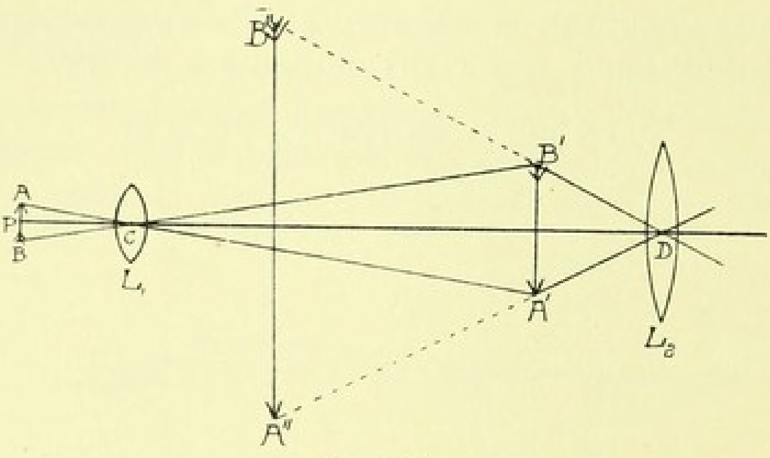



FIG. 184.

the objective and to the ocular and can be very approximately calculated as follows. Let $N$ denote the position of B′A′ on the axis.

The magnification due to the objective is

$$M_1 = B'A'/A\,B = C\,N/C\,P.$$

But $C N$ may be taken as the tube length of the microscope, this value being a variable quantity depending upon the particular maker, while $C P$ is practically equal to the focal length of the objective. Thus $M_1$ may be taken as

$$\text{tube length}/F_O$$

where $F_O$ is the focal length of the objective.

Again, the magnification of the eye-piece can be expressed as $1 + d/F_E$, as shown in Chap. IX., where $F_E$ is the focal length of the eye-piece, but as $F_E$ is always fairly short, the magnification due to the eye-piece may be expressed by

$$d/F_E$$

If we imagine the final virtual image B″ A″ to be projected to the plane of the stage on which the object is placed, the distance $d$ may also be taken as equal to the tube length ; the total magnification therefore is

$$M = M_1 M_2 = \text{tube length}/F_O \times d/F_E = 10^2/F_O F_E,$$

$F_E$ and $F_O$ being expressed in inches.

It must be remembered that this formula is only approximate, but is more accurate the higher the powers are with which we are dealing. To calculate exactly the magnification of the microscope would be laborious, seeing that a knowledge of the principal points of objective and eye-piece, the position of the focal planes, and so forth, is essential.

**Construction of Eye-pieces.**—If two plano Cx. lenses $L_1$ and $L_2$ plane surfaces outward, of, say, 4 in. F each, are separated by $\frac{2}{3}$ F of either, that is, $d = 2\frac{2}{3}$ in.

$$F_B = \frac{(4 - 2\frac{2}{3}) \times 4}{4 + 4 - 2\frac{2}{3}} = 1 \text{ in.}$$

This is the common form of the **Ramsden eye-piece** (Fig. 185). Used in a telescope or other instrument, the real image formed by the objective is 1″

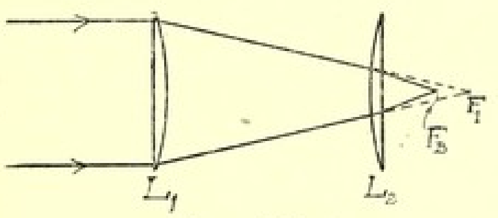



Fig. 185.

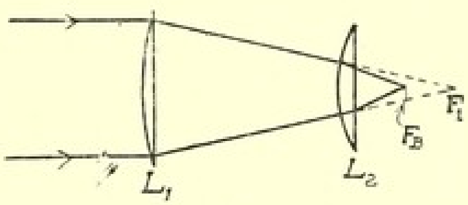



Fig. 186.

in front of $L_1$, and after refraction by $L_1$, its image is 1 1/3″ from it, and therefore in the focal plane of the second lens $L_2$ by which the light is rendered parallel. The front lens of this, and of the two following combinations, is termed the *field lens*, and the back one is the *eye lens.*

Fig. 186 represents the **Huyghen eye-piece**, in which two unequal plano Cx. lenses $L_1$ and $L_2$ (where $F_1 = 3F_2$) of, say, respectively 6 in. and 2 in. focal length are separated by a distance 4 in. equal to the difference between $F_1$ and $F_2$, or half the sum of their focal lengths $(F_1 + F_2)/2$.

When employed with an instrument such as the telescope, the light from the objective is convergent on to $L_1$, which increases the convergence, so that the real image is formed 2″ from $L_2$, and therefore in its focal plane ; then the light finally emerges parallel. Both curved surfaces face the light.

Two equal plano Cx. lenses with their curved faces towards the light, and separated by a distance equal to $F$ of either, constitute the **Kellner eye-piece**. In this case $F_B = 0$, but, when in use, the image formed by the objective lies in the plane of $L_1$, so that the light then diverges to $L_2$ from its focal distance and is, after refraction, parallel.

The utility of the field lens, in all eye-pieces, is to increase the field of view.

**Effectivity of Two Cc. Lenses.**—In Fig. 187 let $L_1$ and $L_2$ be two thin Cc. lenses of, say, 10 in. and 7 in. F respectively; when close together F lies $4\frac{2}{17}''$ in front of them. If $L_1$ is advanced 2 in.

$$F_B = \frac{(-10-2) \times -7}{-10-7-2} = \frac{+84}{-19} = -4\tfrac{8}{19} \text{ in.}$$

As $F_B$ (measured from $F_2$) is lengthened by separation, the effectivity is *decreased* for any plane behind the lenses, although $F_B$ is nearer the front lens

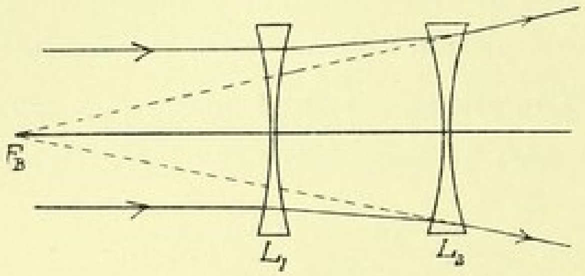



Fig. 187.

As with Cx. lenses, the distance of $F_B$ from the back lens of a combination of two unequal Cc. lenses, separated by an interval, varies as the one or the other lens faces the light. Thus, if the 7 in. Cc. were 2 in. in front of the 10 in., $F_B = 4\frac{14}{19}$ in.

**Effectivity of Cx. and Cc. in Combination.**—If there is an interval between a Cx. and a Cc. of equal focal length, the combination is Cx. If, however, $d$ exceeds $F_1$, the light refracted by the Cx. is brought to a focus, whence it diverges to the Cc., so that $F_B$ is negative.

Thus, in Fig. 188 let $L_1$ be a 10 in. plano-Cx. and $L_2$ a 10 in. Cc. When

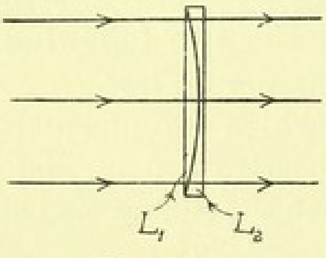



Fig. 188.

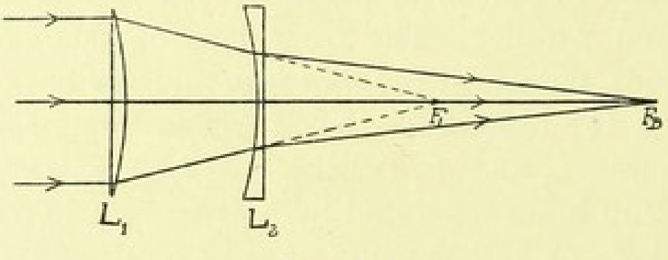



Fig. 189.

placed in contact parallel light is unaltered by them, but if separated by an interval less than $F_1$ (Fig. 189) parallel light incident on the Cx. is converged to the Cc. and if $d = 4$ in.

$$F_B = \frac{(10-4) \times -10}{10-10-4} = \frac{-60}{-4} = 15 \text{ in.}$$

Thus the rays are rendered less convergent, and form a real focus at $15''$ behind $L_2$. This is the principle of the **Unofocal photographic lens**, in which the components are of equal but opposite power.

If the combination be reversed (Fig. 190) so that the light is incident first on the Cc. it is rendered divergent as from $10+4=14''$ from the Cx., and

$$1/F_B = +1/10 - 1/14 = 1/35;$$

thus $F_B$ is at $35''$, or $20''$ further from $L_2$ than the other back focus. Although there is an excess of Cx. power in both cases, $F_B$ is nearer to the back lens when the Cx. faces the light than when the Cc. does so.

If the Cc. lens has a shorter focal length than the Cx. and the two are in contact, the result will be an excess of negative power. If the Cx. be moved towards parallel light, it gains in effectivity, but the total effect is still nega-

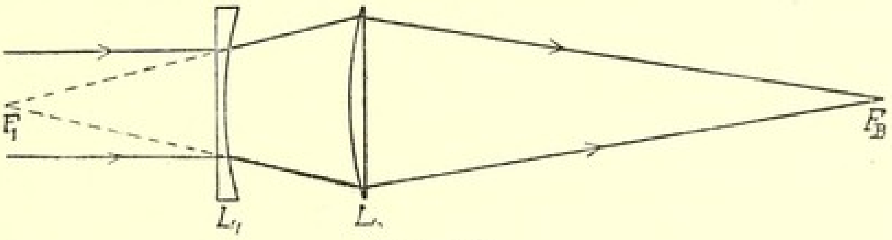



FIG. 190.

tive until, when the separation is equal to the sum of their focal lengths, the back focal length is infinite. Any further increase of separation will give the two lenses a positive focal length, which diminishes as the separation continues, until when $d=F$ of the Cx., the back focal length is zero. Still further separation produces a negative effect. When $d$ is less than $F$ of the Cx., but greater than $F_1+F_2$, the principle of the **Telephoto lens** is illustrated.

In order that a Cx. and Cc. should neutralise each other, and parallel rays emerge from the second lens parallel (Fig. 191) $d$ must be equal to the algebraical sum of their focal lengths, and, further, *the value of $d$ must always be a*

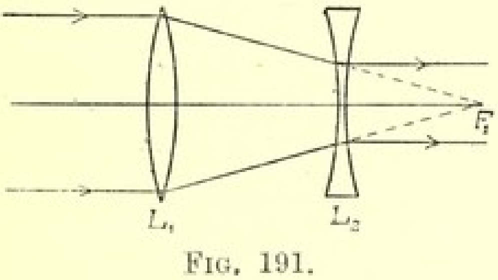



FIG. 191.

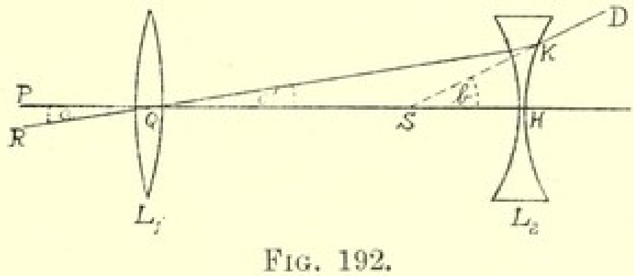



FIG. 192.

*positive quantity*. Thus if $F_1=6$ in. and $F_2=-6$ in., then $6+(-6)=0$; the two lenses must be in contact in order to neutralise. But if $F_1=+6$ and $F_2=-4$ the two lenses must be separated $6+(-4)=2$ in. In this last case the emergent rays are parallel to their axes after refraction, whether the entering rays are first incident on the Cx. or on the Cc. Here by calculation

$$F_B = \frac{(6-2)\times 4}{6-2-4} = \frac{16}{0} = \infty$$

It will be noticed, therefore, in order that a Cx. and a Cc. may neutralise by separation, *the Cx. must always be the weaker*, the principal focus of the Cx. being behind the Cc. as far as that of the latter is in front of it.

**The Opera-Glass** consists of a convex lens, $L_1$, placed in front of a concave lens, $L_2$, of higher power, at a distance equal to the algebraical sum of their focal lengths so that the lenses neutralise each other by separation. Although the *rays* of each pencil emerge parallel from a very distant point after refraction by both lenses, yet the *pencils themselves* are deviated so that an object appears under a larger angle.

The proof of the expression for magnification in the case of the opera-glass can be done very similarly to that of the telescope. In Fig. 192 $Q$ is the optical centre of the objective as before. An extreme axial ray $RQ$ from an object $PR$ at $\infty$ subtending the angle $a$ at $Q$ is incident on the concave, and is refracted away from the axis in the direction $D$ such that $b$ is the angle under which the image is seen. Therefore the magnification is

$$M = b/a = b/a'.$$

The angles, being small, can be replaced by their tangent values, giving

$$M = QH/SH.$$

Now $QH = F_1 - F_2$ the separation necessary for parallel emergent light, and the point $S$ is really the virtual conjugate of $Q$ by refraction at the concave lens. Therefore we have

$$\frac{1}{SH} = \frac{1}{F_2} + \frac{1}{F_1 - F_2} = \frac{F_1}{F_2(F_1 - F_2)}$$

so that

$$SH = \frac{F_2(F_1 - F_2)}{F_1}.$$

Therefore the magnification

$$M = \frac{QH}{SH} = \frac{F_1 - F_2}{\frac{F_2(F_1 - F_2)}{F_1}} = \frac{F_1}{F_2}$$

Thus the magnification of the opera-glass is expressed by $F_1/F_2$ as in the telescope.

When the Cc. is to the front the secondary axial rays of the concave are less divergent after refraction by the convex, and therefore appear to proceed from a smaller object, so that diminution occurs when an opera-glass is turned wrong way round. The magnification in this case is still expressed by $F_1/F_2$, where $F_1$ is that of the Cc.; in practice, of course, the magnification is fractional, indicating a diminution equal to the magnification obtained when the Cx. was to the front.

Thus, if $F_1 = 5$ in. and $F_2 = 2$ in., the magnifying power of the opera-glass is $5/2 = 2\frac{1}{2}$. If the combination is reversed so that the Cc. is to the front, $M = 2/5$, i.e. there is a diminution to $2/5$.

The emmetrope adjusts the glasses so that the separation is exactly $F_1 + F_2$. The hypermetrope, requiring convergent light for clear vision, makes the separation greater, while the myope, needing divergent light, makes it shorter.

**Dioptral Formulæ for Effectivity.**—The formulæ for finding the effective dioptral lens $D_E$ of two separated lenses $D_1$ and $D_2$ are

$$D_E = \frac{D_1}{1-(D_1 d)} + D_2, \text{ or } D_1 + D_2 + \frac{D_1^2 d}{100 - D_1 d}$$

$D_1$ being the power of the front, and $D_2$ that of the back lens; $d$, the interval, is expressed in terms of a metre in the first, and in centimetres in the second, of the above formulæ.

**Separation for Given Effectivity and Opera-Glass Adjustment.**—Suppose an opera-glass, formed of a +10 D objective and a −20 D ocular, has to be adjusted for the vision of a myope of 4 D who requires a back focus of −4 D in order to see clearly through the combination. The distance between the lenses must be such that the +10 D has an effectivity of +16 D in the plane of the −20 D, so that the required −4 D is left over. Then

$$d = 100/10 - 100/16 = 10 - 6{\cdot}25 = 3{\cdot}75 \text{ cm.}$$

Again, suppose a hypermetrope of 4 D similarly desires to see clearly without accommodation through the same combination of +10 D and −20 D. Here a back focus of +4 D is necessary, so that the +10 D must have an effectivity of +24 D in the plane of the Cc. Then

$$d = 100/10 - 100/24 = 10 - 4{\cdot}16 = 5{\cdot}83 \text{ cm.}$$

In order to adjust the distance between two lenses so that the effect is that of a given back focal length the formula for $F_B$ already given may be employed. Let the lenses be 5 Cx. and 2 Cc. and the effect required that of a 20 Cx.; then

$$+20 = \frac{(5-d)\times(-2)}{5-2-d} = \frac{-10+2d}{3-d}$$

$$60 - 20d = -10 + 2d.$$

so that $$-22d = -70, \text{ and } d = 3\tfrac{2}{11} \text{ in.}$$

If the effect required with the same lenses is that of 20 Cc., then

$$-20 = \frac{(5-d)\times(-2)}{5-2-d} = \frac{-10+2d}{3-d}$$

$$-60 + 20d = -10 + 2d$$

so that $$18d = 50, \text{ and } d = 2\tfrac{7}{9} \text{ in.}$$

The value of $d$ can also be obtained from the formula

$$d = \frac{F_1F_2 - F_B(F_1 + F_2)}{F_2 - F_B}$$

**Effectivity when Light is Divergent**.—When incident light is divergent the conjugate focus with respect to the front lens of the combination must be found before the value of $(F_1 - d)$ can be applied. The conditions for neutralising and for obtaining certain effectivities with separated lenses differ for divergent and parallel light. For example, an object is 20 in. in front of an 8 in. Cx. lens behind which, at 2 in., a 13 in. Cc. lens is placed; where is the image? Now, the first conjugate $f_2$ is at

$$1/f_2 = 1/8 - 1/20 = 1/13\tfrac{1}{3}, \text{ and } 1/(13\tfrac{1}{3} - 2) - 1/13 = 1/88\tfrac{2}{5}$$

Therefore I. is at $88\frac{2}{5}$ in. behind the concave.

An object is 40 in. in front of a 7 in. Cx., where should a 5 in. Cc. be placed so that the rays may be rendered parallel? Now

$$1/f_2 = 1/7 - 1/40 = 33/280$$

the image $f_2$ is thus $8\frac{16}{33}$ in. behind the Cx., so that the Cc. must be placed $8\frac{16}{33} - 5 = 3\frac{16}{33}$ in. behind the Cx.

An object is 40 in. in front of a 7 in. Cx. and a 5 in. Cc. The image must be 20 in. behind the back lens; how much must the lenses be separated? $f_2$ is $8\frac{16}{33}$ in. behind the Cx., which must have the effect of $1/5 + 1/20 = 1/4$ in the plane of the Cc. The interval between them must therefore be $8\frac{16}{33} - 4 = 4\frac{16}{33}$.

Suppose a + 3 D lens is placed 20 cm. in front of a screen, where must another equal lens be placed in front of it so that the image of an object 50 cm. from the front lens be focussed on the screen? After refraction by the second lens alone the light is converging to 100 cm., but it must converge to 20 cm. behind the second lens, since the effectivity needed is + 5 D. Therefore, the front lens must act as 5 − 3 = + 2 D in the plane of the second, and the interval between them must be 100 − 50 = 50 cm.

## CHAPTER XVII

## EQUIVALENCE OF THIN LENSES

**Equivalence.**—Any two or more lenses, whether in contact or separated, can be replaced by a single equivalent lens which has the same refracting effect as the component lenses. Or, to put it in another way, since the size of image is proportional to focal length, any number of lenses can always be replaced *by that single thin lens giving the same magnification.*

If two thin lenses are placed in contact the resultant focal length is the same as that of a single lens situated in the same plane and whose power 1/F is that of the sum of the two components $F_1$ and $F_2$. The combined power and F may be written

$$\frac{1}{F}=\frac{1}{F_1}+\frac{1}{F_2} \qquad \text{and} \qquad F=\frac{F_1F_2}{F_1+F_2}$$

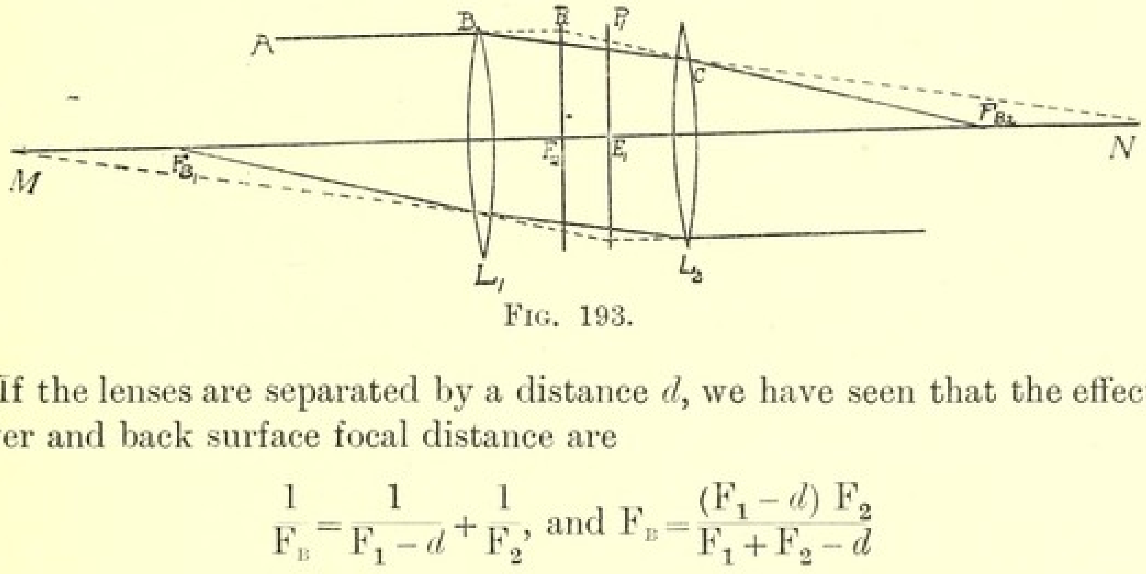



Fig. 193.

If the lenses are separated by a distance $d$, we have seen that the effective power and back surface focal distance are

$$\frac{1}{F_B}=\frac{1}{F_1-d}+\frac{1}{F_2}, \text{ and } F_B=\frac{(F_1-d)\,F_2}{F_1+F_2-d}$$

It now remains to find an expression for the equivalent focal length of two thin separated lenses.

**Equivalent Lens and Focal Length.**—Let $L_1$ and $L_2$ (Fig. 193) be two thin lenses separated by a distance $d$, and let $AB$ be a ray incident on $L_1$ parallel to the principal axis $MN$. This is deviated by $L_1$, and, were it not intercepted by $L_2$, would focus at $N$, but it is refracted still more at $C$ to cross the principal axis in the posterior focus $F_{B2}$.

Now if the incident ray $AB$ be produced, and the final refracted ray $CF_{B2}$ prolonged backwards, the two will meet in the point $P_2$. Through $P_2$ drop the perpendicular $P_2 E_2$. Then if a thin lens of focal length $E_2 F_{B2}$ be introduced into the plane $P_2 E_2$ and the other lenses removed, this single lens would give precisely the same result as the combination $L_1 L_2$. For this reason the plane $P_2 E_2$ is called the *second equivalent plane* and the point $E_2$ the *second equivalent point*.

Similarly if parallel light be incident first on $L_2$, it will pass through the first back focus $F_{B1}$ and $P_1 E_1$ is located in the same way as $P_2 E_2$. The plane $P_1 E_1$ is called the *first equivalent plane* and $E_1$ the *first equivalent point*, and it is here that the equivalent lens must be situated to replace the combination for light coming from the side of $L_2$.

Thus it will be seen that $P_1$ and $P_2$ correspond to the refracting planes of a single thin lens, since all refraction appears to take place on either $P_1$ or $P_2$ depending upon the direction of the light. $E_1$ and $E_2$ likewise correspond to the optical centre, because any ray directed towards $E_1$ will, after refraction, appear to emerge from $E_2$ in a direction parallel to its initial path. This is illustrated in the next diagram.

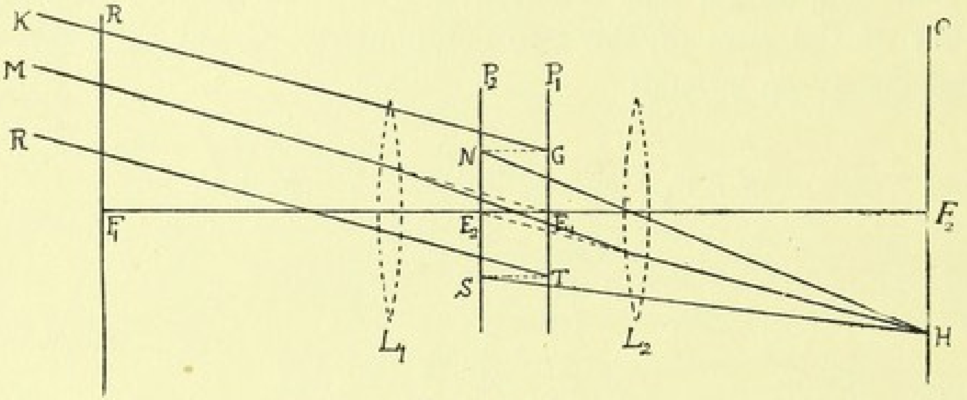



FIG. 194.

In Fig. 194, $P_1$ and $P_2$ are the equivalent planes, $E_1$ and $E_2$ the equivalent points, $F_1$ and $F_2$ the principal foci, $R F_1$ and $Q F_2$ the focal planes. Let an oblique parallel beam, of which $M$ is the secondary axis, fall on $L_1$. The ray $M E_1$ directed towards $E_1$ is bent towards the axis by $L_1$, but is again rendered parallel to its original direction by $L_2$ such that it appears to proceed from $E_2$ towards $H$. Another ray $K G$ after refraction by $L_1$ and $L_2$ is directed towards $H$ in the posterior focal plane, apparently proceeding from a corresponding point $N$ on $P_2$ *such that the distances of G and N from the axis are equal.* Similarly $R T$ is refracted towards $H$, the point of emergence on $P_2$ being $S$, such that $S E_2 = T E_1$. Thus $H$ is the image of the point from which the light originally diverged. Conversely rays diverging from $H$, or any other point in the focal plane, will emerge as a parallel beam.

Since the intrinsic power of a combination is a fixed quantity *the equivalent focal length is the same on each side, and is the distance* $E_2 F_2$ or $E_1 F_1$. The equivalent planes $P_1$ and $P_2$ are always situated symmetrically with respect to the focal planes, and with two ordinarily separated convex lenses $P_1$ and $P_2$ are invariably crossed such that $E_1$ lies nearer to $F_2$ than to $F_1$, and $E_2$ nearer to $F_1$ than to $F_2$, that is to say the 2nd equivalent plane lies nearer to the source of light. Generally, however, the equivalent points and planes are uncrossed. That which is the first equivalent plane when the light is incident on the one lens becomes the second equivalent plane when the lenses are reversed.

The space $E_1E_2$, over which the light apparently jumps, is called the *optical interval* or *equivalent thickness*. Were the two lenses brought together this interval would vanish, so that $E_1$ and $E_2$ merge to form the optical centre of the resultant thin combination, and the united planes $P_1$ and $P_2$ becomes the refracting plane.

**Expression for $F_E$.**—In Fig. 195 $AB$ is a ray parallel to the axis and is refracted through $F_B$, the back focus. Let the focal lengths of $L_1$ and $L_2$ be $F_1$ and $F_2$ respectively, $d$ the separation, and $F_E$ the equivalent focal length.

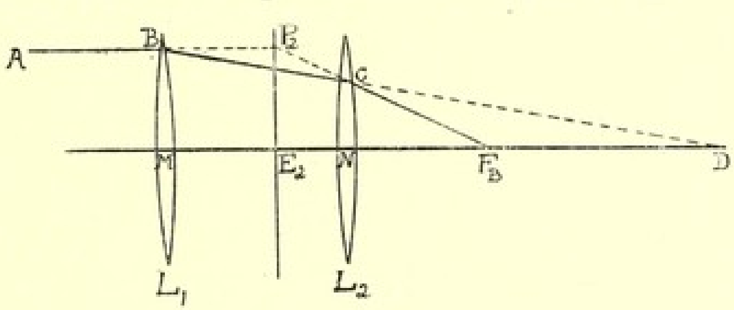



FIG. 195.

Then we have two pairs of similar triangles $CF_BN$, $P_2F_BE_2$, and $CDN$, $BDM$.

Therefore

$$\frac{E_2F_B}{NF_B} = \frac{P_2E_2}{CN} = \frac{BM}{CN} = \frac{MD}{ND} \qquad \text{or} \qquad E_2F_B = \frac{MD \times NF_B}{ND}$$

Now

$$E_2F_B = F_E; \qquad NF_B = \frac{F_2(F_1 - d)}{F_1 + F_2 - d}$$

$$MD = F_1; \qquad ND = F_1 - d$$

$\therefore$

$$F_E = \frac{F_1 \times F_2 (F_1 - d)}{(F_1 + F_2 - d)(F_1 - d)} = \frac{F_1F_2}{F_1 + F_2 - d}$$

This formula, it will be noticed, is independent of the direction of the light.

The distance of the second equivalent point $E_2$ from $L_2$ is found by subtracting the back from the equivalent focal distance, i.e. $F_E - F_B$. Thus

$$E_2 = \frac{F_1F_2}{F_1 + F_2 - d} - \frac{F_2(F_1 - d)}{F_1 + F_2 - d} = \frac{F_2 d}{F_1 + F_2 - d}$$

The corresponding distance of $E_1$ from $L_1$ is

$$E_1 = \frac{F_1F_2}{F_1 + F_2 - d} - \frac{F_1(F_2 - d)}{F_1 + F_2 - d} = \frac{F_1 d}{F_1 + F_2 - d}$$

The equivalent thickness is found from the following equations, the first of which also shows whether the equivalent points are crossed or not.

$$t = d - (E_1 + E_2) = \frac{d^2}{F_1 + F_2 - d}$$

The distance $E_1$ is measured *backwards* from the 1st lens and $E_2$ *forwards* from the second lens, that is in each case *towards the other lens.* If, however, either is a negative quantity, it is measured in the opposite direction or *away from the other lens.*

The positions of $E_1$ and $E_2$ are unchanged in the combination, no matter which lens faces the light; $E_1$ is that which *theoretically* is nearer the source, but actually it may not be so. $E_2$ is that from which the focal length is measured, and if the combination is reversed that which was $E_1$ then becomes $E_2$ and vice versa. When the one lens faces the light $F_E$ is measured from a certain position, and it is measured from another position if the other lens faces the light.

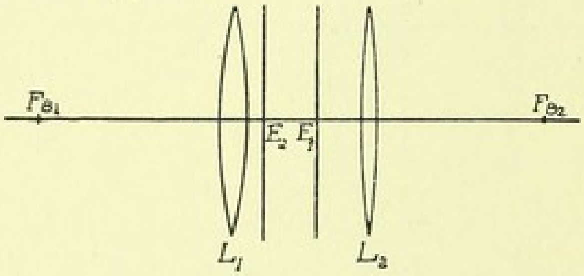


FIG. 196.

**Two Cx. Lenses.**—Suppose a 5 in. Cx. lens is placed 2 in. from a 10 in. Cx. lens. Then (Fig. 196)

$$F_E = \frac{5 \times 10}{10 + 5 - 2} = \frac{50}{13} = 3\tfrac{11}{13} \text{ in.}$$

$E_1 = 5 \times 2/13 = \frac{10}{13}$ in. behind $L_1$ $\qquad E_2 = 10 \times 2/13 = 1\frac{7}{13}$ in. in front of $L_2$

or $\qquad 2 - 1\frac{7}{13} = \frac{6}{13}$ in. behind $L_1$

$$t = 2 - (20 + 10)/13 = -4/13$$

If the 10 in. lens faces the light, the two equivalent points change places, $F_E$ being the same. Since $d = 2$ in., and $E_1$ is 10/13 in. behind $L_1$, while $E_2$ is $1\frac{7}{13}$ in. in front of $L_2$, the distance $t$ is negative, and the two equivalent planes are crossed by 4/13 in.

**Special Cases.**—The following special cases occur with two separated Cx. lenses.

(1) *When* $d = F_1 - F_2$, then $F_E = F_1/2$, and $E_2$ is midway between the two lenses. This is the case of the Huyghen eye-piece. If $F_1 = 3$ in., $F_1 = 1$ in., and $d = 2$ in.

$$F_E = \frac{3 \times 1}{3 + 1 - 2} = 1\tfrac{1}{2} \text{ in.}$$

$E_2$ is 1 in. in front of the back lens, midway between the two lenses; $E_2$, being 3 in. behind the front lens, is 1 in. behind the back lens, and outside the lens system. Here $d > F_2$ but $< F_1$, and if the lens of shorter focus faces the light $F$ lies in front of the back lens.

(2) *When* $d = F_1 + F_2$, then $F_E = \infty$, this being the case of the telescope (q.v.).

(3) *When* $d > F_1$ *but* $< F_1 + F_2$, then $F_E$ is positive and $E_1$ and $E_2$ may be one or both beyond the lenses and crossed (Fig. 197), $d$ being $> F_1$, the light, after refraction by $L_1$, is divergent to $L_2$ as if a single lens were placed further than its focus from a given plane.

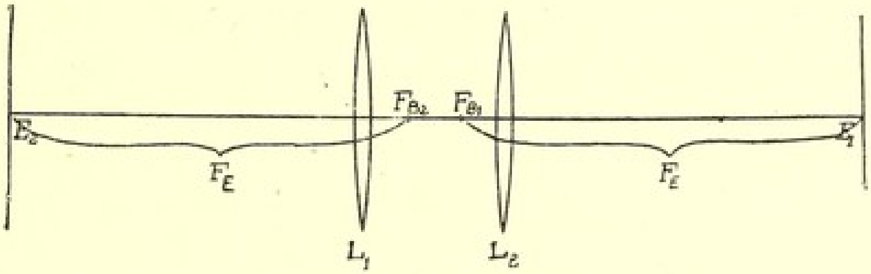



FIG. 197.

Let $F_1 = 4$ in., $F_2 = 4$ in., and $d = 6$ in. Then (Fig. 197)

$$F_E = \frac{4 \times 4}{4 + 4 - 6} = \frac{16}{2} = 8 \text{ in.}$$

The lenses being equal

$$E_1 \text{ or } E_2 = 4 \times 6/2 = 12 \text{ in.}$$

Parallel light incident on $L_1$ comes to a focus at 4 in., whence it diverges

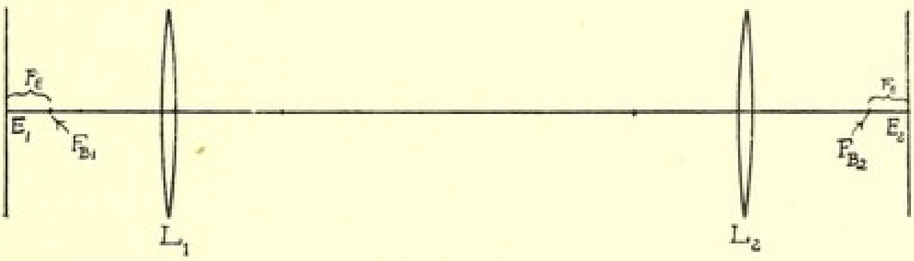



FIG. 198.

to $L_2$, and has its focus, after refraction, at 4 in. in front of $L_2$, or 8 in. behind $E_2$. $T = -18$ inches in this case.

Suppose $F_1 = 7$ in., $F_2 = 16$ in., and $d = 9$ in. Then $F_E = 8$ in., $E_1$ is $4\frac{1}{2}$ in. from $L_1$, and $E_2$ is $10\frac{2}{7}$ in. from $L_2$. The effect is as if an 8 in. lens were placed $10\frac{2}{7}$ in. in front of the plane of $L_2$. Light, refracted by this system, is converged to 7 in., and, after the second refraction, diverges as if from a point $2\frac{2}{7}$ in. in front of $L_2$.

(4) *When* $d > F_1 + F_2$, then $F_E$ is negative, and $E_1$ and $E_2$ are also negative (Fig. 198). Thus if two 4 in. lenses are 20 in. apart, we get

$$F_E = \frac{4 \times 4}{4 + 4 - 20} = \frac{16}{-12} = -1\tfrac{1}{3} \text{ in.}$$

Here, also, the lenses being similar

$$E_1 \text{ or } E_2 = 4 \times 20/-12 = -6\tfrac{2}{3} \text{ in.}$$

Here the equivalent points, being negative, are measured outwards instead of inwards, and $F_E$ lies behind $L_2$ but $1\frac{1}{3}''$ in front of $E_2$.

(5) *When* $d = F_1$, then $F_E = F_1$, and the system illustrates the Kellner eye-piece (q.v.) if the lenses are equal. If the lenses are $3''$ and $1''$ with $d = 3''$, we have $F_E = 3''$, $E_1 = 9''$ and $E_2 = 3''$, that is, in the plane of $L_1$.

(6) *When* $d = F_2$, then $F_E = F_2$, and $E_1 = d$, the image being the same as if the front lens were not there, but its position is shifted. Thus with a $10''$ and a $1''$ lens separated by $1''$ we find $F_E = 1''$, $E_1 = 1''$ and $E_2 = \frac{1}{10}''$ which is the distance that the image is shifted. This illustrates the case of a lens at the anterior focal point of the eye.

(7) *When* $F_1 = F_2$, then $F_B + F_E = F_1$ or $F_2$. This is the case of the Ramsden eye-piece. Let $F_1$ and $F_2$ be each of 4 in. focal length, $d$ being $2/3\ F_1 = 2\frac{2}{3}$ in. Then

$$F_E = \frac{4 \times 4}{4 + 4 - 2\frac{2}{3}} = \frac{16}{5\frac{1}{3}} = 3''$$

$$E_1 \text{ or } E_2 = 4 \times 2\tfrac{2}{3}/5\tfrac{1}{3} = 2''$$

$F_B$ is therefore $3 - 2 = 1''$.

**Equivalence of Two Cc. Lenses.**—If $F_1$ and $F_2$ are both negative, and for example, $F_1 = -8$ in., $F_2 = -10$ in., and $d = 2$ in., then

$$F_E = \frac{-8 \times (-10)}{-8 - 10 - 2} = \frac{80}{-20} = -4 \text{ in.}$$

$$E_1 = -8 \times 2/-20 = \tfrac{4}{5} \text{ in.} \qquad E_2 = -10 \times 2/-20 = 1 \text{ in.}$$

$$t = 2 - (1 + \tfrac{4}{5}) = \tfrac{1}{5} \text{ in.}$$

**Special Cases.**—If $d$ equals the difference between $F_1$ and $F_2$ (both being concave), then $F_E$ is half that of the stronger lens, and the equivalent point measured from the weaker lens is midway between the two.

If $F_1 = F_2$, then $F_B + F_E = F_1$ or $F_2$.

**Equivalence of a Cx. and a Cc. Lens.**—Suppose $F_1 = 10$ cm., $F_2 = -15$ cm., and $d = 2$ cm. Then (Fig. 199)

$$F_E = \frac{10 \times (-15)}{10 - 15 - 2} = \frac{-150}{-7} = 21\tfrac{3}{7} \text{ cm.}$$

$$E_1 = 10 \times 2/-7 = -2\tfrac{6}{7} \text{ in front of } L_1$$

$$E_2 = -15 \times 2/-7 = 4\tfrac{2}{7} \text{ cm. in front of } L_2 \text{ or } 4\tfrac{2}{7} - 2 = 2\tfrac{2}{7} \text{ cm. in front of } L_1$$

$$t = 2 - (-2\tfrac{6}{7} + 4\tfrac{2}{7}) = \tfrac{4}{7} \text{ cm.}$$

If the negative lens is in front (Fig. 200), $E_2$ is $2\frac{6}{7}$ cm. behind the Cx., or $-2\frac{6}{7}-2=-4\frac{6}{7}$ cm. behind the Cc. In the first case $F_E$ lies $17\frac{1}{7}$ cm. behind the back lens, and in the second case $24\frac{2}{7}$ cm. behind it. The combination resembles that of a positive meniscus in which the optical centre lies outside the Cx. surface. Whether a combination, such as this, will have a positive or negative focal length depends, not only on the respective powers of the components, but also, and essentially, on the value of $d$. The weakest Cx. can more than neutralise the strongest Cc. if the separation be great enough.

**Special Cases.**—If the two lenses are separated by $F_1 + F_2$ (the sum of their focal lengths), the negative being of shorter focus, then $F_E = \infty$, and the lenses neutralise each other. This is the case of the *opera-glass*. Thus with 9 in. Cx. and a 4 in. Cc. separated by 5 in.

$$F_E = \frac{9 \times (-4)}{9-4-5} = \frac{-36}{0} = \infty$$

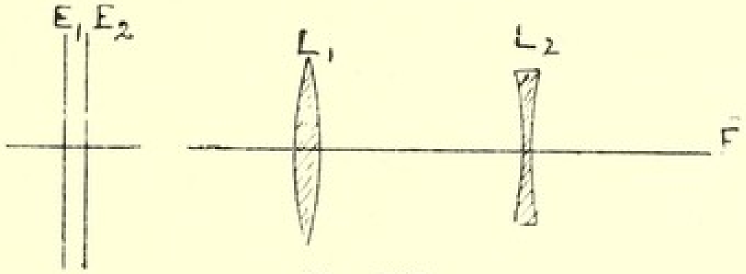



FIG. 199.

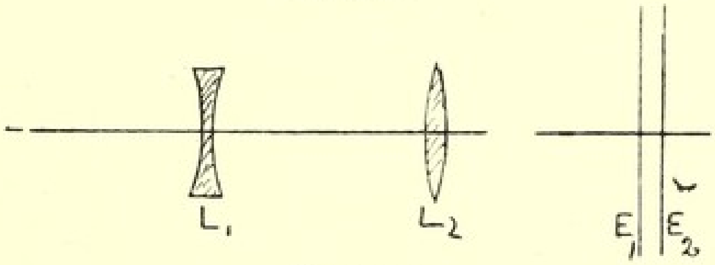



FIG. 200.

If $d < F_1 + F_2$, the combination is negative; if $d > F_1 + F_2$ it is positive.

**When $F_1 = -F_2$.**—If the two lenses have equal focal lengths, $F_B - F_E = F_1$ or $F_2$, and the formula for finding $F_E$ (which is positive) becomes simplified to $F_E = F^2/d$. In this case $E_1$ is negative and both equivalent planes lies beyond the Cx. lens; $E_1 = F_2$ and $E_2 = F_1$; also $t = d$.

**To find d for a given F.**—To find the distance $d$ which should separate two lenses so that they may have a given $F_E$ the following formula serves.

$$d = F_1 + F_2 - F_1F_2/F_E$$

If $d$ results in a negative quantity, it shows that the desired result is impossible. If both lenses are similar the formula may be written $d = 2F - F^2/F_E$, and if the one lens is Cx. and the other Cc. of equal power, the formula simplifies to $d = F^2/F_E$.

Thus when $F_1$ is 10 in. and $F_2$ is $-5$ in., in order that $F_E$ be 12 in.

$$d = 10 - 5 - 10 \times (-5)/12 = 5 - (-4\tfrac{1}{6}) = 9\tfrac{1}{6} \text{ in.}$$

So that the equivalent focal length may be 12 in. Cc. we find that

$$d = 10 - 5 - 10 \times (-5)/-12 = 5 - 4\tfrac{1}{6} = \tfrac{5}{6} \text{ in.}$$

**The Result of Separation.**—Separating two Cx. lenses results always in reduced power or longer $F_E$—indeed if $d$ is great, $F_E$ may become infinite, or even negative. With Cc. lenses the reverse occurs, the power being increased, or $F_E$ shortened. With a Cx. and a Cc. in combination the result varies with the powers of the two components as in the next paragraph, where the results are tabulated.

**Change of $F_E$ for Variation in d.**—As $d$ increases with two Cx. lenses, $F_E$ varies directly, and $t$ varies inversely or becomes negative.

As $d$ increases with two Cc. lenses, $F_E$ varies inversely, and $t$ varies directly.

As $d$ increases with one Cx. and the other Cc., the Cx. being the stronger, $F_E$ varies inversely and $t$ varies directly.

As $d$ increases with one Cx. and the other Cc., the Cc. being the stronger, and $F_E$ being negative, $F_E$ varies directly, and $t$ varies inversely or becomes negative.

As $d$ increases with one Cx. and the other Cc., the Cc. being the stronger, and $F_E$ being positive $F_E$ varies inversely, and $t$ varies inversely.

**Conjugate Foci.**—The equivalent focal length of two separated lenses being that of a single lens substituted for them, the ordinary formulæ for conjugate foci hold good, but the distance of $f_1$ is from $E_1$, and that of $f_2$ is measured from $E_2$, as with thick lenses (q.v.).

**Combination of More than Two Lenses.**—When more than two lenses are separated by intervals, the method of finding $F_E$ of the whole system is to obtain that of the first pair of lenses, and then combine this combination with the third lens, or another pair of lenses, and so on. It must be remembered that the distance $d$ between two combinations is that between *the two theoretically most adjacent equivalent points*, that is, between $E_2$ of the first and $E_1$ of the second combination ; also that the position of the equivalent points $E_1$ and $E_2$ of the whole combination is reckoned respectively from $E_1$ of the first, and $E_2$ of the second combination. In fact the calculations are similar to those required for two thick lenses (q.v.).

## Dioptral Equivalent Thin Lenses.

With dioptral powers, the equivalent power and points of two separated lenses are found from the following formulæ, where $D_1$ and $D_2$ are the powers of the two lenses, $d$ is the interval between them expressed in cm., $D_E$ is the equivalent dioptral lens, $E_1$ and $E_2$ are respectively the first and second equivalent points, and $t$ is the distance between $E_1$ and $E_2$.

$$D_F = D_1 + D_2 - D_1 D_2 d/100$$

$$E_1 = \frac{D_1 D_2 d}{D_1 (D_1 + D_2 - D_1 D_2 d/100)} = \frac{D_1 D_2 d}{D_1 D_E} = \frac{D_2 d}{D_E}$$

$$E_2 = \frac{D_1 D_2 d}{D_2 (D_1 + D_2 - D_1 D_2 d/100)} = \frac{D_1 D_2 d}{D_2 D_E} = \frac{D_1 d}{D_E}$$

$$t = d - (E_1 + E_2)$$

If $d$ is expressed in terms of a metre, we can write:

$$D_E = D_1 + D_2 - D_1 D_1 d$$

If $D_1$ is positive and equal in power to $D_2$, which is negative, then

$$D_E = D^2 d/100$$

The distance between two dioptral lenses so that they may have a certain equivalent dioptral power is found from

$$d = \frac{100 (D_1 + D_2 - D_E)}{D_1 D_2}$$

which, when $D_1$ and $D_2$ are equal, simplifies to

$$d = \frac{100 (2 D - D_E)}{D^2}$$

If the one lens is positive and the other negative and of equal powers, the formula becomes

$$d = 100 D_E / D^2$$

# CHAPTER XVIII

## THICK LENSES AND COMBINATIONS

HITHERTO we have considered all lenses as having no appreciable thickness in relation to their focal length, so that, as described in a previous chapter, all the refraction caused by the two surfaces may be presumed to take place on a single refracting plane passing through the optical centre. Further, this plane may be taken as coinciding with the surfaces and therefore, for practical purposes, all measurements may be taken from the lens itself, and all secondary axes passing through the optical centre assumed to undergo no lateral deviation. With a thick lens, however, these simplifications are not permissible.

Let Fig. 201 represent a thick bi-convex lens of which $X$ and $Y$ are the

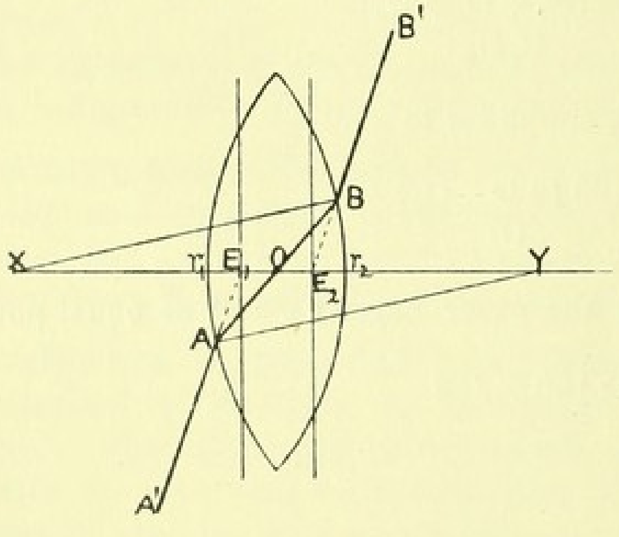



FIG. 201.

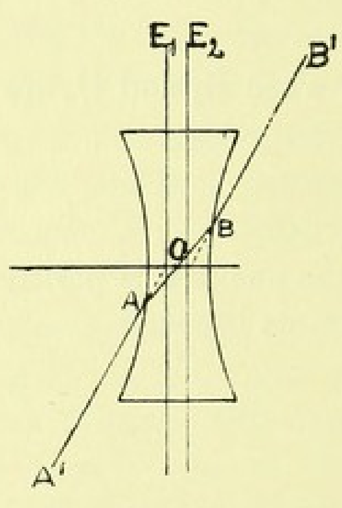



FIG. 202.

centres of curvature. From $X$ and $Y$ let any two parallel radii, such as $XB$ and $YA$ be drawn meeting their respective surfaces in $B$ and $A$; then tangent planes drawn through $A$ and $B$ are parallel, so that at these points the lens acts as a plate, and any ray $A'A$, incident at $A$, after transmission and refraction, emerges as $BB'$ parallel to its original course. As described in Chapter VII., the point $O$ where the ray cuts the axis is the optical centre, which is a fixed point whose position on $XY$ depends only upon the ratio of the radii of curvature.

The point $E_1$, towards which the secondary axis $A'A$ is directed, is the first equivalent point, while $E_2$, from which it apparently emerges, is the second equivalent point. $E_1$ and $E_2$ have precisely the same significance given them as in the previous chapter on thin lens combinations, i.e., they are the points from which the principal and secondary foci are measured, and through which pass the planes where all refraction is presumed to take place. In a single thick Cx. lens, however, $E_1$ and $E_2$ are never crossed as may occur in Cx. lens combinations. Fig. 202 shows the equivalent points and optical centre of a bi-concave lens.

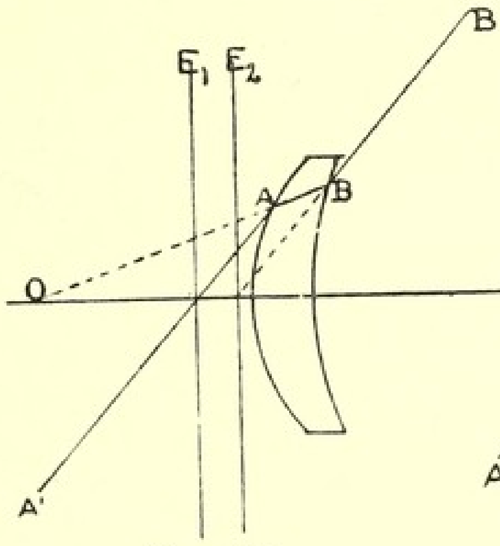



FIG. 203.

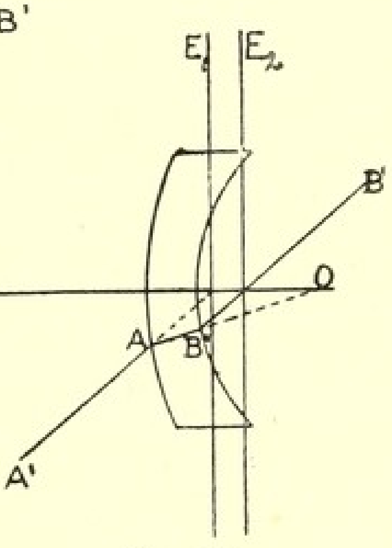



FIG. 204.

In periscopic Cx. or Cc. lenses (Figs. 203 and 204) both $E_1$ and $E_2$ generally lie outside the lens on the Cx. side of the PCx., and on the Cc. side of the PCc., but in some cases the one point may be outside, and the other still within the lens; moreover the optical centre $O$ lies outside the equivalent points. A ray directed to $E_1$ appears, after refraction, to proceed from $E_2$, its course $A\ B$ within the lens being on a line connecting the optical centre $O$, the point of incidence $A$ of the ray at the first surface, and

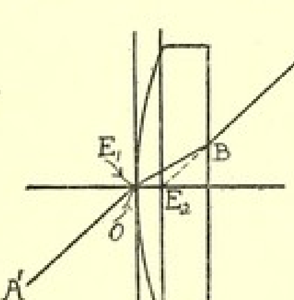



FIG. 205.

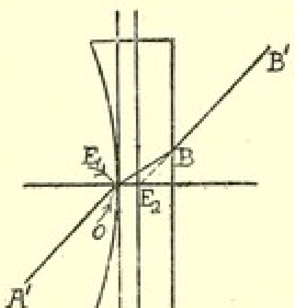



FIG. 206.

the point of emergence $B$ at the second surface. The position of $O$ is therefore determined by producing $B\ A$ to cut the principal axis.

In plano-Cx. and Cc. lenses (Figs. 205 and 206) the only point on the curved surface parallel to any point on the plane surface is at the vertex, through which passes the principal axis. Therefore $E_1$, the first equivalent point, and $O$, the optical centre, coincide at the curved surface.

All the secondary axes proceeding from the various points of a body are directed towards $E_1$, and after refraction appear to diverge from $E_2$, but they cut the principal axis at $O$, either actually or virtually, as in periscopic lenses.

The terms *nodal* or *principal* points are sometimes applied to the equivalent points ; but it is better to reserve the latter term for points that possess the functions of both the former, as they do in lenses where the first and last media—usually air—are similar. Nodal and principal points are discussed in the chapter on *Compound Refracting Systems.*

**The Effect of Thickness.**—This is clearly shown in the foregoing diagrams, and it may be said that a thick lens differs only from a thin one in that it has a plate-like power of laterally displacing all incident light. In other words a thin equivalent Cx. lens can be transformed into a thick lens by splitting it in the refracting plane and cementing the two halves to the opposites sides of a parallel plate. The consequence is that a thick Cx. has a weaker equivalent power than a thin one of similar curvature and $\mu$, while a thick Cc. has a stronger equivalent power than a thin one.

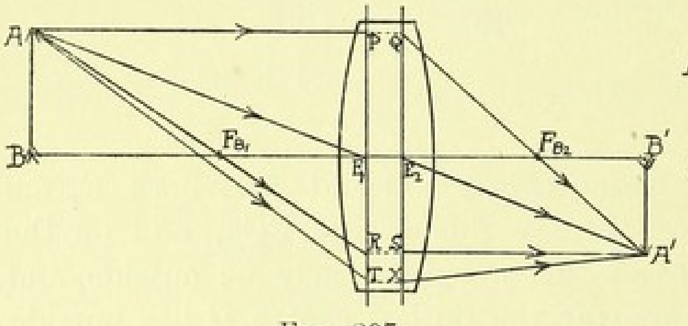


FIG. 207.

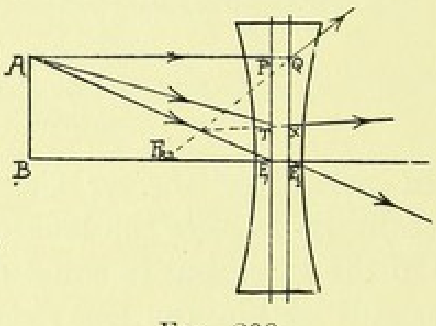


FIG. 208.

**Course of Rays through Thick Lenses.**—Fig. 207 represents a thick Cx. lens in front of which is the object $A\ B$. Any ray $A\ P$ parallel to the axis, takes the course $Q\ A'$ after refraction, and passes through $F_2$. The secondary axis $A\ E_1$, directed to $E_1$, proceeds from $E_2$ parallel to its original course, and a third ray $A\ R$, passing through $F_1$, is refracted as $S\ A'$ parallel to the axis. All three rays meet in the image point $A'$, so that $B'A'$ is the complete eal image of $A\ B$. Any other ray $A\ T$ directed towards the first equivaent plane at $T$ emerges from the second at $X$ and directed toward $A'$ such that $E_1\ T = E_2\ X$.

The construction in the case of thick Cc. is shown in Fig. 208. It is so obvious as not to need any special description.

**Direct Formulæ for a Single Thick Lens in Air.**—We will now proceed to find the back focal length or effectivity, the equivalent focal length, and the positions of the equivalent points in terms of the radii, thickness and index of the lens. Let

$F_E$ be the equivalent focal length.

$E_1$ and $E_2$ be the first and second equivalent points.

T be the distance between $E_1$ and $E_2$ (the optical interval).

$r_1$ and $r_2$ be the radii of curvature of respectively the first and second surfaces.

A and B be the first and second surfaces at the principal axis.

$\mu$ be the index of refraction of the glass.

$t$ be the thickness of the lens on the axis.

Fig. 209 represents a thick bi-convex lens; let $R\,Q$ be a ray incident at $Q$ and parallel to the principal axis $A\,B$; this will be deviated towards the axis by the first surface, and would, if not intercepted by the second surface, cross $A\,B$ in $D$, but is brought to a nearer point $F_B$ by the further refracting effect of the second surface. Then, from definition, $D$ is the posterior focus of the first surface, $F_B$ the principal focus of the lens as a whole, and $B\,F_B$ the back focal distance. Let $F_BC$ be produced backward to meet $R\,Q$ prolonged in $P$. Now a plane perpendicular to the axis, dropped through $P$, will locate the second equivalent plane and, where it cuts the axis, the second equivalent point $E_2$. All the refraction of incident light from the direction $RQ$ (parallel or otherwise) appears to take place on $P\,E_2$. The distance $E_2\,F_B$ is, therefore, the equivalent focal length, since it is the focal length of

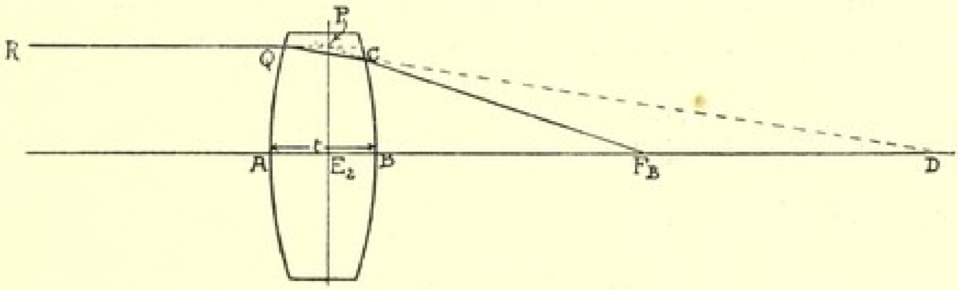



FIG. 209.

the single thin lens which, if placed in the plane of $E_2$, would have the same effect as the original thick lens as a whole.

The distance of $F_B$, the principal focus of a lens, measured from the second surface $B$, is determined by the sum of the anterior focal powers $1/F_A$ and $1/F'_A$ of the two surfaces respectively, that of the first being modified by $t/\mu$, the thickness of the lens, and the index of refraction of the medium through which the light travels, before it meets the second surface. That is

$$1/\mathrm{F}_B = 1/(\mathrm{F}_A - t/\mu) + 1/\mathrm{F}'_A$$

Substituting in the formula $r_1/(\mu-1)$ for $\mathrm{F}_A$, and $r_2/(\mu-1)$ for $\mathrm{F}'_A$ we get

$$\frac{1}{\mathrm{F}_B} = \frac{1}{r_1/(\mu-1) - t/\mu} + \frac{1}{r_2/(\mu-1)} = \frac{\mu(\mu-1)}{\mu r_1 - t(\mu-1)} + \frac{\mu-1}{r_2}$$

$$= \frac{\mu(\mu-1)(r_1+r_2-t(\mu-1)/\mu)}{\mu r_1 r_2 - t\,r_2(\mu-1)}$$

so that

$$\mathrm{F}_B = \mathrm{B\,F}_B = \frac{\mu r_1 r_2 - t\,r_2(\mu-1)}{\mu(\mu-1)(r_1+r_2-t(\mu-1)/\mu)}$$

Similarly the back focus from the other surface is

$$1/F_B = 1/(F'_A - t/\mu) + 1/F_A$$

which becomes

$$F_B = \frac{\mu r_1 r_2 - t r_1 (\mu - 1)}{\mu (\mu - 1) (r_1 + r_2 - t (\mu - 1)/\mu)}$$

These expressions give the back focal distance $F_B$. It remains now to find similar formulæ for the equivalent focal length and the positions of the equivalent points relative to the poles A and B of the surfaces.

In Fig. 209, since the ray $RQ$ is presumed to lie close to the axis, the arcs $QA$ and $CB$ may be taken as straight lines giving two pairs of similar triangles $CF_B B$ and $PF_B E_2$, $CDB$ and $QDA$. Then it follows that

$$E_2F_B/BF_B = PE_2/CB = QA/CB = AD/BD$$

so that

$$E_2F_B = BF_B \times AD/BD$$

But $E_2 F_B$ is the equivalent focal distance, $BF_B$ is the back focal distance, $AD$ the posterior focus of the first surface, and $BD$ is this quantity less the thickness $t$. On substituting these values, therefore, in the above equation we get as the expression for the equivalent focal length

$$F_E = E_2F_B = \frac{\mu r_1 r_2 - t r_2 (\mu - 1)}{\mu (\mu - 1) (r_1 + r_2 - t (\mu - 1)/\mu)} \times \frac{\mu r_1}{\mu - 1} \div \left(\frac{\mu r_1}{\mu - 1} - t\right)$$

which, when worked down, becomes

$$F_E = E_2F_B = \frac{r_1 r_2}{(\mu - 1) (r_1 + r_2 - t (\mu - 1) \mu)}$$

The distance of $E_2$ from the pole $B$ of the second surface, is found by subtracting the back from the equivalent focal distance, which in terms similar to those already used is

$$E_2 = \frac{r_2 t}{\mu (r_1 + r_2 - t (\mu - 1)/\mu)}$$

and the corresponding distance of $E_1$; from $A$ is

$$E_1 = \frac{r_1 t}{\mu (r_1 + r_2 - t (\mu - 1)/\mu)}$$

Now if, as is convenient, we calculate the quantity N which enters into the various formulæ, that is

$$N = r_1 + r_2 - t (\mu - 1)/\mu$$

we have

$$F_E = \frac{r_1 r_2}{(\mu - 1) N}$$

The back surface focal distances from the first and second surfaces respectively are

$$\text{from A} = \frac{\mu r_1 r_2 - t r_1 (\mu - 1)}{(\mu - 1) \mu N} \qquad \text{from B} = \frac{\mu r_1 r_2 - t r_2 (\mu - 1)}{(\mu - 1) \mu N}$$

The distances of the first and second equivalent points measured *inwards* from respectively the first and second surface on the principal axis are found from

$$E_1 = \frac{r_1 t}{\mu N}, \qquad E_2 = \frac{r_2 t}{\mu N}$$

The equivalent thickness T, i.e. the distance between the equivalent points, is

$$T = t - (E_1 + E_2)$$

It should be noted that the formula for $F_E$ is the same as for F of a thin lens, except that the quantity $t(\mu - 1)/\mu$ enters into it.

An approximate formula (accurate when $\mu = 1{\cdot}5$) is

$$F_E = \frac{r_1 r_2}{(\mu - 1)(r_1 + r_2 - t/3)}$$

**Example of a Bi-Cx. Lens.**—If $r_1$ and $r_2 = 10$ cm. and 6 cm. respectively, $\mu = 1{\cdot}5$, $t = 3$ cm., then

$$F_E = \frac{10 \times 6}{{\cdot}5\,(10 + 6 - 3 \times {\cdot}5/1{\cdot}5)} = \frac{60}{{\cdot}5 \times (16 - 1)} = \frac{60}{7{\cdot}5} = 8 \text{ cm}$$

$$E_1 = \frac{10 \times 3}{1{\cdot}5 \times (16 - 1)} = \frac{30}{22{\cdot}5} = 1{\cdot}33 \text{ cm.}$$

$$E_2 = \frac{6 \times 3}{1{\cdot}5 \times (16 - 1)} = \frac{18}{22{\cdot}5} = {\cdot}8 \text{ cm.}$$

$$T = 3 - (1{\cdot}333 + {\cdot}8) = {\cdot}86 \text{ cm.}$$

$F_E$ is anteriorly $8 - 1{\cdot}333 = 6{\cdot}66$ from $A$, and posteriorly $8 - {\cdot}8 = 7{\cdot}2$ cm. from $B$. The optical centre is located at

$3 \times 6/(10 + 6) = 1{\cdot}125$ cm. from B, and $1{\cdot}875$ cm. from A.

A thin lens of same radii and $\mu$ has

$$F = \frac{10 \times 6}{{\cdot}5 \times (10 + 6)} = \frac{60}{8} = 7{\cdot}5 \text{ cm.}$$

Thus we see that *in a bi-Cx. thick lens the true or equivalent focal length is longer than that of the corresponding thin lens, but its back focal length is shorter.* In the case of the thick lens $F = 8$ cm. from $E_2$, but $7{\cdot}2$ cm. from $B$, while if the lens were thin so that $t = 0$, $F$ would be $7{\cdot}5$ cm. from $B$. If two Cx. lenses be made of the same glass and similar curvatures, but the one thicker than the other, the thicker lens is actually the weaker, although its effectivity is greater, i.e. its back focal distance is shorter.

**Example of a Bi-Cc. Lens.**—In Fig. 210 let $r_1$ and $r_2 = -10$ cm. and $-6$ cm. respectively, $\mu = 1{\cdot}5$ and $t = 3$ cm.

$$F_E = \frac{-10 \times (-6)}{{\cdot}5 \times (-10 - 6 - 3 \times {\cdot}5/1{\cdot}5)} = \frac{60}{{\cdot}5 \times (-17)} = -7{\cdot}06 \text{ cm.}$$

$$E_1 = \frac{-10 \times 3}{1{\cdot}5 \times (-17)} = 1{\cdot}18 \text{ cm.} \qquad E_2 = \frac{-6 \times 3}{1{\cdot}5 \times (-17)} = {\cdot}7 \text{ cm.}$$

$$T = 3 - (1{\cdot}18 - {\cdot}7) = 1{\cdot}12 \text{ cm.}$$

Although the true focal length is the same, if the surface *B* faces the

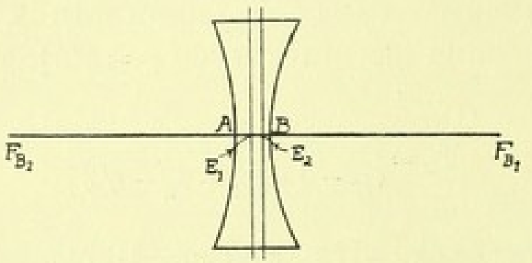



Fig. 210.

light, *F* lies 8·88 cm. from *A*, while if *A* faces the light *F* is 9·36 cm. from *B*. If the lens were thin $F = 7{\cdot}5$ cm., *so that increased thickness causes a Cc. to have a greater equivalent power, but a smaller effectivity.*

**Example with a Plano-Cx. Lens.**—Let $r_1$ (Fig. 211) that of the curved surface $= 6$ cm.; $r_2$ of the plano $= \infty$; $\mu = 1{\cdot}5$, and $t = 3$ cm. Then since $r_2 = \infty$, and this quantity occurs in the upper and lower part of the formula, we can omit it from our calculations as well as the other quantities in the

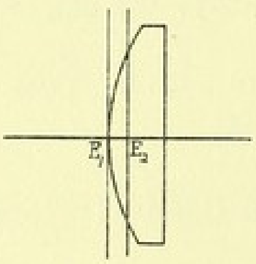



Fig. 211.

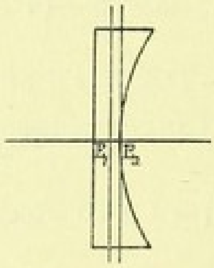



Fig. 212.

bracket containing this value. The formula therefore simplifies to that used for a thin lens, viz. $F_E = r_1/(\mu - 1)$

$$E_1 = 6 \times 3/1{\cdot}5\,\infty = 0, \; E_2 = 3/1{\cdot}5 = 2 \text{ cm.}$$
$$F_E = 6/{\cdot}5 = 12 \text{ cm.} \quad T = 3 - 2 = 1 \text{ cm.}$$

$E_1$ is at the curved surface, and $E_2$ is 2 cm. in front of the plane surface. In the above example, when the Cx. surface is exposed to the light, *F* lies $12 - 2 = 10$ cm. behind the plane, and 13 cm. behind the curved surface. When the plane surface is so exposed, *F* lies 12 cm. behind the curved and 15 cm. behind the plane surface.

**Example with a Plano-Cc. Lens.**—If $r_1$ (Fig. 212) that of the plane surface $= \infty$, as before stated, it may be neglected. Let $r_2$, that of the Cc. $= 6$ cm., $\mu = 1{\cdot}5$, and $t = 3$ cm.

$$E_1 = 3/1{\cdot}5 = 2 \text{ cm.} \qquad E_2 = 6 \times 3/1{\cdot}5\, \infty = 0.$$
$$F_E = -6/{\cdot}5 = -12 \text{ cm.} \qquad T = 3 - 2 = 1 \text{ cm.}$$

$E_1$ is 2 cm. from the plane surface, and $E_2$ is at the Cc. surface. If the curved surface faces the light the focal distance is $12 + 2 = 14$ cm. in front of the plano and 11 cm. from the curved surface. When the light is incident on the plane surface, $F$ lies 12 cm. from the curved and 9 cm. from the plane surface.

**Example of a Positive Meniscus.**—In a periscopic Cx. lens (Fig. 213) let $r_1$ and $r_2$ of the Cx. and Cc. surfaces respectively be $+6$ cm. and $-10$ cm., $\mu = 1{\cdot}5$ and $t = 3$ cm.

$$F_E = \frac{6 \times (-10)}{{\cdot}5\,(6 - 10 - 3 \times {\cdot}5/1{\cdot}5)} = \frac{-60}{{\cdot}5 + (-5)} = 24 \text{ cm.}$$

$$E_1 = 6 \times 3/1{\cdot}5 \times -5 = -2{\cdot}4 \text{ cm.},\ E_2 = -10 \times 3/1{\cdot}5 \times -5 = 4 \text{ cm.}$$

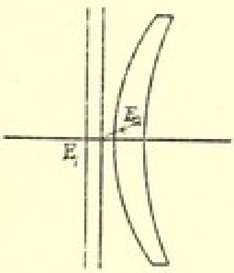



FIG. 213.

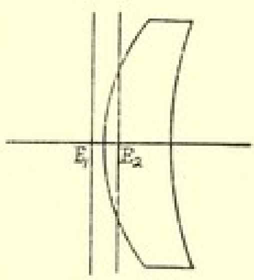



FIG. 214.

$E_1$ being negative must be reckoned outwards, so that the distance of both equivalent points are reckoned the same way, the first outwards from the Cx. surface, the second inwards from the Cc. $E_1$ is $2{\cdot}4$ cm. and $E_2$ is $4 - 3 = 1$ cm. outside the Cx. surface. In this example $T = 3 - (-2{\cdot}4 + 4) = 1{\cdot}4$ cm.

In some cases, with a periscopic Cx. lens, the one equivalent point lies within the Cx. surface, as in Fig. 214. The positions of $E_1$ and $E_2$ depend on the curvatures of the two surfaces; the more nearly equal the two curvatures, the more are $E_1$ and $E_2$ displaced towards the Cx. surface or beyond it. The distance of $F_B$ varies very considerably as the one or the other surface is exposed to the light.

**Example of a Negative Meniscus.**—In a periscopic Cc., as in Fig. 215, let $r_1$ and $r_2$ of the Cx. and Cc. surfaces respectively $= +10$ cm., and $-6$ cm., $\mu = 1{\cdot}5$ and $t = 3$ cm.

$$F_E = \frac{10 \times (-6)}{{\cdot}5\,(+10 - 6 - 3 \times {\cdot}5/1{\cdot}5)} = \frac{-60}{{\cdot}5 \times (3)} = -40 \text{ cm.}$$

$$E_1 = 10 \times 3/1{\cdot}5 \times 3 = 6{\cdot}66 \text{ cm.},\ E_2 = -6 \times 3/1{\cdot}5 \times 3 = -4 \text{ cm.}$$

That is, the distance of both equivalent points are reckoned the same way,

$E_1$ inwards from the Cx. surface, and $E_2$, being negative, outwards from the Cc. surface. The first is 6·66 – 3 = 3·66 cm. outside the Cc. surface, and the second is 4 cm. outside it. T = 3 – (6·66 – 4) = ·33 cm.

As with the Cx. in some cases, the one equivalent plane of a Cc. meniscus lies within the Cc. surface (Fig. 216). Also, the difference in the distance is very marked as $E_1$ or $E_2$ is taken as the first equivalent point.

**Special Cases.**—Certain special cases of menisci are considered in the following articles.

**Afocal Lenses.**—In a meniscus when $r$ of the Cx. is longer than that of the Cc. surface $F_E = \infty$ if

$$r_1 + r_2 = t\,(\mu - 1)/\mu, \text{ or } t = \mu\,(r_1 + r_2)/(\mu - 1)$$

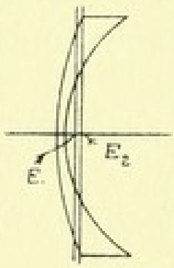



FIG. 215.

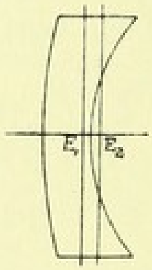



FIG. 216.

Thus, in order that F = ∞ when $r_1 = -1$, $r_2 = +3$, and $\mu = 1·5$.

$$t = 1·5\ (-1 + 3)/·5 = 6 \text{ cm.}$$

This is the principle of the Steinheil cone (Fig. 217), which is practically a fixed focus opera-glass.

If $r_1 = +10$ and $t = 3$ when $\mu = 1·5$, then (Fig. 218) $r_2$ must be – 9 in order that $F_E = \infty$.

Fig. 218 illustrates the form of the *worked* globular or coquille of the

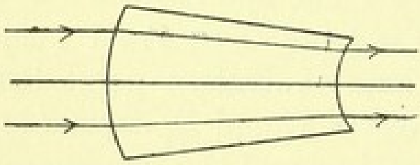


FIG. 217.

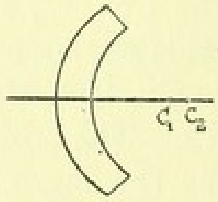



FIG. 218.

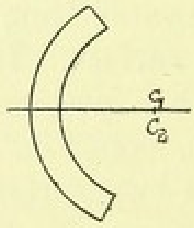



FIG. 219.

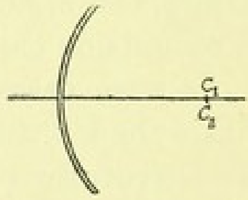



FIG. 220.

optical trade, where a true afocal effect is required. It is evident that to secure this condition the radius of the Cc. surface must be slightly shorter than that of the Cx. by an amount equal to approximately a third the thickness of the lens, $F_E$ being infinite when $r_1 + r_2 = t(\mu - 1)/\mu$. $F_E$ is positive when $r_1 + r_2$ is less than $t(\mu - 1)/\mu$ and negative when $r_1 + r_2$ is greater than $t(\mu - 1)/\mu$. That is to say, when the Cc. surface has the shorter radius, F is positive or negative according as $t$ is sufficiently great or small respectively; and that, when $t$ is of certain value, the power of the Cx. surface neutralises that of the Cc.

**Concentric Lenses.**—If $r_1 + r_2 = t$ ($r_2$ being negative) i.e. if $r_1 - t = r_2$, so that the two centres of curvature coincide, F is negative. Thus (Fig. 219) let $r_1 = 10$ cm., $r_2 = -6$ cm., $t = 4$ cm., and $\mu = 1{\cdot}5$. Then

$$F_E = \frac{10 \times (-6)}{{\cdot}5\,(10 - 6 - 4 \times {\cdot}5/1{\cdot}5)} = \frac{-60}{{\cdot}5\,(2{\cdot}666)} = -45 \text{ cm.}$$

$$E_1 = r_1 = 10\,;\; E_2 = r_2 = -6\,;\; T = 4 - (10 - 6) = 0.$$

The principal points coincide at the common centre of curvature.

If the glass be thin (Fig. 220), and the centres coincide, the concave radius is shorter; we then get a *slightly* concave effect, as is found in the ordinary *unworked* globular or coquille.

If $r_1$, the radius of the Cx., is shorter than $r_2$, that of the Cc. $F_E$ is positive; but if $t$ is greater than F of the first surface, the light is brought to a focus within the lens, and, after crossing, diverges to the Cc. surface.

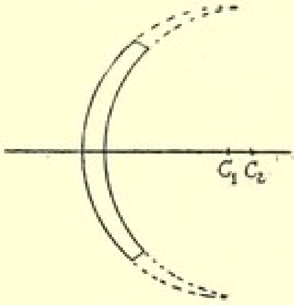



FIG. 221.

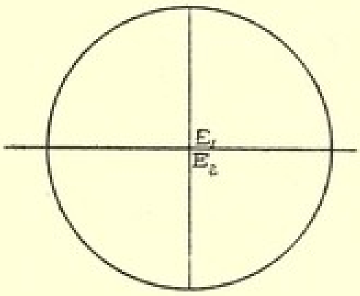



FIG. 222.

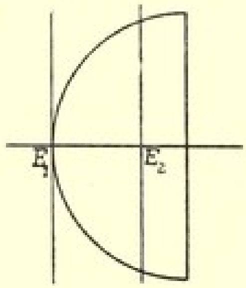



FIG. 223.

**Equi-Curved Lenses.**—If $r_1 = -r_2$ (Fig. 221), $F_E$ is positive. Thus, let $r_1 = +10$ cm., $r_2 = -10$ cm., $t = 3$ cm., and $\mu = 1{\cdot}5$. Then

$$F_E = \frac{10 \times (-10)}{{\cdot}5\,(10 - 10 - 3 \times {\cdot}5/1{\cdot}5)} = \frac{-100}{-{\cdot}5} = 200 \text{ cm.}$$

$$E_1 \text{ or } E_2 = r/(\mu - 1), \text{ in this case } 10/{\cdot}5 = 20 \text{ cm.}$$

If $t$ is greater than F of the Cx., this being towards the light, F lies in front of the second surface. If $t$ is very small $F_E = \infty$.

**The Sphere.**—In a sphere (Fig. 222), $r_1 = r_2$ and $t =$ the diameter $= 2r$. Let $\mu = 1{\cdot}5$, and $r = 6$ cm., so that $t = 12$ cm.

$$F_E = \frac{6 \times 6}{{\cdot}5\,(6 + 6 - 12 \times {\cdot}5/1{\cdot}5)} = \frac{36}{{\cdot}5(8)} = 9 \text{ cm.}$$

$$E_1 \text{ or } E_2 = 6 \times 12/1{\cdot}5 \times 8 = 6 \text{ cm.} \qquad T = 12 - (6 + 6) = 0.$$

Therefore, the equivalent planes of a sphere coincide and pass through the centre of curvature $C$, as in Fig. 222. The formulæ, in the case of a sphere, simplify to

$$F_E = \mu r/2\,(\mu - 1), \text{ and } F_B = r\,(2 - \mu)/2\,(\mu - 1)$$

When the $\mu$ of a sphere is 1·5, $F_E = 1·5\ r$, and $F_B = ·5\ r$. Calculations with a sphere are similar to those of any other thick lens when the object is situated outside the sphere. If, however, the object be within the sphere, the calculations are similar to those connected with a single surface.

**The Hemisphere.**—With the hemisphere (Fig. 223) the Cx. surface to the front, $F_E = r/(\mu - 1)$, $E_1 = r_1 t/\infty = 0$, $E_2 = t/\mu$; from the Cx. surface $F_B = F_E = r/(\mu - 1)$; from the plane surface $F_B = r/\mu(\mu - 1)$. When $\mu = 1·5$, $F_E = 2r$; $F_B = 2r$ from the Cx., and $1\frac{1}{3}r$ from the plane surface.

**Other Calculations.**—What radius must be given to a DCx. lens so that $F_E = 5$ cm. when $\mu = 1·5$ and $t = ·75$ cm. Substituting the known values we have

$$5 = \frac{r^2}{·5\,(2r - ·5 \times ·75/1·5)} = \frac{r^2}{·5\,(2r - ·25)} = \frac{r^2}{r - ·125}$$

then

$$r^2 - 5\,r = -·625.$$

Adding to both sides of the equation $(5/2)^2 = 6·25$

$$r^2 - 5r + 6·25 = -·625 + 6·25$$

Extracting the square root of each side gives

$$r - 2·5 = \pm 2·35$$

so that $r = 2·5 + 2·35 = 4·85$, or $2·5 - 2·35 = ·15$, of which ·15 is the impossible answer. Therefore the required radius is 4·85 cm.

### The Equivalent Power and Points of a Thick Lens by the Dioptric System.

Let $r_1$ be the radius of the first, and $r_2$ that of the second surface, let $D_E$ be the equivalent dioptric power, and $E_1$ and $E_2$ the equivalent points. Then

$$D_E = \frac{100\,(\mu - 1)\,(r_1 + r_2 - t\,(\mu - 1)/\mu)}{r_1 r_2} = \frac{100N\,(\mu - 1)}{r_1 r_2}$$

$$E_1 = r_1 t/\mu N, \qquad E_2 = r_2 t/\mu N, \qquad T = t - (E_1 + E_2)$$

If the distances are expressed in terms of a metre

$$D_E = \frac{(\mu - 1)\,(r_1 + r_2 - t\,(\mu - 1)/\mu)}{r_1 r_2} = \frac{N\,(\mu - 1)}{r_1 r_2}$$

### Calculations of a Thick Lens in Terms of the Foci of its Surfaces.

Instead of expressing the constants of a thick lens in air directly in terms of its radii and index, we may deduce some simpler formulæ involving only the foci of the two surfaces.

Let $f_1$ and $f_2$ represent respectively the anterior and posterior focal distances of the first surface, and $f_2'$ and $f_1'$ represent respectively the anterior and posterior focal distances of the second surface. Let $t$ be the thickness of the lens; then

$$f_1 = r_1/(\mu - 1), \qquad f_2 = \mu r_1/(\mu - 1)$$
$$f_1' = \mu r_2/(\mu - 1), \qquad f_2' = r_2/(\mu - 1)$$

Using the same diagram as for finding the direct formulæ given previously, we see that $D$, as shown on page 215, serves as a virtual object for the second surface $B$ of radius $r_2$, and the distance $B\,F_B$, which is the second back focus $F_B$, is the final image distance with respect to $D$ the virtual object.

Let $B\,D = u$ and $B\,F_B = v$. Now the expression connecting the conjugate foci of the second surface $B$ is

$$1/v + \mu/u = (\mu - 1)/r_2$$

But $1/f_2' = (\mu - 1)/r_2$, and $u = A\,D - t = f_2 - t$, the latter expression being reckoned a negative distance.

Therefore $$1/v - \mu/(f_2 - t) = 1/f_2'$$

whence $$\frac{1}{v} = \frac{1}{f_2'} + \frac{\mu}{f_2 - t} = \frac{\mu f_2' + f_2 - t}{f_2'\,(f_2 - t)}$$

But $$\mu f_2' = \mu r_2/(\mu - 1) = f_1'$$

Therefore $$v = \mathrm{F_B} = \frac{f_2'(f_2 - t)}{f_1' + f_2 - t}$$

The corresponding back focus from A, by similar reasoning, is

$$\mathrm{F_A} = \frac{f_1\,(f_1' - t)}{f_1' + f_2 - t}$$

**Equivalent Focus.**—In Fig. 209 $P$ is the second equivalent plane and $E_2$ the second equivalent point. As before we may consider $C\,B$ as being sensibly straight.

Then $$\mathrm{E_2F_B/B\,F_B = P\,E_2/C\,B = Q\,A/C\,B = A\,D/B\,D}$$

and $$\mathrm{E_2F_B = A\,D \times B\,F_B/B\,D}$$

But $$\mathrm{A\,D} = f_2, \qquad \mathrm{B\,F_B = F_B}, \qquad \text{and } \mathrm{B\,D} = f_2 - t$$

Therefore $$\mathrm{F_E} = f_2 \times \frac{f_2'(f_2 - t)}{f_1' + f_2 - t} \times \frac{1}{f_2 - t}$$

$$= \frac{f_2 f_2'}{f_1' + f_2 - t}$$

It is important to notice that $f_2f_2'=f_1f_1'$, so that $F_E$ may be taken as

$$\frac{f_1f_1'}{f_1'+f_2-t}$$

when the light is incident first on the surface B.

**Equivalent Points.**—The distances of $E_1$ and $E_2$ from the surfaces A and B are found by deducting from the equivalent focal length the respective back foci. Thus

$$E_1=F_E-F_A=\frac{f_1f_1'}{f_1'+f_2-t}-\frac{f_1(f_1'-t)}{f_1'+f_2-t}=\frac{f_1t}{f_1'+f_2-t}$$

Similarly

$$E_2=F_E-F_B=\frac{f_2f_2'}{f_1'+f_2-t}-\frac{f_2'(f_2-t)}{f_1'+f_2-t}=\frac{f_2't}{f_1'+f_2-t}$$

**Example.**—Let $r_1=10$ cm., and $r_2=6$ cm., $\mu=1\cdot5$, $t=3$ cm.; then

$f_1=10/(1\cdot5-1)=20$ cm. $\quad f_2=1\cdot5\times10/(1\cdot5-1)=30$ cm.
$f_1'=1\cdot5\times6/(1\cdot5-1)=18$ cm. $\quad f_2=6/(1\cdot5-1)=12$ cm.

When light is incident first on the surface A

$$F_E=\frac{f_2f_2'}{f_1'+f_2-t}=\frac{30\times12}{18+30-3}=8 \text{ cm.}$$

and when incident on B

$$F_E=\frac{f_1f_1'}{f_1'+f_2-t}=\frac{20\times18}{18+30-3}=8 \text{ cm.}$$

the equivalent foci, of course, being the same in either case.

The equivalent points $E_1$ and $E_2$ are distant from A and B respectively

$$E_1=\frac{f_1t}{f_1'+f_2-t}=\frac{20\times3}{45}=1\cdot33 \text{ cm.}$$

$$E_2=\frac{f_2't}{f_1'+f_2-t}=\frac{12\times3}{45}=\cdot8 \text{ cm.}$$

Thus the back foci from A and B are respectively $8-1\cdot33=6\cdot66$ cm., and $8-\cdot8=7\cdot2$ cm.

## Combination of Thick Lenses.

**Two Thick Lenses in Combination.**—Let *A* be the first and *B* the second lens of a combination of two thick convex lenses separated by an interval.

Let $r_1$ and $r_2$ be the radii of curvature of *A*, and $r_1'$ and $r_2'$ those of *B*.

Let $t_1$ and $t_2$ be, respectively, the actual thicknesses of *A* and *B*.

Let $E_1$ and $E_2$ be, respectively, the first and second equivalent points of *A*.

Let $E_1'$ and $E_2'$ be, respectively, the first and second equivalent points of $B$.

Let $T_1$ and $T_2$ be, respectively, the equivalent thicknesses of $A$ and $B$.

Let $F_1$ and $F_2$ be, respectively, the focal lengths of $A$ and $B$.

Let $d$ be their distance apart, this being the distance between their most adjacent equivalent points, i.e., the distance between $E_2$ and $E'_1$.

Let $E$ and $E'$ be, respectively, the first and second equivalent points of the combination.

Let $F$ be the equivalent focal distance of the combination.

Let $T$ be the equivalent thickness of the combination.

The equivalent focal distance $F$ of two combined lenses is obtained from the formula

$$\mathrm{F}=\frac{\mathrm{F}_1\mathrm{F}_2}{\mathrm{F}_1+\mathrm{F}_2-d}=\frac{\mathrm{F}_1\mathrm{F}_2}{\mathrm{N}}$$

which is the same as that previously proved for two thin lenses in combination. This illustrates the great utility of the equivalent planes in simplifying all thick lens calculations, since, provided we measure from the equivalent planes, a combination can in every way be treated as a simple system.

Similarly the distance of $E$, the first equivalent point of the combi nation, measured from $E_1$, the first equivalent point of $A$, is

$$\mathrm{E}=\frac{\mathrm{F}_1 d}{\mathrm{F}_1+\mathrm{F}_2-d}=\frac{\mathrm{F}_1 d}{\mathrm{N}}$$

The distance of $E'$, the second equivalent point of the combination, measured from $E_2'$, the second equivalent point of $B$, is

$$\mathrm{E}'=\frac{\mathrm{F}_2 d}{\mathrm{F}_1+\mathrm{F}_2-d}=\frac{\mathrm{F}_2 d}{\mathrm{N}}.$$

The distance $T=E\,E'$, between the equivalent points of the combination, is determined by the following

$$\mathrm{T}=d+\mathrm{T}_1+\mathrm{T}_2-(\mathrm{E}+\mathrm{E}') \qquad \text{or} \qquad \mathrm{T}=\mathrm{T}_1+\mathrm{T}_2-d^2/\mathrm{N}.$$

**As an example** let

$r_1=10$ cm., $r_2=8$ cm., and $t_1=2$ cm.
$r'_1=9$ cm., $r'_2=7$ cm,, and $t_2=2$ cm.
$\mu=1{\cdot}5$ and $d=2{\cdot}5$ cm.

Then, when calculated, we obtain

$\mathrm{F}_1=9{\cdot}23$ cm., $\mathrm{E}_1={\cdot}769$ cm., $\mathrm{E}_2={\cdot}615$ cm., $\mathrm{T}_1={\cdot}616$ cm.
$\mathrm{F}_2=8{\cdot}26$ cm., $\mathrm{E}'_1={\cdot}783$ cm., $\mathrm{E}'_2={\cdot}609$ cm., $\mathrm{T}_2={\cdot}608$ cm.

and for the combination

$$F = \frac{9{\cdot}23 \times 8{\cdot}26}{9{\cdot}23 + 8{\cdot}26 - 2{\cdot}5} = \frac{76{\cdot}2398}{15} = 5{\cdot}08 \text{ cm.}$$

$$E = \frac{9{\cdot}23 \times 2{\cdot}5}{9{\cdot}23 + 8{\cdot}26 - 2{\cdot}5} = \frac{23{\cdot}075}{15} = 1{\cdot}538 \text{ cm.}$$

$$E' = \frac{8{\cdot}26 \times 2{\cdot}5}{9{\cdot}33 + 8{\cdot}25 - 2{\cdot}5} = \frac{20{\cdot}65}{15} = 1{\cdot}377 \text{ cm.}$$

$$T = 2{\cdot}5 + {\cdot}616 + {\cdot}608 - (1{\cdot}538 + 1{\cdot}377) = {\cdot}81 \text{ cm.}$$

or $$T = {\cdot}616 + {\cdot}608 - 2{\cdot}5^2/15 = 1{\cdot}224 - 6{\cdot}25/15 = {\cdot}81 \text{ cm.}$$

The combination is of 5·08 cm. focal length and its equivalent planes are ·81 cm. apart.

**Example with a Convex and a Concave Lens.**—Let $F_1 = +12$ in.;

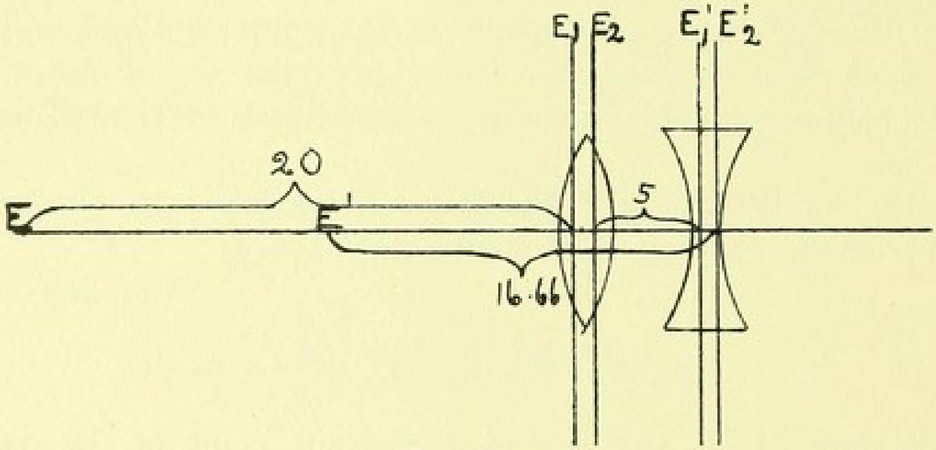



Fig. 224.

$F_2 = -10$ in.; $d = 5$ in.; $T_1 = {\cdot}5$ in.; $T_2 = {\cdot}2$ in.; then combined we obtain (Fig. 224)

$$F = \frac{+12 \times (-10)}{+12 - 10 - 5} = \frac{-120}{-3} = +40 \text{ in.}$$

$$E = 12 \times 5/-3 = -20 \text{ in.} \qquad E' = -10 \times 5/-3 = 16{\cdot}66 \text{ in.}$$

$$T = 5 + {\cdot}5 + {\cdot}2 - (-20 + 16{\cdot}66) = 9{\cdot}03 \text{ in.}$$

or $$T = {\cdot}5 + {\cdot}2 - 5^2/-3 = {\cdot}7 - 25/-3 = {\cdot}7 - (-8{\cdot}33) = 9{\cdot}03 \text{ in.}$$

**Coincidence of E and E′.**—In order that $E$ and $E'$ should coincide, $d$ can be found, for two Cx. or two Cc. lenses, by the following formula.

$$d = \frac{\sqrt{(T_1 + T_2)^2 + 4\,(F_1 + F_2)\,(T_1 + T_2)} - (T_1 + T_2)}{2}$$

Taking as an example a combination where $F_1 = 9$ in., $F_2 = 8$ in., $T_1 =$ ·2 in., and $T_2 = {\cdot}3$ in.

$$d = \frac{\sqrt{(\cdot 2 + \cdot 3)^2 + 4 \times (9 + 8) \times (\cdot 2 + \cdot 3)} - (\cdot 2 + \cdot 3)}{2}$$

$$d = \frac{\sqrt{\cdot 25 + 34} - \cdot 5}{2} = \frac{5\cdot 8524 - \cdot 5}{2} = 2\cdot 6762 \text{ in.}$$

When the lenses are 2·6762 in. apart $T = 0$.

**To find $F_E$ of more than Two Lenses.**—When more than two lenses are in combination the equivalent cardinal points of two of them are determined, and then this combination is again combined with the third lens, or with another equivalent lens as the case might be. Thus, if there are four lenses, A B C D, the equivalent of A and B, also of C and D, are found separately, and these two equivalent combinations again merged into a single one, or the focal length of such a combination can be found directly by the Gauss equation given later.

## The Equivalent Power and Points of Two Thick Lenses by the Dioptric System.

Let $D_1$ and $D_2$ be the powers of the two lenses, $T_1$ and $T_2$ their respective optical thicknesses, and $d$ the distance in cm. between the adjacent equivalent planes of the two lenses.

$$D = D_1 + D_2 - D_1 D_2 d/100$$

If $d$ is expressed in terms of a metre

$$D = D_1 + D_2 - D_1 D_2 d$$

The first equivalent plane E of the combination is distant from $E_1$ of the first lens

$$E = \frac{D_1 D_2 d}{D_1 (D_1 + D_2 - D_1 D_2 d/100)} = \frac{D_1 D_2 d}{D_1 D} = \frac{D_2 d}{D}$$

The second equivalent plane $E_1$ of the combination is distant from $E_2$ of the second lens

$$E' = \frac{D_1 D_2 d}{D_2 (D_1 + D_2 - D_1 D_2 d/100)} = \frac{D_1 D_2 d}{D_2 D} = \frac{D_1 d}{D}$$

$$T = d + T_1 + T_2 - (E + E')$$

**Conjugate Foci.**—It should be noted that, once the equivalent points and foci have been located, calculations of conjugate foci with thick lenses are the same as with thin lenses provided all measurements are taken from the adjacent equivalent planes.

Let $f_1$ be the distance of the object from $E_1$, $f_2$ be the distance of the image from $E_2$, and F the focal length of the combination. The relative sizes of image and object $h_2$ and $h_1$ are proportional to their distances from their adjacent planes, i.e., the image from the second and the object from the first; so that

$$1/F = 1/f_1 + 1/f_2 \qquad \text{and} \qquad h_2/f_2 = h_1/f_1$$

Thus let *A B* be the object 3 cm. long and placed 30 cm. in front of $E_1$ of a combination whose $F_E = 6$ cm.; then

$$1/f_2 = 1/6 - 1/30 = 4/30, \text{ so that } f_2 = 7{\cdot}5 \text{ cm. from } E_2$$

$$h_2 = 3 \times 7{\cdot}5/30 = {\cdot}75 \text{ cm.}$$

Let O be 20 cm. from the surface *A* of the lens calculated on page 224. To find the distance of the conjugate image from *B*, the distance $f_1$ is 20 cm. from *A* and therefore $20 + 1{\cdot}33 = 21{\cdot}33$ cm. from $E_1$, and since *F* is 8 cm. we have $1/f_2 = 1/8 - 1/21{\cdot}33$, whence $f_2 = 12{\cdot}8$ cm. Now $f_2$ is measured from $E_2$, which is ·8 cm. from *B*. Therefore the distance of the image from the second surface of the lens is $12{\cdot}8 - {\cdot}8 = 12$ cms. The calculation for the corresponding thin lens would be $1/f_2 = 1/8 - 1/20$, whence $f_2 = 13{\cdot}33$ cm., and as both surfaces are considered coincident with the optical centre, the distance of the image, in this case, from the lens is 1·33 cm. more than when a thickness of 3 cm. exists. Similar calculations can be made for any type of thick lens or lens combination.

**Construction.**—In constructing images formed by a thick lens (Figs. 207 and 208) or system of lenses, the equivalent planes and points must be made use of in place of the single refracting plane and the optical centre of a thin lens. The course of any ray incident on the plane of $E_1$ is continued from a point on the plane of $E_2$ equally distant from the principal axis, the rays being presumed to pass over the optical interval $E_1E_2$ without further deviation.

As described on page 214, the construction is made by drawing a ray parallel to the axis to the first equivalent plane and continuing its course from the first equivalent plane through $F_2$; drawing another ray to $E_1$ and continuing it from $E_2$ parallel to its original course. These two rays meet at $A'$, which is the image of *A*. $B'A'$ is to *A B* as $B'E_2$ is to $B E_1$.

Similar constructions serve for other forms of thick lenses, also for systems of lenses where the equivalent points are crossed, or where they lie outside the lenses.

**Planes of Unit Magnification.**—At a distance equal to 2 F measured from $E_1$ anteriorly, and from $E_2$ posteriorly there are two points $S_1$ and $S_2$ and their corresponding planes on the principal axis termed the *symmetrical points and planes*, which present the following properties (Fig. 225): (1) An object point situated in the one symmetrical point has its image at the other; (2) any point *A* or *B* on the one symmetrical plane has its image $A'$ or $B'$, respectively, on the other symmetrical plane at an equal distance from the principal axis. Thus, when an object *A B* is situated at the one symmetrical plane, its image *B′ A′* is situated at the other, and the two are of equal size; these are the planes of unit magnification for real images.

The planes of unit *virtual* magnification for thick lenses and lens systems lie in *the equivalent planes* themselves. In other words the equivalent planes are images of each other.

**Construction for the Course of a Ray.**—This is similar to those employed for other surfaces. Let *A B* (Fig. 226) be a ray incident at *B*. Draw the

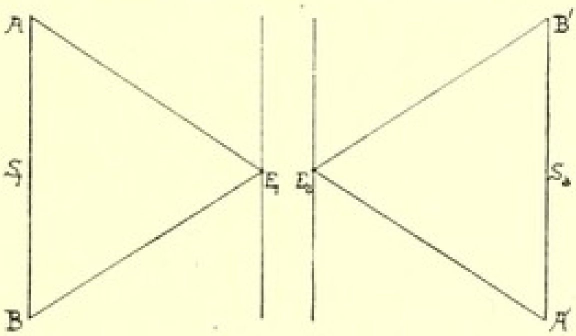


FIG. 225.

normal from $C_1$ and with *B* as centre describe a circle; construct the sines of the angles of incidence and refraction by the method previously shown, and trace *B D*, the course of the ray after the first refraction. To *D* draw the normal from $C_2$, and with *D* as a centre describe a circle; construct the

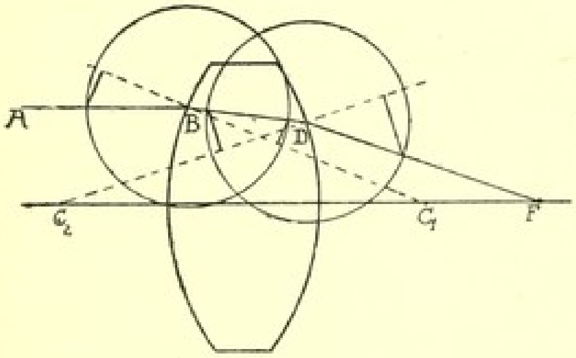


FIG. 226.

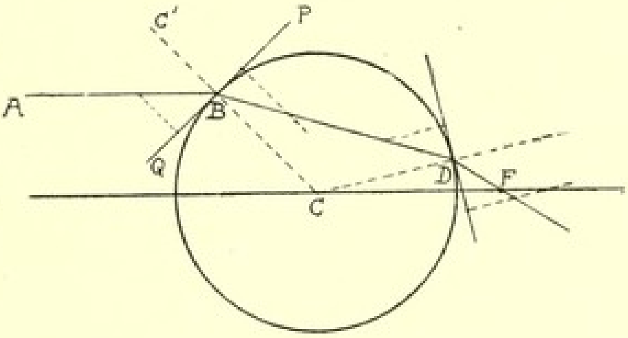


FIG. 227.

second pair of sines of the angles of incidence and refraction, and trace *D F*, the final course of the ray on emergence.

In Fig. 227 there is shown another construction. If *A B* is the incident ray, from *C* draw the normal *C C′* to *B* and the tangent *P Q* to *B*. Then *P Q* may be regarded as the refracting surface, which is divided off as pre viously shown for a surface. *B D* is the course of the ray after the first refraction, and at *D* the process is repeated, the emergent ray being *D F*. Either method serves for lenses or spheres.

## CHAPTER XIX

## COMPOUND REFRACTING SYSTEMS

### The Nodal Points.

THE literal meaning of the word node is "knot," and is applied to the point or points on the principal axis of any system through which the secondary axes pass. Thus the optical centre of a thin lens, and the equivalent points of a thick lens, or system bounded by air or media of the same optical density, have the properties of nodal points. If, however, the first and last media are different, then the equivalent points, although retaining their original property of locating the planes of refraction, no longer act as the crossing or nodal points of the secondary axes. Instead, we have a second point or pair of points—the *nodal points*—displaced towards the denser medium if the system is positive and towards the rarer if it is negative.

This can be illustrated very well in the case of a single refracting surface. Here the refracting plane P is at the vertex of the surface, but the nodal point, i.e. the centre of curvature C, is in the denser medium if the surface is convex, and in the rarer if concave. Also we know that the difference between the foci $F_1$ and $F_2$ is equal to the radius, that is, to the distance of the nodal point from the refracting plane, and, in addition, the ratio of $F_1$ to $F_2$ is also the ratio of the indices of the first and last media. The same occurs when a thin lens is bounded, say, on one side by air and on the other by some medium denser than air. Since all the refraction is presumed to take place in the refracting plane, the position of the latter does not alter, but the posterior focus $F_2$ becomes lengthened. Then we know that the distance of the single nodal point from the refracting plane is $F_2 - F_1$.

For example, suppose a thin 10″ Cx. lens *L L* (Fig. 228) of $\mu$ 1·5 to be bounded on one side by air and on the other by water whose index is 1·33; the effect is to lengthen the posterior focus to 20 in., and the anterior to 15 in. Therefore the distance of the nodal point *N* through which the secondary axes now pass is $F_2 - F_1 = 20 - 15 = 5''$ behind the refracting plane *L L*, which remains unchanged.

**Thick Lens bounded by Different Media.**—Precisely the same arguments apply to a thick lens bounded by different media, but here, since the thickness cannot be neglected, the equivalent planes change their position more or less as well as being separated from the nodal points. The latter are now two in number such that $N_1$ from $E_1$ and $N_2$ to $E_2$ are both equal to the difference $F_2 - F_1$.

As an example, suppose the case of the crystalline lens of the eye with the cornea and aqueous removed (Fig. 229). Let $\mu_1 = 1$, $\mu_2 = 1{\cdot}45$, $\mu_3 = 1{\cdot}33$ $r_1 = 10$ mm. $r_2 = 6$ mm., and $t$, the thickness of the crystalline, 3·6 mm. If light passes from one medium into another and finally into a third, when the thickness of the central medium cannot be ignored, and the

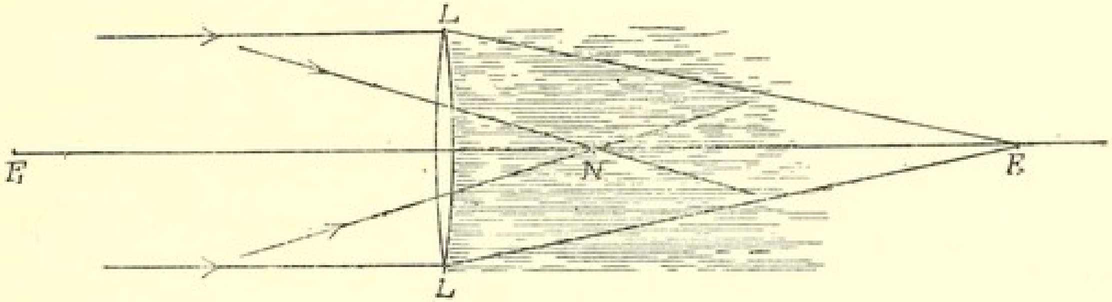



FIG. 228.

bounding surfaces are curved, we have a combination of a thick lens separating different media. Such a combination exists in the present example. In these circumstances a direct, if rather complicated formula can be deduced. Let $r_1$ and $r_2$ be the radii of curvature, and $\mu_1$, $\mu_2$, and $\mu_3$ the three refractive indices. Then

$$F_1 = \frac{\mu_1 r_1 r_2}{r_1(\mu_2 - \mu_3) + r_2(\mu_2 - \mu_1) - t(\mu_2 - \mu_1)(\mu_2 - \mu_3)/\mu_2}$$

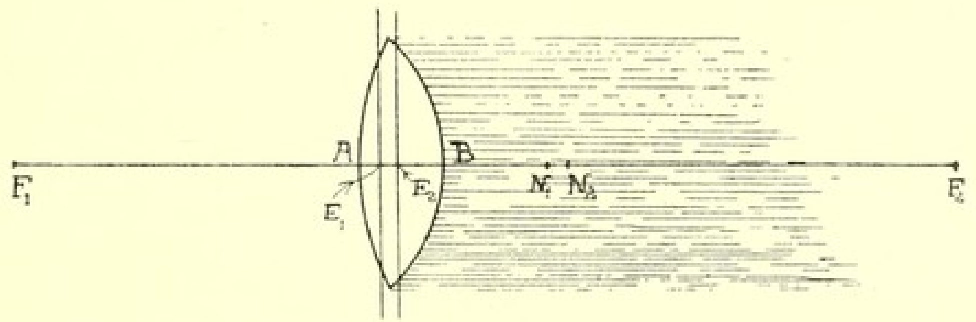



FIG. 229.

Let the denominator of the above be called Q (in the case of a thick lens the corresponding quantity was called N, but as the latter is applied to the nodal point itself, another symbol here prevents confusion.) Then

$$F_1 = \mu_1 r_1 r_2/Q \qquad F_2 = \mu_2 r_1 r_2/Q$$

$$E_1 = \mu_1 r_1 t(\mu_2 - \mu_3)/\mu_2 Q \text{ from A.} \qquad E_2 = \mu_2 r_2 t(\mu_2 - \mu_1)/\mu_2 Q \text{ from B.}$$

Here, the first and last media being different, $F_1$ does not equal $F_2$, but $F_1/F_2 = \mu_1/\mu_3$. The back surface focal distances can be obtained by deducting $E_1$ from $F_1$ and $E_2$ from $F_2$.

Working from the given data we find

$$F_1 = 15 \cdot 93 \quad \text{and} \quad F_2 = 21 \cdot 18$$
$$E_1 = \cdot 79 \text{ from A} \quad \text{and} \quad E_2 = 2 \cdot 37 \text{ from B}$$

The distances of the nodal points from the equivalent points are

$$N_1 = F_2 - F_1 = 21 \cdot 18 - 15 \cdot 93 = 5 \cdot 25 \text{ from } E_1$$
$$N_2 = F_2 - F_1 = 21 \cdot 18 - 15 \cdot 93 = 5 \cdot 25 \text{ from } E_2$$

or $N_1$ is 6·04 mm. from A, and $N_2$ is 2·88 mm. from B

The equivalent thickness or optical interval T = ·44 mm., and the same interval exists between $N_1$ and $N_2$.

**Course of Light through Thick Lens System bounded by Different Media.**—When a refracting body consists of more than one curved surface and is bounded by different media, it has, on its principal axis, six cardinal points, namely, two focal points, two principal points, and two nodal points. These are sometimes called the *Gauss points*, and with their aid the course of

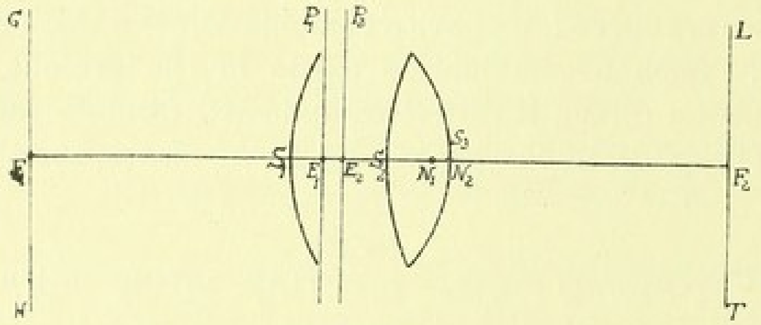



Fig. 230.

a ray can be traced through any compound system of lenses and media. To illustrate the course of light we cannot do better than take the case of the eye itself, which consists of three surfaces $S_1$, $S_2$ and $S_3$, separating four media $\mu_1$, $\mu_2$, $\mu_3$, and $\mu_4$, the first being air and the last vitreous (Fig. 230). As calculated on page 240, the distances from $S_1$ are

$$E_1 = 1 \cdot 96 \text{ mm.}, \quad E_2 = 2 \cdot 39 \text{ mm.}, \quad N_1 = 6 \cdot 96 \text{ mm.}, \quad N_2 = 7 \cdot 39 \text{ mm.}$$
$$E_1F_1 = 15 \text{ mm.}, \quad E_2F_2 = 20 \text{ mm.}, \quad E_1N_1 = E_2N_2 = 5 \text{ mm.}, \quad T = \cdot 43 \text{ mm.}$$

Let $F_1 F_2$ be the principal axis. $F_1$ and $F_2$ are, respectively, the first and second focal points, $G H$ and $L T$ are their corresponding planes.

$E_1$ and $E_2$ are, respectively, the first and second principal points, $P_1$ and $P_2$ their corresponding planes. $N_1$ and $N_2$ are, respectively, the first and second nodal points. $E_1F_1$ is the first, and $E_2F_2$ the second principal focal distance.

Rays which in $\mu_1$ are parallel to the principal axis meet, after refraction, in $\mu_4$ at the second principal focal point $F_2$.

Rays which diverge from $F_1$, the first principal focal point, are after refraction, parallel to the principal axis in $\mu_4$.

Rays which are parallel to the principal axis in $\mu_4$, meet, after refraction, in $\mu_1$ at the first principal focal point $F_1$.

Rays which diverge from $F_2$, the second principal focal point, are after refraction, parallel to the principal axis in $\mu_1$.

A ray directed towards the first principal point, appears after refraction, to proceed from the second, but the direction after refraction *is not parallel to its original course.*

A ray directed to the second principal point appears, after refraction, to proceed from the first. The two principal points are the images of each other.

A ray directed to the first nodal point, after refraction, appears to come from the second, *and its direction is parallel to its original course.* A ray directed to the second, appears, after refraction, to come from the first.

In the case of a single refracting surface a ray directed to its nodal point passes through without deviation; but where, in a compound system, there are two nodal points, a ray must be directed to the first in order to appear to come from the second, or vice versa. The two nodal points are the images of each other.

Rays which in $\mu_1$ are parallel to each other, on any secondary axis are, after refraction, brought to a focus at some point situated on $L\ T$, the second focal plane.

Rays which diverge from a point on $G H$, the first focal plane, are after refraction, parallel to each other in $\mu_4$.

Rays which are parallel to each other on any axis in $\mu_4$ are, after refraction, brought to a focus at some point on $G\ H$, the first focal plane.

Rays which diverge from a point on $L\ T$, the second focal plane, are, after refraction, parallel to each other in $\mu_1$.

A ray directed to any point on $P_1$, the first principal plane, appears after refraction, to proceed from a corresponding point situated on $P_2$, the second principal plane. These two points are *on the same side of the axis and equally distant from it.* A ray directed to a point on $P_2$, the second principal plane, after refraction, appears to proceed from a corresponding point on $P_1$ the first, *equally distant from the axis.* Therefore every point on the one principal plane has its image on the other.

The first principal focal distance of a compound system is $E_1 F_1$, the distance between the first principal point and the first principal focus. The second principal focal distance $E_2 F_2$ is that between the second principal point and the second principal focus.

$\mathbf{E_1 E_2 = N_1 N_2}$.—The distance which separates the two principal points is equal to that which separates the two nodal points.

$\mathbf{N_1 F_1 = E_2 F_2}$.—The distance $N_1 F_1$ between the first nodal point and the first principal focus is equal to the distance $E_2 F_2$ between the second principal point and the second principal focus.

**$N_2F_2 = E_1F_1$.**—The distance $N_2F_2$, between the second nodal point and the second principal focus, is equal to $E_1F_1$, the distance between the first principal point and the first principal focus.

$$E_2F_2 - E_1F_1 = E_1N_1 = E_2N_2$$

$E_1F_1$ and $E_2F_2$ bear to each other the same relationship as the indices of refraction $\mu_1$ of the first medium and $\mu_4$ of the last medium.

$$F_1/\mu_1 = F_2/\mu_4, \text{ or } F_1/F_2 = \mu_1/\mu_4, \text{ or } F_1\mu_4 = F_2\mu_1$$

If the first medium be air and the last medium be vitreous, with an index of refraction of 1·33, then $E_1F_1 : E_2F_2 : : 1 : 1·33$.

**Coincidence of E and N.**—Therefore it follows that, if the first and last media through which rays pass, when refracted by a compound dioptric system, are of the same indices of refraction, the two principal focal distances will be equal, and the nodal and principal points coincide. When these points possess the properties *of both principal and nodal points*, as they do in lenses in media of the same density, they are generally termed *equivalent* points.

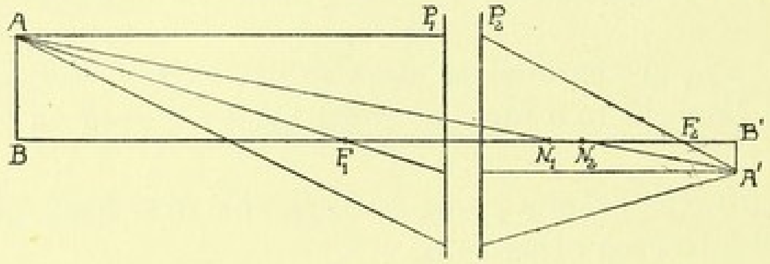



FIG. 231.

**Construction of Image.**—In Fig. 231 let $P_1P_2$ be the equivalent planes, $N_1N_2$ the nodal points, $F_1F_2$ the principal foci, and $AB$ any object in the rarer medium.

A ray $AP_1$ parallel to the axis is refracted at $P_2$ through $F_2$.

A secondary axis $AN_1$ passes on emergence from $N_2$ parallel to its original course.

A ray passing through $F_1$ is, after refraction, parallel to the axis.

Where these rays meet in $A'$ is the image of $A$, so that $B'A'$ is the complete image of $AB$. As will be seen the construction, with the exception of the displacement of $N_1$ and $N_2$, is the same as for any ordinary thick lens or system in air.

Thus we see that, provided the six cardinal points are known, the most complicated system can be reduced to the simplicity of a single thin lens. If the first and last media are the same, or have the same optical density, the equivalent and nodal points coincide so that the relative sizes of image and object are as their distances from the equivalent points; when the media are different the relative sizes of image and object depend upon their distances from the nodal points. In both cases, however, the simple formulæ for single thin lenses, and single refracting surfaces, may be used for calculating conjugate foci, provided all measurement are taken from the appropriate equivalent points.

The Gauss equation, set out in the next chapter, affords a means of calculating the position of the cardinal points for any system.

**Negative System bounded by Different Media**.—This never occurs in practice so that no special discussion is necessary. To anyone who has, however, grasped the principles of a positive system a negative combination would present no additional difficulty.

**Combination of Two Systems when $\mu_1$ differs from $\mu_4$.**—Let $F_1$ and $F_2$ be the anterior and posterior focal distances of the first system, and $F_1'$ and $F_2'$, those of the second system. $E_1$ and $E_2$ pertain to the first, and $E_1'$ and $E_2'$ to the second system. The distance $d$ between the two systems is that between $E_2$ and $E_1'$, i.e. between the two most adjacent points. Let Q be the distance between $F_1'$ and $F_2$, that is, $Q = F_2 + F_1' - d$. $F_A$ and $F_P$ are the anterior and posterior focal lengths of the combined system, $P_1$ and $P_2$ are the principal points, and $N_1$ and $N_2$ are the nodal points.

$$F_A = F_1F_1'/Q \qquad F_P = F_2F_2'/Q$$
$$P_1 = F_1d/Q \text{ from } E_1 \qquad P_2 = F_2'd/Q \text{ from } E'_2$$

$$N_1 = F_P - F_A + P_1 \text{ from } E_1,\ N_2 = F_P - F_A + P_2 \text{ from } E_1, \text{ and } T = P_2 - P_1$$

$$F_P - F_A = P_1 - N_1 = P_2 - N_2, \qquad P_1F_A = N_2F_P, \qquad P_2F_P = N_1F_A$$

Such a system as the above is found in the eye, taking the two components independently; or in a lens placed in front of the eye, the latter, as a whole, being the second system.

## CHAPTER XX

### THE GAUSS EQUATION

By the aid of the Gauss equation every optical system can be so simplified that all problems of conjugate foci, etc., can be worked by the formulæ applicable to single thin lenses. The calculations in the case of more than two surfaces are necessarily long, but they always involve the solution of a continued fraction, so that the difficulties are purely arithmetical.

In using the equation, which serves for any number of surfaces, media and thicknesses, the pencils of light are presumed to be axial and small; in other words, *aberration* is neglected. In order to keep the formulæ as symmetrical as possible and avoid a mixture of signs, the following conventions must be observed, namely, (1) all distances measured to the left of a surface are negative, and to the right positive; (2) all thicknesses are considered negative, and therefore, on substituting actual values, it will be necessary to use the minus sign.

**Thick Lens.**—The following formulæ are deduced from the consideration of the lens having positive radii of curvature according to the above convention, i.e. a periscopic with the concave surface turned towards the right. Let $\mu_1$ be the refractive index of the surrounding medium, $\mu_2$ that of the lens, $t$ the axial thickness, $r_1$ the radius of the first surface, and $r_2$ that of the second. Let $u$ be the object distance, $v_1$ the image distance formed by refraction at the first surface, and $v$ the final image distance after refraction at the second. The fundamental equation connecting $u$ and $v_1$ is

$$\mu_2/v_1 - \mu_1/u = (\mu_2 - \mu_1)/r_1$$

but in order to simplify the formulæ $(\mu_2 - \mu_1)/r_1$ is replaced by $F_1$, while $\mu_2/v_1$ and $\mu_1/u$ are replaced by $1/v_1$ and $1/u$ respectively. These last two are termed *reduced* expressions, i.e. actual distances divided by the $\mu$'s of the media to which they pertain. Similarly in the expression connecting $v_1$ and $v$, given later, $(\mu_1 - \mu_2)/r_2$ and $\mu_1/v$ are replaced by $F_2$ and $1/v$ respectively, while $t$ is also employed reduced, being divided by the $\mu$ in which it is measured. Consequently the values subsequently found are similarly reduced and must be multiplied by the $\mu$, in which each occurs, in order that their true values may be arrived at.

The fundamental formula reduced becomes

$$1/v_1 - 1/u = F_1, \text{ or } 1/v_1 = F_1 + 1/u$$

whence

$$v_1 = \frac{1}{F_1 + 1/u} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The expression connecting $v_1$ and $v$ is

$$\mu_1/v - \mu_2/(v_1 + t) = (\mu_1 - \mu_2)/r_2$$

which, in reduced terms, becomes

$$1/v - 1/(v_1 + t) = F_2, \text{ or } 1/v = F_2 + 1/(v_1 + t)$$

whence

$$v = \cfrac{1}{F_2 + \cfrac{1}{v_1 + t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in (2) the value of $v_1$ in (1) we have

$$v = \cfrac{1}{F_2 + \cfrac{1}{t + \cfrac{1}{F_1 + \cfrac{1}{u}}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

On working out this continued fraction in (3) we get

$$v = \frac{u\,(F_1 t + 1) + t}{u\,(F_1 F_2 t + F_1 + F_2) + F_2 t + 1} \quad . \quad . \quad . \quad . \quad (4)$$

which, for the sake of brevity, is usually written

$$v = \frac{C\,u + D}{A\,u + B} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $A = F_1 F_2 t + F_1 + F_2$; $B = F_2 t + 1$

$C = F_1 t + 1$; $D = t$.

No. (5) connects $v$ and $u$ when both are finite distances. If $u$ is at $\infty$ the quantities D and B disappear and $u$ cancels, so that the focal length measured from the second surface is

$$v = C/A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The value of $v$ in equation (6) *is the back focal distance as measured from the pole of the second surface.*

If $v$ is at $\infty$, then $A\,u + B = 0$, so that by transposition the focal length measured from the pole of the first surface is

$$u = -B/A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Before proceeding further an expression for the total magnification M produced by the lens must be found.

Let $m_1$ be the magnification due to the first surface, and $m_2$ that due to the second ; then the total magnification M is $m_1 \times m_2$.

In Fig. 232 let $AB$ be an object in front of the first surface, and $B'A'$ its corresponding image. A ray from $A$ meeting the vertex in $x$ will be refracted to $A'$ such that $i$ and $r$ are the angles of incidence and refraction respectively. Then

$$m_1 = \mathrm{A'B'/AB}$$

But $i$ and $r$ being small, $AB/u$ may be considered equal to sin $i$, and $A'B'/v_1 = \sin r$, and $\sin r/\sin i = \mu_1/\mu_2$.

Therefore $$m_1 = \mathrm{A'B'/AB} = \mu_1 v_1/\mu_2 u$$

But $u/u_1$ and $v_1/u_2$ are *reduced* quantities and therefore to preserve our notation the refractive indices must be omitted, so that,

$$m_1 = v_1/u$$

Similarly the magnification $m_2$ of the second surface is

$$m_2 = v/(v_1 + t)$$

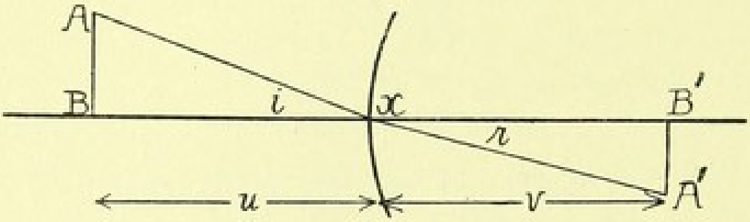



FIG. 232.

Therefore the total magnification

$$\mathrm{M} = v_1/u \times v/(v_1 + t)$$

But from (1) $$v_1/u = 1/(\mathrm{F}_1 u + 1)$$

And from (1) and (2) $$\frac{v}{v_1 + t} = \frac{1}{\mathrm{F}_2\left(\frac{u}{\mathrm{F}_1 u + 1} + t\right) + 1}$$

Therefore $$\mathrm{M} = \frac{1}{\mathrm{F}_1 u + 1} \times \frac{\mathrm{F}_1 u + 1}{u(\mathrm{F}_1\mathrm{F}_2 t + \mathrm{F}_1 + \mathrm{F}_2) + \mathrm{F}_2 t + 1}$$

$$= \frac{1}{u(\mathrm{F}_1\mathrm{F}_2 t + \mathrm{F}_1 + \mathrm{F}_2) + \mathrm{F}_2 t + 1}$$

$$= \frac{1}{\mathrm{A}u + \mathrm{B}} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Now let the magnification be $+1$, i.e., let *virtual* image and object be equal in size. Then

$$\mathrm{A}u + \mathrm{B} = 1$$

whence $$u = \mathrm{P}_1 = (1 - \mathrm{B})/\mathrm{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

this distance being measured from the first surface.

On substituting this value of $u$ in (5), the corresponding value of $v$ is

$$v = P_2 = \frac{C - BC + AD}{A} = \frac{C-1}{A} \qquad . \quad . \quad . \quad (10)$$

because it can be shown that $AD - BC = -1$. This distance is measured from the second surface.

These planes of unit virtual magnification denote the *equivalent planes*, and the points $P_1$ and $P_2$ where they cut the axis are the *equivalent points*. If it were possible to place a small object in the one plane, then its virtual image, identical in all respects to the object, would be situated in the other.

If the magnification be $-1$, then the corresponding values of $u$ and $v$ will locate the *symmetrical planes*, where object and *real* image are equal in size.

To find, therefore, the equivalent focal distances, the values of (9) and (10) must be added to those of $u$ and $v$ in (5); thus

$$v + \frac{C-1}{A} = \frac{C\,(u + (1-B)/A) + D}{A\,(u + (1-B)/A) + B}$$

which simplifies to $$A = 1/v - 1/u \qquad . \quad . \quad . \quad . \quad . \quad (11)$$

This expression (11) should be compared with that of a simple thin lens for the focal length in terms of $u$ and $v$. Then if $u = \infty$

$$v = 1/A \qquad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and if $v = \infty$ $$u = -1/A \qquad . \quad . \quad . \quad . \quad . \quad (13)$$

The principal focal distance given in (12) and (13) are equal when the first and last $\mu$'s are of equal optical density. The values are reduced and must be multiplied by the $\mu$ in which each occurs, so that when in air they are unchanged.

As a simple example, let $r_1 = 6$, $r_2 = 8$, $\mu = 1{\cdot}5$, $\mu_1 = 1$ (air), and $t = 1$; then

$$v = \cfrac{1}{F_2 + \cfrac{1}{t + \cfrac{1}{F_1 + \cfrac{1}{u}}}} \qquad v = \cfrac{1}{{\cdot}0625 + \cfrac{1}{-{\cdot}666 + \cfrac{1}{{\cdot}0833 + \cfrac{1}{u}}}}$$

which works out to $$v = \frac{{\cdot}9445\,u - {\cdot}666}{{\cdot}1423\,u + {\cdot}9584}$$

Then, if $u = \infty$

$$v = \frac{C}{A} = \frac{{\cdot}9445}{{\cdot}1423} = 6{\cdot}63$$

Also $$u = \frac{-B}{A} = \frac{-{\cdot}9584}{{\cdot}1423} = -6{\cdot}73$$

$$P_1 = \frac{1-B}{A} = \frac{1 - {\cdot}9584}{{\cdot}1423} = +{\cdot}29$$

$$P_2 = \frac{C-1}{A} = \frac{{\cdot}9445 - 1}{{\cdot}1423} = -{\cdot}39$$

The equivalent focal distance

$$1/A = 1/{\cdot}1423 = 7{\cdot}02$$

**Multiple Surfaces.**—The Gauss equation may be applied to an optical system having any number of surfaces surrounded by corresponding media of different densities and thicknesses. The equation

$$v = \frac{C\,u + D}{A\,u + B}$$

is universal, although the various values become more complicated as the number of surfaces is increased, but the problem always takes this form, involving the solution of a continued fraction.

Suppose the case of the eye having three surfaces, $F_1$, $F_3$ and $F_5$ with thicknesses $t_2$ and $t_4$, with the following data $r_1 = 8$, $r_3 = 10$, $r_5 = 6$, $t_2 = 3{\cdot}6$, $t = 3{\cdot}6$, $\mu_1 = 1$, $\mu_2 = 1{\cdot}333$, $\mu_3 = 1{\cdot}45$, $\mu_4 = 1{\cdot}333$. Then

$$F_1 = \frac{\mu_2 - \mu_1}{r_1} = \frac{1{\cdot}333 - 1}{8} = {\cdot}0416$$

$$F_3 = \frac{\mu_3 - \mu_2}{r_3} = \frac{1{\cdot}45 - 1{\cdot}333}{10} = {\cdot}0117$$

$$F_5 = \frac{\mu_4 - \mu_3}{r_5} = \frac{1{\cdot}333 - 1{\cdot}45}{-6} = {\cdot}0195.$$

The reduced value of

$$t_2 = -3{\cdot}6/1{\cdot}333 = -2{\cdot}7007$$

and that of

$$t_4 = -3{\cdot}6/1{\cdot}45 = -2{\cdot}4828$$

Then we have

$$v = \cfrac{1}{F_5 + \cfrac{1}{t_4 + \cfrac{1}{F_3 + \cfrac{1}{t_2 + \cfrac{1}{F_1 + \cfrac{1}{u}}}}}} \qquad v = \cfrac{1}{{\cdot}0195 + \cfrac{1}{-2{\cdot}4828 + \cfrac{1}{{\cdot}0117 + \cfrac{1}{-2{\cdot}7007 + \cfrac{1}{{\cdot}0416 + \cfrac{1}{u}}}}}}$$

which becomes, when worked out,

$$v = \frac{{\cdot}7586\,u - 5{\cdot}1050}{{\cdot}0668\,u + {\cdot}8689}$$

That is $A = {\cdot}0668$, $B = {\cdot}8689$, $C = {\cdot}7586$, $D = -5{\cdot}1050$.

The anterior $F = -\mu_1/A = -1/{\cdot}0668 = -15$ mm.

The posterior $F = \mu_4/A = 1{\cdot}333/{\cdot}0668 = 20$ mm.

$$P_1 = \mu_1(1 - B)/A = {\cdot}1311/{\cdot}0668 = 1{\cdot}96 \text{ mm. from } r_1$$

$$P_2 = \mu_4(C - 1)/A = -{\cdot}3128/{\cdot}0668 = -4{\cdot}81 \text{ mm. from } r_2$$

or

$$7{\cdot}2 - 4{\cdot}81 = 2{\cdot}39 \text{ mm. from } r_1.$$

The nodal points $N_1$ and $N_2$, found by subtraction, are, respectively, 6·96 and 7·39 mm. from $r_1$. Neglecting the intervals between $P_1$ and $P_2$ and that between $N_1$ and $N_2$, we have P at 2·2 mm. and N at 7·2 mm. from the cornea.

When working with the Gauss equation two things must be borne in mind; firstly, the convention as to signs upon which the symmetry of the formulæ depend; and secondly, the use of *reduced* instead of absolute distances in order to simplify the formulæ by the inclusion of the refractive indices in other terms. Thus $v$, the final image distance, is always multiplied by the index of the last medium to give the absolute values of the second principal focus and the second equivalent point. On the other hand $u$ which, in the final expression, denotes the anterior focus and first principal point is, except in very rare cases, already reduced, the first medium generally being air. In fact, the same may be said of $v$, as a difference in the indices of the first and last media occurs only in the case of the eye, and in certain instruments as, for instance, the immersion objective of the microscope.

The calculation of a continued fraction for three surfaces being complicated, the results obtained may be checked by the following, which is the continued fraction worked down.

$$v = \frac{\mathrm{U\,N} + \mathrm{R}}{\mathrm{U}\,(\mathrm{F}_5\mathrm{N} + \mathrm{F}_1\mathrm{F}_3 t_2 + \mathrm{F}_1 + \mathrm{F}_3) + \mathrm{F}_5\mathrm{R} + \mathrm{F}_3 t_2 + 1}$$

where $$\mathrm{N} = \mathrm{F}_1\mathrm{F}_3 t_2 t_4 + \mathrm{F}_1 t_2 + \mathrm{F}_1 t_4 + \mathrm{F}_3 t_4 + 1$$

and $$\mathrm{R} = \mathrm{F}_3 t_2 t_4 + t_2 + t_4$$

## CHAPTER XXI

### CURVATURE SYSTEM

THE various formulæ in connection with mirrors, prisms and lenses may also be deduced from a consideration of the actual paths of the waves themselves. The following are elementary examples of the application of this method, which is by some writers preferred to the "ray" theory as representing the actual physical change in shape and direction undergone by the waves in refraction and reflection.

**Plane Surface.**—*A B* (Fig. 233) is a plane wave front incident obliquely on the surface *C D*. If $\mu_1 = 1$ and $\mu_2 = 1{\cdot}5$ the part of the wave which enters at *B* travels in the same time to *F* only 2/3 of the distance *A E*. With *B* as centre and *B F* as radius describe a small arc, a tangent *E F* from *E* showing the inclination of the wave front in the dense medium. At the second

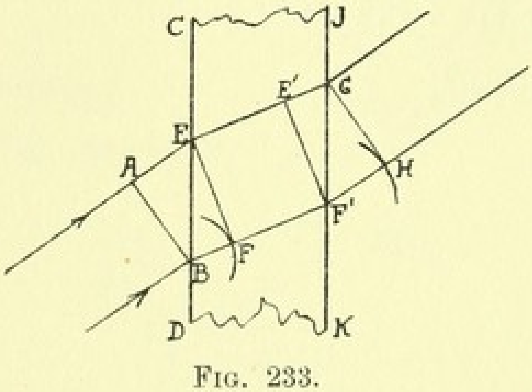



FIG. 233.

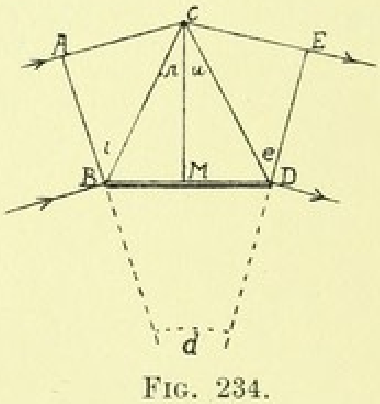



FIG. 234.

surface a similar construction shows the wave front *G H* after emergence, *F'H* being 1·5 times *E'G*.

**Course of a Wave through a Prism at Minimum Deviation.**—Let *CBD* (Fig. 234) be a prism on which is incident the plane wave *AB* at angle of incidence *i*. The portion *B* of the wave meeting the base of the prism is retarded to a greater extent than *A*, the portion in air, so that when the whole wave enters the prism it takes up the position *CM*, *r* being the angle of refraction.

Since the deviation is supposed to be minimum, the total refraction is symmetrical with respect to the surfaces *CB* and *CD*, so that *CM* bisects the principal angle. The wave is then incident on the second surface at the angle *u*, and on emergence it is swung over still more towards the base so that, when completely clear of the prism, it has the position $ED$ making the angle of emergence $e$ with the second surface, $e$ being equal to $i$.

**Curvature.**—The unit of the dioptric curvature system is a curvature having a radius of 1 metre, and the curvature of any wave may be denoted and measured by it. Thus if light diverges from $\frac{1}{2}$ M or 50 Cm. the wave is said to have a divergence of 2 D; at some other distance, say 2 M, the curvature would be ·5 D, and so on. Thus if $R$ denote the actual curvature of a wave it may be expressed either as the reciprocal of its radius $r$ in metres, or simply in diopters $D$. Then $R = 1/r$ or $D$. Now we have seen, from the spherometer formula, that for shallow curvatures $r = d^2/2s$, where $d$ is the semi-chord and $s$ the sagitta of the corresponding arc of radius $r$. In other words, provided the chord remains constant, the radius is inversely proportional to the sag and vice versa, while the curvature of the arc is *directly proportional to the sag on the same chord.* Thus we may say that $1/r$ or $R \propto s$, or $s \propto R$ or $1/r$. It will be seen that the "curvature" formulæ are identical with the "ray" formulæ, only that, with the exception of $\mu$,

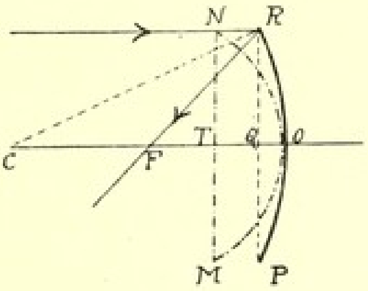



Fig. 235.

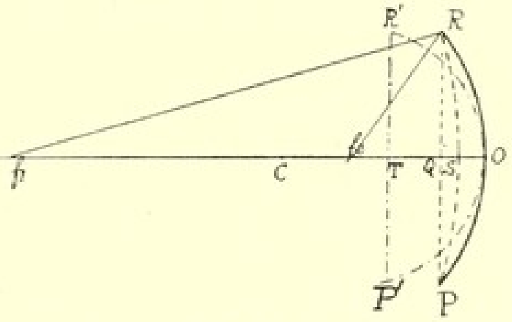



Fig. 236.

all the symbols employed in the one are the reciprocals of those used in the other, and vice versa.

**Cc. Mirror-Plane Incident Wave.**—Let $POR$ (Fig. 235) be any Cc. mirror on which is incident the plane wave $PQR$. If the aperture be small the points $P$ and $R$ of the wave front first meeting the mirror may be considered to be reflected back to $M$ and $N$ while the central point $Q$ is travelling to the vertex $O$. When $Q$ has arrived at $O$ the contour of the reflected wave is $MON$; it remains to find the curvature of $MON$.

Now since $PM = OQ$, $OQ = QT$ and we have $TO = 2OQ$. But, since the curvature of an arc may be taken as proportional to the sag for equal chords, $TQ$ represents the curvature $F$ of the reflected wave $MON$, and $OQ$ the curvature $C$ of the mirror. Thus $TO = 2QO$, or $F = 2C$. In other words the curvature of the reflected wave is double that of the mirror, so that the focal distance $OF$ is half the radius $OC$.

**Cc. Mirror-Divergent Wave.**—Let $PSR$ (Fig. 236) be a wave diverging from a near object $f_1$; then while the vertex $S$ of the wave is travelling to $O$, the extremities $P$ and $R$ are reflected to $P'$ and $R'$ respectively such that $TQ = SO$. Then $P'OR'$ is the reflected wave converging towards $f_2$. $O\,Q$ is the mirror sag, and since $T\,Q = S\,O$ we have $T\,O = 2\,S\,O + Q\,S = 2\,Q\,O - Q\,S$.

Let $F_1$ be the curvature of the object wave $P\,S\,R$, $F_2$ that of the image wave $P'Q\,R'$, while $C$ is the curvature of the mirror, and $F$ its focal curvature or power. Then $Q\,S = F_1$, $O\,Q = C$, $T\,O = F_2$, and $F = 2\,C$. Therefore $T\,O = 2\,Q\,O - Q\,S$, or $F_2 = 2\,C - F_1$, so that $F_1 + F_2 = 2\,C = F$, i.e. the focal power of a Cc. mirror is equal to the sum of the object and image curvatures, and this is the formula for expressing conjugate foci. It will be noticed that $C$, the mirror curvature, is the mean of the object and image curvatures; thus $C = (F_1 + F_2)/2$.

**Convex Mirror-Plane Incident Wave.**—Let $M\,N$ (Fig. 237) be a plane wave incident on the Cx. mirror $P\,O\,R$. $O$ is now the first incident point, and this is reflected to $Q'$, while $M$ and $N$ are travelling to $P$ and $R$, so that $P\,Q'R$ is the reflected wave, which can be shown to have a curvature double that of the mirror, as with a Cc. In other words, since $QQ' = 2\,O\,Q$, $F = 2\,C$.

**Convex Mirror-Divergent Wave.**—When the wave is divergent from a

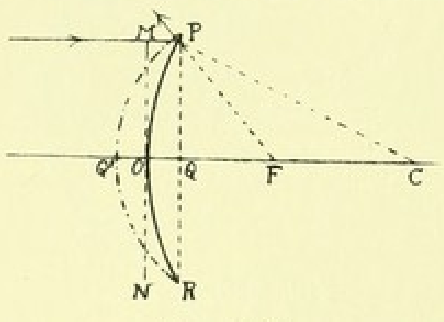



Fig. 237.

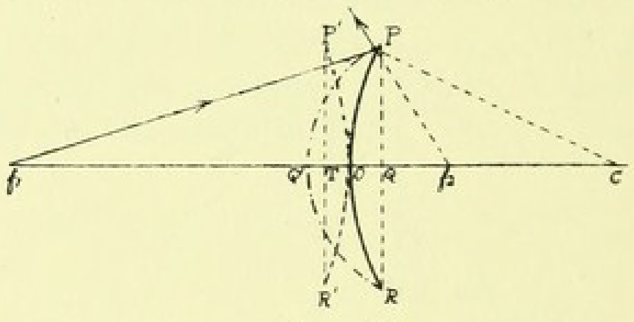



Fig. 238.

near object $f_1$ (Fig. 238), the incident wave is $P'O\,R'$, and the reflected wave $P\,Q'R$ such that $OQ' = OT + OQ$.

$$\therefore\ Q\,Q' = O\,Q + O\,Q' = O\,Q + (O\,Q + O\,T) = 2\,O\,Q + O\,T$$

that is $$F_2 = 2\,C + F_1, \text{ or } F_2 = F + F_1$$

In other words the image curvature is equal to the sum of the object and mirror curvatures, because both are divergent in effect. Employing the usual convention as to signs this expression would be written as for a Cc. mirror, i.e., $F_1 + F_2 = 2\,C = F$, the negative sign being employed when substituting the value of F.

**Single Surface-Cx.**—Let $M\,N$ (Fig. 239) be a plane wave incident on the single Cx. refracting surface $P\,Q\,R$ such that $P\,Q'R$ is the refracted wave convergent towards the posterior principal focus $F_B$. Let $C$ be the curvature of the surface, $F_B$ that of the refracted light, and $\mu$ the index of the medium, the first being air. Then we have $Q\,S = \mu\,Q\,Q'$. But

$$Q\,Q' = C - F_B \text{ and } Q\,S = C$$

$$\therefore \qquad C = (C - F_B)\mu \qquad \text{or} \qquad F_B = C\,(\mu - 1)/\mu$$

Similarly an expression can be found for the anterior principal focus of the same surface *MQN*. Here (Fig. 240) the plane wave advances from the denser medium to meet the surface as *MN*, the retarded wave convergent towards the anterior focus $F_A$, being *PQR*. Then $Q'S = \mu QQ'$. But

$$Q'S = F_A + C, \text{ and } OQ' = C$$

Therefore $\mathbf{F}_A + C = \mu C$.

and $$F_A = C(\mu - 1)$$

For a *concave surface* the formulæ are the same, C being negative.

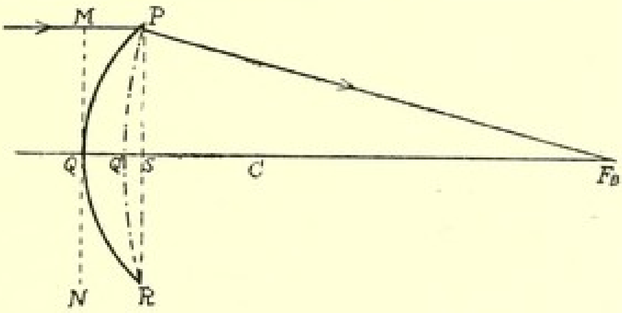



FIG. 239.

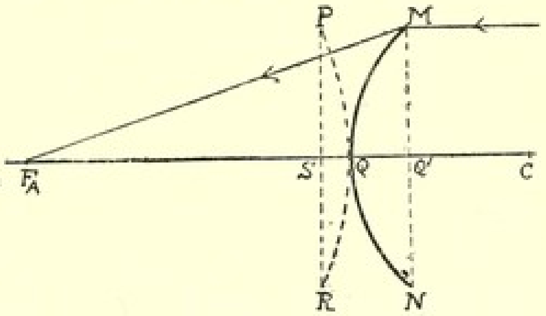



FIG. 240.

**Conjugate Foci-Single Cx. Surface.**—Let $f_1$ (Fig. 241) be any near object from which diverges the wave *MN* to the surface *PQR*, and let $f_2$ be the image formed by the image wave *PQ'R*. Let $TQ = F_1$, $QS = C$ and $Q'S = F_2$. Then

$$TS = \mu QQ' = \mu(SQ - SQ')$$

and $$TS = TQ + QQ' + Q'S$$

$\therefore$ $$\mu(C - F_2) = F_1 + (C - F) + F_2 \quad \text{or} \quad \mu C - \mu F_2 = F_1 + C$$

that is $$F_1 + \mu F_2 = C(\mu - 1)$$

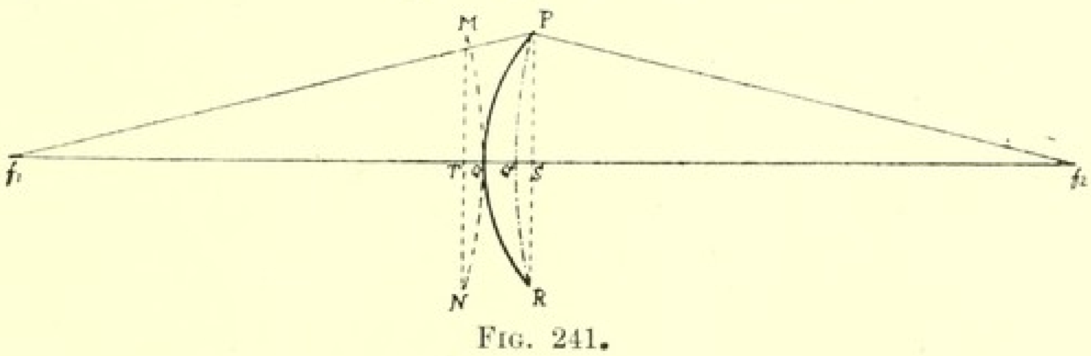



FIG. 241.

Similar formulæ may be deduced for a *concave surface* only here C and $F_2$ are negative.

**Thin Convex Lens.**—With a lens the curvature of each surface is likewise represented by their respective sags, so that in the case of a double Cx. (Fig. 242) *QS* represents the sum of the sags $C_1$ and $C_2$. Let *MN* be a plane wave incident on the lens; then, owing to the greater axial thickness, the centre of the wave is retarded more than the periphery, the resulting wave front taking the form $PSR$ converging to the focus $F_B$. Let $T$ be the united sags of the lens surface and wave fronts; then $T = C_1 + C_2 + F$. But

$$T = \mu QS = \mu (C_1 + C_2)$$

$$\therefore \quad C_1 + C_2 + F = \mu (C_1 + C_2)$$

whence

$$F = (C_1 + C_2)(\mu - 1)$$

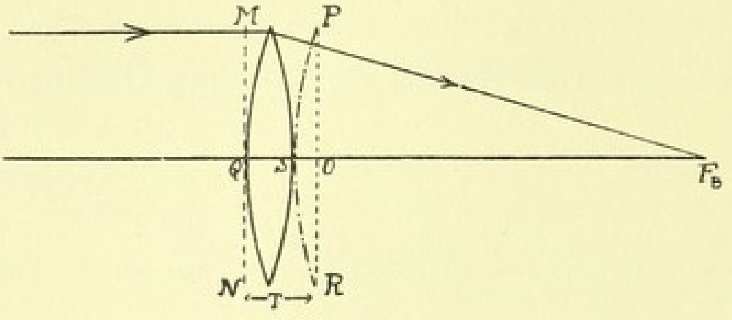



FIG. 242.

In other words the power F of the lens is the product of the united curvatures and the refractivity of the glass.

Similar formulæ in the case of conjugate foci and for concave lenses can be deduced on the same lines; in numerical examples, of course, $C_1$ and $C_2$ of concave lenses are reckoned negative.

# CHAPTER XXII

## COLOUR

**Primary and Secondary Colours.**—There are six or seven distinct colours which can be identified in the solar spectrum, but it was shown by Young, and confirmed by Helmholtz, that every shade of colour in nature can be obtained from the mixture of red, green and violet in certain proportions, whereas these three colours cannot be produced by mixing other colours. For this reason red, green and blue-violet are termed the *primary* colours, while the other spectrum colours are *secondaries*. Thus red and green, in varying proportions, produce orange or yellow, while green and violet produce blue or indigo.

**Complementary Colours.**—If two spectrum colours, when combined, form white light, they are said to be complementary to each other. Hence a *complementary colour* may be defined as that which, when united with another, produces white light. The complement of a primary colour is that secondary colour which results from the mixture of the other two primaries; the complement of a secondary colour is that primary colour which is not contained in it.

| *Spectrum Colour.* | *Complement.* |
|---|---|
| Red. | Green-Blue. |
| Orange. | Blue. |
| Yellow. | Blue-Violet. |
| Green. | Purple-Red. |
| Blue. | Orange. |
| Indigo. | Orange-Yellow. |
| Violet. | Green-Yellow. |

The purple-red is not in the visible spectrum, it being a combination of red and violet. A graphical representation of this table will be found on page 250, where the primary colours of pigments and their complements are discussed.

The nomenclature applied by various authorities to primary and secondary colours differs considerably, but they are here employed as nearly as possible in their popular meaning.

**Colour Sensation.**—According to Young and Helmholtz, there exist in the eye three sets of nerves which convey to the brain the three primary colour sensations of red, green and violet respectively. Each set of nerves conveys, however, not only the sensation of its special colour, but also to a slight extent that of the other two. By stimulating one or more of these nerves, in varying proportions, all colours can be mentally appreciated.

Stimulation of all three produces the sensation of white, and of none of them black. Fig. 243 represents diagrammatically the range of the three colour sensations; the first curve is that of the "red" nerve; the second is that of the "green" nerve, while the third is that of the "violet" nerve. It is thought, however, that the "red" nerve is not stimulated by waves beyond *E* or *F*, the "green" by those beyond *C* on the one side and *G* on the other, while the limit of the "violet" nerve is about *D*. Thus it will be seen that the primary nerve centres have, according to the Young-Helmholtz theory, a sufficient latitude of perception to enable every conceivable secondary colour or combination to be appreciated by the brain.

**Colours of Light.**—Spectrum red and green will, if mixed in certain proportions, produce a sensation of yellow. If spectrum red, green and blue-

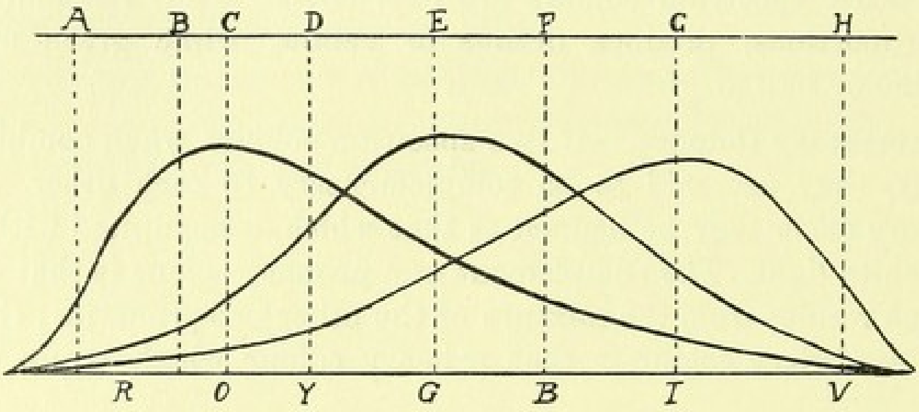



FIG. 243.

violet be mixed in the right proportions white light is formed. If the wave-lengths of red and green be added together the mean will give the wave-length of yellow. Thus, taking the wave-length of orange-red as 656 and that of blue-green as 518, then $656 + 518 = 1174$, and $1174/2 = 587$. Again, taking the wave-lengths of red, green and blue respectively, the sum divided by three will give the wave-length for the brightest part of the yellow, which is the nearest approach to white light which the spectrum affords; thus $748 + 527 + 486 = 1761$, and $1761/3 = 587$. The quantity of light of one colour necessary to mix with any other to produce white light, or a third colour, does not appear to follow any definite law, but the proportions usually remain the same for different observers; occasionally, however, the amount is found to be very different, even among persons who are not colour blind to standard tests. Colours which do not appear in the spectrum are those formed by a combination of two or more non-adjacent wave-lengths, the resultant effect on the eye being, in general, that colour corresponding to the mean wave-length of the components. Purple does not appear, since it is a mixture of the extremes—red and violet—nor do many other colours, as brown or pink.

The following table, according to Helmholtz, shows the effects produced by the addition of any two spectrum colours.

| Colour. | Violet. | Indigo. | Cyan Blue. | Blue-Green | Green. | Greenish-Yellow. | Yellow. |
|---|---|---|---|---|---|---|---|
| Red | Purple | Dark rose | Light rose | White | Whitish-yellow | Golden yellow | Orange |
| Orange | Dark rose | Light rose | White | Light yellow | Yellow | Yellow | — |
| Yellow | Light rose | White | Light green | Light green | Greenish yellow | — | — |
| Greenish-yellow | White | Light green | Light green | Green | — | — | — |
| Green | Light blue | Sea blue | Blue green | — | — | — | — |
| Blue-green | Deep blue | Sea blue | — | — | — | — | — |
| Cyan blue | Indigo | — | — | — | — | — | — |

**Brightness of Colour.**—In a prismatic spectrum the red appears fuller than the violet because the former is more crowded together, while the latter is spread out; this is not the case to the same degree in a diffraction spectrum, in which the extent of colour is about equal on either side of the green-yellow. The latter is the brightest part of the spectrum to the human eye, and in general the intensity rises from zero, at the extreme red, rapidly to the yellow and then, dropping off again, but more slowly, to zero at the extreme violet.

**Colours in Pigments.**—The primary colours in pigments (paints or colouring matter) are so-called red, yellow and blue; any other colour is obtained by mixing two primaries.

The primaries and their complements are shown in Fig. 244, from which it will be seen that *the primaries of pigments are the complements of the primaries of light.* Thus 1, 6 and 10 are the primaries of light, and 4, 7 and 12 are the primaries of pigments. Although the primaries of pigments are popularly known as red, yellow and blue yet the actual tints are not quite those usually associated with the terms.

**Mixing Colours.**—The fundamental difference in the results obtained by mixing spectrum lights and pigment colours lies in the fact that the former is an *additive*, and the latter a *subtractive* process. In other words, the colouration due to mingled lights is due to the *sum* of the separate wave lengths, while the resultant colour given off by a mixture of pigments is that remaining after each pigment has *absorbed* a certain wave or series of wave-lengths. The tendency of added lights is to give increased illumination and to approximate it to white, while with pigments the mixture tends towards black.

Thus, when the primaries of light, i.e. red, green and blue-violet, are mingled—projected, say, from three separate lanterns—the white screen reflects all three impartially to the retina, where their superposition produces the sensation of white. With pigments, however, the final colour is due to that remaining after each pigment, in a certain mixture, has absorbed from the incident white light its own complement. In this way the primary colours of pigments are those capable of absorbing the three primaries of white light, i.e. red, green and blue-violet, whose respective complements are green-blue (peacock blue), purple-red, and yellow. These last three are therefore the primaries of pigments because, when mixed in the right proportion they (theoretically) produce *black*. In practice, however, owing to the inevitable impurities of pigments, and the impossibility of combining the correct proportions, the result is a dark grey. For the same reasons, it is

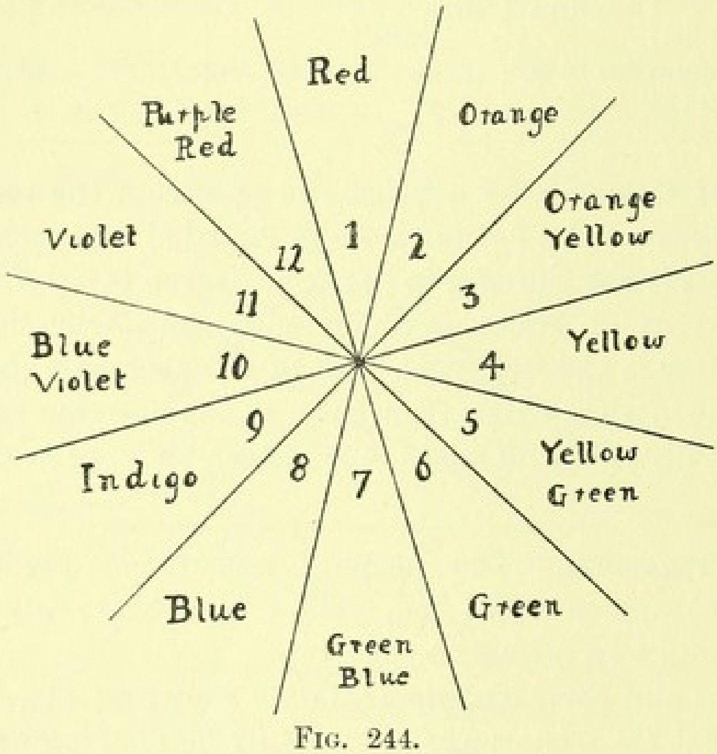



Fig. 244.

impossible accurately to match the spectrum colours by means of pigments, and this is especially the case towards the violet end; in fact we cannot imitate violet by any known pigment or combination of pigment colours.

The additive effect can be roughly imitated by painting yellow and blue sectors alternately on a disc which, when rapidly rotated, gives the impression of white if the proportions of colour are correct. Here the yellow and blue alternately impinge so rapidly on the retina that the sensations caused by alternate sectors have not time to fade away, and therefore become mentally mingled, and give rise to the sensation of white. The experiment must, however, be carried out in white light, but even then the effect is generally far from pure owing to the inevitable muddiness of the pigments. By increasing the number of sectors, and repeating the six spectrum colours in proper proportion all round the disc, a still better white is secured.

In some instances the result of a pigment mixture may be surprisingly different from the result of mingling lights of corresponding colours. If blue and yellow lights are mingled in the right proportion on a white screen they cause the sensation of white. If blue and yellow pigments are combined, the blue absorbs red, the yellow absorbs violet, so that green is produced by such a mixture. Rose red and blue-green are complementary colours which, added to one another, produce white in the case of coloured lights (additive effect), but neutralise each other, i.e. produce black in the case of pigments (subtractive effect). Additive effect can also be produced by the mixture of pigment or coloured powders, where absorption does not occur, but both pigments or powders reflect light. Especially is this so if the two colours are not complementary, or tending to be so; thus red and yellow combined in pigment make orange as they do in the case of lights. Or using the illustration above of blue and yellow pigments combined making green, a blue pigment reflects violet and green, yellow reflects red and green, so that if the two pigments be mixed, there is reflected a certain quantity of violet and of red, and a double quantity of green. The red, the violet, and a portion of the green combine to form white light, so that there is a residue of green light, which gives the nature of the colour to the mixture of the two pigments.

**Qualities of Colours.**—Colours in pigments possess three qualities, viz., tone, brightness and purity. Tone or hue is that quality which differentiates between the various colours—say, red and orange; it depends on wave-length. Brightness, intensity, or luminosity is that quality which represents the strength of a colour; it depends on the amount of light reflected; one which reflects little light is a dark colour, and one which reflects much light is a light colour. Fullness, saturation, tint, or purity is that quality which represents the depth of a colour; the less the admixture of white or black the purer is the colour. Red mixed with white forms pink, whereas red mixed with black makes a kind of maroon. Yellow or orange become straw or brown according as it is mixed respectively with white or black.

**Colours of Bodies.**—A substance is said to be of certain colour when it reflects or transmits rays of certain wave-lengths and absorbs the rest of the spectrum. Thus an object which absorbs the violet and green and reflects the red waves appears red; if it absorbs red waves and reflects green and violet it has a blue colour. A green body absorbs all but the green waves; one which is orange in colour reflects red and green and absorbs violet. The colour reflected by a body is usually the same as that which it transmits, but some bodies transmit the complementary colour to that which they reflect.

A body which reflects light of all wave-lengths is called white; a body which has affinity for all the colours, so that all are absorbed and none reflected, is called black. No body, however, is of a nature so chemically pure as to absorb entirely or reflect all the incident light. An absolutely black body does not exist in nature; even those coated with lamp-black and soot reflect some light, which renders them visible, and allows of their form and solidity being recognised; on the blackest velvet still blacker shadows can be cast. Similarly there is no object which reflects all the light it receives; pure, fresh snow, which is the whitest of all bodies, absorbs some 30 per cent. of the light it receives, and white paper 50 or 60 %.

Colour does not depend entirely upon the body which reflects it, but is also a quality of the illuminating light itself. In order to appear of a certain colour, the object must receive that colour in the light and reflect it, and at the same time absorb all the other colours. Dark colours reflect little light, and slight differences between them are hardly appreciated in dull illumination; similarly, light colours reflect much light, and slight differences are hardly noticed in very bright illumination. The proportion of light reflected varies with the nature and colour of the body. Approximately a coloured body reflects 20 to 50 % of the light which falls on it.

White, grey and black are, in effect, the same and really represent varying degrees of luminosity, the only difference between them being in the total amount of light reflected. By all three the treatment of the different wave-lengths is the same, i.e. there is no selective property as with coloured bodies, but the extent of the light absorption varies in the three cases.

**Coloured Bodies and Lights.**—The real colour of a body is that which it exhibits in white sunlight; it often appears of a different colour in ordinary artificial light. This is due to the fact that some particular colour usually predominates in artificial light, and therefore the mental standard of white is temporarily shifted towards that colour. Thus ordinary gas-light—not incandescent—contains an excess of red and yellow, while blue and violet are the prevailing colours in the electric arc. In this way, the nearer the colour of a body approaches to that of the illuminant the whiter will it—the body—appear; on the other hand should the colour of the body be complementary to that of the illuminant it will appear darker than it would if viewed in white light. Should the light be of a colour exactly corresponding to that which the body absorbs, none will be reflected and the body will consequently appear black.

Of course a white body seen by coloured light is really coloured although it is generally interpretated mentally as white. It certainly is so accepted if the colouration of the luminant is not excessive; thus by gas light a white paper is actually reddish-yellow, but we still call it white. As the illumination becomes progressively feeble all bodies lose their distinctive colours, the latter being replaced by shades varying from light grey to black, and in a very dull illumination all appear equally a dark grey. Painting and matching colours is always difficult in artificial light since the latter is not white; for example, some blues and greens can barely be distinguished by gas light, and still less by lamp or candle light.

**Shadows from Coloured Lights.**—A shadow cast by a body when the light is coloured appears to be tinged with the colour complementary to that of the light. This is due to contrast, because the illuminated ground is coloured by the light, although this fact may be hardly appreciated.

**Coloured Glass.**—Pure neutral or smoke glass absorbs part of all the colours of white light; if not exactly neutral some one colour penetrates it more than the others, and gives a distinct tint to a light seen through it. A glass of definite colour, as red or green, transmits not only its distinctive colour, but also some of the adjacent colours; thus green transmits some yellow and blue. Spectrum blue blocks out both the red and violet ends of the spectrum, and transmits blue, green and a little yellow. Cobalt blue transmits blue and red, but blocks out green and yellow. Orange, amber, yellow and green-yellow glass absorb the violet and ultra-violet light. Smoke glass absorbs a certain quantity of all the colours and therefore to some extent reduces the visual acuity, but it is usually more transmissive to one certain colour—generally red. White crown, and still more, flint glass is absorptive for ultra-violet light, while pebble is specially transmissive for it.

All colours are profoundly modified when viewed through coloured glass, as they are by coloured lights. If a coloured body be viewed through a coloured glass which absorbs the rays reflected by the body, the latter appears black. Thus a red body appears black through a green glass of the proper shade, the red rays reflected by the body not traversing the glass. If the ground be black, the object is barely distinguishable from the ground, or may not be at all, as in the "FRIEND" test.

A body viewed through a glass of the same colour appears almost white, or at least is indistinguishable from a white object seen through the same glass. Thus with red letters on a white ground, seen through red glass, the white background becomes coloured the same as the letters, so that the whole field is of uniform tint; here the colour of the glass is temporarily the *mental standard of white*. On looking at a red object on a green ground, through a piece of red glass, one sees a white object on a black ground, but if on a black ground the object appears redder. Similar phenomena result with other colours.

**Superposition of Coloured Glasses.**—When two coloured glasses are placed together we have an example of the subtractive process similar to that seen in the mixture of pigments. The first glass eliminates from incident white light all but its own colour, and if the second glass is the same as the first, no further alteration takes place, except a slight reduction in intensity. If the second glass is not of the same colour as the first, a certain amount of absorption by *subtraction* takes place in the second, and the more nearly complementary are the two glasses the more nearly will the whole of the incident light be cut off. For example, if a blue-green and a red, or an orange and blue glass, be placed together, the light transmitted by the one is absorbed by the other, and the combination is rendered opaque. Cobalt glass transmits red and blue, ordinary green glass transmits blue and green; on the two being placed together original white light transmitted appears blue, since the blue is transmitted by each, but the remaining colours absorbed. If three pieces of coloured glass corresponding to the summits of the three curves of red, green and blue-violet be superimposed, since each absorbs some of the components of white light, the three will absorb the whole of the visible spectrum, and no light whatever can be seen through the combination.

The natural colours of objects may be imitated by applying the above facts to photography. Three separate photographs are taken of an object or landscape, made up of any number of colours and shades, each through a glass selected to match as nearly as possible one of the three primary colours. A positive is taken from each on a film or paper, stained with the colour complementary to the colour of the glass used for that particular negative, and the three prints are superposed. This may be done either by laying the films exactly over each other and looking through them as a transparency, or each colour may be printed on the same piece of paper, and examined as an opaque object. In this way an approximate facsimile in colour of the original object can be obtained.

**Transmissiveness of Coloured Glasses.**—For the method of measuring this, and the photometry of coloured lights, see Chap. II.

The quantity of light absorbed depends directly on the thickness of the glass, and consequently no ordinary lens which varies in thickness owing to its curvature can have the same depth of tint all over. When a certain tint is selected, by trial with the coloured glasses of the test case, and lenses are required similarly tinted, a modification is necessary. The lens should be ordered of a lower tint if Cx. and of a higher if Cc., the former being thick and the latter thin at the centre. The variation from the No. of the trial glass would necessarily depend on the strength of the lens required.

Equality of tint can be obtained by employing a plano Cx. or Cc. lens cemented to a plano coloured glass. For a sphero-cylindrical combination equality can be obtained by cementing a thin plano spherical and a thin plano cylindrical to the two sides of a thin plano coloured glass; or if one of the components be weak, in comparisón with the other, by employing coloured glass for the weaker and white for the stronger, both being planos and cemented together. In this way practical equality of tint can be obtained.

Since the proportion of incident light transmitted depends on the thickness of the glass, it is not easy to express variations, but approximately the transmission varies inversely as the square of the thickness. Thus a standard No. 4 pure smoke glass transmits 1/5 of the incident light; a second No. 4 placed behind the other, transmits 1/5 of that transmitted by the first—i.e.

$1/5 \times 1/5 = 1/25$ of the total light, originally incident on the first glass, is transmitted by the two together.

Tinted glasses for spectacle work are usually numbered 1 to 6 or A to F. They vary considerably with respect to the quality and quantity of light transmitted, but approximately they transmit light as follows:—

| | Tint. | No. 1 or A. | No. 2 or B. | No. 3 or C. | No. 4 or D. | No. 5 or E. | No. 6 or F. |
|---|---|---|---|---|---|---|---|
| Percentage of light transmitted | Smoke | 60 | 50 | 30 | 20 | 10 | 2 |
| | Blue... | 80 | 70 | 50 | 25 | 20 | 4 |

The O.S. Standard Colour Glasses for spectacle work are:—

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Percentage of light transmitted | 80 | 60 | 50 | 40 | 30 | 20 | 10 | 5 | 2·5 | 1·25 |

**The Eye and Colour.**—White, being the sensation produced by the mixture of all colours, is the sensation of greatest luminosity. It is the standard of colour sensation, but this standard may be displaced, as when coloured illumination is used, or coloured glasses looked through, or by colour fatigue as when the eye is saturated with a certain colour by gazing at it for some time.

Black may be described as a sensation caused by want of colour, but it is very different from what is seen, or rather not seen, in the area occupied by the blind spot, as the head of the optic nerve of the eye is called. The latter is incapable of conveying any sensation of light at all, the resultant absence of sensation being quite different from black, which produces a distinct sensation. That is why the area occupied by the blind spot is unnoticed when we look at the sky, or other extended bright field. Of the specific colours, the human eye is most sensitive to yellow, whether seen in the spectrum or by reflected light. A yellow body will be seen longest as light is reduced and it can be seen further, although its colour may not be distinguishable. Generally speaking, as a characteristic and recognised colour, red is the most persistent of all; owing to its long wave-length it can be recognised at a greater distance than others, it freely penetrates haze, fog or smoke glass, while the penetration of other colours follow more or less in the order of the spectrum. For this reason red is employed as the danger signal, while blue-green is employed as the contrast signal on railways and ships. The sun appears redder at sunrise and sunset than at midday, also in fog, the blue-violet end of the spectrum being absorbed; the colour of light seen through a thick impure smoke glass is generally a brilliant red.

Green, which prevails in nature, fatigues the eye least of all the colours; then blue-grey, purple, yellow, orange and red, the last being the most fatiguing; billiard and card tables are covered with green cloth, and blinds are usually painted that colour. The sea and sky are blue, red and orange occurring in nature only in patches, or occasionally as at sunset; the eye is able to bear those colours best which are most widely distributed in nature, and most likely in consequence of it.

With respect to light, in general, it is more satisfactory the nearer it approaches white, but if coloured lights are used for illumination they should be pale and largely diluted with white. Thus pink and pale orange, or pale green, are pleasant, but the same colours, pure and saturated, would prove extremely fatiguing to the sight.

## CHAPTER XXIII

# CHROMATIC ABERRATION

### Dispersion.

**Dispersion or Chromatism.**—When white light suffers refraction, the component waves are refracted to different extents, so that the various colours become separated, producing what is known as *dispersion*. This is due to the fact that the shorter waves, with rare exceptions, are retarded, by the refracting medium, more than the longer waves. Reflection, unlike refraction, is not accompanied by dispersion, this fact rendering reflecting sometimes preferable to refracting instruments. A body is said to be *chromatic* if it causes dispersion, and *achromatic* if it does not.

**Velocity of Light and Colour.**—The velocity of light in free ether is the same for all colours, and is taken as being so also in air, although this is not quite the case, blue being retarded slightly more than red in its passage through the atmosphere. If $V_1$ be the velocity in air (about 300,000 km. per second) and $V_2$ that in a denser medium, then $V_1/V_2=\mu$. Suppose in a medium $\mu_D=1·5$, $\mu_F=1·51$, $\mu_C=1·49$. Then $V_D=300,000/1·5=200,000$ km., $V_F = 300,000/1·51 = <200,000$ km., and $V_C = 300,000/1·49 = >$ 200,000 km. per sec.

**Dispersive Index.**—Each refracting medium has what may be termed an *index of dispersion*, which represents the differences between the indices of refraction of the lines $A$ and $H$ of the spectrum. Thus, water has an index of refraction for the line $A$ of 1·3289, and for the line $H$ of 1·3434 ; now $1·3434 - 1·3289 = ·0145$, which is the index of dispersion of water. *Mean dispersion* is represented by the difference between the indices of refraction of the lines $C$ and $F$, i.e. between orange-red and blue, and *partial dispersion* is that between the $\mu$'s of any two given lines of the spectrum.

The table on p. 258 gives in the third column the mean dispersion, and in the fourth column the total dispersion of the visible spectrum.

The dispersion of various kinds of glass differs with the materials used in their manufacture, and is independent of their refracting power, some media of high mean refraction having low dispersion and *vice versa ;* generally, however, high refractivity and high dispersivity accompany each other.

TABLE OF DISPERSIONS.

| | | | Mean. | Total. |
|---|---|---|---|---|
| Water ... ... | $\mu_C = 1{\cdot}3317$ | $\mu_F = 1{\cdot}3378$ | ·0061 | ·0145 |
| Alcohol ... ... | $\mu_C = 1{\cdot}3621$ | $\mu_F = 1{\cdot}3683$ | ·0062 | ·0149 |
| Pebble ... ... | — | — | — | ·014 |
| Canada Balsam ... | — | — | — | ·021 |
| Tourmaline ... | — | — | — | ·019 |
| Crown Glass *if* ... | $\mu_C = 1{\cdot}5376$ | $\mu_F = 1{\cdot}5462$ | ·0086 | ·018 |
| Flint Glass *if* ... | $\mu_C = 1{\cdot}6199$ | $\mu_F = 1{\cdot}6335$ | ·0136 | ·026 |
| Diamond ... ... | $\mu_C = 2{\cdot}4102$ | $\mu_F = 2{\cdot}4355$ | ·0253 | ·056 |

**$\nu$ or the Ratio of Refractivity to Dispersion.**—Since refractivity and dispersion are more or less independent of each other, neither the total nor the mean dispersion indicates the optical properties of a medium; for this we must take the ratio between the mean refractivity and the mean dispersion, which ratio is termed the *refractive efficiency*, denoted by the symbol $\nu$ (nu), and is expressed by

$$\nu = \frac{\mu_D - 1}{\mu_F - \mu_C}$$

Here $(\mu_D - 1)$ is the mean refractivity for yellow light ($D$ line) of the medium, i.e. it is the $\mu$ of the medium less the $\mu$ of air $= 1$, while $\mu_F - \mu_C$ is the mean dispersion between the $C$ and $F$ lines of the spectrum produced by the particular substance. The formula, therefore, gives us a value which, when compared with the corresponding $\nu$ of another medium, will show which of the two has the higher refractivity as compared with its mean dispersive power. A high value of $\nu$ denotes a high mean refractivity and a relatively low dispersion, while a low $\nu$ indicates the reverse, i.e. a low refractivity and a relatively high dispersion. Thus if, in a variety of flint glass, $\mu_D = 1{\cdot}6$, $\mu_F = 1{\cdot}61$, and $\mu_C = 1{\cdot}59$ the efficiency is

$$\nu = \frac{1{\cdot}6 - 1}{1{\cdot}61 - 1{\cdot}59} = \frac{{\cdot}6}{{\cdot}02} = 30.$$

Take also a sample of crown where $\mu_D = 1{\cdot}525$, $\mu_F = 1{\cdot}532$ and $\mu_C = 1{\cdot}523$

$$\nu = \frac{{\cdot}525}{{\cdot}009} = 60 \text{ (approx.)}$$

These values of $\nu$, i.e. 30 and 60, show that in the former glass the dispersion is relatively twice as great as in the latter glass; or, as it is better expressed, the crown has twice as much mean refractivity than the flint for the same amount of dispersion. Thus if two glasses have the same $\mu$, but different mean dispersions, the one with the lower dispersion has the higher $\nu$. If two glasses have the same dispersion but different $\mu$'s, the one with the higher $\mu$ has the higher $\nu$. In general crown glass has a higher $\nu$ than flint. The formula enables us to calculate the components of an achromatic prism or lens, and by its aid glasses can be tabulated in the order of their efficiencies, so that a selection can be easily made.

The $\mu_D$ of water is 1·3336, and its mean dispersion is ·0061, so that the $\nu$ is nearly 55. With air $\mu = 1·00029$, and the mean dispersion is ·0000029, so that $\nu = 100$ approx.

**Expression for $\omega$.**—Calculations with respect to chromatism are sometimes based on $\omega$ (omega), the *dispersive power*, which is the reciprocal of $\nu$, and is therefore

$$\omega = \frac{\mu_F - \mu_C}{\mu_D - 1}.$$

**$\delta$ and $\Delta$.**—The difference between the indices of refraction of the F and C lines, i.e., $\mu_F - \mu_C$, is sometimes represented by the symbol $\delta$, and the difference between $\nu_1$ and $\nu_2$ (i.e. the $\nu$'s of two different media), by the symbol $\Delta$.

**Achromatism of a Parallel Plate.**—When light is incident obliquely on a parallel plate of any medium, no colouration of the object is noticed

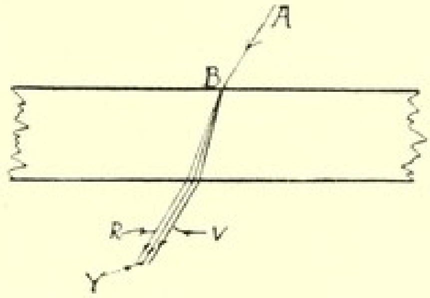



FIG. 245.

because, although dispersion occurs at the first surface, it is neutralised by that of the second.

Let *A B* (Fig. 245) represent a beam of parallel light incident on a plate having parallel surfaces. At *B* dispersion takes place, so that violet is deviated the most and red the least, and were it possible for the eye to receive the beam before it leaves the plate, the object would appear deviated and tinged with colour just as with an ordinary prism. At the second surface, however, all the dispersed rays are rendered parallel to each other and therefore, by their overlapping on the retina, produce the sensation of white. In other words the appearance of the object, so far as dispersion is concerned, is the same as though viewed direct. Thus it will be seen that, in order to cause chromatism or dispersion, a medium must have the power of *altering the angular deviation of the various colours* with respect to each other.

**Achromatic Prism or Lens.**—When a prism or lens is said to be achromatised its action is similar to that of a plate, while the course of light, as a whole, is changed.

## Chromatism of a Prism.

**Crown and Flint Glass.**—Flint glass has a greater refracting power than crown, but its dispersive power is proportionally still greater; the relative refractivities are approximately 1·1 : 1 and their dispersivities 1·4 : 1. A flint prism of 10° and a crown of 11° have each a deviating angle of about 6°, but the spectrum of the flint glass is considerably the longer. If spectra of the same lengths be required, the crown glass prism must be stronger than the flint.

**Real and Virtual Spectra.**—Since a prism refracts the violet waves most and red the least, the real spectrum projected on to a screen exhibits violet nearest the base and red nearest the edge of the prism, as shown in Fig. 246, where *L* is the source of light. If the light be received by the eye, the rays are projected back to form a virtual spectrum, and the violet is then nearest the edge and the red nearest the base. Thus, a disc of light viewed through a prism, base down, exhibits blue above and red below.

The light from a white body, refracted by a prism, causes a series of

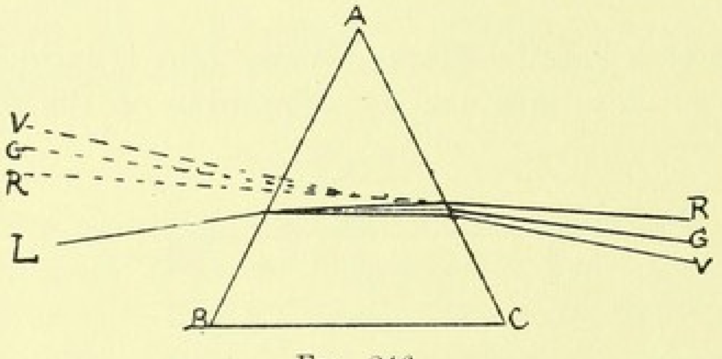



FIG. 246.

separate images of the body, each characterised by a distinctive spectrum colour. These overlap in the centre so that a white virtual image is seen, but the ultimate displacements of blue at the one end, and of red at the other, cause a fringe of blue to appear on that border which is nearest the edge of the prism, and a red-orange fringe on that nearest the base. If the body is black or dark, as compared with its background, the red-orange fringe is towards the edge and the blue fringe towards the base of the prism, these resulting from the dispersion of the light from the space or body surrounding the black. Thus a window bar viewed in daylight, through a prism base down, is blue at the bottom and reddish-yellow on top, but if viewed in artificial light at night the colours are reversed.

**Dispersion of a Prism.**—The wave-front of a beam of light, incident on a prism, is retarded sooner at the base than at the edge, so that the beam is deviated towards the base of the prism, and since the retardation is greater as the wave-length is shorter the blue is, as stated above, more deviated towards the base than the red. Thus when a beam of white light is refracted by a prism, its various components are separated, and form a band of colours called the spectrum. The extent of the dispersion varies with the medium of which the prism is formed, with the angle of the prism, and with the angle of incidence of the light, but the dispersion is not a minimum when the mean ray—yellow—suffers minimum deviation. The dispersion of a prism can be determined by a spectrometer, the difference between the $\mu$'s for the $C$ and $F$ lines being the mean dispersion. In this case the minimum deviation may be obtained for each colour before calculating its index. The position of the prism must be that of minimum deviation for the $D$ line in order that the deviation of the colours, on either side of the yellow, may be observed when the mean deviation is minimum.

**The Refraction Spectrum.**—In order to produce a spectrum by refraction the light should be admitted through a small horizontal aperture $A$ (Fig. 247), preferably about 20 mm. long by ·5 to 1 mm. wide placed parallel to the edge of the prism $P$. The light, being thus admitted, is incident on a prism placed in its path in the position of minimum deviation. The resultant spectrum, however, is said to be *impure* because there is an overlapping between adjacent colours, but if an achromatic bi-convex lens $L$ of, say, 36″ F is 72″ from the slit and close to the prism, a sharp pure spectrum

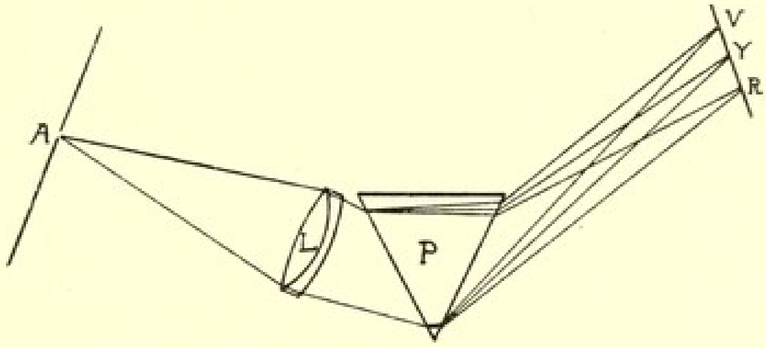



Fig. 247.

$V\,Y\,R$ is formed on a screen situated 72″ beyond the lens. If the prism be placed base up the violet is above and the red below. All the different colours are seen well defined, but the red end of the spectrum is somewhat crowded, while the blue is spread out.

**Pure Spectrum.**—A spectrum is said to be *pure* when the individual colours are isolated to the greatest possible extent, this being secured by having an extremely fine aperture as a source, and an achromatic condensing lens. The effect of the lens is to project a real image of the slit, while that of the prism is to produce, from this single white image, an innumerable series of others corresponding to every different wave-length, and these, lying side by side, result in the ordinary pure spectrum. Actually it is impossible to obtain a theoretically pure spectrum since the source must always be of some definite magnitude, and therefore a certain amount of overlapping always takes place between adjacent colours. The purity, however, reaches a very high standard in the spectroscope where, in addition to the finest possible slit, the light received by the prism from the collimator is parallel, so that prismatic distortion is eliminated.

**The Spectroscope** (Fig. 248) is used for viewing and comparing spectra produced by prisms, and consists of horizontal circle, mounted on a stand, to which are attached a telescope *T* and a collimator *C*, both of which can be rotated around the circle. The collimator *C* is a tube having at one end a Cx. lens and at the other a narrow slit parallel to the refracting edge of the prism *P*. The distance between the slit and the lens is equal to F of the latter, so that light, entering the slit, is rendered parallel by the lens before reaching the prism. In the centre of the circle there is a small table *B* on which the prism is placed.

**The Spectrometer** is a spectroscope with the addition of a horizontal scale of degrees on which the position of the moveable telescope can be indicated, and to which, for accurate readings, a vernier or reading microscope is attached. This enables the principal angle, the deviating angle, and the dispersion of a prism to be measured.

In order to measure the deviating angle of a prism, *C* and *T* are brought into line (Fig. 248) so that the image of the slit appears in the centre of the

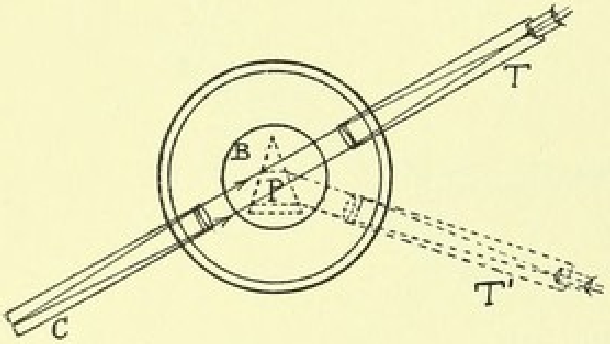



FIG. 248.

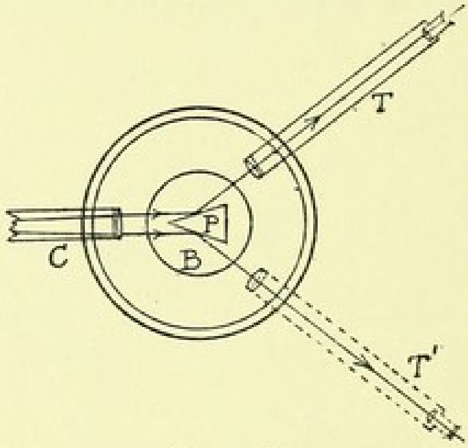



FIG. 249.

field of view, the objective of the telescope forming a real image of the slit in the focal plane of the ocular, through which it is viewed, and a reading is taken on the circle. The prism is then placed in position, and the telescope must be rotated to *T'* until the image of the slit can again be seen. The angular distance through which *T* is moved is the deviating angle of the prism, care being taken that the deviation is a minimum. This can be done by slightly rotating the prism backwards and forwards until a position is found when the slightest movement in *either* direction *increases* the deviation.

The principal angle of a prism is measured by turning the prism until its edge splits into two halves the beam of light issuing from the collimator (Fig. 249). The telescope is rotated to *T* until the image of the slit is seen reflected from the one surface, and then turned to *T'* to receive the image from the other surface of the prism. Half the angle through which the telescope has been rotated gives the principal angle of the prism.

The mean refraction is indicated when the yellow of the spectrum lies on the wire placed in the focus of the ocular. When the principal angle and mean deviating angles are known, the refractive index of the glass, of which the prism is made, can be calculated by the method given elsewhere.

The deviation of a prism, for any colour, can be determined by bringing that colour on to the cross wire and by this means, the total, mean or partial dispersion of the medium, of which a prism is made, can be determined.

The spectrum produced by a given source can be studied and, if necessary, the spectra from two sources can, by suitable arrangement, be formed side by side for comparison. For very accurate determination of refractive index and dispersion, various incandescent gases are employed, which give line spectra, instead of a white source producing a continuous spectrum.

**The Diffraction Spectrum** is purer than that of refraction, and is referred to in Chap. XXVII.

**Refraction and Dispersion.**—Refraction by a simple medium is, so far as known, always accompanied by dispersion or chromatism, and even when a

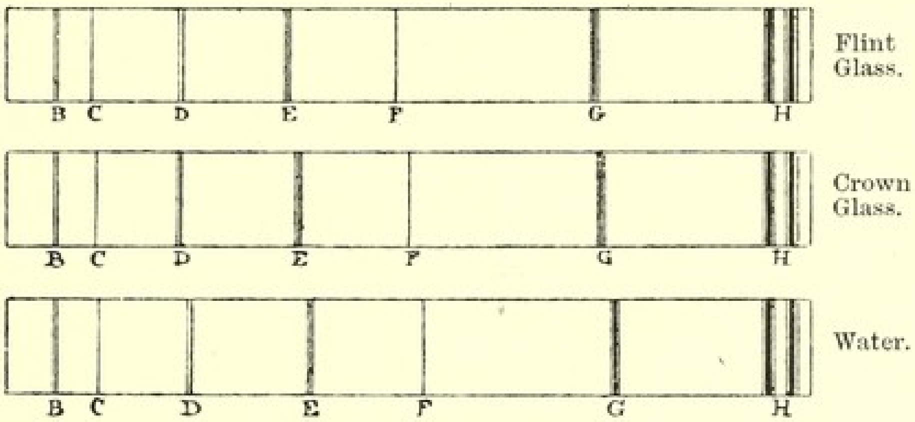



FIG. 250.

prism or lens is, as is termed, fully achromatised by one or more other prisms or lenses, some dispersion still obtains. Although two or even three colours may be brought to a common focus, this can never be done for every colour of the spectrum, and, as will be seen further on, with two lenses only two colours can be brought to a common focus, the coincidence of the others being merely approximate. This want of coincidence of all the colours is due to *irrationality of the spectrum*, which may be defined as the irregularity of sequence of the principal colours in any two spectra produced by different media.

If we take a number of prisms of different substances, but of the same angle, it will be found that those having the higher refractive index usually, but not of necessity, possess the longer spectra These different spectra can be made of the same length by altering the angles or the position of the prisms, or by adjusting the position of the screen. If (Fig. 250) the spectra be placed one under the other so that the *B* lines at the red and the *H* lines at the blue correspond in position, it will be found that the intermediate lines do not do so. This fact renders it difficult to fix the exact position of lines in the spectrum, since a special scale has to be made for each spectroscope.

**Anomalous Dispersion.**—In glass, water and most substances, the order of refrangibility is from the red through the orange, yellow, green, blue, indigo to violet, which is the most refrangible, but certain substances have the property of refracting the normally more refrangible rays less, and the less refrangible more (Fig. 251). This is called *anomalous dispersion.* The substances which exhibit this peculiarity possess what is termed surface colour, i.e., they have a different colour when viewed by reflected light to what they have by transmitted light. As they reflect only a certain colour, the complementary colours are transmitted, and their spectra exhibit an absorption band of more or less considerable dimensions, it being the space which would have been occupied by the reflected colour had it been transmitted. Such substances are termed *dichroic.*

Most metals, except gold and copper, as well as many of the aniline products, possess this abnormal dispersion, the order of colours being

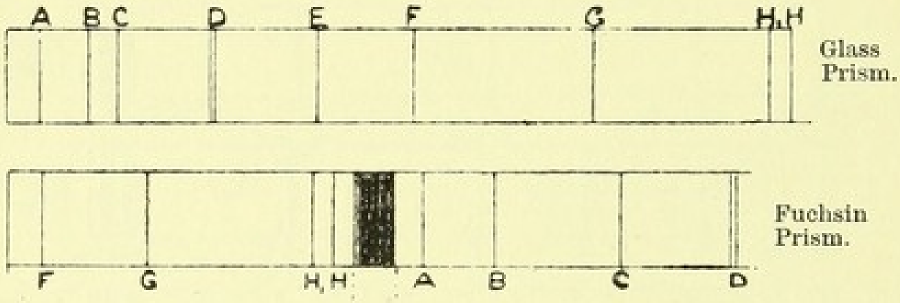



FIG. 251.

changed. Moreover, Kundt found, in the aniline products, the dispersion abnormally increased on the red side of the band, but diminished on the violet side; so that in the case of fuchsin, for example, the red end, usually so short, is actually more extended than the violet end.

**Recomposition of Dispersed Light.**—To recombine the spectrum of a prism in order to form white light we may adopt several methods, as follows:

1. Employing a prism of equal dispersive power. This is placed in the path of the dispersed light, having its base turned in the opposite direction to that of the first prism. (Newton's method.)

2. A series of plane mirrors may be so arranged that each receives a different portion of the spectrum; from each the light is reflected to the same part of a screen, where the colours are re-combined.

3. Receiving the dispersed light on a concave mirror, from which it is reflected on to a screen, and then by rapidly oscillating the mirror or the screen the impression of white light is produced. Or the prism or the screen may be oscillated or rotated to produce a similar effect without the interposition of the concave mirror.

Any mechanical arrangement of rotation or oscillation by which the colours of the spectrum, whether produced by dispersion or by transmission through coloured glasses, or by reflection from pigments, are caused to successively enter the eye with sufficient rapidity, produces the impression of white. Different colour sensations result while others are still existing, and the combination of all results in a sensation of white or grey. Colour tops, or discs divided into sectors of different colours, are examples of this method.

### Achromatic Prism.

**Angular Dispersion.**—The deviating angle of a prism is that of the mean ray (*D* line), and is expressed (in the case of thin prisms) by $d = \text{P}\ (\mu - 1)$, where P is the refracting, and $d$ the deviating angle. Now, since the red ray suffers less, and the blue greater refraction than the *D* line, their angular deviations are respectively

$$d_{\text{C}} = \text{P}\ (\mu_{\text{C}} - 1), \text{ and } d_{\text{F}} = \text{P}\ (\mu_{\text{F}} - 1)$$

$d_{\text{C}}$ and $d_{\text{F}}$ being the deviating angles, and $\mu_{\text{C}}$ and $\mu_{\text{F}}$ being the indices of refraction for red and blue light respectively. The angular dispersion of the prism expressed in degrees is

$$\text{P}\ (\mu_{\text{F}} - 1) - \text{P}\ (\mu_{\text{C}} - 1) = \text{P}\ (\mu_{\text{F}} - \mu_{\text{C}})$$

But, as before stated, it is not sufficient merely to know the mean or angular dispersion of a medium; we must know the amount of dispersion which, in any particular case, accompanies a given amount of deviation, that is, we must know its $\nu$ value. Suppose the $\nu$ of a crown prism is 60, and that of a flint 30, then since $\nu$ of the crown is twice as great as that of the flint we know that, for a given deviation, we have twice as much dispersion in the flint as in the crown. If two prisms of equal deviating angles were worked from the glasses, the spectrum of the flint would be approximately double the length of that of the crown.

**Similar Prisms.**—If two similar prisms, *A* and *B* (Fig. 252) are placed in opposition—base to edge—their angles, refractive indices, and dispersions being the same, both the deviation and dispersion are neutralised, and all the rays emerge parallel to their original course.

In Fig. 253 let the principal angle of a crown prism of $\mu = 1{\cdot}54$ be $11{\cdot}3°$, and that of a flint prism of $\mu = 1{\cdot}61$ be $10°$. Their deviating angles are the same, namely, $6{\cdot}1°$. If in the crown $\mu_{\text{C}} = 1{\cdot}534$, $\mu_{\text{F}} = 1{\cdot}554$, and in the flint $\mu_{\text{C}} = 1{\cdot}586$, $\mu_{\text{F}} = 1{\cdot}62$, their dispersive angles between blue and red are $11{\cdot}3 \times (1{\cdot}554 - 1{\cdot}534) = {\cdot}226°$ for the crown, and $10 \times (1{\cdot}62 - 1{\cdot}586 = {\cdot}34°$ for the flint. The resultant angular dispersion is therefore ${\cdot}34 - {\cdot}226 = {\cdot}114° = 6' \ 50''$. Thus, while no deviation of the mean yellow ray occurs, the red and blue are separated by an angle of nearly $7'$.

**Achromatised Prism.**—If a crown prism of 3°d and $\mu = 1{\cdot}54$, and a flint of 2°d and $\mu = 1{\cdot}61$ (Fig. 254), having efficiencies of 45 and 30 respectively, be placed in opposition, they neutralise each other's dispersion, while there remains 1° deviation. Such prisms are said to be achromatised, i.e., they constitute an achromatic prism which causes deviation without dispersion. The principal angle P of the crown is $3/{\cdot}54 = 5{\cdot}55°$, and of the flint $2/{\cdot}61 = 3{\cdot}28°$. Here, as will be seen from Fig. 254, *every* ray is deviated to the same extent, and the recombination of light is secured as with an ordinary parallel plate.

**To Calculate an Achromatic Prism.**—In order to calculate the data for an achromatic prism, let $d$ represent its deviating angle, and P the principal angle. Let $d_1$, $\nu_1$ and $P_1$ be those of the crown and $d_2$, $\nu_2$ and $P_2$ those of the flint components respectively.

Now we have $d = d_1 + d_2$, and since they have to be in opposition we can regard $d_1$ as positive and $d_2$ as negative. In order that achromatism may result we must have

$$d_1\nu_2 = -d_2\nu_1 \qquad \text{or} \qquad d_1\nu_2 + d_2\nu_1 = 0$$

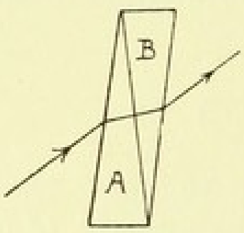


FIG. 252.

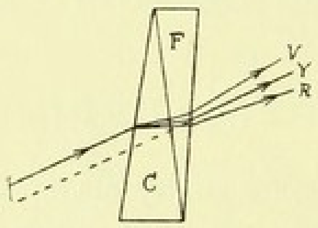


FIG. 253,

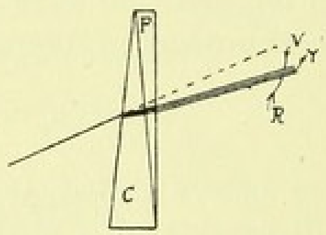


FIG. 254.

To obtain the values of the two components $d_1$ and $d_2$, we must divide $d$, *the deviating angle of the required achromatic prism*, in proportion to the values of $\nu_1$ and $\nu_2$, that is

$$d_1 = \frac{d\,\nu_1}{\nu_1 - \nu_2}, \text{ and } d_2 = \frac{d\,\nu_2}{\nu_2 - \nu_1}$$

It should be noticed that the *deviating*, not the principal angle, of the final prism is divided in the ratio of the $\nu$'s, because P is dependent upon $d$ and the mean refractive index. We find the principal angles of the two components from $P_1 = d_1/(\mu_D - 1)$ and $P_2 = d_2/(\mu_D - 1)$, which, however, hold good only for thin prisms; if strong, $P_1$ and $P_2$ must be found by the exact formulæ previously given.

**As an example**, an achromatic prism of 5°d is needed, the glasses of the component parts being

| | | | |
|---|---|---|---|
| Crown ... ... | $\mu_D = 1{\cdot}53$ | $\mu_C = 1{\cdot}527$ | $\mu_F = 1{\cdot}536$. |
| Flint ... ... | $\mu_D = 1{\cdot}63$ | $\mu_C = 1{\cdot}624$ | $\mu_F = 1{\cdot}644$. |

$$\nu_1 = \frac{1{\cdot}53 - 1}{1{\cdot}536 - 1{\cdot}527} = \frac{{\cdot}53}{{\cdot}009} = 58{\cdot}9$$

$$\nu_2 = \frac{1{\cdot}63 - 1}{1{\cdot}644 - 1{\cdot}624} = \frac{{\cdot}63}{{\cdot}02} = 31{\cdot}5$$

$$d_1 = \frac{5 \times 58{\cdot}9}{58{\cdot}9 - 31{\cdot}5} = \frac{294{\cdot}5}{27{\cdot}4} = 10{\cdot}7^\circ\text{d, and } P_1 = \frac{10{\cdot}7}{{\cdot}53} = 20^\circ$$

$$d_2 = \frac{5 \times 31{\cdot}5}{31{\cdot}5 - 58{\cdot}9} = \frac{157{\cdot}5}{-27{\cdot}4} = -5{\cdot}7^\circ\text{d, and } P_2 = \frac{5{\cdot}7}{{\cdot}63} = 9^\circ$$

$$10{\cdot}7^\circ - 5{\cdot}7^\circ = 5^\circ\text{d.}$$

**To find the Achromatising Prism.**—The power of the flint prism $d_2$ of $\nu_2$, which will neutralise the dispersion of a given crown of $d_1$ and $\nu_1$ is calculated from

$$\nu_1/\nu_2 = d_1/d_2 \qquad \text{or} \qquad d_2 = d_1\nu_2/\nu_1$$

Thus, let the crown be $10{\cdot}7^\circ$d, $\nu_2 = 31{\cdot}5$, and $\nu_1 = 58{\cdot}9$, then

$$d_2 = 10{\cdot}7 \times 31{\cdot}5/58{\cdot}9 = 5{\cdot}7^\circ\text{d}$$

and

$$d_1 + d_2 = 10{\cdot}7 - 5{\cdot}7 = 5^\circ = d$$

### Chromatism of a Lens.

The effect of dispersion, when the refracting body is a lens, is to bring the more refrangible blue and violet to a focus sooner than the less refrangible red and orange. This different focalisation of the various colours is termed *chromatism*, and the confusion of the image caused by it, *chromatic aberration*. The defect, which is made apparent by a fringe of colour on the edge of the real or virtual image projected by the lens, is due to the nature of light, and not to the nature of the lens, although its degree varies with the power of the lens and the kind of glass of which it is made. Indeed, lenses being prismatic in nature produce similar chromatic phenomena to prisms.

If a horizontal white line (Fig. 255) be observed through the marginal portion of a convex lens, a blue-violet fringe will be seen on the side towards the edge of the lens, and a red-orange on the other, the blue being projected back above the red. Viewed through the periphery of a concave, the colours are reversed. Looking at a black line, the fringes are seen in the opposite order to those on a white line, for the reason given in connection with a prism. The centre of the image, whether virtual or real, of a white object, appears white, because the different colours are superposed, so that only at the extremities, where certain colours are not combined with others, is chromatism apparent.

**Longitudinal and Lateral Aberrations of Colour.**—In Fig. 256 a parallel beam of light from a point source is refracted by a convex lens ; the various coloured rays meet at different distances behind the lens, the violet focussing at *V*, the yellow at *Y*, and the red at *R*. If a screen be held at *V*, the diffusion patch has a reddish-yellow fringe ; the red and orange rays, being convergent to a more distant point *R*, impinge on the screen outside the blue and violet. If the screen be placed at *R*, the fringe becomes blue-violet, since these rays, having already met at *V* and crossed, impinge on the screen outside the red and orange. The distance *V R* is the longitudinal aberration, and the diameter *a b* of the disc of confusion in the plane where the extreme violet and red rays cross each other, is the lateral aberration ; this plane is very nearly that of *F X*, where the yellow light is brought to a focus.

**Circle of Least Confusion.**—At *F*, the focus of the most luminous yellow light the circles of diffusion formed by the red and blue are practically of the same size, and therefore no coloured fringe is appreciable.

**F of the Various Colours.**—The index of refraction of a given medium refers to that of the *D* (sodium) line, which is situated in the yellow or most

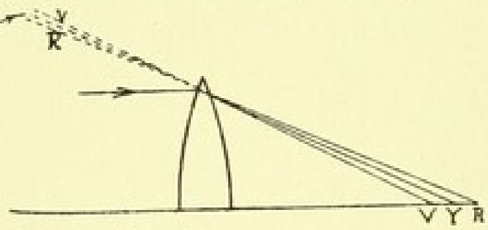



FIG. 255.

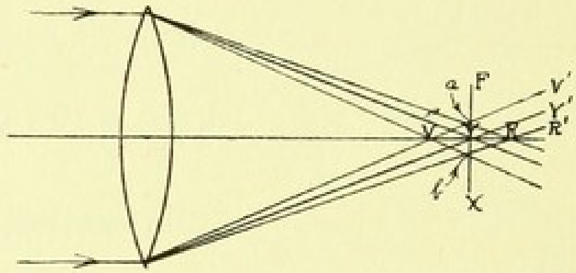



FIG. 256.

luminous part of the spectrum. With such a medium, if $\mu_D = 1{\cdot}54$, the index of refraction for the red (line *A*) might be 1·53 ($\mu_A = 1{\cdot}53$), while for the violet (line *H*) it might be 1·56 ($\mu_H = 1{\cdot}56$). Suppose a thin double convex lens of 10 in. radius, then—

$$F = \frac{r_1 r_2}{(r_1 + r_2)(\mu_D - 1)}$$

that is,

$$F = \frac{10 \times 10}{(10 + 10) \times {\cdot}54} = \frac{100}{10{\cdot}8} = 9{\cdot}26 \text{ in.}$$

which is the mean focal length as well as that for yellow light. But instead of $\mu = 1{\cdot}54$ we must employ $\mu_A = 1{\cdot}53$ and $\mu_H = 1{\cdot}56$ in order to find the focal lengths $F_A$ and $F_H$ for red and violet light respectively ; thus $F_A = 9{\cdot}43$ in. and $F_H = 8{\cdot}93$ in.

The difference in the focal lengths of a lens for red and blue light may be illustrated with a cobalt-blue glass (chromatic disc), which transmits red and blue light, but absorbs the central part of the spectrum, or by focussing with a convex lens light which is rendered monochromatic by being passed through respectively standard red and blue glass. The difference in the focal distances with these two coloured lights is sufficiently well marked to be appreciated.

### Achromatic Lens.

Chromatism can be remedied by making the lens a combination of two different kinds of glass, so chosen that, while the dispersion of the positive component is neutralised by that of the negative one, there shall still be some positive converging power left, so that a real image may be formed. Such a combination is termed an *achromatic* lens, and usually consists of a flint concave and a crown convex. If a concave achromatic combination is required, as occurs sometimes in practice—for instance, in the telephoto lens—then the concave is of crown and the convex of flint.

**Spectrum Lines Combined.**—By an achromatic lens two selected lines of the spectrum, usually the $C$ and $F$ (orange-red and blue) are brought to a focus at the same distance; by uniting these with a third component a third line could also be focussed at the same distance, but for all practical purposes if the $C$ and $F$ lines, which lie near the more central and luminous part of the spectrum, are combined, the combination is one in which chromatism does not cause any appreciable blurring of the image, at least for visual purposes, in which critical definition is not essential. In photographic lenses the lines $D$ and $G$ or $D$ and $H$ are usually selected in order to unite the violet, which is the most chemically active part of the spectrum, with the visual focus. For astro-photographic purposes, in which vision is of little consequence, the lines $F$ and $H$ (or beyond) are brought together.

**Expression for Chromatic Aberration.**—Let $r_1$ and $r_2$ represent the two radii, and $F_D$ the focal length of a thin lens for the $D$ line. Then, if $F_A$ and $F_H$ represent the focal lengths, and $\mu_A$ and $\mu_H$ the indices for extreme red and violet respectively, the chromatic focal difference may be expressed by

$$F_A - F_H = \frac{r_1 r_2}{(r_1 + r_2)(\mu_A - 1)} - \frac{r_1 r_2}{(r_1 + r_2)(\mu_H - 1)}$$

$$= \frac{r_1 r_2 (\mu_H - \mu_A)}{(r_1 + r_2)(\mu_A - 1)(\mu_H - 1)}$$

If instead of $(\mu_H - 1)(\mu_A - 1)$ there be substituted $(\mu_D - 1)^2$, as may be done without sensible error, then

$$F_A - F_H = \frac{r_1 r_2 (\mu_H - \mu_A)}{(r_1 + r_2)(\mu_D - 1)^2} = \frac{F_D (\mu_H - \mu_A)}{(\mu_D - 1)}$$

$$\nu = \frac{\mu_D - 1}{\mu_H - \mu_A} \text{ is the refractive efficiency,}$$

$$\omega = \frac{\mu_H - \mu_A}{\mu_D - 1} \text{ is the dispersive power.}$$

Both of these are similar to those found in the case of a prism. Then the longitudinal aberration is $F_A - F_H = F_D/\nu = F_D\,\omega$.

As an example let $F_D = 10$ in., $\mu_A = 1{\cdot}60$ ; $\mu_D = 1{\cdot}61$ and $\mu_H = 1{\cdot}625$, then

$$F_A - F_H = \frac{10 \times (1{\cdot}625 - 1{\cdot}60)}{1{\cdot}61 - 1} = 10 \times \frac{{\cdot}025}{{\cdot}61} = {\cdot}41 \text{ in.}$$

The lateral chromatic aberration of a lens = diameter of lens/$2\nu$.

Similar calculations can be used for a thick lens.

**Calculation for an Achromatic Combination.**—To calculate an achromatic combination for two lenses in contact, let F and C be the two lines of the spectrum which have to be brought to the same focus. Let F be the focal length of the required combination.

$F_1$ and $\nu_1$ are the focal length and efficiency of the crown component, and $F_2$ and $\nu_2$ those of the flint; $F_2$ is negative, and $1/F = 1/F_1 + 1/F_2$. In order that the achromatism to be eliminated

$$1/F_{1H} - 1/F_{1A} = 1/F_{2H} - 1/F_{2A}$$

or

$$\frac{F_{1A} - F_{1H}}{F_{1A}F_{1H}} = \frac{F_{2A} - F_{2H}}{F_{2A}F_{2H}}$$

But $F_{1A} - F_{1H} = F_1/\nu_1$, and $F_{2A} - F_{2H} = F_2/\nu_2$, and without serious error, $F_{1A}F_{1H} = F_1^2$, and $F_{2A}F_{2H} = F_2^2$, so that the last equation can be written

$$F_1/\nu_1 F_1^2 = F_2/\nu_2 F_2^2$$

$$1/\nu_1 F_1 = 1/\nu_2 F_2, \text{ or } \nu_1 F_1 = -\nu_2 F_2$$

That is

$$F_1\nu_1 + F_2\nu_2 = 0$$

The two components $1/F_1$ and $1/F_2$ are obtained by dividing $1/F$ proportionally to the two efficiencies $\nu_1$ and $\nu_2$ ; that is

$$\frac{1}{F_1} = \frac{1}{F} \times \frac{\nu_1}{\nu_1 - \nu_2} = \frac{\nu_1}{F(\nu_1 - \nu_2)}$$

and

$$\frac{1}{F_2} = \frac{1}{F} \times \frac{\nu_2}{\nu_2 - \nu_1} = \frac{\nu_2}{F(\nu_2 - \nu_1)}$$

or

$$F_1 = F(\nu_1 - \nu_2)/\nu_1, \text{ and } F_2 = F(\nu_2 - \nu_1)/\nu_2$$

**As an example,** let a positive achromatic lens of $6\frac{1}{2}$ in. focal length be required ; if the indices of refraction for the various lines are $\mu_C = 1{\cdot}527$, $\mu_D = 1{\cdot}53$, $\mu_F = 1{\cdot}536$ for the crown, and $\mu_C = 1{\cdot}630$, $\mu_D = 1{\cdot}635$, $\mu_F = 1{\cdot}648$ for the flint

$$\nu_1 = \frac{1{\cdot}530 - 1}{1{\cdot}536 - 1{\cdot}527} = \frac{{\cdot}530}{{\cdot}009} = 58{\cdot}89$$

$$\nu_2 = \frac{1{\cdot}635 - 1}{1{\cdot}648 - 1{\cdot}630} = \frac{1{\cdot}635}{{\cdot}018} = 35{\cdot}28$$

and

$$\nu_1 \sim \nu_2 = 58{\cdot}89 \sim 35{\cdot}28 = \pm 23{\cdot}61$$

Then $F_1 = 6{\cdot}5 \times 23{\cdot}61/58{\cdot}89 = +2{\cdot}61$ in.

and $F_2 = 6{\cdot}5 \times -23{\cdot}61/35{\cdot}28 = -4{\cdot}358$ in.

therefore $1/F = 1/2{\cdot}61 - 1/4{\cdot}358 = 1/6\frac{1}{2}$ or $F = 6\frac{1}{2}$

which is positive and therefore convex.

**To find the Achromatising Cc.**—The F of a Cc. of $\nu_2$ which, with a given Cx. of $\nu_1$, will make the combination achromatic, is found from

$$\nu_1/\nu_2 = F_2/F_1, \text{ or } F_2 = F_1\nu_1/\nu_2$$

Taking the same figures as in the previous example, if the Cx. has F = 2·61 in., then

$$F_2 = 2{\cdot}61 \times 58{\cdot}89/35{\cdot}28 = 4{\cdot}358$$

### Dioptral Formulæ.

With dioptral lenses an achromatic combination is calculated from the following formulæ. Let D represent the power of the combination, $D_1$ and $D_2$ the powers respectively of the Cx. and Cc., $\nu_1$ and $\nu_2$ the respective efficiencies of the crown and flint lenses, $\nu_2$ being negative.

In order to achromatise each other the relationship must be $D_1\,\nu_2 = -D_2\,\nu_1$ or $D_1\,\nu_2 + D_2\,\nu_1 = 0$. The two components must equal D, that is $D = D_1 + D_2$. To obtain the values of $D_1$ and $D_2$, which together equal D, we must divide the latter proportionally to $\nu_1$ and $\nu_2$, that is

$$D_1 = \frac{D\,\nu_1}{\nu_1 - \nu_2} \qquad \text{and} \qquad D_2 = \frac{D\,\nu_2}{\nu_2 - \nu_1}$$

Taking the same glasses as in the previous example where $\nu_1 = 58{\cdot}89$, $\nu_2 = 35{\cdot}28$ and $F = 6\frac{1}{2}''$, or D = 6

$$D_1 = 6 \times 58{\cdot}89/23{\cdot}61 = 14{\cdot}97, \text{ and } D_2 = 6 \times 35{\cdot}28/23{\cdot}61 = -8{\cdot}97$$

$$D_1 + D_2 = +14{\cdot}97 - 8{\cdot}97 = +6 = D$$

Students should note the similarity of these formulæ with those for calculating achromatic prisms.

**To find the Achromatising Cc.**—Since the powers of the two component lenses are proportional to their efficiencies, if $\nu_2 = 60$ and $\nu_2 = 50$, a +6 D and a −5 D will make an achromatic +1 D.

If D, the power of the convex, is known, and it is needed to calculate the concave required to make it achromatic, the formulæ are

$$\nu_1/\nu_2 = D_1/D_2, \text{ or } D_2 = D_1\nu_2/\nu_1$$

As in the foregoing example, if the crown is + 14·97, the flint concave is

$$D_2 = 14{\cdot}97 \times 35{\cdot}28/58{\cdot}88 = 8{\cdot}97$$

**Illustrating Example.**—As an example, give an equi-cx. lens of crown glass whose radius of curvature is 10 in., there is needed to calculate the radius of curvature of a flint Cc. so that the two combined make an achromatic combination.

If in the crown $\mu_D = 1{\cdot}5175$ and $\mu_F - \mu_C = {\cdot}0087$, then

$$\nu_1 = \frac{\mu_D - 1}{\mu_F - \mu_C} = \frac{{\cdot}5175}{{\cdot}0087} = 59$$

If in the flint $\mu_D = 1{\cdot}571$ and $\mu_F - \mu_C = {\cdot}01327$, then

$$\nu_2 = \frac{\mu_D - 1}{\mu_F - \mu_C} = \frac{{\cdot}571}{{\cdot}01327} = 43$$

Now $$\frac{1}{F_1} = {\cdot}5175 \times \left(\frac{1}{10} + \frac{1}{10}\right) = \frac{1}{9{\cdot}662} \text{ for the convex,}$$

and $$\frac{1}{F_2} = \frac{1}{9{\cdot}662} \times \frac{43}{59} = \frac{43}{570{\cdot}058} = \frac{1}{13{\cdot}257} \text{ for the concave,}$$

then $$\frac{1}{F} = \frac{1}{9{\cdot}662} - \frac{1}{13{\cdot}257} = \frac{3{\cdot}595}{128{\cdot}089} = \frac{1}{35{\cdot}63} \text{ for the combination,}$$

or $$F = 35{\cdot}63 \text{ ins.}$$

Now, the radii of curvature of the two adjacent surfaces must be equal, that is, 10 in. Therefore $r$, the second radius of the Cc., is found from

$$-13{\cdot}257 = \frac{-10r}{(-10 + r)\,{\cdot}571}$$

Whence $$r = -31{\cdot}15 \text{ in.}$$

**Second Illustrating Example.**—A plano-Cx. achromatic combination is required of F = 20 in. Let the glasses be

$\mu_C = 1{\cdot}535$, $\mu_D = 1{\cdot}54$, $\mu_F = 1{\cdot}555$ for the crown, and $\mu_C = 1{\cdot}59$, $\mu_D = 1{\cdot}60$, $\mu_F = 1{\cdot}63$ for the flint.

Then $$\nu_1 = {\cdot}54/{\cdot}02 = 27, \text{ and } \nu_2 = {\cdot}60/{\cdot}04 = 15$$

Now $$F_1 = 20 \times 12/27 = 8{\cdot}88 \text{ Cx., and } F_2 = 20 \times -12/15 = 16 \text{ Cc.}$$

The combination will therefore be

$$F = \frac{-16 \times 8{\cdot}88}{-16 + 8{\cdot}88} = +20 \text{ in.}$$

If the one surface of the concave is plano, $F_2 = r/(\mu - 1)$, so that $r = -16 \times {\cdot}6 = -9{\cdot}6$ in. for the curved surface of the negative lens.

Since the Cx. must have one surface of radius 9·6 in.

$$8{\cdot}88 = \frac{r \times 9{\cdot}6}{{\cdot}54 \times (r + 9{\cdot}6)} = \frac{9{\cdot}6\,r}{{\cdot}54\,r + 5{\cdot}184}$$

whence $r$ = 9·6 (approx.), so that the positive lens is a double convex, the combination consisting of a double convex lens and a plano-concave.

**Illustrating the Dioptral Formulæ.**—To work such a calculation by diopters, since 20 in. = 2 D we have $D_1 = 2 \times 27/12 = 4{\cdot}5$ for the crown, and $D_2 = 2 \times -15/12 = -2{\cdot}5$ for the flint.

For a plano $r = (\mu - 1)/D$ in terms of a metre, so that $r = {\cdot}6/-2{\cdot}5 = -{\cdot}24$ M., or $-24$ cm.

Since

$$D = \frac{(\mu - 1)(r + r')}{rr'}$$

substituting, we get for the second radius of the crown

$$4{\cdot}5 = \frac{{\cdot}54 \times ({\cdot}24 + r)}{{\cdot}24\,r} = \frac{{\cdot}1296 + {\cdot}54\,r}{{\cdot}24\,r}$$

That is $1{\cdot}08\,r = {\cdot}1296 + {\cdot}54\,r$, whence ${\cdot}54\,r = {\cdot}1296$, or $r = {\cdot}24$ M. or 24 cm.

As shown previously, the lens is a double convex of 24 cm. or 9·6 in. radius.

**Lens Combinations.**—A combination of lenses having only one achromatised component is not as a rule perfectly achromatic, so that in order that the whole combination may be achromatic, the achromatised component must be suitably overcorrected.

**Achromatism of a Single Lens.**—A single lens cannot be achromatic for a real image ; but when it is used as a magnifier the virtual image is really composed of a series of images formed by every different colour which, being contained within the same visual angle, combine on the retina to form a single impression. This image, however, appears coloured at the edges, owing to the chromatic effects of spherical aberration, which is greater for blue than for red. Then, if spherical aberration is eliminated, as in the Huyghen eyepiece, the virtual image is colourless.

**Separated Lenses.**—If the lenses are not in contact the conditions for achromatism are different. Two lenses made of the same material can be rendered achromatic, for virtual images only, by being separated by a proper distance, which is the case with Huyghen's eye-piece. Those rays which pass through the thin part of the field lens pass through a thicker portion of the eye lens, but as the violet is relatively nearer to the axis than the red, and so is less refracted, all the components of white light form the same visual angle on emergence. Thus, two Cx. lenses of equal $r$ separated by a distance equal to $(F_1 + F_2)/2$ form an achromatic combination for virtual images.

**Chromatic Difficulties.**—Although a combination of lenses may bring different coloured rays to the same focus, the images are not necessarily of the same size. Furthermore, a combination achromatic for an axial pencil of light need not be so for oblique pencils. Finally, if a lens be achromatised for light proceeding from a given plane, it may not be so for light proceeding from other planes. Conversely, if a positive and a negative lens neutralise for the D line, the two, being of different dispersions, may not neutralise for red or violet.

**Irrationality of Dispersion.**—One of the difficulties in optics is to find different glasses so that all the lines of the spectrum will nearly coincide. Thus, if we select two kinds of glass for an achromatic prism or lens, so that the C and F, or D and H, lines coincide, it will be found that other lines will not. This defect is called *irrationality of dispersion*, and the spectrum which remains in an achromatic lens or prism is called *residual* or *secondary*. As before stated, it suffices, for practical purposes, to unite two certain lines of the spectrum according to the use to which the lens is put. With modern glasses and by careful selection it is often possible to unite practically three spectrum lines with two glasses, but a formula for this purpose cannot be made.

**Apochromatic Lens.**—A combination which actually unites three lines of the spectrum is termed *apochromatic*; for such a lens at least three different sorts of glasses must be employed, the residual spectrum still left being so small as to be negligible. For such a combination $1/F = 1/F_1 + 1/F_2 + 1/F_3$, and $F_1\nu_1 + F_2\nu_2 + F_3\nu_3 = 0$.

# CHAPTER XXIV

## ABERRATIONS OF FORM

### Prismatic Aberrations of Form.

**Small Light Pencils.**—A pencil of light parallel before refraction is parallel after refraction by a prism; even if divergent or convergent, any difference may be neglected provided the pencil of light be small and the axial ray suffers minimum deviation; such a pencil may be considered as respectively coming from, or meeting at, a single point. Although, in prisms of small angle, the effects of aberration due to the form of the prism can be ignored, considerable distortion of the image is produced by strong prisms, and this is increased by nearness of the object, largeness of the object, and abnormal position of the prism.

**Large Light Pencils.**—In Fig. 257, which is purposely exaggerated for sake of clearness, let a wide pencil of light diverge from a point $L$, of which

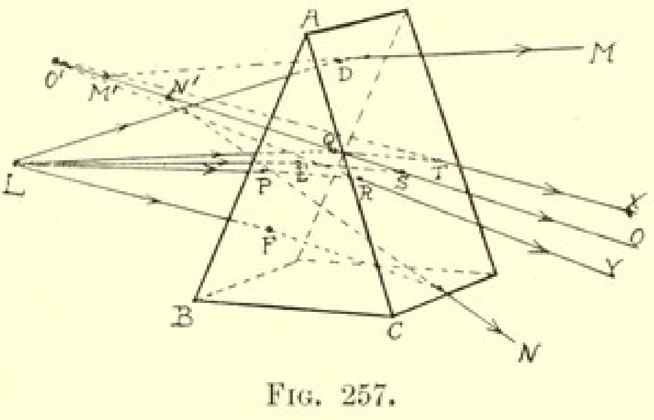



FIG. 257.

$LE$ is the central ray, presumed to suffer minimum deviation, and $LD$, $LF$, $LP$, and $LQ$ extreme rays. Since $LD$, $LE$, and $LF$ are incident on the surface of the prism at different angles in the base-apex plane, they suffer unequal deviation towards $M$, $O$ and $N$ respectively, and if $LE$ is at minimum deviation, the others cannot be also. After refraction at both surfaces, therefore, $LD$ and $LF$ are deviated relatively more than $LE$, and cut this, when produced backwards, in $M'$ and $N'$; $LF$ being more deviated than $LD$, $N'$ is nearer the prism than $M'$. Again, the incidence of rays such as $LQ$ and $LP$ in a plane parallel with the axis also differs, but to a much lesser extent than the incidence in the base-apex plane; notwithstanding, the refracted rays $TX$ and $RY$, when produced backwards, meet in $O'$, which is nearer to the prism than the original source $L$. These two inequalities of incidence are the origin of coma in a lens, while together they are the genesis of radial astigmatism, and for oblique incidence, of distortion as well.

Thus rays in the pencil emanating from a point do not have a point focus, there being two focal lines, the one nearer the prism being parallel to the axis, and the other parallel to the base-apex plane of the prism. The circle of least confusion, which lies between $O'$ and $N'$, may be regarded as the focus. These defects blur the image and cause it to appear nearer than it actually is, and if the prism is in such a position that $LF$ or $LD$ suffers minimum deviation, the whole of the pencil is rendered still more divergent and the image is still more distorted and nearer.

**Distortion due to Inclination.**—If the base-apex line is vertical, say edge upwards, the vertical magnitude of a square object, whether near or distant, appears increased when the edge of the prism is nearer to it than the base. This results because light from the bottom of the object is incident more nearly at minimum deviation than that from the top, so that the latter appears drawn upwards, the effect rapidly increasing as the inclination of the prism is increased. The vertical dimension is lessened if the base is nearer the object, because here the light from the bottom is more deviated, that from the top being more nearly at minimum deviation. Again, the effect increases with the inclination, as does also the *total* deviation of the image in both cases. This distortion is somewhat analogous to that produced by a lens, since both are due to the same causes.

**Distortion due to Thickness.**—Distortion is also caused by the greater thickness of the glass through which the oblique pencils pass from the extremities of an object. These pencils suffer more deviation than the central pencils, and therefore appear to come from points relatively higher and more distant from the centre than those nearer to the centre of the object viewed. Thus a straight line, parallel to the edge, appears curved with its convexity towards the base. A square objec: has its two sides, which are parallel to the edge of the prism, concave to the latter direction.

**Distortion due to Position of Base-Apex Line.**—If a prism be rotated around its base-apex line—i.e. if, say, a vertical prism, edge upwards, be rotated horizontally, so that one side of the prism is nearer the object than the other—the image is lengthened diagonally on the side nearer the object; it is drawn out more towards the edge than the base, so that a square object appears as a distorted parallelogram.

**Distortion due to Size of Object.**—If a narrow pencil of light from the centre of an object enters the eye through a prism, and suffers minimum deviation and but little aberration, the pencils from other points cannot do so; the peripheral portions of a large object are blurred compared with the centre.

### Lens Aberrations of Form.

Apart from chromatism, the image formed by a spherical lens suffers from five distinct aberrations due to its shape, and these must be severally corrected before the lens is capable of forming a geometrically perfect image of an object. The first is *spherical aberration*, the second *coma*, the third *radial astigmatism*, the fourth *curvature of the field*, the fifth *distortion*.

The first three errors mentioned are *point* aberrations, and lenses corrected for them are called *stigmatic* as distinct from *astigmatic*, the literal meaning of which is "without a point"; a lens corrected for spherical aberration is termed *aplanatic* (not wandering). The last two errors are aberrations of a plane, and lenses free from them are termed *rectilinear* or *orthoscopic*.

### Spherical Aberration.

Since a lens may be regarded as consisting of an infinite number of prisms whose angles of inclination increase with the distance from the axis, it follows that the deviation effected by the various zones of a lens depends

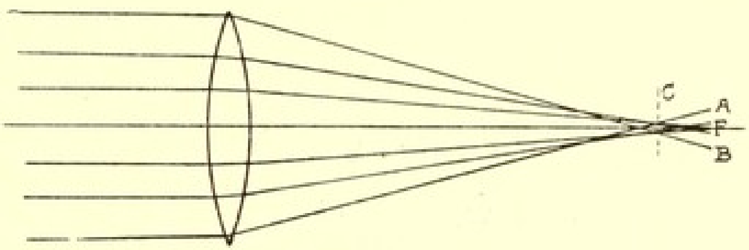



FIG. 258.

on this distance. In a Cx. lens the varying inclination of the different parts of the two surfaces of each meridian causes parallel light to converge, but, actually, the refraction of a spherical lens is such that light from a point is not brought to a focus at a single point, the rays transmitted by the marginal zones of the lens meeting sooner than those transmitted nearer the centre, as depicted in Fig. 258.

Each zone of a lens has its own focal length, varying from the principal focus $F$, for rays refracted in the zones immediately surrounding the principal axis, to a point where rays $A$ and $B$ passing through the most external zones, meet the axis. The inability to unite in a single point all the rays diverging from an object point on the principal axis is called *spherical aberration*, which is due, not to the fact that the deviating power is greater towards the periphery, for that is a natural property of a lens, but to the fact that the deviating power *increases too rapidly* towards the periphery, with the result that wave fronts are not truly spherical after refraction.

**Minimum Deviation.**—In Fig. 259 the opposite points $D$ and $E$ of the lens constitute a portion of a prism $G\ K\ H$, and the ray $A\ D$, incident such that its point of incidence $D$ and its point of emergence $E$ are equi-distant from the edge, therefore suffers minimum deviation. The deviation of the ray *A B C* is not minimum, and is relatively more bent from its course than the ray *A D*. It is mainly owing to the departure from minimum deviation incidence of the light at the periphery that the deviating power there is unduly increased and spherical aberration produced.

**Central and Peripheral Refraction.**—If a piece of black paper, the same size as a lens, be divided by cutting out a disc one half of the diameter, there will be a ring and a disc of equal widths. By gumming first the ring and

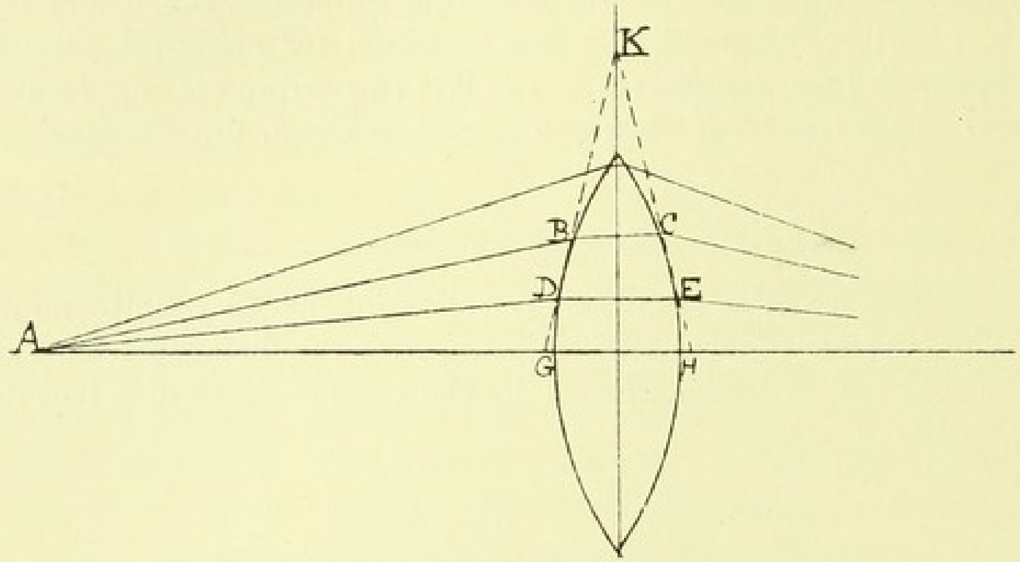



FIG. 259.

then the disc on to the lens we can observe the result of central and peripheral refraction separately. When the peripheral part of the lens (Fig. 260) is blocked out only the central area of the lens is effective, and parallel rays, as a whole, meet slightly within *F*. When the central portion of the lens is covered (Fig. 261) and only the periphery acts on the light, the latter, as a whole, meets still further within *F*.

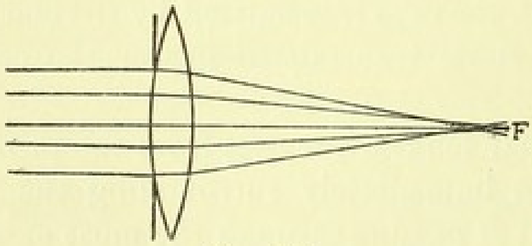



FIG. 260.

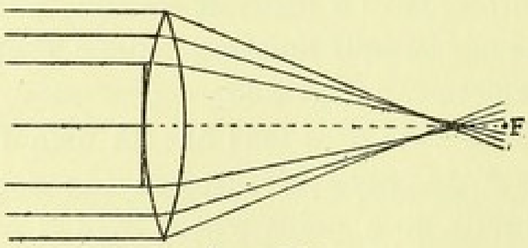



FIG. 261.

**Circle of Least Confusion.**—When the whole lens is exposed to the light (Fig. 258), the converging circles of confusion from the central, and the diverging circles from the peripheral area, are of about the same mean diameter at *C*, where the illumination is greatest and the disc of light of minimum size. At any point either nearer or further the disc is larger than at *C*, but the greatest concentration of light occurs at *F*, where the image of a luminous point is a bright spot surrounded by a halo caused by the diverging light from the periphery of the lens.

The distance of the image from a Cx. lens in the three cases where the periphery only, the centre only, or the whole of the lens is effective, can be shown by experiment, the object being a bright flame placed behind a small aperture covered by a piece of yellow glass in order to make the light more or less monochromatic.

**Longitudinal and Lateral Aberration.**—The distance between the extreme foci is called the *longitudinal* aberration; the diameter of the disc *A B* (Fig. 258), caused by the overlapping of the rays refracted by the margin of the lens when the screen is held in its theoretical focus, is called the *lateral* aberration. The lateral aberration increases more rapidly than the longitudinal with an increase in the aperture of a lens, the latter varying as the square of the aperture, and the former as the cube of the aperture.

**Influencing Factors.**—The definition of an image depends on the smallness of the circles of confusion of which it is constituted, and these circles are dependent on the degree of spherical aberration. The latter is proportional to the incidence of the light, the aperture, form, index, and thickness of the lens; as these factors are changed spherical aberration is increased or decreased.

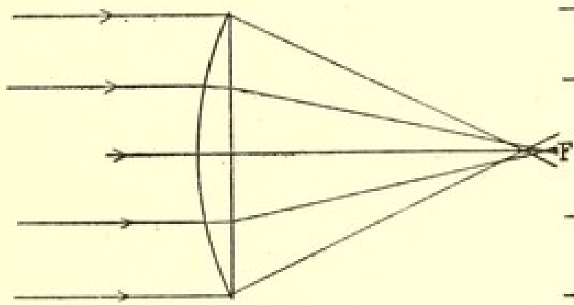



FIG. 262.

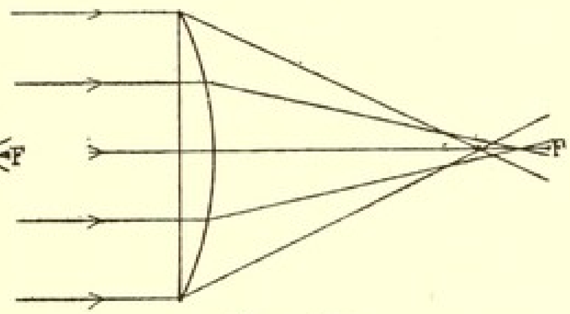



FIG. 263.

**Influence of Form.**—The degree of spherical aberration is least when the rays in general are, after refraction at the first surface, more nearly parallel to the bases of the virtual prisms of which the lens is formed, so that the total refraction is approximately divided between the two surfaces, and therefore the angles of incidence and emergence are equal.

**Best Form of Single Lens.**—As a general rule, for parallel light, the more curved the front and the less curved the back surface of the lens, the smaller is the spherical aberration (Fig. 262); as the object is nearer the lens and the light becomes more and more divergent less curvature is needed for the front, and more for the back surface; in these cases an approach to minimum deviation at the periphery of the lens is obtained. A very high degree of spherical aberration results if the less curved surface is exposed to parallel light (Fig. 263), or the more curved surface to light diverging from the focus of the lens, since a considerable departure from minimum deviation for peripheral rays then occurs. Since the incidence varies with distance of the object, spherical aberration depends not only on the form of a given lens but also on the distance of the luminous point from that lens.

The curvatures needed for a lens having minimum aberration varies with the index of refraction of the glass, the difference between the two radii of curvature increasing directly with $\mu$. A plano-convex, or better, the *crossed* lens, with its more curved surface turned to the light, is the form of single lens which gives the best definition for objects at extreme distances. The same lens turned the other way is the best for very near objects, while the double convex is the best when the incident rays diverge from twice the focal distance, for then, object and image being equi-distant from the lens, the incident and emergent rays form equal angles with the two surfaces. If used for all distances the double Cx. is perhaps the best form of single lens.

The term *crossed* is applied to a lens having unequal radii of curvature, it being usually a bi-Cx. or bi-Cc. whose radii are 6 : 1 approximately. To obtain minimum spherical aberration the radii of the two surfaces of the lens should be in the ratio of $1 + 2\ \mu$, and $1 - 2\ \mu + 4/\mu$. These quantities, when $\mu = 1{\cdot}5$, are as 6 is to 1. When $\mu = 1{\cdot}686$, the value of $1 - 2\ \mu + 4/\mu$ is 0, so that the one surface should be plano, and if the index is higher the lens must be a meniscus, this quantity being then negative.

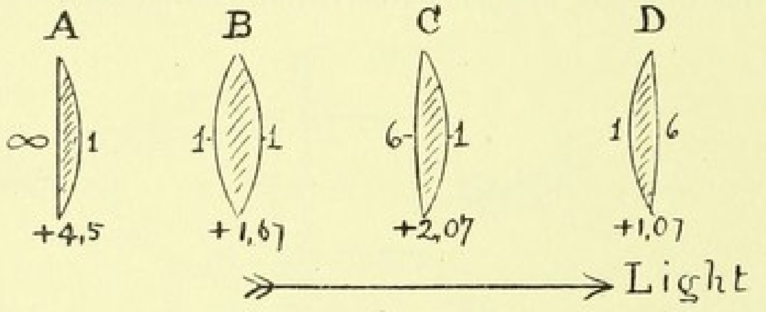



Fig. 264.

**A Numerical Expression for Longitudinal Aberration** is sometimes given, as below, for parallel light and thin lenses, where $\mu = 1{\cdot}5$. The values are in terms of $d^2$ F, where $d$ is the semi-diameter of the lens.

| | |
|---|---|
| A crossed Cx. (Fig. 264) with the more curved surface to the light | 1·07 |
| A plano-Cx., with the curved surface to the light ... ... | 1·17 |
| An equi-Cx. ... ... ... ... ... ... ... ... | 1·67 |
| A crossed Cx. with the less curved surface to the light ... ... | 2·07 |
| A plano-Cx. with the plane surface to the light ... ... ... | 4·5 |

These values would vary with the index of refraction, and for different distances of the source of light; also with the thickness of the lens if this cannot be neglected. The aberration of course increases with the diameter or power of the lens.

**Least Time.**—Since light travels in a straight line it takes the least possible time to reach a given point, and this principal of *least time* holds good for refraction. Thus, various rays diverging from a point in air and passing into another denser medium must arrive at the same point, at the same time, if a focus is to be obtained. With a lens, disregarding spherical aberration, this occurs because although the distance from $A$ to $B$, and thence to $F$, is greater than from $A$ to $C$ and $F$ (Fig. 265), yet the distance traversed in the denser medium is greater in the case of $A\,C\,F$. The law of refraction $\mu_1 \sin i = \mu_2 \sin r$ is in accordance with the principal of least time. If a lens is corrected for spherical aberration all rays diverging from an object point must reach the same image point and in the same time, no matter what course they take. In other words the optical length (which is the actual distance of travel multiplied by the $\mu$ of the medium in which this takes place) must be the same for all rays between the object and image points.

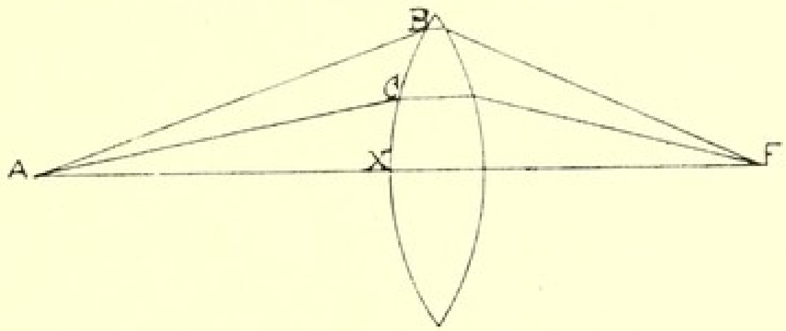



FIG. 265.

Let the distance of $A$ from any point on the refracting surface (Fig. 266) be $d_1$, and the corresponding distance of $F$ be $d_2$; then $d_1\,\mu_1 + d_2\,\mu_2$ is the optical length of any ray diverging from $A$ and refracted to $F$, so that for $A\,X$, $A\,B$ and $A\,C$ to meet at $F$ it would be necessary that $d_1\,\mu_1 + d_2\,\mu_2$ be a constant for any incidence of the light, i.e. $A\,B\,\mu_1 + B\,F\,\mu_2 = A\,C\,\mu_1 + C\,F\,\mu_2 = A\,X\,\mu_1 + X\,F\,\mu_2$. As this cannot occur with spherical surfaces, spherical aberration may be said to be due to the fact that all the rays diverging from a point on the axis cannot reach the same point in a given time, or rather that, within a given time, the rays reach different points of the axis.

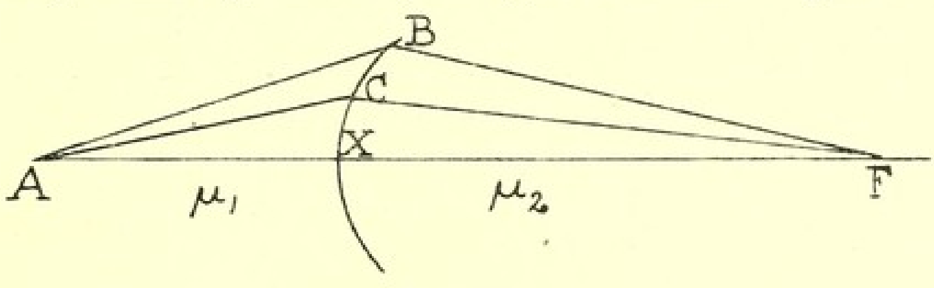



FIG. 266.

In the case of a lens the influence of the two surfaces has to be considered, since each ray travels in three different media. If $d_1$ be its course in the first medium $\mu_1$, $d_2$ its course in the second medium $\mu_2$, and $d_3$ its course in the third medium $\mu_3$, then $d_1\,\mu_1 + d_2\,\mu_2 + d_3\,\mu_3$ would need to be equal for each ray in order that all rays diverging from an object-point may meet, after refraction, at a single image-point.

**Influence of Thickness.**—A ray $A\,B$ traversing a thick lens (Fig. 267) is retarded in the denser medium, and can only reach, in a given time, a point $G$ on the principal axis which lies nearer to the lens than $H$, the point reached by a similar ray passing through a thin lens. Thus, spherical aberration increases with the thickness of the lens.

**Remedies.**—A theoretical remedy would be found if the speed of the light could be increased, or the refractive power of the lens decreased at the periphery. This would necessitate the lens being made of a medium whose index of refraction decreases as the distance from the principal axis increases, which occurs in the crystalline lens of the eye ; or the lens would need to have less curvature at the periphery than at the centre, i.e. one having some conic curve. If the lens be ground down to a smaller size so that only the central area is left, or what amounts to the same thing, if a stop or diaphragm is used in combination with the lens, the marginal rays are cut off and spherical aberration is consequently lessened. In general, also, the aberration is reduced by increasing the number of refracting surfaces. Thus there may be employed two positive lenses, in place of one, by which the curvatures are diminished for the same refractive power, or the positive and negative components of a system separated by an interval, by which

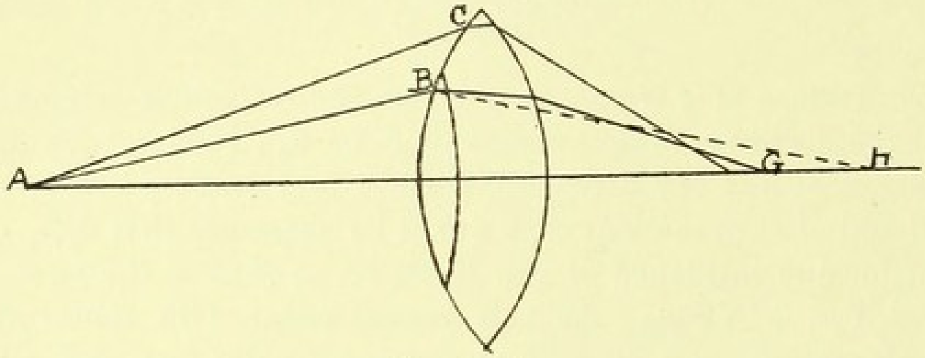



Fig 267.

the positive power of the combination is increased. This last method is sometimes made use of in photographic and microscopic lenses.

Since the defect depends on curvature, a Cx. having a high $\mu$ and low curvature may be combined with a Cc. so that, in an achromatic combination of given focal length, the two surfaces in contact being similar, the aberration of the front surface of the convex is neutralised by that of the back surface of the concave. This method of neutralising the aberration by means of a compensating Cc. is the only true practical means of correcting a lens.

**Aplanatic Lens.**—A lens, or lens combination, corrected for spherical aberration is termed aplanatic, but no combination of lenses can be rendered entirely aplanatic for all distances of the object, nor can it be for other than monochromatic light ; but by employing a stop, as is done in most optical instruments, and a judicious choice of form, it may be rendered so for practical purposes. A single surface may be aplanatic and also a single spherical lens, but only for one distance of object (vide Chap. XXV.).

**Positive and Negative Aberration.**—*Positive* aberration obtains when the marginal rays come to a focus before the central, *negative* aberration if the central rays come to a focus before the marginal.

**Under and Over Correction.**—A lens combination which partially neutralises the positive aberration is *under-corrected*, and if it more than neutralises the positive, it produces *negative* aberration, and is said to be *over-corrected*. In photographic lenses it may happen that spherical aberration is completely eliminated for the axis and periphery, while it may still occur in the intermediate zones.

### The Oblique Aberrations.

A beam of light diverging from a point on the principal axis would, on passing through a lens corrected for central chromatic and spherical aberration, meet again as a point on the principal axis. When, however, the luminous point is situated on a secondary axis, further aberrations are introduced by the oblique incidence of the light, these being the point aberrations *coma* and *radial astigmatism*, and the plane and line aberrations *curvature of field* and *distortion*.

If a bright point of light be placed obliquely below the axis of a lens and a white screen moved behind it, we shall find that the image is blurred at all distances, the image assuming various triangular, comet-shaped, cup-shaped, and pear-shaped figures, which are the result of coma. If coma be reduced by placing a fairly small diaphragm in front of the lens and the screen is held within the focus, and slowly drawn away, the image is seen to form a symmetrical ellipse, and then successively a horizontal line, a horizontal ellipse, an irregular circle, a vertical ellipse, a vertical line, and finally broadens out into a blurred patch. These lines result from radial astigmatism.

### Coma.

**Coma** is an aberration produced by the unequal refracting effect of the different parts of the various meridians of a lens, on an oblique pencil of light; it is spherical aberration for oblique light. Instead of a point image of a point object, situated on a secondary axis, there results a blurred halo of confusion partly surrounding a bright point, and extending therefrom in a direction away from the axis.

Let $d$ and $e$ (Fig. 268) be rays proceeding from a distant point on an oblique axis $A\ B$. The ray $e$ meets the surface of the lens sooner than $d$, and since $e$ departs more from minimum deviation than does $d$, the ray $e\ e'$ cuts the axial ray at $e''$ sooner than does $d\ d'$ at $d''$.

The confusion disc produced by coma presents various forms, as before stated, but it is usually more or less pear- or comet-shaped, the narrow brilliant part being directed towards the principal axis. It is, therefore, non-symmetrical, and in this respect differs from the confusion discs of spherical and chromatic aberration, which are always symmetrical with respect to the axis of the beam of light.

**Influencing Factors.**—Coma is directly proportional to the obliquity of the incident light to the principal axis. It is enhanced in a lens of large aperture, and, in general, whatever tends to increase spherical aberration tends also to increase coma.

**Remedies.**—Coma is reduced in a lens of such form as will cause the incidence of the rays, passing through any meridian of the lens, to be more equal. Thus it is less marked in plano and meniscus lenses than in doubles, for the reason that, in such lenses, less refraction takes place at the second surface. The chief remedy for coma is reduction of the effective aperture of the lens by the employment of a stop, the latter being placed a short distance from the concave surface of the meniscus.

**The Sine Condition.**—In order that coma be eliminated from a lens, the sines of the angles $a$ and $a'$ formed by an incident ray with the axis, before

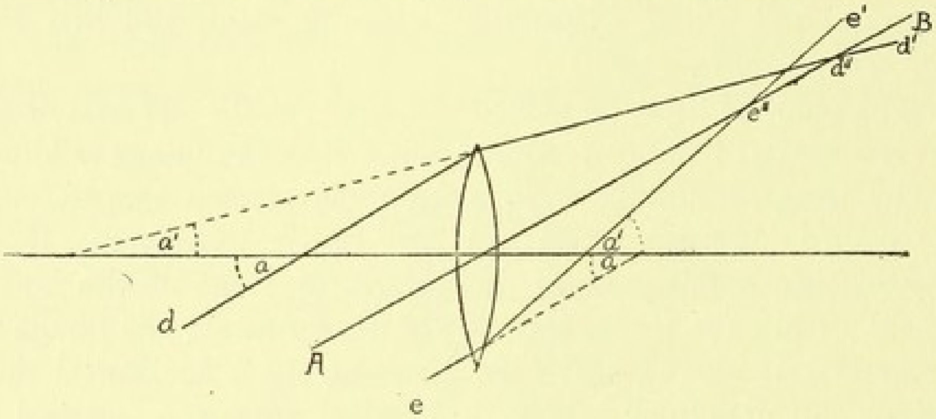


Fig. 268.

and after refraction, should have a constant ratio; that is, $\sin a/\sin a' =$ a constant (Fig. 268).

### Radial Astigmatism.

**Radial Astigmatism** is an aberration which results from the unequal refraction of different meridians of a lens on an oblique pencil of light; instead of a point image of a point object situated on a secondary axis, there is produced two line foci through which pass all the rays contained in the pencil.

For every oblique axis there are two *principal meridians* or planes; the first is that containing the oblique axis and the principal axis, and is termed the *sagittal* plane; the second is that at right angles to the sagittal, and is called the *meridional* plane. The astigmatism is essentially the *distance* between the focal lines produced by the difference in the effective power of the lens in the meridional and sagittal planes of incidence.

When a luminous point is oblique to the principal axis, the effective aperture of the lens is an ellipse in which the sagittal plane of incidence corresponds to the short diameter, and the meridional plane to the long diameter. In the meridional plane the light has to traverse a greater thickness of the lens, and is more oblique than an axial pencil; it is, therefore, rendered more convergent, and has its focus nearer the lens than the focal plane, thus forming the second focal line *R R'* (Fig. 269). In the sagittal plane the light has also a greater thickness to traverse than an axial pencil would have, and it is still more oblique than in the meridional plane. It, therefore, has its focus still nearer the lens, and forms the first focal line *T T'*. Consequently *radial astigmatism is due to the increased angles of incidence of oblique light and increased effective thickness of the lens.*

In Fig. 269 let a pencil of light be incident on a lens from a distant point *A*, situated on the secondary axis *A X*, and let it be presumed to be the central lower point of a body facing the lens; then *S S'* is the sagittal and *M M'* the meridional plane, in this case *S S'* being vertical and *M M'* horizontal. All rays in a sagittal plane as *a* and *c*, *b* and *d*, meet in points along the line *T T'*, which is the first or *tangential* focal line whence, diverging in one direction and converging in the other, they continue to the second or *radial* focal line *R R'*; *R* is the meeting point of *c* and *d*, while *R'* is that of

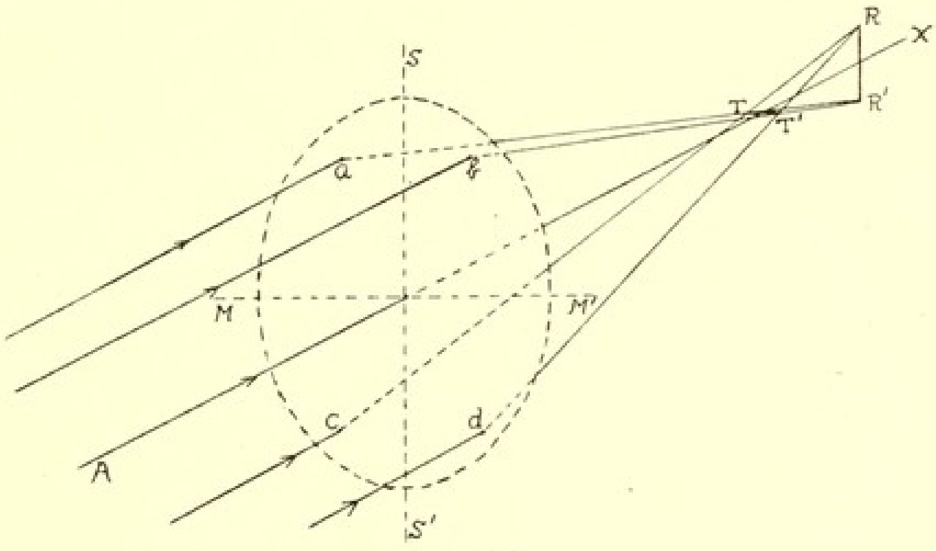



FIG. 269.

*a* and *b*. Thus the *tangential* line is the focus of the *sagittal* plane, while the *radial* line is that of the *meridional* plane, each focal line being at right angles to the plane of which it is the focus.

The radial line is nearer to the lens than the focal plane on the principal axis, and the tangential is still nearer, the distance between them being the astigmatism. Between the two focal lines there is a position where the cross-section of the refracted light is most nearly circular, and this may be regarded as the mean focus of the oblique pencil of light. The calculation for the distances of *T T'* and *R R'* are shown in Chap. XIII., where they are termed $F_1$ and $F_2$ respectively.

To illustrate oblique refraction, let Fig. 270 represent the focal plane of a Cx. lens viewed from behind. Rays parallel to the principal axis and directed before refraction to the points *a b c* and *d*, are refracted towards and meet in the point *F*. If the rays are parallel to an oblique axis, as represented in Fig. 269, *a* meets *c* in *T* and *b* in *R'*, while *d* meets *c* in *T'* and

$b$ in $R$, but $T\,T'$, as already stated, lies nearer the lens than $R\,R'$, and both are nearer than $F$.

Radial astigmatism has been illustrated with the light diverging from a point on the lower edge of an object, so that the resulting tangential focal line is horizontal and the radial line is vertical. If the luminous point is to the right or left of the object, the tangential line is vertical and the radial line horizontal; if the sagittal plane is oblique both lines are oblique, there being a pair of astigmatic lines at right angles for each secondary axis.

The tangential and radial lines of the numberless secondary axes consti-

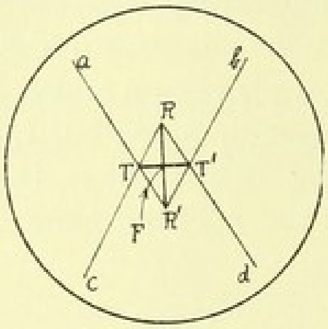


Fig. 270.

tute curved surfaces (Fig. 271), both within the principal focal plane; these curved surfaces meet at the principal axis in the focal plane, where the two focal lines fuse into a point image. The circles of least confusion form a surface $O\,O'$ concave towards the lens lying between $R\,R'$ and $T\,T'$ and this may be regarded as the focal plane of an ordinary lens.

**Influencing Factors.**—Radial astigmatism is in direct proportion to the obliquity of the incident light, and is greater as the lens aperture is larger.

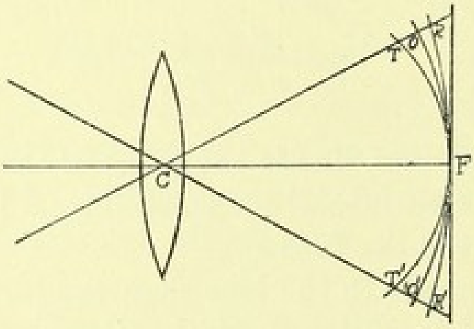


Fig. 271.

It is also greater with certain forms of lenses than others, and, in general, the more nearly a lens is of double Cx. form the more marked it is.

**Remedies.**—Anything that tends to equalise the effective thickness of the lens and the angles of incidence in all meridians will reduce radial astigmatism. Thus a meniscus lens combined with a stop to cut off the extreme peripheral rays is the primary remedy, especially if the stop be placed some little distance—about a fifth the focal length—on the concave side. This has the effect of shortening both focal lines and throwing them back so that the circle of least confusion lies more nearly in the focal plane; by still further displacing the stop away from the lens both lines may even be thrown behind the focal plane. In addition, by combining glasses of high refractive power and low dispersion with those of opposite quality certain conditions are fulfilled which, besides eliminating chromatism, correct astigmatism over a wide area. With the newer varieties of optical glass a degree of correction is secured which was not possible with the older kinds, wherein refractivity and dispersion were more or less proportional.

**Sphero-Cyl. Lens.**—There is a difference between the astigmatism of a sphero-cylindrical and the radial astigmatism of a spherical lens, inasmuch as the former occurs when the object point lies on the principal axis and is due to the varying curvature of the lens; while in the latter the curvature is equal in all meridians and it is due to the oblique incidence of the light, the direction of the focal lines varying with the position of the object point.

## Curvature of the Field.

**Curvature of the Field.**—If $A$ be a point on the lower extremity of an object the light diverging from it, after refraction by the lens, forms two focal lines and between them is situated the circle of least confusion, which may be regarded as the focus of the rays diverging from $A$. On the surface containing the circles of least confusion the sharpest representation of the periphery of the object is formed, and since the effective power of a lens is greater as the light is more oblique, this surface forms a portion of a sphere with its concave surface towards the lens. The image of a convex object would be still more curved than that of a flat object, but a concave object might be so placed as to neutralise curvature of field. (See Figs. 269 and 271.)

While curvature of field is partly due to the same cause which produces radial astigmatism, i.e., the increased power of a lens for oblique light, it is not entirely so, for if $T\ T'$ were made to coincide with $R\ R'$ there would still be curvature. Even if the peripheral foci were at the same distance from the optical centre (or second equivalent point) as the focus on the principal axis they would form radii of a circle and curvature would still remain. Thus, a sphere has equal refracting effect on rays from any point and is therefore entirely free from astigmatism, but the field is nevertheless curved. Therefore, if the image formed by a lens is projected on to a flat screen, either the centre or the periphery may be focussed sharply, but it is impossible to obtain both defined at the same time.

**Condition for a Flat Field.**—In order that an achromatic combination of two lenses may form a flat image, the condition (known as the Petzval condition) which must be satisfied is that $F_1\mu_1 = -F_2\mu_2$, or $F_1\mu_1 + F_2\mu_2 = 0$, where $\mu_1$ and $F_1$ refer to the crown, and $\mu_2$ and $F_2$ to the flint components respectively. In order that this shall not controvert the condition for achromatism, the crown, with less dispersion, must have a higher refractive index than the flint, a condition which has already been referred to in the section on radial astigmatism, and in this case $\nu_1/\nu_2 = \mu_1/\mu_2$.

**Remedies.**—The field can be flattened by placing in front of the lens a stop which, by narrowing the beam, determines at what particular point on a secondary axis the focus, as represented by the disc of least confusion, shall be formed. A distance, dependent on the form of the lens, can be found at which curvature is a minimum, this, for single lenses, generally being about one-fifth the focal length.

If a Cx. and a Cc. of equal power be separated to have convex effect, the distance may be so adjusted as to make the image flat. The oblique rays, after refraction by the convex, meet the concave nearer to the periphery, and the diverging effect is thereby increased; therefore the final convergence is to a point further away, for oblique pencils, than would be the case after refraction by a single Cx. lens, whose power is equal to the effective power of the combination.

Curvature is said to be under-corrected, or *positive*, when the image is concave towards the lens, and *negative* if, by over-correction, the image becomes convex. The image is flat if the focal length of each oblique pencil is equal to F/cos $e$, $e$ being the angle which the oblique axis makes with the principal axis.

An almost perfectly flat and undistorted virtual image is obtained with two equal plano-convex lenses placed with their convex surfaces facing each other, or by two plano-convex lenses whose respective focal distances are as 1 and 3, both curved surfaces facing the same way. Such combinations represent, respectively, the Ramsden and Huyghenian eye-pieces.

## Distortion.

**Distortion** is an aberration in the magnification of the image. There are several forms of distortion known to photographers, but the only kind, dependent on the lens itself, is *curvilinear distortion*, which results because certain natural defects of a lens due to its prismatic formation cause the peripheral image points to be relatively further from or nearer to the axis than their corresponding object points. Distortion of the image is a natural consequence of refraction; even a near object seen through a thick plate appears distorted, or one viewed through a prism, a square appearing to have its two sides, parallel to the edge and base, curved with its concavity towards the edge of the prism. Similarly a square object (Fig. 272) seen through a Cx. lens has its *virtual* image concaved outwards, termed *pincushion* or *positive* distortion (Fig. 273). If viewed through a Cc. lens it appears convex outwards—*barrel* or *negative* distortion (Fig. 274). The image is drawn out in the first, and compressed in the second case towards the periphery. The *real* image of a square formed by a Cx. lens of full aperture is *barrel-shaped.*

**Causes of Distortion.**—Distortion is chiefly the result of spherical aberration, which causes too great a deviation for peripheral pencils of light, but the effect may be varied considerably by alteration of the position of the stop.

Another, but lesser factor, also contributes to distortion, and must be considered. In the formation of a real image each of the incident rays leaves the second equivalent plane at the same distance from the principal axis that it meets the first equivalent plane, so that the refracted rays meet, on the secondary axis, at a point nearer the principal axis than they would if the secondary axis were undeviated as in a lens infinitely thin. As the distance of the object point from the principal axis increases so the disproportion between the distance of the object and image points from the principal axis also increases.

Distortion is therefore an inherent fault of the lens, and is the direct result of spherical aberration and of the increased thickness of glass when the light refracted by it is incident obliquely. The degree of distortion varies directly with the thickness of the lens and the obliquity of the light.

Further, any arrangement of the stop, or separation of the components of a lens system, which causes the light forming the image to be refracted

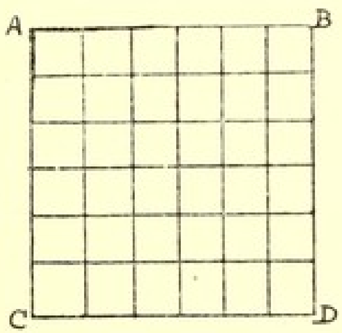


FIG. 272.

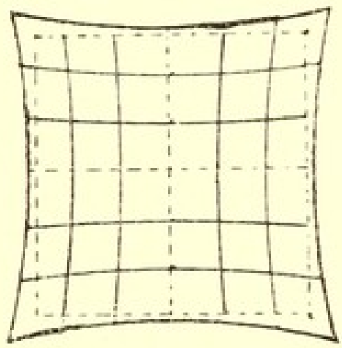


FIG. 273.

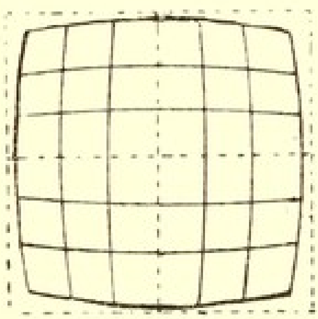


FIG. 274.

by a portion only of the lens, or of one of the component lenses, will produce distortion.

**Influence of a Stop.**—A diaphragm is used with a single lens, or combination, in order to diminish spherical aberration, coma, astigmatism and curvature of field. This accentuates and brings into prominence distortion, so that rectilinear lines of the object near the margin appear curved in the image.

When a stop is in front of a Cx. lens the effective area of the lens for an oblique pencil lies mainly on the opposite side of the principal axis to that of the object point, so that the mean focus lies between $R$ and $R_1$ (Fig. 275) nearer to the axis than if the whole lens were effective. Thus the natural negative distortion of a Cx. lens is enhanced.

When a stop is behind the lens (Fig. 276) the effective area of the latter for an oblique pencil is chiefly on the same side of the lens as the object point, so that the mean focus lies between $R$ and $R_2$ more distant from the principal axis than if there were no stop. The consequence is that the natural distortion of the lens is not only corrected, but positive distortion is produced. The distortion is due to the lens and not to the stop, for if a combination be corrected for distortion the stop may be in front of the lenses, between the lenses or behind them, and no distortion ensues.

**Remedies.**—Distortion is eliminated by employing a combination of lenses with the stop placed between the two components. Then those oblique rays which pass through the one side of the front element must pass through

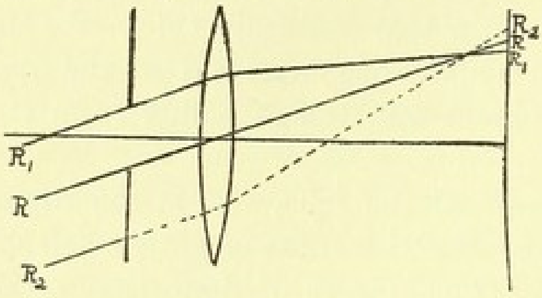



Fig. 275.

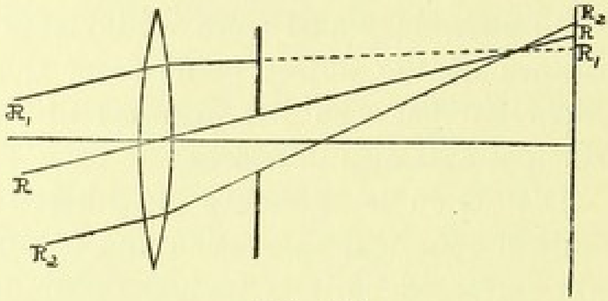



Fig. 276.

the other side of the back element and *vice versa*, so that the distorting effect of the front lens is neutralised by that of the back lens.

Separation of the component parts of a lens system can be utilised for the correction of distortion, and in single lenses it may be reduced somewhat by altering the thickness of the lens and the curves.

**The Tangent Condition.**—A *chief ray X C* or *Y C* (Fig. 277) is one which passes through *C*, the *centre of the stop*. If it be produced forwards and,

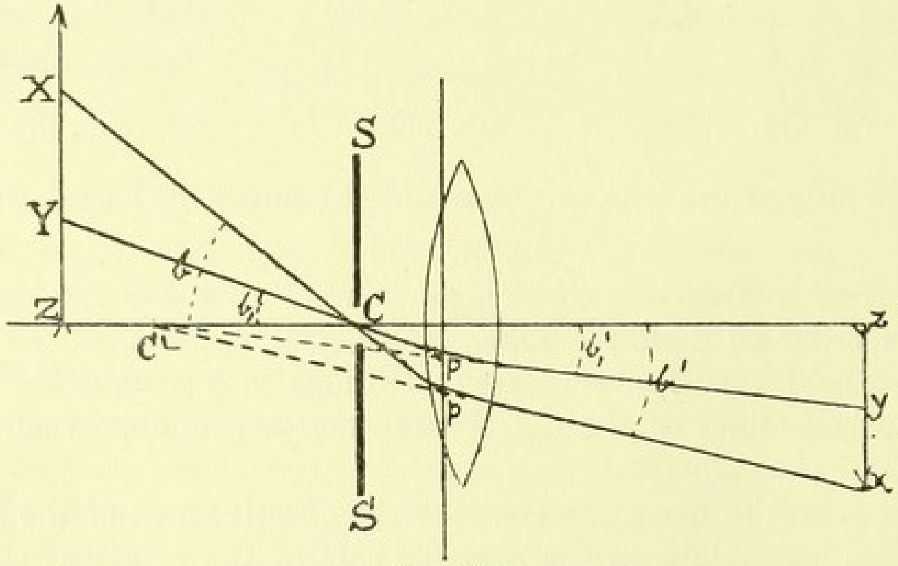



Fig. 277.

after refraction, be produced backwards the point of intersection *p* is a *chief point.*

When the chief points thus formed all lie in a plane perpendicular to the axis, i.e., the refracting plane, the chief rays when produced back to the axis will meet in a single point $C'$. The lens is then said to be spherically corrected with regard to the stop.

Each chief ray makes with the principal axis, before refraction, some angle $b$, and, after refraction, some angle $b'$, and when the foregoing conditions obtain, $\tan b'/\tan b =$ a constant for every chief ray; the image will then be uniformly magnified throughout, i.e. the image will be free from distortion when the *tangent condition* is fulfilled, as in Fig. 277.

### Aberrations in General.

In the brief description of the aberrations contained in the foregoing articles certain points are worthy of special note. All the aberrations of a single lens are reduced by the use of a stop with the exception of distortion, which is generally increased, or anyhow made more apparent. The construction of a good lens also largely depends upon the use of meniscus components without which a wide stigmatic field would be impossible, while, needless to say, crown and flint glasses are essential for achromatism. The nature of the corrections depends largely upon the use to which the lens is to be put, but on the whole, the designer of a photographic objective has a harder task than the maker of telescope and microscope objectives. Of all, perhaps, the photographic objective must be the most generally perfect, since it is required to produce a flat, stigmatic and undistorted image over a wide field whose diameter is not infrequently equal to the focal length of the lens. To secure this a kind of compromise must be effected between central and peripheral definition, since the type of lens—the crossed and plano—giving the best central correction for spherical aberration and chromatism, is useless for eliminating the oblique aberrations.

If a first-class photographic lens designed for wide angled work be examined, it will be found to contain at least one deeply periscopic component, and in all rectilinear objectives both are of meniscus shape. For extreme wide angle work the periscopic type must be still further deepened until we find, in the Hypergon of Busch, a lens consisting of two thin hemispheres with a stop at their common centre. Generally, therefore, the smaller the angular field the flatter are the curves required to produce it.

In the telescope, prism binocular and opera glass only a narrow angular field—not exceeding a few degrees—is required, and therefore the oblique aberrations may be comparatively ignored, and all the attention centred on the correction of spherical aberration and chromatism, which may be done to an exceedingly high degree of perfection. Thus any good telescope or opera-glass objective will be found to be, as a whole, either plano Cx. or bi-Cx. with the greater curvature towards the light, which is practically parallel in all cases.

Rather more care must be bestowed on the microscope objective since here some correction must be given to flatness of field and coma, so that it may be said to occupy an intermediate position between the telescope and photographic objectives, and here, the object being near F, the objective is a plano Cx. lens, or, at any rate, the bottom component is plano-convex, having its plano surface directed outwards.

Again, in visual optics, the deep periscopic and toric is now recognised as being far superior to the double in that the field of sharp definition is greatly extended by the elimination of most of the oblique aberrations.

A plano-Cx. condenser is turned the one way or the other according as the source is near or distant and according as the beam of light projected is large or not.

Spherical aberration is a defect of the image on the principal axis, and, therefore, for best definition it is necessary to distinguish between point objects and objects of definite size. Thus the eye lens of an ocular is always plano Cx. with the curved surface towards the object to be viewed, which is the real image formed by the objective, and notwithstanding that this object lies in the focal plane of the eye lens. This is because the object viewed is of definite size and not merely a point on the axis.

**Aberrations of a Cc. Lens.**—Although in the foregoing articles Cx. lenses have been used in diagrams and examples, it must not be forgotten that

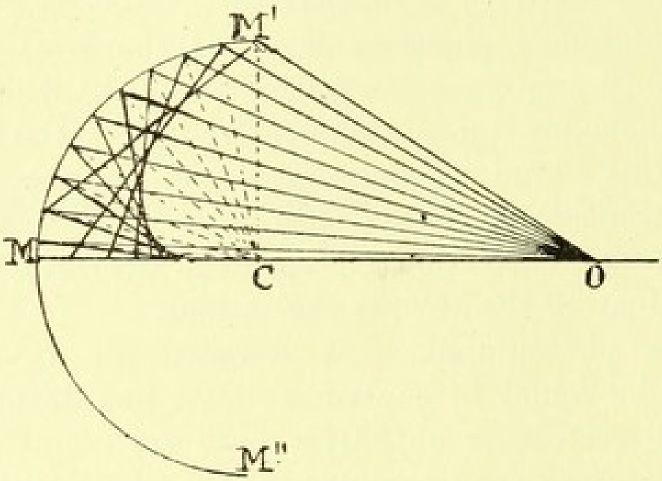



Fig. 278.

Cc. lenses suffer from precisely similar aberrations. They are, of course, opposite to what would be produced in the virtual image of a Cx., *e.g.* the distortion of the virtual image with a Cc. is barrel, whereas it is pincushion with a Cx., so that when two lenses are neutralised in the ordinary way their aberrations are also practically neutralised, unless the lenses are very thick or of deep periscopic form.

**Aberrations of a Mirror.**—If the angular aperture of a spherical mirror be large, rays which diverge from a point *O* on the principal axis (Fig. 278) do not meet in a single conjugate image point after reflection. This is due to spherical aberration, so that the image consists of a series of imperfectly formed foci in the shape of a curve, called a *caustic*, illustrated in Fig. 278. Those rays, however, immediately round the principal axis, having a divergence of only a few degrees mutually unite in a single point, which is taken as the geometrical image since at this spot, i.e. the focus, the condensation of light is greatest. Those rays making larger and larger angles with the axis are reflected to cut the latter in points nearer and nearer to the vertex of the mirror, and their intersection one with another gives rise to the increase of illumination forming the caustic curve, to which every ray is tangential. Caustics by reflection can readily be seen when light from a lamp or the sun falls obliquely on to a cup half filled with milk or tea.

Caustics may, of course, be *virtual* as well as real, which occurs when the object point is either within F of a Cc., or in front of a Cx. mirror. Their effect, however, is never noticed because the pupil of the eye acts as a small stop and limits the divergence of the rays to a minute angle. Thus practically only one point on a virtual caustic may be viewed from one position, and the image thus seen apparently moves when the position of the eye is altered because a new pencil, apparently diverging from another point on the caustic, now enters the eye.

Mirrors suffer also from coma, radial astigmatism, curvature of the field, and distortion, but, as before stated, not from chromatic aberration. The two astigmatic focal lines of a small oblique pencil of parallel light at an angle of incidence $i$, are distant F cos $i$ and F/cos $i$ from the mirror.

## CHAPTER XXV

## CONICS AND APLANATIC REFLECTION AND REFRACTION

### Conic Sections.

CONIC sections, or conics, deal with the figures—and their peculiar properties—obtained by sections, in any direction, through a right circular cone. In this chapter we shall discuss only those geometrical properties of conics as are necessary for the consideration of the optical properties of media having curves which, in section, belong to this class, as distinct from the usual spherical surface.

The cone is generated by the revolution of a line $B B'$ (Fig. 279) around a fixed point $A$, where it intersects a fixed straight line $a a'$. As $B B'$ is revolved around $A$ the figures traced out constitute the surfaces of two similar cones $B A B''$ and $B' A B'''$, of which $A$ is the common apex or vertex, and

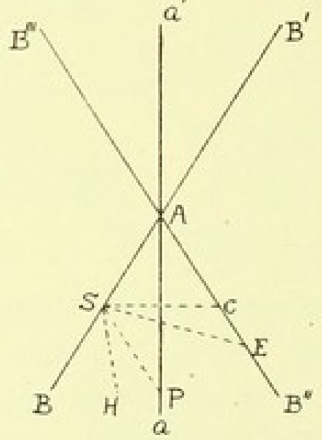



FIG. 279.

$a a'$ is the common axis. Any section $S C$ at right angles to the axis of the cone is *circular*; if the section is through $A$ it is a *point*; a section through $A$ along the surface is a *line*. Any section, as $S E$, oblique to $a a'$, is *elliptical*; any section, as $S P$, parallel to a generating line, is *parabolic*; any section, as $SH$, which, if produced, would cut $B'' B'''$, is *hyperbolic*. These last three are true conics.

Let $D A E$ (Fig. 280) be a conic section of which $A$ is the *vertex*, and $L L'$, a line cutting the vertex, and symmetrical to the two sides, is the *axis*. $F$ is a fixed point called the *focus*, and $R T X$ is a fixed line perpendicular to the axis, called the *directrix*. In all conic curves there is constant ratio, called the *eccentricity*, for every point $P$ on the curve, between $d_1$ the distance of the point from $F$, and $d_2$, its perpendicular distance from $R\,T\,X$. Thus $d_1/d_2$ represents the eccentricity. If $d_1$ is smaller than $d_2$, so that $d_1/d_2 < 1$, the curve is that of an *ellipse* ; if $d_1$ is greater than $d_2$, so that $d_1/d_2 > 1$, the curve is that of a *hyperbola* ; if $d_1 = d_2$, so that $d_1/d_2 = 1$, the curve is that of a *parabola*.

### The Ellipse.

On the line $L\,L'$ (Fig. 281) erect the directrix $R\,T\,X$; take any point $F$ and divide $F\,T$ at $A$ so that $AF/AT$ equals the eccentricity. $P$ is any point

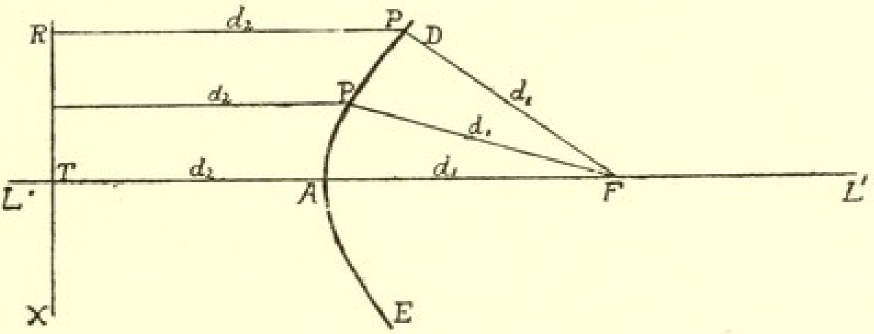



FIG. 280.

on the curve such that $F\,P/P\,R = A\,F/A\,T = d_1/d_2$ = the constant eccentricity, which is less than unity. $F'$ is another point on the axis such that $P\,F + P\,F'$ is a constant equal to $A\,B$ (Fig. 282).

Fig. 282 illustrates the closed elliptical curve of which $A$ and $B$ are the two vertices, $F$ and $F'$ the foci, and $C$ the centre. The chief property with which we are concerned here is that just mentioned, viz. $\mathrm{P\,F + P\,F' = A\,B}$ =

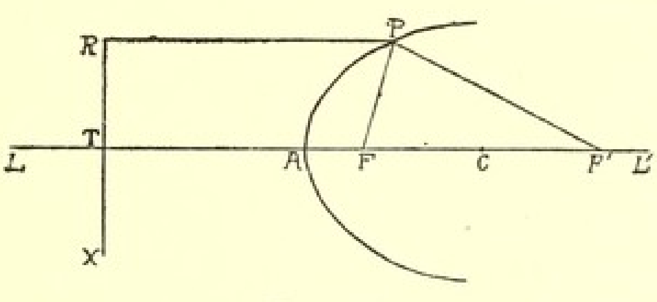



FIG. 281.

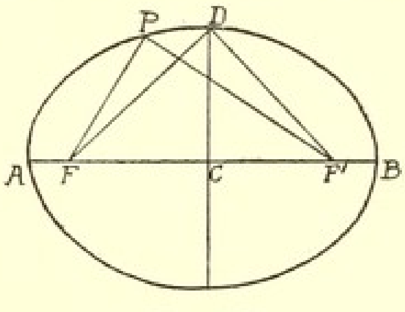



FIG. 282.

a constant. The bounding curve $A\,P\,D\,B$ is the perimeter, $A\,B$ is the long, or major, axis, and that at right angles through $C$ the short, or minor, axis. It will be seen that the ellipse cuts the major axis in two points $A$ and $B$.

**Construction.**—An ellipse may be constructed by putting stout pins through two points, as $F$ and $F'$ in Fig. 282, which become the two foci, and passing over them a suitable length of slack thread. A pencil held upright and pressed against the thread outward from the foci, as $P$ or $D$, on being moved around, describes the ellipse on paper.

If the two axes or the contour of the ellipse are known, to find $F$ and $F'$

we have $DF = DF'$, and $DF + DF' = AF + AF' = AF + BF' =$ a constant, so that $DF = AC$. With a pair of compasses measure $AC$ and connect $D$ with the long diameter $AB$ by lines whose lengths are equal to $AC$; these lines cut $AB$ in the foci $F$ and $F'$. The distance of each focus from the one extremity of the minor axis equals half the major axis.

There is no exact method by which the perimeter of an ellipse can be calculated, but the following formula gives it with a fair degree of exactitude. Let $a$ and $b$ be respectively the long and short axis, and $P$ be the perimeter; then

$$\mathrm{P} = \pi \sqrt{\frac{a^2 + b^2}{2}}$$

When the length of the one axis $a$ is known, that of the other, $b$, is found from

$$b = \sqrt{2\,(\mathrm{P}/\pi)^2 - a^2}$$

If the two axes are equal the figure becomes a circle, so that the peri-

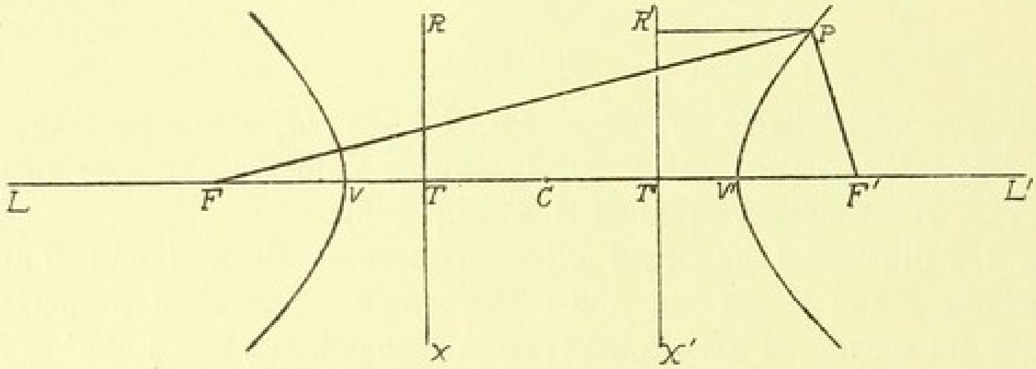



FIG. 283.

meter of the ellipse is analogous to the circumference of a circle. Since the circumference of a circle equals $\pi$ times the diameter, the diameter of the circle whose circumference is equal to the perimeter of a given ellipse is

$$\text{diameter} = \sqrt{\frac{a^2 + b^2}{2}}$$

An ellipsoid is a solid body generated by the revolution of an ellipse about one of its axes; it is *prolate* when formed on the major axis, and *oblate* when formed on the minor axis.

## The Hyperbola.

On the line $LL'$ (Fig. 283) erect $R'T'X'$ and take any point $F'$ on $LL'$. Divide $F'T'$ at $V'$ so that $V'F'/V'T'$ equals the eccentricity. Take $V$ so that $VF'/VT' = V'F'/V'T'$, and mark $F$ so that $VF = V'F'$. Take $T$ so that $VT = V'T'$, and erect $RTX$. The point $C$ is midway between $V$ and $V'$. In the hyperbola $V$ and $V'$ are the two vertices, $F$ and $F'$ are the two foci,

$C$ is the centre, $L\,L'$ is the transverse axis, and $R\ T\,X$ and $R'T'X'$ are the two directrices.

$P$ is any point on the curve such that $F'P/R'P = V'F'/V'T' = d_1/d_2 =$ a constant more than unity. The distance $P\,F - P\,F'$ is a constant, and is equal to $V\,V'$; this latter is the property of the hyperbola with which we are chiefly concerned.

**Construction.**—To the right extremity $A$ of a rod $F\,A$ (Fig. 284), fasten a string whose length $S$ is somewhat less than that of the rod. Draw the

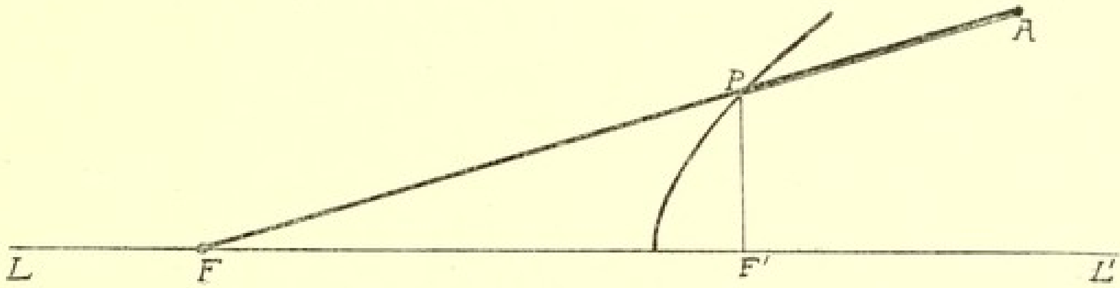



FIG. 284.

axis $L\,L'$, take $F$ and $F'$ as the foci, and put stout pins through them; attach the other end of the string to $F'$. Then a pencil $P$ pressed against the lower edge of the rod, the string being kept taut, will trace the hyperbola as the rod is rotated around $F$. If the string be lengthened to $2\,F\,A - S$, the other branch of the hyperbola can be traced.

### The Parabola.

On the line $L\,L'$ (Fig. 285) erect $R\,T\,X$, and divide $F\,T$ at $V$ such that $V\,F = V\,T$. In the parabola $V$ is the vertex, $F$ is the focus, $L\,L'$ is the

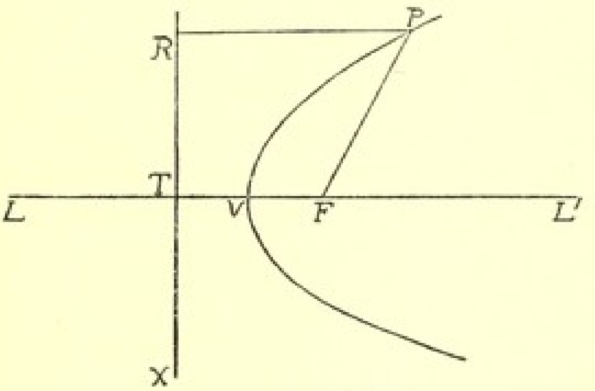



FIG. 285.

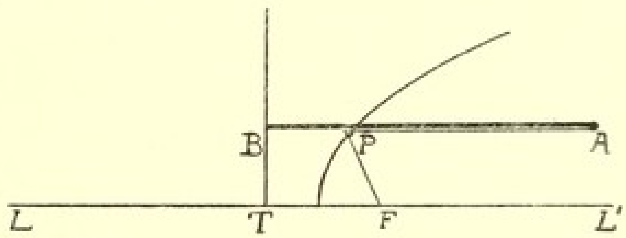



FIG. 286.

axis, and $R\,T\,X$ the directrix. The parabola may be regarded as a special case of either an ellipse or hyperbola whose one vertex and focus is at $\infty$. $P$ is any point on the curve such that $F\,P/R\,P = V\,F/V\,T = d_1/d_2 = 1$, which is the chief property of the parabola with which we have to deal here.

**Construction.**—Fix a string on a rod as for the hyperbola, but let the length $S$ be exactly equal to $A\,B$. Draw the axis $L\,L'$, take some point $F$

as the focus and put a stout pin through it; at $T$ erect the directrix, and attach the string to the pin at $F$. The pencil, used as for the hyperbola, traces the curve as $B$, the base of the rod, is gradually passed along the directrix at $T$ and at right angles thereto.

### Aplanatic Reflection and Refraction.

We have seen that reflection or refraction at single spherical surfaces is always accompanied by more or less spherical aberration; it is possible, however, to conceive surfaces that are *aplanatic*, i.e. capable of producing a point image of a point object situated on the principal axis; such surfaces are, in general, one of the three conic sections briefly described in the foregoing article.

Since light takes the shortest possible path in its course through any medium, if we make the optical length of all rays diverging from the object point equal, the image point will be aplanatic; therefore it is convenient to apply the principal of *least time* in order to determine the nature of the surface in each particular case. As briefly pointed out in Chapter XXIV, the

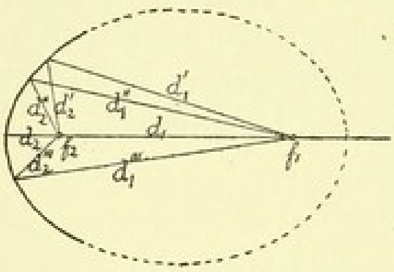


Fig. 287.

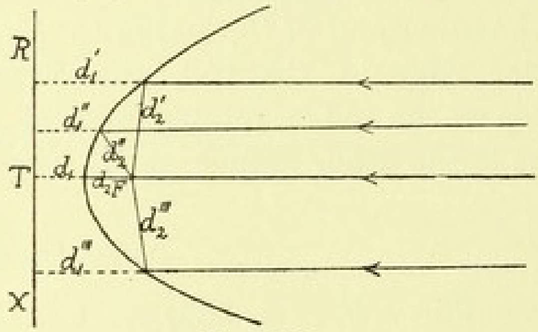


Fig. 288.

optical length of any ray is its actual distance multiplied by the $\mu$ of the medium in which it is travelling.

### Reflection.

**Cc. Surface-Real Conjugates within $\infty$.**—Let it be required to construct a mirror (Fig. 287) capable of producing an aplanatic image, $f_2$ of some point $f_1$ within $\infty$ on the principal axis. Then if $d_1'$ $d_1''$ . . . and $d_2'$ $d_2''$ . . . be the incident and reflected rays, $d_1' + d_2'$ must equal $d_1'' + d_2''$, and likewise for any other incident reflected ray. Thus, in general, $d_1 + d_2$ = a constant, so that *the mirror must be an ellipsoid of revolution with $f_1$ and $f_2$ as the foci.* The object could, of course, be at $f_2$ and the image at $f_1$. The mirror is, however, aplanatic only for these two points, aberration appearing immediately the object point is displaced from either. For every pair of conjugates a different curve is needed, so that ellipsoidal mirrors have no practical utility, as their limited application never occurs.

It should be noted that *spherical* mirrors of any aperture are aplanatic if the light diverges from, or converges to, a point at the centre of curvature.

**Cc. Surface-One Conjugate at ∞.**—If the object point be at ∞ (Fig. 288) *the curve of the reflecting surface becomes that of a parabola* of which $F$ is the focus. Here the directrix $R\,T\,X$ represents a plane wave interrupted by the mirror, and in order that all points on such a wave may meet at a single point, they must be converged to $F$ in precisely equal times, so that, as before, $d_1' + d_2' = d_1'' + d_2''$, etc. If the object point be at $F$, the light is reflected as a parallel beam.

Parabolic mirrors are employed in reflecting telescopes for bringing rays from an infinitely distant object, such as a star, to a sharp focus. Also for projecting a parallel beam of light, as in lighthouse and optical lanterns, microscopic reflectors, etc. Such mirrors possess the advantage over refractors in that all light waves are equally projected, and therefore chromatic aberration does not occur. For this reason also they are preferred to refractors for the photography of celestial bodies.

**Cc. Surface-Virtual Conjugate.**—Let $f_1$ (Fig. 289) be the object point, and $f_2$ its virtual image; the latter is then aplanatic if $d_1' - d_2'$, $d_1'' - d_2''$, etc. be a constant. This results if the curvature of the mirror *is that of a*

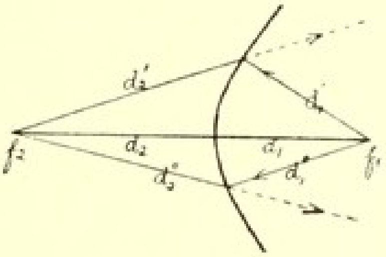


FIG. 289.

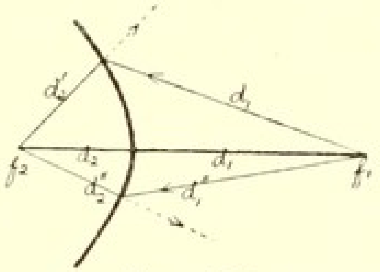


FIG. 290.

*hyperbola*, $f_1$ and $f_2$ being the foci. If the virtual object point be at $f_2$ and the image at $f_1$ the same curvature is required. Like the ellipsoidal, the hyperbolic mirror is of no practical value.

**Cx. Surface.**—An aplanatic *convex* reflecting surface for a near object must be hyperbolic (Fig. 290), while for parallel light it must be parabolic.

## Refraction.

The refracting surface which could be aplanatic for light of a certain wave-length could not be so for other wave-lengths. The conditions to be fulfilled for aplanatism in refraction is the same as for reflection, that is, all the rays diverging from an object point must reach the image point at the same time. It is, however, much more complicated than in reflection for whereas in the latter the light before and after contact with the surface is in the same medium, this is not the case in refraction, the velocity of the light differing before and after contact with the refracting surface.

**Single Cx. Surface-Near Objects.**—Let $f_1$ (Fig. 291) be an object point in air; its real image $f_2$ in the denser medium will be aplanatic if the distances $d_1' + \mu d_2'$, $d_1'' + \mu d_2''$ etc. be equal. The light travels along

$d_1' \, d_1''$ etc. at a velocity $V_1$, while it travels along $d_2'$, $d_2''$ etc., at a lessened velocity $V_2$. The curvature of the surface, where $d_1 + \mu \, d_2$ is a constant, is that of a *Cartesian oval*. If $f_2$ were the object in the dense medium, and $f_1$ the image in air the same conditions apply.

**Single Cx. Surface-One Conjugate at ∞.**—If the object point $f_1$ (Fig. 292) be at ∞, again the condition for aplanatism is that $d_1' + \mu \, d_2'$, $d_1'' + \mu \, d_2''$, etc. be a constant; the curve must then be that of an *ellipsoid*, i.e. all points on a plane wave $R \, T \, X$ must be retarded so that they reach the focus $f_2$ in the same time that each point would have travelled to $R \, T \, X$ if uninterrupted.

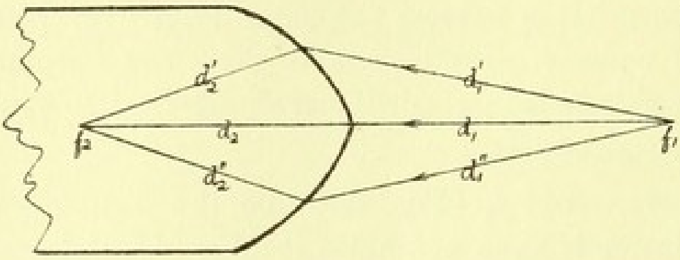


FIG. 291.

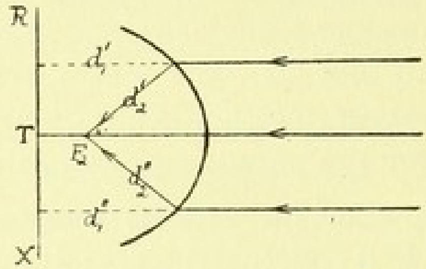



FIG. 292.

If the object be at $f_2$, so that the light is projected parallel, the same surface is required.

**Single Cc. Surface.**—If $f_2$ is the virtual image of $f_1$ (Fig. 293) it is aplanatic if $d_1' - \mu \, d_2'$, $d_1'' - \mu \, d_2''$ etc. is a constant, and the curvature of the surface for this condition is also that of a Cartesian oval. If however $d_1 = \mu \, d_2$ the curve is spherical.

**Cx. Spherical Surface.**—If a luminous point be situated within the dense medium of a single convex refracting surface (Fig. 294), a position on the axis can be found such that the virtual image is aplanatic. The distance of

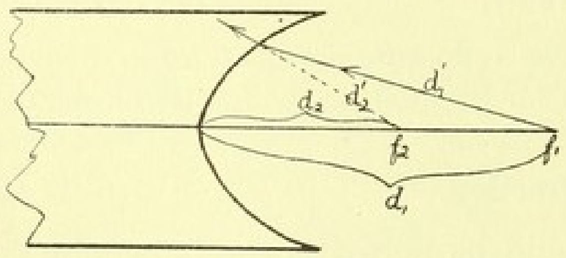


FIG. 293.

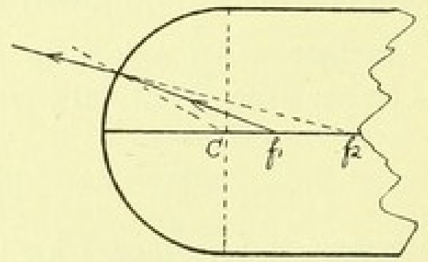



FIG. 294.

$f_1$ from the surface must be $r + r/\mu$, or $r \, (\mu + 1)/\mu$, and therefore the image $f_2$ is formed at $r + \mu \, r$, or $r \, (\mu + 1)$. As the distances of $f_1$ and $f_2$ are respectively $r/\mu$ and $\mu r$ from $C$, the magnification is $f_2/f_1 = \mu^2$. This principle is made use of in Abbé's homogeneous immersion objective employed in high power microscopes.

In this case the bottom lens of the objective is a hemisphere whose plane surface is towards the object, and when immersed in cedar oil of the same index as that of the glass the whole forms a single refracting body as shown in Fig. 294. The object is then placed at $f_1$, and its aplanatic image is formed at $f_2$, which in turn serves as an object for the remainder of the objective components.

**Aplanatic Lens.**—If, in a Cc. periscopic lens (Fig. 295), the object at $f_1$ faces the concave surface, the virtual image at $f_2$ is aplanatic when $d'_1 = \mu d_2'$. In this case $r_1$ the radius of the first surface must be $f_1/(\mu + 1)$, while that of the second surface must be $(f_1 + t)/\mu$ ($t$ being the thickness), for then $f_2$ lies in the centre of curvature of the second surface; $\mu f_2 = f_1$, both measured from the first surface.

**Example.**—Let $f_1 = 15$ cm., $t = 2$ cm., and $\mu = 1{\cdot}5$; then

$$r_1 = 15/(1{\cdot}5 + 1) = 6 \text{ cm., and } r_2 = (15 + 2)/1{\cdot}5 = 11{\cdot}33 \text{ cm.}$$

After refraction at the first surface we have

$$1{\cdot}5/f_2 = -{\cdot}5/6 - 1/15 = -4{\cdot}5/30$$

Therefore $$-4{\cdot}5 f_2 = 45, \text{ or } f_2 = -10$$

An aplanatic Cx. meniscus results when the object faces the Cc. surface (Fig. 296) if $r_1 = f_1$ and $r_2 = \mu(f_1 + t)/(\mu + 1)$. In this case the rays from

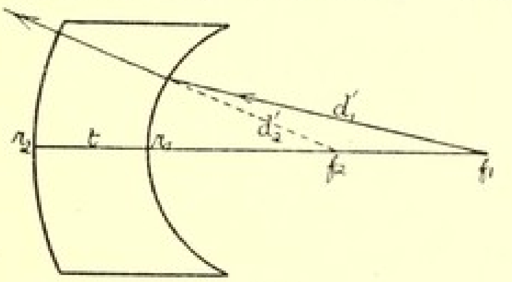


FIG. 295.

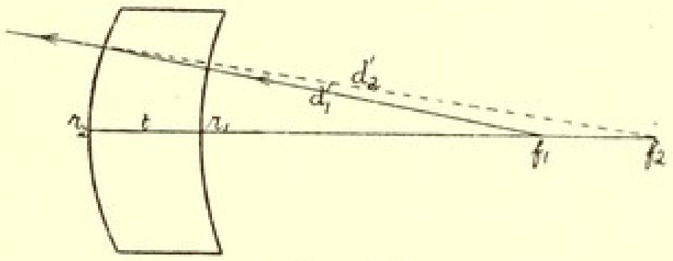


FIG. 296.

$f_1$ are normal to the first surface, and $d_1$ is constant, as is also $d_2$, for all rays; $f_2$ lies in the aplanatic point of the second surface corresponding to the value of $f_1$ given in Fig. 294 illustrating the case of the single surface.

**Example.**—Let $f_1 = 15$ cm., $t = 2$ cm., and $\mu = 1{\cdot}5$; then $r_1 = -15$ cm.,

$$r_2 = 1{\cdot}5\,(15 + 2)/(1{\cdot}5 + 1) = 25{\cdot}5/2{\cdot}5$$

so that $$r_2 = 10{\cdot}2 \text{ cm.}$$

If there are two unknown quantities $r_2$ and $t$, values must be found for them so that $\mu f_1 = f_2$, both measured from the second surface.

These are the only cases where aplanatism can be obtained with lenses; there is no case for parallel light, nor for double Cx. and Cc. lenses, but, as explained under spherical aberration, this can be minimised by employing certain forms of lenses and a stop.

**Curvature.**—If a surface is spherical its curvature is $1/r$, and this applies also to all curved surfaces other than spherical. In the case of the former the curvature is equal at all points on the surface, but this is not so with conic and other curves. The curvature at any point on any refracting or reflecting surface is determined by drawing to it a normal from the axis, the length of this line being the radius of curvature of that particular point.

## CHAPTER XXVI

### POLARISATION AND PEBBLES

**Polarised Light.**—The beam of light transmitted by a homogeneous medium, such as air or glass, is ordinary in the sense that it consists of waves whose transverse vibrations lie in every direction across the line of travel, whereas the vibrations of polarised light are confined to certain directions only. The polarisation of light may be plane, circular, or elliptical. The plane of polarisation of plane-polarised light is that plane from which the vibrations are eliminated, the latter being executed at right angles to the plane of polarisation. Suppose a rope attached to a wall and vibrated at the free end; vibrations or waves will run along the rope in any plane. If, however, the rope be passed between two upright sticks all vibrations will be

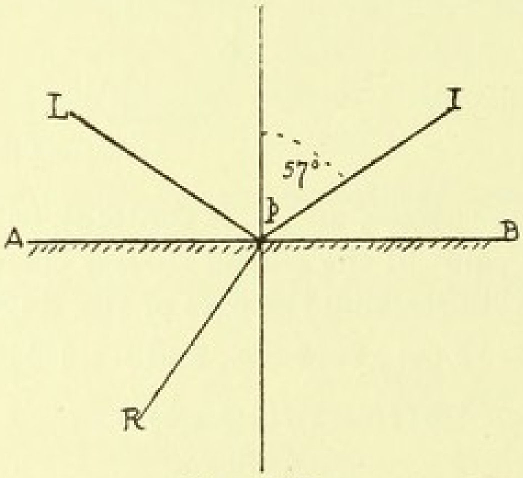


Fig. 297.

stopped except those in the vertical plane. The former illustrates ordinary unpolarised light, and the latter plane polarised light waves.

**Polarisation by Reflection.**—At a certain angle of incidence, which varies with the $\mu$ of the medium, the reflected and refracted beams $L$ and $R$ (Fig. 297) from the glass surface $A B$ are at right angles to each other. The vibrations of the incident light which are perpendicular to the surface penetrate it and are transmitted, while some of those parallel to the surface are reflected. The reflected beam is thus polarised, the vibrations being confined to a plane parallel to the reflecting surface, while the plane of polarisation is perpendicular to the surface, and is therefore the same as the plane of incidence of the light.

The angle of incidence necessary to obtain polarisation of the reflected beam is found by the equation $\mu = \tan P$, where P is the polarising angle. Thus P differs with the optical density of a medium, the polarising angle of water being 53°, that of glass about 57°, and that of a diamond 68°. Differently coloured rays have different polarising angles, so that white light is never completely polarised by reflection. The polished surfaces of metal have no polarising effect.

Polarised reflected light can be best obtained from a sheet of glass blackened on the further side, and, of course, suitably placed with respect to the incidence of the light. The blackening is not essential, but it prevents reflection from the second surface of the glass.

**Polarisation by Refraction.**—The light, incident at the polarising angle on a transparent body, which is refracted and transmitted at right angles to the reflected beam, is partially polarised, the plane of polarisation being at right angles to that of the polarised reflected light. Pure polarised refracted light can only be obtained when a beam is transmitted obliquely through a bundle of thin glass plates bound together, so that, by repeated reflection, all light polarised in the opposite direction is got rid of.

**Double Refraction.**—Most crystals polarise light owing to double refraction, notably calcite (Iceland spar), quartz, and tourmaline. A light wave in air or in any homogeneous body vibrates in every direction across its line of propagation, and its velocity is uniform and inversely proportional to what is termed the optical density of the medium. In a crystal, owing to its molecular structure, the retardation of waves, when incident obliquely to the axis of crystallisation, is greater in one direction than in another, so that the rays are transmitted along two separate paths, the one ray being called *ordinary*, and the other the *extraordinary* ray.

The separated waves caused by double refraction differ in that one of them is *spherical* and the other *elliptical*. The ray corresponding to the spherical wave is said to be *ordinary*, because it obeys the ordinary laws of refraction of light in homogeneous media, but the *extraordinary* ray conforms to no fixed law, since it is not at right angles to the wave front, nor does the refracted ray lie in the same plane as the incident and normal to the point of incidence. Both rays are polarised in planes at right angles to each other and travel at unequal speeds, except in the direction of what is known as the *optic axis*, where both waves have the same velocity and where no double refraction occurs. In planes at right angles to the optic axis there is also no double refraction in the ordinary sense, but the waves are retarded unequally, the one travelling more slowly behind the other.

**Rock Crystal or Pebble.**—Rock crystal or quartz is a pure, usually colourless, crystalline variety of silica, which occurs in nature in the form of a hexagonal (six-sided) prism, terminating in a six-sided pyramid. Its average index of refraction ($\mu = 1{\cdot}54$) is about the same as that of ordinary crown glass, but lower than that of flint glass, its dispersion $(\mu_H - \mu_A) = \cdot 014$ being lower than either. When cut into a slab or ground to form a lens, it is more usually styled a pebble. It is much harder than glass, more brittle, and a better conductor of heat, and it transmits much more readily than glass the ultra-violet rays which lie outside the visible spectrum. Its density is 2·65, that of glass being from about 2·4 to 3·4.

The relative scarcity and greater difficulty of working pebble makes it comparatively dear. Its low dispersive power and freedom from liability to become scratched seem to be its sole advantages, so that, all being considered, pebble is not superior to good optical glass for spectacle lenses, although perhaps for simple spherical convex lenses which are frequently put on and off, and therefore specially liable to become scratched in the centre, it is sometimes to be preferred. As lenses, the pebble should be quite clear and free from striae, specks, and flaws, and should be axis cut.

**Axis-Cut Pebble.**—Axis-cut pebble is that which is cut into slabs at right angles to its line of crystallisation, so that when the surfaces receive their spherical curvatures, the axis of the crystal coincides with the principal axis of the lens. Axis-cut is more expensive than non-axis-cut pebble, because in cutting it there is not so good an opportunity of utilising those parts of the crystal which are free from flaws, as when the slabs are cut without regard to any particular direction.

**To Recognise Pebble.**—Pebble is recognised by (*a*) feeling colder to the tongue than glass, (*b*) by the fact that on account of its hardness a file makes no impression on it, and (*c*) by the polariscope test. By the latter the difference between axis-cut and ordinary pebble can also be seen. As supplied to the optical trade pebble is usually quite colourless, and when in the form of a lens it has a sharper ring than glass.

**Double Refraction in Pebble.**—Pebble possesses the property of double refraction, the refractive index for the ordinary ray being 1·548 and for the extraordinary ray 1·558, and since the index is higher for the extraordinary than for the ordinary way, pebble is described as a *positive* crystal. It is because the difference in the $\mu$'s of the two rays is so small that double refraction by a pebble spectacle lens is not appreciable, the images being too close together to be seen double, the more so since the substance of the lens is thin.

**Tourmaline.**—Tourmaline cut parallel to its axis reduces an incident beam of light to two sets of polarised waves, the one in the plane of the axis of the crystal, the other at right angles to it. By a curious property of tourmaline the former (the ordinary ray) is absorbed almost immediately, and the latter (the extraordinary ray) only is transmitted, so that all the emergent plane polarised light is vibrating in the plane parallel to the axis. The plane of polarisation of a tourmaline plate can be determined by analysing the light polarised by reflection from a plate of glass. If held at the proper angle, the position of complete or partial extinction is found when the axis of the tourmaline is at right angles to the surface of the glass, that is, in the plane of incidence of the light.

**Iceland Spar.**—Spar or calcite, like quartz and tourmaline, has the power of double refraction, but since the ordinary wave has a higher index than the extraordinary, it is termed a *negative* crystal. The index of the ordinary ray is 1·659 and that of the extraordinary 1·486, so that the comparatively large difference between the indices causes a corresponding high degree of double refraction, enabling the doubling of objects to be plainly seen through slabs only a few mm. in thickness.

**Nicol Prism.**—This is a device whereby a beam of pure polarised light is obtained by transmission through a piece of spar. The latter is cleaved obliquely to its axis, and the two segments recemented by balsam whose index of refraction is 1·54, or about midway between the indices of the two rays. Now the angle of cleavage with the axis is so arranged that, when the ordinary ray is incident on the layer of balsam, it does so at an angle greater than the critical angle for indices of 1·659 and 1·540, and is therefore totally reflected to one side. On the other hand, the extraordinary ray, whose index is lower than that of the balsam, is transmitted, and constitutes a plane polarised beam of light which is, however, only half the intensity of the original beam. On account of the scarcity of spar Nicol prisms are now expensive to make, and are largely replaced by reflecting polariscopes of some form or other.

**The Pebble Tester.**—The simple polariscope consists of two plates of tourmaline cut parallel to their axes and suitably mounted. These plates are sometimes fitted to the ends of a wire spring like a pair of sugar tongs and called a *pincette*. If the two plates are placed in such a position that their axes are parallel, the plane polarised beam of light transmitted by the first plate will traverse the second, and if a polariscope, so fixed, is looked through, green or brown light—due to the colour of the tourmaline—can be seen. The combination looks much more opaque than would pieces of glass of the same intensity of colour, because half the light received by it is quenched. The outer plate which polarises the light is called the *polariser*, and the second plate—the one near the eye—is called the *analyser*. If, now, the analyser be rotated, while still looking through the instrument, the light will be found to become less and less bright, until, when it has been turned through a quarter-circle, the two axes being then at right angles to one another, the plane polarised beam transmitted by the polariser is stopped by the analyser. If the axis of the polariser is, say, horizontal, it can transmit only waves whose vibrations are horizontal, while the analyser can transmit only those whose direction is vertical; consequently all the light is blocked out. So long as the two axes are oblique to one another, some light passes through both plates.

It is in the position of *extinction* of the two plates that the polariscope serves as a pebble tester, so that if required for that purpose, it should be looked through and the one plate rotated until the *darkness* is complete. Unless this is done it is useless for the work, although even if it cannot be made quite *dark* there is an appreciable difference in the quantity of light transmitted by glass and by pebble placed between the plates, as explained in the following paragraph. Fig. 298 shows the two tourmalines with their axes parallel, Fig. 299 with their axes oblique, and Fig. 300 with their axes at right angles.

**Recognition of Pebble by Polariscope.**—If an ordinary glass lens, being homogeneous in nature, is placed between the two plates of the polariscope, it has no effect on the plane polarised beam of light transmitted by the polariser, and nothing can be seen through the instrument. A pebble placed in the instrument, by virtue of its double refracting nature, so twists or rearranges the vibrations of the beam transmitted by the first tourmaline plate that the light is incident on the second plate in directions other than at right angles to its axis, and part of it is transmitted. Hence with a pebble tester a pebble can be distinguished from glass, since, when a pebble

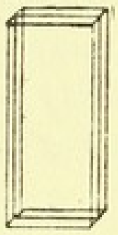


FIG. 298.

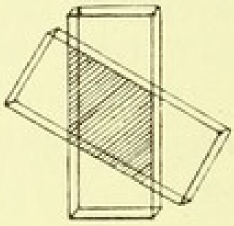


FIG. 299.

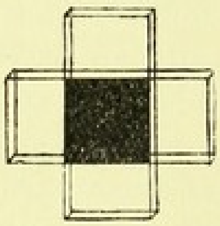


FIG. 300.

placed between the tourmalines, light is seen, while none is seen when glass is so placed.

Also most crystals, when viewed through a polariscope, present arrangements of colour which are characteristic of them. If a pebble cut parallel to the axis of the crystal (non-axis cut) is placed between the dark tourmalines and rotated there are found two positions in which no light passes; the one is where the axis of the pebble is parallel to, or in the same line with, the axis of the polariser, and the other is where it bears the same relation to the axis of the analyser. In either case, the polarised beam of light received by the pebble cannot be made to vibrate so as to be transmitted by the analyser.

**Recognition of Axis-Cut Pebble.**—A ray of light transmitted by quartz cut perpendicular to its axis (axis-cut pebble) is not bifurcated. Such pebble possesses the property of *rotating* the plane of polarisation, so that the vibrations transmitted from the polariser are no longer at right angles to the axis of the analyser. The amount of twisting undergone by the plane of polarisation is proportional to the thickness of the quartz, and, provided monochromatic light were used, extinction could again be obtained by rotating the analyser through a sufficient angle. With white light, however, this is impossible, as the rotation of the plane of polarisation depends also upon the wave-length, i.e. colour of the incident ray, and therefore the angle of extinction differs for each wave-length. In addition, light transmitted obliquely through the polariscope and pebble undergoes double refraction at the latter, and a kind of interference (too complicated to be gone into here) is set up between the ordinary and extraordinary rays having effects somewhat similar to Newton's rings. These are caused by the unequal oblique distances travelled by the two rays within the pebble, and a series of brightly coloured rings are seen (if white light be used) crossed by two dark brushes at right angles to each other. If the analyser be now turned so that its axis is parallel to that of the polariser, the rings will be seen to change to their complementary colours, and clear spaces are substituted for the dark brushes previously formed. A white cloud is the best source in these experiments.

When the pebble is cut nearly, but not quite, perpendicular to its axis, coloured arcs of circles (incomplete rings) are seen; the light also cannot be blocked out, no matter what its position between the plates of tourmaline, because the axis cannot be made parallel to that of either the polariser or analyser. The intensities of the colours and the sizes of the arcs are both dependent on the nearness of the section of the pebble to that of right angles to the axis, i.e., on its nearness to axis-cut.

**Advantage of Axis-Cut Pebble.**—Rock crystal which is axis-cut is preferable for lenses to that which is non-axis cut, because in the former there is no double refraction for light parallel to the axis.

**Unannealed Glass.**—Glass which is unannealed, or has been subjected to pressure, strain, or twisting, polarises light and therefore acts in the polariscope somewhat similarly to a pebble, in that light is transmitted and colours are seen; but the effects produced by unannealed glass can never be mistaken for those of crystals since the patterns of colours, even if not irregular, as is generally the case, are totally unlike those caused by any kind of crystal.

## CHAPTER XXVII

## PHENOMENA OF LIGHT

**Interference.**—If from two adjacent points of light $P_1$ and $P_2$ (Fig. 301) waves of light are propagated, the crests and troughs of the waves from $P_1$ will coincide with those from $P_2$ along certain lines marked $B$, and they reinforce each other, thus causing doubly increased wave motion. Between these lines, marked $D$, the crests from the one source coincide with the troughs of the waves from the other, with the result that the wave motion is neutralised at these spots owing to the *interference* of the one set of waves with the other. Alternate lines of light and darkness, known as interference bands or fringes, are in this way produced. The light bands are along lines so situated that any point on them is a whole number of wave-lengths from $P_1$ and $P_2$. The dark bands are along lines so situated that any point on them is one half wave-length further from the one source than the adjacent

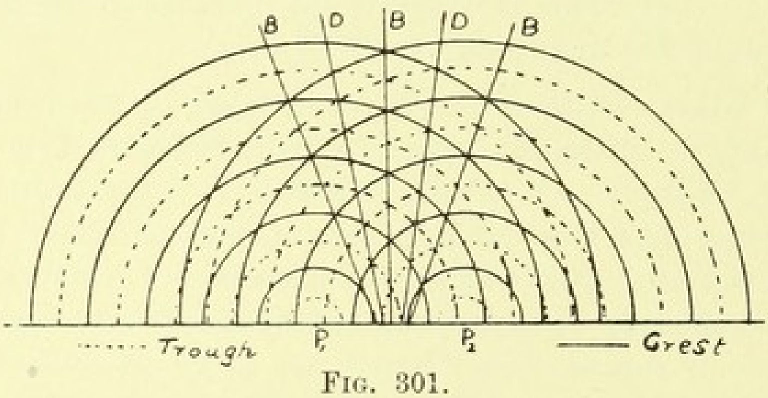


FIG. 301.

white band. The shorter the waves which interfere with each other, the less is the distance between the light and the dark bands. If, as in white light, there are waves of different lengths, the interference bands, instead of being alternately light and dark, take the form of coloured bands which are alternately red, blue, and white, the latter occurring where all the various colour bands coincide.

It must be remembered, however, that in order to secure interference between the light from two sources, the latter must be *exactly similar*, giving out waves of precisely the same length, amplitude and sequence. For preference the sources should be the duplicated images of a single source obtained by means of a double prism or other device. In addition the sources must be as small as possible—in fact the smaller they are the finer are the interference bands—and also they must not be separated by too great a distance; otherwise the bands are so narrow as to be indistinguishable except with artificial magnification.

The colours of thin films, such as soap bubbles, layers of grease on water, etc., are due to interference. Part of the light is reflected from the outer, and part from the inner, surface of the film, and the light reflected from the two surfaces is not in the same phase—i.e., the portion of the wave reflected from the inner surface has to travel over a greater distance than that from the outer surface. If the thickness of the film, therefore, be such that the inner wave emerges half a wave-length, or any odd number of half wave-lengths, behind the outer wave, the two portions of the original waves will interfere, and darkness will result at that spot for that particular wave-length. Should, however, the inner wave emerge in the same phase, i.e., a whole wave-length or any number of wave-lengths behind the outer wave, reinforcement will take place.

**Newton's Rings.**—When two plane, or two similarly curved, surfaces, the one convex and the other concave, are placed in contact, the film of air contained between them is of equal thickness, but if the one surface is not truly plane, or of exactly similar curvature to the other, the film of air is of varying thickness, and colours, due to interference, as explained above, are exhibited. This constitutes a method of determining a true plane or a uniform curvature. If a convex surface is placed in contact with a plane, or another convex surface, the film of air contained between them must be of gradually increasing thickness. At the centre the film is very thin, and a central black spot results, which is surrounded by a series of alternately bright and dark rings if monochromatic light is employed, or by coloured rings if the incident light is white. These are termed *Newton's Rings.* If the rings are viewed by transmitted light the centre is bright and the surrounding rings are alternately bright and dark, or of colours which are complementary to those seen by reflected light.

The width and regularity of the rings afford a very delicate test for similarity between two curves, and is made use of for testing the surfaces of the components of high-class photographic objectives, etc. The standard curve is called a *test plate* on to which is placed the surface to be tested. The absence of coloured rings shows true contact over the whole of the surfaces, but the pres nce of rings proves a difference in curvature and the lens is rejected as incorrect; complete absence of any rings is, however, rare, and the surface is considered satisfactory if the rings are very wide and of a dull colour.

**Diffraction.**—When light reaches the edge of a body owing to its undulatory motion some of the waves bend round the edge of the obstacle and penetrate the shadow cast by it. This phenomenon is known as *diffraction.* If monochromatic light is admitted through a small aperture the edge of the shadow is characterised by a series of alternate light and dark bands or rings, parallel to the edge of the shadow. These bands become less and less distinct as they are progressively further away from it, and they are broader in proportion to the length of the waves. If the source of illumination be white light, the diffraction fringes of the different colours overlap and a series of coloured fringes are seen. It is essential that the aperture be narrow, or small, since otherwise the unimpeded waves so out-number the retarded ones that the diffraction effect is more or less lost—in other words the diffraction effects are lost in the general penumbra (q.v.).

Diffraction bands can also be seen by looking through a narrow slit, at say, the filament of an electric glow lamp, the slit being parallel to the filament. If a very fine obstacle, such as a hair or thin wire, be placed between the light and a screen, a series of fringes can be seen both within and beyond the geometrical shadow. If the obstacle be circular, such as a small round patch on a piece of clear glass, the shadow is seen surrounded by alternate light and dark rings, or, if the source be sunlight, by a series of spectra. These bands encroach on the shadow, at the centre of which a bright dot can be seen. Of course, especially favourable conditions must be chosen to view the diffraction bands on account of the necessary smallness of the source, and the consequent loss of light. A star seen through a perfectly corrected telescope, and small objects seen by the microscope, appear bordered by one or more faint rings. Owing to diffraction, there is a limit to the possible magnifying power of a microscope, since the higher the power of the objective, the smaller the lenses, and consequently the more marked the diffraction phenomena.

The colours of many beetles and of mother-of-pearl are caused by diffraction and interference phenomena, and are not due to pigmentation at all; here the wing-cases of the beetles, or the mother-of-pearl, are very finely striated, which causes them to act like irregular diffraction gratings.

**Diffraction Grating.**—A large number of very fine equidistant lines—some thousands to the inch—ruled parallel to each other on a plate of glass or metal forms a *diffraction grating*.

**Diffraction Spectrum.**—Dispersion can be obtained by reflection from, or transmission through, a glass diffraction grating, or by reflection from a metal grating; the transmitted or reflected light forms a series of spectra which can be thrown on a screen, or be examined by a telescope, and the finer and closer the lines the purer will be the spectrum obtained.

The lines of the grating scatter a small portion of the original waves into fresh and regular series, of which some are quenched by interference. Unlike the spectrum obtained by prismatic refraction, the colours as the direct result of interference are evenly distributed in accordance with their wave-lengths, so that the red end is not condensed, nor the violet end dispersed, while the red and orange occupy more, and the blue or violet occupy less space than in a refraction spectrum; also the most luminous part is more nearly in the centre. Such diffraction gratings afford an accurate means by which to measure the wave-lengths of light and the relative positions of the Fraunhofer lines.

Fig. 303 represents a portion of a highly magnified section of a glass grating, $Q$ and $R$ being the clear spaces between the lines. The distance $QR$, equal to one ruling and one space, forms a *grating element.*

Imagine parallel light falling on the grating from the direction $L$; the bulk of the light passes through uninterrupted, so that an eye placed somewhere in the neighbourhood of $L'$ will see the original source very much as it would through a piece of plane glass. On moving the eye to one side, so that the direction of view is oblique to the grating, colours will commence to appear, these being in the regular spectrum sequence from violet, which makes the smallest angle with the surface, to red, which makes the greatest. A short interval with no colour will occur after the red, but on increasing the obliquity of the eye to the grating, a second series of colours, in the same order as the first, but more drawn out and fainter, will be observed. This is shown diagrammatically in Fig. 302. The first, $VGR$, is the *primary* spectrum; $V'G'R'$ is the *secondary* spectrum, beyond which are others,

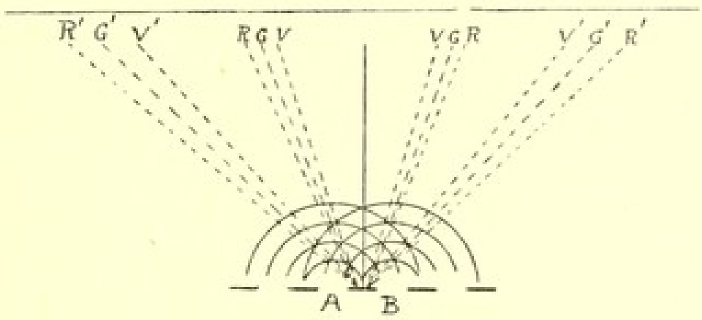



FIG. 302.

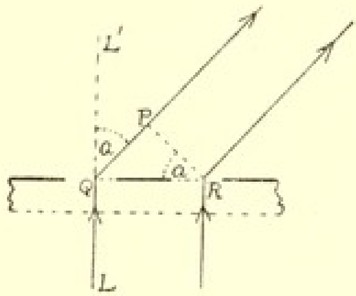



FIG. 303.

provided the grating is not too fine; usually only the primary and secondary spectra can be seen from a grating having about 15,000 lines to the inch. As previously stated, it is by the reinforcement of the wavelets diverging from the grating spaces along certain lines oblique to the surface, aided by certain amount of interference, that the spectra are produced.

In Fig. 303 consider a certain direction $QP$ oblique to the normal $LL'$, making with the latter the angle $a$; or, conversely, suppose the grating itself be tilted through that angle with respect to the incident light. Then the wavelets diverging from $Q$ and $R$ will either reinforce or interfere with each other according as *$QP$ is an even or odd number of half wave-lengths*—in other words, as the difference in the paths of travel of the wavelets is an even or odd number of half wave-lengths. Let $PQ$ be equal to the smallest possible even number, i.e. *two*, of half wave-lengths. Then in the direction $PQ$ there will be reinforcement for that particular colour, giving rise, in the eye or observing telescope, to an image of the original source if the light be monochromatic, or to a spectrum if white light be employed. Now

$$PQ = QR \sin a, \qquad \text{or} \qquad w = E \sin a,$$

where $w$ is one wave-length of the light in question, and E is a grating element. The element E is known, and the angle $a$ can be found by means of a revolving telescope as in the ordinary spectrometer (q.v.); therefore the wave-length $w$ is easily calculated from the above formula.

**Example.**—Let the grating have 15,000 lines to the inch, and suppose the angle $a$ for a particular part of the spectrum, say the yellow ($D$) line, to be 20°. 15,000 lines to the inch corresponds to 25·4/15,000 mm. to every grating element E, and sin 20° = ·342. Therefore $w = 25{\cdot}4 \times {\cdot}342/15{,}000 = {\cdot}000579 = 579\ \mu\mu$, which corresponds very closely with the $D$ line of the spectrum from which the observation was taken.

If the secondary spectrum be employed to obtain the necessary data for the above calculation, $w$ will then represent *two* wave-lengths, so that

$$w = \text{E} \sin a/2$$

the result being the same as in the example given, but $a$ is rather more than 40°, since sin $a$ must be exactly twice as great.

It should be observed that no spectrum is formed when the eye or the observing telescope is normal to the grating, the various reinforcing and interfering wavelets overlapping to form white. As will be seen from the formula, the number of spectra formed is smaller as E is smaller, i.e. as the number of lines to the inch is greater, and vice versa.

By the employment of metal gratings, specially in the form of concave mirrors which focus the spectra direct on to a screen or photographic plate, increased intensity of light is secured. In all such experiments the most suitable source is a fine slit, brightly illuminated, placed parallel to the rulings, the spectrum consisting of an innumerable number of diffracted images of the slit ranged side by side, and representing, as nearly as is possible, a separate image for every wave-length.

## Luminescence.

This is the general name given to the property of a body by which, without sensible rise of temperature, it becomes luminous.

The luminosity of phosphorus, fungi, and decaying vegetable matter is caused by oxidation. Chemical action also (or physiological action) is the cause of the light emitted by shell and deep-sea fishes, fire flies, glow worms, beetles, insects, animalculæ, and the bacteria found in putrefying vegetable and animal matter. Thus the brilliant light observed on tropical seas at night is due to numberless luminescent organisms. The light emitted by various insects is found of almost every colour in one or other species. Luminescence can also be produced by heating fluorspar, quinine, etc., by applying friction to quartz or cane-sugar in the dark, or by cleaving a slab of mica. Fused boric acid or even water when rapidly crystallised or frozen may exhibit this phenomenon.

When a high tension current is passed through a vacuum tube, Röntgen rays are produced, and the walls of the tube emit a greenish luminescence which is assumed to be due to minute electrified particles striking the wall of the tube with immense velocity and producing light and heat by their impact, the colour of the luminescence depending on the nature of the glass. Radium is found to shine perpetually in the dark, and bodies exposed to the radiation of radium become themselves radio-active, i.e. luminescent for a time. Luminescence also includes the following phenomena.

**Phosphorescence** is the term frequently given to the foregoing phenomena of luminescence, but it is more properly applied to the property of a body of being *luminous in the dark after exposure to light.* Some diamonds, fluorspar and various minerals possess this property; chloride or sulphide of calcium or barium, preserved from air in a sealed glass, will shine brilliantly for a long time.

Phosphorescence is excited by rays of high refrangibility, higher than those which produce the phosphorescent light, although the latter may be found of every colour of the spectrum. Phosphorescence is supposed to be due to the absorption of light, and its later radiation, as light of longer wave-length, after the exciting action has been removed.

**Fluorescence.**—Fluorescence is the property possessed by certain bodies of absorbing ultra-violet waves, invisible to the eye, and of emitting, by radiation, light of longer wave-lengths by which they appear self-luminous. This property was first discovered by Stokes in fluorspar, and so named by him fluorescence. The emission of light ceases immediately the original source of light is cut off, and in this fluorescence differs from phosphorescence.

The phenomenon is not confined to the ultra-violet rays, for if a solution of chlorophyll be placed in a dark room and a beam of white light allowed to fall on it, the surface of the solution emits a red fluorescent light. A solution of quinine emits a pale bluish colour in the presence of daylight. The fluorescence increases if the solution is held in the violet end of the spectrum, and is visible when held beyond the limits of the visible spectrum, the invisible ultra-violet rays exciting fluorescence and becoming changed into visible blue-violet rays. Similar effects may be seen with uranium glass, which fluoresces a brilliant green when placed in ultra-violet light. A thick plate of violet glass placed in front of a beam of light from the electric arc will cause the same phenomenon. Æsculine (the juice of the horse-chestnut bark), barium, and many other substances are fluorescent, and so are also the cornea, crystalline lens, and bacillary layer of the retina.

It has been said that the ozone of the atmosphere is fluorescent, and, by converting the ultra-violet into visible rays, makes the sky appear blue.

Fluorescence is generally taken to be the absorption of invisible light and its radiation as visible light *while the exciting cause is present.*

**Calorescence** is the name given by Tyndall to the conversion of the invisible infra-red waves into visible light. This he achieved by focussing an electric light, by a reflector, on to some platinum foil after passing it through substances opaque to visible, but transparent to infra-red light.

## Some Optical Phenomena.

**Blueness of Sky.**—If the air were absolutely transparent and of uniform density, light from the sun would reach the earth without any loss, and the sun, moon, and stars would be set in a sky which would appear black both during the day-time and at night. The air, however, contains a great quantity of aqueous vapour, and the blue colour of the sky is said to be due to reflection from the minute particles of this vapour suspended in the higher layers of the atmosphere. Tyndall showed that when mastic is thrown into water the minute insoluble particles of the mastic emit a deep blue colour similar to that of the unclouded sky. If a cloud of smoke be blown into the air, the smoke particles reflect the short blue waves more freely, and the cloud assumes a blue tint, and if a white screen be held, in bright sunlight, behind the smoke, the screen assumes a reddish brown hue. Large quantities of so-called cosmic dust also are held in suspension in the air, and this is believed, by some scientists, to be a cause. By some the blue of the sky is said to be due to polarisation by oblique reflection from particles of vapour, salt, etc., in the air; by others it is thought to be caused by fluorescence of the ozone.

**Aerial Perspective.**—If two objects, one light and the other dark, be seen at a considerable distance, they lose some of their contrast, the light object becoming darker by absorption of its reflected light by the intervening air, and the dark object becoming lighter by the superadded light diffused through the air. This causes what is known as aerial perspective. If the air is clear and the added light is blue, distant hills throw deep shadows of a purple blue colour in bright sunshine.

**The Horizon.**—When the sun is low down on the horizon its light has to pass through a thicker layer of atmosphere filled with dust particles and moisture; some of its blue and violet rays are absorbed or reflected, and it thus appears reddish, and for the same reason it appears red in a fog.

Near the horizon, the sun and moon appear larger than when higher in the heavens because they are mentally projected beyond the horizon, as compared with terrestrial objects, whereas when seen in the zenith this cannot be done, as they stand alone; they are not really larger as measurements with a telescope show. They also appear slightly flattened vertically, when near the horizon, and appear a trifle higher up than they really are, owing to the refraction of the air and the greater obliquity of the light from their lower edges.

Refraction diminishes the dip of the horizon and so slightly increases its apparent distance. The distance of the horizon can be computed approximately from $d = \sqrt{1 \cdot 5\, h}$, where $h$ is the height in feet of the observer above the sea or earth level, and $d$ is the distance in miles.

**Mirage (Fata Morgana).**—If the layers of the air are of markedly unequal density, as is sometimes the case in hot climates, especially on a desert where the warmest layers are the lowest, the phenomenon known as the *mirage* may be seen. Light from objects above, on its passage to the earth, traverses layers of air which become gradually less refracting, the angles of incidence accordingly increasing so that the light becomes more and more parallel to the surface, until at length the critical angle is reached, beyond which refraction changes to reflection. The light is then reflected in the contrary direction, and ascends to reach the observer's eye as if proceeding from a point below the ground, and objects appear inverted. This is shown in Fig. 304, where light from an object $O$, on reaching the eye at $E$, appears to come from $M$ below the level of the ground.

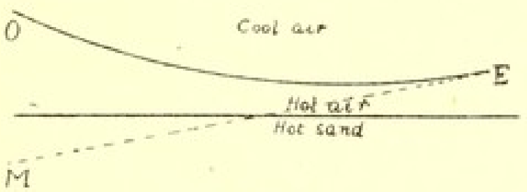



FIG. 304.

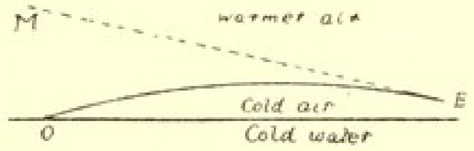



FIG. 305.

If the lowest strata of air are the densest, as in Fig. 305, they give rise to the same phenomenon, but the mirage $M$ is in the contrary direction, so that a landscape, or a ship at sea, may appear above the horizon. This occurs in very cold climates.

**Scintillation.**—The twinkling of a star is due to irregularities in the atmosphere causing variations in the path of the waves, which partially interfere. This produces variations in the apparent brightness and colour of a source of light, subtending a very small angle at the eye, such as a star. It is not observed in the case of a planet, because this has a real magnitude.

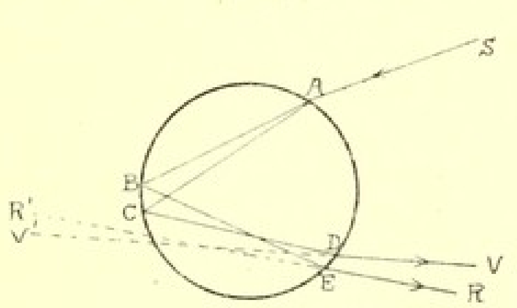



FIG. 306.

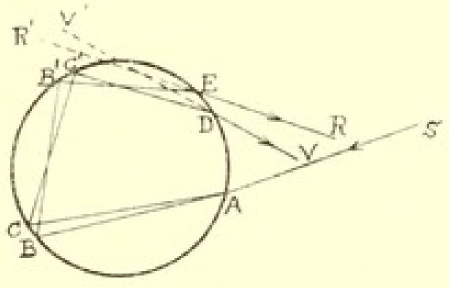



FIG. 307.

**The Rainbow.**—A rainbow is visible when the sun is behind the observer and a shower of rain in front of him, or it may be seen in the spray of a waterfall. Since the sun's rays falling on the raindrops are parallel, the course of light through all the drops must be the same, and it is therefore sufficient to trace the course of a ray through a single drop. Let a pencil of rays from the sun meet the drop at $A$ (Fig. 306). On entering it is refracted and dispersed towards $B$ and $C$ at the back of the drop, thence reflected to $D\,E$, where it is refracted to emerge in the directions $V\,R$ which make an angle with the entering ray. The emergent dispersed light thus diverges to the observer's eye, and the various colours, being unequally refracted, are projected back as $R' V'$, so that the outside of the bow is red and the inside blue-violet. The extent of the bow depends on the position of the sun; when the latter is at the horizon the bow forms a semi-circle to an observer at sea-level. As the sun rises the arc sinks so that its centre is below the horizon, and is smaller.

A secondary larger, broader, and fainter, rainbow is generally seen concentric with the primary. The rays from the sun to a point $A$ (Fig. 307) undergo refraction and are reflected twice at $B\ C$ and $B'\ C'$, and again refracted at $D\ E$. In the emergent light violet is below and red above; these, being reversed on projection, as $V'$ and $R'$, the secondary bow is blue on the outside and red within.

**Eclipses.**—A total eclipse of the sun occurs when the moon is so situated that some portion of the earth lies in the umbra of the shadow cast by it; the eclipse is partial to those portions of the earth in the penumbra of the shadow. An eclipse of the moon occurs when the moon lies in the shadow cast from the sun by the earth.

## CHAPTER XXVIII

### GENERAL EXPERIMENTAL WORK

In order to grasp the various formulæ and the theories underlying them, the student should perform for himself the simpler experiments connected with general optics. Most of the following can be done with quite rough or improvised apparatus, and a complete optical bench, meeting all requirements, can be obtained at a very moderate cost.

**The Optical Bench.**—An optical bench should preferably be scaled in cm. and mm. and be about 2 M long, thus enabling fairly weak lenses, mirrors, etc., to be tested. In addition there should be

(1) A frosted lamp at the zero end of the scale.

(2) A collimator consisting, for preference, of a pinhole fixed in the focal plane of a Cx. lens, the lamp being placed behind the pinhole when the collimator is in use.

(3) A screen of ground glass and another, interchangeable with it, of opaque stiff white card having a central aperture equal in diameter at least to the collimating lens.

(4) A plate with an aperture of same definite size—say 20 mm.—with fine cross wires, to serve as an object, when the lamp is placed behind it.

(5) Three or four carriers for lenses and mirrors—one of these should be universal and capable of holding any diameter lens from the smallest up to one of, say, 3″.

(6) Two or three clips on a single stand capable of taking lenses in contact or combinations of separated lenses. This should also be capable of a *horizontal* rotation round the support as a vertical axis.

(7) A small horizontal astronomical telescope with adjustable eye-piece.

All should be on movable stands and adjustable as to height, since axial alignment is essential in most experiments.

**Parallax** is the term applied to the apparent displacement of an object due to the observer's position. We generally employ the term to indicate the apparent change in the position of one object, in relation to that of another, when the observer changes his point of view. Let an object A be in front of an upright pencil P, and another object B be behind P, and all three in the same straight line in front of the observer. Now on moving the head to, say, the *right*, a gap will be visible between P and A and another between P and B; also A will be to the *left* of P, and B to the *right* of P. If there are two comparatively near small objects P and X, seen close together in the same direct line, the distance of the one P being known, if the head be moved sideways—(*a*) X is actually in the same plane as P, i.e., coincident with it, if no gap between them results; (*b*) X is *nearer* than P if X has apparently moved in the opposite direction to the observer's head; (*c*) X is more *remote* than P if X has moved in the same direction. By placing P respectively nearer, or further away, a position can be found for it such that parallax between them is said to be destroyed, since no apparent separation results from any degree of movement on the part of the observer; the distance of P then equals that of X. This principle is utilised for locating the position of virtual images formed by mirrors and lenses.

### The Refractive Index of Solids.

**Plate Method.**—A parallel plate (Fig. 308) of the medium, say glass, is placed on a sheet of white paper on a drawing board or other smooth surface.

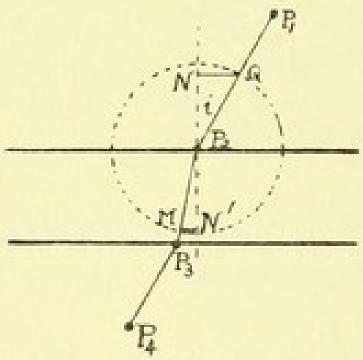



FIG. 308.

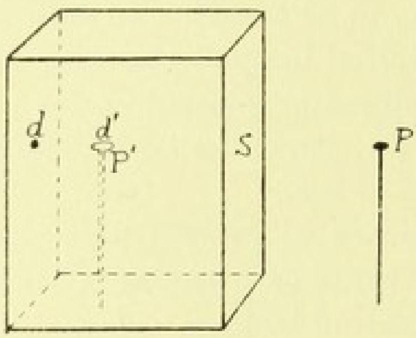



FIG. 309.

A pin $P_1$ is then stuck in any position and a second pin $P_2$ is placed close to the side of the plate and sufficiently to the left of $P_1$ so that a line $P_1 P_2$ makes a fairly large angle $i$ with the normal $N N'$. Now observe through the plate the pins $P_1 P_2$, which will appear displaced towards the right. Stick two more pins $P_3$ and $P_4$ in the board such that all four appear in one straight line. Draw the trace of the plate with a fine pencil, remove it and the pins, and with a compass, with $P_2$ as centre describe any circle—the larger the better—provided it falls within $P_3$. Where this cuts the course of the ray in $Q$ and $M$ drop the perpendiculars $Q N$ and $M N'$, the latter being the sines of $i$ and $r$ respectively; then $\mu = Q N / M N'$. This method is only approximate unless carefully done and therefore three or four readings for different values of $i$ should be taken and the mean result extracted.

**Displacement Method.**—The refractive index of a transparent body, such as glass, can be roughly found as follows:—Make a dot $d$ (Fig. 309) on the back of the block of glass; then find such a position for a pin $P$, placed vertically in front of the glass, that on moving one's head from side to side the virtual image of the pin, reflected from the front surface, appears to be behind that surface at such a distance $P'$ that, owing to absence of parallax, it coincides with the virtual image of the dot seen through the glass. In this case the apparent thickness of the glass is $P'S$ which $= PS$. Then $\mu = dS/PS$.

**Microscopic Method.**—The refractive index can be more accurately determined by means of a low power microscope in the following way. A fine line is focussed and the plate is then placed above the line. Now the microscope must be raised in order that the line be clearly seen, since the rays proceeding from it are divergent as if from a point nearer to the objective. The distance that the microscope objective has to be raised equals the distance between the real position of the line and its apparent position when seen through the plate. Let $t$ be the thickness of the glass, and $d$ the distance that the objective has to be raised; then $\mu = t/(t - d)$. The necessary measurements can be made fairly accurately by means of a mm. scale, some point on the tube being taken as an index. In some cases a fixed scale with a vernier attached to the microscope, or the

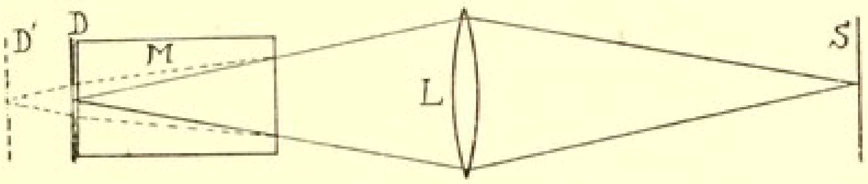


FIG. 310.

scale on the millhead of the fine adjusting screw can be used. Thus, if the thickness of the plate be 1 mm. and the object-glass has to be raised ·38 mm. $\mu = 1/{\cdot}62 = 1{\cdot}61$

**Bench Method.**—Should the medium be in the form of a fairly large body with two parallel surfaces, the index may be found on the bench as follows. Take any Cx. lens $L$ (Fig. 310) of convenient strength and project an image of the cross wires $D$ on to the screen $S$, such that $S$ is somewhere near the second symmetrical plane; carefully note the position of $D$. Then introduce the medium $M$, whose index is to be tested, between $L$ and $D$, when the image on $S$ will be found out of focus owing to the apparent vertical displacement of $D$. In order again to secure a sharp focus on $S$ the disc must be drawn back to some point $D'$ whose position is also read from the bench. Then, if $t$ be the thickness of the medium and $d$ the distance between $D$ and $D'$—the apparent displacement—we have, as for the microscope, $\mu = t/(t-d)$. To secure accurate results the image on $S$ must be well defined, and therefore a small achromatic lens should be used in the experiment. This method serves equally well for liquids if they are enclosed in a tank whose glass surfaces are parallel and whose thickness is very small compared with the depth of the liquid itself.

**Prism Method.**—This is the most accurate of all if a spectrometer is available. The principal and deviating angles being measured as described in Chapter XXIII, the refractive can be found from

$$\mu = \frac{\sin\{(P+d)/2\}}{\sin(P/2)}$$

where P is the principal angle of the prism, and $d$ is the angle of minimum deviation. If the incident light is allowed to fall perpendicularly on to one of the surfaces the formula becomes simplified to $\mu = \sin(P + d)/\sin P$. If white light be used a spectrum will, of course, be formed, but the index for any particular colour can be obtained by bringing the cross wire of the teles cope over that particular colour. Thus the mean index is calculated from the yellow ($D$ line) and the mean dispersion from the difference between the indices of blue-violet ($F$ line) and orange red ($C$ line). Hence $\nu$, the effectivity, may be obtained (Chapter XXIII).

**Example.**—Given a certain prism whose principal angle P is found to be 59° 57′ and the angle of minimum deviation $d$ for the $D$ line is 40° 21′ then

$$(P+d)/2 = (59° \; 57' + 40° \; 21')/2 = 50° \; 9', \text{ and } P/2 = 29° \; 58'$$

so that $\mu = \sin 50° \; 9'/\sin 29° \; 58' = \cdot 76772/\cdot 49949 = 1\cdot 536$

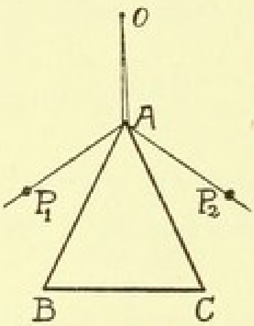



FIG. 311.

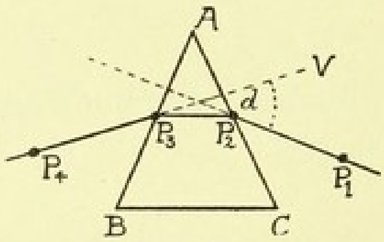



FIG. 312.

**Approximate Prism Method.**—If a spectrometer is not available, the value of P and d can be found roughly as follows. Place the prism (Fig. 311) on the drawing-board and turn the apex towards a window. Now look into the surface $A\,B$, which acts as a plane mirror, and select the image of a vertical window bar; get the image *as near as possible to the apex A* and put the pin $P_1$ in position so that it is in line with $A$. Do the same with the other surface $A\,C$. Draw the trace of the prism, remove it and the pins; then the angle formed by the lines $P_1\,A$ and $P_2\,A$ (i.e. $P_1\,A\,P_2$) is twice the principal angle and an ordinary protractor is used to measure it.

To find the deviating angle $d$, erect, in any convenient position, two pins $P_1$ and $P_2$ (Fig. 312), place the prism with one side in contact with $P_2$; then on looking through the prism somewhere in the direction $V$, the pins will appear displaced towards $A$. Secure minimum deviation by rotating the prism both ways, and finally erect two other pins $P_3$ and $P_4$ such that all four appear in one line. Then, by making the necessary tracings and connections with a fine pointed pencil, the angle of minimum deviation $d$ can be measured on a protractor.

**Lens Method.**—If the substance be in the form of a thin lens, its focal length and radii can be measured, as described elsewhere, and then

$$\mu = \frac{r r'}{(r + r')\,F} + 1$$

**Critical Angle.**—Should it be possible to measure C, the critical angle of a medium—but generally this is neither easy nor accurate—the refractive index is then $\mu = 1/\sin C$. Such a method might be suitable for a substance like butter for which other methods are not suitable. A special apparatus—a refractometer—would be required.

**Polarising Angle—$\mu$ of an Opaque Solid.**—Let $A\,B$ be a ray incident on a smooth or polished body $N$ and reflected in the direction $B\,C$. If the angle

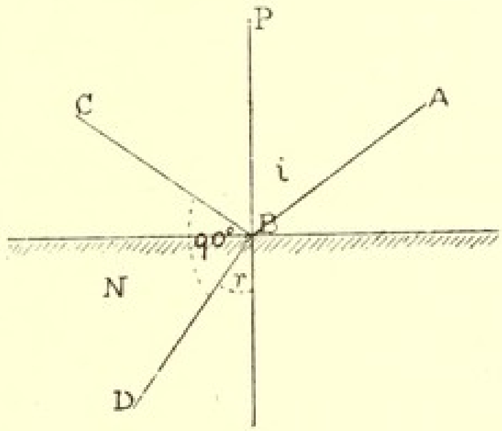



FIG. 313.

of incidence $A\,B\,P$ be the polarising angle of the medium $N$, any light transmitted or absorbed is refracted in the direction $B\,D$ at right angles to $B\,C$. If the polarising angle of an opaque body be known, its refractive index is the tangent of that angle; in Fig. 313, if the angle of incidence is $i$, then the angle of refraction $r = 180 - (90 + i) = (90° - i)$. Now

$$\mu = \sin i/\sin r = \sin i/\sin (90 - i) = \sin i/\cos i = \tan i$$

and since $i = p$, the polarising angle, then $\mu = \tan p$.

The polarising angle can be fairly accurately found as follows. Arrange a small source of light that can be conveniently raised and lowered on the one side, and on the other side similarly arrange a piece of tourmaline from a pebble tester, the axis being vertical. Then on raising or lowering equally both source and tourmaline a position will be found where the reflected image is entirly cut off. Measure the distance $d$ from the point of reflection to the *vertical plane* of the tourmaline, and also the height $h$ of the latter above the horizontal. Then $\mu = \tan p = h/d$.

**The $\mu$ of Metals.**—By making exceedingly thin prisms of less than one minute of arc, Kundt successfully determined the refractive indices of a number of the metals. Thus, if an incident ray fall perpendicularly on to one of the surfaces of such a prism, the refractive index can be quite approximately arrived at by the formula $\mu = d/P + 1$. The results showed a refractive index for silver, gold, copper, magnesium, and sodium as being less than that of a vacuum, and this, no doubt, accounts for the absence of a polarising angle in some substances. The red rays in some cases were found to be more refracted than the blue, so that metals form good examples of anomalous dispersion. The refractive indices of the metals were found to be proportional to their electric conductivities, i.e. those metals which were the best conductors had the lowest refractive index, and vice versa.

## The Refractive Index of Liquids.

In general, experiments similar to those for learning the indices of solids can be employed for liquids, but the arrangement is different in some instances. In all cases, however, the calculations are the same.

**Displacement Method.**—The liquid is placed in a bowl at the bottom of which is some small object *d*. Above the surface S a pinhead or other convenient object P is placed and is raised or lowered until its image, formed by reflection from the surface, apparently coincides, from the absence of parallax, with the image of *d* at P′. Measure *d* S and P S and proceed as with a solid.

**Microscope Method.**—With the microscope first focus the bottom of a small tank, and secondly its image when the liquid has been poured in. Thirdly focus the surface of the liquid, which generally has some conspicuous dusk specks floating about. The difference between the third and first readings gives the real depth, and that between the second and third the apparent depth. Proceed as with a solid.

**Prism Method.**—The liquid is placed in a special hollow glass prism of which each refracting surface consists of a plate with parallel sides. The index is then found as with a solid.

**Lens Method.**—Take a small quantity of the liquid and place it between a thin plate of glass and a Cx. lens of known radius and focal length $F_1$; the liquid then forms a plano-Cc. lens. If now F of the combination be found, that of the Cc. $F_2$ can be learnt. Its radius is also known, it being that of the Cx. lens, so the refractive index $\mu$ can be calculated from $1/F_2 = 1/F - 1/F_1$, and $\mu = (r + F_2)/F_2$.

**Polarising Angle.**—Care being taken to keep the surface of the liquid perfectly clear and steady, the method is the same as with a solid.

**Critical Angle.**—As with a solid.

**Bench Method.**—As described on page 319.

### Plane Surfaces.

**Movement.**—A plano spectacle glass can be determined with sufficient accuracy by observing an object (preferably crossed lines) through it while rotating and moving the glass. If the glass has no power due to curvature the image will appear stationary ; moreover, if the surfaces be true planes no distortion or irregular movements can be detected. If the glass be held obliquely to the eye, so that the direction of vision forms a small angle with the surface, any unevenness of the surface becomes more apparent.

**Contact.**—If one surface be a plane, this can be determined by applying to it a straight edge, or another plano-glass, and observing whether there is contact throughout when holding the applied surfaces against a bright background. Real contact between two surfaces is also quite easily felt.

**Spherometer.**—A plane surface may also be tested by the spherometer (q.v.).

**Whitworth Plane.**—By contact with a Whitworth true plane surface, which has been smeared with some red putty powder, and observing whether any portion has or has not taken an impression.

**Newton's Rings.**—The absence of rings between a known plane surface and the one tested is the most accurate method, which is described generally in Chap. XXVII. See also **Reflection** tests and **Telescopic** tests.

**Reflection Tests.**—A plane surface can be distinguished from a curved one by viewing the reflected image from a bright source of light. If a plane, it acts precisely as a plane mirror, while if a sph. or cyl., the image is altered in size or distorted. If the object viewed is a square, then a Cx. surface will cause it to appear compressed vertically, i.e. in the direction of view, so that it has the appearance of a horizontal rectangle, while a Cc. surface causes vertical extension, giving the appearance of a vertical rectangle. In every instance the lens should be held as close and oblique to the eye as possible.

As the lens is rotated, while still viewing the reflected image, there is no change in the appearance of the latter if the surface is sph. or plane, whereas if cyl. the image does change. If the object viewed be of some definite shape, say a vertical window bar, it is seen quite distinctly when the axis of the cyl. is in line with the direction of view, whereas it is indistinct when the axis is oblique to the plane containing the eye and the bar. The image is most indistinct when the axis is at right angles to the line of vision, the general image being drawn out if the surface is Cc., and compressed if Cx., as with sph. surfaces. This is an extremely delicate test for locating the axis of a cyl.

### Focal Length of Cc. Mirrors.

**Direct Focalisation.**—On the optical bench parallel light is obtained from the collimator *C* (Fig. 314), and passed through the perforated screen *S* on to the mirror *M* whose focal length is to be measured. The mirror is slightly tilted and moved to and fro until the image of the pinhole is thrown sharply on to the screen at *F*. The distance *M F* is the required focal length.

**Conjugate Focalisation.**—If the cross wires be substituted for the collimator such that a real *conjugate* image be formed on the screen *S*, we have $1/F = 1/f_1 + 1/f_2$, where $f_1$ is the distance of the cross wires, and $f_2$ is the conjugate distance *M S* of the screen, to the mirror.

**Symmetrical Planes.**—An especially rapid and accurate way to find *F* is to use the cross wires and the disc containing them as both object and screen. The mirror is advanced towards *S* until the image of the wires appears sharply on the surrounding disc, which must then be at the centre

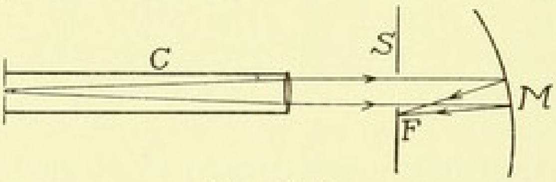


Fig. 314.

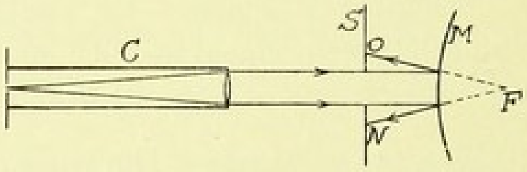


Fig. 315.

of curvature. The radius of curvature is thus directly measured, and equals 2 F, i.e., $F = r/2$.

**Parallax.**—If an object be placed within F, the *virtual* image can be located as described under convex mirrors, and the focal length found from the conjugates, care being taken to reckon the distance of the image as a negative quantity.

**The Spherometer.**—See this method for Cx. mirrors.

### Focal Length of Cx. Mirrors.

**Projection Method.**—Arrange a collimator and perforated screen (Fig. 315) as for a Cc. mirror, the screen being between *C* and *M*. On *S* describe a circle *O N* concentric with the central aperture and of twice the diameter of *C*, the collimator lens. The action of the mirror being divergent it will reflect the parallel beam as a cone apparently diverging from *F*. Move the mirror to and fro until the projected area of illumination on *S* exactly fills the circle *O N*. Then the distance of screen to *M* equals *F* of the mirror.

**Parallax Method.**—Take two rather stiff wires or knitting needles (Fig. 316) and place one $P_1$ represented by the arrow in front of *M* such that its virtual image is *I*, seen on looking into the mirror from the same side as $P_1$. Behind *M* place a second needle $P_2$ such that it approximately coincides with $I$ seen *in* the mirror. Now move the head from side to side, and if there is apparent separation between the virtual image $I$ of $P_1$ and the actual pin $P_2$ the latter must be moved towards or from the mirror until all parallax disappears. Then if $P_1 M$ be $f_1$, and $P_2 M$ be $f_2$, we have, since $f_2$ is a negative quantity, $1/F = 1/f_1 + (-1/f_2)$.

**Convergence towards C. of C.**—Set up the cross wire $D$ (Fig. 317) and in front of it place any convex lens $L$ so that the latter projects a real image at a distance $L C$ greater than the radius of the mirror; the distance $L C$ is measured. On interposing $M$ and moving it to and fro a position will be found where the image of the wires is received back on to the disc $D$.

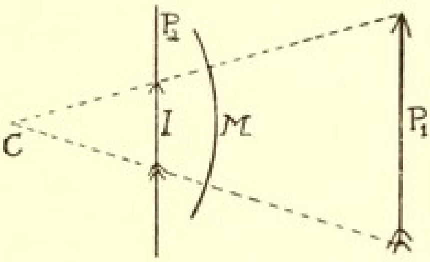



FIG. 316.

When such is the case the convergent light from $L$ must be incident on $M$ directed towards the centre of curvature $C$ because it has returned along its own path. Then the radius of the mirror is the distance $M C$, between the mirror and the real image formed by the lens, and $M C = L C - L M$. $M$ must be slightly tilted to throw the image to the one side of the disc containing the cross wires.

**Spherometer (q.v.).**—The radius of the reflecting surface of a Cx. (or Cc. glass mirror) can be found approximately with the spherometer, but the results are uncertain on account of the amalgam coating. If, however, it

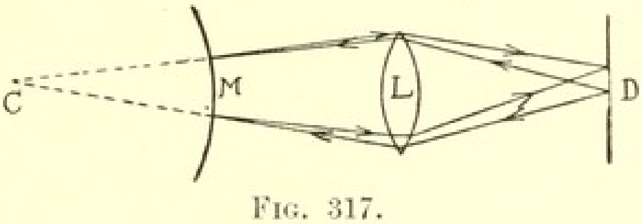



FIG. 317.

has truly parallel surfaces and is thin, the radius of the front surface may be taken to be that of the second or reflecting surface. The latter is slightly shorter in a Cx. (and longer in a Cc.) mirror than when the front surface is measured.

### Single Thin Convex Lenses.

**Direct Focalisation.**—The power of an unknown Cx. lens can be obtained by measuring the distance between the lens and its principal focus. Set up the collimator $C$ (Fig. 318), and in front of, and near to it, place the unknown lens $L$. On the other side of $L$ place the screen $S$, and move the latter to and fro until the image $F$ of the collimator aperture is sharpest possible; then *L F* is the principal focal length. If the lens is weak, or very strong, indirect focalisation (q.v.) is needed. Instead of the collimator, any distant bright source as a window or artificial light, can be employed. As F is short, the image is small, sharp, and bright, so that this method serves very well for fairly strong Cx. sphericals, but is uncertain for weak or very strong ones.

To focalise a periscopic Cx. lens the distance of the optical centre from the lens must be obtained. The distance from the lens to the screen should be taken first with the one face, and then with the other, turned towards the source of light. The mean of the two measured distances is the true focal length. With ordinary periscopic spectacle lenses, the distance of F, from the lens itself, is sufficiently exact in practice.

**Indirect Focalisation.**—If the lens is weak and therefore of long F the image on the screen is large and indistinct, and the exact principal focal distance is difficult to determine. If F is very short, the exact distance also becomes hard to determine with accuracy. For these we employ indirect focalisation. The procedure is to combine, with the *unknown* lens, another

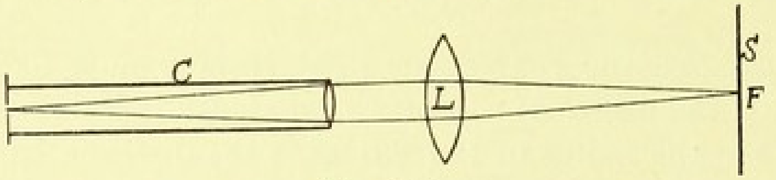



FIG. 318.

*known* lens; find the focal length *of the combination*, and then deduct from the *power* of the combination that of the unknown lens; thus

$$\frac{1}{F_2}=\frac{1}{F}-\frac{1}{F_1} \qquad \text{or} \qquad D_2=D-D_1$$

where F and D are, respectively, the focal length and the power of the two lenses combined, $F_1$ and $D_1$ those of the added lens, and $F_2$ and $D_2$ are those of the unknown lens. The approximate power to be added can be found experimentally, and it is better to divide this power between a pair of lenses, placing one on either side of the unknown lens.

A very strong Cx. lens should be combined with a Cc. lens of sufficient power to lengthen the focal distance to a reasonable extent. For instance, it is difficult to determine whether $F = 2$ in. or $2\frac{1}{8}$ in.; but if the lens be focalised with, say, a 3 in. Cc., the difference between the one and the other is much more marked, it being then about 1 in. Thus if $F = 9$ in. and $F_1 = -3$ in.

$$1/F_2=1/9-(-1/3)=4/9\text{; the lens is } 2\tfrac{1}{4}\text{ in. Cx.}$$

or

$$D_2=4{\cdot}5-(-13)=17{\cdot}5\text{, or say }+18.$$

If $F=6$ in. and $F_1=-3$ in., then

$$1/F_2=1/6-(-1/3)=3/6\text{; the lens is 2 in. Cx.}$$

or

$$D_2=6{\cdot}5-(-13)=19{\cdot}5\text{, or say }+20.$$

To focalise a weak Cx. lens a sufficiently strong Cx., say + 5 D., should be combined with it. Thus, if F = 6 in. and $F_1$ = 7 in. the unknown lens is 1/6 − 1/7 = 1/42 Cx. If D = 6, and $D_1$ = 5, then $D_2$ = 6 − 5 = + 1.

**Conjugate Focalisation.**—Using the cross wires *D* (Fig. 319) as an object, and placing the lens *L* at a reasonable distance from it, a real conjugate image may be formed on the screen *S*. If the distance of the object from the lens be $f_1$, and the distance of its image on the opposite side be $f_2$, then the power of the lens is

$$1/F = 1/f_1 + 1/f_2$$

Suppose $f_1$ be 10 inches from the lens, and $f_2$ at 15 inches; then

$$1/F = 1/10 + 1/15 = 1/6, \text{ or } F = 6 \text{ in.}$$

It is easier to convert each distance into diopters, and calculate by

$$D = d_1 + d_2$$

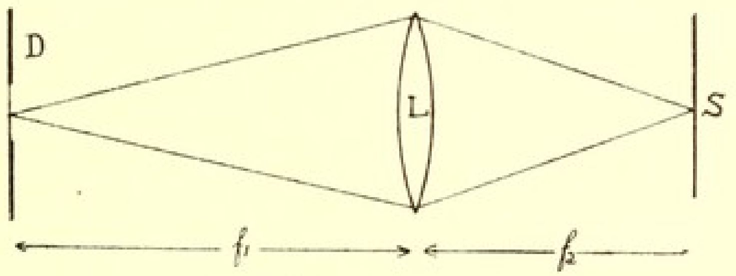



FIG. 319.

Thus, if $d_1$ is 25 cm. and $d_2$ is 20 cm., the lens is

$$100/25 + 100/20 = 4 + 5 = 9 \text{ D.}$$

This method only serves for fairly strong Cx. spherical lenses, but can be applied to weak Cx. lenses by adding another Cx., and calculating as shown in *indirect focalisation*.

**Symmetrical Planes.**—(Donders.)—The method of symmetrical planes is rapid and accurate, and depends on the principle that when image and object are identical in size, the distance of O and I from the lens is 2 F, and the total distance between them is four times the focal length of a thin lens. It is a special case of *conjugate focalisation*, and the cross wires constitute the object.

The lens is placed midway between *D* and *S*, which are moved equally towards or away from the lens until the image on the screen is sharp and of *equal size to the aperture of D.* The experiment is made more accurate if the screen is scaled, and equal movements of the two is facilitated if the carriers are connected by a band suitably arranged for moving them equally. If the lens is weak it should be placed between a *pair* of strong Cx. lenses, if very strong between a pair of Cc. lenses, in order to obtain the symmetrical conjugate foci. The calculation then required is the same as given in *indirect focalisation.* The distance between the symmetrical planes divided by 4 gives F of a thin periscopic Cx. lens.

### Single Thin Concave Lenses.

**Indirect Focalisation.**—The unknown Cc. is combined with a known stronger Cx. spherical, or, better, placed between a pair of Cx. lenses, in order to obtain a real focus. The calculation involved is the deduction of the unknown Cx. from the combined power measured.

Thus if $F = 25$ cm. and $F_1 = 10$ cm. we get for the Cc.

$$1/25 - 1/10 = -15/250 = -1/16, \text{ or } -6 \text{ D}.$$

If $F = 15''$ and $F_1 = 5''$, the Cc. is $1/15 - 1/5 = -1/7{\cdot}5$.

If $D = 4$ and $D_1 = 10$, the Cc. is $4 - 10 = -6$ D.

**Conjugate Focalisation** can be employed for a Cc. by combining it with a stronger Cx. Thus, if $f_1 = 10''$, $f_2 = 15''$, and $F_1 = 5''$, we find the Cc. to be of $30''$ F, for $1/10 + 1/15 = 1/6$ and $1/6 - 1/5 = -1/30$.

Or more easily by diopters; let $d_1 = 50$ cm., $d_2 = 20$ cm., and $D_1 = 10$. Then the Cc. is $100/50 + 100/20 = 2 + 5 = 7$, and $7 - 10 = -3$ D.

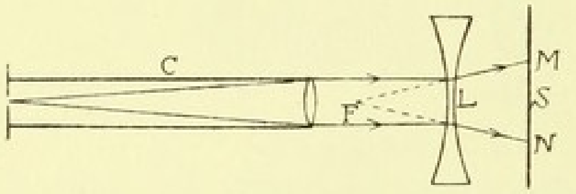


Fig. 320.

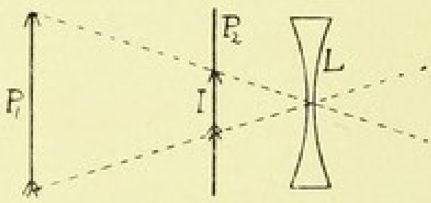


Fig. 321.

**Projection Method.**—This is similar to the projection method for Cx. mirrors.

A parallel beam from $C$ (Fig. 320) is allowed to fall on the unknown Cc. lens, and is diverged by the latter as if proceeding from $F$. If now $S$ be moved back until the luminous area exactly fills the marked circle $MN$—which is twice the diameter of $C$—then the focal length of the lens is equal to $LS$, the distance of lens to screen.

**Parallax Method.**—This is similar to the method for Cc. mirrors, with the exception that both object and image are on the *same* side of the lens, while the observer must be on the *opposite* side. A long pin $P_1$ (Fig. 321) is set up, and its virtual image $I$ is observed through the lens. A second *locating* longer pin $P_2$ is now taken and moved to and fro until, on moving the head, there is an absence of parallax between them, $P_2$, seen above the lens, apparently coinciding with $I$ seen through the lens. Then, if $P_1$ to $L$ be $f_1$, and $P_2$ to $L$ be $f_2$, the latter being a negative quantity,

$$1/F = 1/f_1 + (-1/f_2)$$

Locating the virtual image with a Cc. lens is more difficult and confusing than with a Cx. mirror because the observer sees two objects and two images. If, however, it be remembered that the *more distant image must be made to coincide with the nearer object pin*, no mistake can be made. The actual pin $P_1$ seen *above* the lens, and the image of $P_2$ seen *through* it, must be ignored.

**Reflection.**—The Cc. surfaces of a negative lens may be employed as positive mirrors for measuring their radii of curvature. The image is, however, rather faint, and therefore if the disc holding the cross wires is not already white, a piece of white paper should be stuck to one side of the aperture. The distance of lens to disc is equal to the radius of curvature, and as the focal length of a double Cc. crown lens is very nearly equal to the radius, the measured distance can be taken as the required focal length. If the index of the glass is exactly 1·5, then F is exactly equal to the radius, but should the index be known to be other than 1·5, F can be calculated from the lens formula. More properly each surface should be calculated separately since the lens may not be a double Cc. sph., the refractive F of a lens surface being approximately $= 2r$. If parallel light is employed, as it should be, we get F by reflection, and this equals *r/2 for each surface.* In this case F of each surface by refraction equals 4 F by reflection. For example, if with parallel light F is found to be 4″ for the one surface, and at 8″ for the other, the lens is $1/16 + 1/32 = 1/10$ nearly.

## Cylindrical and Sphero-Cylindrical Lenses.

**Cx. Cyls. and Sph.-Cyls.**—If a Cx. plano-cyl. be at its principal focal distance in front of a screen, parallel light from a point source refracted by it forms on the screen a bright line which corresponds to the direction of the axis of the cyl. By finding the distance at which the line is sharpest and brightest the focal length of the lens can be directly determined. The procedure is the same as for Cx. sph. lenses.

If a Cx. sph.-cyl. be held in front of a screen, parallel light, refracted by it, forms on the screen a line at the focal distance of the sph., and another at the focal distance of the united powers of the sph. and cyl.; the first is at right angles to the cyl. axis, and the latter corresponds to it. By finding these two lines, and measuring the distance between the lens and the screen for each, the focal length and powers of the two principal meridians of the lens can be learnt. Thus, suppose the two distances are 50 and 33 cm., then the combination is + 2 D and + 3 D, or + 2 S. ⊂⊃ + 1 C. If the focal distances are 10 and 8 inches the lens is 1/10 Cx. S. ⊂⊃ 1/40 Cx. C., since $1/8 - 1/10 = 1/40$.

**Conjugate focalisation** can also be employed, using, as before, a point source of light; when both powers are very weak or strong, or the one very weak compared with the other, *indirect focalisation* is indicated.

**Cc. Cyls. and Sph.-Cyls.**—With negative cyls. a Cx. sph. of sufficient power must be added to render the whole positive. The two principal powers are then calculated, and the added sph. deducted from each to give the powers of the unknown lens.

The *projection* method (q.v.) is applicable to Cc. cyls. and sph.-cyls., the size of the luminous disc being twice the diameter of the lens *in the two principal meridians.* The luminous disc is elliptical in shape, so that it suffices to measure its long and short axes.

The *reflection* method (q.v.) can also be employed, the principal meridians corresponding to the directions of the wires. The image of the one wire is seen parallel to the cyl. axis, and twice the measured radius equals F. The other surface, if sph., must be measured separately.

**Indirect Focalisation.**—The procedure is (*a*) find the two foci, of which the weaker power is the sph.; (*b*) deduct the weaker from the stronger, giving the cyl.; (*c*) deduct the added power from the sph. Or, alternatively, (*a*) find the two powers, (*b*) deduct the added power from each, so that (*c*) the weaker power is the sph. and the stronger less the weaker is the cyl. Examples of both procedures are given in the following.

**Examples.**—The added lens is 10″ Cx., and the first focal line $F_1$ is found to be at 7″, and the second $F_2$ at 8″. Then the cyl. is $1/7 - 1/8 = 1/56$, and the sph. is $1/8 - 1/10 = 1/40$, the lens being 1/40 Cx. S. ⊂⊃ 1/56 Cx. C. Or, by diopters, the cyl. is $5{\cdot}75 - 5 = +{\cdot}75$ D, the sph. is $5 - 4 = 1$ D, the lens being +1 S. ⊂⊃ +·75 C.

A +4 D is added to an unknown lens and the foci are found at 33 and 50 cm. The actual powers of the lens are therefore $100/33 - 4 = -1$ D, and $100/50 - 4 = -2$ D, which is equivalent to −1 S. ⊂⊃ −1 C. or any transposition of the same.

If the two focal distances at 15 and 33 cm. and the added lens = +5 D, the actual powers are $100/15 = 6{\cdot}5 - 5 = +1{\cdot}5$ and $100/33 = 3 - 5 = -2$, the lens being +1·5 C. ⊂⊃ −2 C. or a sph.-cyl. possessing similar powers.

If the two foci are at 10 and 6 in., when the added lens is 8″ Cx., the combination is 1/10 Cx. S. ⊂⊃ $(1/6 - 1/10) = 1/15$ Cx. C. and $1/10 - 1/8 = -1/40$. The lens is −1/40 S. ⊂⊃ +1/15 C.

**Telescope Tests.**—More accurate results can be obtained with lenses if the telescope be employed in their focalisation. This is really the reverse of the usual procedure, as will be seen from Fig. 322. The collimator *C* is reversed, so that its lens faces the lamp and the pinhole *P* is away from it. The telescope is adjusted for parallel light by pulling the eye-piece well out, and gradually pushing it in, until some *distant* object is seen sharply through it; the eye-piece is then fixed and the telescope *T* replaced on the bench. The lens to be measured is placed in a clip between *P* and *T* and moved to and fro until the image of *P* is seen sharply through *T*. Then the distance *L P*, from pinhole to lens, will be the focal length of the lens, since only parallel light can have emerged from *L* to enter the telescope and give rise to a sharp image therein. With a cyl. the image will be a line; with a sph.-cyl. there will be two line images at different distances. As in the other tests, a known sph. must be added, if the unknown lens is too strong, too weak, is negative, or the difference between the principal powers insufficiently marked to give accurate results. The smaller the pinhole used in this experiment the sharper will be the lines obtained.

For plane surfaces the telescope is adjusted for infinity. A beam of light rendered parallel by a collimator is allowed to fall obliquely on the surface to be tested and is, after reflection, received in the telescope.

If now, on looking through the telescope, the image seen of the source of light is sharp, the surface is a plane. If the surface is Cx., the eye-piece of

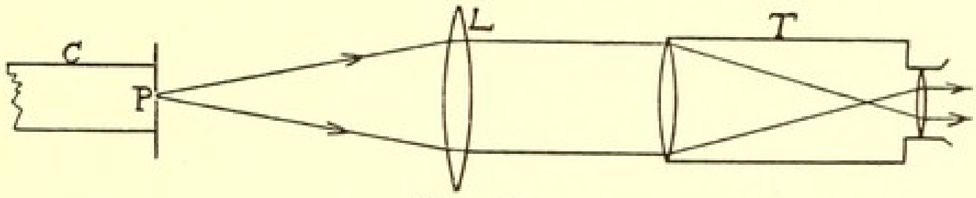



FIG. 322.

the telescope must be pulled out, and if Cc., pushed in, in order to get a sharp image. If the surface is irregular, a sharp image cannot be obtained at any spot. The presence of *astigmatism*, whereby one portion of the image is better defined than the other, is the surest proof of convexity or concavity of a surface.

### Thick Lenses and Combinations of Thin or Thick Lenses.

**Symmetrical Plane Method for a Positive Combination.**—To find experimentally the equivalent focal length of a thick Cx. lens or combination,

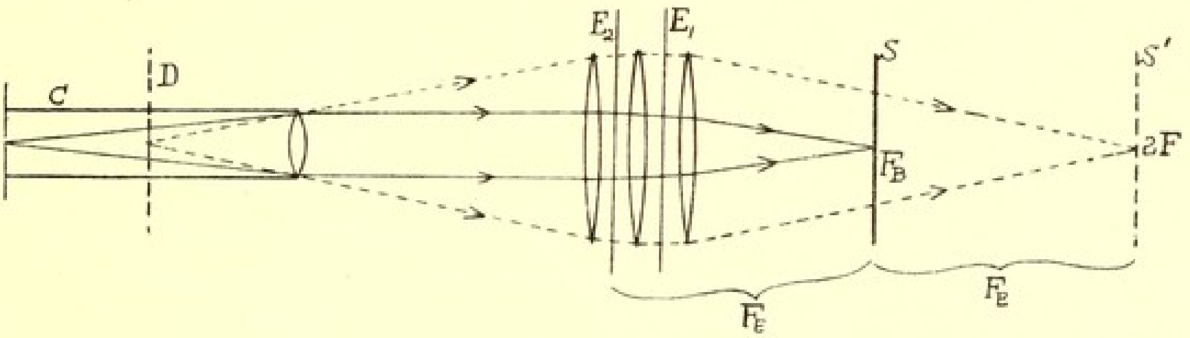



FIG. 323.

it is necessary to locate the equivalent planes, since the focal distances are the distances between these planes and the principal foci.

Let the system of lenses be suitably mounted (Fig. 323). Parallel light from the collimator $C$ is refracted by it, and the principal focus $F_B$ is formed on the screen $S$, *whose position on the bench is noted.* Now substitute the cross wires $D$ for the collimator and move them about until the image formed on $S$, drawn back to $S'$, is the same size. Then $S'$ is the second symmetrical plane, and is therefore at $2F$ *from some plane—the 2nd equivalent plane—not yet located.* But the distance between $2F$ and $F_B$, i.e. *the difference in* *the bench readings of the position of $S'$ and $S$* is the equivalent focal length. Therefore measuring from $F_B$ towards the lens a distance equal to $F$, the second equivalent plane $E_2$ is located. If the combination be turned round, and the process repeated, $F$ will, of course, be found to have the same value, and the first equivalent plane $E_1$ can be located. In some combinations $E_1$ and $E_2$ will be found to be crossed, as illustrated in Fig. 323.

**Rotation—Positive Combination.**—This is, perhaps, the quickest and most accurate method of all for finding the equivalent focal length and equivalent points of a combination, especially a fixed system such as a photographic objective.

Since the secondary axes govern the position and size of the image, and since they all pass out of the system through the second equivalent point, it is obvious that if the combination be rotated horizontally around a vertical axis immediately beneath $E_2$ *the image from originally parallel light will remain stationary*. If the system be rotated around any point other than $E_2$, the image will move. Allow the light from the collimator (Fig. 324) to fall on the lenses so that the screen $S$ locates $F_B$. The combination is mounted in

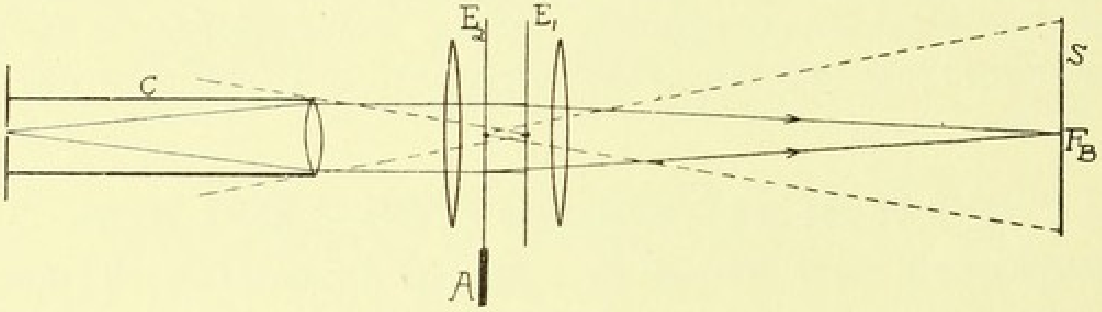



FIG. 324.

a special carrier capable of longitudinal adjustment from and towards $S$, and also rotation round the vertical axis $A$. Then, by a combination of lateral swing and longitudinal movement of the lenses a position can be found where the image on $S$ is motionless. Finally adjust $S$ to secure the sharpest possible image; then the distance from $A$ to $S$ on the bench is the equivalent focal length, and the prolongation of $A$ upwards locates the second equivalent plane $E_2$. By reversing the combination in the carrier $E_1$ can be similarly found.

This method is especially easy with photographic objectives on account of their wide angle of sharp definition; with uncorrected lenses, however, only a small rotation will be found possible before the image rapidly becomes confused from oblique aberration.

**Conjugate Foci Method for a Positive Combination.**—Since $F^2 = A\,B$, where $A$ and $B$ are the distance of O and I beyond F, respectively, on the one and the other side of the lens system, this enables the focal length to be experimentally determined. Thus focus parallel light on the screen, and mark $F_2$ (Fig. 325); repeat the process on the other side and similarly mark

$F_1$. Then place the cross wires at a convenient distance $f_1$ so that its image is, say, $f_2$; measure $F_2 f_2 = B$, also the distance $f_1 F_1 = A$, and calculate from $F = \sqrt{AB}$.

**The Gauss Method for a Positive Combination.**—Let $u$ be the angle subtended at the lens by any two distant objects (Fig. 326) $A$ and $B$, one of which $B$ is situated on the principal axis. This angle can be measured by means of a theodolite, and therefore the angle $u'$ subtended by the image

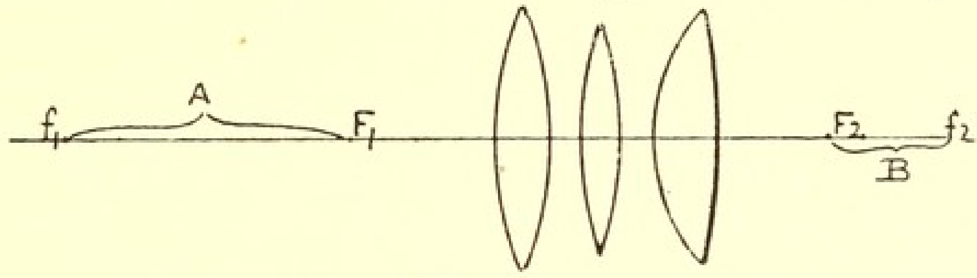



Fig. 325.

$B'A'$ at the second equivalent point $C$ is also known, since it is equal to $u$. Then

$$\tan u' = h'/CB' \qquad \text{or} \qquad CB' = F = h'/\tan u'$$

The image $h'$ can be directly measured on the screen. Since this method is independent of the position of the equivalent planes, these are not shown in the figure, $C$ being the 2nd equivalent point. If $u = 45°$ (Fig. 327), then since $\tan 45 = 1$, $F = h$, i.e., the size of the image $B'A'$ is equal to the focal length of the lens.

**Thin Lens Method for Cx. System.**—If a single thin lens is found which gives on a screen an image equal in size to that formed by a combination,

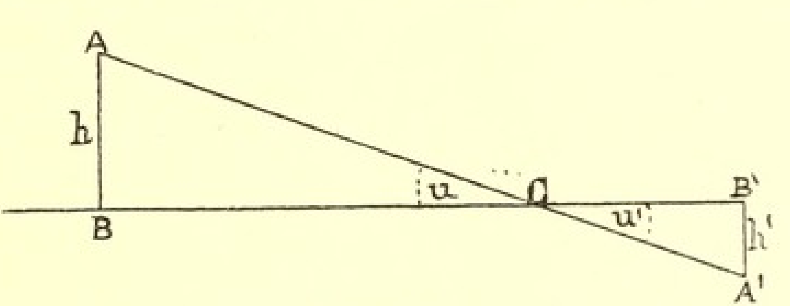



Fig. 326.

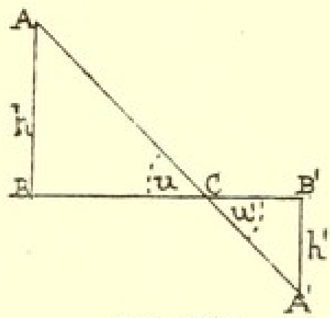



Fig. 327.

the focal distance of the former is that of the latter; also the place at which the single lens is situated determines the second equivalent point of the combination. If the latter is turned so that the original back lens faces the light, the spot at which the single thin lens must be placed in order to give a similar image to that of the combination, fixes the position of the first equivalent point.

**L. Laurance's Method for a Positive Combination.**—Focus sharply for parallel light to locate the principal focus $F_B$; then move the screen back to $f_2$ (Fig. 328) which is $n$ inches from $F$ (say 1/3 of its focal length). Move the cross wires in front of the lens until its image is sharply focussed on the screen at $f_2$ and mark its position $P$. Again withdraw the screen to $f_2'$, which is exactly one (or more) inches further back, so that it is now $n'$ inches from $F$; shift the wires to $P'$ until the image is once more in focus at $f_2'$. Measure the distance $P P'$ through which the object has been moved; call it $d$. Then

$$F = \sqrt{\frac{d\,n\,n'}{n' - n}}$$

If $n'$ be exactly 1 in. longer than $n$, then $n' - n = 1$, and therefore need not be regarded. Further if $n = 1$ and $n' = 2$, the calculation simplifies to $F = \sqrt{2d}$. This is the true focal length, since it is independent of the position of the equivalent planes, which can be found by measuring the focal distance backwards from the principal focus. Thus supposing $d = 3{\cdot}5''$, $F = \sqrt{2 \times 3{\cdot}5} = 2{\cdot}65''$ approx.

**Rotation—Negative Combination.**—The rotation method also serves for a negative combination, but in this case the *virtual* image formed of originally

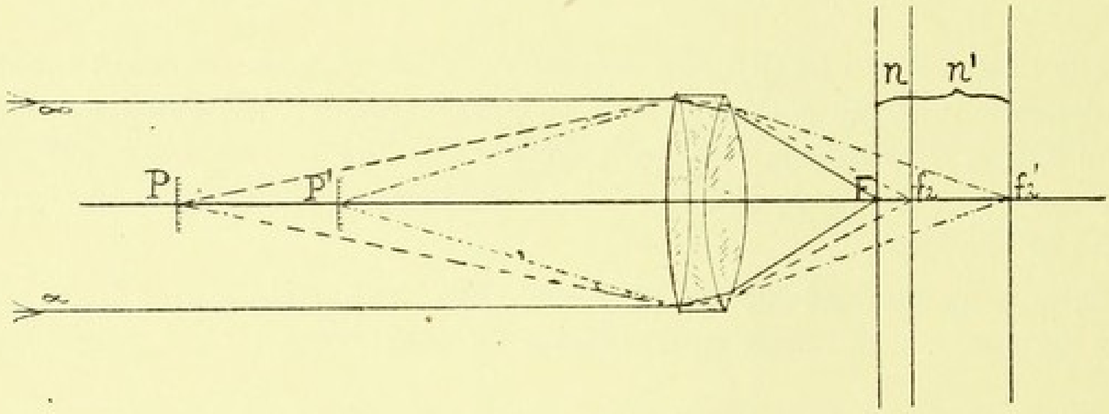



FIG. 328.

parallel light must be observed; to do this the combination must be placed between the telescope and the collimator. Focus carefully on the virtual image formed by the lens by drawing out the eye-piece, and get the image on the vertical cross wire of the telescope. Rotate the combination as described for a positive combination until the image seen through the telescope is stationary and sharp. Remove the combination from the carrier and bring up some object until its image is also seen clearly in the telescope. Then the distance of this object to the standard which originally held the Cc. system will be the focal length of the latter.

**J. R. Dallmeyer's Method for a Negative Combination.**—Take an achromatic positive lens and focus the image of the cross wires on a screen; measure the size of the image formed and let it be $m$ (Fig. 329). Place the negative lens, whose focus is to be found, a short distance within the convergent beam of the positive lens, i.e., between it and the screen. Focus the image formed by the combination and measure its distance $D$ from the back surface or flange of the negative lens; measure the size $m_1$ of the image formed. The size of $m_1$ compared with the size of the image produced by the positive lens alone is $M = m_1/m$.

Now move the negative a little nearer the positive lens (which latter must be kept in a fixed position) and focus a second time on the screen; measure the distance $D'$ of the screen from the back of the negative lens or its flange. The size of the image $m_2$ compared with the size of $m$ is $M' = m_2/m$. Then the focal length F of the negative lens is

$$F = \frac{D' - D}{M' - M}$$

This equation is independent of the position of the equivalent planes, and therefore will hold true for any negative combination of lenses.

**The Radius of Deep Curves and the Focal Length of Very Strong Lenses.**

**Curved Surfaces—Reflection.**—If the object be sufficiently distant compared with that of the image, as is the case with mirrors of small radius,

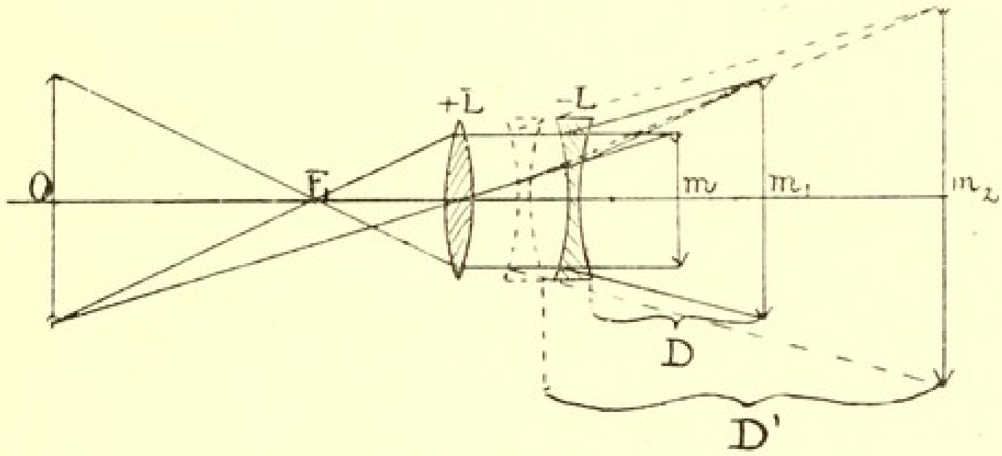



FIG. 329.

when the object is, say, a metre distant, then the radius $r$ of the curved surface bears to the distance of the image from the pole of the mirror, the relationship of $r = 2F$, where F is the focal distance and the distance of the image. Let $h_1$ and $h_2$ be the sizes of, respectively, the object and the image, and $f_1$ the distance of the object from the mirror, while $f_2$ is its focal length. Then

$$f_2 = f_1 h_2/h_1, \text{ and } r = 2f_2$$

The radius of curvature, of strongly curved lenses and mirrors, whether Cx. or Cc., can be measured by employing an instrument like the ophthalmometer. The distance between the two objects being known, that between the two images can be measured by a micrometer scale placed in the focus of the eye-piece of the telescope. $f_1$ is the distance of the objects from the curved surface, $h_1$ is the distance between them, $h_2$ is here the distance between the two images, as seen in the micrometer, and F is the distance between the objective and the micrometer. The relative size of the image formed at $f_2$ and that formed at the micrometer is as $f_1 : \mathrm{F}$, so that the above formula must be multiplied by $f_1/\mathrm{F}$, and we then obtain

$$r = \frac{2f_1^2 h_2}{h_1 \mathrm{F}}$$

**Curved Surfaces—Gauges.**—The radius of small convex lenses is also determined by accurately made gauges, or more generally by glass cups of known curvature, usually known as test-plates. When the curvature of the lens to be gauged does not correspond to that of the cup, interference rings are exhibited, while these are not shown if the two curves exactly correspond; or they are faint, and of slight brilliancy of colour, if the curves nearly correspond. A total absence of colour is, however, in practice, rarely found.

**Curved Surfaces—Dr. C. V. Drysdale's Method.**—Dr. Drysdale explains a method of determining the radius of curvature of small surfaces as follows:

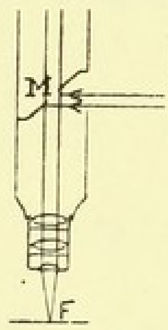


FIG. 330.

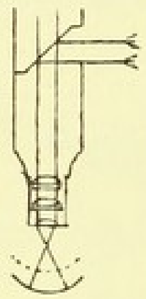


FIG. 331.

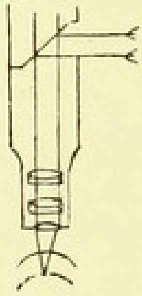


FIG. 332.

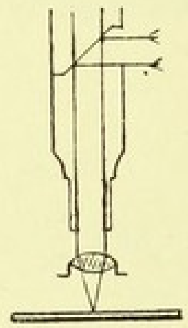


FIG. 333.

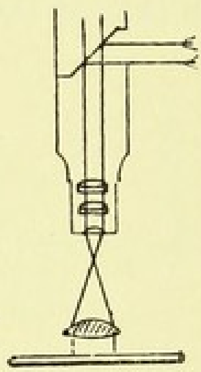


FIG. 334.

A microscope has a portion removed from the tube so that light, from a distant source placed at the side, enters the aperture and falls on a transparent reflecting surface *M* inclined at 45°, so that part of the light is transmitted down the tube towards, and through, the objective, by which it is brought to a focus at *F* as in Fig. 330. If, then, the reflecting surface of a mirror or lens is placed at the focus of the objective, the light is reflected back and seen by the observer, in the field of the eye-piece, as an image of the source. This position or distance of the objective from the reflecting surface is then marked on some part of the microscope. The tube of the latter must be racked upwards, if the surface examined is Cc. (Fig. 331) or downwards if Cx. (Fig. 332), until the image can again be clearly seen. The focus of the objective now coincides with the centre of curvature of the reflecting surface, for the light passing through the objective is incident on the reflecting surface normally and is reflected back along its original course. The distance between the first and second positions of the microscope objective, when the image is clearly seen, is the radius of curvature. The eye-piece is arranged for parallel light by separation of the components, the adjustment being made by turning the reflector so that the light admitted is reflected towards the eye-piece. The curvature of any zone of the surface can be obtained by using a suitable diaphragm.

A later improvement made by Dr. Drysdale on the arrangement of the instrument used in the above method consists of an illuminator immediately above the microscope objective and a lens above the illuminator, which serves as the objective of the telescope and obviates the necessity of separating the eye-piece lenses.

**Focal Length—Dr. C. V. Drysdale's Method.**—By a similar use of the microscope the focal length of small lenses can be found. Employing no objective in the microscope and a plane mirror behind the lens to be tested, this mirror is moved to and fro until the image is sharp in the field of the eye-piece. The mirror is then at the focal length of the lens, the light converged by the latter being reflected back and refracted again as parallel. The lower focal point is thus found, as in Fig. 333.

Replacing the objective (Fig. 334), the lens is moved further back to such a position that it is at its focal length behind the focal point of the objective. Then the light converged by the objective and refracted by the lens is parallel, and falling on the mirror, is again reflected as parallel, to be refracted by the lens to meet at the focal point of the objective, by which it is again refracted as parallel light. The image is sharp in the field of the eye-piece, and the upper focal point is found as in Fig. 334.

The two focal points being marked, the back surface focal lengths are obtained. If, now, the mirror be moved a given distance A downwards, and the objective moved upwards by a distance B until the image is clear, we obtain the equivalent focal length from $F_E = \sqrt{A\,B}$, where A and B are the distances of the conjugates beyond $F_B$ on each side.

Dr. Drysdale has also made an experimental microscope in which the lens under examination can be oscillated around its second equivalent point. This enables the focal length to be determined, and further, by this means, aberrations can be easily detected.

# CHAPTER XXIX

## PRACTICAL SUBJECTS AND CALCULATIONS

**The Vernier.**—*V* (Fig. 335) is an attachment to instruments where great precision of linear or angular measurement is required, and it obviates the necessity of the division of the main scale into very minute parts. It consists of a short scale which slides along the main scale *S* to which it is attached.

The *V* is the same length as a definite number of divisions of *S*, but contains one division more. Thus, if *V* is divided into 10 parts, these equal nine divisions of *S*, or if *V* has 30 divisions they correspond to 29 of *S*. Thus each division of *V* is smaller than each division of *S* by a fraction whose denominator is the number of divisions of *V*, viz., 1/10th or 1/30th, respectively, in the examples just quoted. The greater the number of divisions of *V* the more accurate are the readings, but also the more difficult is its use.

The scale itself may be divided into whole terms of measurement, as

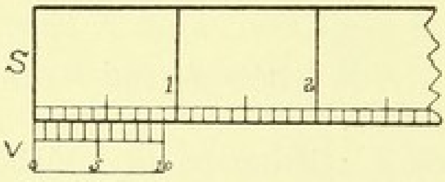


Fig. 335.

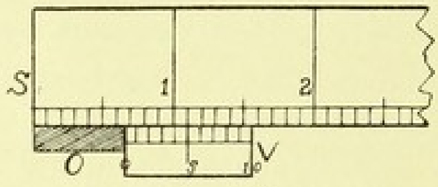


Fig. 336.

mm. or degrees, or more commonly into main fractions of such terms as $\frac{1}{2}$ mm. or $\frac{1}{2}$ degrees. Such whole terms, or main fractions thereof, are read from the *S* itself, the measurement being the last beyond which the zero of the *V* has passed. The minute measurement is obtained from the *V* by finding *that division mark of the V corresponding to, or in exact line with, a division mark of S*. Thus, if 10 $V = 9$ *S*, and the third division mark of *V* is in line with a division of *S*, the exact measurement is 3/10 more than the whole number indicated by *S* itself. If *V* has 60 parts and the 33rd is in line with an *S* division, the fractional reading is $\frac{33}{60}$ plus the whole division indicated on *S*.

Fig. 336 illustrates a reading on a scale *S* directly divided to inches and tenths of inches with a vernier *V* whose 10 divisions = 9 of the scale. The length of an object *O* whose one extremity is at zero of *S* is ·65 in., the 5th division of *V* coinciding with a division of the scale. The ·60 in. is read from the scale itself, where the right-hand extremity of $O$ lies between the 6th and 7th division of $S$; the balance ·05 in. is read from the $V$. The limit of accuracy is $\frac{1}{100}$ in.

As another example, let the scale be divided to inches and tenths of inches, and let 25 $V$ = 24 $S$. If the zero of $V$ showed 5 in. and six spaces of $\frac{1}{10}$ in., plus a certain distance when the fourth division of $V$ is in line with a scale mark, the total measurement would be $5 + \frac{6}{10} + \frac{4}{250}$ or 5·616 in. The accuracy of the reading is carried to $\frac{1}{250}$ in.

Instruments have been made with the $V$ divisions longer than those of $S$, so that, say, 9 $V$ = 10 $S$. The $V$ divisions are then on the near side of the zero, and are read backwards. Verniers for fine straight rules are usually made so that 10 $V$ = 9 $S$, thus measuring to $\frac{1}{10}$ mm. For box sextants and small surveying instruments 30 $V$ = 29 $S$, so that $\frac{1}{2}°$ divisions are subdivided to minutes. For barometers the readings are usually taken to $\frac{1}{10}$ mm. when 10 $V$ = 9 $S$, or to $\frac{1}{250}$ in. when 25 $V$ = 24 $S$. For marine sextants and theodolites 60 $V$ = 59 $S$, measurements being taken to 1/10 of 20′ or of 10′, giving limits of accuracy of, respectively, 20″ or 10″ in the case of these two instruments.

**The Four Leg Spherometer.**—This is an instrument for ascertaining the radius of curvature of a spherical surface. The most accurate form consists of three legs arranged around a common centre, so that their points describe an equilateral triangle, a fourth leg in the centre moving up and down, by means of a fine screw, the head of which supports a round horizontal plate. The latter has its edge almost touching a vertical scale divided into mm. or ·5 mm. as the case may be, and the plate itself is usually divided into 100 parts. The elevation or depression, therefore, of the central leg, from the plane of the other three, can be read with considerable accuracy. Generally the pitch of the thread is so arranged that two complete revolutions of the plate lowers or raises the central leg 1 mm., and as the plate itself is divided in 100 parts, the elevation or depression of the leg can be read to an accuracy of ·005 mm.

Now if two chords of a circle intersect at right angles the product of their respective parts are equal. Thus in Fig. 337 $A\,B$ and $C\,D$ are at right angles, and the line $A\,B$ is divided into two equal parts $d$ and $d$, so that

$$\mathrm{S} \times a = d \times d = d^2$$

But $$a = 2r - \mathrm{S}, \text{ so that } d^2 = \mathrm{S}\,(2r - \mathrm{S})$$

Whence $$r = \frac{d^2 + \mathrm{S}^2}{2\,\mathrm{S}}$$

Now $S$, the *sagitta*, generally referred to as the sag of the curve for any particular chord $A\,B$, is measured by the central leg of the spherometer, and $d$ by the distance between the central leg and an outside leg. Therefore $r$, the radius of curvature, is found from the above formula.

Fig. 338 shows a plan of the instrument, $C$ being the central leg, and $X$, $Y$ and $Z$ the three fixed legs; the distances $C\,X$, $C\,Y$, $C\,Z = d$, and the angles $X\,C\,E$ and $Z\,C\,E$ are each 60°. Let $E$ be the distance between any two of the fixed legs, say $X$ and $Z$; then

$$\mathrm{E}/2 = d \sin 60°, \text{ or } \mathrm{E} = \frac{2d\sqrt{3}}{2} = d\sqrt{3}$$

whence $$d = \mathrm{E}/\sqrt{3}$$

Substituting in the previous formula the value of $d$ in terms of E we get

$$r = \frac{(\mathrm{E}/\sqrt{3})^2 + \mathrm{S}^2}{2\,\mathrm{S}} = \frac{\mathrm{E}^2/3 + \mathrm{S}^2}{2\,\mathrm{S}}$$

that is $$r = \frac{\mathrm{E}^2 + 3\,\mathrm{S}^2}{6\,\mathrm{S}}$$

This formula is applied when, instead of $d$, between a fixed and the central leg, being taken, the distance E between two adjacent fixed legs is measured.

When the sagitta S is very small compared with $r$ (as is nearly always

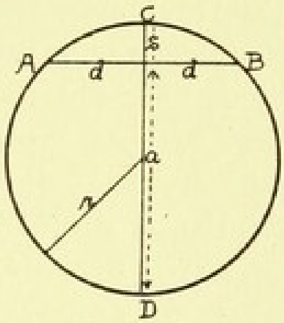


FIG. 337.

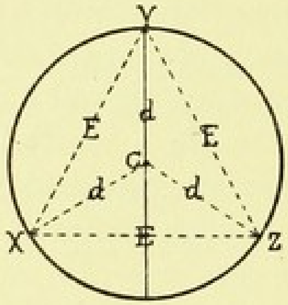


FIG. 338.

the case), the quantity involving $\mathrm{S}^2$ in the formulæ may be neglected, and they become respectively

$$r = d^2/2\mathrm{S} \quad \text{and} \quad \mathrm{E}^2/6\mathrm{S}$$

As an example of the application of the spherometer in calculating the powers of surfaces, suppose the distance between the movable and a fixed leg be 24 mm., and all four legs are brought into contact with a Cx. surface when the central leg is elevated 2·5 mm. Then, using the simplified formula,

$$r = \frac{24^2}{2 \times 2{\cdot}5} = \frac{576}{5} = 115 \text{ mm. (approx.)}$$

Now the *focal length* of the surface of a thin lens bounded by air is given by the formula $\mathrm{F} = r/(\mu - 1)$, or the dioptral power D by $1000\ (\mu - 1)/r$. Therefore, supposing $\mu$ to be 1·6,

$$\mathrm{D} = 1000\,(1{\cdot}6 - 1)/115 = 5{\cdot}25 \text{ D (approx.)}$$

**The Lens Measure** used in the optical trade, is a mechanical instrument, based on the construction of the spherometer. Projecting from the top of a small watch-like case are three metal pins, the centre one of which projects beyond the other two, and is moveable. This latter acts on a spring connected with a pointer which indicates on a dial the dioptral number, or focal length, of the lens. The dial is graduated from known curves whose powers are calculated on an index of 1·52, this being the average $\mu$ of the glass used in ophthalmic lenses.

When the surface of a lens is pressed on the pins, until arrested by the two side ones, the central pin becomes depressed, and causes the pointer to revolve and indicate the power of the lens (as represented by its curvature) in diopters. Care must be taken that the plane of the lens is at right angles to the plane of the pins; it is also important to see that the pointer indicates zero when a plane glass is applied to the instrument. The surface is sph. if, on rotating the lens, while pressed against the pins, the index remains stationary, and it is a plano if zero is then indicated by the pointer. If the index moves to different positions, when the lens is rotated, it indicates a cyl. or toroidal surface, the maximum power being shown by the highest number attained. The axis of a cyl. is indicated when the index points to zero, while the base curve of a toric is indicated by the lowest power registered. The maximum curvature of a cyl., and the highest and lowest curvatures of a toric, are, of course, spherical; the intermediate curvatures, although elliptical, are indicated as if they were spherical, but in all cases the power shown by the lens measure in an intermediate meridian is the same as that obtained by calculation, viz. $D \cos^2 a$, where $a$ is the angular distance between the meridian of greatest power D and the meridian measured.

If the lens be a sph.-cyl., cross-cyl. or toric, the power of each surface is distinct from the other. But when both surfaces are sph., the power of the one must be added to that of the other to obtain the dioptral number of the lens; thus with − 3 D on each surface, the lens is − 6 D. If the one surface is + 2·75 and the other − 1, the lens is + 1·75 D sph.

Should, however, the lens measure be used on a lens not having an index for which it is graduated, the powers registered will naturally be wrong. This can be rectified, provided the lens index is known, in the following way. Let $D_2$ be the true power of the lens, and $D_1$ that given by the lens measure, which is scaled for an index of $\mu_1$; let $\mu_2$ be the index of the lens. Then

$$D_2 = D_1 \frac{(\mu_2 - 1)}{(\mu_1 - 1)}$$

Or, knowing that the measure is gauged for $\mu = 1·52$, we can write

$$D_2 = \frac{D_1(\mu_2 - 1)}{·52}$$

Thus supposing the reading is 5 D, and the lens index to be 1·6; $D_2$, the true power of the surface measured, is

$$D_2 = \frac{5 \times \cdot 6}{\cdot 52} = 5 \cdot 75 \text{ (approx.)}$$

**Surfacing tools** or discs are those employed for grinding the curvature of lenses; they must, of necessity, be gauged for some given refractive index, $\mu_1$, usually 1·52. Now if used on glass whose index $\mu_2$ is higher or lower than this, the lens produced would be respectively stronger or weaker than the indicated power. Let $D_1$ be the power of the surface produced if the index is $\mu_1$, and $D_2$ that produced if the index is $\mu_2$; then, as above,

$$D_2 = D_1 \frac{(\mu_2 - 1)}{(\mu_1 - 1)}$$

The dioptral tool $D_1$ that should be employed to produce a given surface power $D_2$, when the glass is of $\mu_2$ and the tool is gauged for $\mu_1$, is found from

$$D_1 = D_2 \frac{(\mu_1 - 1)}{(\mu_2 - 1)}$$

Thus suppose the tools are made for glass of $\mu_1 = 1 \cdot 52$, and a lens of 10 D has to be made of glass of $\mu_2 = 1 \cdot 54$, we should employ a tool of

$$10 \times \cdot 52 / \cdot 54 = 9 \cdot 5 \text{ D}.$$

If focal lengths are indicated we have

$$F_1(\mu_1 - 1) = F_2(\mu_2 - 1)$$

**Blank Discs for Lenses.**—When a lens of certain power and diameter is required to be worked, it is essential that some idea of the thickness of the necessary blank be obtained in order to avoid undue labour in grinding it down if too thick, or failure to obtain the necessary finished lens if the blank is too thin.

The spherometer formula affords a ready means of calculating the minimum thickness required. S, the sag in the original formula, we can now call $t$ the thickness of the disc, and in place of $d$ we can write $c/2$ which is half the diameter of the lens; $r$ is the radius, and F and D have the usual significance. Therefore

$$t = (c/2)^2 / 2r$$

This formula gives the *minimum* thickness, and about 1 mm. must be added for the bevel of a Cx., or the central thickness of a Cc. lens; $t$ varies directly with the size of the lens, and inversely with $r$. If the radii of the surfaces are given, each is calculated for separately and the two quantities found added together. With sufficient accuracy the radius of a plano Cx. or Cc. lens is half the focal length, and that of a double Cx. or Cc. is equal to the focal length. If, say, a lens of 10 Cm. F were needed in plano Cx. form, $r$ = 5 Cm.; if the lens were double Cx. each $r$ = 10 cm.; therefore if we consider F instead of $r$, the thickness is the same for both forms of lenses. Then, all measurements being in mm., we have

$$t = (c/2)^2/2r = c^2/8r = c^2/4\,F = c^2\,D/4000$$

These formulæ serve for all lenses no matter how the powers are distributed *provided both surfaces are Cx. or both Cc.* For periscopics the surface of greater power only need be reckoned for, but a slightly greater added thickness than 1 mm. is then desirable. The long diameter of an oval lens must be taken for $c$.

**Examples.**—Required a + 20 D. double Cx. lens of 37·5 mm. diameter, that is, ordinary test case size. Then

$$t = \frac{37{\cdot}5^2}{4 \times 50} \quad \text{or} \quad \frac{37{\cdot}5^2 \times 20}{4000} = 6 \text{ mm.}$$

that is 6 + 1 = 7 mm. total thickness of disc.

Suppose a glass without power be required of a radius of 20 cm. and diameter 30 mm.; then, the lens being periscopic,

$$t = \frac{30^2}{200 \times 8} = {\cdot}56 \text{ mm. or say a total of 2 mm.}$$

For ordinary spectacle lenses the approximate thicknesses are

| | | | |
|---|---|---|---|
| 1 eye ... | 1 + 300/F, or 1 + ·3 D. | 0 eye ... | 1 + 350/F, or 1 + ·35 D. |
| 00 eye ... | 1 + 400/F, or 1 + ·4 D. | 000 eye ... | 1 + 450/F, or 1 + ·45 D. |

**Lens Sizes.**—American standard eyes, with their axes, and length of wire needed to make a standard eye wire in mm., are given in the following:

| No. | Axes. | Wire. | No. | Axes. | Wire. | No. | Axes. | Wire. |
|---|---|---|---|---|---|---|---|---|
| 4 | 33·8 × 24·5 | 93·5 | 1 | 36·5 × 27·5 | 103·5 | 000 | 40·9 × 31·9 | 117·5 |
| 3 | 34 × 26 | 95·9 | 0 | 37·8 × 28·8 | 107·5 | 000½ | 42·5 × 33·5 | 122·3 |
| 2 | 35 × 25·5 | 98·6 | 00 | 39·7 × 30·7 | 113·8 | 0000 | 44·3 × 36 | 128·2 |
| | Jumbo ... | | ... | 46 × 38 ... | | ... 134·3 | | |

OPTICAL SOCIETY STANDARDS.

| No. | Length of Periphery. | Corre-sponding American No. | Long Diameters. | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | Oval. | Long Oval. | Round Oval. | Pantos. | ½ Oval. | Round. |
| 1 | 92·5 mm. | 4 | 33·5 | 35 | 31 | 34 | 36 | 29·5 |
| 2 | 94·5 ,, = 92·5 + 2 | 3 | 34 | 35·5 | 31·5 | 34·5 | 36·5 | 30 |
| 3 | 97·5 ,, = 94·5 + 3 | 2 | 35 | 36·5 | 32·5 | 35·5 | 37·5 | 31 |
| 4 | 101·5 ,, = 97·5 + 4 | 1 | 36·5 | 38 | 34 | 37 | — | 32·5 |
| 5 | 106·5 ,, = 101·5 + 5 | 0 | 38 | 39·5 | 35·5 | 38·5 | — | 34 |
| 6 | 112·5 ,, = 106·5 + 6 | 00 | 40 | 41·5 | 37·5 | 40·5 | — | 36 |

This numeration, on the same basis of measurement, applies to all shapes of eyes for spectacles and eye-glasses, the ratio of the long to the short axis of the *oval lens* being approximately 1·3 to 1, and that of the *long oval* 1·5 to 1.

**Power of Cement Bifocals.**—The power of the segment or wafer in a cement bi-focal is that which, added to the main lens, gives the power required for reading. The index of refraction of the Canada balsam, by means of which the wafer is joined to the main lens, is practically the same as that of the glass, so that it need not be considered.

The *free* surface of the segment must be the total reading power less the power of the free surface of the main lens. The power of the contact surface of the segment must be equal to that of the contact surface of the main lens, but of opposite nature, so that no power results from them. Suppose the two powers be + 2 for distance and + 3 for reading (Fig. 339). If the main lens is double Cx. with + 1 on each surface, the segment would need to be − 1 on the contact and + 2 on free surface. If for the same powers the main lens is periscopic Cx., the two surfaces would probably be + 3·25

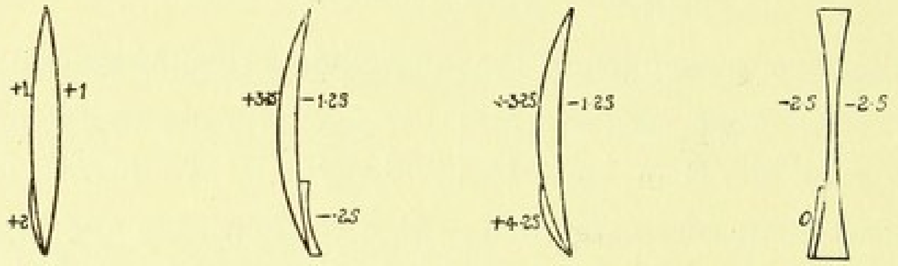



FIG. 339. FIG. 340. FIG. 341. FIG. 342.

and − 1·25 (Fig. 340). The wafer would then be + 1·25 on the contact surface, and − ·25 on the other, the wafer being placed on the Cc. side of the main lens. If placed on the Cx. side, the contact surface of the wafer must be − 3·25, and the free surface + 4·25 D (Fig. 341).

If the main lens is − 5 D Cc. and the reading power is − 2·5, then the segment requires to be a + 2·5 on the contact surface and plano on the other (Fig. 342). If the main lens is − 7 periscopic Cc. with, say, + 1·25 on the one surface, the segment, if placed on the Cc. side, is + 8·25 on the contact, and − 6·25 on the free surface, for a reading power of − 5 D.

When the main lens is a plano-cyl. the segment is attached to the plane surface. When the main lens is a sph.-cyl. the segment is attached to the spherical surface. Thus with, say, + 3 Sph. ⊂⊃ + 2 Cyl. with an addition of + 2 for reading, the wafer must have powers of − 3 and + 5.

**Centering of Cement Bifocals.**—The added segment is always Cx., the lower part being weaker if the upper is Cc., and stronger if the upper is Cx. If the wafer is itself centered, the prismatic effect due to decentration of the main lens remains. For a properly centered lower, the segment of the bifocal must have a prismatic effect contrary to that of the main lens where they are united. This is obtained by decentering the segment to the requisite extent. When the main lens is Cx. the prismatic effect of its lower portion is *base up*, so that the wafer must be base down, its thick part being at the edge of the main lens. If the latter is Cc. its prismatic effect is base down, and therefore the segment must be base up, i.e., its thin part must be at the edge of the main lens.

In Fig. 343 *A* is the geometrical and optical centre of the main lens, and *B* is the optical centre of the reading position; the distance *A B* is usually about 8 mm., but may vary. Let $D_1$ be the total power of the main lens, and $C_1$ be the distance *A B*. Let $D_2$ be the power of the segment by itself, and $C_2$ its needed decentration in cm. Now, in order that there be no prismatic effect at *B* it is necessary that $D_1 C_1 = D_2 C_2$, so that the formula for calculating the needed decentration of the segment is

$$C_2 = D_1 C_1 / D_2$$

$D_1$ is the power of the spherical, or the vertical power of a cyl., or sphero-cyl., whose principal meridians are vertical and horizontal.

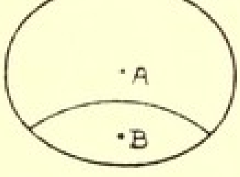



FIG. 343.

**Examples.**—Let the upper be + 4·5 D. and the lower + 6; the segment is + 1·5, so that

$$C_2 = 4{\cdot}5 \times {\cdot}8 / 1{\cdot}5 = 2{\cdot}4 \text{ cm., the thick part } \textit{down}.$$

Let the upper be − 3·5 and the lower − 1, the segment being + 2·5; then

$$C_2 = 3{\cdot}5 \times {\cdot}8 / 2{\cdot}5 = 1{\cdot}1 \text{ cm., the thick part } \textit{up}.$$

The amount of decentering is often very large, and demands either that the blank from which the segment is taken be of extra large dimensions, or the segment be ground on a prism.

It is necessary to place the optical centres of the *lowers* each 1·5 mm., or so, inwards in order to allow for convergence when reading. If the main lenses are Cx. their prism action is base *out*, and that of Cc.'s *in*. To neutralise this the segments must be decentered *in* if the main lenses are Cx., and decentered *out* if they are Cc., such horizontal decentration being considered apart from the placing of the centres of the wafers 1·5 mm. *in* from those of the uppers. The difference between the centres in each eye for distance and reading varies with the interpupillary distance, but 1·5 mm. is a good average distance. In all cases the actual amount of decentering of the wafer required, so that the optical centre of the lower may be in a certain position

—which position should be marked by a dot—can be obtained by sliding the segment over the main lens while viewing the small crossbar as described for centering.

**Inset Bifocals.**—To calculate the curvature of the segment, let $D_1$ be the distance power of the whole lens, and $D_2$ the reading power; let $\mu_1$ be the index of the main lens, and $\mu_2$ the higher index of the segment. The radius of curvature of a surface, separating two dense media, when the focus is finally in air, is

$$r = F(\mu_2 - \mu_1) = 100\,(\mu_2 - \mu_1)/D$$

The following calculation is necessary for finding the tool which, made for producing a certain dioptric power D when the index is $\mu_1$, shall give to the internal surface of the segment of $\mu_2$ the necessary power $D_2$, after allowing for the powers obtained from the two outer surfaces $D_4$ and $D_5$ (Fig. 344). Let $D_3$ be the outer power of the surface containing the segment, $D_4$ the outer power of the segment, $D_5$ the power of the surface not containing the

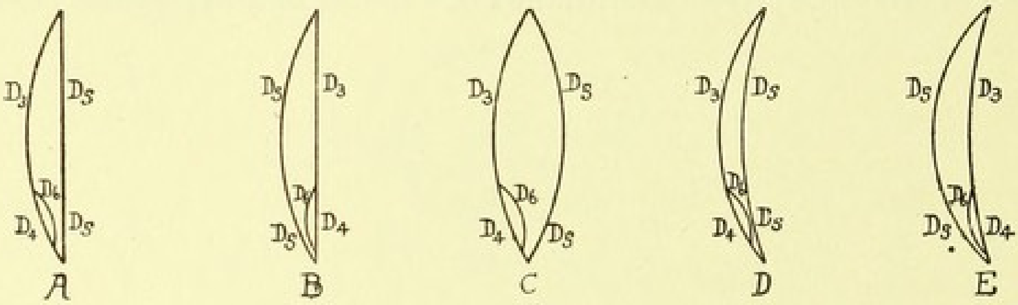



FIG. 344.

segment, and $D_6$ the power of the internal segment surface between the two glasses. Then $D_6 = D_2 - (D_4 + D_5)$

The lens may be of various forms, as shown in Fig. 344.

$$D = \frac{D_6\,(\mu_1 - 1)}{\mu_2 - \mu_1} = \frac{(D_2 - D_4 - D_5)\,(\mu_1 - 1)}{\mu_2 - \mu_1}$$

Now $D_4$ being of higher $\mu$, although of the same curvature as $D_3$, which is known, is of greater power such that $D_4 = D_3\,(\mu_2 - 1)/(\mu_1 - 1)$. Therefore

$$D = \left[D_2 - D_5 - \frac{D_3\,(\mu_2 - 1)}{\mu_1 - 1}\right]\left(\frac{\mu_1 - 1}{\mu_2 - \mu_1}\right)$$

$$= \frac{(D_2 - D_5)\,(\mu_1 - 1) - D_3\,(\mu_2 - 1)}{\mu_2 - \mu_1}$$

The values of $D_2$, $D_3$ and $D_5$ and those of the two $\mu$'s being known, this equation serves for all forms, whether Cx. or Cc., shown in Fig. 344. For a sph.-cyl. form *A* is used and $D_5$ disappears from the equation; for a plano-cyl.

form *B* is employed and $D_3$ disappears. Thus for a plano-cyl. of form *A* having $D_1 = -1{\cdot}5$, and $D_2 = 0$, the $\mu$'s being 1·52 and 1·65, we find

$$D = \frac{0 \times (1{\cdot}52 - 1) - [-1{\cdot}5\,(1{\cdot}65 - 1)]}{1{\cdot}65 - 1{\cdot}52} = \frac{+{\cdot}975}{{\cdot}13} = 7{\cdot}5$$

Let $D_1 = +{\cdot}5$ and $D_2 = +2{\cdot}25$, made periscopic so that $D_3 = +1{\cdot}25$ and $D_5 = -{\cdot}75$; using form (*D*) we get

$$D = \frac{[+2{\cdot}25 - (-{\cdot}75)]\,{\cdot}52 - (1{\cdot}25 \times {\cdot}65)}{1{\cdot}65 - 1{\cdot}52} = \frac{{\cdot}7475}{{\cdot}13} = 5{\cdot}75$$

If the $\mu$'s are 1·52 and 1·65, $D = 4\,(D_2 - D_5) - 5\,D_3$ for forms *C*, *D* and *E*; $D = 4\,D_2 - 5\,D_3$ for form *A*; $D = 4\,(D_2 - D_5)$ for form *B*.

If the $\mu$'s are 1·5 and 1·6, $D = 5\,(D_2 - D_5) - 6\,D_3$ for forms *C*, *D*, and *E*; $D = 5\,D_2 - 6\,D_3$ for form *A*; $D = 5\,(D_2 - D_5)$ for form *B*.

The disc selected must be rather thicker than for ordinary lenses, especially if the segment is of high power. Having insets of known powers the selection of a suitable blank and the curvatures of the two outer surfaces are as follows. Let the two $\mu$'s be 1·65 and 1·52 so that for a given curvature producing $D_3$ we have $D_4 = 5\,D_3/4$, i.e. $D_4$ is 1/4 stronger than $D_3$, and $D_4 - D_3 = D_3/4$. Now part of the additional power for reading is obtained from $D_4 - D_3$, and part from $D_6$, i.e. $D_2 - D_1 = (D_4 - D_3) + D_6$; therefore the powers needed on the two surfaces are $D_3 = 4\,(D_2 - D_1 - D_6)$, and $D_5 = D_1 - D_3$.

It is preferable to select a disc such that $D_6$ is higher than the addition needed for reading, and in that case $D_3$ is Cc. if $D_1$ is Cx. In all cases it is advisable to calculate two or three combinations in order to arrive at the most suitable. A Cc. curvature on the surface of $D_5$ should be avoided; otherwise there is danger of working through to the segment. If $D_6 = D_2 - D_1$ the surface $D_3$ is plano; therefore for sph.-cyls. select $D_6 = D_2 - 1{\cdot}25D_1$, the cyl. being ground on to the side of $D_5$, and the sph. on that of $D_3$.

The proportional increase of power of $D_4$ over $D_3$ is found from

$$\frac{(\mu_2 - \mu_1) - (\mu_1 - 1)}{(\mu_1 - 1)}$$

so that if the two $\mu$'s are other than those given above, the factor 4 in the value of $D_3$ would vary accordingly.

As examples, for $D_1 = +2{\cdot}25$, and $D_2 = +3{\cdot}5$ select $D_6 = 1{\cdot}5$; then $D_3 = 4 \times (1{\cdot}25 - 1{\cdot}5) = -1$, and $D_5 = 2{\cdot}25 + 1 = +3{\cdot}25$.

For $D_1 = -3{\cdot}5$, and $D_2 = -2{\cdot}25$ select $D_6 = 2{\cdot}5$; then $D_3 = 4 \times (1{\cdot}25 - 2{\cdot}5) = -5$ and $D_5 = -3{\cdot}5 + 5 = 1{\cdot}5$.

For +6 S. ⊂⊃ −2 C. with +8 sph. for reading, $D_6 = +8 - 1{\cdot}25 \times 6 = {\cdot}5$.

For −10 S. ⊂⊃ −3 C. with −7 sph. for reading $D_6 = -7 + 1{\cdot}25 \times 10 = 5{\cdot}5$.

**To Construct Test Types after Snellen.**—Each letter at a certain distance

$d$ must subtend an angle of 5′ and each limb of such letter an angle of 1′. In small angles as these, the arc, chord, sine and tangent may be considered equal. The general formula is then

$$S = d \tan V$$

where S is the size of the letter, $d$ is the distance in mm., and V is the visual angle. Since tan 5′ = ·001455 and tan 1′ = ·000291, for use at any metric distance, the diameter of each letter, being square, is the same each way and is obtained from

$$S = 1000 \times \cdot 001455 = 1 \cdot 455\, d \text{ (} d \text{ being in M. and S in mm.).}$$

The diameter of each limb is similarly obtained from ·291 $d$, but it is quite accurate to divide the letter dimension by 5 in order to obtain the limb dimension.

If the letters and distances are in Imperial measure, the diameter of letter in inches = 12 × ·00145 = ·0174 $d$ ($d$ being in feet).

Thus for 6 M. the types are 6 × 1·455 = 8·75 mm., those for 12 M. are 17·5 mm., and so on for every other distance. If the visual angle is other than 5′ the required size of type in mm. is

$$S = M \times \cdot 000291 \times \text{visual angle in minutes of arc.}$$

The size of types can also be deduced from circular measure. The radian = 57·3° = 3438′, the arc subtending it being equal to the radius or, in this case, the distance; if an angle is smaller the arc, subtending it, is proportionately smaller, so that

$$V/3438 = S/d \qquad \text{or} \qquad S = V\,d/3438$$

Suppose the types be required for 18 M. under a visual angle of 4′; then

$$S = 4 \times 18000/3438 = 21 \text{ mm. (approx.)}$$

**To Construct Tangent Scales.**—For prism diopters the card must be scaled so that each division shall be 1 cm. for each M distance at which it is used, *e.g.* the divisions are each 2 cm. for 2 M., 6 cm. for 6 M. and so on. For distances and spaces in Imperial measure each division is 2·4 inches for 20 ft., and for other distances in proportion.

In order that equal divisions should accurately indicate equal increase of angular deviating power of prisms, the scale should be on an arc at the centre of which the prism is held. This, in fact, was the basis of the *Centrad* notation which, however, owing to the inconvenience of such an arrangement, did not come into general use.

**For Degrees of Deviation.**—The divisions should be $d$ tan1°, $d$ tan2° etc., where $d$ is the distance at which the chart is used; that is, the successive spaces should increase in size from zero, since equal increases in the angles of deviation correspond to greater increases in the tangents; but for small angles the error is negligible, so that it is customary and sufficiently accurate to make each division in cm. = 1·745 $d$, where $d$ is the distance in metres. Thus for use at 3 M. each division would be 3 × 1·745 = 5·25 cm. approx. and so on for every other distance. For distances in Imperial measure the divisions are ·01745 × 12 = ·21 in. for each foot; that is, 4·2 in. for 20 ft.

**For Degrees.**—If $\mu$ is taken as 1·5 the division should be ·875 cm. for each M.; if $\mu = 1{\cdot}52$, they should be ·9 cm.; if $\mu = 1{\cdot}54$ they should be ·94 cm. In practice the △ scale serves for degrees.

**Combined Scale.**—A scale as shown in Fig. 148 can be made to indicate both prism diopters and degrees of deviation. If used at 2 M. the numbers give degrees of deviation, at 3·5 M. they indicate prism diopters, the divisions being each 3·5 cm. Thus if a given prism at 2 M. indicates, say, 4 it is a 4°d; if held at 3·5 M. it will indicate 7△, which is the equivalent of 4°d.

**Mirror for Reversed Test Types.**—The necessary size S of the mirror depends on the size C of the chart, and the distances M and $d$ respectively of chart and subject from the mirror; it should be just large enough to be

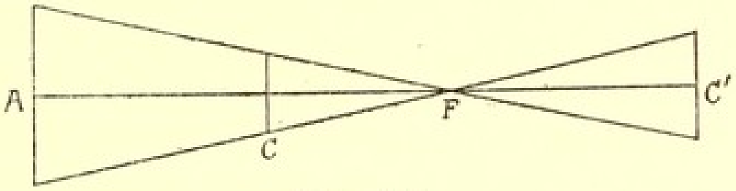



FIG. 345.

filled entirely by the image of the chart. S/C = $d/(d + \text{M})$ so that S = C$d/(d + \text{M})$. If, as is generally the case, $d + \text{M} = 6$ M., the subject and the chart being at the same distance, i.e. 3 M. from the mirror, the latter is just one half the size of the chart in both dimensions.

**Confusion Discs.**—The size of a disc of confusion $C$ (Fig. 345) depends on its distance $b$ from the focus, the distance $f$ from the lens to $F$, and on $A$ the aperture of the lens; thus A/C = $f/b$, or C = A $b/f$.

For instance, with a + 4 D lens the disc of confusion at 15 cm. from the lens, which is 25 − 15 = 10 cm. from $F$, is 10/25 of $A$. It would be the same size at $C'$ if 35 cm. from the lens, and also 10 cm. from $F$. If $C'$ is 40 cm. from $A$, then $C'/A = 15/25$. In these cases the source of light is presumed to be distant. If the source of light is not large the same calculation gives very approximately the diameter of the whole cone of light. If the object is near, the conjugate distance $f_2$ must be taken instead of $F$.

If a screen be held close behind a Cx. lens facing a distant bright source, the emergent light is similar in size to the lens aperture, and it becomes smaller as the screen is receded, the minimum being reached at the focus, after which it again increases in size.

With cylindrical lenses the two diameters must be calculated, the confusion disc being elliptical. These two dimensions $C$ and $C'$ at any distance are found from

$$C = A\,a/F_1 \qquad \text{and} \qquad C' = A\,b/F_2$$

where $a$ and $b$ are the distances respectively from $F_1$ and $F_2$.

Thus what is the size of the confusion disc formed at 30 cm. by a +4 S. ⊂⊃ +2 C. Ax. 90°, the diameter of the lens $A$ being 5 cm.? Now $F_1 = 16{\cdot}66$, and $a = 30 - 16{\cdot}66 = 13{\cdot}33$ cm.; $F_2 = 25$, and $b = 30 - 25 = 5$ cm., so that

$$C = 5 \times 13{\cdot}33/16{\cdot}66 = 4 \text{ cm.} \qquad \text{and} \qquad C' = 5 \times 5/25 = 1 \text{ cm.}$$

The disc is 4 cm. horizontally and 1 cm. vertically. The size and distance of the circular disc where $C$ and $C'$ are equal are shown in "The Interval of Sturm," Chap. X., and from the above we find at 20 cm. both dimensions to be 1 cm. It is difficult to show the two diameters in one diagram, but a separate one for each clearly shows the principles involved.

**Reflecting Refractors.**—If a lens be used so that light refracted by the first surface is reflected back from the second surface, and again refracted by the first we have a reflecting refractor, known as a Mangin mirror.

A Cx. surface is a positive refractor and a negative reflector, while a Cc. surface is a negative refractor and positive reflector, but the second surface of a lens, used in this way, if Cx. as a refractor, becomes Cc. as a reflector and is positive in both cases; if the second surface of the lens is a Cc. refractor, it becomes a Cx. reflector, and is negative in both cases.

To calculate the focal length of a reflecting lens it is necessary to add the reflective power of the second surface to the refractive power of the first surface as the light enters and emerges. The lens is treated as thin, and $r$ is the radius of curvature of a plano or double Cx. or Cc. F is the resultant focal length.

For a plano Cx. or Cc. with the plane surface as a reflector

$$\frac{1}{F} = \frac{\mu - 1}{r} + \frac{\mu - 1}{r} = \frac{2(\mu - 1)}{r} \qquad \text{or} \qquad F = \frac{r}{2(\mu - 1)}$$

For a plano Cx. or Cc. with the curved surface as a reflector

$$\frac{1}{F} = \frac{2\mu}{r} \qquad \text{or} \qquad F = \frac{r}{2\mu}$$

For a double Cx. or Cc.

$$\frac{1}{F} = \frac{\mu - 1}{r} + \frac{2\mu}{r} + \frac{\mu - 1}{r} = \frac{4\mu - 2}{r} \qquad \text{or} \qquad F = \frac{r}{4\mu - 2}$$

For doubles of unequal radius and periscopic lenses, $r_1$ being the radius of the first, and $r_2$ that of the second or reflecting surface, we get

$$\frac{1}{F}=\frac{\mu-1}{r_1}+\frac{2\mu}{r_2}+\frac{\mu-1}{r_1}=\frac{2\mu(r_1+r_2)-2r_2}{r_1r_2} \quad \text{or} \quad F=\frac{r_1r_2}{2\mu(r_1+r_2)-2r_2}$$

the − sign being prefixed when $r_1$ or $r_2$ has a diverging effect.

Since a curved surface has more reflecting than refracting power, the effect of a Cx. or Cc. periscopic may be positive or negative as the one or the other surface faces the light. In order that incident parallel light emerge as parallel it is necessary that

$$r_2=\frac{-r_1\mu}{\mu-1} \quad \text{or} \quad -r_2=\frac{r_1\mu}{\mu-1}$$

or approximately $r_2 = -3r_1$, i.e. when the radius of the reflecting surface is equal to the posterior focal length of the first surface, parallel light retraces its own course; or the power of the mirror must equal twice the power of the lens, that is, $2/r_2=2(\mu-1)(1/r_1+1/r_2)$ care being taken that the proper signs be affixed for converging and diverging effects.

When $\mu=1{\cdot}5$, let F′ be the focal length and D′ the dioptric power of the lens; then, used as a reflecting refractor, a plano Cx. or Cc. with the plano surface reflecting

$$F=r=F'/2 \quad \text{or} \quad D=2\,D'$$

With a plano Cx. or Cc. with the curved surface reflecting

$$F=r/3=F'/6 \quad \text{or} \quad D=6\,D'$$

With a double Cx. or Cc.

$$F=r/4=F'/4 \quad \text{or} \quad D=4\,D'$$

The above formulæ show that, in all cases, the power of the whole system is equal to the power of the reflector plus twice the power of the lens. As a reflector a surface has about four times as much power as it has as a refractor, so that $1/F=2/F_1+6/F_2$, or $D=2\,D_1+6\,D_2$, where $F_1$ and $D_1$ refer to the first, and $F_2$ and $D_2$ to the second surface. Thus with a lens of $D_1=+1$ and $D_2=+2$, we have $D=+14$; if $D_1=+4$ and $D_2=-2$, we get $D=-4$; if this lens were turned the other way $D=+20$.

**Images Formed by cyl. Lenses and Mirrors.**—Hitherto the formation of images by cyls. has only been considered so far as the production of focal lines from point sources is concerned. Nevertheless, a plano cyl. can produce an image of sorts, although naturally very ill defined and distorted, from an ordinary object; such images, even when real, are best examined by the eye, because the pupil of the latter acts as a stop, and cuts down the excessive confusion caused by the absence of point foci.

**Cx. cyl.**—The real image produced on a screen by a plano Cx. cyl. is made up of focal lines approximately equal to the axial diameter of the lens;

in consequence the real image is an infinite number of streaks parallel to the axis. Viewed from behind by the eye, the image seen is partly real, partly virtual, is equal in size to the object along the axis, and may be diminished, magnified, or of the same size across the axis. With the axis, say, horizontal, the image is not laterally reversed, but is inverted; with the axis vertical the image is reversed but not inverted. When the object is within F the image is wholly virtual, there is neither reversal nor inversion, there being unit magnification along the axis, and enlargement across it.

**Cc. cyl.**—Here the image is always virtual, is equal in size to the object along the axis, and diminished across the axis. There is neither reversal nor inversion.

**Cyl. Mirrors.**—Remembering that a Cc. cyl. mirror acts similarly to a Cx. cyl. lens, and a Cx. cyl. mirror to a Cc. cyl. lens, what has been said in the foregoing paragraphs with regard to lenses applies equally to mirrors.

## Optical Glass.

Glass is a hard, generally transparent or translucent substance, made by the fusion of silica with potash, soda, lime, lead and other substances, such as pearlash, arsenic, manganese, saltpetre, chalk, etc. It is brittle, sonorous, ductile when heated, and fusible only at a very high temperature. It is usually not soluble, but is acted on by hydrofluoric acid, and is a very bad conductor of heat. There are many varieties of glass, and the process of manufacture, as regards the ingredients used and the treatment after complete fusion of the various components, depends on the nature of the glass produced.

If suddenly cooled, glass becomes extremely brittle owing to the state of tension produced by the cooling of the outer portions while the inner are still in a molten condition; annealing tends to reduce brittleness. Glass used for optical purposes must be homogeneous, *i.e.*, of equal density and refractive power throughout, and perfectly transparent; it is therefore carefully mixed and gradually cooled. It should also be free from air bubbles, striae and colour for spectacle lenses, although a few air bubbles, if small, may be of little or no consequence in a camera lens. The solid block of glass is usually polished on two sides, so as to allow of the detection of defects, and from it clear discs of appropriate size are cut.

Lenses are made of crown glass, which contains lime, or of flint glass, which contains lead. Flint has generally a higher refractivity and chromativity; the greater the proportion of lead in the glass the greater, usually, are the refractive and dispersive powers. It is denser, heavier, and softer than crown, and is almost perfectly colourless. Crown glass has the advantage of lower dispersion and is harder, so that it does not so easily become scratched, but it is more brittle than flint. It has sometimes a decided greenish tint, due to the presence of iron. The pinkish tint found in some glass results from the admixture of manganese.

According to its component ingredients and manufacture, the indices of refraction of glass vary for the various lines of the spectrum. The mean $\mu$ of different kinds of glass made for optical purposes can be taken as 1·574, that of the crowns being 1·524, and of the flints 1·624.

The following may be taken as very rough examples of the proportions of the materials entering in the manufacture of optical glass:—

**Flint Glass** (100 parts).—Silica 50, lead 30, potash 10, other ingredients 10.

**Crown Glass** (100 parts).—Silica 70, soda 10, lime 10, other ingredients 10.

In the following table some examples (not actual kinds) are given to illustrate the refraction, dispersion and specific gravity of different kinds of optical glass, and the method generally employed in arranging them in the order of their $\nu$ values or efficiencies.

TABLE OF OPTICAL GLASSES.

| Description. | $\mu_D$ | $\nu = \frac{\mu_D - 1}{\delta\mu}$ | Dispersion. | | | | Specific Gravity |
|---|---|---|---|---|---|---|---|
| | | | Medium. $C - F = \delta\mu$ | $A - D$ | $D - F$ | $F - G$ | |
| Very light Crown | 1·48 | 66 | ·0073 | ·0050 | ·0055 | ·0040 | 2·25 |
| Light ,, | 1·50 | 62 | ·0081 | ·0055 | ·0065 | ·0045 | 2·50 |
| Ordinary ,, | 1·52 | 60 | ·0087 | ·0060 | ·0070 | ·0050 | 2·75 |
| Heavy ,, | 1·56 | 55 | ·0102 | ·0065 | ·0075 | ·0055 | 3 |
| Very heavy ,, | 1·60 | 52 | ·0115 | ·0070 | ·0085 | ·0065 | 3·5 |
| Very light Flint | 1·54 | 48 | ·0123 | ·0075 | ·0090 | ·0070 | 3 |
| Light ,, | 1·58 | 43 | ·0135 | ·0085 | ·0095 | ·0080 | 3·25 |
| Ordinary ,, | 1·62 | 40 | ·0155 | ·0095 | ·0115 | ·0100 | 3·50 |
| Heavy ,, | 1·68 | 35 | ·0194 | ·0105 | ·0130 | ·0110 | 4 |
| Very heavy ,, | 1·85 | 24 | ·0354 | ·0185 | ·0280 | ·0250 | 5·5 |

REFRACTIVE INDICES OF VARIOUS MEDIA.

| | |
|---|---|
| Air ... ... ... | $\mu_D = 1·000$ |
| Ice ... ... ... | $\mu_D = 1·310$ |
| Water (distilled) ... | $\mu_D = 1·336$ |
| Sea-water ... ... | $\mu_D = 1·343$ |
| Blood ... ... ... | $\mu_E = 1·354$ |
| Albumen ... ... | $\mu_F = 1·360$ |
| Absolute Alcohol ... | $\mu_D = 1·366$ |
| Oil of Bergamot ... | $\mu_D = 1·464$ |
| Olive Oil ... ... | $\mu_E = 1·470$ |
| Oil of Cassia ... ... | $\mu_E = 1·618$ |
| Oil of Fennel ... ... | $\mu_D = 1·544$ |
| Anilin Oil ... ... | $\mu_D = 1·580$ |
| Oil of Cloves ... ... | $\mu_D = 1·533$ |
| Oil of Cinnamon ... | $\mu_E = 1·508$ |
| Cedar Oil (Lens immersion oil) ... ... | $\mu_D = 1·512$ |
| Naphtha ... ... | $\mu_E = 1·475$ |
| Turpentine ... ... | $\mu_E = 1·478$ |

REFRACTIVE INDICES OF VARIOUS MEDIA—*Continued.*

| | |
|---|---|
| Glycerine ... ... | $\mu_D = 1{\cdot}460$ |
| Gum Arabic ... ... | $\mu_E = 1{\cdot}512$ |
| Spermaceti ... ... | $\mu_E = 1{\cdot}444$ |
| Bisulphide of Carbon ... | $\mu_D = 1{\cdot}687$ |
| Alum ... ... ... | $\mu_D = 1{\cdot}457$ |
| Sugar ... ... ... | $\mu_D = 1{\cdot}535$ |
| Rock Salt ... ... | $\mu_D = 1{\cdot}555$ |
| Salt Solution ... ... | $\mu_E = 1{\cdot}375$ |
| Phosphorus ... ... | $\mu_D = 2{\cdot}224$ |
| Diamond ... ... | $\mu_D = 2{\cdot}470$ |
| Chromate of Lead ... | $\mu_D = 2{\cdot}500$ to $2{\cdot}970$ |
| Canada Balsam (liquid) | $\mu_D = 1{\cdot}520$ |
| ,, ,, (hard) | $\mu_D = 1{\cdot}535$ |
| Rock Crystal Pebble (ordinary ray) ... | $\mu_D = 1{\cdot}544$ |
| Rock Crystal Pebble (extraordinary) ... | $\mu_D = 1{\cdot}553$ |
| Tourmaline (ordinary ray) ... ... | $\mu_D = 1{\cdot}636$ |
| Tourmaline (extraordinary) ... ... | $\mu_D = 1{\cdot}620$ |
| Iceland Spar or Calcite (ordinary ray) ... | $\mu_D = 1{\cdot}659$ |
| Iceland Spar or Calcite (extraordinary) ... | $\mu_D = 1{\cdot}486$ |
| Felspar ... ... | $\mu_E = 1{\cdot}764$ |
| Fluor Spar ... ... | $\mu_D = 1{\cdot}434$ |

REFRACTIVE INDICES OF SOME METALS (KUNDT).

| | Red. | Yellow (D.). | Blue. |
|---|---|---|---|
| Silver ... ... ... | — | 0·27 | — |
| Gold ... ... ... | 0·38 | 0·58 | 1·00 |
| Copper ... ... ... | 0·45 | 0·65 | 0·95 |
| Platinum ... ... ... | 1·76 | 1·64 | 1·44 |
| Iron ... ... ... ... | 1·81 | 1·73 | 1·54 |
| Nickel ... ... ... | 2·17 | 2·01 | 1·85 |
| Cobalt ... ... ... | 2·61 | 2·26 | 2·16 |

**The Transmissiveness** of various transparent media to different parts of the visible and invisible spectrum varies considerably. Thus crown and flint glass are comparatively opaque to heat rays and equally transparent to light rays, but while crown is rather opaque to the ultra-violet, flint is exceedingly so. Most crystals, as fluor spar and pebble, are exceedingly transparent to the ultra-violet, and fluor spar also to the infra-red rays. Rock salt and iodine are very transparent, while alum is very opaque, to the infra-red rays.

**Opacity.**—The cause of opacity may be said to be due to the restraining influence exerted by bodies—or rather, their composition—on the passage through them of waves of certain lengths. The light is not, however, lost, but is converted into some other form of energy—perhaps generally heat—but the rise in temperature would be slight. It is due to the infra-red or heat radiations accompanying light that an opaque body becomes markedly heated when exposed to general radiation. A rise due to opacity to ethereal vibration must, however, be distinguished from that caused by the nature of the surface, i.e., its absorptive power, which has a much more powerful influence in raising the temperature of a body. Thus polished and blackened metal may be equally opaque, but the latter would be rendered much the hotter by freer absorption of heat. In fact it would be difficult to eliminate the factor of absorption in the measurement of the rise of temperature produced by opacity to light.

# APPENDIX

## LENS SCALE

Of the Approximately Relative Values of Lenses Numbered by the Dioptric System and by Focal Length in Mm. and Inches.

| Diopters. Refractive Power. | Focal Length in Mm. | Focal Length in Inches. | No. of Cc. Old English System. (These Nos. have a very Uncertain Value.) |
|---|---|---|---|
| 0·125 | 8000 | 320 | — |
| 0·25 | 4000 | 160 | 0000 |
| 0·375 | 2666 | 100 | — |
| 0·50 | 2000 | 80 | 000 |
| 0·625 | 1600 | 60 | 00 |
| 0·75 | 1333 | 52 | — |
| 0·875 | 1143 | 48 | 0 |
| 1·00 | 1000 | 40 | — |
| 1·125 | 888 | 36 | 1 |
| 1·25 | 800 | 32 | — |
| 1·375 | 727 | 30 | — |
| 1·50 | 666 | 26 | — |
| 1·625 | 616 | 24 | 2 |
| 1·75 | 570 | 22 | — |
| 1·875 | 533 | 21 | — |
| 2·00 | 500 | 20 | 2½ |
| 2·125 | 470 | 19 | — |
| 2·25 | 444 | 18 | 3 |
| 2·375 | 421 | 17 | — |
| 2·50 | 400 | 16 | 3½ |
| 2·625 | 381 | 15 | — |
| 2·75 | 363 | 14 | 4 |
| 3·00 | 333 | 13 | 4½ |
| 3·25 | 308 | 12 | 5 |
| 3·50 | 286 | 11 | 6 |
| 4·00 | 250 | 10 | 7 |
| 4·50 | 222 | 9 | 8 |
| 5·00 | 200 | 8 | 9 |
| 5·50 | 182 | 7 | 10 |
| 6·00 | 166 | 6½ | — |
| 6·50 | 154 | 6 | 11 |
| 7·00 | 142 | 5½ | — |
| 7·50 | 133 | 5¼ | — |
| 8·00 | 125 | 5 | 12 |
| 8·50 | 118 | 4¾ | 13 |
| 9·00 | 111 | 4½ | 14 |
| 9·50 | 105 | 4¼ | 15 |
| 10·00 | 100 | 4 | 16 |
| 10·50 | 95 | 3¾ | 17 |
| 11·00 | 90 | 3½ | 18 |
| 12·00 | 83 | 3¼ | 19 |
| 13·00 | 77 | 3 | 20 |
| 14·00 | 71 | 2¾ | 21 |
| 16·00 | 62 | 2½ | 22 |
| 18·00 | 55 | 2¼ | 23 |
| 20·00 | 50 | 2 | 24 |
| 22·00 | 45 | 1¾ | — |
| 26·00 | 38 | 1½ | — |
| 32·00 | 31 | 1¼ | — |
| 40·00 | 25 | 1 | — |

## METRIC MEASURE.

1 Kilometre (K.) = 1000 metres = $\frac{3}{5}$ mile. The K. = 1 billion $\mu\mu$.
1 Metre (M.) = 10 decimetres = $\frac{1}{1000}$ Kilometre = 39·37 inches.
1 Decimetre (Dm.) = 10 centimetres = $\frac{1}{10}$ Metre = 3·937 inches.
1 Centimetre (cm.) = 10 millimetres = $\frac{1}{100}$ Metre = 0·3937 inch.
1 Millimetre (mm.) = 1000 microns = $\frac{1}{1000}$ Metre = 0·03937 inch.
1 Micron ($\mu$) = 1000 micromillimetres = $\frac{1}{1000}$ mm. = $\frac{1}{25.400}$ inch.
1 Micromillimetre ($\mu\mu$) = 10 Ångstrom units = $\frac{1}{1.000.000}$ mm. = $\frac{1}{25.400.000}$ inch.
1 Ångstrom unit (Å.) = $\frac{1}{10}$ $\mu\mu$ = $\frac{1}{10.000}$ $\mu$ = $\frac{1}{10.000.000}$ mm.

---

To convert mm. to inches multiply by ·03937 or divide by 25·4.
,, ,, cm. to inches multiply by ·3937 or divide by 2·54.
,, ,, M. to inches multiply by 39·37 or divide by ·0254.
,, ,, M. to feet multiply by 3·28 or divide by ·3048.

---

### Approximate Conversions.

Feet to M. multiply by 3 and divide by 10.
Feet to cm. multiply by 30.
Inches to cm. multiply by $2\frac{1}{2}$ or multiply by 10 and divide by 4.
Inches to mm. multiply by 25 or multiply by 100 and divide by 4.
M. to feet multiply by 10 and divide by 3.
M. to inches multiply by 40.
Cm. to inches divide by $2\frac{1}{2}$ or multiply by 4 and divide by 10.
Mm. to inches divide by 25 or multiply by 4 and divide by 100.

### Equivalents of Standards of Measurement..

| | |
|---|---|
| 1 millimetre (mm.) ... ... ... ... = | ·03937 English inch. |
| ,, ... ... ... ... = | ·03694 Paris inch. |
| ,, ... ... ... ... = | ·03824 Prussian inch. |
| ,, ... ... ... ... = | ·03796 Austrian inch. |
| 1 English inch, also U.S. and Russian ... = | 25·4 mm. |
| 1 Paris inch ... ... ... ... ... = | 27·07 ,, |
| 1 Prussian inch, also Danish and Norwegian = | 26·15 ,, |
| 1 Austrian inch ... ... ... ... = | 26·3 ,, |
| 1 Swiss inch ... ... ... ... ... = | 30 ,, |
| 1 Swedish inch ... ... ... ... = | 29·7 ,, |
| 1 English inch .. ... ... ... = | ·94 Paris inch. |
| 1 Paris inch ... ... ... ... ... = | 1·07 English inches. |

## COMMERCIAL NOTATION FOR OPERA GLASSES AND TELESCOPES.

The French inch is divided into 12 lignes or lines, in which the diameter of the object glass of ordinary opera glasses and small telescopes is expressed.

| Lines ... | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| M.M. (approx.) | 11 | 13 | 15 | 17 | 20 | 22 | 24 | 26 | 29 | 31 | 33 | 36 | 38 | 40 | 43 | 45 | 47 | 49 |

| Lines ... | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| M.M. (approx.) | 52 | 54 | 56 | 58 | 61 | 63 | 65 | 67 | 70 | 72 | 74 | 76 | 78 | 81 | 83 | 85 | 87 | 90 |

## THE GREEK ALPHABET, WITH THE ENGLISH EQUIVALENTS AND THE PRONUNCIATION OF THE LETTERS.

| | | |
|---|---|---|
| Α α ... alpha *a* | Ι ι ... iōta ... *i* | Ρ ρ ... rho ... *r* |
| Β β ... bēta ... *b* | Κ κ ... kappa... *k* | Σ σ ς ... sigma *s* |
| Γ γ ... gamma *g* (hard) | Λ λ ... lamda... *l* | Τ τ ... tau ... *t* |
| Δ δ ... delta... *d* | Μ μ ... mu ... *m* | Υ υ ... ūpsīlon *u* or *y* |
| Ε ε ... epsīlon *e* (short) | Ν ν ... nu ... *n* | Φ φ ... phi ... *ph* |
| Ζ ζ ... zēta ... *z* | Ξ ξ ... xi ... *x* | Χ χ ... chi ... *ch* (hard) |
| Η η ... ēta ... *e* (long) | Ο ο ... ōmicron *o* (short) | Ψ ψ ... psi ... *ps* |
| Θ θ ϑ ... thēta *th* | Π π ... pi ... *pi* | Ω ω ... ōmĕga *o* (long) |

## SINES, TANGENTS, ETC.

**Small Angles.**—The sin or tan of $1° = \cdot01745$; those of any angle smaller than 1° can be found roughly by subdividing ·01745. Thus the sin or tan of $1' = \cdot0003$; of $5' = \cdot00145$; of $10' = \cdot0029$; of $15' = \cdot00435$, and so on. The cosine of these small angles may be taken as 1.

For an angle intermediate to those in the table, the value of the sine or tangent may be found by adding to the next lower the proportional difference between the next higher and lower values. Thus suppose sin 14° 40′ be needed; now sin 14° 30′ = ·2504, and sin 15° = ·2588. The difference for $30' = \cdot2588 - \cdot2504 = \cdot0084$ so that

$$\sin 14°\ 40' = \cdot2504 + \frac{\cdot0084 \times 10}{30} = \cdot2504 + \cdot0028 = \cdot2532$$

For the cosine and cotangent, which decreases as the angle increases, the proportional value of the intermediate angle must be subtracted from the next higher. Thus

$$\cos 52°\ 18' = \cdot6157 - (\cdot6157 - \cdot6088) \times \tfrac{18}{30} = \cdot6157 - \cdot0041 = \cdot6116$$

# TABLE OF NATURAL SINES, ETC.

In these tables read downwards from 0° to 45°, and upwards from 45° to 90°. *For angles less than* 1° and *intermediate angles*, see opposite page.

| ° | ′ | Sines. | Tangents. | Cotangents. | Cosines. | ° | ′ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0·0000 | 0·0000 | ∞ | 1·0000 | 90 | 0 |
| 0 | 30 | 0·0087 | 0·0087 | 114·5887 | 1·0000 | 89 | 30 |
| 1 | 0 | 0·0175 | 0·0175 | 57·2900 | 0·9998 | 89 | 0 |
| 1 | 30 | 0·0262 | 0·0262 | 38·1885 | 0·9997 | 88 | 30 |
| 2 | 0 | 0·0349 | 0·0349 | 28·6363 | 0·9994 | 88 | 0 |
| 2 | 30 | 0·0436 | 0·0437 | 22·9038 | 0·9990 | 87 | 30 |
| 3 | 0 | 0·0523 | 0·0524 | 19·0811 | 0·9986 | 87 | 0 |
| 3 | 30 | 0·0610 | 0·0612 | 16·3499 | 0·9981 | 86 | 30 |
| 4 | 0 | 0·0698 | 0·0699 | 14·3007 | 0·9976 | 86 | 0 |
| 4 | 30 | 0·0785 | 0·0787 | 12·7062 | 0·9969 | 85 | 30 |
| 5 | 0 | 0·0872 | 0·0875 | 11·4301 | 0·9962 | 85 | 0 |
| 5 | 30 | 0·0958 | 0·0963 | 10·3854 | 0·9954 | 84 | 30 |
| 6 | 0 | 0·1045 | 0·1051 | 9·5144 | 0·9945 | 84 | 0 |
| 6 | 30 | 0·1132 | 0·1139 | 8·7769 | 0·9936 | 83 | 30 |
| 7 | 0 | 0·1219 | 0·1228 | 8·1443 | 0·9925 | 83 | 0 |
| 7 | 30 | 0·1305 | 0·1317 | 7·5958 | 0·9914 | 82 | 30 |
| 8 | 0 | 0·1392 | 0·1405 | 7·1154 | 0·9903 | 82 | 0 |
| 8 | 30 | 0·1478 | 0·1495 | 6·6912 | 0·9890 | 81 | 30 |
| 9 | 0 | 0·1564 | 0·1584 | 6·3138 | 0·9877 | 81 | 0 |
| 9 | 30 | 0·1650 | 0·1673 | 5·9758 | 0·9863 | 80 | 30 |
| 10 | 0 | 0·1736 | 0·1763 | 5·6713 | 0·9848 | 80 | 0 |
| 10 | 30 | 0·1822 | 0·1853 | 5·3955 | 0·9833 | 79 | 30 |
| 11 | 0 | 0·1908 | 0·1944 | 5·1446 | 0·9816 | 79 | 0 |
| 11 | 30 | 0·1994 | 0·2035 | 4·9152 | 0·9799 | 78 | 30 |
| 12 | 0 | 0·2079 | 0·2126 | 4·7046 | 0·9781 | 78 | 0 |
| 12 | 30 | 0·2164 | 0·2217 | 4·5107 | 0·9763 | 77 | 30 |
| 13 | 0 | 0·2250 | 0·2309 | 4·3315 | 0·9744 | 77 | 0 |
| 13 | 30 | 0·2334 | 0·2401 | 4·1653 | 0·9724 | 76 | 30 |
| 14 | 0 | 0·2419 | 0·2493 | 4·0108 | 0·9703 | 76 | 0 |
| 14 | 30 | 0·2504 | 0·2586 | 3·8667 | 0·9681 | 75 | 30 |
| 15 | 0 | 0·2588 | 0·2679 | 3·7321 | 0·9659 | 75 | 0 |
| 15 | 30 | 0·2672 | 0·2773 | 3·6059 | 0·9636 | 74 | 30 |
| 16 | 0 | 0·2756 | 0·2867 | 3·4874 | 0·9613 | 74 | 0 |
| 16 | 30 | 0·2840 | 0·2962 | 3·3759 | 0·9588 | 73 | 30 |
| 17 | 0 | 0·2924 | 0·3057 | 3·2709 | 0·9563 | 73 | 0 |
| 17 | 30 | 0·3007 | 0·3153 | 3·1716 | 0·9537 | 72 | 30 |
| 18 | 0 | 0·3090 | 0·3249 | 3·0777 | 0·9511 | 72 | 0 |
| 18 | 30 | 0·3173 | 0·3346 | 2·9887 | 0·9483 | 71 | 30 |
| 19 | 0 | 0·3256 | 0·3443 | 2·9042 | 0·9455 | 71 | 0 |
| 19 | 30 | 0·3338 | 0·3541 | 2·8239 | 0·9426 | 70 | 30 |
| 20 | 0 | 0·3420 | 0·3640 | 2·7475 | 0·9397 | 70 | 0 |
| 20 | 30 | 0·3502 | 0·3739 | 2·6746 | 0·9367 | 69 | 30 |
| 21 | 0 | 0·3584 | 0·3839 | 2·6051 | 0·9336 | 69 | 0 |
| 21 | 30 | 0·3665 | 0·3939 | 2·5386 | 0·9304 | 68 | 30 |
| 22 | 0 | 0·3746 | 0·4040 | 2·4751 | 0·9272 | 68 | 0 |
| 22 | 30 | 0·3827 | 0·4142 | 2·4142 | 0·9239 | 67 | 30 |
| ° | ′ | Cosines. | Cotangents. | Tangents. | Sines. | ° | ′ |

| ° | ′ | Sines. | Tangents. | Cotangents. | Cosines. | ° | ′ |
|---|---|---|---|---|---|---|---|
| 23 | 0 | 0·3907 | 0·4245 | 2·3559 | 0·9205 | 67 | 0 |
| 23 | 30 | 0·3987 | 0·4348 | 2·2998 | 0·9171 | 66 | 30 |
| 24 | 0 | 0·4067 | 0·4452 | 2·2460 | 0·9135 | 66 | 0 |
| 24 | 30 | 0·4147 | 0·4557 | 2·1943 | 0·9100 | 65 | 30 |
| 25 | 0 | 0·4226 | 0·4663 | 2·1445 | 0·9063 | 65 | 0 |
| 25 | 30 | 0·4305 | 0·4770 | 2·0965 | 0·9026 | 64 | 30 |
| 26 | 0 | 0·4384 | 0·4877 | 2·0503 | 0·8988 | 64 | 0 |
| 26 | 30 | 0·4462 | 0·4986 | 2·0057 | 0·8949 | 63 | 30 |
| 27 | 0 | 0·4540 | 0·5095 | 1·9626 | 0·8910 | 63 | 0 |
| 27 | 30 | 0·4617 | 0·5206 | 1·9210 | 0·8870 | 62 | 30 |
| 28 | 0 | 0·4695 | 0·5317 | 1·8807 | 0·8829 | 62 | 0 |
| 28 | 30 | 0·4772 | 0·5430 | 1·8418 | 0·8788 | 61 | 30 |
| 29 | 0 | 0·4848 | 0·5543 | 1·8040 | 0·8746 | 61 | 0 |
| 29 | 30 | 0·4924 | 0·5658 | 1·7675 | 0·8704 | 60 | 30 |
| 30 | 0 | 0·5000 | 0·5774 | 1·7321 | 0·8660 | 60 | 0 |
| 30 | 30 | 0·5075 | 0·5890 | 1·6977 | 0·8616 | 59 | 30 |
| 31 | 0 | 0·5150 | 0·6000 | 1·6643 | 0·8572 | 59 | 0 |
| 31 | 30 | 0·5225 | 0·6128 | 1·6319 | 0·8526 | 58 | 30 |
| 32 | 0 | 0·5299 | 0·6249 | 1·6003 | 0·8480 | 58 | 0 |
| 32 | 30 | 0·5373 | 0·6371 | 1·5697 | 0·8434 | 57 | 30 |
| 33 | 0 | 0·5446 | 0·6494 | 1·5399 | 0·8387 | 57 | 0 |
| 33 | 30 | 0·5519 | 0·6619 | 1·5108 | 0·8339 | 56 | 30 |
| 34 | 0 | 0·5592 | 0·6745 | 1·4826 | 0·8290 | 56 | 0 |
| 34 | 30 | 0·5664 | 0·6873 | 1·4550 | 0·8241 | 55 | 30 |
| 35 | 0 | 0·5736 | 0·7002 | 1·4281 | 0·8192 | 55 | 0 |
| 35 | 30 | 0·5807 | 0·7133 | 1·4019 | 0·8141 | 54 | 30 |
| 36 | 0 | 0·5878 | 0·7265 | 1·3764 | 0·8090 | 54 | 0 |
| 36 | 30 | 0·5948 | 0·7400 | 1·3514 | 0·8039 | 53 | 30 |
| 37 | 0 | 0·6018 | 0·7536 | 1·3270 | 0·7986 | 53 | 0 |
| 37 | 30 | 0·6088 | 0·7673 | 1·3032 | 0·7934 | 52 | 30 |
| 38 | 0 | 0·6157 | 0·7813 | 1·2799 | 0·7880 | 52 | 0 |
| 38 | 30 | 0·6225 | 0·7954 | 1·2572 | 0·7826 | 51 | 30 |
| 39 | 0 | 0·6293 | 0·8098 | 1·2349 | 0·7771 | 51 | 0 |
| 39 | 30 | 0·6361 | 0·8243 | 1·2131 | 0·7716 | 50 | 30 |
| 40 | 0 | 0·6428 | 0·8391 | 1·1918 | 0·7660 | 50 | 0 |
| 40 | 30 | 0·6494 | 0·8541 | 1·1708 | 0·7604 | 49 | 30 |
| 41 | 0 | 0·6561 | 0·8693 | 1·1504 | 0·7547 | 49 | 0 |
| 41 | 30 | 0·6626 | 0·8847 | 1·1303 | 0·7490 | 48 | 30 |
| 42 | 0 | 0·6691 | 0·9004 | 1·1106 | 0·7431 | 48 | 0 |
| 42 | 30 | 0·6756 | 0·9163 | 1·0913 | 0·7373 | 47 | 30 |
| 43 | 0 | 0·6820 | 0·9325 | 1·0724 | 0·7314 | 47 | 0 |
| 43 | 30 | 0·6884 | 0·9490 | 1·0538 | 0·7254 | 46 | 30 |
| 44 | 0 | 0·6947 | 0·9657 | 1·0355 | 0·7193 | 46 | 0 |
| 44 | 30 | 0·7009 | 0·9827 | 1·0176 | 0·7133 | 45 | 30 |
| 45 | 0 | 0·7071 | 1·0000 | 1·0000 | 0·7071 | 45 | 0 |
| ° | ′ | Cosines. | Cotangents. | Tangents. | Sines. | ° | ′ |

## TABLE OF SINE SQUARED AND COSINE SQUARED.

Read downwards from 0° to 45° and upwards from 45° to 90°.

| Degrees. | Sin.² | Cos.² | Degrees. | Degrees. | Sin.² | Cos.² | Degrees. |
|---|---|---|---|---|---|---|---|
| 0 | 000000 | 1000000 | 90 | 23 | 152646 | 847354 | 67 |
| 1 | 000306 | 999694 | 89 | 24 | 165405 | 834595 | 66 |
| 2 | 001218 | 998782 | 88 | 25 | 178591 | 821409 | 65 |
| 3 | 002735 | 997265 | 87 | 26 | 192195 | 807805 | 64 |
| 4 | 004872 | 995128 | 86 | 27 | 206116 | 793884 | 63 |
| 5 | 007586 | 992414 | 85 | 28 | 220430 | 779560 | 62 |
| 6 | 010920 | 989080 | 84 | 29 | 235031 | 764969 | 61 |
| 7 | 014859 | 985141 | 83 | 30 | 250000 | 750000 | 60 |
| 8 | 019377 | 980623 | 82 | 31 | 265225 | 734775 | 59 |
| 9 | 024461 | 975529 | 81 | 32 | 280794 | 719206 | 58 |
| 10 | 030137 | 969863 | 80 | 33 | 296589 | 703411 | 57 |
| 11 | 036405 | 963595 | 79 | 34 | 312605 | 687395 | 56 |
| 12 | 043183 | 956817 | 78 | 35 | 329017 | 670983 | 55 |
| 13 | 050625 | 949375 | 77 | 36 | 345499 | 654501 | 54 |
| 14 | 058516 | 941484 | 76 | 37 | 362163 | 637837 | 53 |
| 15 | 066977 | 933023 | 75 | 38 | 379086 | 620914 | 52 |
| 16 | 075995 | 923005 | 74 | 39 | 396018 | 603982 | 51 |
| 17 | 085498 | 914502 | 73 | 40 | 413192 | 586808 | 50 |
| 18 | 095481 | 904519 | 72 | 41 | 430467 | 569533 | 49 |
| 19 | 106015 | 893985 | 71 | 42 | 447695 | 552305 | 48 |
| 20 | 116964 | 883036 | 70 | 43 | 465124 | 534876 | 47 |
| 21 | 128450 | 871550 | 69 | 44 | 482608 | 517392 | 46 |
| 22 | 140325 | 859675 | 68 | 45 | 500000 | 500000 | 45 |
| Degrees. | Cos.² | Sin.² | Degrees. | Degrees. | Cos.² | Sin.² | Degrees. |

## USEFUL DATA.

The circumference of a circle $= 2\pi r$. The diameter of a circle $= C/\pi$.

The area of a circle $= \pi r^2$. The surface of a sphere $= \pi d^2$.

The volume of a sphere $= \pi d^3/6$. The area of a triangle $=$ Per $\times$ base/2.

The length of an arc $= \cdot 01745 r \times$ No. of degrees.

The area of a sector of circle $=$ area of circle $\times$ degrees of arc/360.

The area of a circular ring $= \pi/4 \times (d_1^2 - d_2^2)$.

The perimeter of an ellipse $= \pi \sqrt{(a^2 + b^2)/2}$. The area of an ellipse $= \pi ab/4$.

The value of $\pi = 3\cdot 14159$ or approx. 22/7.

The radian is an angle subtended by an arc equal to its radius $= 360/2\pi = 180/\pi = 57\cdot 3°$.

# INDEX

*Those numbers in heavy type indicate chapters or main articles on particular subjects. The subheadings to the articles serve as further guides.*

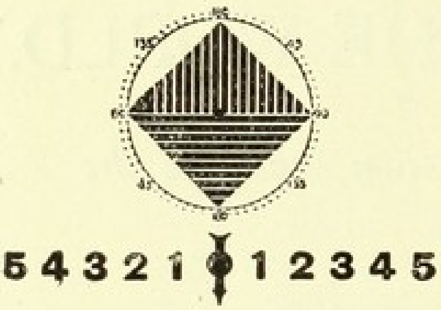
THE END

BILLING AND SONS, LTD., PRINTERS, GUILDFORD

# ORTHOPS SPECIALITIES.


## THE ORTHOPS CHART.

A complete self-contained chart designed for the most modern system of sight testing. Every detail has been determined or calculated with a definite object, and on scientific principles.




## THE ORTHOPS NEAR MUSCLE-TEST.

It is now generally recognized that a complete knowledge of the muscular imbalance cannot be obtained from the distance-test alone. The Orthops near muscle-test, which has been designed to afford such a near test at the customer's ordinary reading point, is small and inexpensive, and no refractionist should be without it. It is employed in the ordinary way with the Maddox rod or groove, enabling the imbalance for distance and near to be compared almost simultaneously.

## THE ORTHOPS TRIAL-FRAME.

This frame combines all the simplicity and rapidity in use of the best variety of drop frame combined with all the necessary facial adjustments. In it the clumsy and painful hook sides have been abolished in favour of long powerful straight sides which keep the frame firmly yet comfortably in place. It is hand-made, and therefore light and rigid ; it holds three lenses in each eye, and the broad swinging bridge adapts itself to any type of nose.

## THE ORTHOPS FRAME AND FACE-RULE.

This well-known rule is practically unique in that with it facial and frame measurements can be taken with equal ease and accuracy. Made in boxwood or xylonite. Full descriptive pamphlet on application.

---

*Orthops Specialities can be obtained from all Wholesale Optical Houses.*

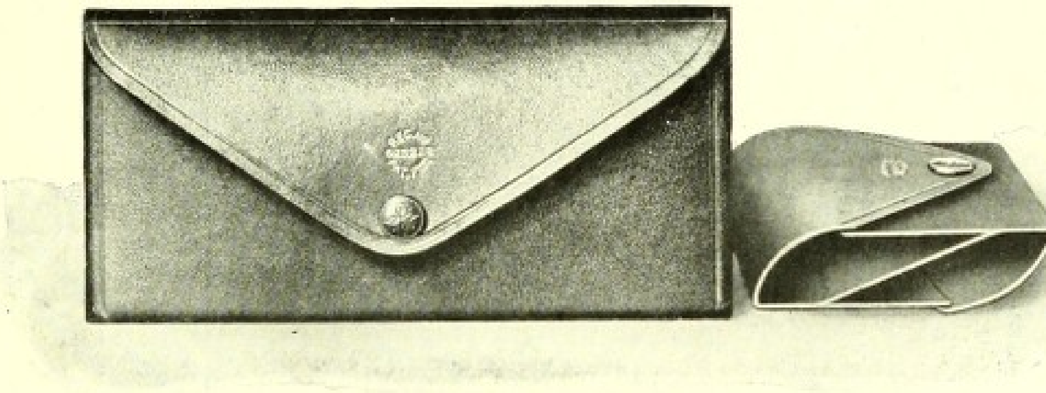
# "DUPLEX" CASES.

(Patent No. 3240/13.)

Designed to accommodate two pairs of eye-glasses, or two pairs of spectacles, or one of each.

The accompanying illustration indicates how a diagonal partition separates the overlapping two pairs of glasses, with the most important result that

**THE "DUPLEX" CASE IS CAPABLE OF CONTAINING TWO PAIRS OF GLASSES WHILE BEING SCARCELY MORE BULKY THAN A SINGLE CASE.**



As every optician knows, customers complain of the inconvenience of carrying two cases when, as very frequently happens, they require to have about them two pairs of glasses.

**THE "DUPLEX" CASE REMOVES THIS CAUSE OF COMPLAINT.**

In many an instance two pairs of glasses would be purchased if they could be carried conveniently.

**THE "DUPLEX" CASE ALLOWS THIS TO BE DONE.**

Surely no optician will neglect the opportunity of selling two pairs of glasses instead of one.

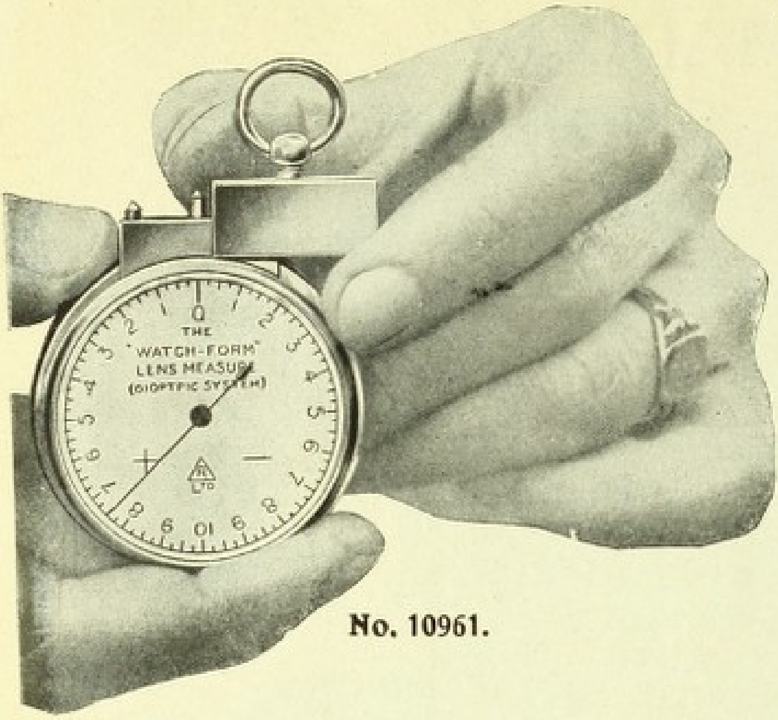
**THE "DUPLEX" CASE VERY STRONGLY OFFERS THIS OPPORTUNITY.**

"Duplex" cases are made in various qualities and in two sizes—for two pairs of eye-glasses or for two pairs of spectacles.

They can be obtained from all wholesale optical houses, or direct from the manufacturers :

## WILLMOTTS, LTD., EVESHAM, Worcs.

And 55, HATTON GARDEN, LONDON, E.C.

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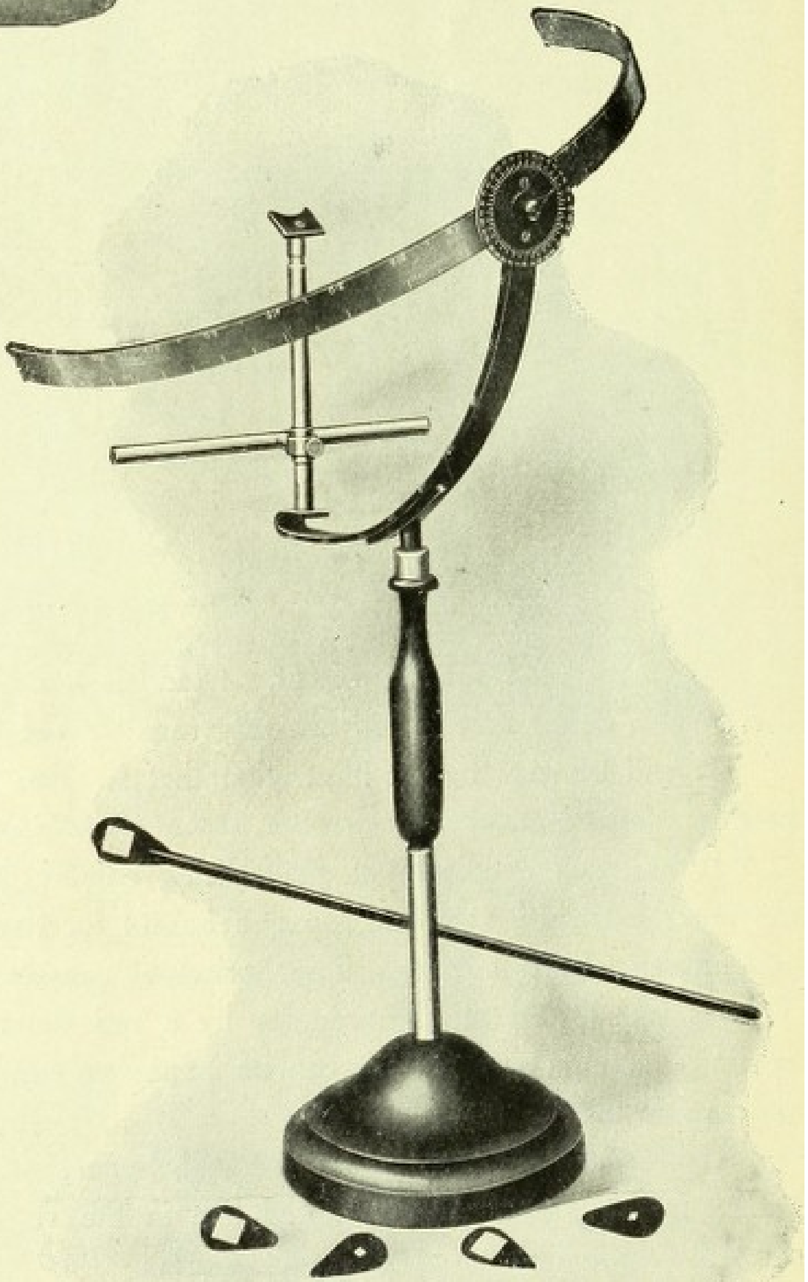

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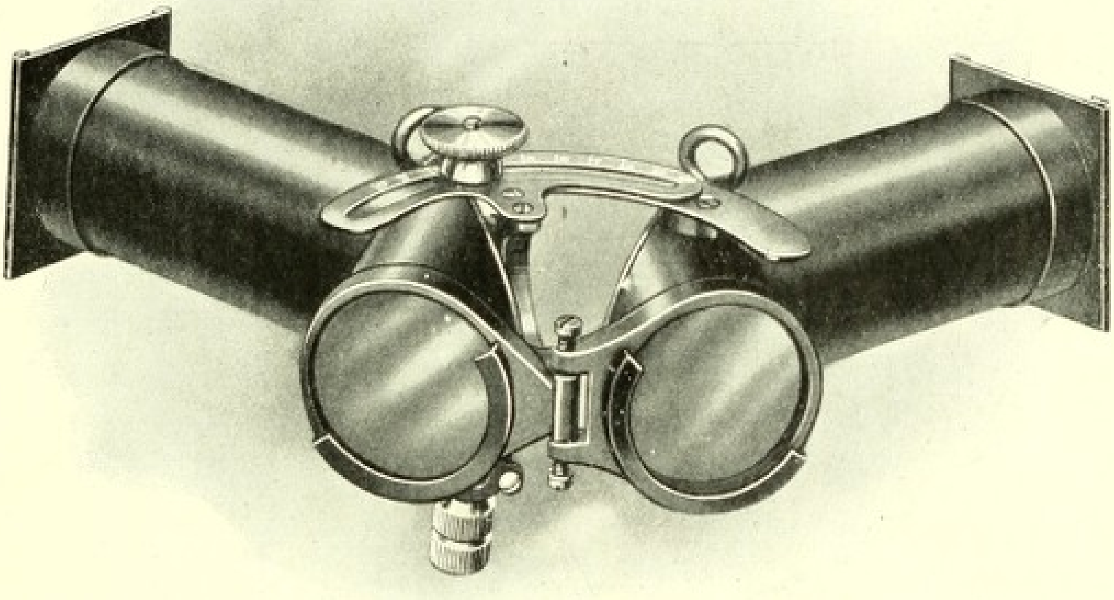

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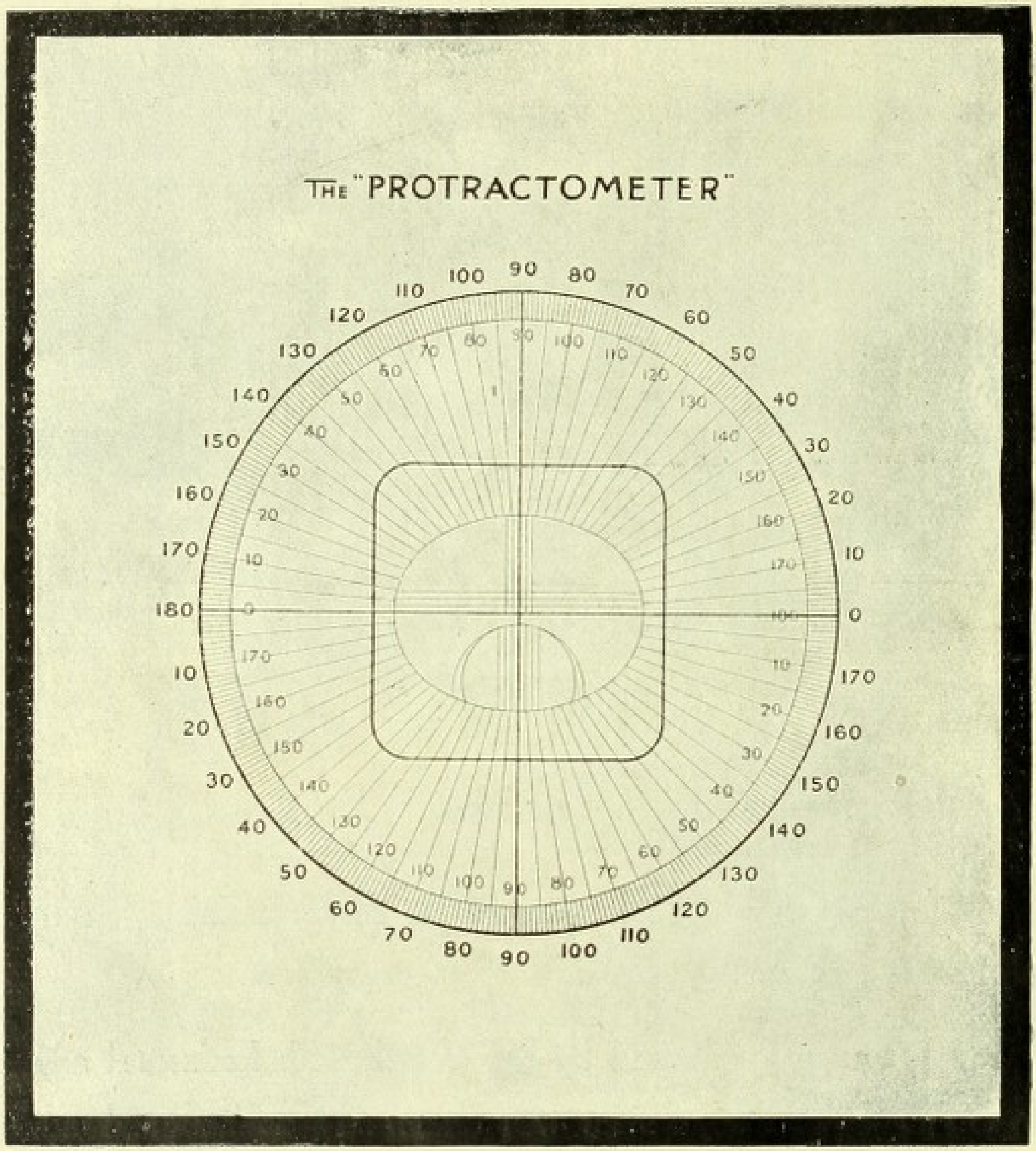


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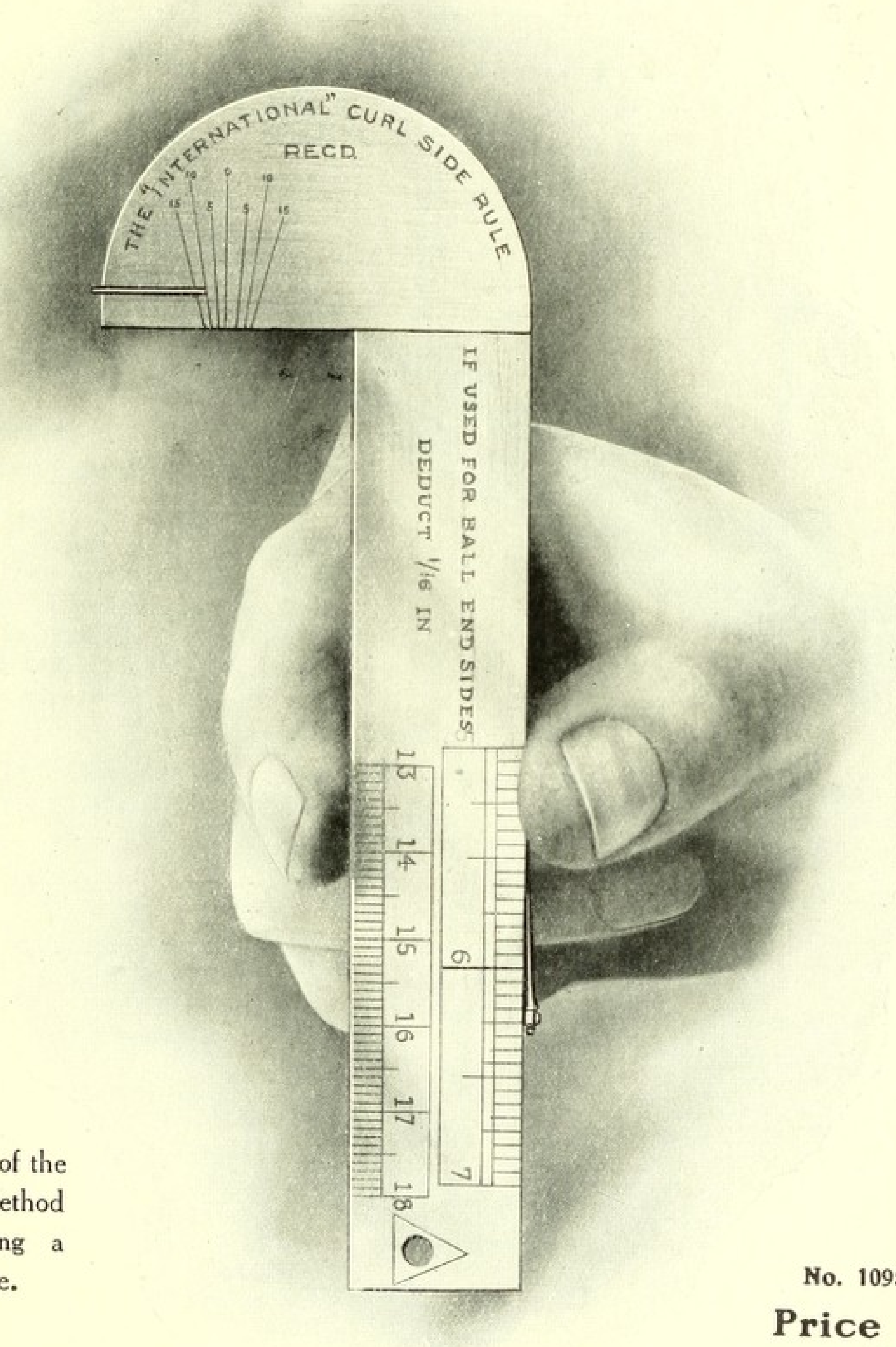


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