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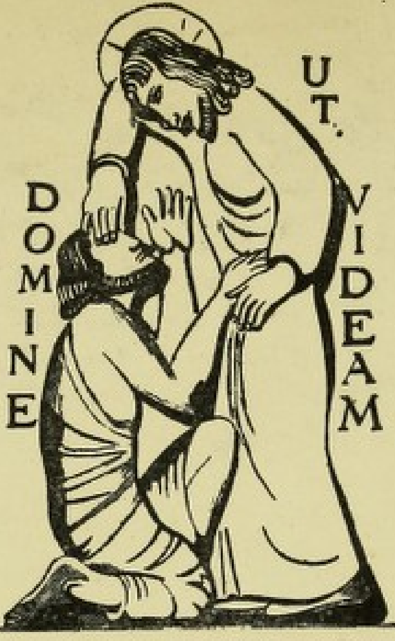

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THE PRINCIPLES AND METHODS OF GEOMETRICAL OPTICS

ESPECIALLY AS APPLIED TO THE THEORY OF OPTICAL INSTRUMENTS

BY

JAMES P. C. SOUTHALL
Professor of Physics in the Alabama Polytechnic Institute

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TO

HENRY C. LOMB, Esq.

WHOSE KIND ENCOURAGEMENT AND EFFECTUAL AID
WILL ALWAYS BE REMEMBERED

THIS VOLUME IS GRATEFULLY INSCRIBED BY

THE AUTHOR

PREFACE.

From time to time, almost like a voice crying in the wilderness, some one is heard to lament the apathy with which Geometrical Optics is regarded in this country and in England;[1] although it is sufficient merely to call the roll of such names as Barrow, Newton, Cotes, Smith, Blair, Young, Airy, Hamilton, Herschel, Rayleigh, etc., in order to be reminded that this domain of science was once at any rate within the sphere of British influence. At present, however, it can hardly be gainsaid that the great province of applied optics is almost exclusively German territory; so that not only is it a fact that nearly all of the extraordinary developments of modern times in both the theory and construction of optical instruments are of German origin, but it is equally true also that until at least quite recently[2] there was actually no treatise on Optics in the English language where the student could find, for example, hardly so much as a reference to the remarkable theories of Petzval, Seidel and Abbe—to mention only such names as are inseparably associated with the theory of optical imagery. Partly with the object of supplying this deficiency, and partly also in the hope (if I may venture to express it) of rekindling among the English-speaking nations interest in a study not only abundantly worthy for its own sake and undeservedly neglected, but still capable, under good cultivation, of yielding results of far-reaching importance in nearly every field of scientific research, I have prepared the following work, wherein my endeavour has been to lay before the reader a connected exposition of the principles and methods of Geometrical Optics, especially such as are applicable to the theory of optical instruments; and although I am regretfully aware of many shortcomings in the execution of this task, I cling to the hope that they will be perhaps not so apparent to many of my critics as they are to myself.

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[1] Referring to Czapski's *Theory of Optical Instruments* (the first edition of which was published in 1893) and to the volume on *Optics* in the ninth (1895) edition of Mueller-Pouillet's *Physics*, Professor Silvanus P. Thompson, in the preface of his valuable translation of Dr. O. Lummer's *Contributions to Photographic Optics* (London, 1900), writes as follows:

"Both these works are in German, and most unfortunately no translation of either has appeared—most unfortunately, for there is no English work in optics that is at all comparable to either of these. I say so deliberately, in spite of the admirable article by Lord Rayleigh on 'Optics' in the *Encyclopædia Britannica*, in spite of the existence of those excellent treatises, Heath's *Geometrical Optics*, and Preston's *Theory of Light*. No doubt such books as Heath's *Geometrical Optics* and Parkinson's *Optics* are good in their way. They serve admirably to get up the subject for the Tripos; but they are far too academic, and too remote from the actual modern applications. In fact, the science of the best optical instrument-makers is far ahead of the science of the text-books. The article of Sir John Herschel 'On Light' in the *Encyclopædia Metropolitana* of 1840 marks the culminating point of English writers on optics."

[2] H. Dennis Taylor's *A System of Applied Optics* (London, 1906) is a most valuable work by the inventor of the celebrated "Cooke" lenses for photography.

I have not hesitated to use, especially in connection with the geometrical theory of optical imagery in Chapters V and VII, the elegant and direct methods of the modern geometry, but these applications are always so simple and elementary that it is hardly to be feared that any readers will be deterred thereby.

In the theory of optical imagery developed by GAUSS with such rare analytical skill, it is assumed that both the aperture and the field of view of the optical system of centered spherical surfaces is exceedingly small, so that all the rays concerned in the production of the image are comprised within a narrow cylindrical region immediately surrounding the optical axis. In the design of telescopes with objectives of considerable diameter, the necessity of taking account of the so-called spherical aberration due to the increase of the aperture was first recognized; which led to the well known investigations on this subject of EULER, BESSEL, AIRY, GAUSS, SEIDEL and others. With the development of the microscope and the birth and growth of photography, new requirements had to be filled in order to portray parts of the object which were not situated on the optical axis, so as to correct, if possible, the aberrations due not only to increase of the aperture but also to increase of the field of view. This difficult problem, undertaken first by PETZVAL with only partial success, was investigated by SEIDEL, professor of mathematics in the University of Munich, in a series of papers contributed to the *Astronomische Nachrichten* in the year 1855; wherein, by an extension of GAUSS's methods so as to include in the series-developments the terms of the next higher order, elegant and entirely general formulæ are derived in a comparatively simple way, which enable one to perceive almost at a glance how the faults in an image formed by a centered system of spherical refracting surfaces are due partly to the size of the aperture and partly also to the extent of the field of view. These methods and theories are treated at length in Chapter XII.

Prism-Spectra and the Chromatic Aberrations of Dioptric Systems are the subjects that are included under the head of "Colour-Phenomena" in Chapter XIII.

One of the most important divisions is Chapter XIV, wherein the reader will find a fairly complete treatment of ABBE's theory of the limiting of the ray-bundles by means of perforated diaphragms or "stops," which has so much to do with the practical efficiency of an actual optical instrument.

Without entering more fully into the contents of the various chapters, it may be stated that the work as a whole is designed as a general introduction to the special theory of optical instruments (telescope, microscope, photographic objective, etc., including also the eye itself). To discuss properly and fully each of these types would require a separate and extensive volume, which I may be induced to undertake at some future time as a sequel to the present work.

A complete system of notation which is free from objection is difficult to devise; and, in spite of the pains I have bestowed on the matter and the importance which I have attached to it, I do not doubt that fault will be found not only with the plan which I have adopted but with many of the characters which I have introduced. My object has been to make the work convenient as a book of reference, so that the meaning of a symbol and of the marks that distinguish it would be immediately obvious as far as possible; but in order to aid the reader still further in this respect, the principal uses of the letters in both the diagrams and the formulæ are quite fully explained in an appendix at the end of the volume. In some instances the same letter or sign has been employed deliberately in two or more totally different senses, but only where there seemed to be no chance of confusion, and because also I have tried carefully to avoid resorting to strange and uncouth symbols which often make a mathematical work appear to be far more difficult and uncanny than it really is.

The original sources from which I have borrowed have been given, as far as possible, either in the text or in the foot-notes. I am especially aware of how much I have derived in one way or another from Dr. CZAPSKI's epoch-making book, *Die Theorie der optischen Instrumente nach* ABBE, and from *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904) edited by Dr. M. VON ROHR under the auspices of CZAPSKI himself and in collaboration with the staff of optical engineers connected with the world-famous establishment of CARL ZEISS in Jena. This latter work—which is, in fact, the offspring of the former, and in whose praise one might well exclaim, "*O matre pulchra filia pulchrior!*"—is a vast treasury of optical theory amassed by experts in the various branches of Geometrical Optics which will remain for many years to come the standard book of reference on this subject.

I gladly take this opportunity of expressing my thanks to Professor CHARLES HANCOCK, of the University of Virginia, who made the drawings of the diagrams, and to my colleague Professor A. H. WILSON and my assistant Mr. C. D. KILLIBREW who have helped me with the proof-reading. I esteem it a privilege to be permitted to dedicate the work to HENRY C. LOMB, Esq., of Rochester, N. Y.

JAMES P. C. SOUTHALL.

AUBURN, ALA.,
December 1, 1909.

PREFACE TO THE SECOND EDITION.

The kindness with which this book has been received has so far exceeded my expectations that I am encouraged to bring out a new edition. Several sections (for example, the whole of Art. 15 and § 102) have been entirely re-written, and various omissions have been supplied. In particular, I hope that the appendices which have been added at the ends of Chapters XI and XII will be found to contain new material of value.

To readers both in Europe and in America I am indebted for numerous suggestions and especially for bringing to my attention quite a number of slips here and there which I have diligently sought to correct in this new edition. It is not possible to acknowledge here all these kindnesses, but at least I avail myself of this opportunity of expressing my thanks to Dr. REGINALD S. CLAY, of London, for a very complete list of *errata* which he compiled.

More than one of my critics have justly reproached me for making no allusion to E. T. WHITTAKER's excellent and original little volume on *The theory of optical instruments* (Cambridge, 1907). Equally reprehensible, I may add, was my failure to mention also J. G. LEATHEM's *Elementary theory of the symmetrical optical instrument* (Cambridge, 1908).[1]

JAMES P. C. SOUTHALL.

AUBURN, ALA.,
March 1, 1913.

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[1] The following is a partial list of some of the more notable works on Optics which have been published in recent years:

A. GLEICHEN: *Leitfaden der praktischen Optik* (Leipzig, 1906).

M. VON ROHR: *Die optischen Instrumente* (Leipzig, 1906).

M. VON ROHR: *Die binokularen Instrumente* (Berlin, 1907).

H. VON HELMHOLTZ: *Handbuch der physiologischen Optik*, dritte Auflage ergaenzt u. herausgegeben in Gemeinschaft mit Prof. Dr. A. GULLSTRAND u. Prof. Dr. J. VON KRIES von Prof. Dr. W. NAGEL. (The first volume of this work which contains notable contributions by GULLSTRAND appeared in 1909.)

E. ABBE: *Die Lehre von der Bildentstehung im Mikroskop:* bearbeitet und herausgegeben von O. LUMMER u. F. REICHE (Braunschweig, 1910).

C. HESS: *Die Refraktion und Akkomodation des menschlichen Auges und ihre Anomalien.* GRAEFE-SAEMISCHS *Handbuch der Augenheilkunde.* 3. Aufl., Kap. XII (Leipzig, 1910).

A. PELLETAN: *Optique appliquée* (Paris, 1910).

A. S. PERCIVAL: *The Prescribing of Spectacles* (Bristol, 1910).

A. GLEICHEN: *Die Theorie der modernen optischen Instrumente* (Stuttgart, 1911).

CHR. VON HOFE: *Fernoptik* (Leipzig, 1911).

M. VON ROHR: *Die Brille als optische Instrument* (Leipzig, 1911).

R. W. WOOD: *Physical Optics*, new and revised edition (New York, 1911).

R. S. CLAY: *Treatise on Practical Light* (London, 1912).

P. DRUDE: *Lehrbuch der Optik.* 3. Aufl. (Leipzig, 1912).

P. G. NUTTING: *Outlines of Applied Optics* (Philadelphia, 1912).

M. VON ROHR: *Das Auge und die Brille* (Leipzig, 1912).

A. S. PERCIVAL: *Geometrical Optics* (London, 1913).

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GEOMETRICAL OPTICS.

CHAPTER I.

METHODS AND FUNDAMENTAL LAWS OF GEOMETRICAL OPTICS.

ART. 1. THE THEORIES OF LIGHT.

1. According to the Corpuscular or Emission Theory of Light, maintained and developed by NEWTON, the sensation of Light is due to the impact on the retina of very minute particles, or *corpuscles*, projected from a luminous body with enormous speeds and proceeding in straight lines. Thus, in NEWTON's famous work on Optics,[1] published in 1704, he asks: "Are not rays of light very small bodies emitted from shining substances? For such bodies will pass through uniform mediums in right lines without bending into the shadow, which is the nature of the rays of light." Opposed to this view was the Undulatory Theory of Light, which, notwithstanding the speculations that have been found in the writings of earlier philosophers, such as LEONARDI DA VINCI, GALILEO and others, must beyond doubt be attributed to HUYGENS as its author, whose work, entitled *Traité de la lumière* (Leyden, 1690), was based on the assumption that the phenomena of light were dependent on an hypothetical *Ether*, or very subtle, imponderable and exceedingly elastic medium, which not only pervaded all space but penetrated freely all material bodies solid, liquid and gaseous. According to this theory, the something that was emitted from a luminous body was not matter at all but a kind of *Wave-Motion* which was propagated through the all-pervading ether with a finite speed which is different according to the different circumstances in which the ether through which the disturbance advanced is conditioned by the presence of ordinary gross matter. This remarkable and ingenious theory encountered at first great difficulties, and even HUYGENS himself was not able to give satisfactory explanations of some of the most familiar phenomena of light. In the end, however, it was destined to triumph, and, in the hands of such advocates as YOUNG and above all of FRESNEL (who, in order to account for the

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[1] I. NEWTON: *Opticks: or a treatise of the reflexions, inflexions, and colours of light* (London, 1704); see Book iii., Qu. 29.

Polarisation of Light, was led to assume that the ether-vibrations were transversal), the Wave-Theory won its way to the front rank of science, where it remains to-day more firmly established than ever.

The Electromagnetic Theory of Light, which is a development from the Wave-Theory, is a monument to the genius and mathematical insight of MAXWELL, but the experimental basis of this theory is to be found in the investigations of HERTZ, who showed that electrical energy also was propagated by means of ether-waves which, under certain circumstances, obeyed the laws of Reflexion and Refraction and travelled with the speed of light.

2. But, independent of any of the theories as to the real nature of light, there are certain well-ascertained facts about the mode of propagation of light which may themselves be made the basis of a certain science of light, and which—provided we are careful to confine our investigations along these lines within justifiable limits—will lead us often by the easiest route to a true knowledge, at any rate, of the behaviour and effects of light. Moreover, these cardinal facts, which we may call the fundamental characteristics of the mode of propagation of light, are so few and so simple and suffice to explain such a large class of important phenomena, especially those phenomena on which the design and construction of optical instruments chiefly depend, that the advantage of this method has been long recognized.

ART. 2. THE SCOPE AND PLAN OF GEOMETRICAL OPTICS.

3. The fundamental characteristics of the mode of propagation of light may be enumerated under three heads as follows:

(1) The Law of the Rectilinear Propagation of Light, from which we derive the ideas of "rays" of light; (2) the assumption that the parts of a beam of light are mutually independent; so that, for example, the effect produced at any point is to be attributed only to the action of the so-called "rays" which pass through that point; and, finally, (3) The Laws of Reflexion and Refraction of Light.

These laws, inasmuch as they are concerned essentially only with the direction of the propagation of light, are purely geometrical; and, hence, the science which is based upon them, and which seeks, by their means, to explain the phenomena of light, either as they occur in nature or as they are produced by the agency of optical instruments, is called *Geometrical Optics*.

4. But while it is the peculiar office of Geometrical Optics to give as far as possible explanations of such phenomena of light as depend simply on changes in the directions in which the light is propagated, it does not pretend to be able to explain *all* such phenomena; and, especially, it excludes as outside of its province all cases in which the light is propagated in anisotropic or crystalline media.

Moreover, also, although the fundamental laws above-mentioned are sufficient in themselves to construct a very complete and satisfactory system of explanation of a large class of optical phenomena, it must not be supposed that Geometrical Optics is willing to dispense entirely or even partially with the more accurate ideas and conceptions which are to be derived only by the consideration of the real and essential nature of light. If such were to be our procedure, we should often go astray, and, indeed, we know by experience that when Geometrical Optics has ignored or even lost sight of the notions of the Wave-Theory of Light, and pushed too far the geometrical consequences of the fundamental theorems on which it is based, erroneous results have been obtained. On the contrary, the wave-phenomena of interference and the like must be kept throughout constantly in view even when they are not paraded to the front, and every result should be subjected to the test of the methods of Physical Optics. Viewed in this way, Geometrical Optics is not to be regarded as a mere mathematical discipline—as is sometimes said by way of reproach—but it takes its rank as a useful and important branch of Physics.

ART. 3. THE RECTILINEAR PROPAGATION OF LIGHT.

5. *In an isotropic medium light travels in straight lines*, is the statement of a fact, which, if not absolutely and unexceptionally true, certainly cannot be far from the truth; and, indeed, until comparatively recent times this statement had never been called in question. The fact is confidently assumed not merely in the ordinary affairs of life but in the most exact measurements both in Geodesy and in Astronomy, and, so far as these sciences are concerned, its validity has never been doubted. In order to view a star through a long narrow tube, the axis of the tube must be pointed so that it coincides with the straight line which joins the (real or apparent) position of the star with the eye of the observer. In aiming a rifle or in any of the processes that we call "sighting" the method is based with certainty upon this commonest fact of experience. The most conclusive proof that a line is straight consists in showing that it is the path which light pursues. The greatest difficulty that HUYGENS encountered in his wave-theory of light was to explain its apparent rectilinear propagation. It was from this law that the idea of a "ray of light" originated.

Nevertheless, the law is only approximately true, as has been well ascertained now for more than a century. For when we proceed to subject it to as rigid a test as possible, and try, by means of screens with very narrow openings, to separate from a beam of light the so-called "rays" themselves, we discover that these latter have in reality no physical existence; and that the narrower we succeed in making the opening, the less do we realize the idea conveyed by the term "ray". When the light arrives at the narrow opening, it does not merely pass through it without changing its direction, but it spreads out laterally as well, utterly misbehaving itself so far as the law of rectilinear propagation is concerned. Thus, although the straight line joining a point-source of light with an eye may pierce an interposed screen at an opaque part of the screen, a narrow slit in another part of the screen may enable the eye to perceive the source. When an opaque object is interposed between a point-source of light and a screen, the shadow on the screen will be found to correspond less and less with the geometrical shadow in proportion as the dimensions of the opaque body are made smaller and smaller, and, in fact, the very places where, on the hypothesis of the rectilinear propagation of light, we should expect shadows often prove to be places of quite contrary effects, and *vice versa*. The fact is, light is propagated not by "rays" but by waves, and the rectilinear propagation of light is practically true in general because the wave-lengths of light are so minute. But when we have to do with narrow apertures and obstacles whose dimensions are comparable with those of the wave-lengths, we have the so-called Diffraction-effects which are treated at great length in works on Physical Optics and which can only be alluded to here.

6. However, in order to arrive at a clear comprehension of the matter, let us consider briefly the explanation afforded by the wave-theory of the mode of propagation of light in an isotropic medium. We may begin by giving **Huygens's Construction of the Wave-Front,** which enables us to see how HUYGENS himself tried to explain the assumed rectilinear propagation of light.

Let O (Fig. 1) be a point-source[1] of light, or a luminous point, from which as a centre or origin ether-waves proceed with equal speeds in all directions. At the end of a certain time the disturbances will have arrived at all the points which lie on a spherical surface σ described around the centre O, which is the locus of all the points in the isotropic medium that are in this particular initial phase of vibration, and which is the *Wave-Front* at this instant. According to HUYGENS, every point P in the wave-front, from the instant that the disturbance reaches it, will become a new source or centre of disturbance, from which secondary waves will be propagated in all directions. Moreover, HUYGENS assumed that these secondary waves, originating at all the points affected by the principal wave, interfere with each other in such fashion that their resultant sensible effects are produced only at the points of the surface which envelops at any given instant all the secondary wave-fronts, and that this enveloping surface is, therefore, the wave-front at that instant. Obviously, in an isotropic medium, such as is here supposed, this surface will be a sphere described around O as centre.

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[1] An actual " point-source " of light by itself cannot be physically realized. A " luminous point " is an infinitely small bit of luminous surface. Nevertheless, exactly as in Mechanics we are accustomed to speak of " particles of matter", and similarly in all branches of Theoretical Physics, we may make use in Optics of this convenient and useful conception, whether it be actually realizable or not.

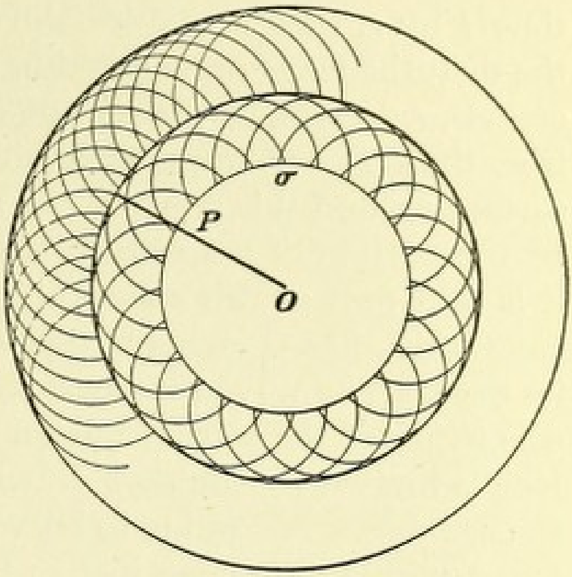

FIG. 1.
HUYGENS'S CONSTRUCTION OF THE WAVE-FRONT.

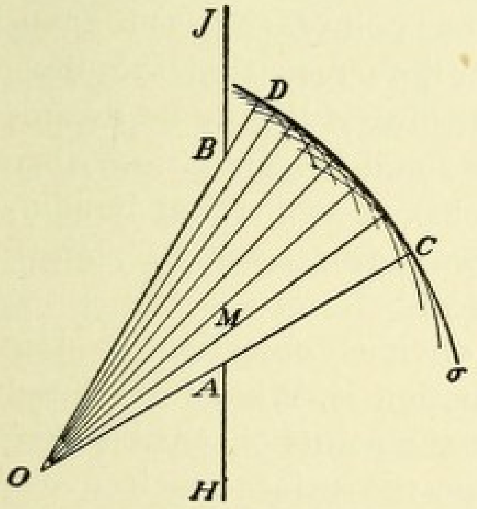

FIG. 2.

HUYGENS'S CONSTRUCTION OF THE WAVE-FRONT. Spherical waves diverging from the point-source O and passing through an opening AB in the opaque screen HJ. The arc DC is a section of the portion of the spherical wave-front σ which contains the points beyond the screen which have been reached by the disturbance.

Accordingly, if waves diverge from a luminous point O (Fig. 2), and if an opaque plane screen HJ with an opening AB is interposed in front of the advancing waves, the wave-front at any time t may be constructed as follows: Consider all the points, such as M, which lie in the plane of the screen at the place where the aperture is made. As soon as the disturbance arrives at one of these points, it will become a new centre of disturbance, from which will diverge, therefore, secondary spherical waves. In general, the radii of these secondary waves will be different. Thus, in the diagram, as here drawn, the point designated by A is nearer the source O than the point designated by M, so that the disturbance must arrive at A first, and hence the secondary wave proceeding from A will have had time to travel farther than the secondary wave originating at M. If we put $OM = x$, and if we denote the radius of the secondary spherical wave around M at the time t by r, then $d = x + r$ will denote the distance from O to which the disturbance is propagated in the time t, which shows that as x increases, r decreases; that is, the greater is the distance of the point M from the source O, the smaller will be the radius of the secondary wave-surface around this point M. Thus, the enveloping surface is seen to be the portion of a spherical surface of radius d around O as centre: it is that part of this spherical surface which is comprised within the cone which has O for its vertex and the opening AB of the screen for its base. The wave proceeds, therefore, from O into the space on the other side of the screen, but on this side of the screen the wave-surface is limited by the rays drawn from O to the points in the edge of the opening. According to HUYGENS's view, the disturbance is propagated within this cone just as though the screen were not interposed at all, whereas points on the far side of the screen but outside this cone of rays are not affected at all. This mode of explanation leads to the theory of the rectilinear propagation of light.

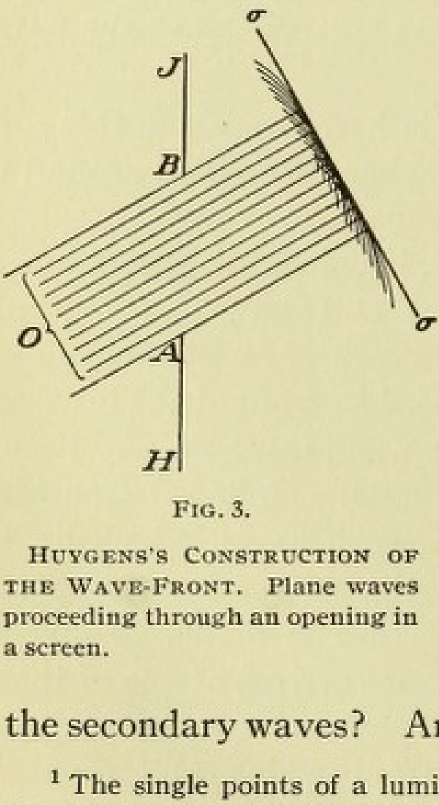

FIG. 3.

HUYGENS'S CONSTRUCTION OF THE WAVE-FRONT. Plane waves proceeding through an opening in a screen.

If the luminous point O (Fig. 3) is at such a distance from the screen that the dimensions of the opening AB may be regarded as vanishingly small in comparison therewith, we shall have a cylindrical bundle of rays, and the wave-fronts will be plane instead of spherical.[1]

The most obvious objection to HUYGENS's construction is, What right has he to assume that the points of sensible effects are the points on the surface which envelops the secondary waves? And why is the light not propagated backwards as well as forwards? Moreover, if the opening in the screen is very narrow, this construction does not correspond at all with the observed facts.

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[1] The single points of a luminous heavenly body are to be regarded as at an infinite distance in comparison with the dimensions of our apparatus, so that the wave-fronts of the disturbances emitted from such points are plane. But the rays which come from different points of a celestial body cannot be regarded as parallel unless the parallax of the star is sufficiently small. This angle has a right considerable magnitude in the cases of both the sun and the moon, so that the divergence of the rays which come from opposite ends of the diameters of these bodies may amount to more than half a degree. For most experiments in Optics this divergence is negligible, and a beam of sunlight may be regarded as consisting of parallel rays. We may obtain bundles of parallel rays from terrestrial sources of light by means of lenses, etc.

7. Fresnel's Extension of Huygens's Method. In place of HUYGENS's arbitrary assumption that the places where there are sensible effects are to be found only on the surface which envelops the elementary waves, FRESNEL insisted that these secondary waves, encountering each other, must therefore be regarded as interfering with each other, and thus he conceived that the disturbance at any point P (Fig. 4) must be due to the superposition of the component disturbances propagated to P from all the points of the wave-surface σ. According to FRESNEL, therefore, light-effects are to be found, not on the enveloping surface, but at all points where the secondary waves combine to reinforce each other.

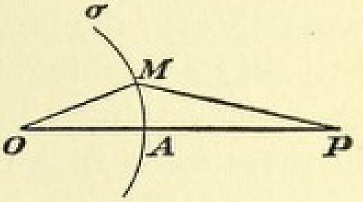

FIG. 4.

FRESNEL'S METHOD. The effect at the point P of a disturbance originating at O is to be attributed almost entirely to the disturbance that is propagated along the straight line OP; provided the wave-lengths are very small.

On investigation—which we do not attempt to show here—it appears that the disturbances which arrive at P along all the straight lines joining P with points on the wave-front in great measure neutralize each other, and the result is (assuming that the wave-length is small) that *the actual effect at P may be considered as due wholly to the action of a very small element of the wave-front situated at the point A where the straight line joining O with P intersects the wave-front σ.* (This point A is called the "pole" of the wave with respect to the point P; it is the point of the wave-front that is nearest to P, so that the disturbance from this point arrives at P before the disturbance from any other point of the wave-front.) Hence, if between O and P we interpose a small opaque screen which exactly shuts off from P the effect due to the small "zone" around A, there will be darkness at P; moreover, what is true of this point P is true also of any point which, like P, is situated on the straight line OP. On the other hand if a plane screen is placed tangent to the wave-surface at A, with a small circular opening in it the centre of which is at A, so that the point P is screened from the entire wave-surface except the very small "effective zone" immediately around A, the effect at P, as also at all points along the straight line AP, is found to be precisely the same as though the screen had not been interposed. It is thus that the idea of HUYGENS as developed by FRESNEL leads, as we see, to the theory of the approximate rectilinear propagation of light—that is, light does in fact behave very nearly as if it were propagated in straight lines.

8. Although, therefore, this fundamental law has always to be stated with certain reservations, and, as a matter of fact, is never strictly true, yet even when it is regarded from the standpoint of the wave-theory, the law of the rectilinear propagation of light loses very little of its meaning. On the contrary, in agreement with experience, that theory shows that in the cases which ordinarily occur, especially in those cases where we have to do with beams of light of finite dimensions, the effects at any rate are for all practical purposes the same as if these beams of light were composed of separate rays, each independent of the others, along which the light is propagated in straight lines. But, however useful and generally safe this simple and convenient rule may be, it must be borne in mind that it is inexact and we must be prepared, therefore, to meet here and there exceptional cases where the rule is plainly inadmissible. It is only in this way that the methods of Geometrical Optics can be approved.

ART. 4. RAYS OF LIGHT.

9. A self-luminous point is said to emit "rays of light" in all directions. In an isotropic medium (§10) the ray-paths are straight lines proceeding from the centre of the expanding spherical wave-surface; and whether the medium is isotropic or not, the direction of the ray-path at any point is to be considered as being always along the normal to the wave-surface that goes through that point (see §42). What are called "rays of light" in Geometrical Optics are in fact those shortest paths, optically speaking (§38), along which the ether-disturbances are propagated. Employed in this sense, the word "ray" is a purely geometrical idea. However, there is a certain sense in which we can attach a physical meaning also to these so-called "light-rays". For, as a rule, it is approximately true that the ether-disturbance at any point of the path of a ray of light is due to disturbances which have occurred successively at all points along the ray that are nearer to the source than the point in question; so that, according to this view, the effect at any point P is to be considered as in no degree arising from disturbances at other points which do not lie on a ray passing through P. This is, in fact, the **Principle of the Mutual Independence of the Rays of Light,** which is also one of the fundamental laws of Geometrical Optics, and which assumes that each ray in a beam of light is somehow separate and distinct from its fellows, and has, therefore, a certain physical existence. Thus, for example, if we have a wide-angle cone of rays incident on a screen and producing there a comparatively large light-spot, and if we interpose an opaque object so as to intercept a considerable fraction of the rays before they reach the screen, a corresponding portion of the light on the screen will vanish; and, hence, it can be inferred that we may suppress some of the rays in a beam without altering, apparently, the effect produced by the remaining rays.

Here also, however, when this principle is examined from the standpoint of the wave-theory, we find that it, too, has to be stated with reservations. According to FRESNEL, the disturbance at the point P (Fig. 4) is to be considered as the resultant of an infinite number of partial disturbances propagated to P from all points situated on the wave-front σ; so that in a certain sense P may be considered as being at the vertex, or "storm-centre", of a cone of rays which are by no means independent of each other. Every point, such as P, which lies ahead of the advancing wave-front is in similar circumstances. But, as has been stated (§7), the resultant effect at the point P is due in the main to the disturbance that is propagated along the central ray of the cone of rays that converge to P; and, thus, the law of the Mutual Independence of Rays, if it is true at all, can only be said to be true of these central rays of all such cones of rays as are here meant. In point of fact, the resultant effect at the point P is to be ascribed not merely to the disturbance propagated along this central ray from the pole A of the wave, but to a zone of the wave-surface of very small, but finite, dimensions, with its vertex at A. And the moment we attempt to isolate physically the ray AP by screening P from the effects of this zone, the effect at P vanishes entirely and the ray ceases to exist.

10. It is best, therefore, without any reference to its physical meaning, to define a ray of light as a line or path along which the ether-disturbance is propagated. An optical medium is any space, whether filled or not with ponderable matter, which may be traversed by rays of light. In Geometrical Optics, where we have to do only with isotropic media, the rays of light are straight lines (Art. 3). At a surface of separation of two media the direction of the ray will usually be changed abruptly, either when the ray passes from one medium into the next or is bent away at the surface of a body; so that under such circumstances the ray-path will consist of a series of straight line-segments. If, for example, B_k designates the point where the ray meets the kth surface, then the straight line-segment $B_{k-1}B_k$ will represent the path of the ray in the kth medium: and here it may be remarked that, so long as we are speaking of this portion of the ray-path, any point P lying on the straight line determined by B_{k-1} and B_k is to be considered as situated in the kth medium, even though the substance of which the medium is composed does not extend out to the point P. If the point P is situated on the straight line $B_{k-1}B_k$ between these two incidence-points, we say that the ray in this medium passes *really* through the point P; otherwise, we say that the ray goes *virtually* through the point P.

ART. 5. THE BEHAVIOUR OF LIGHT AT THE SURFACE OF SEPARATION OF TWO ISOTROPIC MEDIA.

11. In order to have clear ideas of certain matters mentioned in the preceding articles, it will be necessary to know how the rays of light are affected when they arrive at the boundary-surface separating two adjoining optical media. At such a surface the "*incident*" light (as it is called) will, in general, be divided into two portions, which are propagated from the places where the light falls on the surface in abruptly changed directions:

(1) One portion of the light is turned back or "*reflected*" at the surface, and pursues its progress in the first medium along new ray-paths (except under special conditions).

(2) The remaining portion, crossing the surface and entering the second medium, makes its way, in general, in this new region; this is the so-called "*refracted*" light.

12. However, here also a closer study of these phenomena reveals the fact that neither of the above statements is an entirely accurate description. Thus, it will be found that even that part of the light which is said to be reflected and which ultimately returns into the first medium had crossed the boundary-surface and penetrated a little way into the second medium. This is the explanation of the colour of a body as seen by reflected light: the incident light falling on the body and penetrating to a slight extent below the surface is there, according to the "Theory of Selective Absorption" (into which we cannot enter here), robbed of certain of its constituent parts, and only the remainder is finally reflected. The depth of penetration depends on the qualities of the two media and in a very great degree on the character of the separating surface. Thus, for example, if the second medium is glass, this question will involve the knowledge of whether the glass is in a compact (solid) state or in the form of a fine powder; and if the glass were solid, the next question would be as to the surface, whether it was highly polished or not, etc.

When a beam of sunlight is admitted through an opening in a shutter into an otherwise dark room, and is allowed to fall, for example, on a metallic surface, the reflected light itself consists of two portions, viz., one part (in this case the greater part) which leaves the metallic surface in a perfectly definite direction, and which is said therefore to be *regularly reflected*, and another part which leaves the surface in countless different directions, and which is said to be "scattered". This scattering or "diffusion" of the reflected light is due to the inequalities or rugosities of the surface; it may be greatly diminished by cleaning and polishing the surface. If the reflecting surface is geometrically regular and physically smooth, the reflected light will be nearly all regularly reflected. And even in those cases where the light is irregularly reflected or diffused, as, for example, when a beam of sunlight is reflected from a ground-glass surface, it would be more correct to attribute the irregularity not so much to the behaviour of the rays of light as to the peculiarity of the surface itself. Perhaps, if we knew precisely the arrangement and orientation of the elements of such a reflecting surface, we should discover that the reflexion was quite regular after all. However, the actual dimensions of these rugosities of the surface will also affect the phenomenon, inasmuch as when these dimensions are sufficiently small, the assumptions which lie at the foundation of Geometrical Optics will cease to be valid.

It is in consequence of this fact, that the light which is incident on a rough surface is subjected to different experiences at the different places in the surface, that these irregularities are made visible to us as themselves sources of rays of light; whereas if the reflecting surface were perfectly smooth, so that the rays were regularly reflected all according to the same law, we should not be able to see the surface at all, we should see merely the images of objects from which the rays had come—objects which were either self-luminous or else illuminated by diffusely reflected light. Moreover, in order to view the images, the eye would have to be placed somewhere along these special routes of the reflected rays; otherwise, none of these rays would enter the eye and nothing would be visible by the reflected light. Most objects are seen by diffusely reflected light, and no matter where the eye is situated, it will intercept some of the rays that are scattered from the surface of the body.

13. In large measure the above observations concerning the portion of the light that is reflected apply also to the other portion that is refracted. If the surface of separation of the two media is smooth, the directions of the refracted rays will, in general, depend only on the directions of the incident rays according to the so-called Law of Refraction; and in this case the light is said to be *regularly refracted.* But if the boundary-surface is rough, the rays will be *diffusely refracted* in all directions ("irregular refraction").

The light which enters the second medium may be modified in various ways. A greater or less portion of it, depending on the character and peculiarity of the medium, will be *absorbed;* that is, the ether loses some of its energy and ordinary matter gains it. Invariably, a fraction of the light-energy will be transformed into heat, possibly also into chemical and electrical forms of energy. If the medium is perfectly *transparent*, the rays of light traverse it without being absorbed at all; whereas if the medium absorbs all the light-rays, it is said to be perfectly *opaque.* No medium is absolutely transparent on the one hand or absolutely opaque on the other. A perfectly transparent body would be quite invisible, although we may easily be made aware of the presence of such a body by the distortion of the images of bodies viewed through it. As a rule, the absorptive power of a medium will depend on the colour (or wave-length) of the light. Thus, a piece of green glass will allow only certain kinds of light to pass through it, and therefore when the rays of the sun fall on it, it will absorb some of these rays and be transparent to others, and the transmitted light falling on the retina of the eye, will produce a sensation which we describe vaguely as green light. An interesting phenomenon occurs called *Fluorescence*, whereby the colour of the light undergoes a change in the second medium.

Again, there are some media which, although they cannot be called transparent, nevertheless permit light to pass through them in a more or less irregular and imperfect fashion; for example, such substances as porcelain, milk, blood, moist atmospheric air, which contain suspended or imbedded in them particles of matter of a different optical quality from that of the surrounding mass. The light undergoes internal diffused reflexion at these particles. Objects viewed through such media can be discerned, perhaps, but always more or less indistinctly. These so-called "cloudy media" are said, therefore, to be *translucent*, but not transparent.

It is usually assumed in Geometrical Optics that the media are not only homogeneous, but perfectly transparent; and also that the surfaces of separation between pairs of adjoining media are perfectly smooth.

ART. 6. THE LAWS OF REFLEXION AND REFRACTION.

14. Let $\mu\mu$ (Fig. 5) be the trace in the plane of the diagram of the smooth reflecting or refracting surface separating two transparent isotropic media. Let PB represent the rectilinear path of a ray of light in the first medium (a). The ray PB is called the *incident ray*, the point B where this ray meets the boundary-surface between the two media (a) and (b) is called the *incidence-point*, the normal NN' to the surface at the point B is called the *incidence-normal*, and the plane PBN determined by the incident ray and the incidence-normal (which is here the plane of the paper) is called the *plane of incidence*.

In general, to an incident ray PB there will correspond two rays, viz., a *reflected ray* BR, which remains in the first medium (a) and a *refracted ray* BQ, which shows the path taken by the light in the second medium (b). The acute angles at the incidence-point B between the incidence normal NBN' and the rays PB, BR and BQ are called the *angles of incidence, reflexion and refraction*, respectively. Each of these angles is defined as *the acute angle through which the incidence-normal has to be turned in order to bring it into coincidence with the straight line which shows the path of the ray in question*. Thus, in the diagram the angles of incidence, reflexion and refraction are $\angle NBP$, $\angle NBR$ and $\angle N'BQ$, respectively; where the order in which the letters are written indicates the sense of rotation. These angles are to be reckoned as positive or negative according as the sense of rotation is counter-clockwise or clockwise.

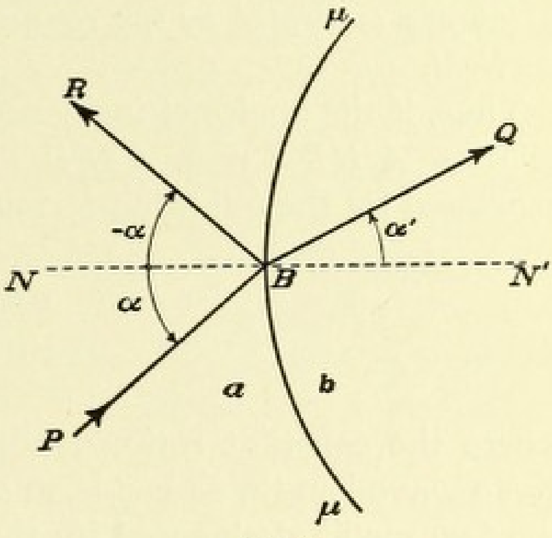

Fig. 5.

Laws of Reflexion and Refraction. $\mu\mu$ is a section in the plane of incidence (plane of paper) of the surface separating the first medium (a) from the second medium (b). The point B is the point of incidence, and NN' is the normal to the surface at this point. PB, BR and BQ are the incident, reflected and refracted rays, respectively.

$\angle NBP = \alpha$, $\angle NBR = -\alpha$, $\angle N'BQ = \alpha'$.

15. The Laws of Reflexion and Refraction, as determined by experiment, may now be set forth in the following statements:

(1) *Both the reflected and the refracted rays lie in the plane of incidence.*

(2) *The reflected ray in the first medium and the refracted ray in the second medium lie on the opposite side of the normal from the incident ray in the first medium.* Or if we prolong the refracted ray backwards

into the first medium, we see that in this medium the straight lines belonging to the incident and refracted rays lie on the same side of the incidence-normal, whereas the incident and reflected rays lie on opposite sides of the normal. Thus, while the angles of incidence and refraction have always like signs, the angles of incidence and reflexion have opposite signs.

(3) *The magnitudes of the angles of incidence and reflexion are equal;* that is,

$$\angle NBP = \alpha = - \angle NBR = \angle RBN.$$

(4) *The sines of the angles of incidence and refraction are in a constant ratio, the value of which depends only on the nature of the two media which are separated by the refracting surface and on the wave-length of the light.*

Thus, if the angles of incidence and refraction are denoted by α, α', so that $\angle NBP = \alpha$, $\angle N'BQ = \alpha'$, the law of refraction may be expressed by the following formula:

$$\frac{\sin \alpha}{\sin \alpha'} = n_{ab}; \tag{1}$$

where the constant ratio, denoted here by n_{ab}, which for light of a given wave-length, as has been stated, depends only on the nature of the two media designated by the letters a and b, is called *the relative index of refraction from the medium* (a) *into the medium* (b), or the index of refraction of medium (b) with respect to medium (a). The order in which the subscripts are written is the same as the order in which the media are traversed by the light.

16. The best experimental proof of the law of reflexion is obtained by the use of a theodolite or meridian circle to observe the light reflected from an artificial mercury-horizon. This is the actual method employed in the astronomical measurement of the altitude of a star, and is capable of a very high degree of accuracy.

The law of refraction may be regarded as completely verified by the methods which are employed in the determinations of the values of the indices of refraction for different pairs of media, and, above all, in the design and construction of optical instruments, by the complete agreement between the actual performances of such apparatus and the calculations based on the law of refraction.

17. The law of the reflexion of light is very ancient. The earliest precise statement of the law is to be found in a work on optics attributed to Euclid (300 B. C.). On the other hand, the law of refraction is much more modern. Claudius Ptolemæus, the great astronomer, who flourished during the reigns of the Antonines, published a treatise on optics ('Οπτικὴ πραγματεία) in which he describes a number of experiments whereby he measured the angles of incidence and refraction, without, however, discovering the law. The next experiments along this line of which we have any record are those of Alhazen who died in Cairo in 1038; he repeated the experiments of Ptolemæus, but added nothing to the previous knowledge of the matter. Kepler also made experiments, but was equally unsuccessful. The real discoverer of the law was Willebrord Snell, of Leyden, who announced it some time prior to 1626. It was first published by Descartes[1] in 1637; who seems undoubtedly to have obtained it from Snell, although he failed to mention his name in connection with it.

18. In the case of Reflexion, it is obvious that the directions of the incident and reflected rays may be reversed, so that if PBR (Fig. 5) represents the path pursued by the light in going from P to R, undergoing reflexion at the incidence-point B, then RBP will represent the path which the light takes in going from R to P under the same circumstances, that is, *via* the incidence-point B. Experiment shows that the same rule holds good also for the ray refracted at B; so that if PBQ is the route followed by a ray in going from a point P in the medium (a) to a point Q in the medium (b), undergoing refraction at the incidence-point B, the same route will be pursued in the reverse sense QBP by a ray whose direction in the medium (b) is from Q towards B. And, hence, since

$$\frac{\sin\alpha}{\sin\alpha'} = n_{ab}, \quad \frac{\sin\alpha'}{\sin\alpha} = n_{ba},$$

we have obviously, the relation:

$$n_{ab} \cdot n_{ba} = 1. \tag{2}$$

This general law of optics, known as the **Principle of the Reversibility of the Light-Path,** may be stated as follows:

If a ray of light, undergoing any number of reflexions and refractions, pursues a certain route from one point A to another point A', and if at A' it is incident normally on a mirror so that it is reflected back from A' in the direction exactly opposite to that by which it arrived, it will return over precisely the same route in the reverse order and arrive finally at A again.

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[1] René Du Perron Descartes: *Discours de la méthode pour bien conduire sa raison et chercher la verité dans les sciences; plus la Dioptrique, les Météores et la Géometrie* (Leyden, 1637).

19. The Laws of Reflexion and Refraction as derived by the Wave-Theory (Huygens's Construction). A plane wave travelling in an isotropic medium advances with uniform speed and without change of form along the direction of the normal to its plane, so that the rays are parallel to each other and perpendicular to the plane wave-front. If the wave-front arrives at a smooth geometric surface separating the first medium (a) from another isotropic optical medium (b), the refracted wave proceeding into this second medium will, in general, be changed both in form and in direction. At the same time also a

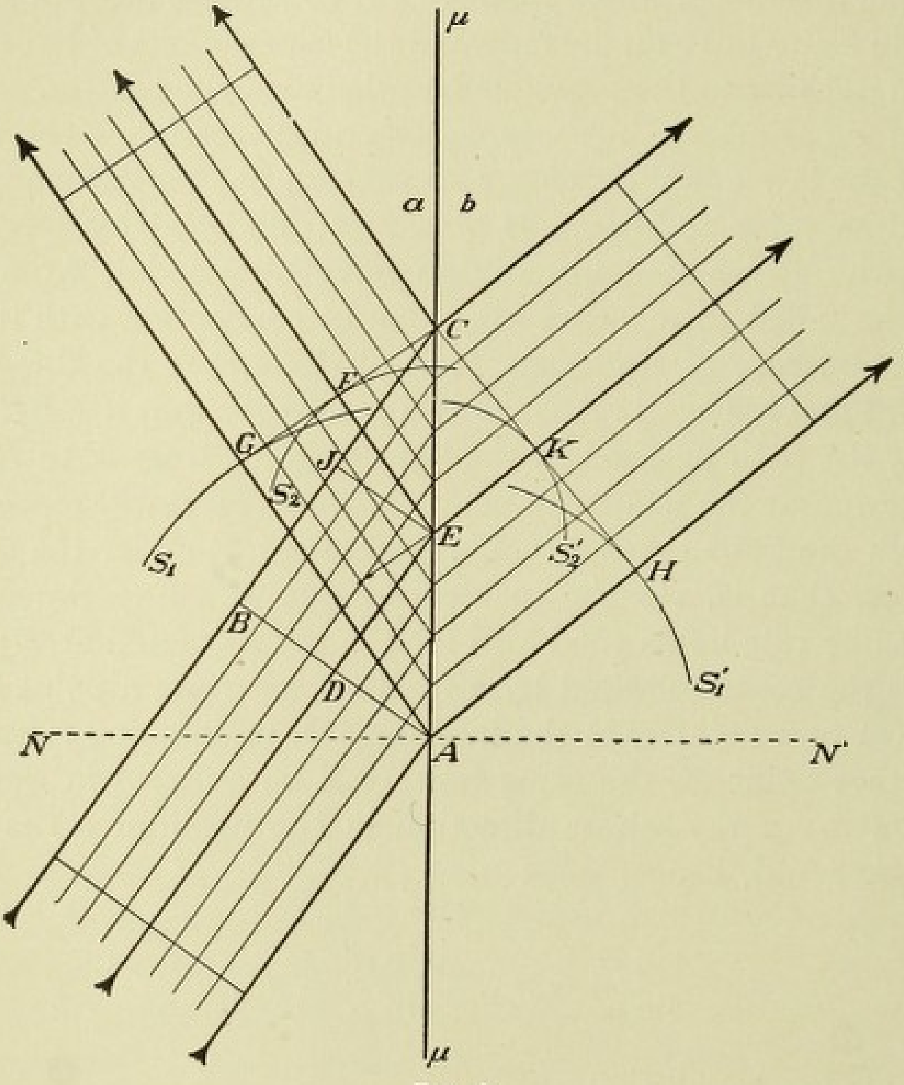

Fig. 6.

Huygens's Construction of the Reflected and Refracted Wave-Fronts in the Case of a Plane Wave Incident on a Plane Surface. $\mu\mu$ is a section in the plane of incidence (plane of paper) of a plane surface separating the first medium (a) from the second medium (b). The straight lines AB, CG and CH are the traces in the plane of the paper of the incident, reflected and refracted wave-fronts.

wave will be reflected back from the boundary-surface into the first medium, which likewise may be changed both in form and in direction. But the speed of propagation of the reflected wave will be the same as that of the incident wave; whereas the speed of propagation of the refracted wave in the new medium (*b*) will be different from that of the incident wave in the medium (*a*).

For the sake of simplicity, let us suppose that the two media (*a*) and (*b*) are separated by a *plane surface.* We proceed to give **HUYGENS's Construction of the Reflected and Refracted Wave-Fronts** for this case. In the diagram (Fig. 6) $\mu\mu$ represents the trace in the plane of the paper of the plane surface separating the media (*a*) and (*b*); and AB is the trace in the same plane of a portion of the advancing incident plane wave; so that the incident rays in the plane of the paper will be represented by straight lines perpendicular to AB, such as BC and DE. At the instant when we begin to reckon time the incident wave-front is supposed to be in the position shown by AB, and hence at this moment the disturbance will have just arrived at the point A of the plane surface $\mu\mu$. From this moment, therefore, according to HUYGENS's idea, this disturbed point A is itself to be regarded as a centre of disturbance, and from it as centre elementary hemispherical waves are propagated not only into the second medium (*b*) but also back into the first medium (*a*). Exactly the same condition will be true at this instant ($t = 0$) of every point in the plane surface situated on the straight line perpendicular to the plane of the paper at the point A. The envelope of each of these two sets of equal hemispherical surfaces will be a semi-cylinder, whose axis is the straight line just mentioned. A little later the disturbance which was initially at D will reach the point E in the line $\mu\mu$; and if v_a denotes the speed with which the disturbance is propagated in the medium (*a*), the moment when it arrives at E will be $t = DE/v_a$. Beginning from this moment the two sets of semi-cylindrical surfaces which have for their common axis the straight line perpendicular at E to the plane of the paper will begin to be formed. And, thus, at successively later and later instants, the disturbance will arrive in turn at all the points in $\mu\mu$ which lie between A and C; until, finally, at the time $t = BC/v_a$ the disturbance reaches the extreme point C. Meanwhile, around all the straight lines perpendicular to the plane of the diagram at the points on $\mu\mu$ which lie between A and C two sets of co-axial semi-cylindrical elementary wave-surfaces have been forming, one set being propagated back into the first medium (*a*) and the other set being propagated forward into the second medium (*b*). The nearer one of these points between A and C is to the point C, the smaller will be the radius of the corresponding semi-cylinder.

20. Let us consider, first, the **Reflected Wave.** At the moment $t = BC/v_a$, when the point C begins to be disturbed, the semi-cylindrical wave S_1 whose axis passes through A will have expanded in the first medium until its radius is equal to BC. At this same instant the semi-cylindrical wave S_2 whose axis is determined by the point E will have been expanding into the first medium during the time $BC/v_a - DE/v_a$, so that the disturbance will have been propagated a distance $BC - DE = JC$, which is therefore the radius of this cylindrical surface.

According to HUYGENS's Principle, the surface which at any instant is tangent to all the elementary semi-cylindrical reflected waves will be the required reflected wave-front at that instant. We shall show that this reflected wave-front is a plane surface which at the moment when the disturbance reaches C contains this point; or, what amounts to the same thing, we shall show that if the line CG in the plane of the diagram touches at G the semi-circle in which the plane cuts the semi-cylinder S_1, it will be the common tangent of all such semi-circles; for example, it will be tangent to the semi-circle S_2 around any point E as centre. From C draw CG tangent to S_1 at G and CF tangent to S_2 at F. Draw AG and EF. The triangles CGA and ABC are congruent, since the angles at B and G are both right angles and $AG = BC$. Hence, $\angle GCA = \angle BAC$. Similarly, from the congruence of the triangles CFE and CEJ, it follows that $\angle FCE = \angle JEC$. And since $\angle BAC = \angle JEC$, we have $\angle GCA = \angle FCE$; and, consequently, the tangent-lines CG and CF coincide. Hence, the trace in the plane of the paper of the reflected wave-front is the straight line CFG. This reflected plane wave will be propagated onwards, parallel with itself, in the direction shown in the diagram by the reflected rays AG, EF, etc. It is evident from the construction that the ray incident at A, the normal AN to the reflecting surface at A and the corresponding reflected ray AG are all situated in the same plane, viz., here the plane of the paper which is the plane of incidence for the ray in question. It only remains therefore to show that the angles of incidence and reflexion are equal. This is obvious also from the congruence of the triangles CGA and CBA.

21. **The Refracted Wave.** If the velocity of propagation of the wave in the second medium (b) is denoted similarly by v_b, it is plain that at the moment $t = BC/v_a$ when the disturbance reaches the point C, the secondary disturbance which proceeds from A as centre will have been propagated into the medium (b) to a distance $AH = v_b t = v_b \cdot BC/v_a$; and, similarly, the disturbance at any intermediate point, as E, between A and C, will have been propagated in the second medium to a distance $EK = (BC - DE)v_b/v_a = EJ \cdot v_b/v_a$. Thus, the radii of the elementary semi-cylindrical refracted waves S'_1 and S'_2, whose axes are perpendicular to the plane of the paper at A and E, are $BC \cdot v_b/v_a$ and $EJ \cdot v_b/v_a$, respectively. The refracted wave-front at any instant will be the surface which is tangent to all these elementary cylindrical surfaces at this instant. Exactly the same method as we used in finding the reflected wave-front can be employed here; and we shall find that at the instant when the disturbance reaches C the refracted wave-front is the plane containing the point C which is perpendicular to the plane of the paper and tangent to the elementary wave S'_1 at H.

SNELL's law of refraction may be deduced at once by observing that in the figure $AG = AC \cdot \sin\alpha$, where $\alpha = \angle ABC$ is equal to the angle of incidence of the parallel incident rays, and $AH = AC \cdot \sin\alpha'$, where $\alpha' = \angle ACH$ is equal to the angle of refraction of the parallel refracted rays; and, consequently:

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{AG}{AH} = \frac{v_a}{v_b} = \text{constant.} \tag{3}$$

22. In the figure the case is represented where the disturbance is propagated faster in the first medium (a) than in the second medium (b), that is, v_a is greater than v_b. In this case the angle of refraction α' is less than the angle of incidence α, and hence the refracted rays are bent towards the normal, as, for example, when light is refracted from air into glass. According to the Wave-Theory of Light, therefore, the velocity of propagation in the optically denser of the two media is less than it is in the other medium. Now the NEWTONian or Emission Theory of Light leads to precisely the opposite conclusion. The two theories are here in direct conflict with each other, and experiment has decided in favor of the Wave Theory. ARAGO, in 1838, suggested the method of measuring the speed of propagation of light which was afterwards (1865) successfully employed by FOUCAULT. FOUCAULT's experiments demonstrated that light travelled faster, for example, in air than in water. These experiments were subsequently repeated by MICHELSON, with an improved form of apparatus, and MICHELSON found that the speed of light in air was 1.33 times as great as that in water, which agrees with the value of the relative index of refraction of air and water. The same experimenter found that the speed in air was 1.77 times the speed in carbon bisulphide, whereas the value of n for these two substances is about 1.63, so that in this case the agreement was not so close.

ART. 7. ABSOLUTE INDEX OF REFRACTION OF AN OPTICAL MEDIUM.

23. According to the Wave-Theory, therefore, the relative index of refraction of two media (a) and (b) is equal to the ratio of the speeds of propagation of light in the two media. And, hence, if we know the indices of refraction of a medium (c) with respect to each of two media (a) and (b), we can easily compute the value of the relative index of refraction of the two media (a) and (b) with respect to each other. For, according to formulæ (1) and (3), we shall obtain:

$$n_{ab} = \frac{v_a}{v_b}, \quad n_{ac} = \frac{v_a}{v_c}, \quad n_{bc} = \frac{v_b}{v_c};$$

and therefore:

$$n_{ab} = \frac{n_{ac}}{n_{bc}};$$

which, according to (2), may be written also:

$$n_{ab} = \frac{n_{cb}}{n_{ca}}.$$

For example, suppose that the substances designated by the letters a, b and c are water, glass and air, respectively, and that we know the values of the relative indices of air and water and of air and glass, viz., $n_{ca} = 4/3$ and $n_{cb} = 3/2$; then the value of the relative index of refraction from water to glass will be $n_{ab} = (3/2):(4/3) = 9/8$.

Generally, it may be shown that if the letters $a, b, c, \ldots i, j, k$ are employed to designate a number of optical media, then:

$$n_{ab} \cdot n_{bc} \cdot n_{cd} \cdots n_{ij} \cdot n_{jk} = n_{ak}. \tag{4}$$

And, in particular, if the last medium (k) is identical with the first medium (a), the continued product of the relative indices of refraction will be equal to unity; formula (2) states this law for the case where there are only two media (a) and (b).

24. The fact that $n_{ab} = n_{cb} : n_{ca}$ suggests the idea of employing some *standard optical medium* (c) with respect to which the indices of refraction of all other media could be expressed. The medium that is selected for this purpose is that of empty space or vacuum, and the index of refraction of a medium with respect to empty space is called, therefore, the **absolute index of refraction** of the medium, or, simply, *the refractive index* of the medium. Accordingly, the absolute index of refraction of empty space is itself equal to unity, and if n_a, n_b denote the absolute indices of two media (a) and (b), then evidently:

$$n_{ab} = \frac{n_b}{n_a}. \tag{5}$$

The absolute indices of refraction of all known transparent media are greater than unity. However, KUNDT[1] determined, in 1888, the indices of refraction of a number of metallic substances, using very thin prisms of the materials which he subjected to investigation; and the values of n which he obtained in the case of silver, gold and copper were all less than unity: which implies that light travels faster in each of these metals than it does *in vacuo*. See also more recent experiments with such substances as these, especially those of DRUDE and MINOR in 1903.

The index of refraction of air, at 0° C. and under a pressure of 76 cm. of mercury for light corresponding to the FRAUNHOFER D-line has been found to be equal to 1.000293; it is usually taken as equal to unity.

25. With every isotropic optical medium there is associated, therefore, a certain numerical constant n; and thus when a ray of light is refracted from a medium of index n into another medium of index n', the trigonometric formula of the law of refraction may be written in the following symmetrical form:

$$n \cdot \sin \alpha = n' \cdot \sin \alpha'; \tag{6}$$

which may be stated by saying:

At every refraction of a ray of light from one medium to another, the product of the refractive index of the medium and the sine of the acute angle between the ray and the incidence-normal remains unchanged.

This product

$$K = n \cdot \sin \alpha \tag{7}$$

is sometimes called the *Optical Invariant*.

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[1] A. KUNDT: Ueber die Brechungsexponenten der Metalle: *Ann. der Phys.* (3), xxxiv. (1888), 469–489.

26. Reflexion considered as a Special Case of Refraction. Whereas the angles of incidence and refraction have like signs always, on the contrary the signs of the angles of incidence and reflexion are always opposite. In order, therefore, that formula (6) may be applicable also to the case of reflexion as well as to that of refraction, the values of n and n' in the former case must be such that $\alpha' = -\alpha$ is a solution of the equation in question; and the condition that we shall have this solution is evidently:

$$n' = -n, \quad \text{or} \quad n'/n = -1.$$

Thus, it will not be necessary to investigate separately and independently each problem of reflexion; for so soon as we have discovered in any special case the relation between the incident ray and the corresponding refracted ray, we have merely to impose the condition

$$n' = -n$$

in order to ascertain directly the relation which under the same circumstances exists between the incident ray and the corresponding reflected ray. This procedure, which will be frequently employed in the following pages, will be found to be exceedingly convenient and serviceable, besides saving much needless labour.

Here, also, we take occasion to say that hereafter whenever we speak of the "direction of a straight line"—that is, *the positive direction* of the line—we shall mean always *the direction from a point on the line in the medium of the incident rays towards the point where the line meets the reflecting or refracting surface.* If the straight line is itself the *path of an incident or refracted ray* of light, the positive direction as thus defined will be *the direction along the line in which the light goes;* but if the straight line is the *path of a reflected ray*, the positive direction in this case (assuming that there is only one reflecting surface) will be opposite to that which the light actually follows. It will be well to bear this in mind, especially in deriving reflexion-formulæ from the corresponding refraction-formulæ by the method above mentioned. (See §176; see also §251.)

ART. 8. THE CASE OF TOTAL REFLEXION.

27. The formula

$$\sin\alpha' = \frac{n}{n'}\sin\alpha$$

enables us to calculate the magnitude of the angle of refraction α', so soon as we know the values of the indices n, n' of the two media and the magnitude of the angle of incidence α; and thus we can determine the direction of the refracted ray corresponding to a given incident ray. However, the solution of the above equation is not always possible, for if the magnitudes denoted by the symbols n, n' and α are such that the expression on the right-hand side of the equation turns out to have a value greater than unity, evidently there will be no angle α' that can satisfy the equation, and hence in such a case there will be no refracted ray corresponding to the given incident ray. In order to make this matter clear, let us distinguish here two cases as follows:

(1) The case when $n' > n$; as, for example, when the light is refracted from air to water ($n'/n = 4/3$). In this case the second medium is said to be more highly refracting, or "optically denser", than the first medium. The angle of incidence α will be greater than the angle of refraction α', so that a ray, entering the second medium from the first, will be *bent towards the incidence-normal.* Under these circumstances, the value of the expression on the right-hand side of the above equation will be always less than unity, so that there is always a certain angle α' whose sine has this value. *Provided the second medium is optically denser than the first, to every incident ray there will always be a corresponding refracted ray.*

(2) The case when $n' < n$; as, for example, when the light is refracted from water to air ($n'/n = 3/4$); in which case the first medium is the optically denser of the two. The angle of incidence α now will be less than the angle of refraction α', so that the refracted ray will be *bent away from the incidence-normal.* When n is greater than n', the expression on the right-hand side of the above equation may be less than, equal to or greater than unity, depending on the value of the incidence-angle α. For a certain limiting value $\alpha = A$ of the angle of incidence, we shall have $n \cdot \sin\alpha/n' = 1$, and hence $\alpha' = 90°$. In this case, therefore, the refracted ray corresponding to an incident ray which meets the refracting surface at an angle of incidence A such that

$$\sin A = \frac{n'}{n} = n_{ab}, \tag{8}$$

will lie in the tangent-plane to the refracting surface at the point of incidence. If the two media (a) and (b) are separated by a plane surface, the refracted ray in this limiting case will proceed along the surface, or, as we say, just "graze" the surface. This angle A between the incidence-normal and the direction of the ray in the denser of the two media is called the *critical angle* for the two media (a) and (b). In formula (8) n denotes always the refractive index of the denser of the two media ($n_{ab} < 1$); so that, for example, if the two media are air and water, the water corresponds to medium (a) and the air to medium (b), and hence we have $n_{ab} = 3/4$, for which we find $A = 48° 27' 40''$. For air and glass, $n_{ab} = 2/3$ and $A = 42° 37'$.

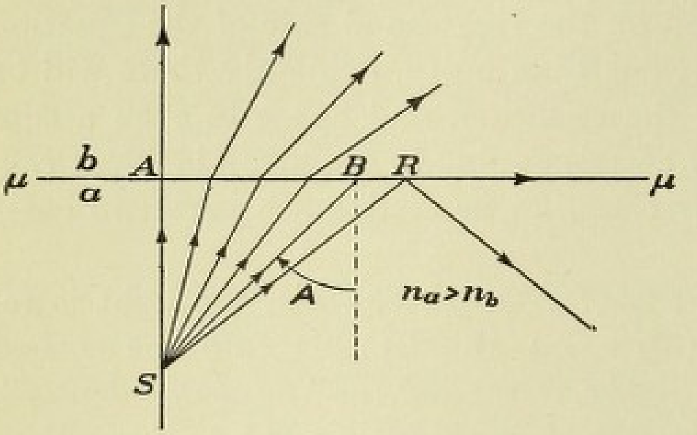

Fig. 7.

Total Internal Reflexion.

In this case ($n > n'$), if the incident ray meets the refracting surface at an angle of incidence α greater than the critical angle A, the expression $n \cdot \sin\alpha / n'$ will be greater than unity, which means that there will be no real value belonging to the angle α', and hence to such an incident ray there will be no corresponding refracted ray. The ether-disturbance propagated in the denser medium in such a direction as this will not cross the boundary-surface between the two media, but will be **totally reflected** there. Consider, for example, the diagram (Fig. 7), where the point designated by S represents a point-source of light supposed to be situated in a medium (a) which is optically denser than the medium (b) from which it is separated by a plane refracting surface, the trace of which in the plane of the paper is the straight line $\mu\mu$. Rays are emitted from S in all directions, but only those rays are refracted into the rarer medium (b) that are comprised within the conical surface whose vertex is at S, whose axis is the perpendicular SA let fall from S on $\mu\mu$, and whose semi-angle is $\angle ASB = \angle A = \sin^{-1} n_{ab}$. The ray SB is refracted along the plane refracting surface in the direction $B\mu$, as shown by the arrow-head; whereas a ray SR which has an angle of incidence α greater than the critical angle A is not refracted at all.

Another way of regarding this diagram is to suppose that an eye were placed at the point S, and that the rays were being refracted from the medium (b) into the denser medium (a); so that in this case the directions of the arrow-heads on the rays in the figure should all be reversed. All the rays entering the eye at S will be comprised within the cone generated by revolving the right triangle SAB around

SA as axis. For example, suppose that the media (a) and (b) are water and air, respectively, so that $\mu\mu$ represents, therefore, the horizontal free surface of tranquil water, and suppose that S marks the position below the water of the eye of an observer. An object situated on the horizon (determined by the water-surface) would be made visible by means of the ray BS, and the eye under water would locate the object as being in the air in the direction SB. A ray coming from a star and falling on the surface of the water between A and B might enter the eye at S, but the apparent zenith-distance of the star would always be less than its actual zenith-distance, except when the star was actually at the zenith-point of the celestial sphere.

The phenomenon of total reflexion of light at the boundary-surface between water and air is beautifully exhibited in the luminous fountains and cascades that in recent years have been spectacular features at expositions and places of amusement.

Incidentally, it may be remarked here that the ratio of the intensity of the reflected light to that of the refracted light increases steadily with increase of the angle of incidence, from the least value of this angle when the rays are normally incident to its greatest value when the rays are totally reflected. The rays that are totally reflected from the inside of one of the faces of an equilateral triangular glass prism placed in the sunlight are seen at a glance to be brighter than the rays reflected at the outside face of the prism.

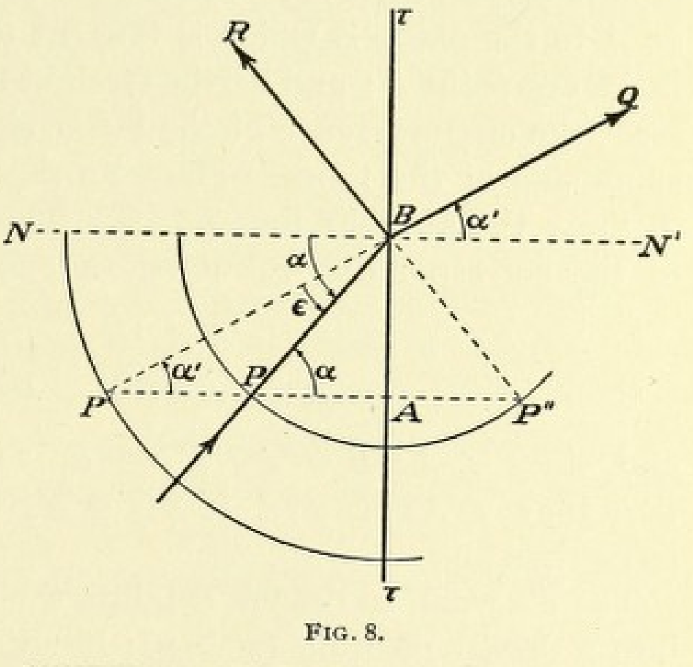

FIG. 8.

CONSTRUCTION OF REFLECTED AND REFRACTED RAYS. The straight line $\tau\tau$ is the trace in the plane of the paper of the tangent-plane at the incidence-point B to the reflecting or refracting surface. $BP' = n' \cdot BP/n$; $PA = AP''$; PB, BR and BQ represent the paths of the incident, reflected and refracted rays, respectively.

ART. 9. GEOMETRICAL CONSTRUCTIONS, ETC.

28. Construction of the Reflected Ray. In the diagram (Fig. 8) the straight line PB represents the path of an incident ray meeting a reflecting surface at the incidence-point B, and NN' represents the normal to this surface at B; so that, if α denotes the angle of incidence, $\angle NBP = \alpha$. The straight line perpendicular to NN' at the point B is the trace in the plane of incidence of the tangent-plane $\tau\tau$ to the reflecting surface at B. In order to construct the corresponding reflected ray, we draw from any point P of the incident ray the straight line PA perpendicular to $\tau\tau$ at A, and prolong this perpendicular to P'' until $AP'' = PA$, and from P'' draw the straight line $P''BR$; then BR will represent the path of the corresponding reflected ray. The proof of the construction is obvious from the figure, since we have:

$$\angle NBR = \angle PP''R = \angle BPP'' = \angle PBN = -\alpha;$$

according to the law of reflexion.

29. Construction of the Refracted Ray. Let n, n' denote the absolute indices of refraction of the two isotropic media separated by a smooth refracting surface, and let B (Fig. 8) designate the point where the given incident ray PB meets this surface. With the incidence-point B as centre, and with any radius $r = BP$ describe in the plane of incidence the arc of a circle cutting the incident ray in a point P; and in the same plane describe also the arc of a concentric circle of radius equal to $n'r/n$. Through P draw a straight line perpendicular at A to the plane $\tau\tau$ which is tangent to the refracting surface at the incidence-point B; and let the straight line AP, produced if necessary, meet the circumference of the latter circle in a point P' lying on the same side of the tangent-plane as the point P. Through the point B draw the straight line $P'BQ$. Then BQ will represent the path of the corresponding refracted ray. For

$$\frac{\sin\angle APB}{\sin\angle AP'B} = \frac{BP'}{BP} = \frac{n'}{n},$$

and, since $\angle APB = \angle NPB = \alpha$, it follows from the law of refraction that $\angle AP'B = \angle N'BQ = \alpha'$, where α' denotes the angle of refraction.

The diagram, as drawn, exhibits the case when the ray is refracted into a denser medium $(n' > n)$; but the construction given above is equally applicable to the other case also.

Assuming that $n' > n$, we see from Fig. 8 that when the angle of incidence $\angle NBP = 90°$, the incident ray PB will be tangent to the refracting surface at the incidence-point B, and then $BA = BP$, so that AP' will be tangent to the construction-circle of radius BP. In this case we shall have:

$$\alpha' = \angle PP'B = \sin^{-1}\frac{PB}{P'B} = \sin^{-1}\frac{n}{n'} = A,$$

where A denotes the magnitude of the so-called critical angle of the two media (§ 27).

30. The Deviation of the Refracted Ray. The angle between the directions of the incident and refracted rays is called the *angle of deviation*, and will be denoted here by the symbol ϵ. Thus, if PB (Fig. 9) represents the path of a ray incident on a refracting surface at the point B, and if $P'B$ (constructed as explained in § 29) shows the direction of the corresponding refracted ray, then $\angle\ P'BP = \epsilon$; that is, ϵ denotes the acute angle through which the direction of the refracted ray has to be turned to bring it into the same direction as that of the incident ray. If $\alpha = \angle\ NBP = \angle\ APB$, $\alpha' = \angle\ NBP' = \angle\ AP'B$ denote the angles of incidence and refraction, the angle of deviation is defined by the following relation:

$$\epsilon = \alpha - \alpha'. \tag{9}$$

The diagram is drawn for the case when $n' > n$, for which the sign of the angle ϵ is positive. By merely interchanging the letters P, P' in the figure, we obtain the case when $n' < n$, for which the angle denoted by ϵ is negative.

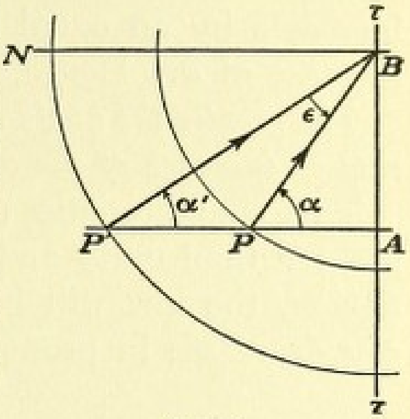

FIG. 9.

DEVIATION OF THE REFRACTED RAY. The straight lines PB, $P'B$ show the directions of the incident and refracted rays.

$\angle\ APB = \alpha$, $\angle\ AP'B = \alpha'$,
$\angle\ P'PB = \epsilon$, $\epsilon = \alpha - \alpha'$.

It is apparent from the figure that the intercept $P'P$ included between the circumferences of the two construction-circles, which remains always parallel to the normal BN, increases in length as the angle of incidence α increases; and, since the other two sides BP and BP' of the triangle BPP' have constant lengths, it follows that *the deviation of the refracted ray increases with increase of the angle of incidence.* This is true both for $n' > n$ and for $n' < n$.

Differentiating equation (6), we obtain (after eliminating n, n'):

$$\frac{d\alpha'}{d\alpha} = \frac{\tan \alpha'}{\tan \alpha}; \tag{10}$$

and, since from the figure $\tan \alpha' / \tan \alpha = AP/AP'$, we have therefore the following relations:

$$d\alpha' : d\alpha : d\epsilon = AP : AP' : PP';$$

so that in the triangle $PP'B$ the side PP' opposite the angle ϵ is divided externally at A into segments which are inversely proportional to the corresponding variations of the angles at P and P'.

Moreover, since

$$\frac{d\epsilon}{d\alpha}=\frac{PP'}{AP'}=\frac{1}{1+AP/PP'},$$

and since as the angle α increases, not only does AP decrease but PP' increases by an equal amount, it follows that $d\epsilon/d\alpha$ increases with increase of α. Hence,

The greater the angle of incidence, the greater will be the corresponding rate of increase of the angle of deviation.

This characteristic property of refraction is true both for n'/n greater than unity and for n'/n less than unity. In the case of reflexion, the law will be different, for the deviation of the reflected ray decreases in proportion as the angle of incidence increases.

ART. 10. CERTAIN THEOREMS CONCERNING THE CASE OF SO-CALLED OBLIQUE REFRACTION (OR REFLEXION).

31. The plane of incidence containing the normal to the refracting (or reflecting) surface at the point of incidence is a *normal section* of the surface at that point; and, whenever feasible, it will be convenient to select this plane as the plane of the diagram. In the following pages, however, we shall often have occasion to investigate the path of a ray which is incident in succession on a series of refracting (or reflecting) surfaces; in which case the plane of incidence with respect to one such surface will, in general, make with the plane of incidence with respect to the next following surface an angle different from zero. Accordingly, in our diagrams it may happen that the normal section of the refracting (or reflecting) surface which lies in the plane of the paper may not coincide with the normal section which contains the ray incident on that surface and its corresponding refracted (or reflected) ray. In such a case as this the ray is said to be "*obliquely*" incident on the surface; and in this connection the following theorems will be found useful.

We remark that it will be necessary to treat here only the problem of refraction; as the corresponding theorems relating to reflexion, which may easily be proved independently also, may be derived immediately by merely putting $n'=-n$, according to the general principle explained in § 26.

32. In the diagram (Fig. 10) the straight line NN' is the normal at the point B to the refracting surface, so that the plane of the diagram is therefore the plane of a normal section of the surface with respect to the point B; and the straight line $\tau\tau$ is the line of intersection of the plane of the paper with the plane tangent to the refracting surface at B. Let RB represent the path of a ray incident on the surface at the point B, and from any point R of this ray draw RN perpendicular at N to the normal NN'. The plane of incidence RBN is also the plane of a normal section of the surface; but we shall suppose here that RB is "obliquely" incident on the refracting surface, so that the plane of incidence is not the same as the plane of the paper. The corresponding refracted ray BR' will lie in the plane of incidence. On this refracted ray take a point R', such that

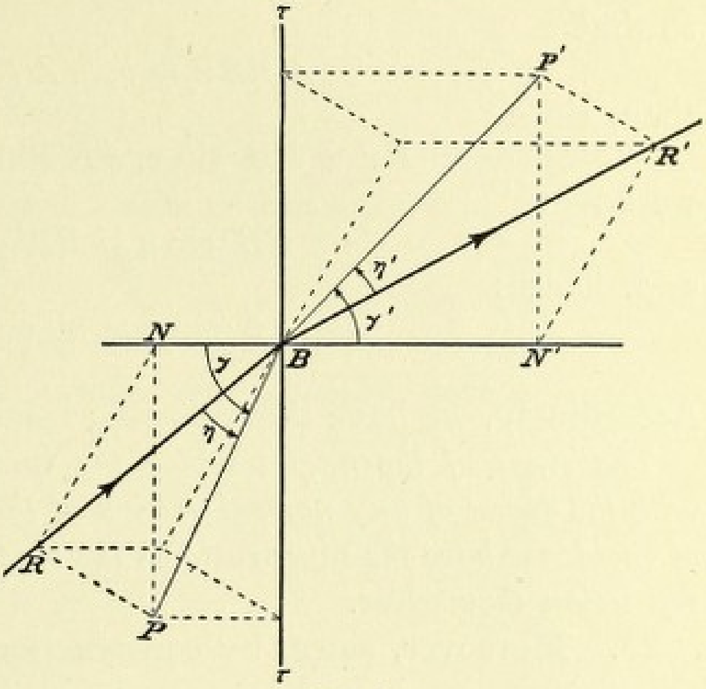

FIG. 10.

OBLIQUE REFRACTION. RB, BR' represent the paths of the incident and refracted rays. $\angle NBR = \alpha$, $\angle N'BR' = \alpha'$, $\angle RBP = \eta$, $\angle R'BP' = \eta'$, $\angle NBP = \gamma$, $\angle N'BP' = \gamma'$.

$$RB : BR' = n : n',$$

and from R' draw $R'N'$ perpendicular to NN' at N'. Draw also RP, $R'P'$ perpendicular to the plane of the paper. Then the two planes RPN, $R'P'N'$ will be parallel to the tangent-plane at B. By the law of refraction:

$$n \cdot \sin \alpha = n' \cdot \sin \alpha',$$

where $\angle NBR = \alpha$, $\angle N'BR' = \alpha'$. By the construction:

$$RN = RB \cdot \sin \alpha, \; R'N' = R'B \cdot \sin \alpha';$$

and, therefore:

$$RN = N'R'.$$

Since RN and $N'R'$, lying both in the plane of incidence, are equal and parallel, PN and $N'P'$, which are the projections of RN and $N'R'$ in the plane of the paper, are also equal and parallel; so that the triangles NPR and $N'P'R'$ are congruent, and $RP = P'R'$.

If the symbols η, η' are employed to denote the angles made by the incident and refracted rays RB, BR' with their projections PB, BP' in the plane of the normal section which is the plane of the paper, so that

$$\angle\, RBP = \eta, \quad \angle\, R'BP' = \eta',$$

then, since

$$RP = RB \cdot \sin\eta, \quad R'P' = R'B \cdot \sin\eta',$$

we have:

$$RB \cdot \sin\eta = BR' \cdot \sin\eta';$$

and, hence:

$$n \cdot \sin\eta = n' \cdot \sin\eta'. \tag{11}$$

Accordingly, we have the following result:

The sines of the angles which the incident and refracted rays make with the plane of any normal section of the refracting surface at the point of incidence have the same ratio as the sines of the angles of incidence and refraction themselves.

33. Moreover, since, by construction (Fig. 10),

$$n' \cdot RB = n \cdot BR',$$

and

$$PB = RB \cdot \cos\eta, \quad BP' = BR' \cdot \cos\eta',$$

we have:

$$n' \cdot PB \cdot \cos\eta = n \cdot BP' \cdot \cos\eta'.$$

Putting

$$\angle\, NBP = \gamma, \quad \angle\, N'BP' = \gamma',$$

so that

$$PB \cdot \sin\gamma = BP' \cdot \sin\gamma',$$

we find:

$$n \cdot \cos\eta \cdot \sin\gamma = n' \cdot \cos\eta' \cdot \sin\gamma'; \tag{12}$$

a result which may be stated as follows:

The projections of the incident and refracted rays on a plane of a normal section of the refracting surface at the point of incidence are also subject to a law of refraction, the absolute indices of refraction $n \cdot \cos\eta$ and $n' \cdot \cos\eta'$ being dependent on the angles η and η' made by the incident and refracted rays with the plane of the normal section.

If we put

$$n_\eta = n \cdot \cos\eta, \quad n_\eta' = n' \cdot \cos\eta',$$

and bear in mind that we have also the relation:

$$n \cdot \sin\eta = n' \cdot \sin\eta',$$

we can derive easily the formula:

$$\frac{n_\eta'}{n_\eta} = \frac{\sqrt{n'^2 + (n'^2 - n^2)\tan^2\eta}}{n}$$

in the form given by CORNU.[1]

34. The following is a convenient method of constructing a drawing representing the path of a ray obliquely refracted at the surface of separation of two isotropic optical media.

Let the plane of the paper (Fig. 11) be designated as the *xy*-plane and let the tangent-plane to the refracting surface at the incidence-point *B* be designated as the *yz*-plane, which is represented as making an acute angle with the plane of the paper. From *B*, in a plane *xz* perpendicular to the plane of the paper, draw *BN* normal at *B* to the tangent-plane *yz*, and draw *BM* normal at *M* to the plane of the paper. Suppose, for example, that the real length of *BM* is twice its length as shown in the figure. Let *P* designate the position of the point where the given incident ray meets the plane of the paper, so that *BP* shows the direction of the incident ray lying in the plane of incidence *BPN*. If the triangle *BPN* is revolved around *PN* as axis until it comes into the plane of the paper, the point *B* will arrive at a point *C* on the straight line drawn from *M* perpendicular to *NP*,

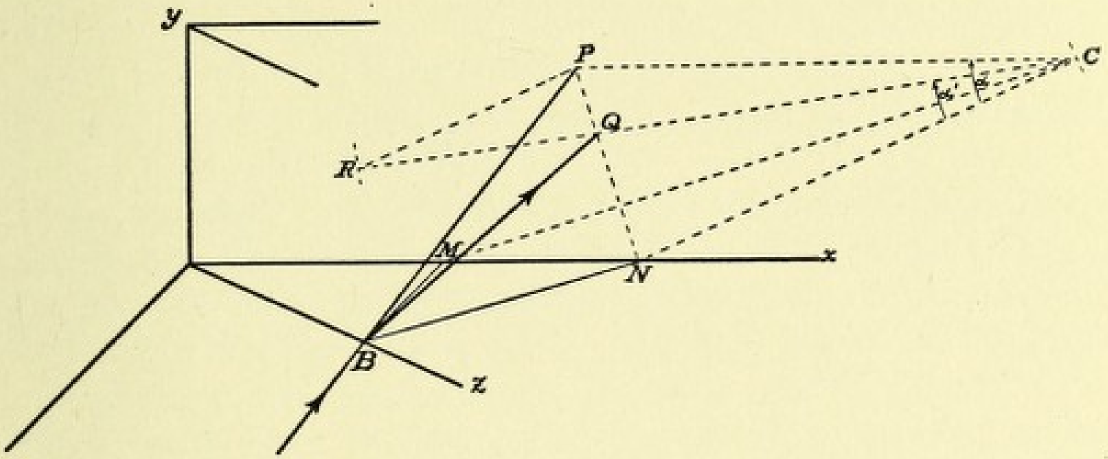

FIG. 11.

CONSTRUCTION OF OBLIQUELY REFRACTED RAY.

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[1] A. CORNU: De la réfraction à travers un prisme suivant une loi quelconque: *Ann. éc. norm.* (2), i. (1872), 237. See also E. REUSCH: Die Lehre von der Brechung u. Farbenzerstreuung des Lichts an ebenen Flaechen und in Prismen, die in mehr synthetischer Form dargestellt: POGG. *Ann.* cxvii. (1862), 247; and A. BRAVAIS: Notice sur les parhélies qui sont situés a la même hauteur que le soleil: *Journ. éc. polyt.*, xviii., cah. 30 (1845), 79; and Mémoire sur les halos: *Journ. éc. polyt.*, xviii., cah. 31 (1847), 27.

and $\angle NCP = \alpha$. Hence, with C as centre and with radius equal to $n' \cdot CP/n$ describe in the plane of the paper the arc of a circle meeting in a point R the straight line drawn through P parallel to CN; evidently, as in §29, $\angle NCR = \alpha'$. Therefore, the straight line BQ joining the incidence-point B with the point Q where CR meets NP will represent in the diagram the direction of the refracted ray.

CHAPTER II.

CHARACTERISTIC PROPERTIES OF RAYS OF LIGHT.

ART. 11. THE PRINCIPLE OF LEAST TIME (LAW OF FERMAT).

35. FERMAT[1] (1608–1665), arguing from an assumed law of the economy of nature that light must be propagated from one point to another in the shortest time, was able to deduce the law of refraction in the case of a ray refracted from one isotropic medium to another across a *plane boundary-surface;* or, conversely, that the time required by the light to be transmitted from any point P on the incident ray to any point Q on the corresponding refracted ray is less than it would be along any other route between the points P and Q. A corresponding law in regard to light reflected at a plane mirror dates back to HERO of Alexandria (150 B. C.).

If the boundary-surface separating the two media is *curved*, the time taken by the light to be transmitted from P to Q along the actual path may not, however, be always a minimum; on the contrary, in certain cases it may be a maximum. A simple illustration is given by Sir WM. ROWAN HAMILTON,[2] who instances the fact that "if an eye is placed in the interior, but not at the centre, of a reflecting hollow sphere, it may see itself reflected in two opposite points, of which one indeed is the nearest to it, but the other on the contrary is the farthest; so that of the two different paths of light, corresponding to these two opposite points, the one indeed is the shortest, but the other is the longest of any."

36. A characteristic property of a ray of light may be stated quite generally as follows:

If a ray of light, undergoing any number of reflexions and refractions, connects two points P and Q, the time taken by the light to be transmitted from P to Q along the actual path of the ray is either a minimum or a maximum.

It will be entirely sufficient if we prove the truth of this statement merely for the case of a single refraction; as it can then be extended immediately to the case where the ray suffers any number of reflexions and refractions.

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[1] P. FERMAT: *Litteræ ad* P. MERSENUM *contra Dioptricam Cartesianam* (Paris, 1667).

[2] W. R. HAMILTON: On a General Method of expressing the Paths of Light and of the Planets: *Dublin University Review*, October, 1833.

In the diagram (Fig. 12) the point designated by P represents the starting point in the first medium and the point designated by Q represents the terminal point in the second medium; and $\mu\mu$ is the trace of the refracting surface in a plane containing the two points P and Q (which is represented here as the plane of the paper). The problem to be solved is, What must be the position on the refracting surface of the incidence-point B, in order that the time taken by the light to be transmitted from P to Q, viz.,

$$t = \frac{PB}{v} + \frac{BQ}{v'},$$

where v and v' denote the speeds of propagation of light in the first and second medium, respectively, shall be either a minimum or a maximum? Let us suppose that this point B is also situated in the plane of the paper, which will be therefore the plane of incidence of the ray PB. Evidently, this critical position of the incidence-point B will be such that an infinitely small variation from this position would not alter the time taken by the light in going from P to Q. It will suffice to consider a variation of the position of B in the plane of incidence; accordingly, let us designate by B_1 the position of a point on the normal section $\mu\mu$ of the refracting surface infinitely close to the critical point of incidence B. If the light travelled from P to Q along the route PB_1Q, the time taken would be:

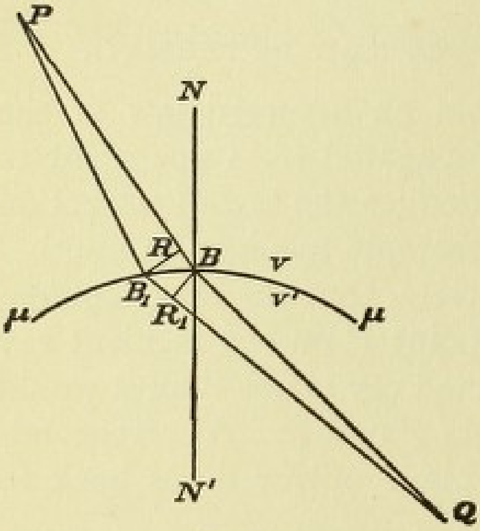

FIG. 12.

FERMAT'S LAW OF LEAST TIME. PB, BQ represent paths of incident and refracted rays. PB_1Q is another hypothetical route from the point P in the first medium to the point Q in the second medium, which differs infinitesimally from the actual route PBQ.

$$\frac{PB_1}{v} + \frac{B_1Q}{v'};$$

and, consequently, the condition which has to be imposed is that

$$\frac{PB - PB_1}{v} + \frac{BQ - B_1Q}{v'} = 0.$$

From B and B_1 draw BR_1 and B_1R perpendicular to B_1Q and PB at R_1 and at R, respectively; then

$$PB - PB_1 = RB \quad \text{and} \quad B_1Q - BQ = B_1R_1;$$

and, hence, the condition above becomes:

$$\frac{RB}{v} - \frac{B_1R_1}{v'} = 0.$$

Draw NBN' normal to the curve $\mu\mu$ at the point B, and put $\angle NBP = \alpha$, $\angle N'BQ = \alpha'$; then

$$RB = BB_1 \cdot \sin\alpha, \quad B_1R_1 = BB_1 \cdot \sin\alpha'.$$

The condition may be written, therefore:

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{v}{v'},$$

which will be recognized as the law of refraction (§ 21). But the actual path of the ray from P to Q is according to this law. Consequently, the time t along this path will be either a minimum or a maximum.

Whether in any given special case the time is a minimum or a maximum, can be determined only by investigating the form of the refracting (or reflecting) surface.

37. This result, as was stated above, can be immediately extended to the case where the ray is compelled in its progress from P to Q to traverse any number of media or to bend away at certain surfaces of separation between two bodies, that is, where the ray is constrained to undergo a certain prescribed series both of refractions and of reflexions. If we denote by t_k the time occupied by the light between two successive adventures of this kind, the analytical expression of the so-called *Principle of Least Time* may be written in the following form:

$$\delta(\Sigma t_k) = 0; \tag{13}$$

that is, the time taken by the light to be transmitted, under certain prescribed conditions, from one point P to another point Q along the actual path of the ray differs from the time which would be taken along any other hypothetical route, which is infinitely near to the actual route, by an infinitesimal of an order higher than the first order.

38. The Optical Length of a Ray; and the Principle of the Shortest Route. The sum of the products of the length of the path of a ray in each medium by the refractive index of that medium is called the *Optical Length*, sometimes also the *reduced length*, of the ray. Thus if l_1, l_2, etc., denote the actual lengths of the ray-path in the media whose indices of refraction are denoted by n_1, n_2, etc., respectively, the optical length of the ray is:

$$n_1l_1 + n_2l_2 + \cdots = \Sigma n_k l_k,$$

where n_k, l_k denote the values of the magnitudes n, l for the kth medium. When the ray is reflected at a body i, we must put here $n_i = -n_{i-1}$, according to the rule given in § 26; so that the definition given above applies to reflexions as well as to refractions.

Since $n_k = V/v_k$, where V and v_k denote the speeds of propagation of light *in vacuo* and in the kth medium of the series, respectively, and $l_k = v_k t_k$, we have $n_k l_k = V t_k$; whence we see that the optical length of the ray $(= V \cdot \Sigma t_k)$ is equal to the distance that light would travel *in vacuo* in the same length of time as it takes to go over its actual path. This explains the use of the term "reduced length".

We also see that equation (13) is equivalent to the following:

$$\delta(\Sigma n_k l_k) = 0; \tag{14}$$

whence is derived the so-called *Principle of the Shortest Route*, which may be stated as follows:

When light is transmitted from one point P to another point Q, undergoing during its progress any prescribed series of reflexions and refractions, the optical length measured along the actual path of the ray is a minimum or a maximum.

ART. 12. HAMILTON'S CHARACTERISTIC FUNCTION.

39. The statement at the end of the last article recalls MAUPERTUIS's celebrated "Principle of Least (or Stationary) Action", afterwards developed by EULER and other great mathematicians; provided we define the vague term "action" in the case of a ray of light to mean the optical length of the ray. The function

$$L = \Sigma n_k l_k$$

is the so-called *Characteristic Function*, the idea of which was first introduced into mathematical optics by Sir W. R. HAMILTON, and which reduces the solution of all problems, in theory at least, to one common process.[1]

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[1] Professor P. G. TAIT in his book on *Light* (Edinburgh, 1889) says (Art. 189): "HAMILTON was in possession of the germs of this grand theory some years before 1824, but it was first communicated to the Royal Irish Academy in that year, and published in imperfect instalments some years later." HAMILTON's papers on this subject published under the title "Theory of systems of rays" are to be found in the *Transactions of the Royal Irish Academy*, xv. (1828), 69–174; xvi. (1830), 3–62; and 93–126; and xvii. (1837), 1–144.

In the application of this method the co-ordinates of the two terminal points $P(a, b, c)$ and $Q(a', b', c')$ which are connected by the ray are to be regarded as known, and, therefore, invariable. The equations of the reflecting and refracting surfaces must likewise be given. But the co-ordinates of the points where the ray meets these surfaces are the variables in the problem. The equation of the kth surface may be written:

$$z = f_k(x, y),$$

and since the co-ordinates x_k, y_k, z_k of the point where the ray meets this surface must satisfy this equation, we may regard z_k as a known function of x_k, y_k. The actual length of the ray-path between the $(k-1)$th and the kth surfaces will be:

$$l_k = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2 + (z_k - z_{k-1})^2}.$$

Since $\delta L = 0$, we must have:

$$\frac{dL}{dx_k} = 0, \quad \frac{dL}{dy_k} = 0,$$

where z_k is to be considered as the dependent variable, so that:

$$\frac{dL}{dx_k} = \frac{\partial L}{\partial x_k} + \frac{\partial L}{\partial z_k}\frac{dz_k}{dx_k}, \quad \frac{dL}{dy_k} = \frac{\partial L}{\partial y_k} + \frac{\partial L}{\partial z_k}\frac{dz_k}{dy_k}.$$

40. In order to illustrate the use of the method in a simple case, let us suppose that there is only one refracting surface separating two media of refractive indices n and n'; then:

$$l_1 = l = \sqrt{(x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2},$$
$$l_2 = l' = \sqrt{(a' - x_1)^2 + (b' - y_1)^2 + (c' - z_1)^2},$$

and

$$L = nl + n'l';$$

and, according to the equations above, we derive here:

$$\frac{dL}{dx_1} = n\frac{(x_1 - a) + (z_1 - c)\dfrac{dz_1}{dx_1}}{l} - n'\frac{(a' - x_1) + (c' - z_1)\dfrac{dz_1}{dx_1}}{l'} = 0,$$

$$\frac{dL}{dy_1} = n\frac{(y_1 - b) + (z_1 - c)\dfrac{dz_1}{dy_1}}{l} - n'\frac{(b' - y_1) + (c' - z_1)\dfrac{dz_1}{dy_1}}{l'} = 0.$$

If the incidence-point is taken as the origin of co-ordinates, then $x_1 = y_1 = z_1 = 0$. Moreover, if the incidence-normal is taken as the z-axis, then also $dz_1/dx_1 = dz_1/dy_1 = 0$. Introducing these simplifying values, we find:

$$\frac{na}{l} + \frac{n'a'}{l'} = 0, \quad \frac{nb}{l} + \frac{n'b'}{l'} = 0.$$

If, further, we take the plane of incidence for the yz-plane, we must put $a = 0$; whence it follows from the first of the two equations above that $a' = 0$ also; and hence the point Q, and therefore also the refracted ray, must lie in the plane of incidence, in accordance with a fundamental law of refraction. Finally, if α, α' denote the angles of incidence and refraction, it is evident that:

$$\frac{b}{l} = -\sin\alpha, \quad \frac{b'}{l'} = \sin\alpha',$$

and hence the second equation above is equivalent to the other fundamental law of refraction:

$$n \cdot \sin\alpha = n' \cdot \sin\alpha'.$$

Thus we see how this process leads to the ordinary laws of refraction.

41. If the characteristic function of a system is known, it is possible in theory to deduce from it all the optical properties of the system. In some comparatively simple cases this process enables us to get results with almost magical facility. It must be admitted, however, that the method, so fascinating on account of its generality, is difficult in its applications, involving as it does the theories of the higher analytical geometry and demanding mathematical knowledge and skill of the highest order. In addition to HAMILTON, a number of other investigators, among whom may be mentioned especially MAXWELL[1] and THIESEN[2] and BRUNS,[3] have developed in one way or another the theory of the characteristic function in optics. But the greatest difficulty is encountered in turning the theory to account, and, so far as the practical optician is concerned, the HAMILTONian method has not been found to smooth his way.

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[1] J. C. MAXWELL: A dynamical theory of the electromagnetic field, *Proc. Roy. Soc.*, xiii. (1864), 531–536; *Phil. Trans.*, clv. (1865), 459–512; *Phil. Mag.*, (4) xxix. (1865), 152–157. Also, On the application of HAMILTON's characteristic function to the theory of an optical instrument symmetrical about an axis: *Proc. of London Math. Soc.*, vi. (1874–5), 117–122; and On HAMILTON'S characteristic function for a narrow beam of light; *Proc. London Math. Soc.*, vi. (1874–'5), 182–190.

[2] M. THIESEN: Beitraege zur Dioptrik: *Berl. Ber.*, 1890, 799–813. Also, Ueber vollkommene Diopter, WIED. *Ann. der Phys.* (2), xlv. (1892), 821–'3. See also, Ueber die Construction von Dioptern mit gegebenen Eigenschaften, WIED. *Ann. der Phys.* (2), xlv., 823–'4.

[3] H. BRUNS: Das Eikonal: *Saechs. Ber. d. Wiss.*, xxi. (1895), 321–436. See also F. KLEIN: Ueber das BRUNSche Eikonal; and, also, Raeumliche Kollineation bei optischen Instrumenten: *Zft. f. Math. u. Phys.*, xlvi. (1901).

ART. 13. THE LAW OF MALUS.

42. The wave-front at any instant due to a disturbance emanating from a point-source is the surface which contains all the farthest points to which the disturbance has been propagated at that instant. Thus, the wave-surface may be defined as the totality of all those points which are reached in a given time by a disturbance originating at a point. In a single isotropic medium the wave-surfaces will be concentric spheres described around the point-source as centre; but if the wave-front arrives at a reflecting or refracting surface μ, at which the directions of the so-called rays of light are changed, the form of the wave-surface thereafter will, in general, be spherical no longer; and even in those cases when the refracted (or reflected) wave-front is spherical, the centre (except under certain very special circumstances) will not coincide with the centre of the incident wave-surfaces. The function Σnl (§ 38) has the same value for all actual ray-paths between one position of the wave-surface and another position of it; so that knowing the form of the wave-front at any instant and the paths of the rays, we may construct the wave-front at any succeeding instant by laying off equal optical lengths along the path of each ray. It follows that the ray is always normal to the wave-surface. For, suppose that the straight line PB represents the path of a ray incident at the point B on a surface μ separating two media, and that the straight line BQ represents the path of the corresponding refracted (or reflected) ray; and let σ designate the wave-surface whereon the point Q lies. From the incidence-point B draw any straight line BR meeting the wave-surface σ in the point designated by R. Then, by the minimum property of the light-path, the route PBQ is less than the route PBR, because the natural route from P to R is not *via* the incidence-point B; and hence the straight line BQ must be shorter than the straight line BR, and therefore BQ is the shortest line that can be drawn from the incidence-point B to the wave-surface σ. It follows that BQ meets the wave-surface σ normally. The same reasoning will be applicable also in the case of every other refraction or reflexion, so that we may state generally:

The light-rays meet the wave-surface normally; and, conversely, *the system of surfaces which intersect at right angles the rays emanating originally from a point-centre is a system of wave-surfaces.*

43. The fact which has just been proved is equivalent to the law enunciated by MALUS,[1] in 1808, which may be stated as follows: *An orthotomic system of rays remains orthotomic, no matter what refractions (or reflexions) the rays may undergo in traversing a series of isotropic media.* (An orthotomic system of rays is one for which a surface can be constructed which will cut all the rays at right angles.)

A proof of this law which does not contain any reference to the ideas of the Wave-Theory is given by HEATH[2] as follows:

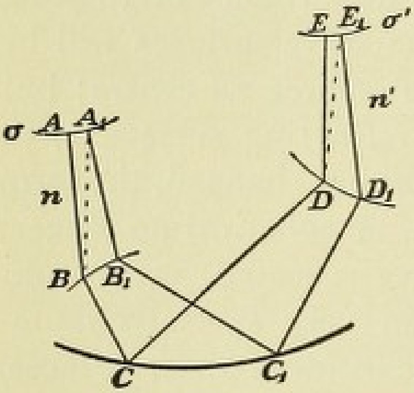

FIG. 13.
LAW OF MALUS.

Let $ABCDE$ (Fig. 13) and $A_1B_1C_1D_1E_1$ be two infinitely near ray-paths, and suppose that they cross normally at A and A_1 a certain surface σ. On each ray of the system, reckoning from the points A, A_1, etc., where the rays cross the surface σ, let a series of points E, E_1, etc., be determined such that the optical lengths from A to E, from A_1 to E_1, etc., are all equal. We propose to show that the surface σ' which contains the terminal points E, E_1, etc., of these rays will cut the rays at right angles.

In order to prove this, we draw the straight lines A_1B and DE_1 as shown in the figure. The optical length Σnl measured along the infinitely near hypothetical route A_1BCDE_1 is, by FERMAT's Law, equal to Σnl along $A_1B_1C_1D_1E_1$ or along $ABCDE$. Hence, subtracting from each the part BCD which is common to the routes $ABCDE$ and A_1BCDE_1, we have:

$$n \cdot AB + n' \cdot DE = n \cdot A_1B + n' \cdot DE_1,$$

where n, n' denote the refractive indices of the first and last medium, respectively. But since AB is normal to the surface σ, ultimately $A_1B = AB$; and, hence, ultimately also $DE = DE_1$; that is, DE must be normal to the surface σ'. In the same way we can show that any other ray D_1E_1 will likewise be normal to σ'.

ART. 14. OPTICAL IMAGES.

44. In case we do not wish to utilize all the rays emitted from a luminous body, we may interpose a screen with a suitable opening in it, whereby some of the rays are intercepted, while others, called the "effective rays", are permitted to pass through the opening. Thus, each separate point of a luminous body is to be regarded as the vertex of a cone or *bundle of rays*. In every bundle of rays there is always a certain central or representative ray, usually coinciding with the axis of the cone, or distinguished in some special way, called the *chief ray*[1] of the bundle. A *pencil of rays* is obtained from a bundle of rays by passing a plane through the axis or chief ray of a bundle. This use of this term is convenient and is also in accordance with the usage of some writers on geometry.

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[1] E. L. MALUS: Optique: *Journ. de l'École Polyt.*, vii. (1808), 1–44; 84–129.

[2] R. S. HEATH: *A Treatise on Geometrical Optics* (Cambridge, 1887), Art. 87.

An optical system is a combination of isotropic media arranged in a certain sequence so that they are traversed by the effective rays all in the same order. In this case the effective rays emitted by a luminous point P are those rays coming from P which succeed finally in passing through the system from one end to the other without being intercepted at any point on the way. In general, through any point P', within the region reached by the bundle of emergent rays which had their origin at the luminous point P, one ray, and one ray only, will pass, since the optical route between P and P', for a given disposition of the optical media, will usually be *uniquely* determined. However, within this region there may be found a number of points P' where two or more rays intersect; and under certain circumstances it may indeed happen that all of the effective rays emanating from the point P will, after traversing the optical system, meet again in one point P'; and in this exceptional case the point P' is said to be the *optical image* of the point P, and the two points P and P', object-point and image-point, are called *conjugate points* or *conjugate foci.* If the rays actually pass through P', the image is said to be *real;* whereas if it is necessary to produce backwards the actual portions of the rays in order to make them intersect in P', the image is said to be *virtual*. Thus, in the case of a *perfect image*, all of the "emergent rays" corresponding to the rays of a given bundle of "incident rays" proceeding from the object-point P will intersect in the image-point P'.

45. In order, therefore, to have an image in the sense above defined, the optical system must transform a train of spherical waves with the object-point P as centre into another train of spherical waves with the image-point P' as centre. The optical lengths along all the ray-paths between P and P' will be equal, so that the disturbances arrive at P' along all these different routes all in the same phase, and hence conspire to produce at P' a maximum effect. According to the notions of Geometrical Optics, there will be no light-effects whatever at points which lie outside of the cone of rays which meet in P'; but when the matter is investigated by the surer methods of Physical Optics, we discover that this conclusion is not justified, and that there are light-effects at points which are not comprised within this geometric cone. In fact, instead of a single image-point P', we find that we have around P' a so-called *diffraction-pattern*. But the wider the cone of rays that meet in P', the more nearly will the distribution of light around P' approach as its limit the ideal image-point of Geometrical Optics; and this is the only meaning which Physical Optics can attach to the idea of an image-point.

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[1] The term "chief ray" is a happy rendering of the German *Hauptstrahl* which has been introduced into English Optics by Professor SILVANUS P. THOMPSON in his translation of Dr. O. LUMMER'S *Photographic Optics* (London, 1900).

ART. 15. CHARACTER OF A BUNDLE OF OPTICAL RAYS.

46. Caustic Surfaces. According to the Law of MALUS (Art. 13), the characteristic property of a bundle of optical rays emanating originally from a point-centre is that the rays are normal to the so-called wave-surface; and hence the investigation of the constitution and general laws of a bundle of optical rays will involve nothing more and nothing less than the theory of a bundle of normals to a curved surface. Only in a few simple and comparatively unimportant cases can the surface be represented by an algebraic equation, and thus, as a rule, it is necessary to study the surface in the immediate vicinity of a selected point and the bundle of rays in the immediate vicinity of a selected ray. In an optical instrument the ray-bundles are limited by the transversal dimensions of the apparatus, or indeed, to speak more exactly, by a suitably disposed "stop" or perforated screen the function of which is to intercept all the rays except such as are useful and convenient for the imagery (see Chapter XIV). The *chief ray* of a bundle of rays, emanating originally from an object-point, is that ray which in traversing the medium where the stop is situated (called sometimes the "stop-space") passes through the *stop-centre*. The totality of the chief rays constitutes a homocentric bundle of rays in the stop-space, which behaves exactly as if the light radiated from the stop-centre.

In the special case when the surface is spherical, the normals all meet in one point at the centre of the sphere; but if the curved surface has any other form, the normals drawn at two different points of the surface will, in general, not intersect at all. The curved line which is traced on the surface by a plane containing the normal to the surface at a point O is called a *normal section* of the surface at this point, and the curvatures of these sections through O will, in general, vary from one azimuth to another. EULER has shown that *at every point O of a curved surface the normal sections of greatest and least curvatures are at right angles to each other;*[1] and, accordingly, the two normal sections thus distinguished are called the *principal sections* of the surface at the point O. The principal sections are not only at right angles to each other, but they are characterized also by a remarkable law, as follows: *The normals at consecutive points of a curved surface will intersect each other, provided those points are taken along the curves of greatest and least curvature; although, in general, the normals at consecutive points do not intersect.* Thus, on the normal u drawn to the surface at the point O, two points C_1, C_2, the so-called *centres of principal curvature* of the surface with respect to the point O, can be determined by finding the two points in which the normal u is met by two consecutive normals taken in the planes of the principal sections through O. (Only in the special case when the point O is an "umbilic" or "circular point" will the two points designated by C_1, C_2 be coincident.)

A *line of curvature* on a surface is a curve traced on it such that the normals at any two consecutive points of the curve intersect each other. Therefore, through every ordinary point of the surface two such lines of curvature will pass intersecting each other at right angles; and the totality of each of these two systems of lines of curvature completely covers the entire surface. Ordinarily, a line of curvature will not be a plane curve, and even in the special case when it is plane, it need not coincide with a principal normal section at the given point

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[1] Taking the tangent-plane as the xy-plane of a rectangular system of co-ordinates, and the origin at the point of contact, the equation of the curved surface may be written:

$$2z = rx^2 + 2sxy + ty^2 + \text{etc.},$$

where

$$r = \left(\frac{\partial^2 z}{\partial x^2}\right)_0, \quad s = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_0, \quad t = \left(\frac{\partial^2 z}{\partial y^2}\right)_0.$$

A plane parallel to the tangent-plane and very near it will cut the surface in an ellipse, hyperbola or two parallel straight lines according as $(s^2 - rt)$ is less than, greater than, or equal to zero. In the first case ($s^2 - rt < 0$, viz., when the principal curvatures have the same sign), the surface is "synclastic" at the point considered; and if $s = 0$ and $r = t$, the curvatures of all normal sections are equal. Such points are called "umbilics." In the second case ($s^2 - rt > 0$, viz., when the principal curvatures have opposite signs, as, for example, at the summit of a mountain-pass or in the case of a saddle), the surface is "anticlastic" at the point considered. In this case the curvature is a maximum in both principal sections, but on account of the difference of sign we may regard one curvature as the greatest and the other as the least. And, finally, when $s^2 - rt = 0$, the surface has only one curvature at the point in question, and the point is called a "cylindrical" (or "parabolic") point of the surface.

O, though it must be tangent to it there; for the principal section must be normal to the surface, whereas the line of curvature may be oblique. These statements are illustrated very instructively in the case of a *surface of revolution.* At any point O of a surface generated by the revolution of a plane curve about an axis in its plane, one line of curvature is the meridian curve which passes through O; for all the normals to this curve are also normals to the surface, and, being in one plane, they intersect. One of the principal sections at O is this meridian line, because it contains the normal and touches a line of curvature. The other line of curvature is the circular section made by a plane through O perpendicular to the axis; but this is not a principal section because it does not contain the normal. The other principal section of the surface at O will be the section made by a plane containing the normal to the surface at O and the tangent to the circle described by O; and the centre of curvature corresponding to this principal section will be at the point where the normal to the surface at O meets the axis of revolution.

If a line is traced on a curved surface and the normals to the surface drawn at all points along this line, the ruled surface thus generated will, in general, be a *skew surface* (or "scroll"), because the consecutive generating normals will usually not intersect But if the line traced on the curved surface is a line of curvature, the ruled surface in this case will be a so-called *developable surface* (or "torse") in which each generating normal is intersected by the next consecutive one. The locus of the points where two consecutive generators of a developable surface intersect is a curve called the *cuspidal edge* (or "edge of regression") of that developable. The cuspidal edge of the developable surface generated by the normals belonging to all the points along a line of curvature of the given surface is the evolute of that line of curvature or the envelope of this system of normals. Thus, the normal of the surface at the point O touches the two cuspidal edges corresponding to the two lines of curvature through O, and the points of contact are the two centres of principal curvature C_1, C_2.

The *surface of centres* is a surface of two sheets which is the locus of the centres of principal curvature of every point of the given surface. The cuspidal edges corresponding to one of the two systems of the lines of curvature of the surface lie all on one sheet, and the cuspidal edges of the other system lie on the other sheet of the surface of centres. The normal to the surface at any point O touches both sheets of the surface of centres, for since it touches at C_1 and C_2 the cuspidal edges corresponding to the lines of curvature through O, which are traced on the surface of centres, it must be tangent to this surface also at both C_1 and C_2.

The cuspidal edge with respect to the wave-surface is called in Optics a *caustic curve*, and the totality of the caustic curves corresponding to one system of the lines of curvature of the wave-surface form a *caustic surface*, which is one sheet of the so-called surface of centres. Thus, there are, in general, two caustic surfaces, one for each of the two systems of the lines of curvature of the wave-surface, and every ray is a common tangent of the two caustic surfaces.

In the special case when the wave-surface is a surface of revolution, so that the orthotomic system of rays is therefore symmetrical with respect to the axis of revolution, it is easy to obtain a clear idea of the caustic surfaces. For here one system of lines of curvature are the meridian curves of the surface, and consequently the caustic surface corresponding thereto is generated by the revolution about the axis of symmetry of the evolute of the meridian curve. And the other system of lines of curvature are circles with their centres ranged along the axis of symmetry, and, since the rays which cross the wave-surface at points lying in the circumference of one of these circles will all lie in the surface of a right circular cone whose vertex is on the axis of revolution, the caustic surface corresponding to this system of lines of curvature reduces to a segment of the axis of revolution itself.

Another simple case is afforded by a bundle of optical rays which is symmetric with respect to two rectangular planes which are the principal sections of the bundle of rays with respect to the central ray which coincides with the line of intersection of these planes. The procedure of the rays in the planes of symmetry is exhibited in the two diagrams, Fig. 14*a* and Fig. 14*b*. The curves a_1a_2 and b_1b_2 (Fig. 14*a*) which are met by the central ray in the points C_1 and C_2, respectively, are the sections of the two caustic surfaces made by the plane of the first principal section of the bundle; just as the curves b_1b_2 and a_1a_2 (Fig. 14*b*) are the sections of the two caustic surfaces made by the other principal section. The a-curves with cusps at C_1 and C_2 are the envelopes of the rays lying in the planes of symmetry. An idea of the forms of the two caustic surfaces here may be obtained by placing the two figures together with their planes at right angles in such manner that C_1 and C_2 in one figure coincide with C_1 and C_2, respectively, in the other figure; then if one figure is held fixed while the other, remaining always in a plane at right angles to the former, is moved so that the cusp of the a-curve of this section slides along the *b*-curve of the other section, at the same time without altering the direction of the tangent at the cusp, the *a*-curve will thus generate one of the caustic surfaces.

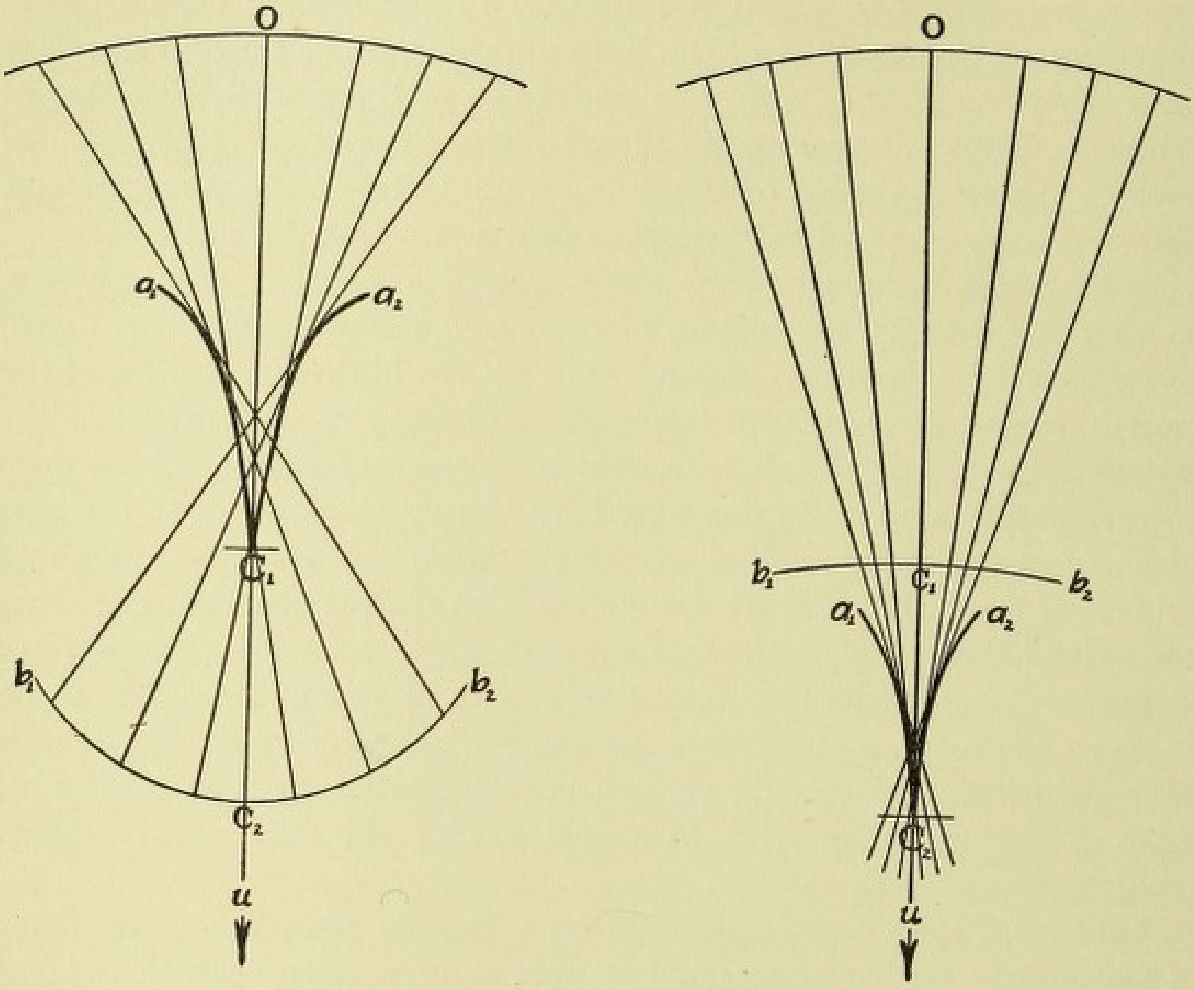

FIG. 14*a*. FIG. 14*b*.

CAUSTIC SURFACES IN CASE OF ASTIGMATIC BUNDLE OF RAYS WHICH IS SYMMETRIC WITH RESPECT TO THE CENTRAL RAY.

In Chapter VI of HEATH's *Geometrical Optics* (Cambridge, 1887) the reader who wishes to pursue this subject further will find an extensive investigation of the forms and properties of caustic lines and surfaces in a number of interesting special cases. WOOD's *Physical Optics* (the first edition of which was published in 1905), wherein the caustic surfaces are studied especially from the standpoint of the Wave-Theory, and experimentally rather than mathematically, contains also much on this subject that is both novel and suggestive. However, so far as the theory and design of optical instruments is concerned, it is doubtful whether it will repay us here to attempt to investigate these surfaces in detail; although in the next chapter, by way of illustration, we shall study briefly the caustic in the case of the refraction of a spherical wave at a plane surface (§ 54).

47. Proceeding now to apply these results from the theory of curved surfaces, we remark, in the first place, that a bundle of optical rays, originally *homocentric* (or *monocentric*, as it is also called, that is, emanating from one and the same point or "focus") is, in general, transformed by reflexion or refraction into a non-homocentric or *astigmatic* bundle of rays—a term which will be more precisely explained in what follows. Since the rays are normal to the wave-surface, these rays form two sheaves of developable surfaces which intersect each other at right angles along each ray and whose tangent-planes are the *principal sections* of the bundle with respect to the ray which coincides with the line of intersection of these planes. The ray u which crosses the wave-surface at the point O touches the cuspidal edges of the two developable surfaces to which the ray belongs in two points C_1 and C_2, which with respect to the point O are the centres of principal curvature, but which with respect to the ray u we shall call the *image-points* (or *foci*) lying on this ray. All the cuspidal edges of one of the two sheaves of developable surfaces form together one caustic surface which is touched by the other sheaf of developable surfaces. The tangents to the two caustic surfaces which lie in the planes drawn through C_1 and C_2 perpendicular to the ray u are the so-called *image-lines* (or *focal lines*) of the bundle with respect to this ray. Thus, if one of the two principal sections is denominated the *primary* (or I.) *principal section* and the other one the *secondary* (or II.) *principal section* (without implying by these terms any difference of rank or importance in the two sections), we can say that the I. image-line is perpendicular to the I. principal section at the I. image-point and lies in the II. principal section. In the special case when the two caustic surfaces touch each other at the common point of tangency of the ray u, the bundle of rays is said to be *anastigmatic* (or *stigmatic*) along this particular ray, but with few exceptions along this ray only. Thus a bundle of rays may be astigmatic along one ray and stigmatic along another ray. Along any arbitrarily selected ray the bundle of rays is generally astigmatic with two image-points.

If a line is drawn on the wave-surface, the rays which cross this surface at points lying along this line form a ruled surface or *ray-surface*, as we may call it. In general, a ray-surface is a skew surface; but if the line traced on the wave-surface is a line of curvature, the ray-surface will be developable and will be characterized by the fact that the tangent-plane to such a ray-surface will remain invariably the same from point to point along any chosen ray. Such a ray-surface is perhaps the nearest approach that we have in the case of actual optical rays to a plane *pencil of rays* of finite dimensions. Gullstrand,[1] whose recent investigations of the general characteristics of a bundle of optical rays are destined to become classic, calls this a *focal ray-surface.*

48. Theory of an Infinitely Narrow Bundle of Optical Rays; Sturm's Conoid. The theory of an infinitely narrow bundle of rays is inseparably associated with a celebrated theorem due to Sturm,[2] which may be stated as follows: All the normals to a surface in the neighbourhood of a point converge to or diverge from two focal lines at right angles to each other. We proceed therefore to describe Sturm's *Conoid* in the case when the narrow bundle of rays has two planes of symmetry. Consider an element of the wave-surface in the neighbourhood of the point O (Fig. 15*a*) where the chief or central ray u of the astigmatic bundle of rays crosses the surface. This surface-element may be regarded as a curvilinear rectangle $KLMN$ traced on the surface with its sides parallel to the lines of curvature P_1OQ_1 and P_2OQ_2 which pass through O. All rays crossing the wave-surface at points lying along the arc P_1OQ_1 intersect the chief ray u at the image-point C_1, and, similarly, all rays crossing the wave-surface at points lying along the arc P_2OQ_2 intersect the chief ray u at the image-point C_2. The rays which cross the element of the surface at points lying along any arc drawn parallel to the arc P_1OQ_1 will all meet (as a first approximation) in a point of the infinitely short image-line $A_1C_1B_1$ which lies in the plane of the principal section $P_2C_2Q_2$; and such rays will also cross the plane of the principal section $P_1C_1Q_1$ at points which (to the same degree of approximation) lie in the other infinitely short image-line $A_2C_2B_2$. By interchanging the subscripts 1 and 2 affixed to the letters A, B, C, P

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[1] A. Gullstrand: Ueber Astigmatismus, Koma und Aberration: Drude's *Ann. d. Phys.*, xviii (1905), 941–973.—Die reele optische Abbildung: *Kungl. Svenska Vetenskaps-akademiens Handligar.*, xli, No. 3 (1906), 119 pages.—Tatsachen und Fiktionen in der Lehre von der optischen Abbildung: A. Gleichen's *Archiv f. Optik*, i (1907), 2–41, 81–97. See also in the first volume of the third edition of Helmholtz's *Handbuch der physiolog. Optik* (Hamburg u. Leipzig, 1909) the chapter written by Gullstrand entitled "Die optische Abbildung," pages 226–258. In the articles above mentioned Gullstrand refers also to numerous other contributions, one of them dating as far back as 1890, which deal with the subject here under consideration.

[2] J. C. Sturm: Mémoire sur l'optique: Liouville's *Journ. de Math.*, iii (1838), 357–384. Also, Mémoire sur la théorie de la vision: *Comptes rend.*, xx (1845), 554–560; 761–767; 1238–1257. (This latter paper was translated and published in Pogg. *Ann. der Phys.*, lxv (1845).)

See also the following writers, among others, on this subject:

E. E. Kummer: Allgemeine Theorie der gradlinigen Strahlensysteme: Crelle's *Journ.*, lvii (1860), 189–230.—Modelle der allgemeinen unendlich duennen, gradlinigen Strahlenbuendel: *Berl. Akad. Ber.*, 1860, 469–474.—Ueber die algebraischen Strahlensysteme, in's Besondere ueber die der ersten und der zweiten Ordnung: *Berl. Akad. Monatsber.*, 1865, 288–293.—*Berl. Akad. Abh.*, 1866, No. 1, 1–120.

H. Helmholtz: *Handbuch der physiolog. Optik*, ii. Thl. (1860), 246.

A. F. Moebius: Geometrische Entwickelung der Eigenschaften unendlich duenner Strahlenbuendel: *Sitzungsber. d. Saechs. Akad., math.-phys. Cl.*, xiv (1862), 1–16.

F. Lippich: Ueber Brechung und Reflexion unendlich duenner Strahlensysteme an Kugelflaechen: *Denkschr. d. Wien. Akad., math.-phys. Cl.*, xxxviii (1878), 163–192.

C. Neumann: *Sitzungsber. d. Saechs. Akad., math.-phys. Cl.*, 1879, 42.

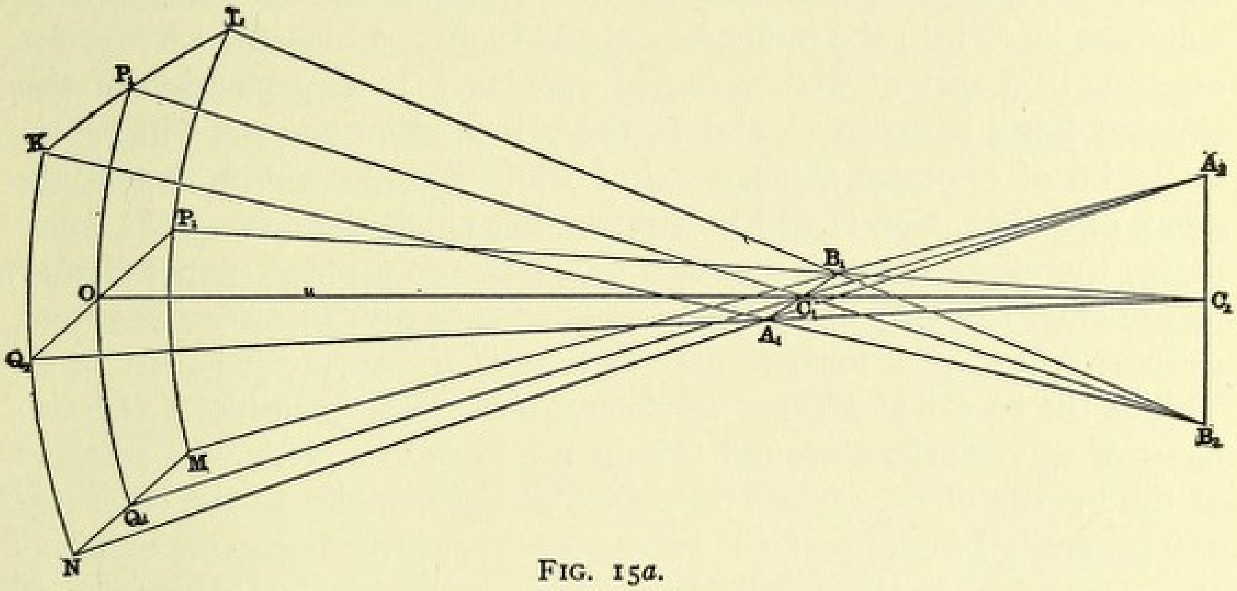

FIG. 15*a*.
STURM'S CONOID.

and Q in the above statement, we obtain a corresponding statement for the rays which cross the element of the wave-surface at points lying along an arc parallel to P_2OQ_2. Thus, the entire bundle of rays may be regarded as composed of a sheaf of pencils of rays in either of two ways as follows: First, the bundle of rays may be considered as arising from the rotation of the pencil of rays $P_2C_2Q_2$ about $A_1C_1B_1$ as axis, so that the element of arc P_2Q_2 generates the element of the wave-surface; or, second, we may regard the bundle of rays as generated by the rotation of the pencil of rays $P_1C_1Q_1$ about $A_2C_2B_2$ as axis.

If, however, we trace on the surface-element at O any line passing through O which does not coincide with one of the lines of curvature, and consider the system of rays which cross the element of wave-surface at points lying along this elementary arc, we find that, in general, these rays will not intersect at all.

The image-line at C_1 (or C_2) contains the vertices of all those pencils of rays which have their planes perpendicular to the plane of principal curvature for which C_2 (or C_1) is the centre. If we are given the chief ray u and the two image-lines, we can construct (according to STURM's theory) the entire bundle of rays by joining each point of one image-line with all the points of the other image-line.

With the passage of time, the element of the wave-front advances in the direction of the wave-normal u, each point of this element proceeding along the normal belonging to it. Approaching the image-line at C_1, the element shrinks in dimensions, until arriving at C_1, it collapses into the image-line $A_1C_1B_1$. Thereafter, the element of wave-front begins to open out again, and then again contracts until it collapses at C_2 into the image-line $A_2C_2B_2$; after which the wave-front expands in a sort of wedge-shaped opening. In any position of the element lying between C_1 and C_2 the principal curvatures will necessarily be of opposite signs, so that while the element is expanding along one dimension it will be contracting along the other. At some place, therefore, between the two image-points a plane perpendicular to the chief ray will cut the narrow bundle of rays in a section whose contour will have a form similar to that of the surface-element at O. This is the so-called *place of least confusion*. For example, if the element of wave-surface around O is in the form of a circle, the sections of the bundle of rays made by planes at right angles to the chief ray will generally be elliptical, the two axes of the ellipse becoming equal at the place where we have a "circle of least confusion." On one side of this place the major axes of the elliptical sections will be parallel to one of the image-lines, whereas on the other side the major axes will be parallel to the other image-line. From the optical standpoint the importance of this place of least confusion consists in the fact that we have here in a certain sense the best convergence of the rays. (In connection with this statement, however, see the important remark at the end of § 49).

49. The Image-Lines. In the above discussion it has been assumed that the lines of curvature at the different points of the element of the wave-surface are parallel to the lines of curvature at the point O; which is true, however, only in case we neglect magnitudes of the second order of smallness. Hence, the results which are given above as to the constitution of an infinitely narrow bundle of optical rays, known as STURM's Theory, are valid only to that degree of approximation. Taking account of magnitudes of the second order of smallness, we shall find that, instead of image-lines going through image-points, we have bits of image-surfaces, which, however, in special cases may collapse into image-lines having any inclinations to the chief ray.[1]

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[1] See, for example, LUDWIG MATTHIESSEN: Ueber die Form der unendlich duennen astigmatischen Strahlenbuendel und ueber die KUMMER'schen Modelle: *Sitzungsber. der math.-phys. Cl. der koenigl. bayer. Akad. der Wiss. zu Muenchen*, xiii (1883), 35–51.

In the case when the magnitudes of the second order of smallness are neglected, the question has been raised also, especially by MATTHIESSEN,[1] as to the part of STURM's Proposition which asserts that the two image-lines are perpendicular to the chief ray. If, for example, the wave-surface is a surface of revolution, and if we draw two infinitely near normals in the plane of the meridian curve and rotate the meridian plane through a small angle about the axis of revolution, we obtain for the II. image-line the piece of the axis intercepted by the two normals, which may not, and generally will not, be perpendicular to the chief normal u. The other image-line here will be the infinitely small arc of a circle described about the axis by the point of intersection of the two normals. (Cf. § 46. See also § 232.)

GULLSTRAND distinguishes three principal types of astigmatic bundles of optical rays according as the bundle has one or two planes of symmetry or none at all, which last is the general case. Thus,

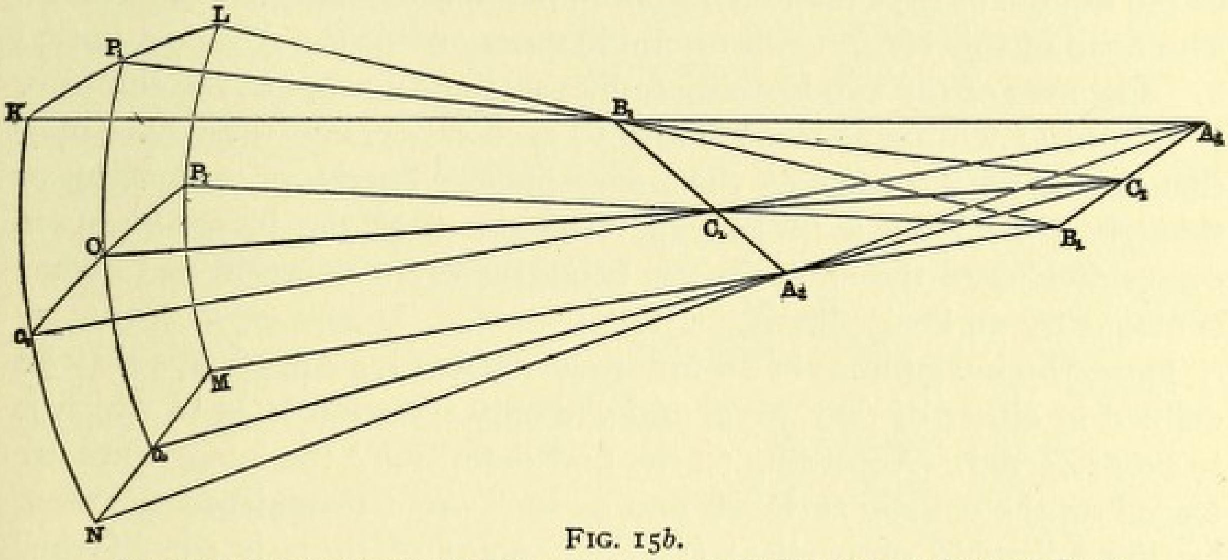

FIG. 15*b*.
GENERAL CONSTITUTION OF INFINITELY NARROW BUNDLE OF OPTICAL RAYS.

STURM's Conoid, with the image-lines at right angles to the central ray, represents a special case of the doubly symmetric bundle of optical rays. When a narrow homocentric bundle of incident rays is refracted at a surface of revolution, it will generally be transformed into an astigmatic bundle; and if the chief ray lies in the same plane as the axis of revolution, the bundle of rays will be symmetric with respect to this plane, in which case one of the image-lines (viz., the so-called primary image-line which is perpendicular to the plane of symmetry) will be perpendicular to the chief ray, whereas the other image-line will not intersect the chief ray at right angles. Finally, if the chief ray does not lie in a meridian plane of the surface of revolution, the bundle of refracted rays will have no plane of symmetry, and then neither image-line will be perpendicular to the chief ray. Thus, the general constitution of an infinitely narrow astigmatic bundle of rays is represented in Fig. 15*b*, where $KLMN$ is an element of the wave-surface.

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[1] L. MATTHIESSEN: Neue Untersuchungen ueber die Lage der Brennlinien unendlich duenner copulirter Strahlenbuendel gegen einander und gegen einen Hauptstrahl: *Acta Math.*, iv (1884), 177–192. (This paper was also published in the Supplement of *Zft. f. Math. u. Phys.*, xxix (1884), 86.) See also by same author: Untersuchungen ueber die Constitution unendlich duenner astigmatischer Strahlenbuendel nach ihrer Brechung in einer krummen Oberflache: *Zft. f. Math. u. Phys.*, xxxiii (1888), 167–183.

The discussion in regard to the directions of the image-lines of an infinitely narrow bundle of rays depends essentially, as CZAPSKI[1] has shown, on how those lines are defined. All the rays of the bundle may be regarded as intersecting both of the image-lines $A_1C_1B_1$ and $A_2C_2B_2$, provided we neglect infinitesimals of the second order. But with this same proviso, we may also consider as image-line any section of the bundle of rays made by a plane passing through either C_1 or C_2. The form of this section will resemble more or less the Arabic numeral 8. The axes of the two lemniscate-like sections normal to the chief ray at C_1 and C_2 will be at right angles to each other, and these axes may themselves be regarded as the image-lines. Therefore, according to this view, we have a perfect right to say that the image-lines are perpendicular to the chief ray; it being merely a question as to what is meant by an image-line.

Thus, the image-lines of an infinitely narrow bundle of rays may be defined in either of two ways, and the only question is as to which is to be preferred. According to the first definition,[2] the image-lines are traced on the caustic surfaces, and as such are distinguished therefore by the following properties: (1) Each point of them is the "focal" point or meeting place of elementary wave-trains (or rays) of equal optical lengths; so that assuming that these waves have a common origin, they will reinforce each other at this convergence-point on the image-line, and hence at this point there will be a maximum light-effect. (This latter item is not mentioned by MATTHIESSEN, but CZAPSKI directs attention to it as a matter not to be overlooked in this discussion.) And (2), as MATTHIESSEN very particularly remarks, in these lines the section of the bundle is a minimum, in some cases indeed an infinitesimal of order higher than that of any other section.

However, from a practical point of view, the STURM image-lines perpendicular to the chief ray of the bundle possess also certain advantages; by their very definition they have the distinguishing property of being the places in the bundle where the element of wave-surface is smallest. Accordingly, after a thorough discussion of the relative merits of the two modes of defining the image-lines, CZAPSKI reaches the conclusion that he can see no advantage in abandoning the "classical" image-lines of STURM. (See also § 232.)

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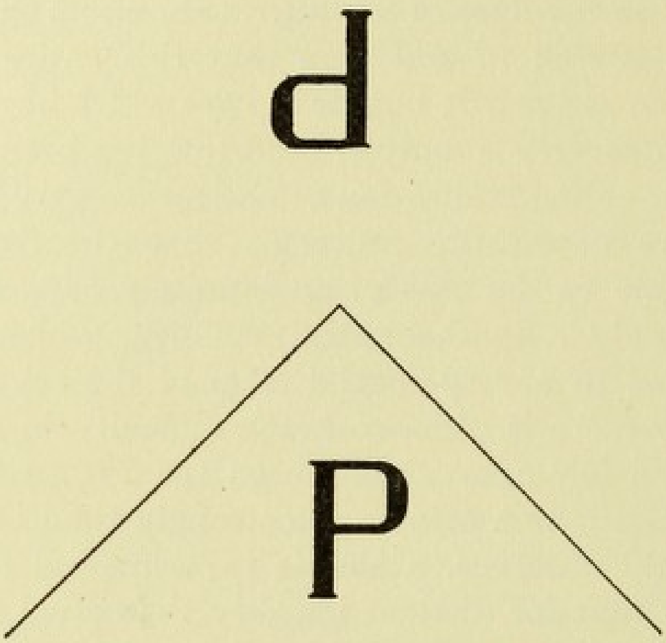
[1] S. CZAPSKI: Zur Frage nach der Richtung der Brennlinien in unendlich duennen optischen Buescheln: WIED. *Ann. der Phys.*, xlii (1891), 332–337.

[2] See L. MATTHIESSEN: *Berlin-Eversbusch. Zft. f. vergl. Augenheilk.*, vi (1889), 104.

However, it is well to add here that this entire question of the image-lines of an astigmatic bundle of rays which has been the subject of so much controversy in optical literature has been confused by the fact that, so far as the realization of optical imagery is concerned, the ray-bundles are necessarily finite; so that the STURM image-lines which apply only to the case of an infinitely narrow bundle of rays have at best only a mathematical validity, which moreover is in reality restricted to a very special form of the ray-bundle. This is the reason why there is often so much difficulty in obtaining experimentally even a fair approximation to the STURM image-lines. No one has emphasized this point more strongly than GULLSTRAND. As this writer aptly observes, a simple experimental investigation will suffice to show that for distinct imagery the actual size of the cross-section of the bundle of rays is of secondary importance as compared with the mode of distribution of the light inside this section; and hence it is necessary to investigate the form of the caustic surface, since it is the section of this surface which shows how the light-intensity is distributed within the bundle of rays.

Note to § 51, concerning the image after two reflexions in a pair of plane mirrors. (See Chapter III, page 55.) One other application of these theorems about inclined mirrors which is of practical importance in the construction of certain modern optical instruments deserves to be noted here. Rays emanating from an object lying in the dihedral angle θ between a pair of plane mirrors and undergoing two reflexions, first at one mirror and then at the other, will give rise to an image which is so related to the object that the position and orientation of the image can be determined merely by revolving the object bodily (in the direction towards the second mirror) around the line of intersection of the planes of the two mirrors as axis through an angle equal to 2θ. If this revolution is executed, the object will be found to be completely congruent with the image here in question. In particular, if the planes of the mirrors are at right angles, the angle of revolution will amount to 180°; so that in this case the image is completely inverted with respect to the object. This is the principle which was so ingeniously utilized first by Porro in 1852 and more successfully afterwards by Abbe in 1893 in the construction of prism-telescopes, made familiar to every one nowadays in the modern binocular glasses (field glasses, opera glasses, etc.). In these instruments the image, which would naturally be inverted, is rectified by the simple device of contriving that the rays after traversing the objective shall be made to undergo successive reflexions at a series of plane surfaces which are so disposed with respect to each other that the reflecting system, so far as the orientation of the image is concerned, is precisely equivalent to a pair of plane mirrors at right angles to each other.



Complete Inversion of Image by Double Reflexion at Two Plane Mirrors at Right Angles to each other.

CHAPTER III.

REFLEXION AND REFRACTION OF LIGHT-RAYS AT A PLANE SURFACE.

ART. 16. THE PLANE MIRROR.

50. In the diagram (Fig. 16) the plane reflecting surface or *mirror* is supposed to be perpendicular to the plane of the paper; the straight line AB showing its trace in this plane. The reflected ray BQ corresponding to an incident ray LB will lie in the plane of incidence (which is here the plane of the paper), and the path of the reflected ray will be such that, if it is produced backwards to meet at L', the straight line drawn from L perpendicular to the plane of the mirror at the point A, we shall have $L'A = AL$ (see § 28). Moreover, since the position of the point L' is independent of the position on the plane mirror of the incidence-point B, all incident rays which go through the point L will be reflected along paths which, prolonged backwards, will meet in the point L' (Fig. 17); so that *to a homocentric bundle of incident rays reflected at a plane mirror there corresponds also a homocentric bundle of reflected rays.*

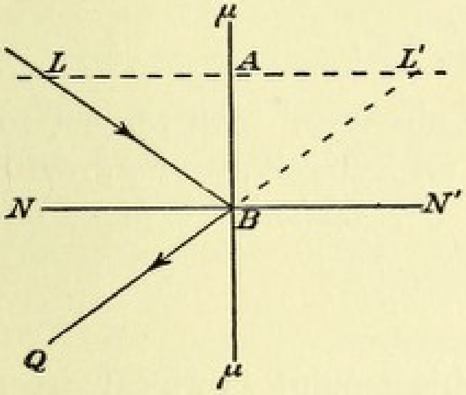


FIG. 16.

PATH OF A RAY REFLECTED AT A PLANE MIRROR.

$AL = L'A$.

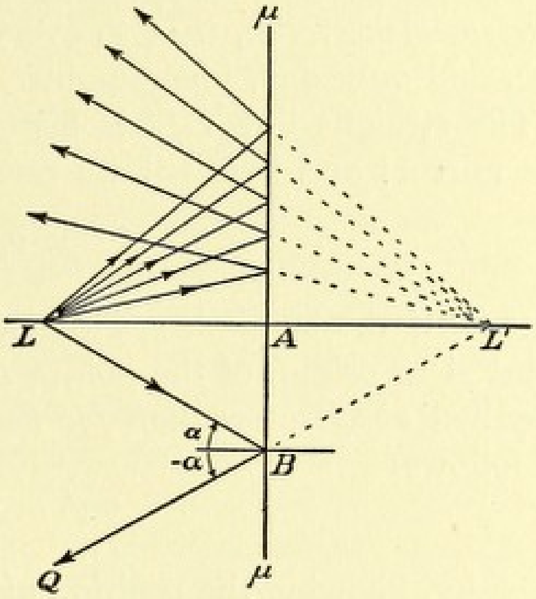


FIG. 17.

HOMOCENTRIC BUNDLE OF RAYS REFLECTED AT A PLANE MIRROR.

$AL = v$, $AL' = v'$.

The points designated by L and L', which are the vertices of these two corresponding homocentric bundles of rays, are a pair of *conjugate* points with respect to the plane mirror (§ 44). In this case the point L' is a *virtual image* of the object-point L. But if the bundle of incidence rays converge towards a "virtual object-point" L situated behind the mirror, we shall have then a *real image* at a point L' in front of the mirror. Thus, the object-point L may be situated anywhere in infinite space, and there will be always a corresponding image-point L'. It may be remarked here that the plane mirror is the only optical system which, without any restrictions whatever as to the angular apertures of the bundles of rays concerned in the formation of the image, satisfies perfectly the geometrical condition of *collinear correspondence*, viz., that to every object-point there shall correspond one, and only one, image-point.

The straight line LL' is bisected at right angles by the plane of the mirror; and hence if we put

$$AL = v, \quad AL' = v',$$

that is, if the symbols v, v' denote the abscissæ,[1] with respect to the point A as origin, of the points L, L', respectively, we may write the so-called *abscissa-equation for the case of reflexion at a plane mirror*, as follows:

$$v' = -v; \tag{15}$$

whereby, knowing the position of the object-point L, we can ascertain the position of the corresponding image-point L'.

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[1] The word *abscissa* will be employed throughout this book (unless otherwise specifically stated) to describe the position of the point where a ray crosses the *optical axis* xx' of a refracting or reflecting surface with respect to the *vertex* A of the surface as origin. This optical axis (which will be particularly defined in a following chapter) is identical here with the straight line drawn from the luminous point perpendicular to the surface. Thus, for example, in Fig. 16, the abscissa of the object-point L is AL, which is *always to be reckoned in the sense in which the letters are written*, so that if $v = AL$, then $-v = LA$.

So far as our immediate purposes in this chapter are concerned, it is entirely immaterial which direction along the axis we take as the positive direction; the opposite direction will, of course, have to be reckoned as negative. Subsequently, we shall see that, as a rule, it will be convenient to reckon *the positive direction along any ray of light as the direction which the light pursues along that ray;* and we may, therefore, use this method here (cf. § 26). Generally, in all our diagrams the incident light will be represented as travelling from left to right.

In this place we take occasion, also, to say expressly that, if $A, B, C, D \dots J, K$ designate the positions of a number of points ranged all along a straight line, in any order whatever, we have always the following relations:

$$AB + BA = 0, \quad \text{or} \quad AB = -BA;$$
$$AB + BC + CA = 0;$$
$$AB + BC + CD + \cdots + JK = AK; \text{ etc., etc.} \quad \text{(See Appendix.)}$$

If, instead of a single luminous point L, we have an *extended object* consisting of an aggregation or system of luminous points, to each point of the object there will correspond one image-point, and the image of such an object will be formed by the system of image-points. Thus, if L_1, L_2, etc. (Fig. 18), designate the positions of the points of the object, the positions of the corresponding image-points L'_1, L'_2, etc., will be determined by the fact that the plane of the mirror must bisect at right angles the straight lines joining each pair of conjugate points. It is obvious that the image in this case will have exactly the same dimensions as the object. In general, however, the two will not be congruent; that is, image and object, although similar and equal, cannot be superposed, because, being symmetrically situated with respect to the mirror, their corresponding parts face opposite ways; so that, for example, the situation of the object with respect to right and left is reversed in the image. The object and image will be congruent only in case the object is a plane figure, as shown in the diagram.

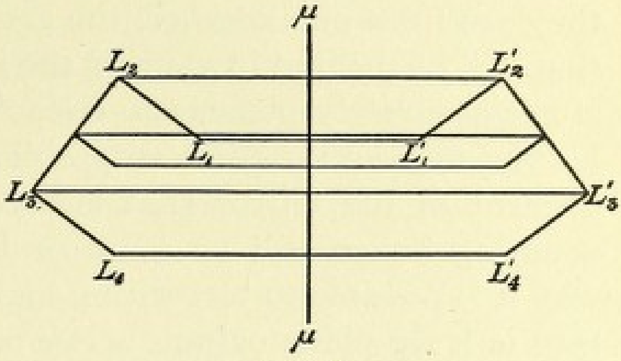

FIG. 18.

IMAGE OF EXTENDED OBJECT IN A PLANE MIRROR.

The extent of the portion of the mirror that is actually utilized will depend on the magnitude and position of the object whose image is to be viewed and also on the position of the eye of the observer. Thus, for example, in order that a man standing erect in front of a vertical plane mirror may be able to view his image from head to foot, the height of the mirror must be at least half the height of the man, and then the lower edge of the mirror must be placed at a level half-way between the levels of the eyes and feet; as may easily be verified.

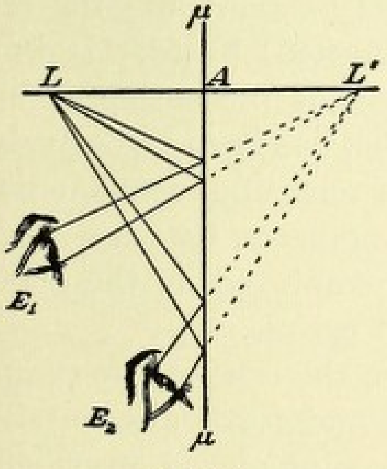

FIG. 19.

The two bundles of reflected rays having equal angular apertures intercept unequal pieces of the plane mirror.

Wherever the eye is placed in front of a plane mirror, the image of an object will appear always at the same place and of the same dimensions. The more inclined towards the mirror is the cone of rays that enter the eye, the greater will be the piece of the mirror utilized by these rays. For example (Fig. 19), the cone of rays which enters the eye at E_1 intercepts on the mirror a shorter piece of the mirror than will be intercepted when the eye is placed at E_2 in the figure. The nearer the object and the eye are to the mirror, and the farther they are from one another, the greater will be the piece of the mirror that will be utilized in viewing the image. If the surface of the mirror is not accurately plane, any irregularities in it will be made apparent by viewing the image at very oblique incidence; for in this case each element of the mirror that is used will produce an image, and the resulting image will be more or less blurred or indistinct. In this way it is possible to test with a high degree of accuracy whether a surface is truly plane or not. The method has been employed to show the curvature, due to the spherical form of the earth's surface, of the free surface of tranquil mercury.

51. A number of the most important uses of the plane mirror depend on the fact that *when the mirror is turned through any angle about an axis perpendicular to the plane of incidence, the reflected ray will be turned through an angle just twice as great.* This follows immediately from the law of reflexion. For if the plane of the mirror is turned through any angle θ, the normal to the mirror will be turned through an equal angle, and hence the angle between a given incident ray and the normal at the point of incidence will be changed by the amount θ, and therefore the angle between the incident and the reflected rays will have been increased (or diminished) by 2θ. It was POGGENDORFF who first suggested the method of measuring small angles which depends on this principle, and which has been extensively employed for this purpose in a great variety of scientific instruments, such as the reflexion-lever, the mirror galvanometer, GAUSS's magnetometer, etc. Essentially the same idea is employed in the goniometer in measuring the angles of crystals and prisms.

In this connection, we may mention here also the case of two plane mirrors at which the rays are reflected back and forth alternately. The incident rays emanating from a luminous point placed anywhere in the dihedral angle between the planes of the two mirrors which fall on mirror No. 1 will give rise to one series of images, while the incident rays which fall on mirror No. 2 will give rise to a second series of images. The images of both series will evidently all be ranged on the circumference of a circle whose centre is at the point of the line of intersection of the planes of the mirrors determined by a plane through the luminous point perpendicular to this line, and whose radius is equal to the length of the straight line joining the centre with the luminous point. The last image of each of the two series will be the first image of that series which is so situated as to be behind both mirrors, and which lies, therefore, in the equal dihedral angle formed by continuing the planes of the mirrors backwards beyond their common line of intersection. The total number of images in any case will depend on the angle included between the planes of the two mirrors, and also on the position of the object-point with respect to the mirrors. If θ denotes the angle between the two plane mirrors, and if the angular distances of the object-point from the two mirrors are denoted by ω and φ, so that $\theta = \omega + \varphi$, the total number of images may be shown to be as follows:

(1) If the angle θ is contained a whole number of times, say i, in 180°, so that $180/\theta = i$, the number of images in this case will be $2i - 1$, no matter what may be the values of the angles denoted by ω, φ.

(2) But if the angle θ is contained in 180° a whole number of times i with a remainder $\epsilon < \theta$, so that $180/\theta = i + \epsilon/\theta$, there are four cases here to be distinguished as follows:

(*a*) If $\epsilon > \theta/2$, the number of images in this case $= 2i + 2$;

(*b*) If $\epsilon = \theta/2$, the number of images in this case $= 2i + 1$;

(*c*) If $\epsilon < \theta/2$, but $> \omega$, the number of images $= 2i + 1$;

and

(*d*) If $\epsilon < \varphi$ and also $< \omega$, the number of images $= 2i$.

See HEATH's *Geometrical Optics* (Cambridge, 1887), Art. 32.

This theory explains Sir DAVID BREWSTER's *Kaleidoscope*, in which multiple images are formed by two plane mirrors inclined to each other. When the mirrors are parallel and facing each other ($\theta = 0$), the number of images will be infinite.

Another theorem of inclined mirrors given in HEATH's *Geometrical Optics*, Art. 14, which is applied in the instrument known as the *Sextant*, is as follows:

When a ray of light is reflected an even number of times ($2i$) in succession at two plane mirrors (the reflexions occurring in a plane at right angles to the planes of the mirrors), the total deviation is equal to $2i$ times the angle of inclination of the mirrors.

ART. 17. TRIGONOMETRIC FORMULÆ FOR CALCULATING THE PATH OF A RAY REFRACTED AT A PLANE SURFACE. IMAGERY IN THE CASE OF REFRACTION OF PARAXIAL RAYS AT A PLANE SURFACE.

52. Let L (Fig. 20) designate the position of a point on a ray incident at the point B on a plane refracting surface which separates two isotropic optical media of absolute indices of refraction n and n'. The straight line LA or x which, going through the point L, meets the surface normally at the point A is called the *axis* of the refracting plane with respect to the point L. The magnitudes

$$AL = v, \quad \angle ALB = \alpha,$$

which determine completely the position of the incident ray, may be called the *ray-co-ordinates*. Similarly, if L' designates the position of the point where the refracted ray, produced backwards from the incidence-point B, crosses the axis x, then

$$AL' = v', \quad \angle ALB = \alpha'$$

will be the ray-co-ordinates of the refracted ray $L'B$. The problem is, being given the incident ray (v, α), to determine the refracted ray (v', α').

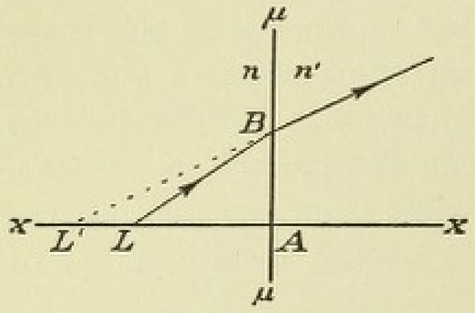

FIG. 20.

RAY OF LIGHT REFRACTED AT A PLANE.

$AL = v$, $AL' = v'$, $\angle ALB = \alpha$, $\angle AL'B = \alpha'$.

From the figure we derive immediately:

$$\frac{v'}{v} = \frac{\tan \alpha}{\tan \alpha'},$$

moreover, by the law of refraction:

$$n \cdot \sin \alpha = n' \cdot \sin \alpha';$$

whence are obtained the following formulæ:

$$\left.\begin{aligned} v' &= \frac{v}{n} \frac{\sqrt{n'^2 - n^2 \cdot \sin^2 \alpha}}{\cos \alpha}, \\ \sin \alpha' &= \frac{n}{n'} \sin \alpha. \end{aligned}\right\} \qquad (16)$$

These equations enable us to find the magnitudes v', α' and to determine, therefore, the refracted ray.

For a given value of v, we see that the value of v' will depend on the angle of incidence α. Only those rays emanating from the point L which meet the plane refracting surface at equal angles of incidence (and which lie, therefore, on the surface of a right circular cone generated by the revolution of the straight line LB about LA as axis) will, after refraction, all intersect at a point L' on the axis x. So that if L is a luminous point emitting rays in all directions, an eye placed in the second medium (n') will, in general, not see a distinct, but only a blurred and distorted, image of the object-point at L; as will be more fully explained in the section which treats of the caustic by refraction at a plane surface (Art. 18).

53. Refraction of Paraxial Rays at a Plane Surface. In one special case, however, the imagery produced by refraction at a plane surface is ideal. Let MA (Fig. 21) be the axis of the plane refracting surface μ with respect to the object-point M; and let us suppose that all the points of the refracting plane are screened from M except those points which are infinitely near to the point A where the axis meets the surface; so that of all the rays proceeding from M only those whose paths lie very close to the axis can meet the refracting surface. We shall have thus an infinitely narrow bundle of *paraxial* incident rays (enormously exaggerated in the diagram) whose chief ray coinciding with the axis of the refracting plane meets this plane normally. The angles of incidence and refraction of the chief ray are both equal to zero; whereas in the case of all the other rays these angles will both be infinitely small. If we suppose that the angles α, α' are so small that we may neglect the second and higher powers of these angles, the angle α disappears entirely from the first of equations (16); and if the abscissæ with respect to the point A of the conjugate axial points M, M' are denoted by u, u', respectively, that is, if here we put

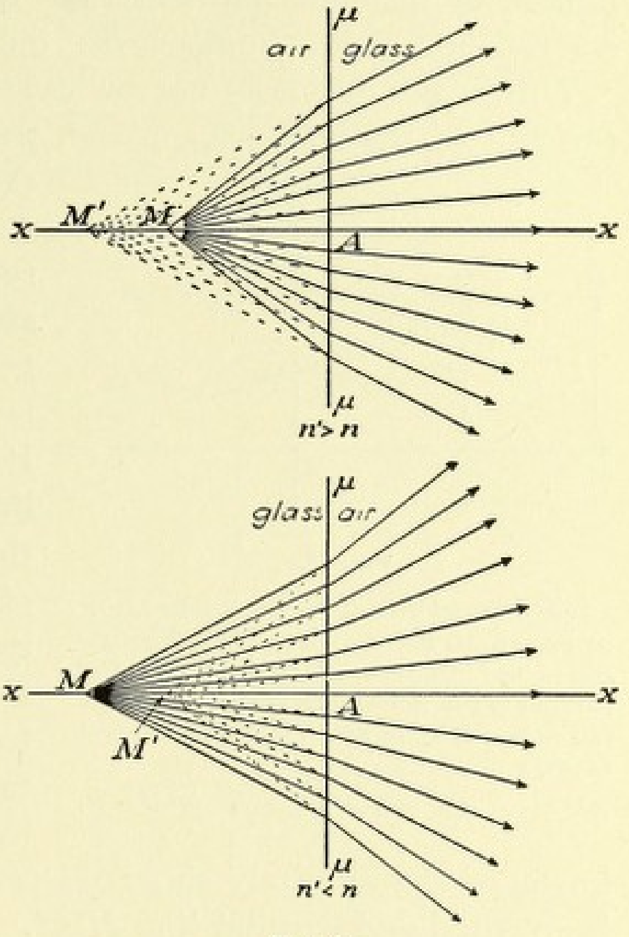

FIG. 21.

REFRACTION OF PARAXIAL RAYS AT A PLANE. In these diagrams the incident rays are supposed to meet the Refracting Plane almost normally. The angular apertures of the cones of rays are in reality infinitely small, although they are here enormously magnified. Paraxial Rays diverging from a point M are refracted at the Plane Surface as though they came from M'.

$AM = u, \quad AM' = u'$.

$$AM = u, \quad AM' = u',$$

we have evidently the following relation:

$$u' = \frac{n'}{n} u; \tag{17}$$

which is the so-called *abscissa-equation for the refraction of paraxial rays at a plane surface.* Provided we know the position on the axis of the object-point M, this equation enables us to determine the position of the corresponding image-point M'. Thus, to a homocentric bundle of incident paraxial rays refracted at a plane surface there corresponds also a homocentric bundle of refracted rays.

Within the infinitely narrow cylindrical region immediately around the axis of the refracting plane, we have, therefore, a point-to-point correspondence of object and image. According to (17), since u, u' have the same signs, the points M, M' lie always on the same side of the refracting plane, that is, the point M' is a virtual image of the point M. If the object is an infinitely short line MQ (Fig. 22) perpendicular to the axis at M, obviously, the image of the point Q will be a point Q' lying on the straight line drawn through Q perpendicular to the refracting plane and at the same distance from this plane as the axial image-point M'. Consequently, the image of the infinitely short object-line MQ at right angles to the axis is an equal and parallel line $M'Q'$. The ratio y'/y, where $MQ = y$, $M'Q' = y'$ is called the *Lateral Magnification* or the *Linear Magnification*, and will be denoted here by the symbol Y. Thus, in the case of the imagery produced by the refraction of paraxial rays at a plane surface, we have:

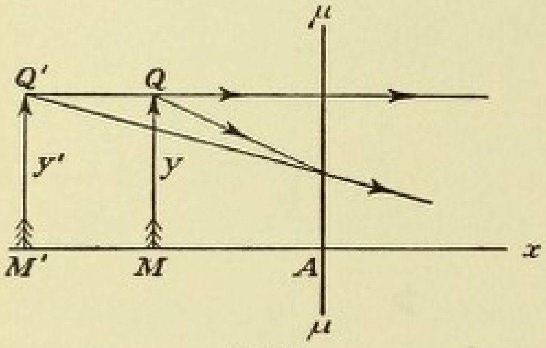

Fig. 22.

Imagery in the Case of Refraction of Paraxial Rays at a Plane. The image of the infinitely small object-line MQ parallel to the Refracting Plane is an equal image-line $M'Q'$ having the same direction as MQ.

$AM = u$, $AM' = u'$, $MQ = y$, $M'Q' = y'$.

$$Y = \frac{y'}{y} = +1. \tag{18}$$

The two equations (17) and (18) show that the image is always virtual and erect and of the same size as the object, provided the latter is a line at right angles to the axis. If $\angle AMQ$ is not a right angle, the image-line will not be parallel to the object-line nor of the same length as the object-line. We have here, in fact, a special case of collinear correspondence, known as *Central Collineation*, the refracting plane being itself the plane of collineation and the centre of collineation being the infinitely distant point of a straight line perpendicular to the refracting plane. It is the relation that in geometry is called *affinity*.

ART. 18. CAUSTIC SURFACE IN THE CASE OF A HOMOCENTRIC BUNDLE OF RAYS REFRACTED AT A PLANE SURFACE.

54. In general, as we saw (§ 52), to a homocentric bundle of rays incident on a plane refracting surface there corresponds a system of refracted rays which is not homocentric. It will be an instructive exercise to investigate in this comparatively simple case the form of the caustic surface (§ 46), especially as this example will afford a very good illustration of the general principles explained in Art. 15 of the preceding chapter.

Let the vertex of the homocentric bundle of incident rays be designated by S (Fig. 23), and let the straight line marked μ show the trace in the plane of the paper of the refracting plane. Since everything is symmetrical with respect to the normal SA, drawn from S to the plane refracting surface, it will be sufficient to investigate the form of the refracted wave-surface in the plane of the paper. Let the straight line SB drawn in the plane of the paper and meeting the refracting plane in the point B represent the path of an incident ray, and let L' designate the point where the corresponding refracted ray, produced backwards, intersects the straight line SA. In the case which we shall consider here the first medium (n) is supposed to be optically

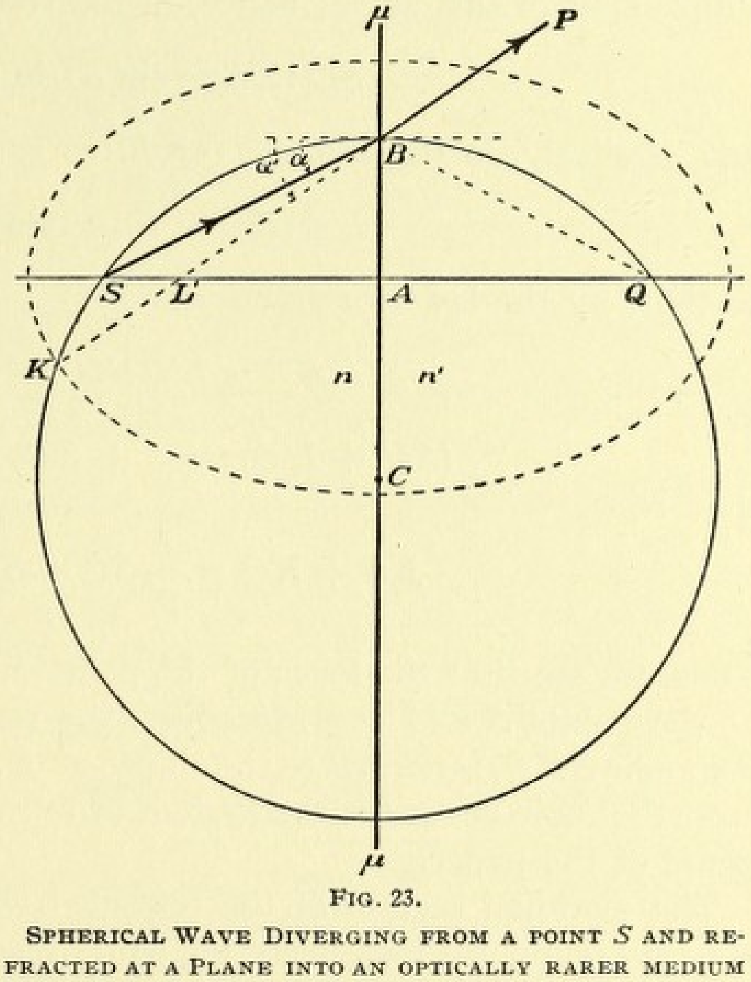

FIG. 23.

SPHERICAL WAVE DIVERGING FROM A POINT S AND REFRACTED AT A PLANE INTO AN OPTICALLY RARER MEDIUM ($n' < n$). SB is ray incident on refracting plane at the point B, and BP is the corresponding refracted ray.

denser than the second medium (n'), as, for example, when the rays are refracted from water into air; hence, $n > n'$, where n, n' denote the absolute indices of refraction of the two media. In this case, therefore, the point L' will lie between S and A, as shown in the figure.

Produce the normal SA into the second medium to a point Q so that $AQ = SA$, and pass a circle through the points S, B and Q, and produce the refracted ray backwards to meet the circumference of this circle in the point designated in the figure by K. The angle SKQ is evidently bisected by the straight line KB, and we have:

$$\angle SKB = \angle BKQ = \angle BSQ = \alpha,$$

since these inscribed angles stand on equal arcs of the circle. The two angles at L' are equal to the angle of refraction α' and to the supplement of this angle; hence, in the triangle $SL'K$ we have:

$$SL':KS = \sin\alpha:\sin\alpha';$$

and, similarly, in the triangle QKL':

$$L'Q:KQ = \sin\alpha:\sin\alpha';$$

so that, by the law of refraction:

$$SL':KS = L'Q:KQ = n':n,$$

or

$$(SL' + L'Q):(KS + KQ) = n':n;$$

that is,

$$KS + KQ = \frac{n}{n'}SQ = \text{constant}.$$

Thus, we see that the locus of the point K is an ellipse with its foci at the points S and Q. Moreover, the refracted ray, which bisects the angle SKQ, is normal to the ellipse at K. The ellipse is, therefore, an orthotomic curve for the system of refracted rays which lie in the plane of the paper.

The meridian section of the refracted wave-front at any moment may be found by measuring off equal distances from the points of this ellipse along each refracted ray; that is, the refracted wave-fronts are parallel curves to this orthotomic ellipse (Fig. 24). These curves will not be themselves ellipses, since the parallel to a conic is, in general, a curve of the eighth degree.[1] But the parallel to a conic has the same evolute as the conic itself; so that the *caustic curve* which is the evolute of the wave-line will in the present case be the evolute of an ellipse.

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[1] See SALMON's *Conic Sections*, 6th edition, Art. 372, Ex. 3.

If here we put $AS = c$ (Fig. 23), and if the centre A of the ellipse is taken as origin of a system of rectangular axes (SA, AB being the directions of the positive axes of x, y, respectively), the Cartesian equation of the ellipse will be:

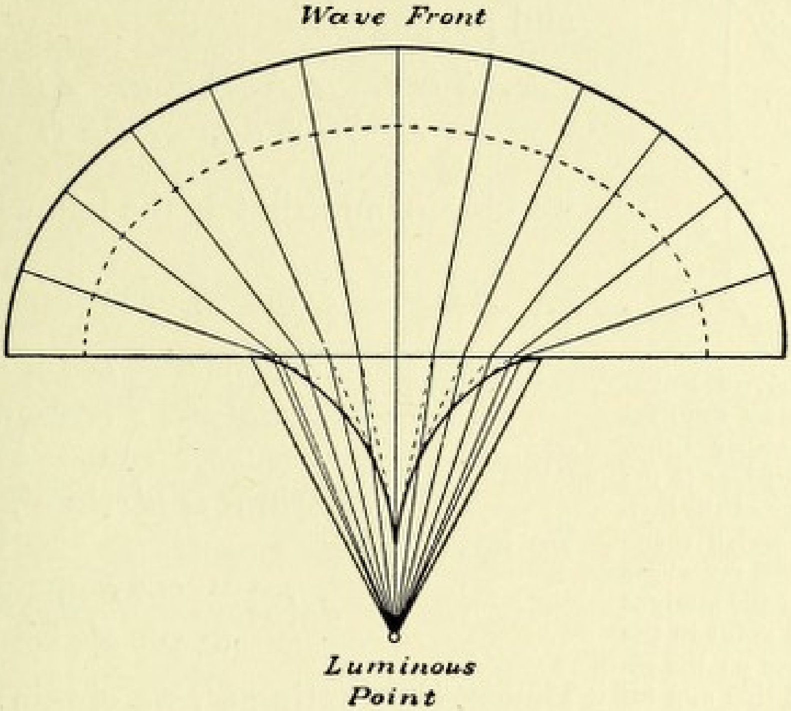

FIG. 24.

CAUSTIC CURVE AND WAVE-LINE IN THE CASE OF REFRACTION OF CIRCULAR WAVES AT A STRAIGHT LINE. Rays refracted from water into air.

$$\frac{x^2}{n^2} + \frac{y^2}{n^2 - n'^2} = \frac{c^2}{n'^2},$$

and the rationalized equation of the evolute of this ellipse is:

$$\{n^2x^2 + (n^2 - n'^2)y^2 - n'^2c^2\}^3 + 27c^2n^2n'^2(n^2 - n'^2)x^2y^2 = 0.$$

This is, therefore, the equation of the caustic curve in the case here considered. The caustic here is a "virtual" caustic.

It has been assumed above that the first medium was optically denser than the second. In the opposite case, viz., $n < n'$, the orthotomic curve for the system of meridian refracted rays proves to be a hyperbola with the same foci as the ellipse above, so that the caustic curve for this case will be the evolute of the hyperbola.

55. The equation of the caustic by refraction at a straight line may also be deduced directly, as follows:

Taking, as above, the point A (Fig. 25), which is the foot of the perpendicular let fall from the object-point S on the refracting straight line, as the origin of a system of rectangular axes, where SA and AB are the positive directions of the axes x and y, respectively, and putting:

$$AS = c, \quad \angle ASB = \alpha, \quad \angle HS'B = \alpha',$$
$$\angle BSG = d\alpha, \quad \angle BS'G = d\alpha',$$

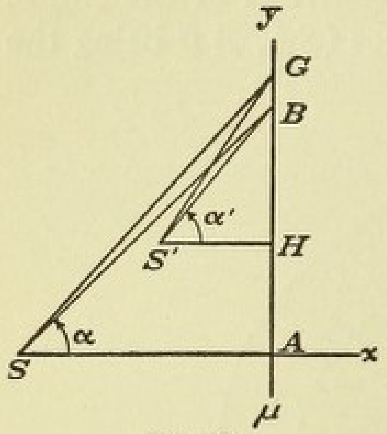

FIG. 25.

USED IN DERIVING EQUATION OF CAUSTIC BY REFRACTION AT A STRAIGHT LINE. The plane of the paper is the plane of incidence of the incident ray SB, to which corresponds the refracted ray whose direction is along the straight line $S'B$. G is a point in the plane of incidence on the refracting straight line and infinitely near to the incidence-point B. SG, $S'G$ incident and refracted rays.

we obtain immediately the following relations:

$$AB = -c \cdot \tan\alpha; \quad BG = -c\frac{d\alpha}{\cos^2\alpha};$$

$$BS' = \frac{GB \cdot \cos\alpha'}{d\alpha'} = \frac{\cos\alpha' \cdot d\alpha}{\cos^2\alpha \cdot d\alpha'} c.$$

Since

$$n \cdot \sin\alpha = n' \cdot \sin\alpha',$$

we have:

$$d\alpha' = \frac{n}{n'} \frac{\cos\alpha}{\cos\alpha'} d\alpha.$$

Hence, eliminating α', we obtain:

$$BS' = \frac{c(n'^2 - n^2 \cdot \sin^2\alpha)}{nn'\cos^3\alpha}.$$

Now

$$\begin{aligned} x &= HS' = BS' \cdot \cos\alpha' \\ &= \frac{c}{nn'^2}\{n'^2 - (n^2 - n'^2)\tan^2\alpha\}^{\frac{3}{2}}. \end{aligned}$$

Again, since

$$AB = -c \cdot \tan\alpha, \quad BH = BS' \cdot \sin\alpha, \quad \text{and} \quad AH = AB + BH,$$

we find after several reductions:

$$AH = y = -\frac{c}{n'^2}(n^2 - n'^2)\tan^3\alpha.$$

Eliminating tan α from these expressions for x and y, we obtain the Cartesian equation of the locus of the primary image-point S' (§ 47) corresponding to the object-point S, as follows:

$$\left(\frac{n}{n'c}x\right)^{\frac{2}{3}} + \left(\frac{\sqrt{n^2 - n'^2}}{n'c}y\right)^{\frac{2}{3}} = 1;$$

which, being rationalized, gives precisely the same equation as is given at the end of § 54.

The caustic turns its convex side towards the refracting straight line and touches it at a point designated by V in Fig. 26 whose distance from the point $A = n'c/\sqrt{n'^2 - n^2}$, and which is therefore the extreme point of incidence on the positive side of the y-axis. Of course, there is also another point of tangency at an equal distance from A on the negative side of the y-axis. Thus, for example, if the radiant point S is in water, and if the rays emerge from water into air ($n/n' = 4/3$), we shall find $AV = 1.14 \cdot AS$. Putting $y = 0$ in the equation of the caustic curve, we find the cusp of the caustic at a point M' on the normal to the refracting surface at A, such that $AM' = n' \cdot AS/n$, and hence (§ 53) this point M' is the image-point by paraxial rays of the object-point S.

If the diagram is revolved around SA as axis, the caustic curve will generate the caustic surface of the refracted rays corresponding to the homocentric bundle of incident rays emanating in all directions from the radiant point S. There are always two caustic surfaces, but in case the refracting surface is a surface of revolution, one of the caustic surfaces collapses into a piece of the axis of symmetry, which in this case is the segment SM' (see § 46).

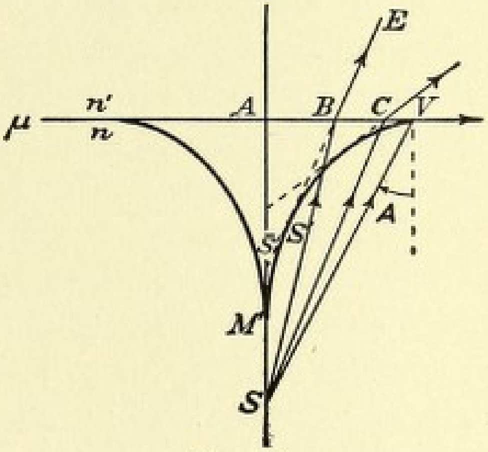

Fig. 26.

Caustic by Refraction at a Plane Surface for case when $n > n'$. Diagram is drawn for the case when the first medium is water and the second medium is air. The horizontal surface of the still water is the refracting plane, the rays being refracted from below upwards.

The refracted ray BE, corresponding to the incident ray SB, when produced backwards is tangent to the caustic curve at the point marked S' and meets the normal SA at the point marked $\bar{S}'$. These points S', $\bar{S}'$ are the positions of the I. and II. Image-Points of the astigmatic bundle of refracted rays that enter the eye at E.

If an eye were placed at the point E in air (Fig. 26), and if at a point S below the surface of still water there were situated a radiant point, the primary image of the radiant point S would be located at the point of tangency S' of the tangent to the caustic curve drawn from the point E. If the eye is placed vertically above the radiant point S, the image will be seen at M', that is, at a depth one-fourth nearer to the surface of the water than the object-point actually is. We see therefore how it is that an object under water viewed by an eye in the air above will, in general, appear not only to be raised towards the surface, but also to be displaced sideways more and more towards the observer, the more obliquely he regards the object. Obviously, incident rays meeting the water-surface at points beyond the extreme point V will be totally reflected (§ 27).[1]

ART. 19. ASTIGMATIC REFRACTION OF AN INFINITELY NARROW BUNDLE OF RAYS AT A PLANE SURFACE.

56. In the diagram (Fig. 27) the refracting plane designated by μ is supposed to be perpendicular to the plane of the paper. The point S in the first medium (n) is the vertex of an infinitely narrow homocentric bundle of incident rays, whose chief ray, viz., the ray SB or u, meets the refracting plane at the incidence-point B. The section of this bundle of rays made by the refracting plane will be a small closed curve $GJGJ$; which will be elliptical in case the cone of incident

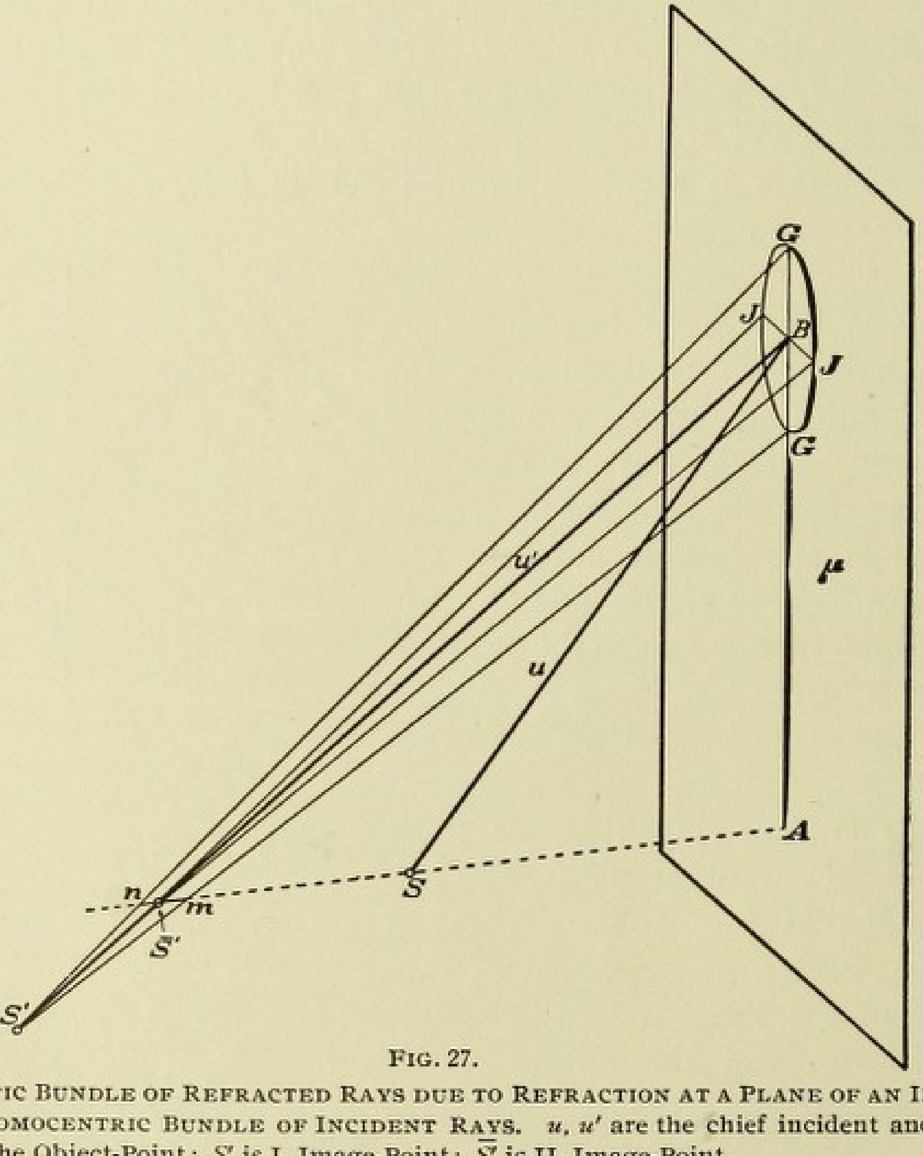

Fig. 27.

Astigmatic Bundle of Refracted Rays due to Refraction at a Plane of an Infinitely Narrow Homocentric Bundle of Incident Rays. u, u' are the chief incident and refracted rays. S is the Object-Point; S' is I. Image-Point; $\bar{S}'$ is II. Image-Point.

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[1] See in connection with this section L. Matthiessen: Das astigmatische Bild des horizontalen, ebenen Grundes eines Wasserbassins: *Ann. der Phys.* (1901), 347–352.

rays has a circular cross-section. The chief refracted ray u', corresponding to the chief incident ray u, will, if produced backwards from B, meet in the point designated by $\overline{S}'$ the straight line SA which is normal to the refracting plane at the point A. In general, the other rays of the bundle of refracted rays will not intersect the chief ray u', but will pass from one side of it to the other both above and below the plane of the paper. The refracted rays will constitute an astigmatic bundle of rays (§ 47), on whose chief ray u' the two image-points will lie. In order to ascertain the positions of these image-points, we must investigate those rays of the astigmatic bundle which meet the chief ray u'.

57. The Meridian Rays. The three points S, B and A determine the plane of incidence of the chief ray u; in the diagram this is the plane of the paper. This is likewise the plane of incidence of all the rays of the bundle which meet the refracting plane at points lying along the diameter GG of the closed curve $GJGJ$. The rays, therefore, which are refracted at the points G, G will necessarily meet the refracted chief ray u'; and, since we assume that the bundle of rays is infinitely narrow, the rays refracted at the points G, G will meet the chief refracted ray u' in one and the same point S', provided we neglect infinitesimal magnitudes of the second order; and the same thing will be true of all the rays refracted at points lying along the line-element GG. The plane of incidence of the chief ray u which contains this pencil of rays is one of the principal planes of curvature (§ 46) of the refracted wave-surface at the incidence-point B, and these rays are the so-called *meridian rays* of the bundle. The meridian rays of the refracted bundle of rays all intersect at the I. Image-Point S'. In this statement it is assumed that we neglect magnitudes of the second order of smallness, and hence the convergence of the rays at S' is said to be a "convergence of the first order" only.

Moreover, to a pencil of incident rays proceeding from the radiant point S and meeting the refracting plane at points lying along a chord of the curve $GJGJ$ which is parallel to the diameter GG there corresponds a pencil of refracted rays lying in the same plane as the pencil of incident rays (the plane determined by the chord and the radiant point) whose vertex will be a point infinitely close to the point S', above or below it, lying in the I. Image-Line at S' which is perpendicular to the plane of incidence of the chief ray u (§ 47).

58. The Sagittal Rays. Let us next consider the rays of the infinitely narrow bundle which meet the refracting plane at points lying along a diameter JJ of the curve $GJGJ$ which is at right angles to the diameter GG. The rays of this pencil which are incident at the end-points J, J of the diameter JJ are symmetrical with respect to the normal SA, so that after refraction they will intersect the chief refracted ray u' in the point $\overline{S}'$ where u' meets SA. This can be made clearer, if necessary, by imagining that the right triangle SBA is rotated through an infinitely small angle above and below the plane of the paper around SA as axis, so that the point B traces the line-element JJ, and the chief incident ray u coincides in succession with all the rays of the pencil SJJ. It is obvious that all the rays of this pencil will, after refraction, intersect the chief refracted ray u' at the II. Image-Point $\overline{S}'$. The plane $\overline{S}'JJ$ which is the plane of this pencil of refracted rays and which is perpendicular to the plane of incidence of the chief incident ray u is the other principal plane of curvature of the refracted wave-surface at the incidence-point B. This plane determines the sagittal section of the bundle of refracted rays, and the pencil of rays $\overline{S}'JJ$ contains the *sagittal rays* after refraction; these refracted rays correspond to the incident sagittal rays belonging to the pencil SJJ. The rays of the sagittal section of the bundle of refracted rays intersect in $\overline{S}'$ not merely approximately, but exactly, because in the sagittal section there is symmetry with respect to the plane of incidence of the chief ray u, so that rays from the radiant point S which make equal angles with the plane SAB on opposite sides of this plane will, after refraction, all pass through $\overline{S}'$; so that at the II. Image-Point the convergence is of the second order.

To a pencil of incident rays which meet the refracting plane at points lying along a chord of the closed curve $GJGJ$ which is parallel to the diameter JJ there corresponds a pencil of refracted rays which meet all at one point of the II. Image-Line. This latter lies in the plane of incidence of the chief ray u, and, according to Sturm, is perpendicular at $\overline{S}'$ to the chief refracted ray u'. However, we may also consider as II. Image-Line of the astigmatic bundle of refracted rays, not the line-element in the plane of incidence of the chief ray u that is perpendicular to the refracted chief ray u' at the point $\overline{S}'$, but the element mn of the normal to the refracting plane at the point A which is intercepted on this normal by the two extreme rays of the meridian pencil of refracted rays. Through this bit of the normal all the rays of the bundle of refracted rays must pass, as may easily be seen by rotating the plane of the paper around SA as axis through a small angle above and below this plane. In the course of this rotation, the rays of the meridian section will trace out all the other rays of the bundle, but the element mn of the normal SA will remain unchanged in magnitude and in position. As to this matter, see § 49.

We proceed now to determine the positions of the two image-points S' and $\bar{S}'$ of the astigmatic bundle of refracted rays.

59. Position of the Primary Image-Point S'. In the diagram (Fig. 28) the straight line $A B$ shows the trace in the plane of the paper of the refracting plane. The straight line SB represents the chief ray u of an infinitely narrow homocentric bundle of incident rays emanating from the object-point S. The plane of the paper is the plane of incidence of the chief ray u. Infinitely near to the incidence-point B of the chief ray and in the plane of the paper let us take the point G, so that SG represents a secondary ray of the pencil of meridian incident rays. The I. Image-Point S' will be at the point of intersection of the refracted rays corresponding to the incident rays SB and SG. Let α, α' denote the angles of incidence and refraction of the chief ray; so that, referring to the figure, we may write:

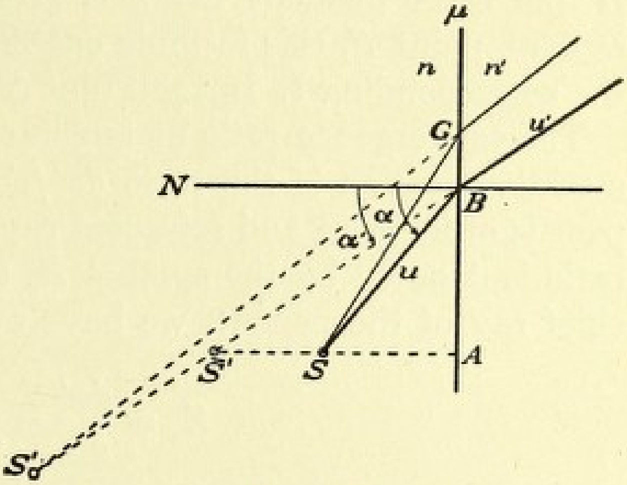

FIG. 28.

REFRACTION OF NARROW BUNDLE OF RAYS AT A PLANE. Figure for the determination of the positions of the I. and II. Image-Points S' and $\bar{S}'$ on the chief refracted ray u' corresponding to the object-point S on the chief incident ray u.

$\angle NBS = \alpha$, $\angle NBS' = \alpha'$, $\angle BSG = d\alpha$,
$\angle BS'G = d\alpha'$, $BS = s$, $BS' = s'$, $B\bar{S}' = \bar{s}'$.

$$\angle NBS = \alpha, \quad \angle NBS' = \alpha', \quad \angle BSG = d\alpha, \quad \angle BS'G = d\alpha'.$$

Then in the triangles BSG, $BS'G$ we have:

$$\frac{BG}{SB} = \frac{d\alpha}{\cos\alpha}, \quad \frac{BG}{S'B} = \frac{d\alpha'}{\cos\alpha'};$$

and, therefore:

$$\frac{BS'}{BS} = \frac{\cos\alpha'}{\cos\alpha} \frac{d\alpha}{d\alpha'}.$$

And since

$$n \cdot \sin\alpha = n' \cdot \sin\alpha',$$

we obtain finally:

$$\frac{BS'}{BS} = \frac{n' \cdot \cos^2\alpha'}{n \cdot \cos^2\alpha};$$

or if we put here:

$$BS = s, \quad BS' = s',$$

the formula above may be written:

$$s' = \frac{n' \cdot \cos^2 \alpha'}{n \cdot \cos^2 \alpha} s. \tag{19}$$

If the chief incident ray u is given and the position on it of the radiant point S, this formula enables us to determine the position of the corresponding I. Image-Point S' on the chief refracted ray u'.

The **convergence-ratio** or **angular magnification** of the rays of the meridian section is the ratio $d\alpha'/d\alpha$ of the angular apertures of the pencils of incident and refracted rays in the meridian section. If this ratio is denoted by the symbol Z_u (where the subscript indicates the chief ray of the pencil), we have evidently:

$$Z_u = \frac{d\alpha'}{d\alpha} = \frac{n \cdot \cos \alpha}{n' \cdot \cos \alpha'}. \tag{20}$$

60. Position of the Secondary Image-Point $\overline{S}'$. In order to determine the position of the II. Image-Point $\overline{S}'$, which is at the point of intersection of the straight line drawn through the homocentric object-point S perpendicular to the refracting plane with the chief refracted ray u' of the astigmatic bundle of refracted rays, we have from the triangle $SB\overline{S}'$:

$$\frac{BS}{B\overline{S}'} = \frac{\sin \angle B\overline{S}'S}{\sin \angle BS\overline{S}'} = \frac{\sin \alpha'}{\sin \alpha},$$

and, therefore:

$$\frac{B\overline{S}'}{BS} = \frac{n'}{n}.$$

If we put

$$BS = s, \quad B\overline{S}' = \bar{s}',$$

we shall have the following equation:

$$\bar{s}' = \frac{n'}{n} s. \tag{21}$$

Thus, if we know the position of the homocentric object-point S on the chief incident ray u, this formula enables us to locate the position of the II. Image-Point on the corresponding chief refracted ray u'.

All incident rays lying on the surface of the cone generated by the revolution of the ray SB around the normal SA as axis will after refraction at the plane refracting surface lie on the surface of a cone generated by the revolution of $\overline{S}'B$ around the same axis; as is evident immediately from the formula just obtained.

If SJ is a ray of the sagittal section of the homocentric bundle of incident rays which meets the plane refracting surface at a point J infinitely near to the incidence-point B of the chief incident ray u, $\bar{S}'J$ will show the direction of the corresponding ray of the sagittal section of the astigmatic bundle of refracted rays; and the ratio of the angles $B\bar{S}'J$ and BSJ is the **convergence-ratio** or **angular magnification** of these corresponding pencils of sagittal rays. If here we put:

$$\angle BSJ = d\bar{\lambda}, \quad \angle B\bar{S}'J = d\bar{\lambda}',$$

and if the symbol $\bar{Z}_u$ denotes the convergence-ratio of the pencils of incident and refracted sagittal rays with the chief incident ray u, we have evidently:

$$\bar{Z}_u = \frac{d\bar{\lambda}'}{d\bar{\lambda}} = \frac{BS}{B\bar{S}'} = \frac{s}{\bar{s}'} = \frac{n}{n'}. \tag{22}$$

61. The **Astigmatic Difference** of the bundle of refracted rays is the piece of the chief refracted ray u' comprised between the II. and I. Image-Points of the astigmatic bundle of rays; that is,

$$\bar{S}'S' = \bar{S}'B + BS' = s' - \bar{s}'.$$

In the case of an infinitely narrow homocentric bundle of incident rays refracted at a plane, we obtain from formulæ (19) and (21):

$$\frac{\bar{s}'}{s'} = \frac{\cos^2\alpha}{\cos^2\alpha'}; \tag{23}$$

and for the astigmatic difference of the bundle of refracted rays:

$$\bar{S}'S' = \frac{n's}{n}\left(\frac{\cos^2\alpha'}{\cos^2\alpha} - 1\right). \tag{24}$$

The astigmatic difference vanishes only in case $\alpha = \alpha' = 0$; that is, when the chief incident ray u is normal to the refracting plane; which is the case of paraxial rays (§ 53).

62. Refraction at a Plane Surface of an Infinitely Narrow Astigmatic Bundle of Incident Rays. If the bundle of incident rays is astigmatic, and if we designate by S and $\bar{S}$ (Fig. 29) the vertices of the pencils of incident meridian and sagittal rays, respectively, the bundle of refracted rays will, in general, be astigmatic also, and the I. and II. Image-Points S' and $\bar{S}'$, lying on the chief refracted ray u', will correspond to the points S and $\bar{S}$, respectively, lying on the chief incident ray u. We may call the point S the I. Object-Point and the point $\overline{S}$ the II. Object-Point. The pencil of meridian incident rays emanating from the I. Object-Point S and lying in the plane of incidence of the chief incident ray u will be transformed by refraction into a pencil of meridian refracted rays lying in the same plane with its vertex at the I. Image-Point S'. Hence, putting

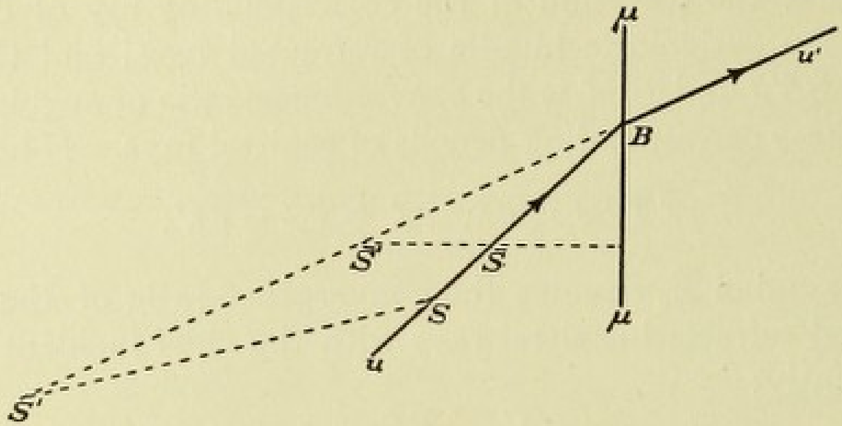

FIG. 29.

ASTIGMATIC BUNDLE OF INCIDENT RAYS REFRACTED AT A PLANE. The chief rays of the astigmatic bundles of incident and refracted rays are designated by u and u'. S, $\overline{S}$ designate the positions on u of the I. and II. Object-Points. To S on u corresponds the I. Image-Point S' on u', and to $\overline{S}$ on u corresponds the II. Image-Point $\overline{S}'$ on u'. In the diagram the plane of the paper coincides with the plane of the meridian rays.

$$BS = s, \quad BS' = s',$$

we have, according to formula (19):

$$s' = \frac{n'}{n} \frac{\cos^2 \alpha'}{\cos^2 \alpha} s.$$

Similarly, putting $B\overline{S} = \bar{s}, \quad B\overline{S}' = \bar{s}',$

we have by formula (21) $\bar{s}' = \frac{n'}{n} \bar{s}.$

The bundle of incident rays will have been rendered astigmatic in consequence, for example, of previous refractions.

ART. 20. REFRACTION OF INFINITELY NARROW BUNDLE OF RAYS AT A PLANE: GEOMETRICAL RELATIONS BETWEEN OBJECT-POINTS AND IMAGE-POINTS.

63. If on a given incident ray u (Fig. 30) we take a range of object-points $P, Q, R, S, \cdots$, whereto on the refracted ray u' correspond the range of I. Image-Points $P', Q', R', S', \cdots$ and the range of II. Image-Points $\overline{P}', \overline{Q}', \overline{R}', \overline{S}', \cdots$, then, according to formula (19), we must have:

$$\frac{BP'}{BP} = \frac{BQ'}{BQ} = \frac{BR'}{BR} = \frac{BS'}{BS} = \cdots;$$

which means that the straight lines $PP', QQ', RR', SS', \cdots$, joining the object-points on the incident ray u with their corresponding I. Image-Points on the refracted ray u' are a system of parallel straight lines; and, hence, the point-ranges $P, Q, R, \cdots$ and $P', Q', R', \cdots$ are *similar ranges* of points. And, since the straight lines $P\overline{P}'$, $Q\overline{Q}'$, $R\overline{R}'$, $\cdots$

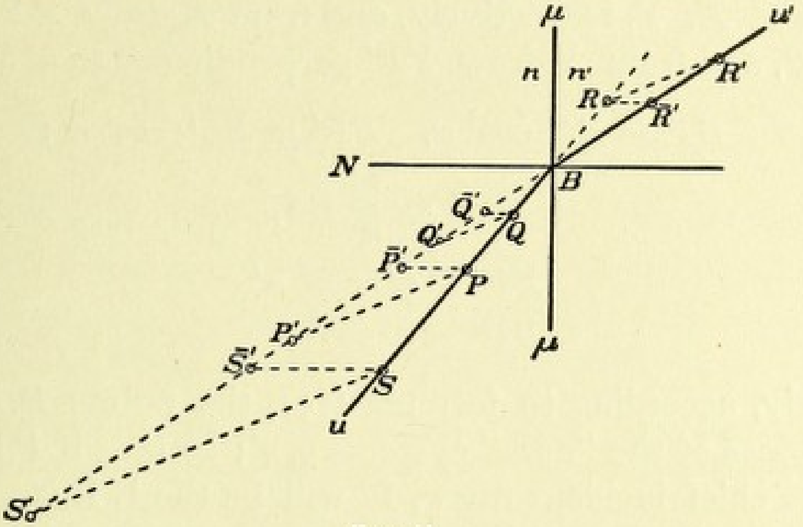

FIG. 30.

REFRACTION OF NARROW BUNDLE OF RAYS AT A PLANE. The range of Object-Points $P, Q, \cdots$ lying on the chief incident ray u is similar to the range of I. Image-Points $P', Q', \cdots$ and also to the range of II. Image-Points $\overline{P}', \overline{Q}', \cdots$ lying on the chief refracted ray u'.

which connect the Object-Points on the incident ray u with their corresponding II. Image-Points on the refracted ray u' are all perpendicular to the refracting plane, and therefore parallel to each other, it follows that the point-ranges $P, Q, R, \cdots$ and $\overline{P}', \overline{Q}', \overline{R}', \cdots$ are also *similar ranges* of points.

Conjugate to any object-point X, lying in the plane of incidence of the incident ray u, there will be on the refracted ray x', which corresponds to an incident ray x parallel to u and going through the object-point X, the I. Image-Point X' and the II. Image-Point $\overline{X}'$; and the range of Object-Points lying along the incident ray x is similar to the ranges of I. and II. Image-Points lying along the refracted ray x'. Thus, the plane system η of the Object-Points $X, \cdots$, which lie in the plane of incidence of the incident ray u, is in *affinity* with the plane-system η' of the I. Image-Points $X', \cdots$ and also with the plane-system $\overline{\eta}'$ of the II. Image-Points $\overline{X}', \cdots$: hence, also, the plane-systems η' and $\overline{\eta}'$ are in affinity with each other. The straight line in which the plane refracting surface meets the plane of incidence is the *affinity-axis*[1] for all three of these "*affin*" plane-systems.

64. Construction of the I. Image-Point. Let u (Fig. 31) be an incident ray meeting the plane refracting surface μ at the point B, and let u' be the corresponding refracted ray. Corresponding to an Object-Point S on u we find the II. Image-Point $\overline{S}'$ on u' at the point of intersection with u' of the perpendicular SA drawn from S to the refracting plane. Draw SX, $\overline{S}'Y$ perpendicular to the incidence-normal BN at X, Y, respectively, and from X draw XP perpendicular to u at P, and from Y draw YP' perpendicular to u' at P'. Then

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[1] The affinity-axis of two plane "*affin*" systems is the straight line common to the two systems which corresponds with itself point by point. Obviously, any pair of corresponding straight lines of the two systems will meet in the affinity-axis.

$$BP = BS \cdot \cos^2 \alpha, \quad BP' = B\overline{S}' \cdot \cos^2 \alpha';$$

and since

$$\overline{B}S' = \frac{n'}{n} BS,$$

we have:

$$\frac{BP'}{BP} = \frac{n' \cdot \cos^2 \alpha'}{n \cdot \cos^2 \alpha}.$$

Consequently, according to formula (19), the points P, P' are corresponding points of the "*affin*" systems η, η'; so that if P is an Object-Point of the chief incident ray u, P' will be the I. Image-Point lying on the chief refracted ray u'. The I. Image-Point S' corresponding to the Object-Point S of the chief incident ray u is found by drawing through S a straight line parallel to PP' which, by its intersection with the chief refracted ray u', will determine the required point S'. This is essentially the construction given by Reusch.[1]

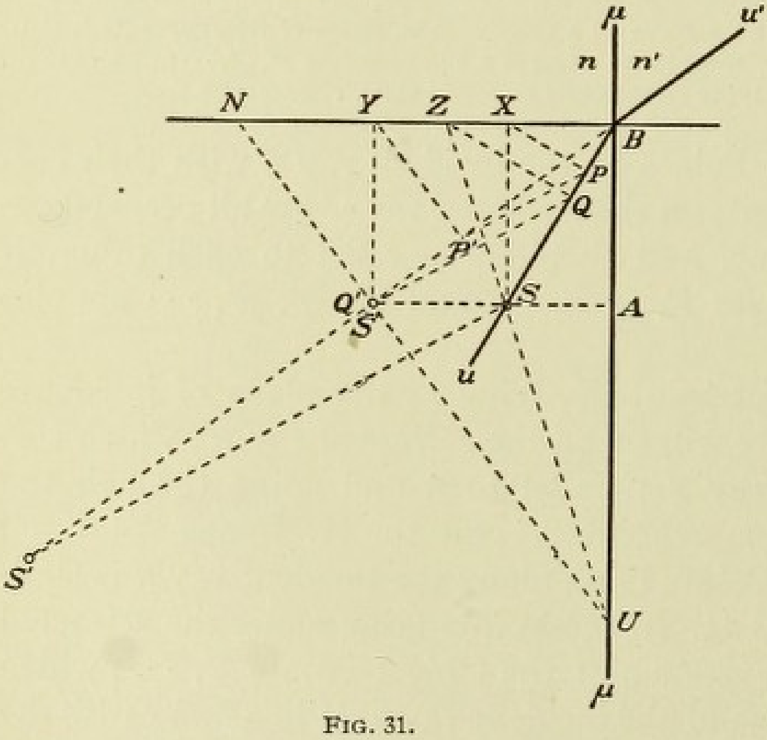

Fig. 31.

Refraction of Narrow Bundle of Rays at a Plane. Construction of I. Image-Point.

Another construction of the two corresponding points of the "*affin*" systems η, η' is as follows: Through $\overline{S}'$ draw a straight line perpendicular to the refracted ray u' and meeting the straight line BA in the point U; draw the straight line US meeting the incidence-normal BN in the point Z; and from Z let fall a perpendicular ZQ on the incident ray at the point Q. Then the point $\overline{S}'$ (or Q') is the I. Image-Point corresponding to the Object-Point Q

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[1] E. Reusch: Reflexion und Brechung des Lichts an sphaerischen Flaechen unter Voraussetzung endlicher Einfallswinkel: Pogg. *Ann.*, cxxx. (1867), 497–517.

on the chief incident ray u; as we proceed to show. Let the straight line joining U and $\overline{S}'$ meet the incidence-normal BN in the point N; then

$$\frac{B\overline{S}'}{BP'} = \frac{BN}{A\overline{S}'} = \frac{BU}{AU} = \frac{BZ}{AS} = \frac{BZ}{BX} = \frac{BQ}{BP},$$

and, therefore:

$$\frac{B\overline{S}'}{BQ} = \frac{BP'}{BP};$$

and, hence, $Q\overline{S}'$ (or QQ') is parallel to PP'. Therefore, $\overline{S}'$ (or Q') is the point of η' which corresponds to the point Q of η.[1]

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[1] For other methods of construction of the I. Image-Point see F. KESSLER: Beitraege zur graphischen Dioptrik: *Zft. f. Math. u. Phys.*, xxix. (1884), 65-74.

See also the construction of the I. Image Point in the case of refraction at a plane considered as a special case of refraction at a sphere, as given in § 249.

Since a plane surface may be regarded as a spherical surface with its centre at an infinite distance, obviously, all the problems treated in this chapter can be considered as special cases of the problem of refraction at a spherical surface, as will be seen hereafter. According to this view, this entire chapter might be regarded as superfluous.

CHAPTER IV.

REFRACTION THROUGH A PRISM OR PRISM-SYSTEM.

ART. 21. GEOMETRICAL CONSTRUCTION OF THE PATH OF A RAY REFRACTED THROUGH A PRISM IN A PRINCIPAL SECTION OF THE PRISM.

65. In optics the term **Prism** is applied to a portion of a transparent, isotropic substance included between two non-parallel plane refracting surfaces called the *faces* or *sides* of the prism. These are distinguished as the first and second faces of the prism in the order in which the light-rays arrive at them. The straight line in which the two plane faces meet is called the *edge* of the prism, and the dihedral angle between the two faces is called the *refracting angle.* This angle, which will be denoted by the symbol β, may be defined more precisely as *the angle through which the first face of the prism has to be turned, around the prism-edge as axis, in order to bring this face into coincidence with the second face.* A *principal section* of the prism is made by any plane perpendicular to the edge of the prism. At first we shall consider only such rays as lie in a principal section of the prism or infinitely narrow bundles of rays whose chief rays lie in a principal section.

In the general case of the problem of refraction through a single prism we have to do with as many as three optical media, viz.: the medium of the incident rays or the first medium, the medium of which the prism-substance is composed and the medium of the emergent rays. The absolute indices of refraction of these media will be denoted by n_1, n'_1 and n'_2 in the order named. In most cases the third medium is identical with the first, as, for example, in the case of a glass prism surrounded by air; and unless the contrary is expressly stated, we shall assume that this is the case. Thus, we shall have $n_1 = n'_2$; and the symbol

$$n = \frac{n'_1}{n_1} = \frac{n'_1}{n'_2}$$

will be employed to denote the relative index of refraction of the medium of the prism-substance with respect to the surrounding medium.

66. The following construction of the path of a ray refracted through a prism in a principal section was published by Reusch[1] in 1862; the same construction was published by Radau[2] in the following year.

In the diagrams (Figs. 32 and 33) the plane of the paper represents a principal section of the prism, and the point V in this plane shows where the prism-edge meets the plane of the principal section. The two plane faces of the prism, designated by μ_1, μ_2 are shown therefore by two straight lines meeting in the point V. The straight line L_1B_1 (or u_1) represents the path of the given incident ray meeting the first face of the prism at the incidence-point B_1; and the problem is to construct the remainder of the ray-path both within the prism and after emergence from the prism. The method is in fact the same as that given in §29.

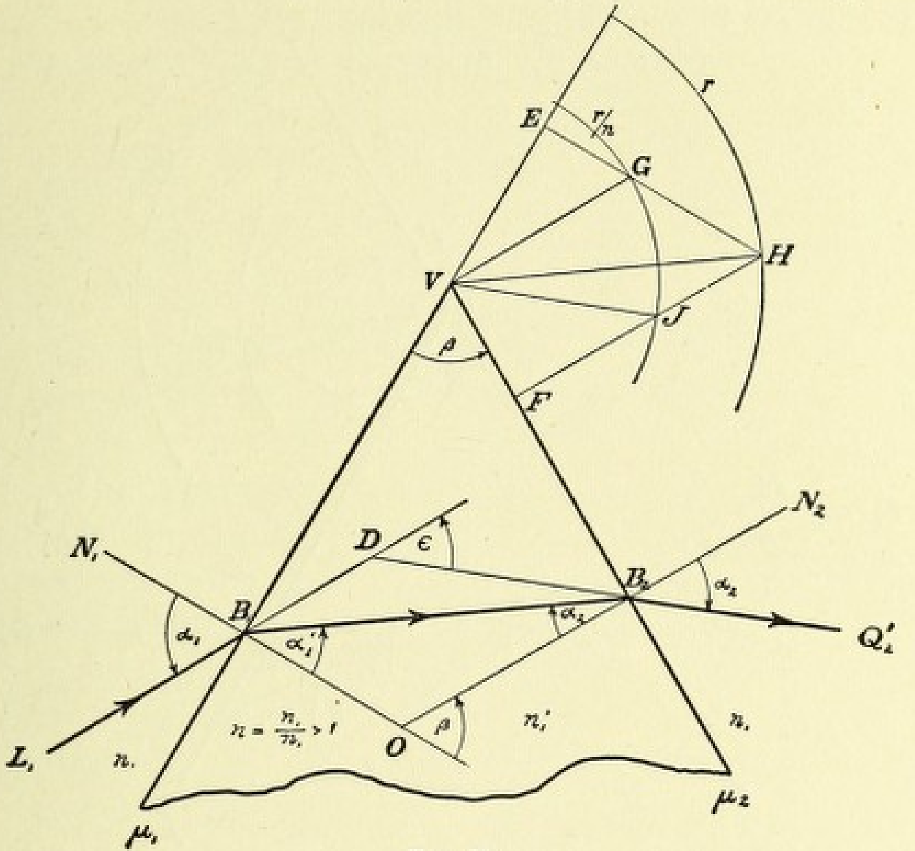

Fig. 32.

Construction of the Path of a Ray Refracted through a Prism in a Principal Section. Case when $n_2' = n_1$ and $n_1' > n_1$.

With the point V as centre, and with radii equal to r and r/n

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[1] E. Reusch: Die Lehre von der Brechung und Farbenzerstreuung des Lichts an ebenen Flaechen und in Prismen in mehr synthetischer Form dargestellt: Pogg. *Ann.*, cxvii. (1862), 241–262.

[2] R. Radau: Bemerkungen ueber Prismen: Pogg. *Ann.*, cxviii. (1863), 452–456. The method was obtained independently by Radau and it is often called by his name; but he himself in Carls *Rep.*, iv. (1868), p. 184, acknowledges Reusch's priority in the matter.

(where r may have any value), describe the arcs of two concentric circles. Through V draw a straight line parallel to the given incident ray L_1B_1 meeting the circumference of circle r/n in a point G, and through G draw a straight line perpendicular at E to the first face of the prism (produced, if necessary), and let H designate the position of the point where this straight line meets the circumference of the circle r. Then the straight line B_1B_2 drawn parallel to the straight line VH will show the path of the ray within the prism. For if $\alpha_1 = \angle N_1B_1L_1$ and $\alpha_1' = \angle OB_1B_2$ denote the angles of incidence and refraction at the first face of the prism, then, by the law of refraction:

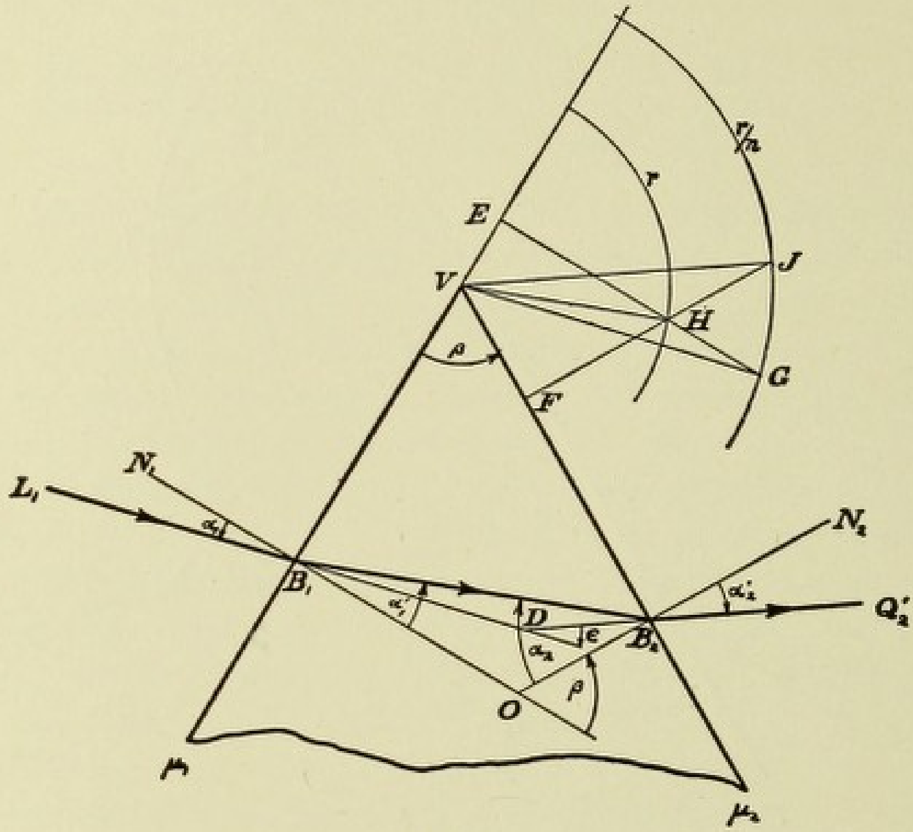

FIG. 33.

CONSTRUCTION OF PATH OF RAY REFRACTED THROUGH A PRISM IN A PRINCIPAL SECTION. Case when $n_2' = n_1$ and $n_1' < n_1$.

$$n_1 \cdot \sin \alpha_1 = n_1' \cdot \sin \alpha_1'.$$

According to the construction, we have:

$$\frac{\sin \angle EGV}{\sin \angle EHV} = \frac{VH}{VG} = \frac{n_1'}{n_1};$$

and since, by construction, $\angle EGV = \alpha_1$, it follows that $\angle EHV = \alpha_1'$, and hence the path of the ray within the prism must be parallel to VH.

Again, from the point H let fall a perpendicular on the second face of the prism, meeting this face in the point designated by F and meeting the circumference of the circle r/n in the point designated by J; then the straight line B_2Q_2' drawn parallel to the straight line VJ will represent the path of the ray after refraction at the second face of the prism back into the first medium. For if $\alpha_2 = \angle OB_2B_1$ and $\alpha_2' = \angle N_2B_2Q_2'$ denote the angles of incidence and refraction (emergence) at the second face of the prism, we must have:

$$n_1'\cdot\sin\alpha_2 = n_1\cdot\sin\alpha_2'.$$

We have:

$$\frac{\sin\angle FJV}{\sin\angle FHV} = \frac{VH}{VJ} = \frac{n_1'}{n_1};$$

and, since by construction $\angle FHV = \alpha_2$, it follows that $\angle FJV = \alpha_2'$, and hence the path of the emergent ray is parallel to VJ.

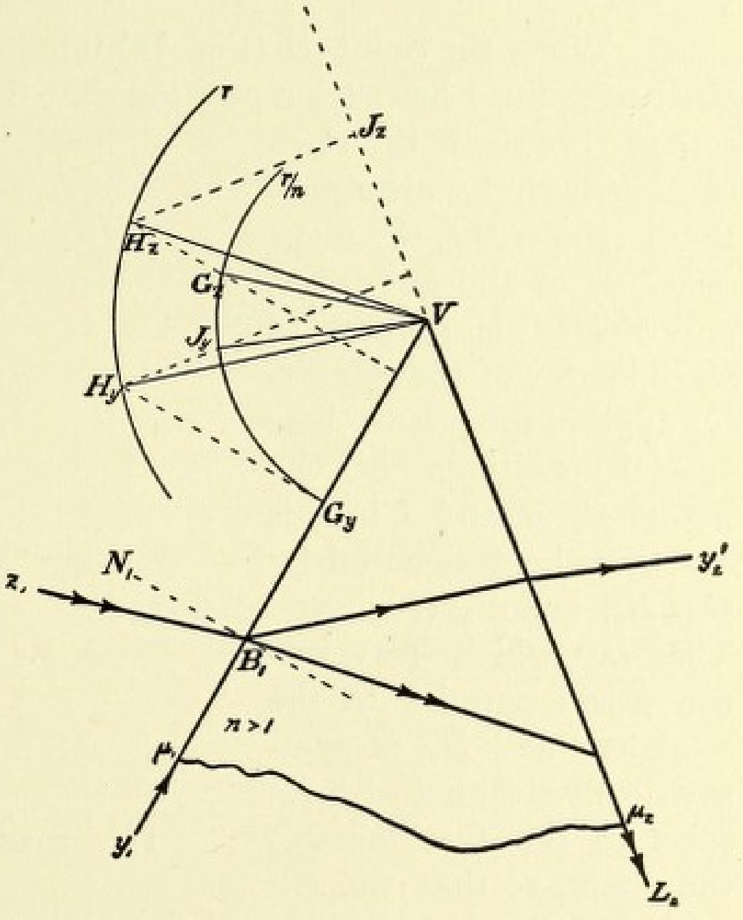

Fig. 34.

Showing the Paths of the Two Limiting Rays in a Principal Section of a Prism. ($n > 1$.)

In Fig. 32 the prism-medium is more highly refracting than the surrounding medium ($n = n_1'/n_1$ greater than unity; as, for example, a glass prism in air). The opposite case ($n = n_1'/n_1$ less than unity, as, for example, an air prism embedded in glass) is shown in Fig. 33.

If the medium of the emergent rays is not the same as that of the incident rays, the construction is practically the same; only, around V as centre we must describe now the arcs of *three* concentric circles with radii $VH = r$, $VG = n_1r/n_1'$, and $VJ = n_2'r/n_1'$.

67. The angle GHJ between the normals to the two faces of the prism is equal to the refracting angle β; and hence for a given prism this angle is constant. If the direction of the incident ray L_1B_1 is varied, the vertex H of this angle will move along the circumference of the circle of radius r, the sides of the angle having the fixed directions of the normals to the prism-faces. The two extreme positions of this point H which are reached when one or other of the sides of the $\angle GHJ$ is tangent to the circle of radius r/n (which can occur only when n is greater than unity, because then only will the point H lie outside the circle of radius r/n) are shown in Fig. 34. The two incident rays which correspond to these two extreme positions of the point H are the ray y_1B_1, which, entering the first face of the prism at "grazing" incidence ($\alpha_1 = 90°$) at the point B_1, and traversing the prism as shown in the figure, emerges as the ray y'_2, and the ray z_1, which, entering the prism at the point B_1, and arriving at the second face at the critical angle of incidence (§ 27), emerges only by "grazing" this face. In order that a ray incident at the point B_1 may not be *totally reflected* at the second face of the prism, it must lie within the $\angle z_1B_1y_1$.

68. When the point H (Fig. 32) lying on the circumference of the circle of radius r has such a position that the sides of the $\angle GHJ$ intercepted between the two concentric circles are equal, that is, $HG = HJ$, the diagonal VH of the quadrilateral $VGHJ$ is normal to the bisector of the refracting angle of the prism. The special positions of the points G, H and J in this case may be designated by G_0, H_0, and J_0 (Fig. 35). The ray which traverses the prism parallel to the straight line VH_0 is symmetrically situated with respect to the two faces of the prism, so that the triangle $VB_1B_{2,0}$ is isosceles, and the angles of incidence at the first face and emergence at the second face are equal.

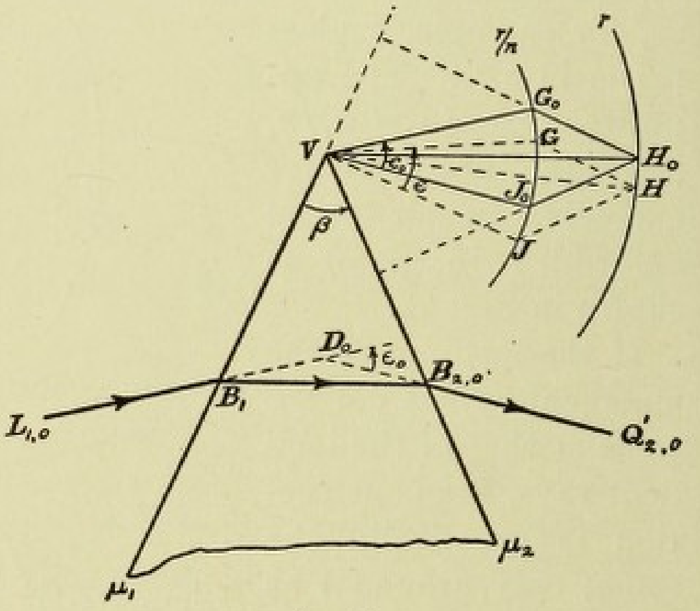

FIG. 35.
PATH OF THE RAY OF MINIMUM DEVIATION.

The angle JVG, denoted by ϵ, between the directions of the incident and emergent rays is called the *angle of deviation*, and it may be shown that when the ray traverses the prism symmetrically, this angle has its least value. Let H (Fig. 35) designate the position of a point on the circumference of the circle of radius r which is infinitely near to the point H_0, and draw HG, HJ normal to the two faces of the prism and meeting the circumference of the other construction-circle in the points G, J, respectively. In the diagram the point H is taken below the point H_0, and it is obvious that the two parallels H_0J_0, HJ meet the circumference of the circle of radius r/n more obliquely than do the parallels H_0G_0, HG; so that the infinitely small arc JJ_0 is greater than the infinitely small arc GG_0; and, hence, $\angle JVJ_0$ is greater than $\angle GVG_0$; and, consequently:

$$\angle JVG > \angle J_0VG_0.$$

The angle J_0VG_0 is the angle of deviation of the ray which traverses the prism symmetrically; it may be denoted by ϵ_0. According to the above, we have always (for we shall arrive at the same result if we take the point H on the other side of H_0 from that shown in the figure):

$$\epsilon > \epsilon_0.$$

Thus, we see that *the ray which traverses the prism symmetrically is also the ray which is least deviated by its passage through the prism* (**Ray of Minimum Deviation**).[1]

69. *When a ray of light passes through a prism the material of which is more highly refracting than the surrounding medium* ($n > 1$), *the deviation is always away from the edge towards the thicker part of the prism.* If the angles of the triangle VB_1B_2 (Fig. 32) at B_1 and B_2 are both acute angles, the incident and emergent rays lie on the sides of the normals at B_1 and B_2 away from the prism-edge, so that at each refraction the ray will be bent away from the edge. If one of the angles, say, the angle at B_2, is a right angle, there will be no deviation at emergence, but at the other incidence-point B_1 the ray will be bent away from the prism-edge. And, finally, if one of the angles of the triangle VB_1B_2 at B_1 or B_2 is obtuse, for example, the angle at B_2, the deviation at emergence will, it is true, be towards the prism-edge, but this will not be so great as the previous deviation at B_1 which was away from the edge; as will be easily seen by examining a diagram for this case. So that in every case, provided $n > 1$, the total deviation will be away from the prism-edge.

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[1] On the subject here considered optical literature is very extensive. A complete list of references to all the writers will be found in H. KAYSER'S *Handbuch der Spectroscopie*, Bd. I. (Leipzig, 1900), pages 258–'9. Among the more important synthetic proofs of the fact that the ray which traverses the prism symmetrically is the least deviated, the following contributions may be specially mentioned:

E. LOMMEL: Ueber die kleinste Ablenkung im Prisma: POGG. *Ann.*, clix. (1876), p. 329.

F. KESSLER: *Jahresbericht der Gewerbeschule zu Bochum für 1880.* Also, Das Minimum der Ablenkung eines Lichtstrahls durch ein Prisma: WIED. *Ann.*, xv. (1882), 333.

R. H. SCHELLBACH: Das Minimum der Ablenkung eines Lichtstrahles im Prisma: WIED. *Ann.*, xiv. (1881), 367.

FR. C. G. MUELLER: Der Satz vom Minimum der Ablenkung beim Prisma: *Zft. f. den phys. u. chem. Unterr.*, iii. (1889-90), 247.

J. H. KIRKBY: Refraction through a prism: *Nature*, xliv. (1891), 294.

A. KURZ: Die kleinste Ablenkung im Prisma: *Zft. f. Math. u. Phys.*, xxxvii. (1892), 317 and xxxviii. (1893), 319.

H. VEILLON: Elementare geometrische Behandlung des Minimums der Ablenkung beim Prisma: *Zft. f. den phys. u. chem. Unterr.*, xii. (1899), 150–'2.

If $n < 1$, all these effects are reversed.

ART. 22. ANALYTICAL INVESTIGATION OF THE PATH OF A RAY REFRACTED THROUGH A PRISM IN A PRINCIPAL SECTION.

70. The angles of incidence and refraction at the first and second faces of the prism, denoted by α_1, α_1' and α_2, α_2', respectively, are, by definition (§ 14), the acute angles through which the normal to the refracting surface at the incidence-point has to be turned in order to bring it into coincidence with the incident and refracted rays at each face of the prism. The angle of deviation or the total deviation of a ray refracted through a prism, denoted by the symbol ϵ, is the angle between the directions of the emergent and incident rays, or the angle through which the emergent ray must be turned around the point D (Fig. 32) in order that it may coincide with the incident ray in both position and direction. Thus, $\epsilon = \angle JVG$.

Assuming that the prism is surrounded by the same medium on both sides, we have obviously the following system of equations:

$$\left.\begin{aligned} n = n_1'/n_1, \quad n_2' = n_1;\\ \sin\alpha_1 = n\cdot\sin\alpha_1', \quad n\cdot\sin\alpha_2 = \sin\alpha_2';\\ \alpha_1' - \alpha_2 = \beta;\\ \epsilon = \alpha_1 - \alpha_1' + \alpha_2 - \alpha_2' = \alpha_1 - \alpha_2' - \beta. \end{aligned}\right\} \tag{25}$$

Combining these formulæ, we obtain:

$$\sin\alpha_2' = \sin\alpha_1\cdot\cos\beta - \sin\beta\cdot\sqrt{n^2 - \sin^2\alpha_1}; \tag{26}$$

whence, knowing the relative index of refraction ($n = n_1'/n_1 = n_1'/n_2'$) and the refracting angle (β) of the prism, we can compute the angle of emergence (α_2') corresponding to any given value of the angle of incidence (α_1) at the first face of the prism.

The total deviation (ϵ) of a ray refracted through a prism depends only upon the values of the magnitudes α_1, β and n: for given values of these magnitudes, the angle ϵ will be uniquely determined by formulæ (25). So long as n is different from unity and β is different from zero, the value of ϵ cannot be zero. On the other hand, to each value of ϵ there corresponds always two values of the angle α_1; for a second ray incident on the first face of the prism at an angle equal to the angle of emergence of the first ray will evidently emerge at the second face at an angle equal to the angle of incidence of the first ray at the first face; and hence it is obvious that these two rays will undergo equal deviations in traversing the prism. For example, suppose that the values of the angles of incidence and emergence of the first ray were $\alpha_1 = \theta$, $\alpha'_2 = \theta'$: a second ray incident on the first face of the prism at the angle $\alpha_1 = -\theta'$ will emerge at the second face at an angle $\alpha'_2 = -\theta$, and both of these rays will have the same deviation, viz., $\epsilon = \theta - \theta' - \beta$.

71. Analytical Investigation of the Case of Minimum Deviation. We have just seen that there is always a pair of rays for which the deviation (ϵ) has a given value. One pair of rays for which the deviation is the same are the two identical rays determined by the relation:

$$\alpha_1 = \theta = -\alpha'_2.$$

In this case the course of the ray through the prism is symmetrical with respect to the two faces of the prism; that is, the ray crosses at right angles the plane which bisects the dihedral angle β between the two faces of the prism.

Inasmuch as the deviation (ϵ) is a symmetrical function of α_1 and $-\alpha'_2$, it must be either a maximum or a minimum when the ray within the prism is equally inclined to both faces of the prism ($\alpha_1 = -\alpha'_2$). We shall show that as a matter of fact the deviation in this case is a minimum.

For a critical value of the angle ϵ, we must have $d\epsilon/d\alpha_1 = 0$. Differentiating the prism-formulæ above, we obtain:

$$\cos\alpha_1 = n\cdot\cos\alpha'_1\frac{d\alpha'_1}{d\alpha_1}, \quad n\cdot\cos\alpha_2\frac{d\alpha_2}{d\alpha_1} = \cos\alpha'_2\frac{d\alpha'_2}{d\alpha_1},$$

$$\frac{d\alpha'_1}{d\alpha_1} - \frac{d\alpha_2}{d\alpha_1} = 0, \quad \frac{d\epsilon}{d\alpha_1} = 1 - \frac{d\alpha'_2}{d\alpha_1}.$$

These latter give:

$$\frac{d\epsilon}{d\alpha_1} = 1 - \frac{\cos\alpha_1\cdot\cos\alpha_2}{\cos\alpha'_1\cdot\cos\alpha'_2},$$

and, hence, putting $d\epsilon/d\alpha_1 = 0$, we have:

$$\frac{\cos\alpha_1}{\cos\alpha_1'} = \frac{\cos\alpha_2'}{\cos\alpha_2}.$$

Now the first side of this equation is a function of α_1 and n, whereas the second side of the equation is the *same* function of α_2' and n; and therefore we must have:

$$\alpha_1 = \pm\,\alpha_2'.$$

The upper sign is inadmissible here, as the value $\alpha_1 = +\,\alpha_2'$ would make the refracting angle of the prism equal to zero, which cannot be in the case of a prism. Hence, the critical value of the angle ϵ occurs when we have:

$$\alpha_1 = -\,\alpha_2';$$

and, consequently, also:

$$\alpha_1' = -\,\alpha_2.$$

As we saw above, this was the value of α_1 when the ray crossed the prism symmetrically.

In order to determine whether this critical value of ϵ is a maximum or minimum, we shall have to investigate the sign of the second derivative of ϵ. Differentiating the formula above for the first derivative, we obtain:

$$\cos^2\alpha_1'\cdot\cos^2\alpha_2'\,\frac{d^2\epsilon}{d\alpha_1^2} = \cos\alpha_1'\cdot\cos\alpha_2'\left(\cos\alpha_1\cdot\sin\alpha_2\,\frac{d\alpha_2}{d\alpha_1} + \sin\alpha_1\cdot\cos\alpha_2\right)$$
$$-\cos\alpha_1\cdot\cos\alpha_2\left(\cos\alpha_1'\cdot\sin\alpha_2'\,\frac{d\alpha_2'}{d\alpha_1} + \cos\alpha_2'\cdot\sin\alpha_1'\,\frac{d\alpha_1'}{d\alpha_1}\right).$$

Now when $d\epsilon/d\alpha_1 = 0$, we have:

$$\frac{d\alpha_2}{d\alpha_1} = \frac{d\alpha_1'}{d\alpha} = \frac{\cos\alpha_2'}{n\cdot\cos\alpha_2},\quad \frac{d\alpha_2'}{d\alpha_1} = 1,\quad \alpha_1 = -\,\alpha_2' \quad\text{and}\quad \alpha_1' = -\,\alpha_2;$$

moreover,

$$\alpha_1' = -\,\alpha_2 = \beta/2;$$

hence, substituting these values in the above, we obtain finally, when $d\epsilon/d\alpha_1 = 0$:

$$n\cdot\cos\alpha_1\cdot\cos\alpha_1'\,\frac{d^2\epsilon}{d\alpha_1^2} = (n^2 - 1)\cdot\sin\beta.$$

Since all the values of the angles α_1 and α_1' are comprised between $+\,\pi/2$ and $-\,\pi/2$, and since the angle β is positive, the sign of $d^2\epsilon/d\alpha_1^2$

depends on the value of n. If $n > 1$, $d^2\epsilon/d\alpha_1^2$ will be positive, and hence for $\alpha_1 = -\alpha_2'$ the deviation ϵ has a *minimum* value. But if $n < 1$, $d^2\epsilon/d\alpha_1^2$ will be negative, so that we obtain the rather unexpected result that, under these circumstances, the deviation has a *maximum* value. The explanation is apparent; for if we recall that the angle ϵ is negative when $n < 1$, as will be seen by an inspection of Fig. 33, it is evident that a maximum value of ϵ in this case corresponds to a minimum absolute value.

If the critical value of the angle ϵ is denoted by the symbol ϵ_0, we have, therefore, for the **Position of Minimum Deviation** the following set of equations:

$$\left.\begin{aligned} &\alpha_1 = -\alpha_2', \quad \alpha_1' = -\alpha_2, \\ &2\alpha_1' = \beta, \qquad \alpha_1 = \frac{\beta + \epsilon_0}{2}; \\ &n = \frac{\sin\dfrac{\beta + \epsilon_0}{2}}{\sin\dfrac{\beta}{2}}. \end{aligned}\right\} \tag{27}$$

The last of the above formulæ is the basis of the FRAUNHOFER-method of determining the refractive index n, the angles β and ϵ_0 being capable of easy measurement.

72. Other Special Cases. If the emergent ray is normal to the second face of the prism, we must put $\alpha_2 = \alpha_2' = 0$; and, thus, for the case of **perpendicular emergence at the second face,** we have $\alpha_1' = \beta$, $\alpha_1 = \beta + \epsilon$, so that we obtain:

$$n = \frac{\sin(\beta + \epsilon)}{\sin\beta}, \quad \alpha_2 = \alpha_2' = 0.$$

This also is a convenient formula for the experimental determination of the value of the refractive index n. The procedure is described in treatises on physics.

Case of a **Thin Prism** (*Prism with very small Refracting Angle*). If the refracting angle of the prism is so small that we may put $\sin\beta = \beta$, $\cos\beta = 1$, the deviation ϵ will also be a small angle of the same order of magnitude. In this case, therefore, since

$$\alpha_2' = \alpha_1 - (\beta + \epsilon),$$

we have:

$$\sin\alpha_2' = \sin\alpha_1 - (\beta + \epsilon)\cos\alpha_1.$$

Moreover, since

$$\sin \alpha_2' = n \cdot \sin \alpha_2, \quad \alpha_2 = \alpha_1' - \beta,$$

we obtain

$$\sin \alpha_2' = n (\sin \alpha_1' - \beta \cdot \cos \alpha_1') = \sin \alpha_1 - n \cdot \beta \cdot \cos \alpha_1'.$$

Therefore, equating these two values of sin α_2', we obtain in the case of finite value of the incidence-angle α_1:

$$\epsilon = \beta \left(n \frac{\cos \alpha_1'}{\cos \alpha_1} - 1 \right).$$

In case the angle of incidence α_1 is also a very small angle, we obtain the following approximate formula for the deviation:

$$\epsilon = (n - 1)\beta. \tag{28}$$

In these formulæ the angles are all measured in radians. According to (28), for small values of both β and α_1, the deviation ϵ is proportional to the refracting angle β, and is independent of the incidence-angle α_1.

The Case of **Total Reflexion at the Second Face** *of the Prism.* If the angle of emergence at the second face of the prism is a right angle, that is, if $\alpha_2' = -90°$, the emergent ray B_2Q_2' will proceed along (or "graze") the second face μ_2. When this occurs, we have $\alpha_2 = -A$, where the symbol A, defined by the formula:

$$\sin A = \frac{n_1}{n_1'} = \frac{1}{n}, \quad (n > 1),$$

denotes the magnitude of the so-called "critical angle" (§ 27) for the two media whose relative index of refraction = n. If the absolute value of the angle α_2 is greater than this critical angle A, the ray will be *totally reflected* at the second face of the prism, and there will be no emergent ray. We proceed to discuss this case in some detail.

For a prism of given refracting angle (β), there is a certain *limiting value* ι of the angle of incidence α_1 at the first face of the prism for which we shall have $\alpha_2 = -A$ and $\alpha_2' = -90°$; so that *a ray which is incident on the first face of the prism at an angle less than this limiting angle* ι *will not pass through the prism, but will be totally reflected at the second face.* Putting $\alpha_1 = \iota$, $\alpha_2 = A$ in formulæ (25), we obtain the formula:

$$\sin \iota = n \cdot \sin (\beta - A); \tag{29}$$

whereby the limiting angle of incidence (ι) for a given prism can be computed. Examining this formula, we derive the following conclusions:

(1) If $\beta > 2A$, then (since $\sin A = 1/n$) $\sin \iota$ is greater than unity; which means that for such a prism there is no limiting angle ι. Accordingly, *if the refracting angle of the prism is more than twice as great as the "critical angle" (A), it will be impossible for any ray whatever to be transmitted through the prism.* For instance, for a crown-glass prism in air, the angle $A = 40° 50'$, and hence a prism of this material with a refracting angle greater than $81° 40'$ will not permit any ray to emerge at its second face.

(2) If $\beta = 2A$, we obtain, by formula (29), $\iota = 90°$. In this case the limiting ray I_1B_1 (Fig. 36) will "graze" the first face of the prism.

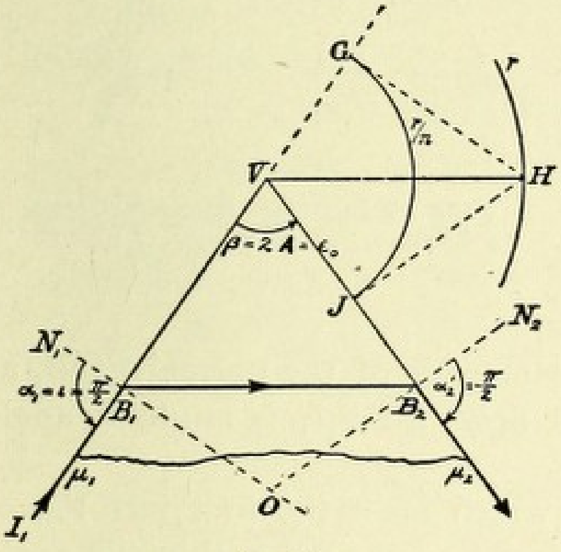

FIG. 36.

REFRACTING ANGLE OF PRISM EQUAL TO TWICE THE CRITICAL ANGLE. The only ray that can pass through the prism is $I_1B_1B_2\mu_2$.

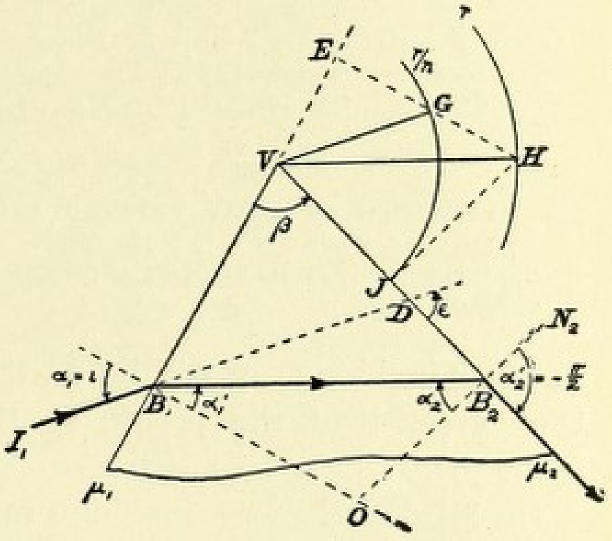

FIG. 37.

REFRACTION OF A RAY THROUGH A PRISM. Limiting Ray in the case when $2A > \beta > A$ ($n > 1$).

This is the only ray that can pass through the prism, and it will emerge at B_2 and proceed along the second face of the prism in the direction $B_2\mu_2$. Since here we have $\alpha_1 = \iota = 90° = -\alpha'_2$, evidently this is also the ray of minimum deviation (§ 71) for this prism ($\epsilon_0 = \beta$).

(3) If the refracting angle β is greater than A, but less than $2A$ (that is, $2A > \beta > A$), the value of ι, as determined by (29), will be comprised between 90° and 0°. Hence, for a prism with a refracting angle such as this, the limiting ray I_1B_1 will have a direction between the directions μ_1B_1 and N_1B_1; that is, the $\angle VB_1I_1$ (Fig. 37) will be an obtuse angle.

(4) If $\beta = A$, we find $\iota = 0$; so that for such a prism the limiting incident ray I_1B_1 will be in the direction of the normal at B_1 (Fig. 38).

In this case, therefore, $\angle VB_1I_1 = 90°$. The ray which "grazes" the first face of this prism will meet the second face normally.

(5) Finally, if $\beta < A$, formula (29) gives in this case a negative value of the angle ι; and, hence, for a prism with such a refracting angle the limiting incident ray will lie on the same side of the normal at B_1 as the vertex V of the prism, so that now $\angle VB_1I_1$ is an acute angle (Fig. 39).

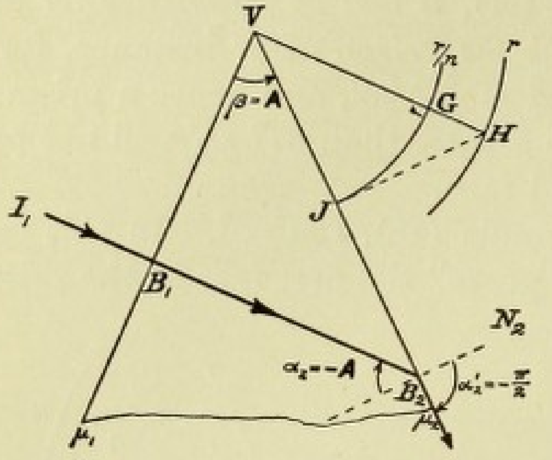

FIG. 38.

REFRACTION OF A RAY THROUGH A PRISM. Case when $\beta = A$ $(n > 1)$. Limiting incident ray meets first face of prism normally.

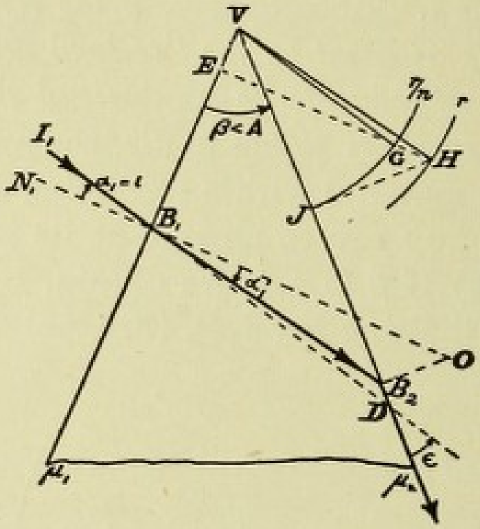

FIG. 39.

PATH OF LIMITING RAY FOR PRISM OF REFRACTING ANGLE $\beta < A$ $(n > 1)$.

In all these cases incident rays which meet the first face $V\mu_1$ of the prism at the point B_1, and which are comprised within the angle $I_1B_1\mu_1$ will be transmitted through the prism; whereas all rays incident at B_1 and lying within the angle VB_1I_1 will be totally reflected at the second face of the prism.

If $\beta = 0°$, the prism is a *Slab with Parallel Faces*, and then we have $\iota = -90°$ and $\angle VB_1I_1 = 0°$. All incident rays will be transmitted through such a slab.

In KOHLRAUSCH's method of measuring the relative index of refraction (n), the prism is adjusted so that the incident ray "grazes" the first face of the prism; in which case the value of n will be given by the formula:

$$\sqrt{n^2 - 1} = \frac{\cos\beta - \sin\alpha_2'}{\sin\beta}, \quad (\alpha_1 = 90°).$$

The Total-Reflexion Principle is made use of also in the so-called Total Refractometers of ABBE and PULFRICH for the determination of the refractive index.

73. For convenience of reference the following collection of formulæ for calculating the path of a ray refracted through a prism in a principal section are placed here.

PRISM FORMULÆ.

$$\sin \alpha_1 = \cos \beta \cdot \sin \alpha_2' + \sin \beta \sqrt{n^2 - \sin^2 \alpha_2'};$$
$$\sin \alpha_2' = \cos \beta \cdot \sin \alpha_1 - \sin \beta \sqrt{n^2 - \sin^2 \alpha_1};$$
$$\alpha_2 = \alpha_1' - \beta; \quad \epsilon = \alpha_1 - \alpha_2' - \beta.$$

Minimum Deviation:

$$\alpha_1 = - \alpha_2', \quad \alpha_1' = - \alpha_2 = \frac{\beta}{2};$$
$$\sin \alpha_1 = n \cdot \sin \frac{\beta}{2}; \quad \epsilon_0 = 2\alpha_1 - \beta.$$

Grazing Incidence:

$$\alpha_1 = 90^\circ, \quad \sin \alpha_2' = \cos \beta - \sin \beta \sqrt{n^2 - 1};$$
$$\alpha_1' = A, \quad \alpha_2 = A - \beta, \quad \epsilon = 90^\circ - \alpha_2' - \beta.$$

Grazing Emergence:

$$\alpha_2' = - 90^\circ, \quad \sin \alpha_1 = \sin \beta \sqrt{n^2 - 1} - \cos \beta;$$
$$\alpha_1' = \beta - A, \quad \alpha_2 = - A, \quad \epsilon = 90^\circ + \alpha_1 - \beta.$$

Normal Incidence:

$$\alpha_1 = 0, \quad \sin \alpha_2' = - n \cdot \sin \beta;$$
$$\alpha_1' = 0, \quad \alpha_2 = - \beta, \quad \epsilon = \beta - \alpha_2'.$$

Normal Emergence:

$$\alpha_2' = 0, \quad \sin \alpha_1 = n \cdot \sin \beta;$$
$$\alpha_2 = 0, \quad \alpha_1' = \beta, \quad \epsilon = \alpha_1 - \beta.$$

The subjoined table gives the results of the calculations of the values of these angles for a prism of flint glass, designated in the glass catalogue of SCHOTT u. Gen., Jena, as O.103, the refractive index of which for rays of light corresponding to the FRAUNHOFER *D*-line is $n = 1.620\,2$. The refracting angle of the prism is taken as 30°. The value of the critical angle A ($= \sin^{-1} 1/n$) is 38° 6′ 45″.

First Face.		Second Face.		Deviation.
Angle of Incidence. α_1	Angle of Refraction. α_1'	Angle of Incidence. α_2	Angle of Refraction. α_2'	ϵ
90° 0′ 0″	38° 6′ 45″	8° 6′ 45″	13° 13′ 1″	46° 46′ 59″
54 6 20	30 0 0	0 0 0	0 0 0	24 6 20
24 47 34	15 0 0	−15 0 0	−24 47 34	19 35 8
0 0 0	0 0 0	−30 0 0	−54 6 20	24 6 20
−13 13 1	− 8 6 45	−38 6 45	−90 0 0	46 46 59

ART. 23. PATH OF A RAY REFRACTED ACROSS A SLAB WITH PARALLEL FACES.

74. If the two plane refracting surfaces of the prism are parallel ($\beta = 0$), we no longer call it a prism, but a **slab** or *plate with parallel faces*. The path of a ray refracted across such a slab may evidently be constructed as follows:

Around the incidence-point B_1' (Fig. 40), where the incident ray meets the first surface of the slab, describe in the plane of incidence three concentric circles of radii r, $n_1 r/n_1'$, $n_2' r/n_1'$, where the radius r has any arbitrary length, and where n_1, n_1' and n_2' denote the absolute indices of refraction of the first, second and third medium, respectively. Let G designate the point where the incident ray L_1B_1 meets the circle of radius $n_1 r/n_1'$, and draw GE perpendicular to the first face of the slab at the point E, and let this perpendicular, produced if necessary, meet the circle of radius r in the point H; then the straight line HB_1 will evidently give the direction of the ray after refraction at the first face. Moreover, if J designates the point where the straight line GE meets the circle of radius $n_2' r/n_1'$, and if B_2 designates the incidence-point of the ray at the second face of the slab, the straight line B_2Q_2' drawn parallel to JB_1 will give the direction and path of the emergent ray.

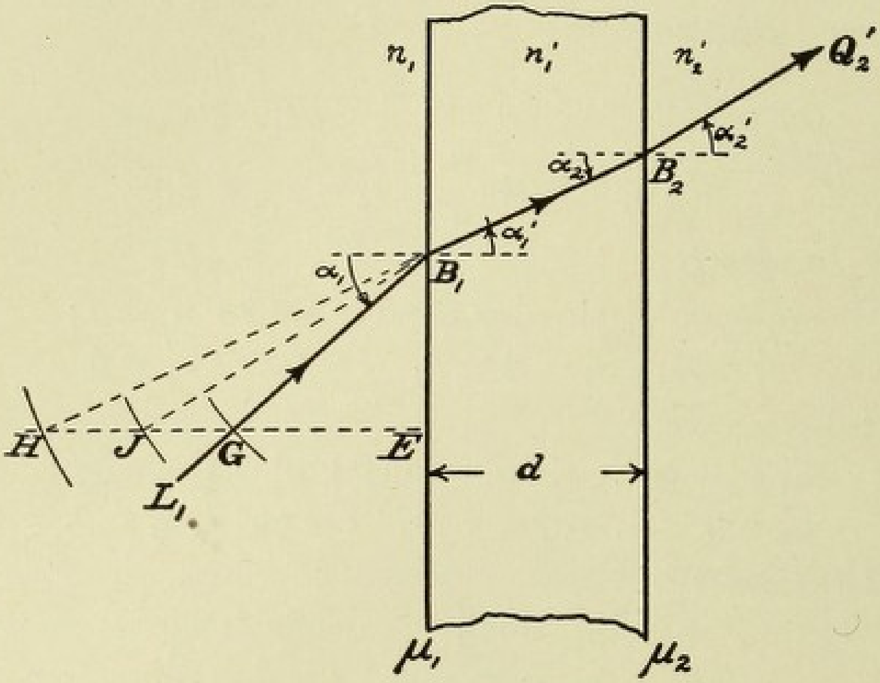

FIG. 40.

CONSTRUCTION OF PATH OF RAY REFRACTED ACROSS A SLAB WITH PLANE PARALLEL FACES.

In the special case when the last medium is identical with the first, so that we have $n'_2 = n_1$, two of the circles used in the above construction will coincide, and accordingly the points designated by G and J will be coincident. And hence in this case the emergent ray will be parallel to the incident ray. The perpendicular distance between the parallel paths of the incident and emergent rays is equal to

$$\frac{\sin(\alpha - \alpha')}{\cos\alpha'}d,$$

where d denotes the thickness of the slab and α, α' denote the angles of incidence and refraction at the first face.

75. We may also investigate here *the path of a ray which is refracted in succession at a series of parallel refracting plane surfaces* μ_1, μ_2, μ_3, etc. If n_1, n'_1, n'_2, etc., denote the absolute indices of refraction of the media traversed in succession by the ray, and if α_1, α'_1; α_2, α'_2; etc., denote the angles of incidence and refraction at the series of parallel refracting planes, then, supposing that we have, say, m such planes, we shall have the following set of equations:

$$\begin{aligned} n_1 \cdot \sin\alpha_1 &= n'_1 \cdot \sin\alpha'_1, \\ n'_1 \cdot \sin\alpha_2 &= n'_2 \cdot \sin\alpha'_2, \\ &\cdots\cdots\cdots\cdots \\ n'_{m-1} \cdot \sin\alpha'_m &= n'_m \cdot \sin\alpha'_m. \end{aligned}$$

Multiplying together the expressions on each side of these equations, and remarking that, since the refracting planes are parallel, we must have:

$$\alpha'_1 = \alpha_2,\ \alpha'_2 = \alpha_3, \quad \ldots,\ \alpha'_{m-1} = \alpha_m,$$

we obtain immediately:

$$n_1 \cdot \sin\alpha_1 = n'_m \cdot \sin\alpha'_m.$$

Thus, without knowledge of either the number or the nature of the intervening media, this formula enables us to find the direction α'_m of the ray which emerges into the last medium. The effect of the intervening media between the first and the last is merely to produce a parallel displacement of the emergent ray; otherwise, everything is the same as if the ray had been refracted from the first to the last medium across a single refracting plane. If the last medium is identical with the first, so that we have $n'_m = n_1$, then $\alpha'_m = \alpha_1$, and the direction of the emergent ray will be parallel to that of the incident ray, as we saw above in the case of a slab surrounded by the same medium on both sides. For example, if we interpose in front of the object-glass of a telescope pointed towards a star a plate of glass with plane parallel sides, the image of the star will not be deviated thereby. This fact is employed in a simple method of testing with a high degree of precision whether or not two faces of a plate of glass are accurately parallel.

ART. 24. REFRACTION, THROUGH A PRISM, OF AN INFINITELY NARROW HOMOCENTRIC BUNDLE OF INCIDENT RAYS, WHOSE CHIEF RAY LIES IN A PRINCIPAL SECTION OF THE PRISM.

76. If an infinitely narrow homocentric bundle of incident rays is refracted through a prism, and if the chief ray lies in the plane of a principal section of the prism, the meridian sections of the incident and refracted bundles of rays will lie in the principal section which contains the chief rays of the bundles, and which is the plane of incidence of the chief incident ray u_1; whereas the planes of the sagittal sections of the bundles of incident and refracted rays will intersect in straight lines parallel to the edge of the prism.

We give, first, the **construction of the I. and II. Image-Points corresponding to a homocentric object-point.** Let S_1 (Fig. 41) be the radiant point or homocentric object-point of the bundle of incident rays, the

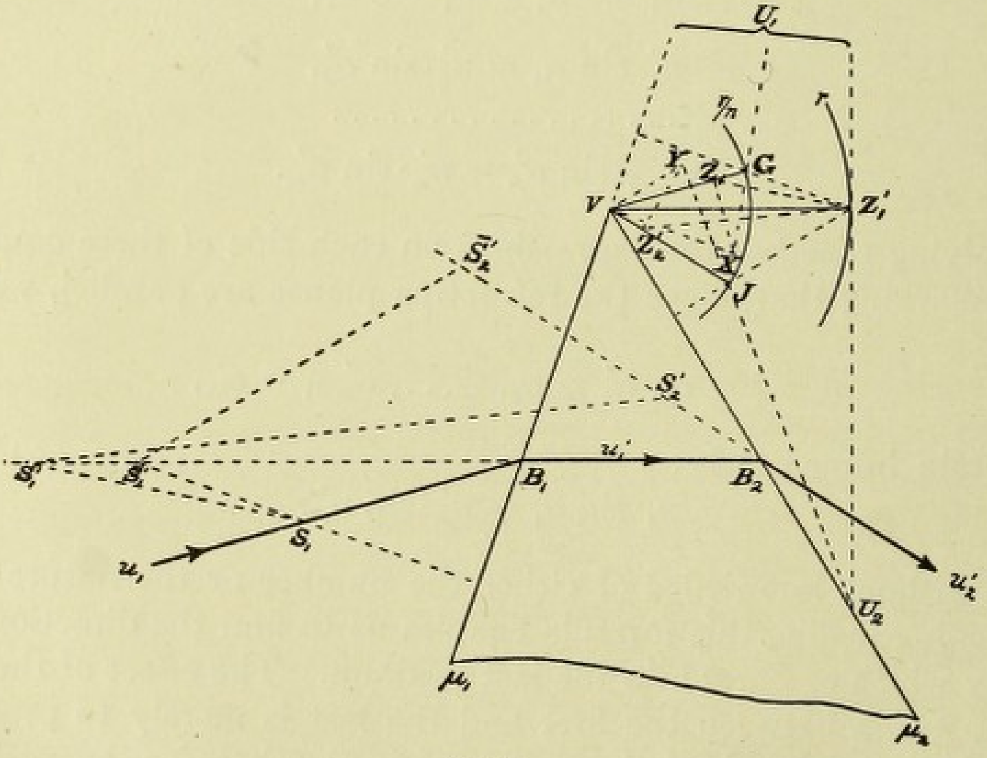

FIG. 41.

REFRACTION, THROUGH A PRISM, OF AN INFINITELY NARROW BUNDLE OF RAYS WHOSE CHIEF RAY LIES IN A PRINCIPAL SECTION OF THE PRISM. Construction of the I. and II. Image-Points S_2', $\bar{S}_2'$ on the chief emergent ray u_2' corresponding to Object-Point S_1 on chief incident ray u_1.

chief ray of which (u_1) is incident on the first face of the prism at the point B_1. The path of this ray within the prism and after emergence from it is constructed by REUSCH's Construction (§ 66). We shall assume that the medium of the emergent rays is identical with that of the incident rays, so that $n = n'_1/n_1 = n'_1/n'_2$. In the figure $VG = VZ'_1/n$. The straight line VG is drawn through V parallel to the chief incident ray u_1; the straight line GZ'_1 is drawn through G normal to the first face μ_1 of the prism. The path within the prism of the chief ray u'_1 of the astigmatic bundle of rays refracted at the first face will be along the straight line B_1B_2 drawn from B_1 parallel to the straight line VZ'_1. And if Z'_1J is normal to the second face of the prism, the chief ray u'_2 of the astigmatic bundle of emergent rays will be along the straight line $B_2u'_2$ parallel to VJ.

On the ray VG which is incident on the first face of the prism at the point V where the principal section intersects the edge of the prism and which, by construction, is parallel to the incident ray u_1 there is a point Z_1 to which the point Z'_1 on the ray VZ'_1 resulting from the refraction of the ray VG at the first face of the prism, corresponds as I. Image-Point. This point Z_1 may be constructed according to the second of the two constructions given in § 64, as follows: Through Z'_1 draw a straight line perpendicular to VZ'_1, and let U_1, U_2 designate the positions of the two points where this straight line meets the prism-faces μ_1, μ_2, respectively. Let X designate the point of intersection of the normal at V to the first face of the prism with the straight line U_1G. The point Z_1 is the foot of the perpendicular let fall from X on the incident ray VG. In the same way the I. Image-Point Z'_2 corresponding to the point Z'_1 on the ray VZ'_1 incident on the second face of the prism at V will be found to lie on the emergent ray VJ at a point which is determined by drawing UJ to meet at Y the normal at V to the second face of the prism, and dropping from Y a perpendicular on VJ, the foot of which will be the required point Z'_2.

Hence, according to the relations of affinity which were shown in § 63 to exist between the object-points and the image-points in the case of the refraction of parallel rays at a plane surface, the I. Image-Point S'_1 corresponding to the homocentric object-point S_1 on the chief ray u_1 of the bundle of incident rays will be the point of intersection of the straight line drawn through S_1 parallel to $Z_1Z'_1$, with the chief ray u'_1 of the astigmatic bundle of rays refracted at the first face of the prism. This point S'_1 is the vertex of the pencil of meridian rays after refraction at the first face of the prism. Considered with respect to the refraction at the second face of the prism, this point is the vertex of the pencil of meridian rays which are incident on this face; so that it might also be designated as the point S_2. From S_1' draw a straight line parallel to $Z_1'Z_2'$ meeting the emergent chief ray u_2' in the point S_2', which is accordingly the vertex of the pencil of emergent meridian rays of the bundle, and which is therefore the I. Image-Point on the emergent chief ray u_2' corresponding to the object-point S_1 on the chief incident ray u_1.

Again, the normal to the first face of the prism drawn through the object-point S_1 on the chief incident ray u_1 will meet the chief ray u_1' of the astigmatic bundle of rays refracted at this face in the II. Image-Point $\overline{S}_1'$; which is the vertex of the pencil of sagittal rays after refraction at the first face of the prism. This point may also be designated as the point $\overline{S}_2$ by regarding it as the vertex of the pencil of sagittal rays which are incident on the second face of the prism. And, finally, if through S_1' we draw a normal to the second face of the prism, this normal will meet the chief emergent ray u_2' in the point $\overline{S}_2'$ which is the II. Image-Point on the chief emergent ray u_2' corresponding to the object-point S_1 on the chief incident ray u_1.

Applying here the results of § 63, we can say:

Corresponding to a range of homocentric object-points P_1, Q_1, R_1, $\cdots$ *on an incident chief ray* u_1, *which is refracted through a prism in a principal section, we have a similar range of I. Image-Points* P_2', Q_2', R_2', $\cdots$ *and a similar range of II. Image-Points* $\overline{P}_2'$, $\overline{Q}_2'$, $\overline{R}_2'$, $\cdots$ *both lying on the emergent chief ray* u_2'.

77. Formulæ for Calculation of the Positions on the Chief Emergent Ray of the I. and II. Image-Points. Let α_1, α_1' and α_2, α_2' denote the angles of incidence and refraction of the chief ray of the bundle at the first and second faces of the prism, respectively; so that if n_1, n_1' and n_2' denote the absolute indices of refraction of the three media traversed by the ray in succession, we shall have:

$$n_1'\cdot\sin\alpha_1' = n_1\cdot\sin\alpha_1, \quad n_2'\cdot\sin\alpha_2' = n_1'\cdot\sin\alpha_2.$$

Referring to the figure (Fig. 41), let us employ the following symbols:

$$B_1S_1 = s_1, \quad B_1S_1' = s_1', \quad B_1\overline{S}_1' = \bar{s}_1', \quad B_2S_2' = s_2', \quad B_2\overline{S}_2' = \bar{s}_2';$$

then, by formulæ (19) and (21), we have:

For the *Meridian Rays:*

$$s_1' = \frac{n_1'\cdot\cos^2\alpha_1'}{n_1\cdot\cos^2\alpha_1}\cdot s_1, \quad s_2' = \frac{n_2'\cdot\cos^2\alpha_2'}{n_1'\cdot\cos^2\alpha_2}\cdot B_2S_1';$$

and for the *Sagittal Rays:*

$$\bar{s}'_1 = \frac{n'_1}{n_1}\bar{s}_1, \quad \bar{s}'_2 = \frac{n'_2}{n'_1} B_2\bar{S}'_1;$$

If we put $B_1B_2 = \delta_1$, then:

$$B_2S'_1 = B_2B_1 + B_1S'_1 = s'_1 - \delta_1,$$
$$B_2\bar{S}'_1 = B_2B_1 + B_1\bar{S}'_1 = \bar{s}'_1 - \delta_1;$$

and substituting these values in the equations above, and eliminating s'_1 and $\bar{s}'_1$, we obtain finally the following formulæ for determining the positions of the I. and II. Image-Points on the chief emergent ray:

Meridian Rays:

$$s'_2 = \frac{n'_2}{n_1}\frac{\cos^2\alpha'_2}{\cos^2\alpha_2}\left(\frac{\cos^2\alpha'_1}{\cos^2\alpha_1}s_1 - \frac{n_1}{n'_1}\delta_1\right); \tag{30}$$

Sagittal Rays:

$$\bar{s}'_2 = n'_2\left(\frac{\bar{s}_1}{n_1} - \frac{\delta_1}{n'_1}\right). \tag{31}$$

Thus, knowing the position of the object-point S_1 on the chief incident ray u_1 of the homocentric bundle of incident rays, and knowing also the optical and geometrical constants of the prism, we can calculate by means of formulæ (30) and (31) the positions of the I. and II. Image-Points S'_2 and $\bar{S}'_2$ on the chief ray u'_2 of the astigmatic bundle of emergent rays.

78. Convergence-Ratios of the Meridian and Sagittal Rays. Let G_1 and J_1 (not shown in the figure) designate the positions of two points on the first face of the prism infinitely near to the incidence-point B_1: the point G_1 lying in the plane of the meridian section and the point J_1 in the plane of the sagittal section of the infinitely narrow homocentric bundle of incident rays which emanate from the object-point S_1. The straight lines S_1G_1, S_1J_1 will be the paths of a secondary meridian ray and a secondary sagittal ray of the bundle of incident rays. Also, let G_2, J_2 designate the points where these two rays, after refraction at the first face of the prism, meet the second face. If the angles which these secondary incident rays make with the chief incident ray u_1 and the angles which the corresponding secondary emergent rays make with the chief emergent ray u'_2 are denoted as follows:

$$\angle B_1S_1G_1 = d\alpha_1, \quad \angle B_1S_1J_1 = d\bar{\lambda}_1, \quad \angle B_2S'_2G_2 = d\alpha'_2, \quad \angle B_2\bar{S}'_2J_2 = d\bar{\lambda}'_2,$$

the ratios:

$$Z_u = \frac{d\alpha'_2}{d\alpha_1}, \quad \bar{Z}_u = \frac{d\bar{\lambda}'_2}{d\bar{\lambda}_1},$$

are called the *Convergence-Ratios* of the Meridian and Sagittal Rays, respectively. Applying formulæ (20) and (22) to the refractions at the two faces of the prism, we obtain immediately:

Meridian Rays:

$$Z_u = \frac{d\alpha'_2}{d\alpha_1} = \frac{n_1}{n'_2} \frac{\cos\alpha_1 \cdot \cos\alpha_2}{\cos\alpha'_1 \cdot \cos\alpha'_2}; \tag{32}$$

Sagittal Rays:

$$\overline{Z}_u = \frac{d\overline{\lambda}'_2}{d\overline{\lambda}_1} = \frac{n_1}{n'_2}. \tag{33}$$

79. If the prism is surrounded by the same medium on both sides, so that $n'_2 = n_1$, the formulæ above (30), (31), (32) and (33) may be simplified by putting $n = n'_1/n_1 = n'_1/n'_2$. In this case the convergence-ratio of the sagittal rays will be equal to unity for all directions of the chief incident ray. Moreover, if when $n'_2 = n_1$ the chief incident ray has the direction of the ray of minimum deviation, so that (§ 71) $\cos\alpha_1 \cdot \cos\alpha_2 = \cos\alpha'_1 \cdot \cos\alpha'_2$, the convergence-ratio of the meridian rays will likewise be equal to unity; that is, $Z_{u,0} = 1$.

In general, therefore, the image of a luminous point as seen through a prism, viewed either by the naked eye or through a telescope, will not be a point. Depending on how the eye or telescope is focussed, the image of a point-source of light will appear through the prism as a small straight line parallel to the prism-edge (I. Image-Line), or a disc of light, or, finally, a small straight line lying in the plane of the principal section of the prism (II. Image-Line). See description of L. BURMESTER's Experiment, § 85. In a prism-spectroscope the source of light is usually a narrow illuminated slit with its length parallel to the prism-edge. If, as is usually done in this case, we focus the telescope on the II. Image-Line, the slit-image, except near its ends, will be clear and distinct, so that here we encounter practically no serious disadvantage on account of astigmatism (§ 86).

80. The Astigmatic Difference. If the bundle of incident rays is itself astigmatic, instead of a homocentric object-point, we shall have a I. Object-Point S_1 and a II. Object-Point $\overline{S}_1$; and the astigmatic difference (see § 61) of the bundle of incident rays will be:

$$\overline{S}_1 S_1 = s_1 - \overline{s}_1,$$

where $s_1 = B_1 S_1$, $\overline{s}_1 = B_1 \overline{S}_1$; and the astigmatic difference of the corresponding bundle of emergent rays will be:

$$\overline{S}'_2 S'_2 = s'_2 - \overline{s}'_2.$$

Accordingly, from formulæ (30) and (31), we obtain the following general formula for the astigmatic difference of the emergent rays:

$$\overline{S}'_2 S'_2 = \frac{n'_2 \cos^2 \alpha'_1 \cdot \cos^2 \alpha'_2}{n_1 \cos^2 \alpha_1 \cdot \cos^2 \alpha_2} s_1 - \frac{n'_2}{n_1} \overline{s}_1 - \frac{n'_2}{n'_1} \delta_1 \left(\frac{\cos^2 \alpha'_2}{\cos^2 \alpha_2} - 1 \right).$$

In case the bundle of incident rays is homocentric ($s_1 - \overline{s}_1 = 0$), the astigmatic difference of the bundle of emergent rays is given by the following formula:

$$\overline{S}'_2 S'_2 = \frac{n'_2}{n_1} \left(\frac{\cos^2 \alpha'_1 \cdot \cos^2 \alpha'_2}{\cos^2 \alpha_1 \cdot \cos^2 \alpha_2} - 1 \right) s_1 - \frac{n'_2}{n'_1} \left(\frac{\cos^2 \alpha'_2}{\cos^2 \alpha_2} - 1 \right) \delta_1. \qquad (34)$$

According to this formula, therefore, the magnitude of the astigmatic difference of the bundle of emergent rays depends not only on the direction of the chief ray of the homocentric bundle of incident rays, but also on the length of the ray-path $B_1 B_2 = \delta_1$ within the prism. Moreover, for a given incident chief ray u_1, the astigmatic difference $\overline{S}'_2 S'_2$ depends on the position on u_1 of the homocentric object-point S_1; for it increases in proportion as s_1 increases; that is, for a given prism and a given incident chief ray u_1, the astigmatic difference of the bundle of emergent rays is proportional to the distance of the homocentric object-point S_1 from the incidence-point B_1.

81. Magnitude of the Astigmatic Difference in Certain Special Cases.

(1) If the object-point S_1 on the chief incident ray u_1 coincides with the incidence-point B_1 at the first face of the prism, that is, if $s_1 = 0$, the astigmatic difference of the bundle of emergent rays will have the value given by the following expression:

$$\overline{B}'_2 B'_2 = \frac{n'_2}{n'_1} \left(1 - \frac{\cos^2 \alpha'_2}{\cos^2 \alpha_2} \right) \delta_1,$$

where B'_2, $\overline{B}'_2$ designate the I. and II. Image-Points, respectively, corresponding to a homocentric object-point B_1 coinciding with the incidence-point of the chief ray u_1 at the first face of the prism. Evidently, if the incidence-point B_1 is regarded as an object-point on u_1, the I. and II. Image-Points B'_1, $\overline{B}'_1$ on the chief ray u'_1 of the bundle of rays refracted at the first face of the prism will coincide with each other at the point B_1. If, therefore, through B_1 we draw two straight lines, one parallel to $Z'_1 Z'_2$ (see Fig. 42) and one perpendicular to the second face of the prism, these two straight lines will determine by their intersections with the chief emergent ray u'_2 the I. Image-Point

B_2' and the II. Image-Point $\overline{B}_2'$, respectively, corresponding to the homocentric object-point B_1 on u_1.

(2) If the object-point is a point Z_1 lying on an incident chief ray z_1 which meets the refracting edge of the prism, so that the ray goes through the point V in the principal section of the prism, in this limiting case the ray-length within the prism is vanishingly small, and, hence, putting $\delta_1 = 0$ in formula (34), we find:

$$Z_2'\overline{Z}_2' = \frac{n_2'}{n_1}\left(\frac{\cos^2\alpha_1'\cdot\cos^2\alpha_2'}{\cos^2\alpha_1\cdot\cos^2\alpha_2} - 1\right)\cdot VZ_1.$$

(3) The condition that the astigmatic difference shall be independent of the distance of the homocentric object-point S_1 from the point B_1 where the chief incident ray meets the first face of the prism, that is, the condition that the astigmatic difference shall be independent of the magnitude s_1, is evidently:

$$\cos\alpha_1'\cdot\cos\alpha_2' = \cos\alpha_1\cdot\cos\alpha_2;$$

which, in the general case, leads to an equation of the eighth degree for calculating the value of the angle of incidence α_1 in order to ascertain what must be the direction of the chief incident ray. In the special case, however, when the prism is surrounded on both sides by the same medium ($n_2' = n_1$), the equation above will be recognized as the condition that the ray shall traverse the prism with *minimum deviation* (§ 71). Accordingly, in case the chief ray $u_{0,1}$ of the homocentric bundle of incident rays has the direction of the ray of minimum deviation, we have here the following special formulæ:

Minimum Deviation ($n = n_1'/n_1 = n_1'/n_2'$):

$$\overline{S}_{0,2}'S_{0,2}' = \frac{\delta_1}{n}\left(1 - \frac{\cos^2\alpha_2'}{\cos^2\alpha_2}\right), \quad Z_{0,u} = \overline{Z}_{0,u} = 1.$$

In this special case, since $2\alpha_2 + \beta = 0$, the formula for the magnitude of the astigmatic difference in the case of minimum deviation may be written also in the following form:

$$\overline{S}_2'S_2' = \frac{\delta_1}{n}(n^2 - 1)\tan^2\frac{\beta}{2};$$

which shows clearly that the magnitude of the astigmatic difference, which in every case depends on the length of the ray-path within the prism, is in the special case of minimum deviation of the chief ray directly proportional to this magnitude δ_1. The nearer to the edge of the prism the chief ray $u'_{0,1}$ is, the smaller will be the astigmatic difference; and in the limiting case when the chief incident ray coincides with the ray $z_{0,1}$, which is the ray of this system of parallel incident rays that meets the refracting edge of the prism, the astigmatic difference vanishes altogether, so that the I. and II. Image-Points $Z'_{0,2}$, $\overline{Z}'_{0,2}$ corresponding to any object-point $Z_{0,1}$ on the chief incident ray $z_{0,1}$ are coincident, as is also evident from the formula given under (2) above.

These results may be stated as follows:

To every actual object-point lying on an incident chief ray, which traverses a prism in its principal section in a direction such that

$$\cos\alpha_1\cdot\cos\alpha'_1=\cos\alpha_2\cdot\cos\alpha'_2,$$

there corresponds an astigmatic difference of the astigmatic bundle of emergent rays which is independent of the position of the object-point on the incident chief ray, and which is proportional to the distance of the chief ray from the refracting edge of the prism. In particular, on the incident chief ray which meets the refracting edge of the prism, to every object-point there corresponds on the emergent ray a homocentric image-point. (This latter statement, however, has merely geometric significance, since in this limiting case the length of the ray-path within the prism vanishes.)[1]

(4) The special case when the astigmatic difference = 0 will be considered at length in the following article (Art. 25).

ART. 25. HOMOCENTRIC REFRACTION, THROUGH A PRISM, OF NARROW, HOMOCENTRIC BUNDLE OF INCIDENT RAYS, WITH ITS CHIEF RAY LYING IN A PRINCIPAL SECTION OF THE PRISM.

82. In general, as we have seen (§ 81), the astigmatic difference of the bundle of emergent rays arising from the refraction through a prism of an infinitely narrow homocentric bundle of incident rays, having its chief ray in a principal section of the prism, will not be zero; that is, the I. and II. Image-Points S'_2, $\overline{S}'_2$ will not coincide in one point Σ'_2 on the chief emergent ray u'_2. If this does happen, then the image of a point-source S_1 as seen through a prism will be a point Σ'_2. In certain cases, as has been very beautifully shown by Prof. Dr. L.

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[1] This law in its general form, as here stated, is given by L. BURMESTER: Homocentrische Brechung des Lichtes durch das Prisma: *Zft. f. Math. u. Phys.*, xl. (1895), 65–90. For the case when the prism is surrounded by the same medium on both sides, the law was obtained also by A. GLEICHEN: Ueber die Brechung des Lichtes durch Prismen: *Zft. f. Math. u. Phys.*, xxxiv. (1889), 161–176.

BURMESTER,[1] this can occur; although until the publication of BURMESTER's investigations on this subject, the laws of the homocentric refraction of light through a prism appear not to have been clearly formulated except for certain special cases.[2]

In the following discussion we shall show how the main results obtained by BURMESTER by purely geometrical methods may be deduced from the general formula (34) for the astigmatic difference, as is done by LOEWE;[3] and we shall give also an outline of the elegant geometrical method used by BURMESTER himself.

83. Analytical Method. In the first place we may remark that when the incident rays are an infinitely narrow bundle of parallel rays ($s_1 = \infty$), the ratio $(s_2' - \bar{s}_2')/s_1$ will be vanishingly small; that is, the astigmatic difference will be practically equal to zero in comparison with the distance from the prism of the point-source of light. This is the essential advantage of using parallel incident rays in working with a prism-spectroscope.

The condition that the astigmatic difference $\overline{S}_2'S_2'$ of the bundle of emergent rays shall vanish, that is, that the I. and II. Image-Points on the emergent chief ray u_2' shall coincide in a single point Σ_2', is found immediately by putting $s_2' - \bar{s}_2' = \overline{S}_2'S_2' = 0$ in formula (34), and is as follows:

$$B_1\Sigma_1 = \frac{n_1\delta_1}{n_1'} \frac{\cos^2\alpha_1(\cos^2\alpha_2 - \cos^2\alpha_2')}{\cos^2\alpha_1\cdot\cos^2\alpha_2 - \cos^2\alpha_1'\cdot\cos^2\alpha_2'}, \tag{35}$$

where Σ_1 designates the homocentric Object-Point on the incident chief ray u_1 to which corresponds the homocentric Image-Point Σ_2' on the emergent chief ray u_2'. This distance $B_1\Sigma_1$ is determined by this equation as a unique function of α_1 and δ_1; the two magnitudes which, for a given prism, define completely the incident chief ray u_1. Hence, equation (35) shows that:

On every incident chief ray u_1, refracted through a prism in a principal section, there is in general one, and only one, Object-Point Σ_1 to which on the emergent chief ray u_2' there corresponds a homocentric Image-Point Σ_2'.

Moreover, for a given value of the angle of incidence α_1, the length of the ray-path within the prism, denoted by δ_1, is proportional to the distance VB_1 of the incidence-point from the refracting edge of the prism, and if B_1 coincides with V, $\delta_1 = 0$; and, therefore:

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[1] L. BURMESTER: Homocentrische Brechung des Lichtes durch das Prisma: *Zft. f. Math. u. Phys.*, xl. (1895), 65–90.

[2] See H. HELMHOLTZ: *Wissenschaftliche Abhandlungen*, Bd. II (Leipzig, 1883), S. 167. A. GLEICHEN: Ueber die Brechung des Lichtes durch Prismen: *Zft. f. Math. u. Phys.*, xxxiv. (1889), 161–176. J. WILSING: Zur homocentrische Brechung des Lichtes durch das Prisma: *Zft. f. Math. u. Phys.*, xl. (1895), 353–361.

[3] F. LOEWE: Die Prismen und die Prismensysteme: Chapter VIII of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), herausgegeben von M. VON ROHR. See p. 433.

Object-Points lying upon parallel incident chief rays which are refracted through the prism in a principal section, which have homocentric Image-Points, are all contained in a certain plane which passes through the edge of the prism.

Thus, a bundle of parallel incident chief rays determines a remarkable plane passing through the edge of the prism which is characterized by the property that each Object-Point in this plane is imaged in the prism by a homocentric Image-Point. For another bundle of parallel incident chief rays we shall have another such plane passing through the refracting edge of the prism and characterized in the same way. Moreover, it is easy to show similarly that the locus of the homocentric image-points for such a system of parallel incident chief rays is also a plane passing through the refracting edge of the prism. The angles made with the faces of the prism by each of these two corresponding planes, one of which is the locus of the Object-Points and the other the locus of the corresponding homocentric Image-Points, for a given direction α_1 of the parallel incident chief rays, can also be easily calculated, as is done by LOEWE.

A rather exceptional case, however, is the case when the incident chief ray has a direction such that

$$\cos\alpha_1 \cdot \cos\alpha_2 = \cos\alpha_1' \cdot \cos\alpha_2';$$

for then the denominator of the fraction on the right-hand side of equation (35) vanishes, and we have therefore $B_1\Sigma_1 = \infty$; in which case we have also $B_2\Sigma_2' = \infty$; so that for this particular direction of the chief incident ray both the Object-Point and the corresponding homocentric Image-Point lie at infinity. If the prism is surrounded on both sides by the same medium, the equation above is the condition that the chief ray shall traverse the prism with *minimum deviation* (§ 71, § 81).

84. Geometrical Investigation (according to Burmester). In a principal section of the prism let us consider a system of parallel incident chief rays of which, for example, u_1 is one ray. Corresponding to a range of Object-Points

$$P_{u,1},\ Q_{u,1},\ R_{u,1},\ \cdots$$

on the incident chief ray u_1, we have, according to § 63, a *similar* range of I. Image-Points

$$P'_{u,1},\ Q'_{u,1},\ R'_{u,1},\ \cdots;$$

and a *similar* range of II. Image-Points

$$\bar{P}'_{u,1},\ \bar{Q}'_{u,1},\ \bar{R}'_{u,1},\ \cdots;$$

both lying on the chief ray u'_1 of the bundle of rays refracted at the first face of the prism. And so corresponding to a range of Object-Points on each incident chief ray of the system of parallel rays we shall have two similar ranges lying on the corresponding chief ray of the bundle of rays refracted at the first face of the prism. This system of Object-Points lying on parallel incident chief rays, such as u_1, v_1, etc., may be referred to as a whole as the system η_1; and to this system of Object-Points η_1 there corresponds, as explained in § 63, a system of I. Image-Points η'_1 and a system of II. Image-Points $\bar{\eta}'_1$ lying on the parallel chief rays of the bundles of rays which are refracted at the first face of the prism. Each of these systems η'_1 and $\bar{\eta}'_1$ is in *affinity* with the system of Object-Points η_1.

Again, corresponding to the system η'_1, we have a system of I. Image-Points η'_2 which lie on the rays of the pencil of parallel emergent chief rays, which is likewise in affinity with the system η'_1. And, similarly, corresponding to the system $\bar{\eta}'_1$, we have a system of II. Image-Points $\bar{\eta}'_2$ which lie on the rays of the pencil of parallel emergent chief rays, which is likewise in affinity with the system $\bar{\eta}'_1$.

Since the system of Object-Points η_1 is in affinity with the systems η'_1 and $\bar{\eta}'_1$, and since η'_1 is in affinity with η'_2, and $\bar{\eta}'_1$ is in affinity with $\bar{\eta}'_2$, it follows that the system η_1 is in affinity with both η'_2 and $\bar{\eta}'_2$; and, hence, also the systems η'_2, $\bar{\eta}'_2$ are in affinity with each other.

The three systems η_1, η'_1 and $\bar{\eta}'_1$, which are in affinity each with the other, have a common affinity-axis, viz., the straight line VB_1 in which the plane of the principal section meets the first face of the prism.

The straight line VB_2 in which the plane of the principal section meets the second face of the prism is the affinity-axis of the two systems η'_1 and η'_2; and this straight line is also the affinity-axis of the two systems $\bar{\eta}'_1$ and $\bar{\eta}'_2$. The point V in which the plane of the principal section meets the refracting edge of the prism is on both of these affinity-axes; so that for each pair of the five systems η_1, η'_1, $\bar{\eta}'_1$, η'_2, $\bar{\eta}'_2$ which are in affinity with each other the point V is a self-corresponding point.

The three points S_1, S'_1 and S'_2 on the corresponding chief rays u_1 u'_1 and u'_2 (Fig. 41), or the three points Z_1, Z'_1 and Z'_2 on the corresponding chief rays z_1, z_1' and z_2' parallel to u_1, u_1' and u_2', respectively, determine the three systems η_1, η_1' and η_2' which are in affinity each with the other.

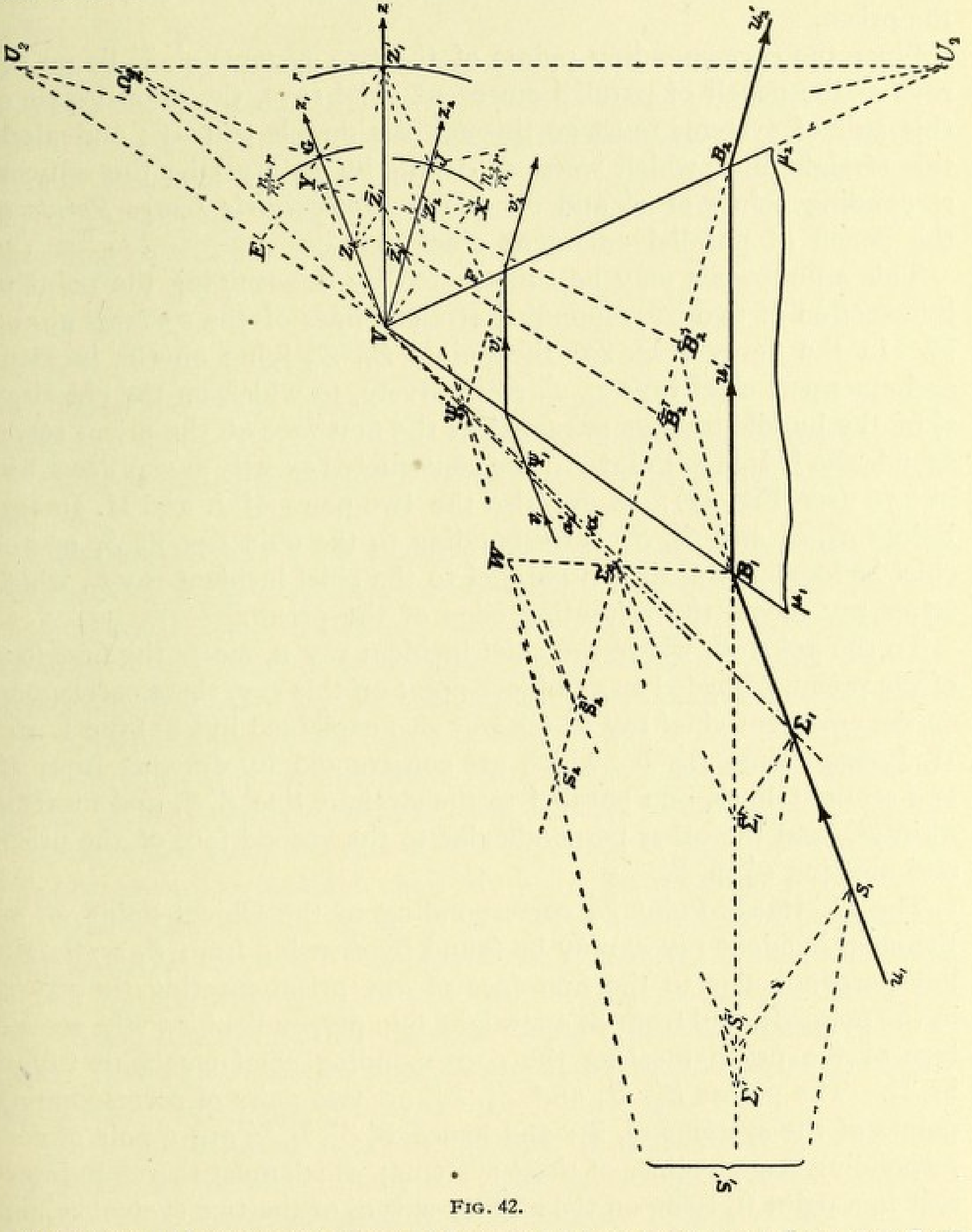

Fig. 42.

Homocentric Refraction, through a Prism, of Infinitely Narrow Bundle of Rays, with chief ray (as u_1) lying in a Principal Section of Prism. To the object-point Σ_1 on the chief incident ray u_1 corresponds the homocentric image-point Σ_2' on the chief emergent ray u_2'.

In the same way, the three corresponding chief rays u_1, u_1' and u_2' or z_1, z_1' and z_2' are sufficient to determine the three systems η_1, $\overline{\eta}_1'$ and

$\overline{\eta}'_2$ which are in affinity each with the other; for the corresponding points of η_1, $\overline{\eta}'_1$ lie on the normals to the first face of the prism, and the corresponding points of $\overline{\eta}'_1$, $\overline{\eta}'_2$ lie on the normals to the second face of the prism.

Since the corresponding points of the two systems η'_2, $\overline{\eta}'_2$ lie on the rays of the pencil of parallel emergent chief rays, the affinity-axis of this pair of systems must go through the double-point V; and along this straight line, which we shall denote by a'_2, must lie the self-corresponding points of η'_2 and $\overline{\eta}'_2$, or the *Homocentric Image-Points* of the pencil of parallel emergent chief rays.

This affinity-axis may be constructed by determining the point of intersection of two corresponding straight lines of the systems η'_2 and $\overline{\eta}'_2$. In the figure (Fig. 42) the points Z_1, Z'_2 lying on the incident and emergent chief rays z_1, z'_2, respectively, to which on the chief ray z'_1 of the bundle of rays refracted at the first face of the prism corresponds the I. Image-Point Z'_1, are constructed exactly as was described in § 76 (see Fig. 41); as are also the two pairs of I. and II. Image-Points S'_1, $\overline{S}'_1$ and S'_2, $\overline{S}'_2$ corresponding to the object-point S_1 on the chief incident ray u_1 drawn parallel to the chief incident ray z_1, which latter ray meets the refracting edge of the prism.

To the point B_1 where the chief incident ray u_1 meets the first face of the prism, regarded as an object-point on this ray, there correspond on the emergent chief ray u'_2 (as was also explained in § 81) the I. and II. Image-Points B'_2, $\overline{B}'_2$, which are constructed by drawing from B_1 two straight lines, one parallel to the straight line $Z'_1Z'_2$ and meeting u'_2 in B'_2, and the other perpendicular to the second face of the prism and meeting u'_2 in $\overline{B}'_2$.

The II. Image-Point $\overline{Z}'_2$ corresponding to the Object-Point Z_1 on the chief incident ray z_1 may be found by drawing from Z_1 a straight line perpendicular to the first face of the prism meeting the ray z'_1 in the point $\overline{Z}'_1$, and from $\overline{Z}'_1$ a straight line perpendicular to the second face of the prism meeting the corresponding chief emergent ray z'_2 in $\overline{Z}'_2$. The points B'_2, $\overline{B}'_2$ and Z'_2, $\overline{Z}'_2$, are two pairs of corresponding points of the systems η'_2, $\overline{\eta}'_2$; and hence $B'_2Z'_2$, $\overline{B}'_2\overline{Z}'_2$ are a pair of corresponding straight lines of these systems; which must therefore intersect in a point Ω'_2 lying on the affinity-axis a'_2 of the two systems η'_2, $\overline{\eta}'_2$. Accordingly, the affinity-axis a'_2 is the straight line $V\Omega'_2$. Instead of using here the pair of corresponding points B'_2, $\overline{B}'_2$, we may use also any other pair as S'_2, $\overline{S}'_2$ on u'_2, in conjunction with the pair Z'_2, $\overline{Z}'_2$ on z'_2, which will determine some other point Ψ'_2 on the affinity-axis a'_2.

The point Σ'_2 where the emergent ray u'_2 meets the affinity-axis a'_2

is the double-point, or *homocentric Image-Point*, of the two similar ranges of I. and II. Image-Points lying on the emergent chief ray u'_2 which correspond to a similar range of Object-Points lying on the incident chief ray u_1.

This point Σ'_2 may also be constructed by producing $S'_1S'_2$ and $\overline{S}'_1\overline{S}'_2$ to meet, say, in a point W; then the point where the straight line WB_1 meets the emergent chief ray u'_2 will be the double-point Σ'_2 of the two similar ranges of I. Image-Points (S'_2, B'_2, $\cdots$) and II. Image-Points ($\overline{S}'_2$ $\overline{B}'_2$, $\cdots$) lying along the emergent chief ray u'_2. For, evidently, according to this construction, we have:

$$\frac{\Sigma'_2\overline{B}'_2}{\Sigma'_2\overline{S}'_2} = \frac{\Sigma'_2B_1}{\Sigma'_2W} = \frac{\Sigma'_2B'_2}{\Sigma'_2S'_2};$$

and since the point ranges S'_2, B'_2, $\cdots$ and $\overline{S}'_2$, $\overline{B}'_2$, $\cdots$ are similar, obviously the point Σ'_2 must be the self-corresponding or double-point of these two ranges of points lying together on the emergent chief ray u'_2.

The Object-Point Σ_1 on the incident chief ray u_1, to which on the emergent chief ray u'_2 corresponds the homocentric Image-Point Σ'_2 may be constructed in either of two ways as follows: From the point Σ'_2 draw a straight line parallel to $Z'_2Z'_1$ meeting u'_1 in the point Σ'_1; and from this point Σ'_1 draw a straight line parallel to Z'_1Z_1 which will determine by its intersection with u_1 the required Object-Point Σ_1. Or, from Σ'_2 draw a straight line perpendicular to the second face of the prism meeting u'_1 in the point $\overline{\Sigma}'_1$; and from this point $\overline{\Sigma}'_1$ draw a straight line perpendicular to the first face of the prism which will likewise meet the incident chief ray u_1 in the required point Σ_1.

These Object-Points Σ_1, V_1, Ψ_1, Ω_1, $\cdots$ of the pencil of parallel incident chief rays to which correspond the homocentric image-points Σ'_2, V, Ψ'_2, Ω'_2, $\cdots$ all lying, as we saw, on the affinity-axis a'_2 of the systems η'_2, $\overline{\eta}'_2$, will themselves also lie on a straight line a_1 meeting a'_2 in the point V, the two straight lines a_1 and a'_2 corresponding to each other as incident and emergent rays.

From the foregoing results it follows that the point Σ_1 is the only Object-Point on the incident chief ray u_1 to which on the emergent chief ray u'_2 there corresponds a homocentric Image-Point Σ'_2; and, hence, on every such chief ray refracted through a prism in a principal section there is always one, and only one, pair of points which are, so to speak, in *Homocentric Correspondence* with each other; exactly as we saw also in § 83. Moreover, Object-Points, lying on the rays of a pencil of parallel incident chief rays in a principal section of the prism, to which on the rays of the pencil of parallel emergent rays correspond Homocentric Image-Points, are ranged along a straight line Va_1 which goes through the refracting edge of the prism; and the corresponding Homocentric Image-Points are ranged likewise along a straight line Va'_2 which may be regarded as the emergent ray corresponding to the incident ray Va_1. All these results are in agreement with those found in § 83. These laws were distinctly formulated first by BURMESTER, and for a further account of his investigations the reader is referred to his original paper on this subject.

85. In order to verify his results, BURMESTER employed a glass prism of refracting angle 60°, for which the value of n for the FRAUNHOFER D-line was $n = 1.7$. The prism is shown in section in the diagram (Fig. 43) which gives the disposition of the apparatus. On a certain incident ray u_1B_1 the Object-Point Σ_1 to which corresponds a homocentric Image-Point Σ'_2 on the emergent chief ray u'_2 was constructed; and, moreover, the I. and II. Image-Points S'_2 and $\overline{S}'_2$ corresponding to an arbitrary Object-Point S_1 on u_1 were also constructed by the methods given above. The prism was supported on a block, which was movable in parallel guides in a direction parallel to the straight line u'_2. On this same block was placed a glass cube with two of its parallel faces perpendicular to the incident chief ray u_1, the face nearest to the prism being at the distance $B_1\Sigma_1$ from it. This face was covered with lamp-black except at the point Σ_1 where a small opening was made with a needle. A sodium-flame F was placed on the block with the prism and the glass cube. Finally, an ABBE's Focometer A, for which the distance from the objective O of a distinctly visible object is equal to 110 mm., was placed in a fixed position with its axis coinciding with the emergent chief ray u'_2, and the

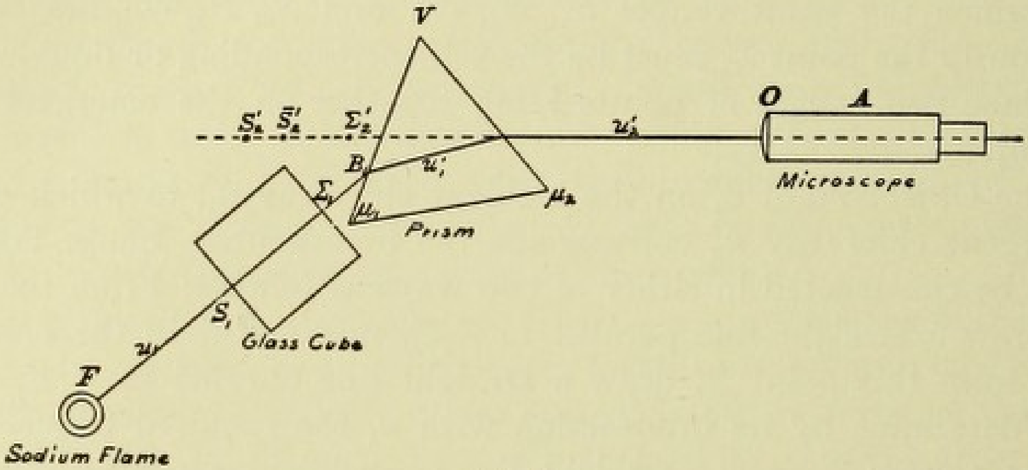

FIG. 43.

SHOWING THE PLAN OF BURMESTER'S EXPERIMENT.

movable block was displaced with respect to the fixed focometer so that the distance $\Sigma_2'O = 110$ mm. These adjustments having been completed, the homocentric Image-Point at Σ_2' resulting from the refraction through the prism of the narrow homocentric bundle of incident rays proceeding from the point-source at Σ_1 was seen through the focometer as a "small bright opening so distinctly that even the roughness of the contour of the hole in the layer of lamp-black could be clearly recognized."

Moreover, when the glass cube was placed on the movable block with its blackened face at the distance B_1S_1 from the prism, so that the pin-hole opening in the layer of lamp-black was at the Object-Point S_1, and when the block was displaced with respect to the fixed focometer so that the distance $S_2'O = 110$ mm., the image of the point-source S_1 as seen through the focometer was a small straight line parallel to the (vertical) edge of the prism; whereas if the block was displaced so that $\overline{S}_2'O = 110$ mm., the image was a small horizontal line.

ART. 26. APPARENT SIZE OF IMAGE OF ILLUMINATED SLIT AS SEEN THROUGH A PRISM.

86. If the source of the incident rays of light is a narrow illuminated slit, with its length parallel to the prism-edge, as is the usual arrangement in the prism-spectroscope, and if we adjust the eye (or telescope) to view the image of the slit formed by the sagittal rays, a clear and distinct image, practically unaffected by astigmatism, will be seen, as was mentioned in § 79.

If the apparent breadth and the apparent height of the slit as seen from the point of incidence B_1 at the first face of the prism are denoted by db and dh; and if the apparent breadth and the apparent height of the II. Image of the slit as seen from the point of emergence B_2 at the second face of the prism are denoted by db' and dh', then, evidently:

$$\frac{db'}{db} = Z_u = \frac{d\alpha_2'}{d\alpha_1}, \quad \frac{dh'}{dh} = \overline{Z}_u = \frac{d\overline{\lambda}_2'}{d\overline{\lambda}_1};$$

where the magnitudes denoted by Z_u, $\overline{Z}_u$ are determined by formulæ (32) and (33). If the prism is surrounded by the same medium on both sides, so that $n_2' = n_1$ then $\overline{Z}_u = 1$; so that in this case the apparent height of the slit-image is equal to the apparent height of the slit itself.

But the apparent breadth of the slit-image will, in general, be different from that of the slit; for, since

$$Z_u = \frac{\cos\alpha_1 \cdot \cos\alpha_2}{\cos\alpha'_1 \cdot \cos\alpha'_2}, \quad (n'_2 = n_1),$$

it appears that the value of Z_u will depend on the angle of incidence α_1. If, for example, either α'_1 or $\alpha'_2 = 90°$, the value of Z_u will be infinite, and hence $db' = \infty$. On the other hand, if one of the angles in the numerator, for example, $\alpha_1 = 90°$ (case of so-called "grazing incidence"), we have $Z_u = 0$ and, therefore, also $db' = 0$. Thus, the image of the slit may appear infinitely broad or infinitely narrow, and may have any apparent breadth between these two extremes depending on the value of the angle of incidence α_1.

When the rays proceed through the prism with minimum deviation ($n'_2 = n_1$, $\cos\alpha_1 \cdot \cos\alpha_2 = \cos\alpha'_1 \cdot \cos\alpha'_2$), we have $Z_u = 1$; so that then both the apparent height and breadth of the slit-image are equal to the apparent height and breadth of the slit itself.

ART. 27. ASTIGMATIC REFRACTION OF INFINITELY NARROW, HOMOCENTRIC BUNDLE OF INCIDENT RAYS ACROSS A SLAB WITH PLANE PARALLEL FACES.

87. As we have seen, a Slab or Plate, with parallel plane refracting faces, may be treated as a prism whose refracting angle is equal to zero; so that the methods and formulæ of the preceding articles can be adapted to this problem by treating the slab as a special case of the prism. For the sake of generality, let us assume that the media of the incident and emergent rays are different; and let us denote the absolute indices of refraction of the three media, in the order in which they are traversed by the rays of light, by n_1, n'_1 and n'_2. We shall give, first, *the construction of the I. and II. Image-Points corresponding to an Object-Point on a given incident chief ray* u_1.

In the figure (Fig. 44) the plane of the paper represents the plane of incidence of the incident chief ray u_1 which meets the first face of the slab at the point B_1. With B_1 as centre and with radii equal to r, $n_1 r/n'_1$, and $n'_2 r/n_1$ (where r denotes any arbitrary length) describe the arcs of three concentric circles; and through the point G where the circle of radius $n_1 r/n'_1$ meets the incident chief ray u_1, draw a straight line normal to the first face of the slab, and let this straight line meet the circles of radii r and $n'_2 r/n'_1$ in the points S'_1 and J, respectively. Then, exactly as in § 74, the straight line $S'_1 B_1$ will determine the path of the ray $B_1 B_2$ or u'_1 after refraction at the first face of the slab, and the path of the emergent ray u'_2 is determined by drawing through B_2 a straight line parallel to JB_1.

Through S_1' draw a straight line perpendicular to $S_1'B_1$ meeting the first face of the slab in the point U_1. From U_1 draw the straight lines U_1G and U_1J meeting at X and Y, respectively, the incidence-normal B_1N_1, and from X and Y let fall on GB_1 and JB_1 the perpendiculars XS_1 and YZ, respectively; and, finally, draw $S_1'Z$ meeting the emergent chief ray u_2' in the point S_2'. Then S_2' is the I. Image-Point on the chief emergent ray u_2' corresponding to the Object-Point S_1 on the chief incident ray u_1; as is evident, since the construction given here is merely a special case of the construction in the case of a prism (§ 76).

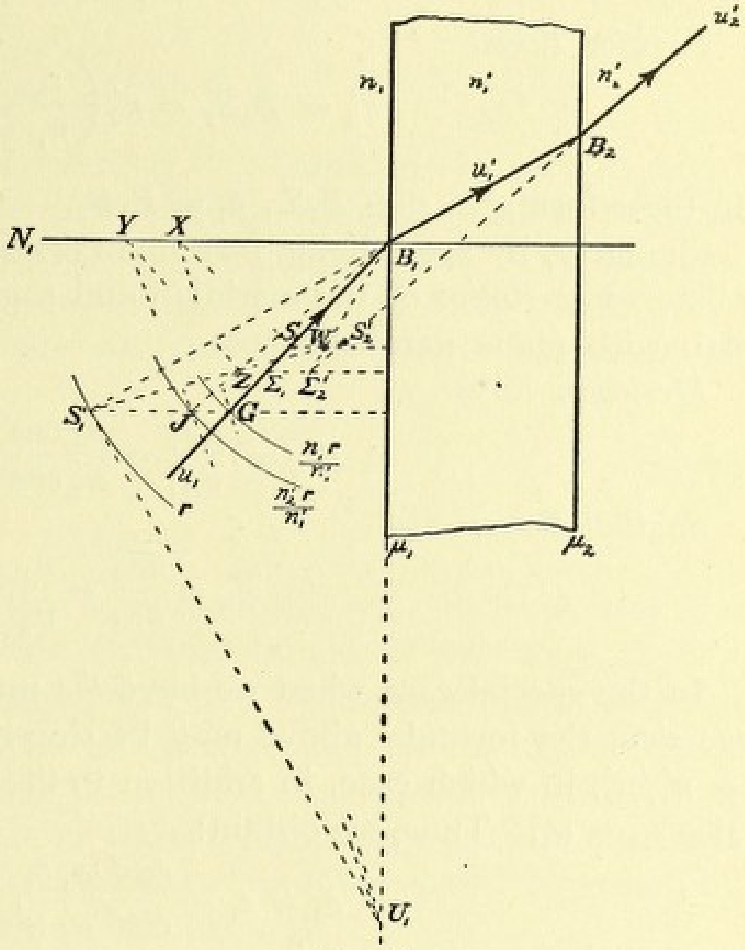

FIG. 44.

REFRACTION OF NARROW BUNDLE OF RAYS ACROSS A SLAB WITH PLANE PARALLEL FACES. Construction of the homocentric image-point on the chief emergent ray corresponding to a given chief incident ray.

The II. Image-Point $\overline{S}_2'$ corresponding to the Object-Point S_1 on the chief incident ray u_1 is found by drawing through S_1 a straight line perpendicular to the first face of the slab, which will meet the emergent chief ray u_2' in the required point.

88. Formulæ for the Determination of the Positions of the I. and II. Image-Points. Since in the case of a slab with plane parallel faces we have $\alpha_1' = \alpha_2$, we have merely to introduce this condition in the formulæ (30) and (31) in order to obtain the corresponding formulæ for this special case. Thus, employing also the relations:

$$n_1 \cdot \sin \alpha_1 = n_1' \cdot \sin \alpha_1', \quad n_1' \cdot \sin \alpha_2 = n_2' \cdot \sin \alpha_2',$$

we derive easily the following relations for the determination of the positions of the Image-Points in the case of an infinitely narrow, homocentric bundle of incident rays refracted across a slab with plane parallel faces:

Meridian Rays:

$$s'_2 = B_2S'_2 = \frac{n_2'^2 - n_1^2 \cdot \sin^2\alpha_1}{n'_2}\left\{\frac{s_1}{n_1 \cdot \cos^2\alpha_1} - \frac{n'_1\delta_1}{n_1'^2 - n_1^2 \cdot \sin^2\alpha_1}\right\}; \qquad (36)$$

Sagittal Rays:

$$\bar{s}'_2 = B_2\bar{S}'_2 = n'_2\left(\frac{s_1}{n_1} - \frac{\delta_1}{n'_1}\right). \qquad (37)$$

In these formulæ, $s_1 = B_1S_1$, $\delta_1 = B_1B_2$.

Similarly, by specializing formulæ (32) and (33), we obtain for the *Convergence-Ratios* of the meridian and sagittal rays in the case of a slab with plane parallel faces:

Meridian Rays:

$$Z_u = \frac{d\alpha'_2}{d\alpha_1} = \frac{n_1 \cos\alpha_1}{n'_2 \cos\alpha'_2}; \qquad (38)$$

Sagittal Rays:

$$\bar{Z}_u = \frac{d\bar{\lambda}'_2}{d\bar{\lambda}_1} = \frac{n_1}{n'_2}. \qquad (39)$$

In the special case when we have *the same medium on both sides of the slab*, the formulæ above may be simplified by putting $n = n'_1/n_1 = n'_1/n'_2$, in which case, in addition to the condition $\alpha'_1 = \alpha_2$, we have also $\alpha'_2 = \alpha_1$. Thus, we obtain:

$$\left.\begin{aligned} s'_2 &= s_1 - \frac{\cos^2\alpha_1}{\cos^2\alpha'_1}\frac{\delta_1}{n}, \\ \bar{s}'_2 &= s_1 - \frac{\delta_1}{n}, \\ Z_u &= \bar{Z}_u = 1. \end{aligned}\right\} \qquad (40)$$

In case the slab is at the same time very thin, so that B_1B_2 is practically negligible, we have approximately $s_1 = s'_2 = \bar{s}'_2$.

89. Astigmatic Difference in Case of a Slab. The formula for the astigmatic difference of the bundle of emergent rays corresponding to an infinitely narrow homocentric bundle of incident rays refracted across a slab with plane parallel faces may be obtained by combining formulæ (36) and (37), or, perhaps more simply still, by introducing in formula (34) the condition $\alpha'_1 = \alpha_2$; thus, we obtain:

$$\bar{S}'_2S'_2 = s'_2 - \bar{s}'_2 = \frac{n'_2}{n_1}\left(\frac{\cos^2\alpha'_2}{\cos^2\alpha_1} - 1\right)s_1 - \frac{n'_2}{n'_1}\left(\frac{\cos^2\alpha'_2}{\cos^2\alpha_2} - 1\right)\delta_1. \qquad (41)$$

In general, therefore, the astigmatic difference for a given chief incident ray will depend on the position on this ray of the radiant point S_1.

The condition that the astigmatic difference of the bundle of emergent rays shall be independent of the ray-distance (s_1) of the Object-Point S_1 from the incidence-point B_1 is evidently:

$$\frac{\cos^2 \alpha_2'}{\cos^2 \alpha_1} - 1 = 0;$$

which implies here that we have $\alpha_2' = \alpha_1$; and this in turn involves either (1) that we have the same medium on both sides of the slab ($n_2' = n_1$), or (2) that the chief ray u_1 is incident normally on the slab, so that $\alpha_2' = \alpha_1 = 0$. We consider briefly each of these cases.

(1) *Slab surrounded by same medium on both sides.* We obtain in this case:

$$\overline{S}_2'S_2' = \frac{\delta_1}{n}\left(1 - \frac{\cos^2 \alpha_1}{\cos^2 \alpha_1'}\right), \tag{42}$$

where $n = n_1'/n_1 = n_1'/n_2'$.

(2) *Normal Incidence* ($\alpha_1 = \alpha_1' = \alpha_2 = \alpha_2' = 0$). In this case the astigmatic difference vanishes, and the bundle of emergent rays is homocentric. To every object-point on a normally incident chief ray there corresponds a homocentric Image-Point. If on the normally incident chief ray N_1B_1 (Fig. 44) we take any Object-Point M_1 (not marked in the figure) to which on the corresponding normally emergent chief ray there corresponds the homocentric image-point M_2', then, by formula (36) or formula (37), we obtain:

$$B_2M_2' = \frac{n_2'}{n_1} B_1M_1 - \frac{n_2'}{n_1'} d_1,$$

where d_1 denotes the thickness of the slab ($d_1 = B_1B_2$); which may also be put in the following form:

$$B_2M_2' = \frac{n_2'}{n_1} B_1M_1 + \left(1 - \frac{n_2'}{n_1'}\right).$$

If $B_1M_1 = B_1M_2' = B_1O$, so that the homocentric Image-Point coincides at the point O with the Object-Point M_1, this double-point O of the two similar ranges of Object-Points and Image-Points lying on a straight line perpendicular to the faces of the slab can be located by the following formula:

$$B_1O = \frac{n_1(n_1' - n_2')}{n_1'(n_1 - n_2')} d_1.$$

Hence, on a normally incident chief ray refracted across a slab with plane parallel faces there can always be found a certain point O at which the Object-Point and its Homocentric Image-Point coincide with each other. If the slab is surrounded by the same medium on both sides, this point O lies at infinity.

When a luminous point M_1 is viewed normally through a transparent slab with parallel plane faces, the displacement in the line of vision of the Homocentric Image-Point M_2' with respect to the Object-Point M_1 is:

$$M_1M_2' = M_1B_1 + B_1M_1' = \left(\frac{n_2'}{n_1} - 1\right)B_1M_1 + d_1\left(1 - \frac{n_2'}{n_1'}\right),$$

and if the slab is surrounded by the same medium on both sides ($n = n_1'/n_1 = n_1'/n_2'$), we obtain:

$$M_1M_2' = \frac{n-1}{n}d_1;$$

a formula which, according to a method suggested by Duc DE CHAULNES (1767), is employed for the determination of the relative index of refraction (n), the lengths M_1M_2' and d_1 being both capable of easy measurement.

90. Exactly as in the case of refraction through a prism (§ 84), we can construct on every incident chief ray u_1 of a narrow bundle of incident rays refracted across a slab with parallel plane faces the Object-Point Σ_1 to which on the chief ray u_2' of the bundle of emergent rays there corresponds the Homocentric Image-Point Σ_2'. Thus, drawing through S_1 (Fig. 44) a straight line perpendicular to the first face of the slab and meeting the straight line $S_1'S_2'$ in the point W, we find the Homocentric Image-Point Σ_2' at the point of intersection of B_1W with the emergent chief ray u_2'. A straight line drawn through Σ_2' perpendicular to the first face of the slab will determine by its intersection with the incident chief ray u_1 the Object-Point Σ_1 which corresponds to the Homocentric Image-Point Σ_2'.

The formula for the determination of the position on a given incident chief ray u_1 of the Object-Point Σ_1 which has a Homocentric Image-Point can be obtained from formula (41) by equating to zero the right-hand side of this equation. Thus, writing here $B_1\Sigma_1$ in place of s_1, and also employing the relations:

$$n_1 \cdot \sin\alpha_1 = n_1' \cdot \sin\alpha_1', \quad n_1' \cdot \sin\alpha_2 = n_2' \cdot \sin\alpha_2',$$

we obtain:

$$B_1\Sigma_1 = \frac{n_1^3}{n_1'^3}\frac{n_2'^2 - n_1'^2}{n_2'^2 - n_1^2}\frac{\cos^2\alpha_1}{\cos^2\alpha_1'}\delta_1,$$

whence it is seen that the distance of the Object-Point Σ_1 from the incidence-point B_1 is proportional to the length of the ray-path δ_1 within the slab, that is, is proportional to the thickness of the slab, since $d_1 = \delta_1 \cdot \cos \alpha_1'$. BURMESTER,[1] employs this formula to obtain a very simple geometrical construction of the Object-Point Σ_1.

In the case of normal incidence, where the angles of incidence and refraction at both faces of the slab are equal to zero, the value of s_1 as given by formula (41) is indeterminate; that is, to every Object-Point on a normally incident chief ray there corresponds a Homocentric Image-Point; as we saw also above.

If the slab is surrounded by the same medium on both sides, we find from the formula just derived that $B_1\Sigma_1 = \infty$ for all angles of incidence. In this special case, therefore, the Homocentric Image-Points and the Object-Points to which they correspond are both at an infinite distance.

ART. 28. PATH OF A RAY REFRACTED THROUGH A SYSTEM OF PRISMS, IN THE CASE WHEN THE REFRACTING EDGES OF THE PRISMS ARE ALL PARALLEL, AND THE RAY LIES IN A PRINCIPAL SECTION COMMON TO ALL THE PRISMS.

91. A series of transparent optical media separated from each other by plane refracting surfaces constitutes a system of prisms; the second face of one prism being at the same time the first face of the following prism of the series. If there are $m + 1$ plane refracting surfaces, we shall have a system of m prisms. We shall assume here (as is almost invariably the case in actual practice) that *the edges of the prisms are parallel straight lines;* accordingly, any plane perpendicular to this system of parallel lines will be *a principal section common to all the prisms.*

In the diagram (Fig. 45) the plane of the paper is supposed to be a plane of a principal section of the prism-system. The straight line L_1B_1 represents the path of a ray incident at the point B_1 on the first refracting plane μ_1 of the series of m refracting planes. The problem is to determine the path of the emergent ray B_mB_{m+1} after the ray has been refracted in succession at each of the m refracting planes.

The absolute indices of refraction of the successive media traversed by the ray will be denoted by n_1, n_1', n_2', etc., so that n_{k-1}' will denote the index of refraction of the kth medium, and, consequently, n_m' will denote the index of refraction of the medium into which the ray emerges after the mth refraction. In actual prism-systems it is usually the case that every other medium of the series is air, and almost invariably the first and last media are air, so that $n_1 = n'_2 = n'_4 = \cdots = n'_m$; but

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[1] L. BURMESTER: Homocentrische Brechung des Lichtes durch das Prisma: *Zft. f. Math. u. Phys.*, xl. (1895), 65–90.

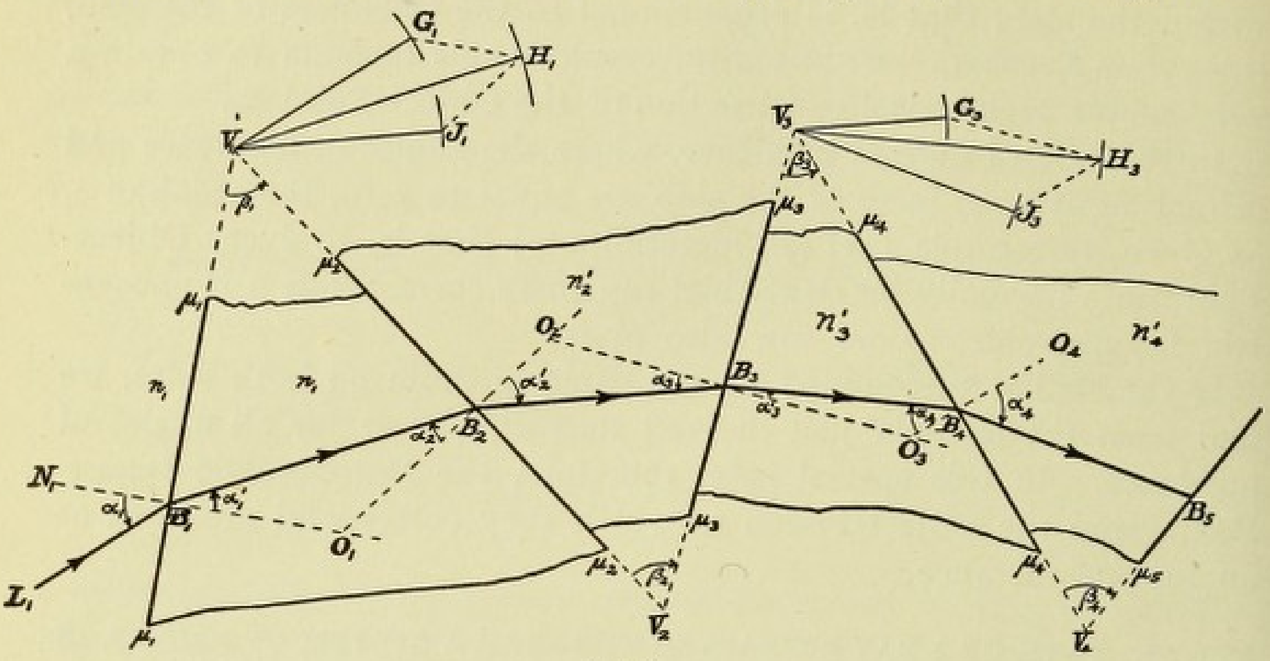

FIG. 45.

PATH OF A RAY IN A COMMON PRINCIPAL SECTION OF A SYSTEM OF PRISMS WITH THEIR REFRACTING EDGES ALL PARALLEL.

for the sake of generality we shall not assume here (except in special cases) that any two of the series of media are the same.

The points in the diagram where the refracting edges of the prisms meet the common principal section are designated as V_1, V_2, etc.; thus, the point where the refracting edge of the kth prism (that is, the straight line in which the two refracting planes μ_k and μ_{k+1} intersect) meets the principal section will be designated as the point V_k.

92. Construction of the Path of the Ray. The path of a ray through a system of prisms can be constructed geometrically by repeated applications of the construction of the path of a ray through a single prism (§ 66). Thus, with centre at the point V_1 and with radii r_1, $n_1 r_1/n'_1$ and $n'_2 r_1/n'_1$ (where r_1 denotes any arbitrary length) describe the arcs of three concentric circles; and through V_1 draw a straight line parallel to the given incident ray $L_1 B_1$ meeting the circle of radius $n_1 r_1/n'_1$ in the point G_1. Through G_1 draw a straight line perpendicular to the first refracting plane μ_1 and meeting the circle of radius r_1 in the point H_1; and from H_1 draw a straight line perpendicular to the second refracting plane μ_2 and meeting the circle of radius $n'_2 r_1/n'_1$ in the point J_1; and draw the straight lines $V_1 H_1$ and $V_1 J_1$. Similarly, with V_3 as centre and with radii r_2, $n'_2 r_2/n'_3$ and $n'_4 r_2/n'_3$ describe the arcs of three concentric circles, and through V_3

draw a straight line parallel to V_1J_1 meeting the circle of radius n'_2r_2/n'_3 in the point G_3. The points H_3 and J_3 are determined by drawing the straight lines G_3H_3 and H_3J_3 perpendicular to the refracting planes μ_3 and μ_4, respectively. Finally, draw the straight lines V_3H_3 and V_3J_3. Having performed this construction as often as necessary, we can construct the path of the ray through each prism in succession. Thus, we must draw B_1B_2 parallel to V_1H_1, B_2B_3 parallel to V_1J_1 (or to V_3G_3), B_3B_4 parallel to V_3H_3, B_4B_5 parallel to V_3J_3, etc.; where B_k designates the point of incidence of the ray at the kth refracting plane μ_k.

93. Formulæ for the Trigonometrical Calculation of the Path of the Ray through the System of Prisms. The refracting angle of the kth prism of a system of prisms is the angle through which the refracting plane μ_k has to be turned about the refracting edge of the prism in order that this plane shall be brought to coincide with the plane μ_{k+1}. This angle will be denoted by β_k; thus,

$$\angle V_{k-1}V_kV_{k+1} = \beta_k.$$

The angles of incidence and refraction at the kth refracting plane μ_k will be denoted by α_k, α'_k. Thus, if the straight line $O_{k-1}B_kO_k$ is the normal to the plane μ_k at the incidence-point B_k, then

$$\angle O_{k-1}B_kB_{k-1} = \alpha_k, \quad \angle O_kB_kB_{k+1} = \alpha'_k.$$

The angle of deviation at the kth refracting plane μ_k is the acute angle through which the straight line B_kB_{k+1} must be turned about the point B_k in order that B_kB_{k+1} may have the same direction as $B_{k-1}B_k$. This angle will be denoted by the symbol ϵ_k. The total deviation, denoted by the symbol ϵ (without any subscript), is the angle through which the emergent ray must be turned in order that its direction may be the same as that of the incident ray. Thus, in case there are m refracting surfaces,

$$\epsilon = \sum_{k=1}^{k=m} \epsilon_k.$$

All these angular magnitudes are reckoned positive or negative according as the rotation is counter-clockwise or clockwise.

Accordingly, for calculating the path of a ray through a system of prisms, consisting of m refracting planes, the refracting edges of the prisms being all parallel, and the ray lying in a principal section common to all the prisms, we have the following system of equations:[1]

$$\left.\begin{array}{lll}
\text{I} & \text{II} & \text{III} \\
n_1' \cdot \sin \alpha_1' = n_1 \cdot \sin \alpha_1, & & \epsilon_1 = \alpha_1 - \alpha_1', \quad (\mu_1) \\
& \alpha_2 = \alpha_1' - \beta_1, & \\
n_2' \cdot \sin \alpha_2' = n_1' \cdot \sin \alpha_2, & & \epsilon_2 = \alpha_2 - \alpha_2', \quad (\mu_2) \\
& \alpha_3 = \alpha_2' - \beta_2, & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot\ \cdot \\
& \cdot\ \cdot\ \cdot\ \cdot & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot\ \cdot \\
& \alpha_k = \alpha_{k-1}' - \beta_{k-1}, & \\
n_k' \cdot \sin \alpha_k' = n_{k-1}' \cdot \sin \alpha_k, & & \epsilon_k = \alpha_k - \alpha_k', \quad (\mu_k) \\
& \cdot\ \cdot\ \cdot\ \cdot & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot\ \cdot \\
\multicolumn{3}{r}{\text{Total Deviation} = \epsilon = \sum_{k=1}^{k=m} \epsilon_k = \alpha_1 - \alpha_m' - \sum_{k=1}^{k=m-1} \beta_k.}
\end{array}\right\} \quad (43)$$

Here the term $\sum_{k=m}^{k=m-1} \beta_k$ = angle between the first and last (or mth) refracting planes; and if we denote this angle by Ω, we can write:

$$\epsilon = \alpha_1 - \alpha_m' - \Omega.$$

94. Condition that the Total Deviation shall be a Minimum. The total deviation ϵ of a ray refracted through a given system of prisms will be a minimum when the ray is incident on the first refracting plane at an angle α_1 determined by the condition $d\epsilon/d\alpha_1 = 0$; it being assumed that the conditions $d^2\epsilon/d\alpha_1^2 > 0$ and $\epsilon > 0$ are also fulfilled. According to the equation above, the condition $d\epsilon/d\alpha_1 = 0$ is equivalent to:

$$d\alpha_1 = d\alpha_m';$$

and in order to express $d\alpha_m'$ as a function of $d\alpha_1$, we employ the equations in columns I and II of the system of equations (43). Thus, differentiating each of these equations, we obtain:

$$\begin{array}{ll}
d\alpha_1' = \dfrac{n_1}{n_1'} \dfrac{\cos \alpha_1}{\cos \alpha_1'} d\alpha_1, & \\[2ex]
d\alpha_2' = \dfrac{n_1'}{n_2'} \dfrac{\cos \alpha_2}{\cos \alpha_2'} d\alpha_2, & d\alpha_2 = d\alpha_1', \\[2ex]
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\[2ex]
d\alpha_k' = \dfrac{n_{k-1}'}{n_k'} \dfrac{\cos \alpha_k}{\cos \alpha_k'} d\alpha_k, & d\alpha_k = d\alpha_{k-1}', \\[2ex]
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}$$

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[1] See A. GLEICHEN: Ueber die Brechung des Lichtes durch Prismen: *Zft. f. Math. u. Phys.*, xxxiv. (1889), 161–176. Also, S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), S. 137. H. KAYSER: *Handbuch der Spectroscopie*, Bd. I (Leipzig, 1900), S. 272. F. LOEWE: Die Prismen und die Prismensysteme: Chapter VIII of *Die Theorie der optischen Instrumente*, Bd. I, herausgegeben von M. VON ROHR (Berlin, 1904), S. 421.

Combining these equations, we obtain for a system of m refracting planes:

$$d\alpha'_m = \frac{n_1}{n'_m} \frac{\cos\alpha_1 \cdot \cos\alpha_2 \cdots \cos\alpha_m}{\cos\alpha'_1 \cdot \cos\alpha'_2 \cdots \cos\alpha'_m} d\alpha_1.$$

Hence, putting $d\alpha_1 = d\alpha'_m$, we obtain as the *Condition of Minimum Deviation:*

$$\frac{\cos\alpha_1 \cdot \cos\alpha_2 \cdots \cos\alpha_m}{\cos\alpha'_1 \cdot \cos\alpha'_2 \cdots \cos\alpha'_m} = \frac{n'_m}{n_1}.$$

This formula may be written in abbreviated form as follows:

$$n_1 \prod_{k=1}^{k=m} \cos\alpha_k = n'_m \prod_{k=1}^{k=m} \cos\alpha'_k, \tag{44}$$

where the symbol Π is used to denote the product of a series of terms. In the special case when the first and last media are the same ($n_1 = n'_m$), the condition of Minimum Deviation is:

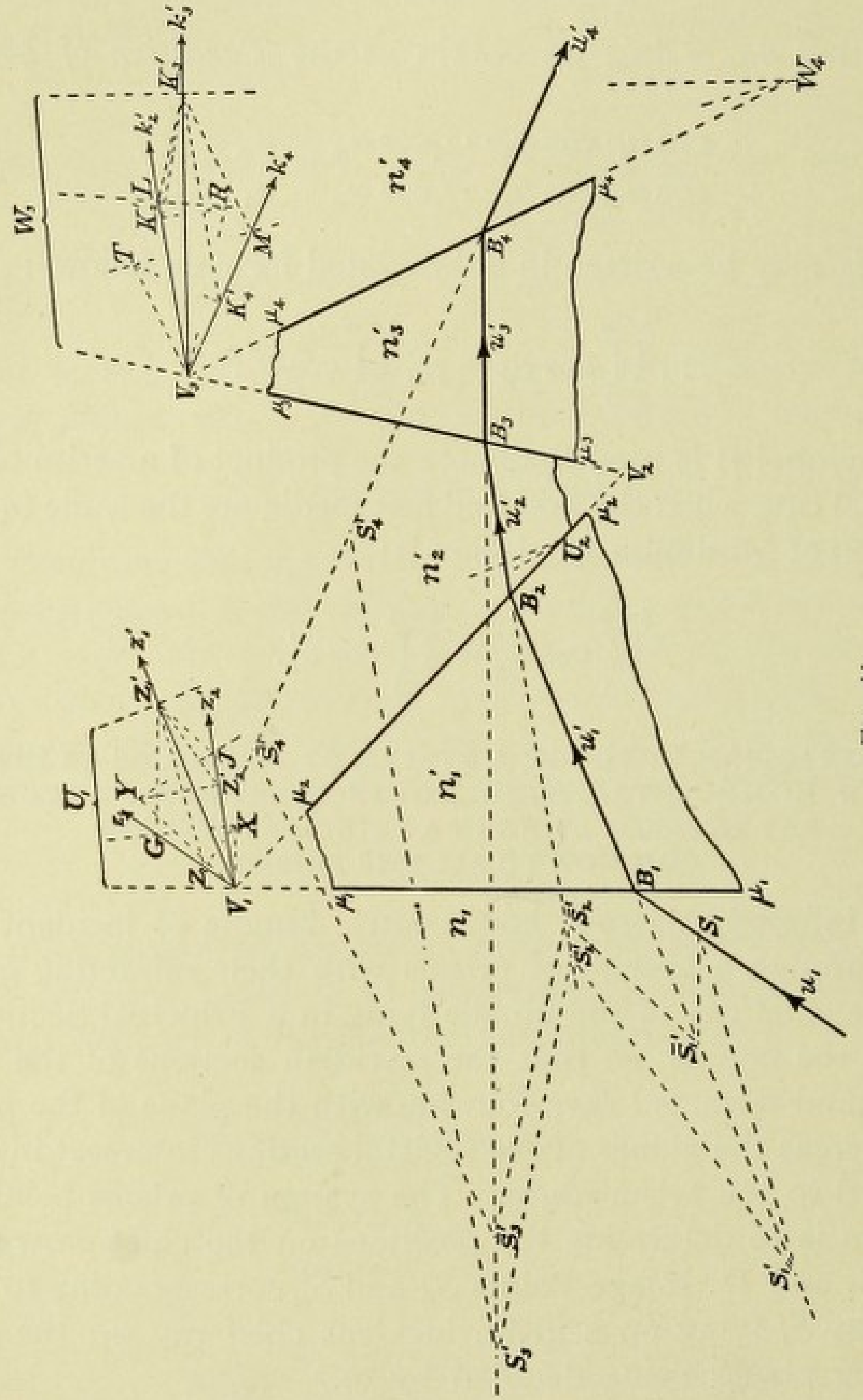
$$\prod_{k=1}^{k=m} \cos\alpha_k = \prod_{k=1}^{k=m} \cos\alpha'_k.$$

ART. 29. REFRACTION, THROUGH A SYSTEM OF PRISMS, OF AN INFINITELY NARROW, HOMOCENTRIC BUNDLE OF INCIDENT RAYS: THE CHIEF RAY THEREOF LYING IN A PRINCIPAL SECTION COMMON TO ALL THE PRISMS.

95. An infinitely narrow, homocentric bundle of incident rays is refracted through a system of prisms with their refracting edges all parallel, the chief ray of the bundle lying in a Principal Section common to all the prisms; so that the meridian sections of the bundles of incident and refracted rays coincide with the plane of the principal section, whereas the planes of the sagittal sections intersect in straight lines parallel to the prism-edges. The system of prisms being given, the problem is to determine the positions on the chief emergent ray u'_m of the I. and II. Image-Points S'_m and $\overline{S}'_m$ corresponding to a given Object-Point S_1 lying on a given incident chief ray u_1; the number of refracting planes being denoted by m.

Geometrical Construction of the I. and II. Image-Points. In the diagram (Fig. 46) only the first three prisms of the system are represented. The chief ray u_1 of the bundle of incident rays meets the first refracting plane at the point B_1; and the path of this ray through the system of prisms must be constructed as explained in § 92. If on the straight line $V_1z'_1$ drawn parallel to the straight line B_1B_2 (or u'_1) we take a point

Z_1', and construct, as in § 76, the corresponding points Z_1 and Z_2' lying on V_1z_1 and V_1z_2' drawn parallel to u_1B_1 and B_2B_3 (or u_2'), respectively; and if, in the same way, to a point K_3' taken on the straight line V_3k_3'



V_1
V_2
U_1
W_1
W_4'
U_2'
z_1'
z_2'
Z_1'
Z_2'
X
Y
G
J
Z
k_2'
k_3'
k_4'
K_2'
K_3'
K_4'
L
T
R
M'
u_1
u_1'
u_2'
u_3'
u_4'
B_1
B_2
B_3
B_4
μ_1
μ_2
μ_3
μ_4
n_1
n_1'
n_2'
n_3'
n_4'
S_1
S_1'
S_2'
S_3'
S_4'

Fig. 46.

Refraction, through a System of Prisms, of an Infinitely Narrow, Homocentric Bundle of Incident Rays. Construction of the I. and II. Image-Points of the Astigmatic Bundle of Emergent Rays.

drawn parallel to the straight line B_3B_4 (or u_3'), we construct the corresponding points K_2' and K_4' lying on the straight lines V_3k_2' and V_3k_4' drawn parallel to B_2B_3 (or u_2') and B_4u_4', respectively; we can construct the I. Image-Point S_4' on the chief ray u_4' of the bundle of rays refracted at the fourth refracting plane which corresponds to any assumed Object-Point S_1 lying on the chief incident ray u_1, as follows: Through S_1 draw a straight line parallel to $Z_1Z'_1$ meeting u'_1 in the point S'_1; through S'_1 draw a straight line parallel to Z'_1Z_2 meeting u'_2 in the point S'_2; through S'_2 draw a straight line parallel to $K'_2K'_3$ meeting u'_3 in the point S'_3; and, finally, through S'_3 draw a straight line parallel to $K'_3K'_4$ meeting the chief ray u'_4 in the I. Image-Point S'_4 corresponding to the Object-Point S_1 on the chief incident ray u_1.

In order to construct the II. Image-Point $\overline{S}'_4$, we draw $S_1\overline{S}'_1$ perpendicular to the first refracting plane μ_1 and meeting u'_1 in the point $\overline{S}'_1$; and draw $\overline{S}'_1\overline{S}_2$ perpendicular to the second refracting plane μ_2 and meeting u'_2 in the point $\overline{S}'_2$, and draw $\overline{S}'_2\overline{S}_3$ perpendicular to the third refracting plane μ_3 and meeting u'_3 in the point $\overline{S}'_3$; and, finally, we draw $\overline{S}'_3\overline{S}'_4$ perpendicular to μ_4 and meeting the emergent ray u'_4 in the II. Image-Point $\overline{S}'_4$ corresponding to the Object-Point S_1 on the chief incident ray u_1.

If we have more than four refracting planes, we have merely to continue the construction as above-indicated until we have constructed the I. and II. Image-Points S'_m and $\overline{S}'_m$ lying on the chief emergent ray corresponding to the homocentric Object-Point S_1 on the chief incident ray u_1.

Applying here also the results which were found in § 63, we can say: *Corresponding to a Range of Homocentric Object-Points $P_1, Q_1, R_1, \cdots$ on an incident chief ray u_1 which is refracted, in a principal section, through a System of Prisms, we have on the chief emergent ray u'_m a Similar Range of I. Image-Points $P'_m, Q'_m, R'_m, \cdots$ and a Similar Range of II. Image-Points $\overline{P}'_m, \overline{Q}'_m, \overline{R}'_m, \cdots$.*

96. Formulæ for Calculation of the Positions on the Chief Emergent Ray of the I. and II. Image-Points. The linear magnitudes in the following equations will be denoted, in accordance with our previous notation, by the following system of symbols:

The distances from the incidence-point B_k (Fig. 47), measured along the chief ray, before and after refraction at the kth surface, of the I. and II. Image-Points will be denoted as follows:

$$B_kS'_{k-1} = s_k, \quad B_kS'_k = s'_k, \quad B_k\overline{S}'_{k-1} = \overline{s}_k, \quad B_k\overline{S}'_k = \overline{s}'_k.$$

The length of the ray-path within the kth prism, or the distance measured along the chief ray u'_k between the kth and the $(k+1)$th refracting planes, will be denoted as follows:

$$\delta_k = B_kB_{k+1}.$$

The definitions and symbols of the angular magnitudes are the same as those given in § 93.

We shall assume that the system of prisms is formed by m plane refracting surfaces and that the edges of the $(m-1)$ prisms are all parallel.

According to formulæ (19) and (20) of § 59, we have for

The Meridian Rays After Refraction at the kth Plane:

$$\left.\begin{aligned} s'_k &= \frac{n'_k}{n'_{k-1}}\frac{\cos^2\alpha'_k}{\cos^2\alpha_k}s_k, &\quad (k &= 1, 2, \cdots, m),\\ s_k &= s'_{k-1}-\delta_{k-1}, &\quad (k &= 2, 3, \cdots, m),\\ \frac{d\alpha'_k}{d\alpha_k} &= \frac{n'_{k-1}}{n'_k}\frac{\cos\alpha_k}{\cos\alpha'_k}, &\quad (k &= 1, 2, \cdots, m). \end{aligned}\right\} \qquad (45)$$

In the first and third of these formulæ we must give k in succession all integral values from $k = 1$ to $k = m$; and in the second all integral

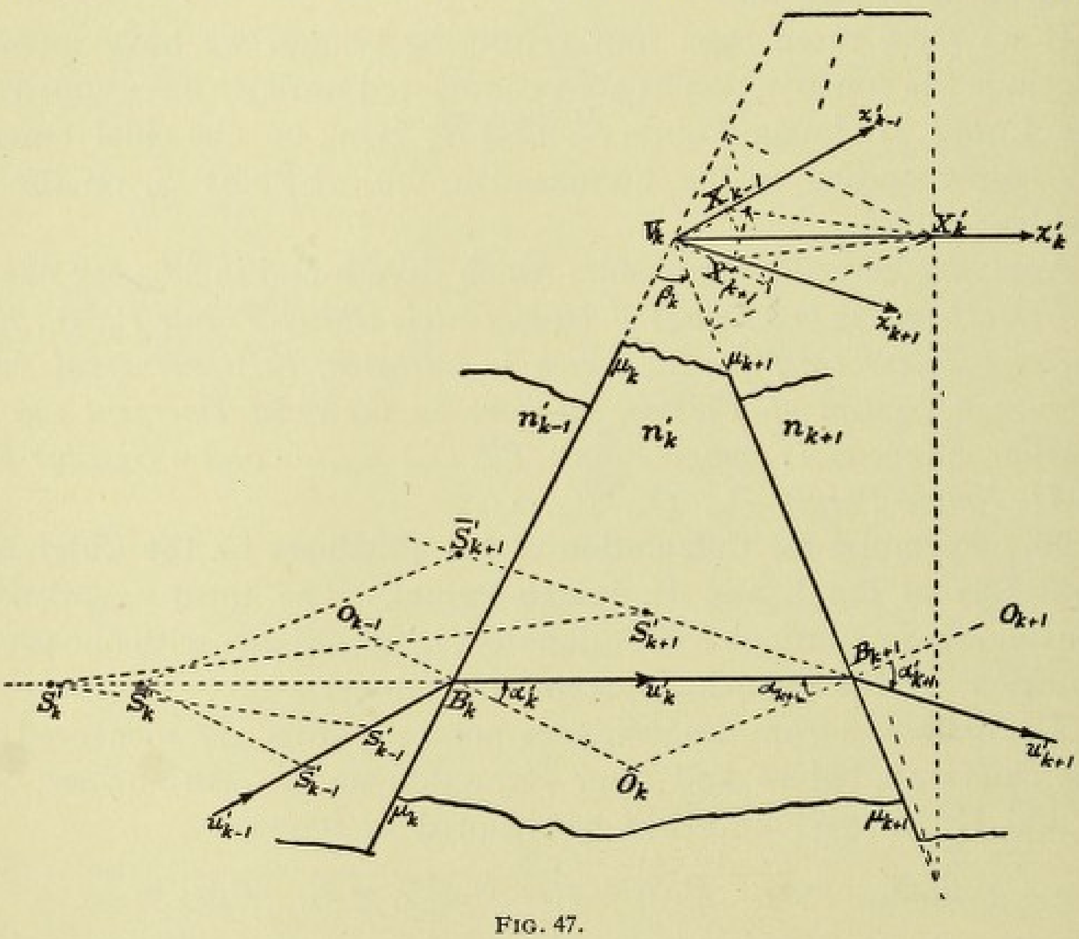

x'_{k-1}
X'_{k-1}
V_k
X'_k
x'_k
β_k
X'_{k+1}
x'_{k+1}
μ_k
μ_{k+1}
n'_{k-1}
n'_k
n_{k+1}
$\bar{S}_{k+1}$
O_{k-1}
S'_{k+1}
O_{k+1}
β_{k+1}
α'_{k+1}
S'_k
$\bar{S}_k$
B_k
α_k
u'_k
α_{k+1}
S'_{k-1}
$\bar{S}_{k-1}$
O_k
u'_{k+1}
u'_{k-1}
μ_k
μ_{k+1}

FIG. 47.

SHOWING THE PATH OF THE CHIEF RAY THROUGH THE *k*TH PRISM OF A SYSTEM OF PRISMS.

values from $k = 2$ to $k = m$. It may be observed also that $n'_0 = n_1$.

Eliminating s_k from the first two of equations (45), and abbreviating by writing:

$$R_k = \frac{n'_k}{n_{k-1}}\frac{\cos^2\alpha'_k}{\cos^2\alpha_k},$$

we obtain:

$$s'_k = R_k(s'_{k-1} - \delta_{k-1});$$

and by successive applications of this formula:

$$\begin{aligned}
s'_k &= R_k \cdot s'_{k-1} - R_k \cdot \delta_{k-1} \\
&= R_{k-1} \cdot R_k \cdot s'_{k-2} - (R_{k-1} \cdot R_k \cdot \delta_{k-2} + R_k \cdot \delta_{k-1}) \\
&= R_{k-2} \cdot R_{k-1} \cdot R_k \cdot s'_{k-3} - (R_{k-2} \cdot R_{k-1} \cdot R_k \cdot \delta_{k-3} + R_{k-1} \cdot R_k \cdot \delta_{k-2} + R_k \cdot \delta_{k-1}) \\
&= \text{etc., etc.}
\end{aligned}$$

Thus,

$$\begin{aligned}
s'_k &= (R_1 \cdot R_2 \cdots R_k) s_1 \\
&\quad - (R_2 \cdot R_3 \cdots R_k \cdot \delta_1 + R_3 \cdot R_4 \cdots R_k \cdot \delta_2 + \cdots + R_{k-1} \cdot R_k \cdot \delta_{k-2} + R_k \cdot \delta_{k-1});
\end{aligned}$$

and, hence:

$$s'_m = s_1 \cdot \prod_{k=1}^{k=m} R_k - \sum_{k=1}^{k=m-1} \left\{ \delta_k \cdot \prod_{r=k+1}^{r=m} R_r \right\}.$$

(It should be remarked that $s'_0 = s_1$ and $\delta_0 = 0$.)
Now

$$\prod_{k=1}^{k=m} R_k = \frac{n'_m}{n_1} \prod_{k=1}^{k=m} \frac{\cos^2 \alpha'_k}{\cos^2 \alpha_k};$$

and

$$\prod_{r=k+1}^{r=m} R_r = \frac{n'_m}{n'_k} \prod_{r=k+1}^{r=m} \frac{\cos^2 \alpha'_r}{\cos^2 \alpha_r};$$

accordingly, we obtain finally the following formula for calculating the position of the I. Image-Point S'_m on the chief emergent ray u'_m:

$$s'_m = \frac{n'_m}{n_1} s_1 \prod_{k=1}^{k=m} \frac{\cos^2 \alpha'_k}{\cos^2 \alpha_k} - n'_m \sum_{k=1}^{k=m-1} \left\{ \frac{\delta_k}{n'_k} \prod_{r=k+1}^{r=m} \frac{\cos^2 \alpha'_r}{\cos^2 \alpha_r} \right\}. \qquad (46)$$

Similarly, according to formulæ (21) and (22) of § 60, we have for *The Sagittal Rays After Refraction at the kth Surface:*

$$\left.\begin{aligned}
\overline{s}'_k &= \frac{n'_k}{n'_{k-1}} \overline{s}_k, && (k = 1, 2, \cdots, m), \\
\overline{s}_k &= \overline{s}'_{k-1} - \delta_{k-1}, && (k = 2, 3, \cdots, m), \\
\frac{d\overline{\lambda}'_k}{d\overline{\lambda}_k} &= \frac{n'_{k-1}}{n'_k}, && (k = 1, 2, \cdots, m).
\end{aligned}\right\} \qquad (47)$$

Eliminating $\overline{s}_k$ from the first two of these equations, we obtain:

$$\overline{s}'_k = \frac{n'_k}{n'_{k-1}} \overline{s}'_{k-1} - \frac{n'_k}{n'_{k-1}} \delta_{k-1};$$

and by successive applications of this formula:

$$\begin{aligned}\bar{s}'_k &= \frac{n'_k}{n'_{k-2}}\,\bar{s}'_{k-2} - \frac{n'_k}{n'_{k-2}}\,\delta_{k-2} - \frac{n'_k}{n'_{k-1}}\,\delta_{k-1}\\ &= \frac{n'_k}{n'_{k-3}}\,\bar{s}'_{k-3} - n'_k\left(\frac{\delta_{k-3}}{n'_{k-3}} + \frac{\delta_{k-2}}{n'_{k-2}} + \frac{\delta_{k-1}}{n'_{k-1}}\right)\\ &= \text{etc., etc.,}\\ &= \frac{n'_k}{n_1}\,s_1 - n'_k\left(\frac{\delta_1}{n'_1} + \frac{\delta_2}{n'_2} + \cdots + \frac{\delta_{k-1}}{n'_{k-1}}\right);\end{aligned}$$

and, hence, the formula for calculating the position of the II. Image Point $\overline{S}'_m$ on the chief emergent ray u'_m is as follows:

$$\bar{s}'_m = \frac{n'_m}{n_1}\,s_1 - n'_m \sum_{k=1}^{k=m-1} \frac{\delta_k}{n'_k}\,. \tag{48}$$

97. The Convergence-Ratios of the Meridian and Sagittal Rays.

The Convergence-Ratio of the Meridian Rays,

$$Z_u = \frac{d\alpha'_m}{d\alpha_1},$$

is easily obtained from the third of formulæ (45); for, since

$$d\alpha'_{k-1} = d\alpha_k,$$

we find immediately:

$$Z_u = \frac{d\alpha'_m}{d\alpha_1} = \frac{n_1}{n'_m} \prod_{k=1}^{k=m} \frac{\cos\alpha_k}{\cos\alpha'_k}\,. \tag{49}$$

In the same way the Convergence-Ratio of the Sagittal Rays,

$$\overline{Z}_u = \frac{d\bar{\lambda}'_m}{d\bar{\lambda}_1},$$

is found immediately by the third of formulæ (47); for, since

$$d\bar{\lambda}'_{k-1} = d\bar{\lambda}_k,$$

we obtain:

$$\overline{Z}_u = \frac{d\bar{\lambda}'_m}{d\bar{\lambda}_1} = \frac{n'_m}{n_1}. \tag{50}$$

In viewing through a system of prisms the II. Image of an illuminated slit, with its length adjusted parallel to the prism-edges, we see by formula (50) that, provided $n'_m = n_1$, the apparent length of the slit will not be altered; whereas, in general, the apparent breadth of the slit-image will be different from that of the slit itself, depending on the direction of the chief incident rays, according to formula (49); exactly as was found to be the case in viewing the slit-image through a single prism (§§ 79 and 86).

98. Formula for the Astigmatic Difference. The expression for the astigmatic difference of the bundle of emergent rays corresponding to an infinitely narrow, homocentric bundle of incident rays refracted, in a principal section, through a system of prisms consisting of m refracting planes may be found from formulæ (46) and (48), as follows:

$$\begin{aligned}\bar{S}'_m S'_m = s'_m - \bar{s}'_m = \frac{n'_m}{n_1}\left\{\prod_{k=1}^{k=m}\frac{\cos^2\alpha'_k}{\cos^2\alpha_k} - 1\right\}s_1 \\ - n'_m\sum_{k=1}^{k=m-1}\left[\frac{\delta_k}{n'_k}\left\{\prod_{r=k+1}^{r=m}\frac{\cos^2\alpha'_r}{\cos^2\alpha_r} - 1\right\}\right]. \qquad (51)\end{aligned}$$

The magnitude of the astigmatic difference depends therefore not only on the constants of the prism-system and on the direction of the chief incident ray and on the position of the Object-Point on this ray, but also on the lengths of the ray-paths within the prisms; in general, it will be different from zero.

The condition that the astigmatic difference shall be independent of the ray-distance (s_1) of the Homocentric Object-Point S_1 from the incidence-point B_1 of the chief ray at the first refracting plane is evidently:

$$\prod_{k=1}^{k=m}\cos\alpha_k = \prod_{k=1}^{k=m}\cos\alpha'_k;$$

which, provided the medium of the emergent rays is the same as that of the incident rays, is the condition that the chief ray shall traverse the prism-system with minimum-deviation (§ 94).

The magnitude of the astigmatic difference depends essentially, as was remarked above, on the lengths δ_k of the ray-paths within the prisms. If the condition expressed by the last equation is fulfilled, and if, in addition, each of the $(m - 1)$ values of δ_k is infinitesimally small (a condition which is hardly practicable geometrically, especially if there is more than one prism), the astigmatic difference vanishes.

For finite values of δ_k, the magnitude of the astigmatic difference is of comparatively less and less importance, the farther the Homocentric Object-Point S_1 is removed from the incidence-point B_1. Thus, when the incident rays are a bundle of parallel rays ($s_1 = \infty$), the ratio $(s'_m - \bar{s}'_m)/s_1$ is vanishingly small; so that, practically, the astigmatism is equal to zero in this case. These results are exactly analogous to those which were obtained in the case of a single prism (§§ 80 and foll.).

99. Homocentric Refraction through a System of Prisms. If the astigmatic difference of the bundle of emergent rays is equal to zero, the I. and II. Image-Points S'_m and $\overline{S}'_m$ will coincide in a single point Σ'_m on the chief emergent ray u'_m; and in this case the image of a point-source Σ_1 on the chief incident ray u_1 will be a point Σ'_m. Thus, exactly as in Art. 25, where the special case of homocentric refraction through a single prism was investigated, the condition that the astigmatic difference shall vanish is found by putting $s'_m - \overline{s}'_m = 0$ in formula (51); whereby we obtain:

$$B_1\Sigma_1 = \frac{n_1 \sum\limits_{k=1}^{k=m-1} \left[\frac{\delta_k}{n'_k}\left\{\prod\limits_{r=k+1}^{r=m}\left(\frac{\cos^2\alpha'_r}{\cos^2\alpha_r}\right) - 1\right\}\right]}{\prod\limits_{k=1}^{k=m}\left(\frac{\cos^2\alpha'_k}{\cos^2\alpha_k}\right) - 1}, \tag{52}$$

where Σ_1 designates the Homocentric Object-Point on the chief incident ray u_1 to which corresponds the Homocentric Image-Point Σ'_m on the chief emergent ray u'_m. This formula gives the distance $B_1\Sigma_1$ as a unique function of the angle of incidence α_1 of the chief incident ray u_1 and the ray-lengths δ_k from one surface to the next; so that precisely as in the case of a single prism (§ 83), we have also here the following statement:

On every incident chief ray u_1, refracted in a principal section through a system of prisms with their edges all parallel, there is, in general, one, and only one, Object-Point Σ_1 to which on the chief emergent ray u'_m there corresponds a Homocentric Image-Point Σ'_m.

It is easy to show likewise by the same methods as were used in the case of a single prism (§§ 83, foll.) that *Object-Points, lying on parallel incident chief rays, refracted, in a principal section, through a system of prisms with their edges all parallel, to which on the parallel emergent chief rays correspond Homocentric Image-Points, are ranged along a certain straight line a_1; and the Homocentric Image-Points are ranged also along a straight line a'_m, which may be regarded as the emergent ray corresponding to the incident ray a_1.*

The construction of the Homocentric Image-Point Σ'_m on the chief emergent ray u'_m and of the corresponding Homocentric Object-Point Σ_1 on the chief incident ray u_1 is performed by a method entirely analogous in every detail to the method given for the case of a single prism (§ 84). The system of Object-Points $P_{u,1}$, $P_{v,1}$, etc., lying on parallel incident chief rays u_1, v_1, etc., may be denoted as the system

η_1; and this system of Object-Points is in *affinity* with the system of corresponding I. Image-Points (system η'_m) and with the system of corresponding II. Image-Points (system $\bar{\eta}'_m$); and the straight line a'_m, whereon lie the double-points of the systems η'_m, $\bar{\eta}'_m$, or the Homocentric Image-Points of this system of parallel chief rays, is the affinity-axis of the systems η'_m and $\bar{\eta}'_m$. This straight line a'_m can be constructed, therefore, by finding the points of intersection of any two pairs of corresponding straight lines of the systems η'_m and $\bar{\eta}'_m$.[1]

ART. 30. PATH OF A RAY REFRACTED OBLIQUELY THROUGH A PRISM.

100. Construction of the Path of the Ray. In the figure (Fig. 48) the prism is shown in oblique parallel projection with its refracting edge Vy vertical and lying in the plane of the paper which is supposed to coincide with the plane of the second face of the prism. The construction of the ray refracted at the first face, corresponding to a ray L_1B_1 incident obliquely on this face at the point B_1, is exactly similar to the construction given in § 34. The plane containing the incidence-point B_1 and perpendicular to the prism-edge will be the plane of a principal section. The straight lines B_1M and B_1N normal to the two faces μ_1 and μ_2, respectively, will lie in this principal section. Let the incident ray L_1B_1 prolonged meet the second face of the prism in the point designated by D; so that B_1DM is the plane of incidence at the first face. If the triangle B_1DM is revolved about MD as axis until it comes into the plane of the paper, the point B_1 will arrive at a point C lying on the straight line drawn from N perpendicular to MD and at a distance from M which will be the real distance of B_1 from M.

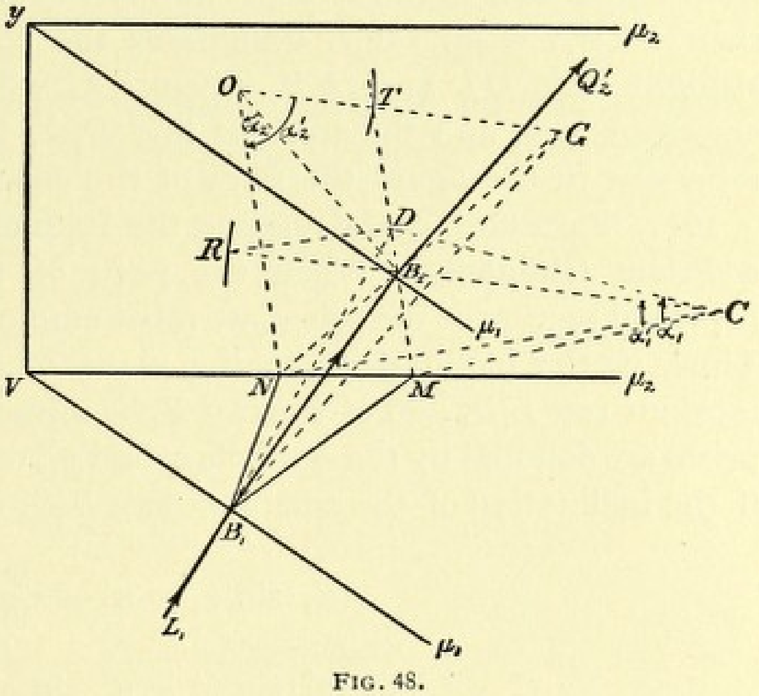


FIG. 48.

CONSTRUCTION OF THE PATH OF A RAY REFRACTED OBLIQUELY THROUGH A PRISM. *Vy* represents the refracting edge of the prism, and the plane of the paper is supposed to coincide with the second face of the prism.

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[1] See L. BURMESTER: Homocentrische Brechung des Lichtes durch das Prisma: *Zft. f. Math. u. Phys.*, xl. (1895), 65–90.

In the diagram, as drawn, the real distance of B_1N is supposed to be twice the length of B_1N in the diagram. With the point C as centre and with radius equal to $n'_1 \cdot CD/n_1$, describe, in the plane of the paper, the arc of a circle meeting the straight line drawn through D parallel to the straight line CM in the point R; and draw the straight line CR meeting MD in the point B_2; then $\angle MCD = \alpha_1$, $\angle MCB_2 = \alpha'_1$; and the straight line B_1B_2 will represent in the diagram the path of the ray within the prism.

The plane of incidence at the second face of the prism is the plane B_1B_2N, and if we revolve the triangle B_1NB_2 about NB_2 as axis until it comes into the plane of the paper, the point B_1 will arrive at a point O on the straight line NO perpendicular to NB_2, the length of ON being the real length of B_1N, and $\angle B_2ON = \alpha_2$. With the point O as centre and with radius $OT = n'_2 \cdot OB_2/n'_1$, describe in the plane of the paper the arc of a circle meeting the straight line drawn from B_2 parallel to the straight line NO in the point designated by T; then $\angle TON = \alpha'_2$. If G designates the point of intersection of the straight lines OT and NB_2, then B_1G will be the direction of the emergent ray, and the straight line $B_2Q'_2$ drawn parallel to B_1G will represent in the figure the path of the emergent ray.

101. Formulæ for Calculating the Path of a Ray Refracted through a Prism Obliquely. When the path of the ray does not lie in a principal section of the prism, we must employ the formulæ of §§ 32, 33. Thus, if the inclinations to the plane of a principal section of the incident ray L_1B_1 and of the ray B_1B_2 refracted at the first face of the prism are denoted by the symbols η_1 and η'_1, respectively; and, similarly, if the inclination of the emergent ray $B_2Q'_2$ is denoted by η'_2, we have:

$$n_1 \cdot \sin \eta_1 = n'_1 \cdot \sin \eta'_1,$$

$$n'_1 \cdot \sin \eta'_1 = n'_2 \cdot \sin \eta'_2;$$

and, hence,

$$n_1 \cdot \sin \eta_1 = n'_2 \cdot \sin \eta'_2. \tag{53}$$

If, as is usually the case, $n_1 = n'_2$, we shall have $\eta_1 = \eta'_2$; accordingly, *when the prism is surrounded on both sides by the same medium, the incident and emergent rays are equally inclined to the plane of the principal section.*

Moreover, if, as in § 33, the symbols γ_1, γ'_1 and γ_2, γ'_2 denote the angles which the normals to the two faces of the prism make with the projections in a principal section of the incident and refracted rays at the first and second faces, respectively, we have also:

$$\left.\begin{aligned} n_1\cdot\cos\eta_1\cdot\sin\gamma_1 &= n_1'\cdot\cos\eta_1'\cdot\sin\gamma_1', \\ n_1'\cdot\cos\eta_1'\cdot\sin\gamma_2 &= n_2'\cdot\cos\eta_2'\cdot\sin\gamma_2', \\ \gamma_2 &= \gamma_1' - \beta. \end{aligned}\right\} \tag{54}$$

These formulæ (53) and (54) enable us to determine completely the emergent ray corresponding to a given incident ray.

If the prism is surrounded by the same medium on both sides, and if we write $n = n_1'/n_1 = n_1'/n_2'$, the formulæ will be simplified as follows:

$$\left.\begin{aligned} \sin\eta_1 &= n\cdot\sin\eta_1', \\ \cos\eta_1\cdot\sin\gamma_1 &= n\cdot\cos\eta_1'\cdot\sin\gamma_1', \\ \gamma_2 &= \gamma_1' - \beta, \\ n\cdot\cos\eta_1'\cdot\sin\gamma_2 &= \cos\eta_1\cdot\sin\gamma_2'. \end{aligned}\right\} \tag{55}$$

102. Deviation (*D*) of Ray Obliquely Refracted through a Prism. If E denotes the deviation of the so-called "Projected Ray," that is, the angle between the projections, on the plane of a principal section, of the incident and emergent rays, then evidently:

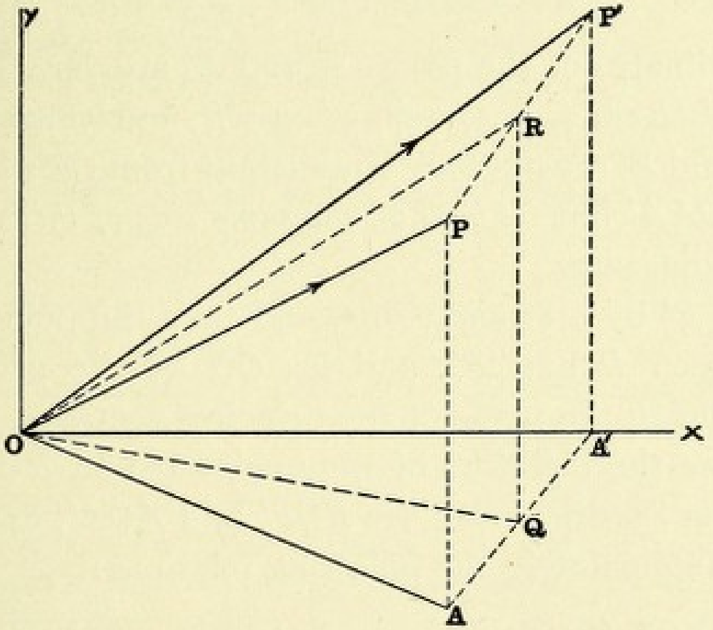

Fig. 49.
Deviation (*D*) of Ray Refracted Obliquely through a Prism.

$$E = \gamma_1 - \gamma_2' - \beta; \tag{56}$$

an equation which is precisely similar to the last of equations (25). We proceed now to determine the total deviation (D) of a ray obliquely refracted through a prism in terms of this angle E.

Let O (Fig. 49) designate the position of a point in the prism-edge Oy. From O draw two vectors OP and OP' to represent the directions of the incident and emergent rays, respectively; and take the points P and P' at equal distances from O, so that $OP = OP' = r$ (say). In the diagram the plane of the paper is the plane determined by the straight lines Oy and OP'. The principal section of the prism through the point O will be, therefore, a plane xOz perpendicular to the plane of the paper; and the line-segments OA and OA' are the projections in this plane of OP and OP', respectively. Thus, employing the symbols D and E to denote the angles above mentioned, we have:

$$\angle P'OP = D, \quad \angle A'OA = E.$$

Now if we assume also that the prism is surrounded by the same medium on both sides, then (see § 101)

$$\angle POA = \eta_1 = \eta_2' = \angle P'OA'.$$

If the y-co-ordinates of the points P and P' are denoted by y and y', respectively, then $\sin \eta_1 = y/r$, $\sin \eta_2' = y'/r$, and hence $y' = y$; so that the points P and P' are not only equidistant from the plane of the principal section OAA' but *lie both on the same side of this plane*, as represented in the diagram.

Join P and P' by a straight line, and take the point R in this line midway between P and P'; and let OQ be the projection of the straight line OR in the plane of the principal section. Evidently, OR and OQ will be the bisectors of the angles D and E at the common vertex O of the isosceles triangles POP' and AOA', respectively; and we can write immediately the following relations:

$$\sin\frac{D}{2} = \frac{P'R}{OP'} = \frac{A'Q}{r}, \quad \sin\frac{E}{2} = \frac{A'Q}{OA'}, \quad \cos\eta_1 = \frac{OA}{OP} = \frac{OA'}{r};$$

and hence finally:

$$\sin\frac{D}{2} = \sin\frac{E}{2}\cdot\cos\eta_1. \tag{57}$$

This relation, together with equation (56), enables us to compute the total deviation D of a ray obliquely refracted through a prism. It is obvious that *the angle D is always smaller than the angle E*, except in the limiting case when $D = E$.[1]

Corresponding to any given value of the angle η_1, there will be certain definite minimum values D_0 and E_0 of the angles denoted by D and E, respectively; which will be connected by the equation:

$$\sin\frac{D_0}{2} = \sin\frac{E_0}{2}\cdot\cos\eta_1. \tag{57a}$$

The conditions that the angle E shall be a minimum are given by the relations:

$$\gamma_1 = -\gamma_2{}' = \frac{E_0 + \beta}{2}, \quad \gamma_1{}' = -\gamma_2 = \frac{\beta}{2};$$

The usual formula given in all the text-books on Optics, and unfortunately given also in the first edition of this work, is:

$$\cos\frac{D}{2} = \cos\frac{E}{2}\cdot\cos\eta_1;$$

see, for example, R. S. HEATH: *A Treatise on Geometrical Optics* (Cambridge, 1887), Art. 29. This formula would be true, and the angle D would be greater than the angle E, as HEATH and nearly all the other writers state, provided the points P and P' were on *opposite* sides of the plane of the principal section. The author's attention was first directed to this widespread error, of which he had also been guilty, by Dr. H. S. UHLER, who published the correct formula and a complete discussion of it in a paper entitled "On the Deviation of Rays by Prisms" in *The Amer. Journ. of Science*, xxvii (1909), 223–228. Subsequently, the mistake was again pointed out in a letter to the author from Prof. G. F. C. SEARLE, F.R.S., of Cambridge, England, who writes that the correct formula was given by LARMOR, *Proc. Cambr. Phil. Soc.*, ix (1898), p. 108, and that although KAYSER in his *Handbuch der Spectroscopie* gives a bibliography which refers to LARMOR'S paper, he himself follows HEATH! Prof. SEARLE adds that the correct formula is given in HERMAN'S *Geometrical Optics* and also in MASCART'S *Traité d'optique* (Paris, 1889), i, 84. It would appear therefore that the latter writer was entitled to the priority in the matter; but although MASCART obtains formulæ (57) and (57*a*) in the text, he does not show (as UHLER does) that $\epsilon_0 < D_0$. UHLER'S paper in *The American Journal of Science* was written without knowledge of the fact that the correct formula had been obtained by others.

Since the above was written, Dr. UHLER has published another paper "On the Deviation Produced by Prisms" (*Amer. Journ. Sci.*, xxxv (1913), 389–423), in which the entire subject here under discussion is dealt with extensively and exhaustively. It is impossible here to do more than refer to this valuable and original contribution. Incidentally, it may be remarked that Dr. UHLER recalls the fact that A. BRAVAIS as long ago as 1845 appears to have discerned clearly the correct relations in the case of prism-deviation (see reference to BRAVAIS'S work in § 103).

which are derived exactly in the same way as the analogous equations (27) for the case of an actual ray traversing the prism symmetrically in the plane of a principal section were obtained; only, we must observe that here for the so-called "projected ray," instead of n, we have to introduce the "artificial" relative index of refraction (see § 33), viz.:

$$n_\eta = n\frac{\cos\eta_1'}{\cos\eta_1}, \quad \left(n = \frac{n_1'}{n_1} = \frac{n_1'}{n_2'}\right);$$

so that the angle E_0 will be determined from the formula:

$$n_\eta = \frac{\sin\dfrac{E_0+\beta}{2}}{\sin\dfrac{\beta}{2}};$$

which is entirely similar to the third of formulae (27), viz.:

$$n = \sin\frac{\epsilon_0+\beta}{2} : \sin\frac{\beta}{2};$$

where ϵ_0 denotes the angle of minimum deviation of a ray traversing a principal section of the prism. If we assume that $n > 1$, we find $n_\eta > n$; whence it is obvious that the angle E_0 (which is the projection on the plane of a principal section of the angle of minimum deviation D_0 of a ray refracted obliquely through the prism, corresponding to a given value of η_1) is always greater than the angle of minimum deviation ϵ_0 of a ray refracted through the prism in a principal section.

Since (as may be easily verified)

$$\cos\eta_1 = \sqrt{\frac{n^2-1}{n_\eta^2-1}},$$

we find:

$$\cos\eta_1 = \sqrt{\frac{\sin^2\dfrac{\epsilon_0+\beta}{2} - \sin^2\dfrac{\beta}{2}}{\sin^2\dfrac{E_0+\beta}{2} - \sin^2\dfrac{\beta}{2}}} = \sqrt{\frac{\sin\dfrac{\epsilon_0}{2}\cdot\sin\dfrac{\epsilon_0+2\beta}{2}}{\sin\dfrac{E_0}{2}\cdot\sin\dfrac{E_0+2\beta}{2}}};$$

and obviously, therefore, formula (57*a*) may be written:

$$\sin\frac{D_0}{2} = \sin\frac{\epsilon_0}{2}\sqrt{\frac{\sin\frac{E_0}{2}\cdot\sin\frac{\epsilon_0+2\beta}{2}}{\sin\frac{\epsilon_0}{2}\cdot\sin\frac{E_0+2\beta}{2}}}.$$

None of these angles can be greater than 180°, and it is easy to show that

$$\frac{\sin\frac{E_0}{2}}{\sin\frac{\epsilon_0}{2}} > \frac{\sin\frac{E_0+2\beta}{2}}{\sin\frac{\epsilon_0+2\beta}{2}},$$

and, hence, the expression under the radical in the last equation above must be greater than unity; whence it follows that

$$D_0 > \epsilon_0.$$

In general, therefore,

$$E_0 > D_0 > \epsilon_0;$$

and the following statement, as usually made in all the books, is correct:

Of all the rays which go through a prism, that one which, lying in a principal section, traverses the prism symmetrically will be the least deviated.

The case when $n < 1$ may be discussed exactly in the same way; cf. § 71.

103. The formulæ given in this article for the path of a ray obliquely refracted through a prism may properly be attributed to BRAVAIS,[1] although the same results, in a more general form, were afterwards derived by geometric methods by REUSCH[2] and CORNU[3] and, analytically, by STOKES[4] and HOORWEG.[5]

[1] A. BRAVAIS: Notice sur les parhelies qui sont situés à la même hauteur que le soleil: *Jour. de l'éc. polyt.*, xviii., cah. 30 (1845), 79. Mémoire sur les halos, etc.: *Journ. de l'éc. polyt.*, xviii., cah. 31 (1847), 27.

[2] E. REUSCH: Die Lehre von der Brechung und Farbenzerstreuung des Lichts an ebenen Flaechen und in Prismen in mehr synthetischer Form dargestellt: POGG. *Ann.*, cxvii. (1862), 241–284.

[3] A. CORNU: De la réfraction à travers un prisme suivant une loi quelconque: *Ann. éc. norm.*, (2) I. (1872), 255–257.

[4] G. G. STOKES: In a " Note " on a paper by TH. GRUBB: *Proc. Roy. Soc.*, xxii. (1874), 309.

[5] J. L. HOORWEG: Ueber den Gang der Lichtstrahlen durch ein Spectroscop: POGG. *Ann.*, cliv. (1875), 423–430.

See, also, A. ANDERSON: On the maximum deviation of a ray of light by a prism: *Camb. Proc.*, ix. (1896–'8), 195–197.

The explanation of the *curvature of the lines of the spectrum*, as observed through a prism-spectroscope, which appears to have been remarked for the first time in GEHLER's *Physikalisches Woerterbuch*, is to be found in the fact that the function denoted above by n_η depends on the inclination (η_1) of the incident ray to the principal section of the prism. BRAVAIS[1] derived a formula for the radius of curvature at the vertex of the image-line which is given in KAYSER's *Handbuch der Spectroscopie*, Bd. I (Leipzig, 1900), Art. 321; also, in F. LOEWE's treatise *Die Prismen und die Prismensysteme.*[2]

ART. 31. HOMOCENTRIC REFRACTION THROUGH A PRISM OF AN INFINITELY NARROW, HOMOCENTRIC BUNDLE OF OBLIQUELY INCIDENT RAYS.

104. We propose now to investigate the conditions that must be satisfied in order that to a narrow, homocentric bundle of obliquely incident rays refracted through a prism there shall correspond a homocentric bundle of emergent rays. The solution of this problem was given first by BURMESTER,[3] whose geometrical method is the one given here. An analytical deduction of the same results, based on HELMHOLTZ's formulæ for the passage of light through a prism as given in his *Handbuch der physiologischen Optik*, has been given by WILSING.[4]

When a ray of light is refracted through a prism, the plane of incidence at the first face and the plane of emergence at the second face will, in general, not be coincident; in fact, this will be the case only when the incident ray lies in the plane of a principal section of the prism, as we have seen. To a homocentric bundle of incident rays emanating from an Object-Point Σ_1 on the chief incident ray u_1 there corresponds within the prism an astigmatic bundle of refracted rays whose chief ray is designated by the symbol u'_1, and which, therefore, we may speak of as the "bundle u'_1". If the incidence-point of the chief ray at the first prism-face is designated by B_1, the II. Image-Plane of the astigmatic bundle u'_1 will be the plane of incidence $u_1B_1u'_1$, and the I. Image-Plane will be the plane which contains u'_1 and which is perpendicular to the plane $u_1B_1u'_1$.

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[1] A. BRAVAIS: Mémoire sur les halos, etc.: *Journ. de l'éc. polyt.*, xviii., cah. 31 (1847), 1–280.

[2] See *Die Theorie der optischen Instrumente:* Herausgegeben von M. VON ROHR, Bd. I (Berlin, 1904), p. 429.

In the same connection, see also the following:

L. DITSCHEINER: Ueber die Kruemmung von Spectrallinien: *Wien. Ber.*, li., II. (1865), 368–383. Notiz zur Theorie der Spectralapparate: POGG. *Ann.*, cxxix. (1866), 336–340.

J. L. HOORWEG: as cited above.

H. v. JETTMAR: Zur Strahlenbrechung im Prisma; Strahlengang und Bild von leuchtenden zur Prismenkante parallelen Geraden: 35. *Jahresb. ueber das k. k. Staatsgymn. im Bez. Wiens*, 1885.

J. v. HEPPERGER: Ueber Kruemmungsvermoegen und Dispersion von Prismen: *Wien. Ber.*, xcii., II. (1885), 261–300.

A. CROVA: Étude des aberrations des prismes et de leur influence sur les observations spectroscopiques: *Ann. chim. et phys.*, 5, xxii. (1881), 513–520.

W. H. M. CHRISTIE: Note on the curvature of lines in the dispersion spectrum, etc.: *Monthly Notices of the Roy. Astr. Soc.*, xxxiv. (1874), 263–5.

W. SIMMS: Note on a paper by Mr. CHRISTIE: *Monthly Not.*, xxxiv. (1874), 363–'4.

See also KAYSER's *Handbuch der Spectroscopie*, Bd. I (Leipzig, 1900), Arts. 260, 321, 322 and 323.

[3] L. BURMESTER: Homocentrische Brechung des Lichtes durch das Prisma: *Zft. f. Math. u. Phys.*, xl. (1895), 65–90.

[4] J. WILSING: Zur homocentrischen Brechung des Lichtes durch das Prisma: *Zft. f. Math. u. Phys.*, xl. (1895), 353–361.

On the other hand, let us consider an astigmatic bundle of rays within the prism whose chief ray may be designated as the ray v'_1, and which, therefore, we shall call the "bundle v'_1". Let B_2 designate the incidence-point of this chief ray v'_1 at the second prism-face. Moreover, let us assume that the bundle of emergent rays corresponding to the bundle v'_1 is a homocentric bundle of rays with its vertex at a point designated by Σ'_2. The II. Image-Plane of the astigmatic bundle v'_1 coincides with the plane of incidence $v'_1B_2\Sigma'_2$ of the ray v'_1 at the second face of the prism, and the I. Image-Plane of this bundle is the plane which contains the ray v'_1 and which is perpendicular to the plane $v'_1B_2\Sigma'_2$.

Now, if these two astigmatic bundles u'_1 and v'_1 within the prism are identical, then the point Σ'_2 is the homocentric Image-Point on the chief emergent ray u'_2 which corresponds to the homocentric Object-Point Σ_1 on the chief incident ray u_1. Now in order that these two astigmatic bundles of rays shall be identical, it is necessary, in the first place, that the I. and II. Image-Planes of the two bundles shall be coincident; which may happen in either of two ways: (1) The I. Image-Planes of the two astigmatic bundles of rays may be identical, and also the II. Image-Planes; in which case the chief rays will lie in the plane of a principal section of the prism; which was the case investigated in Art. 25; or (2) The I. Image-Plane of one bundle of rays may coincide with the II. Image-Plane of the other bundle, and this is the case that interests us at present. In this latter case, if also the I. Image-Point of one bundle of rays coincides with the II. Image-Point of the other bundle, and *vice versa*, the two astigmatic bundles of rays u'_1 and v'_1 will be identical (provided we neglect infinitesimals of the second order, as is here assumed). Therefore, in order that, corresponding to an Object-Point lying on a chief incident ray which is obliquely refracted through the prism, we shall have on the chief emergent ray a homocentric Image-Point, it is necessary, first of all, that *the planes of incidence and emergence shall be at right angles;* that is, if u_1, u'_2 designate the chief incident ray and the corresponding chief emergent ray, respectively, and if the straight line B_1B_2 represents the path of the chief ray from the first face of the prism to the second face, the two planes $u_1B_1B_2$ and $B_1B_2u'_2$ must be perpendicular.

105. In the accompanying diagram (Fig. 50) the refracting edge of the prism is represented by the vertical straight line Vy lying in the plane of the paper, which, as in the similar diagram (Fig. 48), is supposed to be the plane of the second face of the prism. From the point B_1 in the first face of the prism draw the straight line B_1M normal to this face and meeting the second face in the point designated by M and the straight line B_1N normal to the second face at the point designated by N; so that B_1MN will be the plane of the principal section of the prism which is passed through the point B_1. On the straight line MN as diameter, describe in the plane of the paper a circle, only half of which is shown in the figure; and in the circumference of this circle take any point B_2, and draw the straight lines MB_2, NB_2, B_1B_2. If the straight line B_1B_2 represents the path within the prism of the chief ray of a bundle of rays, to which corresponds the chief incident ray u_1B_1 and the chief emergent ray $B_2u'_2$, then B_1B_2M will be the plane of incidence at the first face, and B_1B_2N will be the plane of emergence at the second face; and these two planes will be at right angles to each other, according to the essential condition which was found in § 104 above.

In order to construct the chief incident ray u_1 and the chief emergent ray u'_2 corresponding to a ray B_1B_2 (or u'_1) within the prism, we proceed almost exactly as in § 100. First, we revolve the triangle B_1B_2M around the straight line MB_2 as axis until it comes into the plane of the paper; so that the point B_1 falls at a point C in the straight line NB_2 whose real distance from M will depend on the scale of the oblique parallel projection. In the figure as here drawn, the real length of B_1N is twice its length as actually shown. With the point C as centre, and with radius equal to $n_1 \cdot CB_2/n'_1$, describe in the plane of the paper the arc of a circle meeting the straight line drawn from B_2 parallel to the straight line MC in a point designated by E, and let D designate the point of intersection of the straight lines CE and MB_2; then the straight line B_1D will give the direction of the chief incident ray u_1 to which within the prism corresponds the ray B_1B_2.

Again, revolve the triangle B_1NB_2 around the straight line NB_2 as axis until it comes into the plane of the paper, and let the point designated by O be the impression in this plane of the point B_1, so that NO is the real length of the straight line B_1N, and $B_2O = B_2C$; and

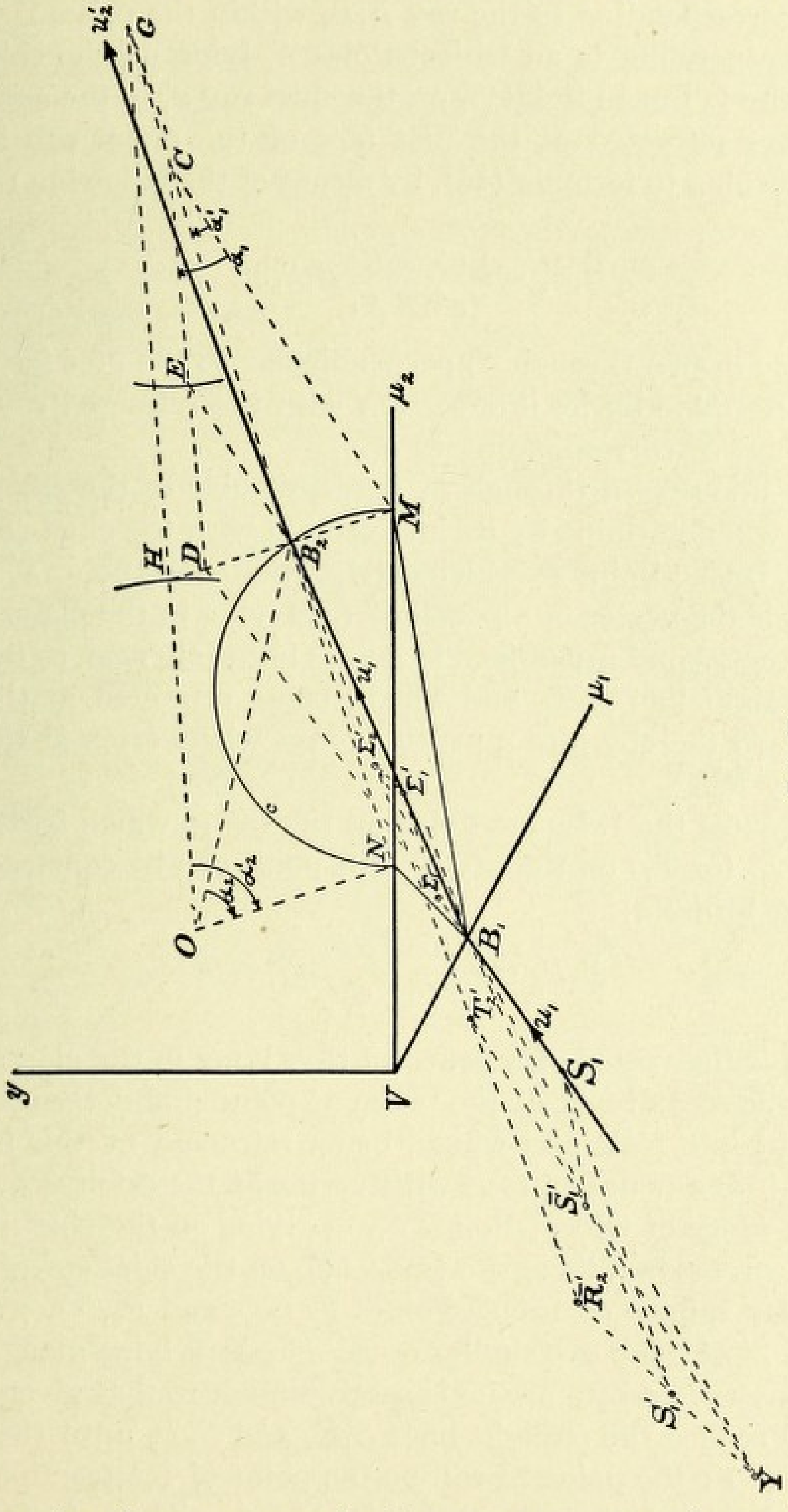
u'_2
G
C
L'_1
α'_1
α_1
E
μ_2
H
D
M
B_2
u'_1
μ_1
L
L'_1
N
E
B_1
O
α_1
α'_2
α_2
T'_2
u_1
S_1
y
V
S'_2
$\bar{S}'_1$
R'_2
S'_1
Y

Fig. 50. Homocentric Refraction through a Prism. Case when the chief ray is obliquely refracted through the prism.

with the point O as centre and with radius equal to $n'_2 \cdot OB_2/n'_1$, describe in the plane of the paper the arc of a circle meeting the straight line

MB_2 in the point designated by H, and draw the straight line OH meeting NB_2 in a point designated by G; then the straight line $B_2u'_2$ drawn parallel to the straight line B_1G_1 will show the path of the emergent ray u'_2 corresponding to the ray B_1B_2 within the prism.

106. Corresponding to an Object-Point S_1 lying on the chief incident ray u_1, the I. Image-Point S'_1 on the chief ray u'_1 of the astigmatic bundle of rays refracted at the first face of the prism can be constructed, according to formula (19), by means of the following relation:

$$B_1S'_1 = \frac{n'_1}{n_1}\frac{\cos^2\alpha'_1}{\cos^2\alpha_1}B_1S_1.$$

A straight line drawn through S_1 perpendicular to the first face of the prism will determine by its intersection with the refracted ray u'_1 the II. Image-Point $\overline{S}'_1$ corresponding to S_1.

A straight line drawn through S'_1 perpendicular to the second face of the prism will determine by its intersection with the chief emergent ray u'_2 the II. Image-Point $\overline{R}'_2$ which corresponds to the point S'_1 (or $\overline{R}'_1$). The point S'_1 is the vertex of a pencil of rays lying in the plane B_1B_2M which meet the second prism-face in points infinitely near to the point B_2 in the straight line MB_2, and which, being refracted at this face, are transformed thereby into a pencil of rays with vertex at the point designated by $\overline{R}'_2$.

If T'_2 designates the I. Image-Point on the ray u'_2 which corresponds to the point $\overline{S}'_1$ (or T'_1) on the ray u'_1, this point can be constructed by the following formula:

$$B_2T'_2 = \frac{n'_2}{n'_1}\frac{\cos^2\alpha'_2}{\cos^2\alpha_2}B_2\overline{S}'_1.$$

The point $\overline{S}'_1$ is the vertex of a pencil of rays lying in the plane B_1B_2N which meet the second face of the prism at points infinitely near to B_2 in the straight line NB_2, and which, being refracted at this face, are transformed into a pencil of rays with vertex at the point T'_2.

Thus, to a range of Object-Points $S_1, \cdots$ lying on the chief incident ray u_1 there correspond, therefore (see § 63), on the chief emergent ray u'_2 two similar ranges of Image-Points $T'_2, \cdots$ and $\overline{R}'_2, \cdots$; and the double-point Σ'_2 of these two similar ranges of points lying along the ray u'_2 may be constructed by a method exactly similar to that given in § 84, as follows: Produce the straight lines $S'_1\overline{R}'_2$ and $\overline{S}'_1T'_2$ until they intersect in a point Y; the point Σ'_2 will be the point of intersection of the straight line YB_1 with the emergent chief ray u'_2. Through this point Σ'_2 draw a straight line parallel to B_1N meeting u'_1 in the point Σ'_1;

then the straight line drawn through Σ_1' parallel to the straight line $S_1'S_1$ will determine by its intersection with the chief incident ray u_1 the Object-Point Σ_1 to which corresponds the Homocentric Image-Point Σ_2'.

The results of this investigation may be summarized as follows:

On every incident chief ray that meets the first face of the prism at a point B_1 *and that is refracted through the prism along a generating line of the conical surface* B_1c (where c designates the circle described on the straight line MN as diameter—see Fig. 50), *there is one single Object-Point to which on the emergent chief ray there corresponds a Homocentric Image-Point.*

107. The analytical expression for the position of this unique Object-Point Σ_1 on such an incident chief ray u_1 may be easily obtained as follows:

According to formulæ (19) and (21), we have:

$$B_1S_1' = \frac{n_1'}{n_1}\frac{\cos^2\alpha_1'}{\cos^2\alpha_1}\cdot B_1S_1; \quad B_1\overline{S}_1' = \frac{n_1'}{n_1}\cdot B_1S_1;$$

$$B_2T_2' = \frac{n_2'}{n_1'}\frac{\cos^2\alpha_2'}{\cos^2\alpha_2}\cdot B_2\overline{S}_1'; \quad B_2\overline{R}_2' = \frac{n_2'}{n_1'}\cdot B_2S_1'.$$

If we put

$$\delta_1 = B_1B_2 = B_1S_1' - B_2S_1' = B_1\overline{S}_1' - B_2\overline{S}_1',$$

we obtain:

$$B_2T_2' = \frac{n_2'}{n_1'}\frac{\cos^2\alpha_2'}{\cos^2\alpha_2}\left(\frac{n_1'}{n_1}\cdot B_1S_1 - \delta_1\right);$$

$$B_2\overline{R}_2' = \frac{n_2'}{n_1}\frac{\cos^2\alpha_1'}{\cos^2\alpha_1}\cdot B_1S_1 - \frac{n_2'\delta_1}{n_1'};$$

whence we find:

$$\overline{R}_2'T_2' = B_2T_2' - B_2\overline{R}_2' = \frac{n_2'}{n_1}\left(\frac{\cos^2\alpha_2'}{\cos^2\alpha_2} - \frac{\cos^2\alpha_1'}{\cos^2\alpha_1}\right)B_1S_1 - \frac{n_2'}{n_1'}\left(\frac{\cos^2\alpha_2'}{\cos^2\alpha_2} - 1\right)\delta_1.$$

Putting $\overline{R}_2'T_2' = 0$, we obtain finally:

$$B_1\Sigma_1 = \frac{n_1}{n_1'}\frac{\cos^2\alpha_1\,(\cos^2\alpha_2 - \cos^2\alpha_2')}{\cos^2\alpha_1'\cdot\cos^2\alpha_2 - \cos^2\alpha_1\cdot\cos^2\alpha_2'}\cdot\delta_1. \tag{58}$$

CHAPTER V.

REFLEXION AND REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE.

ART. 32. INTRODUCTION. DEFINITIONS, NOTATIONS, ETC.

108. In nearly all forms of optical apparatus the reflecting and refracting surfaces are spherical; for a plane may also be regarded as a spherical surface of infinite radius. In our diagrams the centre of the reflecting or refracting sphere will be designated by the letter C (Fig. 51). The straight line determined by this point C and another point M is called the *axis* of the spherical surface with respect to the point M, and the point A where the straight line MC meets the refracting (or reflecting) surface is called the *vertex* of the surface with respect to the point M. Evidently, a spherical surface will be symmetrical with respect to such an axis, and the plane of the diagram which contains the axis is a meridian section of the spherical surface.

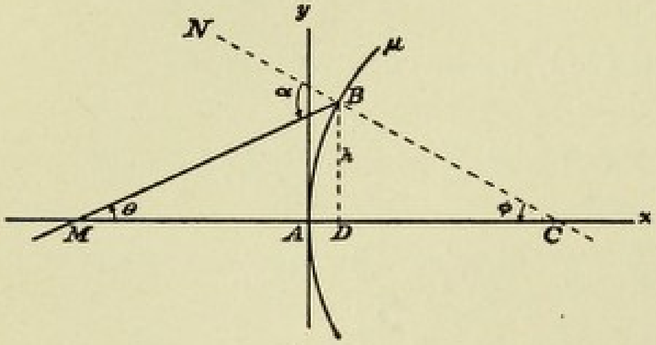

FIG. 51.

RAY INCIDENT ON A SPHERICAL SURFACE.

Consider now an incident ray lying in this plane, and crossing the axis, either really or virtually (see § 10), at the point M. If the point M is situated in front of the vertex A (that is, to the left of A), as in the figure, the intersection of an incident ray with the axis at the point M will be a "real" intersection; whereas if the point M lies beyond A (in the sense in which the incident light is propagated, which in our diagrams is represented always as being from left to right), the intersection of an incident ray with the axis at the point M will be a "virtual" intersection. If B designates the position of the point where the ray meets the spherical surface, and if on the straight line CB we take a point N in the medium of the incident ray, the angle of incidence, defined as in § 14, will be $\angle NBM = \alpha$. In the figure the plane of the paper represents the plane of incidence, and after reflexion or refraction at the point B, the path of the ray will still lie in this plane.

It will be convenient to take the vertex A of the spherical surface as the origin of a system of plane rectangular co-ordinates; the axis of the spherical surface, defined as the straight line AC, being taken as the x-axis, and the tangent to the surface at its vertex A, in the incidence-plane, being taken as the y-axis. *The positive direction of the x-axis is the same as the direction which light would pursue if this line were the path of an incident ray* (see § 26). The positive direction of the y-axis is found by rotating the positive half of the x-axis about the point A through an angle of 90° in a sense opposite to that of the motion of the hands of a clock; so that in our diagrams where the x-axis is represented as a horizontal line with its positive direction from left to right, the positive direction of the y-axis will be vertically upwards.

The abscissa of the centre C, which we shall call the *radius* of the spherical surface, will be denoted by the symbol r; thus, $AC = r$. The radius r is positive or negative according as the centre C lies beyond or in front of the vertex A; and according as the sign of r is positive or negative, the spherical surface is said to be "convex" or "concave".

From the incidence-point B draw BD perpendicular to the x-axis at the point D; the ordinate $h = DB$ is called the *incidence-height* of the ray which meets the spherical surface at the point B. It will be positive or negative according as the incidence-point B is above or below the x-axis.

The *slope* of the ray is the acute angle through which the x-axis has to be turned about the point M in order that it may coincide in position (but not necessarily in direction) with the rectilinear path of the ray. This angle will be denoted by the symbol θ; thus, in the figure $\angle AMB = \theta$. Here, as always in the case of angular magnitudes, counter-clockwise rotation is to be reckoned as positive. The sign of the angle θ may always be determined from the following relation:

$$\tan\theta = -\frac{h}{DM}. \tag{59}$$

The acute angle at the centre C of the spherical surface subtended by the arc AB will be denoted by the symbol φ. This angle is defined as the angle through which the radius drawn to the incidence-point B must be turned in order that the straight line BC may coincide with AC; thus, $\varphi = \angle BCA$. According to this definition, we shall have always $\sin\varphi = h/r$.

From the diagram, we derive at once the following important relation:

$$\alpha = \theta + \varphi. \tag{60}$$

109. From the diagram, also, we obtain easily the following relations:

$$BM = \frac{DM}{\cos\theta} = \frac{DC + CA + AM}{\cos\theta} = \frac{r(\cos\varphi - 1) + AM}{\cos\theta}.$$

In the special case when the point of incidence B is very near to the vertex A of the spherical surface, the angle of incidence α will be correspondingly small, as will be also the angles denoted by θ and φ. Now if these angles α, θ and φ are all so small that we may neglect the second and higher powers thereof, and write therefore in place of the sines of these angles the angles themselves and also put

$$\cos\alpha = \cos\theta = \cos\varphi = 1,$$

obviously, we obtain in this case $BM = AM$. Under these circumstances, the ray MB is called a **Paraxial Ray.**

A Paraxial Ray is a ray which proceeds very near to the axis of the Spherical Surface, which, therefore, meets this surface at a point very close to the vertex and at nearly normal incidence; the angles α, θ *and* φ *being all so small that we can neglect the second powers of these angles.*

The ray which proceeds along the x-axis is called the *axial ray.*

In this chapter, as well as in several chapters following, we shall be concerned with the special case of paraxial rays only; that is, we shall consider only such rays as proceed within a very narrow cylindrical region immediately surrounding the axis of the spherical surface which is also the axis of the cylinder. The only part of the spherical surface that will be utilized for reflexion or refraction will be the small zone which has the vertex A for its summit. We may imagine, therefore, that, physically speaking, the rest of the spherical surface is abolished entirely, or that it is rendered opaque and non-reflecting by being painted over with lamp-black; or we may suppose that a screen with a small circular opening is placed at right angles to the axis with the centre of the opening on the axis just in front of the vertex A of the spherical surface, so that only such rays as proceed through this opening and are incident on the spherical surface at points very close to the point A will undergo reflexion or refraction at the spherical surface.

I. REFLEXION OF PARAXIAL RAYS AT A SPHERICAL MIRROR.

ART. 33. CONJUGATE AXIAL POINTS IN THE CASE OF RELEXION OF PARAXIAL RAYS AT A SPHERICAL MIRROR.

110. In the accompanying diagrams (Figs. 52 and 53) the axis of the spherical mirror is shown by the straight line MC. The straight line MB represents an incident paraxial ray meeting the spherical reflecting surface at the point B. In Fig. 52 the spherical surface is convex, and in Fig. 53 it is concave. At the point B the ray is reflected in a direction BW, such that $\angle NBM = \angle WBN$, where BN is the normal to the surface at the point B drawn in the medium of the incident and reflected rays. Designating by M' the point where the reflected ray crosses the axis, either really (as in Fig. 53) or virtually (as in Fig. 52), let us denote by the symbols u and u' the abscissæ, with respect to the vertex A as origin, of the two points M and M' where the ray crosses the axis before and after reflexion, respectively; thus $AM = u$, $AM' = u'$. Also, as in § 108, $AC = r$.

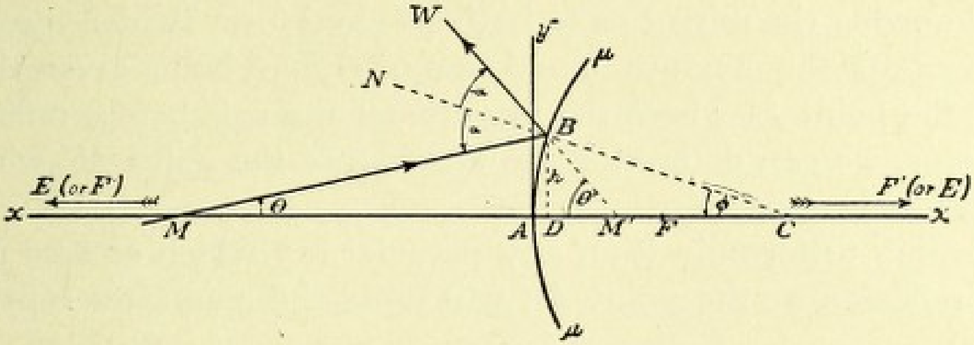

FIG. 52.

REFLEXION OF PARAXIAL RAYS AT A SPHERICAL MIRROR. Convex Mirror.

$AM = u$, $AM' = u'$, $AC = r$, $AF = r/2 = -f$, $FM = x$, $FM' = x'$,
$\angle AMB = \theta$, $\angle AM'B = \theta'$, $\angle BCA = \phi$, $\angle NBM = \angle WBN = \alpha$.

Since the normal BN bisects the (interior or exterior) angle at B of the triangle MBM', we have:

$$\frac{CM}{BM} = \frac{M'C}{BM'};$$

and since the point B is very close to A, this proportion may be written:

$$\frac{CM}{AM} = \frac{M'C}{AM'},$$

where, as we saw above (§ 109), magnitudes of the second order of smallness are neglected.[1] Now

$$CM = CA + AM = u - r, \quad M'C = M'A + AC = r - u';$$

and, therefore,

$$\frac{u-r}{u} = \frac{r-u'}{u'},$$

or

$$\frac{1}{u} + \frac{1}{u'} = \frac{2}{r}. \tag{61}$$

Thus, knowing the mirror as to both size and form (which means that we know both the magnitude and sign of r), and being given the position of the point M where the ray crosses the axis before reflexion at the mirror, we can determine the abscissa of the point M' where the ray crosses the axis after reflexion.

According to formula (61), any paraxial ray which crosses the axis before reflexion at the point M, will cross the axis after reflexion at the point M'. Thus, *a homocentric bundle of paraxial rays incident on a spherical mirror remains homocentric after reflexion at the mirror.* According as the sign of the abscissa u' is positive or negative, the point M' will lie to the right or left of the vertex A. In the former case the image at M' is a *virtual image* (Fig. 52), whereas in the latter case we have a *real image* at M' (Fig. 53); see § 44.

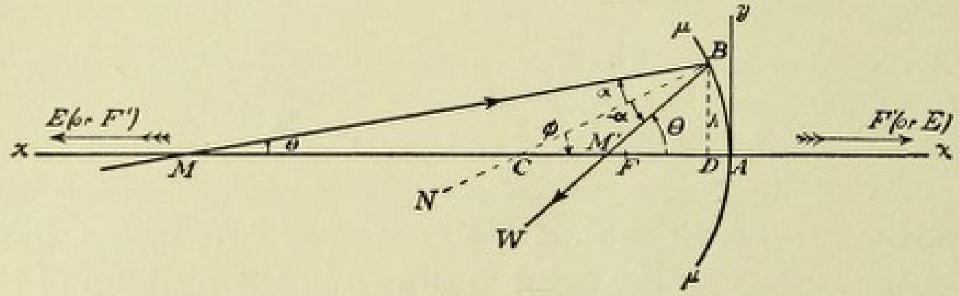

Fig. 53.

Reflexion of Paraxial Rays at a Spherical Mirror. Concave Mirror.

$AM = u$, $AM' = u'$, $AC = r$, $AF = r/2 = -f$, $FM = x$, $FM' = x'$,
$\angle AMB = \theta$, $\angle AM'B = \theta'$, $\angle BCA = \phi$, $\angle NBM = \angle WBN = \alpha$.

Moreover, since the formula is symmetrical with respect to u and u', so that the equation remains unaltered when we interchange the letters u and u', it follows that if M' is the image of M, then M will also be the image of M'; which is merely, of course, an illustration of the general law of Optics known as the Principle of the Reversibility of the Light-Path (§ 18). But the symmetry of the equation implies more than is involved in this principle. It indicates also that, in the case of Reflexion, Object-Space and Image-Space coincide completely: the paths of the incident and reflected rays both lying in front of the mirror; so that an Object-Ray and an Image-Ray are always so related that when either is regarded as the Object-Ray, the other will be the corresponding Image-Ray.

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[1] In writing this proportion, we must be careful that the two members of it shall have like signs. Thus, in the diagrams, as here drawn, CM and AM have the same directions, so that for these diagrams the ratio CM/AM is positive. Hence, if the ratio $M'C/AM'$ is equal to CM/AM, it must be positive also; that is, $M'C$ and AM' must likewise have the same directions.

The magnitudes denoted by u and u' are the radii of the incident and reflected wave-fronts at the moment when the disturbance arrives at the vertex A of the mirror; and hence the relation given by formula (61) may also be expressed as follows: *The algebraic sum of the curvatures of the incident and reflected waves at the instant when the disturbance arrives at the vertex of the spherical mirror is equal to twice the curvature of the mirror.*

The convergence of paraxial rays after reflexion (or refraction) at a spherical surface is said to be a "convergence of the second order"; which means that the second and higher powers of the incidence-angle α are neglected. When we neglect magnitudes of this order, the spherical surface will coincide with every surface of revolution which has the same curvature at the vertex; so that the formula (61) applies to the reflexion of paraxial rays at a surface of revolution of any form, where r denotes the radius of curvature at the vertex of a meridian section of the surface.

111. Since $CM : AM = M'C : AM'$, it follows that

$$\frac{CM}{CM'} : \frac{AM}{AM'} = -1,$$

that is, the anharmonic or double ratio of the four axial points C, A, M and M' is

$$(CAMM') = -1; \tag{62}$$

consequently, the points C, A, M, M' are a *harmonic range of points*, the Object-Point M and its Image-Point M' being harmonically separated by the centre C and the vertex A of the spherical mirror. Accordingly, we have the following simple **construction of the Image-Point** M' due to the reflexion at a spherical mirror of paraxial rays emanating originally from an axial Object-Point M:

On any straight line, supposed to represent the axis of the spherical mirror, take three points A, C, M (Fig. 54), ranged along the line in any order whatever; the letters A and C designating the positions of the vertex and centre, respectively, of the spherical mirror, and the letter M designating the position of the given axial Object-Point. On any other straight line drawn through the point M take two points H and J; and draw CH and AJ intersecting in a point P and AH and CJ intersecting in a point Q. The straight line connecting the points P and Q will meet the axis in the Image-Point M' conjugate to the Object-Point M. In making this construction a straight-edge is the only drawing-instrument that will be needed. The proof of the construction is obtained at once from the complete quadrangle $ACHJ$. If the points A and C in the diagram are interchanged, the points M and M' will evidently be a pair of conjugate points also with respect to this new spherical surface; thus, if the points M, M' are a pair of axial conjugate points with respect to a spherical mirror with its centre at C and its vertex at A, these same points will be conjugate to each other with respect to a spherical mirror of the opposite kind with its centre at A and its vertex at C.

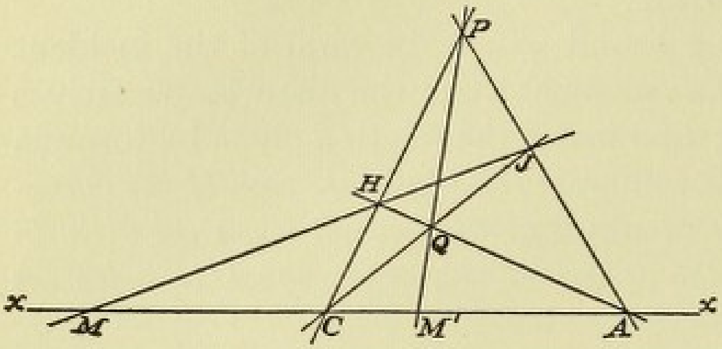

FIG. 54.

REFLEXION OF PARAXIAL RAYS AT A SPHERICAL MIRROR. Construction of Conjugate Axial Points M, M'. Centre of Mirror at C, vertex at A.

112. Focal Point and Focal Length of Spherical Mirror. In the special case when the axial Object-Point M coincides with the infinitely distant Object-Point E of the x-axis, the conjugate point M' will in this case be situated at a point E', such that

$$(CAEE') = \frac{AE'}{CE'} = -1:$$

that is,

$$AE' = E'C.$$

Hence, a cylindrical bundle of incident paraxial rays parallel to the axis will be transformed by reflexion at a spherical mirror into a homocentric bundle of rays with its vertex at a point E' lying midway between the vertex and centre of the mirror.

On the other hand, if the Image-Point M' coincides with the infinitely distant Image-Point F' of the x-axis, the corresponding Object-Point M will be situated on the axis at a point F, such that

$$(CAFF') = \frac{AF}{CF} = -1;$$

or

$$AF = FC;$$

that is, this point F, which is the vertex of a bundle of incident paraxial rays to which corresponds a cylindrical bundle of reflected rays all parallel to the axis, is also situated midway between the vertex and the centre of the spherical mirror. Thus, in the case of a spherical mirror the two points F and E' are coincident.

The points designated by F and E' are called the *Focal Points* of the optical system.

The *Focal Length* of a Spherical Mirror may be defined as the abscissa of the vertex A with respect to the Focal Point F; thus, if the Focal Length is denoted by the symbol f, we have:

$$FA = f = -r/2.$$

If the abscissæ, with respect to the Focal Point F, of the conjugate axial points M, M' are denoted by x, x', respectively; that is, if we put

$$FM = x, \quad FM' = x',$$

then we have at once:

$$u = x - f, \quad u' = x' - f,$$

and substituting these values in formula (61), we obtain:

$$xx' = f^2; \tag{63}$$

a most convenient and simple form of the abscissa-relation of conjugate axial points, which contains the whole theory of the reflexion of paraxial rays at a spherical mirror.

According as the Focal Length f is positive or negative, the mirror is convex or concave. Thus, in a concave mirror the Focal Point F lies in front of the mirror, so that incident paraxial rays parallel to the axis will be converged by reflexion at a concave mirror to a real focus at F; whereas in a convex mirror the Focal Point F lies beyond the mirror (to the right of the vertex A), so that a bundle of incident paraxial rays which are parallel to the axis will be transformed by reflexion at a convex mirror into a bundle of rays diverging as if they had come from a virtual focus at the point F.

Whether the mirror is convex or concave, and whether the bundle of incident rays is convergent or divergent, *the conjugate axial points M,*

M' lie always on the same side of the Focal Point of the Spherical Mirror; as is readily seen from formula (63).

ART. 34. EXTRA-AXIAL CONJUGATE POINTS AND THE LATERAL MAGNIFICATION IN THE CASE OF THE REFLEXION OF PARAXIAL RAYS AT A SPHERICAL MIRROR.

113. Graphical Method of Showing the Imagery by Paraxial Rays. Let M, M' designate the positions on the axis of a spherical mirror of a pair of conjugate points, constructed according to the method given in § 111; and connect both of these points by straight lines with a point V on the surface of the reflecting sphere. In the plane of these lines draw Ay tangent to the sphere at its vertex A, and let B and G designate the points where the straight lines MV, $M'V$ meet the straight line Ay. Also, join the point B with the point M' by a straight line. If the point V were very close to the vertex A, then the straight line MV would be the path of an incident paraxial ray proceeding from M, and the path of the corresponding reflected ray would be VM'. In this case, however, the points designated here by the letters V, B and G would all be so near together that, even when we cannot regard V as coincident with A, we can regard V, B and G as coincident with one another; and therefore we might take the straight line BM' as representing the path of the reflected ray.

In the construction of diagrams exhibiting the procedure of paraxial rays a practical difficulty is encountered due to the fact that, whereas in reality such rays are comprised within the very narrow cylindrical region immediately surrounding the axis of the spherical surface, it is obviously impossible to show them in this way in a figure, because we should have to take the dimensions of the figure at right angles to the axis so small that magnitudes of the second order of smallness in such directions would no longer be perceptible. On the other hand, if we were to represent these magnitudes as larger than they actually are, the relations which we have found above would no longer be true in the case of such lines; thus, for example, the rays in the drawing would not intersect in the places demanded by the formulæ.

In order to overcome this difficulty, REUSCH suggested a method of drawing these diagrams which has been very generally adopted, and which in large measure is entirely satisfactory. Without altering the dimensions parallel to the axis, the dimensions at right angles to the axis are all magnified in the same proportion. Thus, for example, if the ordinate $h = DB$ (Fig. 51) is a magnitude of the order $1/k$, it is shown in the figure magnified k times; whereas an ordinate of magnitude of the order $1/k^2$, that is, of the second order as compared with h, even in the magnified diagram would be shown as a magnitude of the order $1/k$; so that if k is infinite, such ordinates as h, which are of the first order of smallness, will be shown in the figure by lines of finite length, whereas magnitudes of the second order of smallness will disappear completely in the magnified diagram.

Of course, one effect of this lateral enlargement will be to misrepresent to some extent the relations of the lines and angles in the figure. Thus, for example, the circle in which the spherical surface is cut by the plane of a meridian section will thereby be transformed into an infinitely elongated ellipse with its major axis perpendicular to the axis of the spherical surface, that is, into a straight line Ay tangent to the circle at its vertex A. The minor axis of this ellipse remains unchanged and equal to the diameter of the circle, and, moreover, the centre of the ellipse remains at the centre C of the circle. The most apparent change will be in the angular magnitudes which will be completely altered. Thus, for example, every straight line drawn through the centre C really meets the circle normally, but in the distorted figure the axis will be the only one of such lines which meets normally the straight line which takes the place of the circle. Angles which in reality are equal will appear unequal, and *vice versa.* However—and this after all is the really essential matter—the *relative* dimensions of the ordinates and the *absolute* dimensions of the abscissæ will not be changed at all; and, therefore, lines which are really straight lines will appear as straight lines in the figure, and straight lines which are really parallel will be shown in the figure as parallel straight lines. The abscissa of the point of intersection of a pair of straight lines as it appears in the figure will be the real abscissa of this point.

In such a figure, therefore, any ray, no matter what slope it may have, nor how far it may be from the axis, is to be considered as a paraxial ray. The meridian section of the spherical surface will be represented in the figure by the straight line Ay (the y-axis), and the position of the centre C with respect to the vertex A will show whether the surface is convex or concave.

114. If we suppose that the axis of the spherical mirror is rotated about the centre C through a very small angle MCQ so that the axial point M moves along the infinitely small arc of a circle to a point Q, the conjugate axial point M' will likewise describe an infinitely small arc of a concentric circle, and will determine a point Q' on the straight line joining Q with C, such that if U designates the point where the straight line QC meets the spherical surface, the points Q, Q' will be harmonically separated (§ 111) by the points C, U; that is, $(CUQQ') = -1$. Thus, the point Q' is evidently the image-point conjugate to the Extra-Axial Object-Point Q. If the Object-Points lie on the element of a spherical surface which is concentric with the reflecting sphere, the corresponding Image-Points will likewise be found on an element of another concentric spherical surface, and any straight line going through the centre C will determine by its intersections with this pair of concentric surfaces, of radii CM and CM', a pair of conjugate points such as Q, Q'. If, as we assume here, the angle MCQ is infinitely small, the arcs MQ, $M'Q'$ may be regarded as very short straight lines perpendicular to the axis at M, M', respectively. Accordingly, on the supposition that the only rays concerned in the production of the image are such rays as meet the reflecting surface at very nearly normal incidence, the following conclusions may be drawn:

(1) *The image, in a spherical mirror, of a plane object perpendicular to the axis is likewise a plane perpendicular to the axis;* (2) *A straight line passing through the centre of the spherical mirror intersects a pair of such conjugate planes in a pair of conjugate points;* and (3) *To a homocentric bundle of incident paraxial rays proceeding from a point Q in a plane perpendicular to the axis of the spherical mirror there corresponds a homocentric bundle of reflected rays with its vertex Q' lying in the conjugate Image-Plane.*

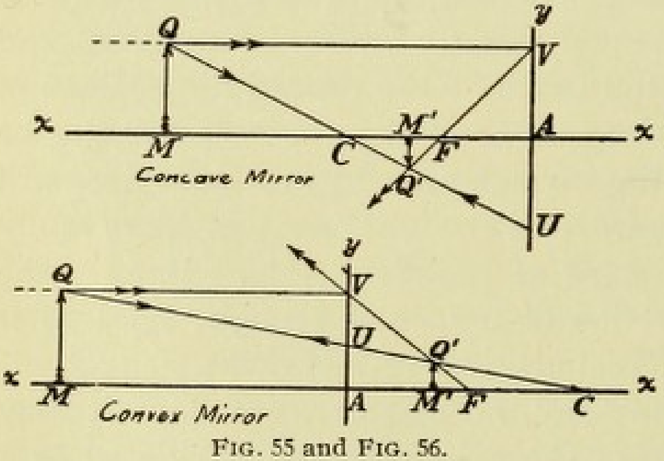

Fig. 55 and Fig. 56.

Reflexion of Paraxial Rays at a Spherical Mirror. Construction of point Q' conjugate to extra-axial Object-Point Q. In the diagrams the meridian section of the mirror is represented by a straight line Ay perpendicular to the axis of the mirror at the vertex A. The straight line $M'Q'$ perpendicular to the axis is the image of the straight line MQ also perpendicular to the axis.

115. In order to *construct the Image-Point Q' of the Extra-Axial Object-Point Q*, we have merely to find the point of intersection after reflexion at the spherical mirror of any two rays emanating originally from the point Q. The two diagrams (Figs. 55 and 56), which are drawn according to the method described above (§ 113), exhibit this construction in the case of both a concave and a convex mirror. Of the incident rays proceeding from Q it is convenient to select the following pair for this construction: the incident ray QC which proceeding towards the centre of the mirror C meets the spherical surface normally at the point U, whence it is reflected back along the same path, and the incident ray QV which proceeding parallel to the axis and meeting the mirror in the point designated by V is reflected at V along the straight line joining V with the Focal Point F. The Image-Point Q' will be the point of intersection of this pair of reflected rays. Moreover, having located the position of Q', we can draw QM and $Q'M'$ perpendicular to the axis at M and M', respectively; then $M'Q'$ will be the image of the straight line MQ perpendicular to the axis at M. In Fig. 55 the case is shown where the image $M'Q'$ is *real* and *inverted;* whereas in Fig. 56 the image $M'Q'$ is *virtual* and *erect.* Whether the image is real or virtual and erect or inverted will depend on the position of the object with respect to the mirror as well as on whether the mirror is convex or concave.

116. The Lateral Magnification. If the ordinates of the pair of extra-axial conjugate points Q, Q' are denoted by y, y', respectively, that is, if $MQ = y$, $M'Q' = y'$, the ratio y'/y is called the *Lateral Magnification* at the axial point M. This ratio will be denoted by Y; thus, $Y = y'/y$. The sign of this function Y indicates whether the image is erect or inverted; if Y is positive, as in Fig. 56, the image will be erect; whereas if Y is negative, as in Fig. 55, the image will be inverted. The absolute value of Y depends on the relative heights of the object and its image; it will be greater than, equal to, or less than, unity, according as the height of the image is greater than, equal to, or less than, that of the object. A very simple investigation shows how Y is a function of the abscissa of the axial point M. Since the triangles MCQ, $M'CQ'$ (Figs. 55 and 56) are similar,

$$M'Q' : MQ = M'C : MC;$$

and since

$$M'C = r - u', \quad MC = r - u,$$

and, by formula (61), we have:

$$\frac{r - u'}{r - u} = -\frac{u'}{u},$$

we derive the following formula for the Lateral Magnification in the case of a Spherical Mirror:

$$Y = \frac{y'}{y} = -\frac{u'}{u}. \tag{64}$$

Or, in case we wish to obtain Y as a function of the abscissa x

($x = FM$), we obtain from the diagrams directly:

$$\frac{M'Q'}{MQ} = \frac{M'Q'}{AV} = \frac{FM'}{FA},$$

and putting $FM' = x'$, $FA = f$, and using also formula (63), we have:

$$Y = \frac{y'}{y} = \frac{f}{x} = \frac{x'}{f}; \tag{65}$$

which of course is likewise easily deducible from (64).

Either of the two pair of formulæ (61) and (64) or (63) and (65) determine completely the Imagery in the case of the Reflexion of Paraxial Rays at a Spherical Mirror.

117. If the axial Object-Point M is supposed to travel along the axis of the spherical mirror, and if at the same time the point Q is supposed to travel with an equal velocity along a line parallel to the axis, the corresponding manœuvres of the image $M'Q'$ will be easily perceived by an inspection of the diagram (Fig. 57), which shows the case of a concave mirror with its Focal Point F in front of the mirror. Let us suppose that the Object moves from left to right starting from an infinite distance in front of the mirror. The numerals 1, 2, 3, etc., are used to designate a number of successive positions of the Object-Point Q, whereas the same numerals with primes show the correspond-

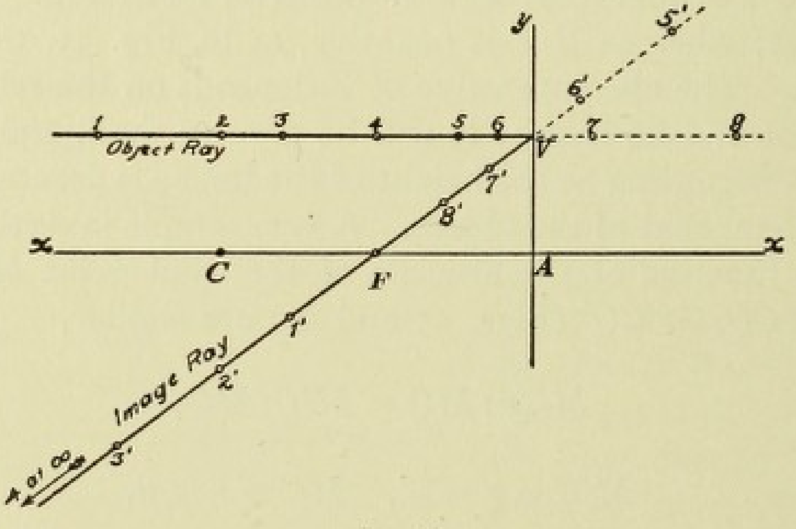

Fig. 57.

Reflexion of Paraxial Rays at a Spherical Mirror. The numerals 1, 2, 3, etc., ranged from left to right along a straight line parallel to the axis of the mirror indicate the successive positions of an object-point, and the numerals 1′, 2′, 3′, etc., show the corresponding positions of the image-point ranged along the straight line VF. The case shown in the figure is for a Concave Mirror. The straight lines 11′, 22′, 33′, etc., all intersect at the centre C of the mirror. If the object-point is virtual (as at 7 or 8), the image in a concave mirror will be real.

ing positions of the Image-Point Q'. Evidently, all the straight lines 11′, 22′, 33′, etc., will pass through the centre C of the mirror. So long as the Object MQ lies in front of the Focal Point F the image $M'Q'$ in the concave mirror is real and inverted. As MQ advances towards the centre C, the Image $M'Q'$ proceeds between F and C also towards the centre C, and Object and Image arrive together in the plane perpendicular to the axis at C, the Image being then of the same size as the Object, but inverted. As the Object proceeds past C towards F, the real and inverted Image proceeds in the opposite direction towards infinity; so that when the point Q arrives at the point marked 4 in the Focal Plane, the point Q' is the infinitely distant point of the straight line VF. As the Object continues its journey from the Focal Point F towards the vertex A of the mirror, the Image, which is now virtual and erect, travels from infinity towards the vertex A, and Object and Image arrive together at the vertex and coincide with each other there. If the Object proceeds beyond the vertex, we shall have then the case of a *virtual Object*, to which there corresponds a real erect Image lying between the vertex A and the Focal Point F. The Image, it will be observed, travels always in a direction opposite to that taken by the Object; which is a characteristic property of reflexion. Moreover, it will be noted that Object and Image lie always on the same side of the Focal Plane.

II. REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE.

ART. 35. CONJUGATE AXIAL POINTS IN THE CASE OF THE REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE.

118. In the diagrams (Figs. 58 and 59) the plane of the paper represents the meridian section of a spherical refracting surface separating two isotropic optical media of absolute refractive indices n and n'. In Fig. 58 the centre C lies in the second medium (n'), so that the spherical surface is convex; whereas in Fig. 59 the centre C lies in the first medium (n), and the spherical surface is concave. The axis of the refracting sphere is the straight line xx which joins the centre C with the vertex A. The letters in these figures have the same meanings as in the corresponding diagrams for the reflexion of paraxial rays at a spherical mirror.

An incident ray meeting the spherical surface will be refracted in a direction such that, if α and α' denote the angles of incidence and refraction, then, by the Law of Refraction:

$$n \cdot \sin \alpha = n' \cdot \sin \alpha'.$$

If M, M' designate the points where the ray crosses the axis before and after refraction at the spherical surface, and if BN is the normal to the spherical refracting surface at the incidence-point B drawn from B into the first medium, then in the figures:

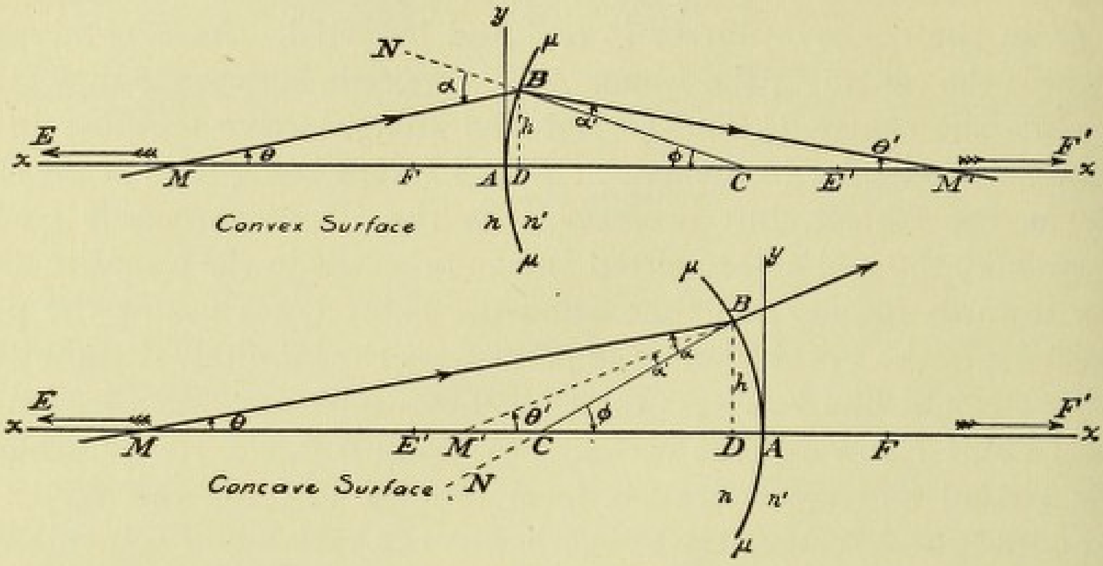

FIG. 58 and FIG. 59.

REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE SEPARATING TWO MEDIA OF INDICES n, n'.

$AM=u$, $AM'=u'$, $AC=r$, $FA=f$, $E'A=e'$, $FM=x$, $E'M'=x'$, $DB=h$,
$\angle NBM=\alpha$, $\angle CBM'=\alpha'$, $\angle AMB=\theta$, $\angle AM'B=\theta'$, $\angle BCA=\phi$.

$$\angle NBM = \alpha, \quad \angle CBM' = \alpha';$$

also if φ denotes the angle subtended at the centre C by the arc AB, then, according to the definition of φ given in § 108, $\angle BCA = \varphi$.

In the triangles MBC and $M'BC$ we have:

$$CM:BM = \sin\alpha:\sin\varphi, \quad CM':BM' = \sin\alpha':\sin\varphi,$$

and, hence, dividing one of these equations by the other, we obtain:

$$\frac{CM}{CM'}:\frac{BM}{BM'} = \frac{n'}{n}.$$

Since the incidence-point B is supposed to be so near A that we can neglect magnitudes of the second order of smallness, we may write A in this equation in place of B; and thus we obtain for the refraction of a paraxial ray at a spherical surface:

$$\frac{CM}{CM'}:\frac{AM}{AM'} = \frac{n'}{n},$$

or

$$(CAMM') = \frac{n'}{n}; \tag{66}$$

that is, *The Double (or Anharmonic) Ratio of the four axial points C, A, M, M' is constant, and equal to the relative index of refraction of the two media.*

Thus, for a given spherical refracting surface, the axial point M' corresponding to a given axial point M is a perfectly definite point, and accordingly we derive the following result:

To a homocentric bundle of incident paraxial rays with its vertex lying on the axis of the spherical refracting surface there corresponds also a homocentric bundle of refracted rays with its vertex lying on the axis.

Thus, if M designates the position of an axial Object-Point, its image produced by the refraction of paraxial rays at a spherical surface will be at a point M' on the axis. In Fig. 58 we have at M' a real image of the Object-Point M; whereas in Fig. 59 the image is virtual. The four points M, M', A and C may be ranged along the axis in any order whatever, depending on the form of the spherical refracting surface and on whether n is greater or less than n'. If the incident rays converge towards a point M lying on the axis beyond (or to the right of) the vertex A, the point M will be a virtual Object-Point; but in this case, as in all cases, the corresponding Image-Point M' can be found by formula (66).

Moreover, if $(CAMM') = n'/n$, then also $(CAM'M) = n/n'$. Thus, if a ray proceeding from an axial point M in the first medium crosses the axis after refraction at the spherical surface at the point M' in the second medium (see § 10), then also a ray proceeding from the point M' in the second medium will be refracted at the spherical surface so as to cross the axis at the point M. This is in accordance with the general Principle of the Reversibility of the Light-Path (§ 18). If M' is the image of M, M will be likewise the image of M'.

119. Construction of the Image-Point M' conjugate to the Axial Object-Point M. The following is a simple method of constructing the Image-Point M' corresponding to an Object-Point M lying on the axis of a spherical refracting surface. Through the centre C draw any straight line, and take on it two points so situated that their distances from C are in the ratio $n' : n$. Instead of drawing *any* straight line through C, it will be convenient to take this line perpendicular to the axis at C, as is done in Fig. 60. Let G, G' designate the positions on this line of two points whose distances from C are such that we have:

$CG : CG' = n' : n$. Through the vertex A of the spherical refracting surface draw the straight line Ay parallel to the straight line drawn through C; if this latter is perpendicular to the axis, the straight line

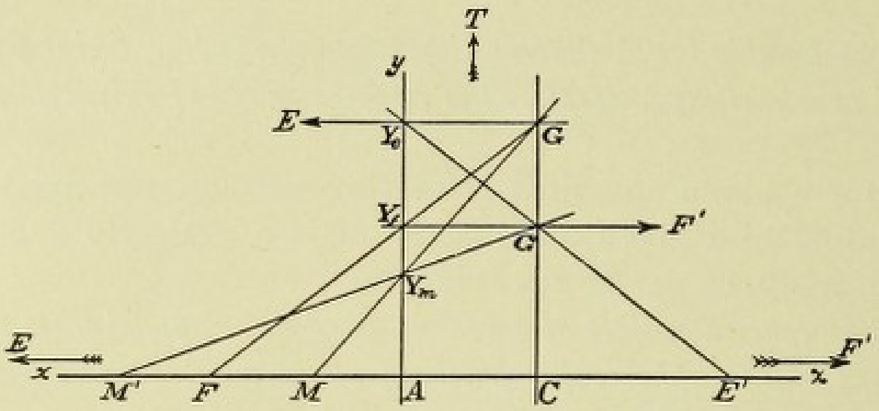

FIG. 60.

REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE. Construction of axial point M' conjugate to axial Object-point M. C is centre, A is vertex and xx axis of Spherical Refracting Surface. $CG'GT$ is perpendicular to axis; $CG : CG' = n' : n$. T is infinitely distant point of y-axis.

Ay will be tangent to the spherical surface at the point A. Join the axial Object-Point M with the point G by a straight line, and let Y_m designate the point where this line meets Ay; then the straight line joining the points Y_m and G' will meet the axis in the required point M'. For, evidently, the point Y_m is the centre of perspective of the two projective point-ranges C, A, M, M' and C, T, G, G', where T designates the infinitely distant point of the straight line which intersects the axis at C; and since, by construction,

$$(CTGG') = \frac{CG}{CG'} : \frac{TG}{TG'} = \frac{CG}{CG'} = \frac{n'}{n},$$

it follows that we must have:

$$(CAMM') = \frac{n'}{n},$$

in accordance with formula (66).

120. The Focal Points F and E' of a Spherical Refracting Surface. Evidently, the vertex A and the centre C of the spherical refracting surface are two self-corresponding points of the two projective ranges of Object-Points and Image-Points lying along the axis. Let us distinguish these two ranges of corresponding points by the letters x and x', and let E and F' designate the infinitely distant points of x and x', respectively. Thus E is the infinitely distant axial Object-Point and F' is the infinitely distant axial Image-Point. In order to find the Image-Point E' conjugate to the infinitely distant Object-Point E, we must draw through the point G (Fig. 60) a straight line parallel to the axis meeting the straight line Ay in the point designated by Y_e; and then the straight line Y_eG' will determine by its intersection with the axis the required point E'. Similarly, in order to find the position on the axis of the Object-Point F corresponding to the infinitely distant Image-Point F', we must draw through the point G' a straight line parallel to the axis and meeting the straight line Ay in a point Y_f; and then the straight line GY_f will determine by its intersection with the axis the required point F.

Thus, a paraxial ray which before refraction is parallel to the axis of the spherical surface will, after refraction, cross the axis (really or virtually) at the point designated by E'; and, also, a paraxial ray which before refraction crosses the axis (really or virtually) at the point designated by F will, after refraction, be parallel to the axis. These points F and E' are called the *Focal Points;* the point F is called the Focal-Point of the Object-Space or the Primary Focal Point, and the point E' is called the Focal Point of the Image-Space or the Secondary Focal Point. The two Focal Points of an optical system are always of the highest importance.

A mere inspection of the diagram (Fig. 60) shows that the Focal Points F and E' of a spherical refracting surface are situated so that

$$FA = CE', \quad E'A = CF; \tag{67}$$

and, hence, we have the following rule:

The Focal Points of a Spherical Refracting Surface are so situated on the axis that the step from one of them to the vertex A is identical with the step from the centre C to the other one.

This result may also be stated in a different way; for, since

$$FA = CE' = CA + AE',$$

we have also the following relation:

$$AF + AE' = AC; \tag{68}$$

that is, *The algebraic sum of the distances of the Focal Points from the vertex of the spherical refracting surface is always equal to the distance of the centre from the vertex.*

Another useful relation, obtained from the two similar triangles $AY_f F$ and $AY_e E'$ is the proportion:

$$\frac{FA}{AE'} = \frac{CG'}{CG},$$

or

$$\frac{AE'}{AF} = -\frac{n'}{n}; \tag{69}$$

which may be put in words as follows: *The two Focal Points F and E′ of a spherical refracting surface lie on opposite sides of the vertex, and at distances from it which are in the ratio n : n′.*

The answer to the question, Which of the two Focal Points lies in the first medium, and which in the second medium? will depend on each of two things, viz.: (1) Whether the spherical surface is convex or concave, and (2) Whether n' is greater or less than n. Thus, for example, if the rays are refracted from air to glass ($n'/n = 3/2$), we find from formulæ (68) and (69) $AF = 2CA$, $AE' = 3AC$; so that, starting at the vertex A and taking the step CA twice, we can locate the Primary Focal Point F, and returning to the vertex A and taking the step AC three times, we can locate the Secondary Focal Point E'. The two diagrams I and II (Fig. 61) show the positions of the Focal Points in the case when the rays are refracted from air to glass at a convex and at a concave spherical surface. It will be seen that for this case the Primary Focal Point of the concave surface lies in the second medium (virtual focus), whereas the Primary Focal Point of the convex surface lies in the first medium (real focus). On the other hand, in case the rays are refracted from glass to air ($n'/n = 2/3$), we have $AF = 3AC$, $AE' = 2CA$, and now the Primary Focal Point of a convex spherical refracting surface will lie in the second medium and the Primary Focal Point of a concave surface will lie in the first medium, as is shown in the diagrams III and IV (Fig. 61).

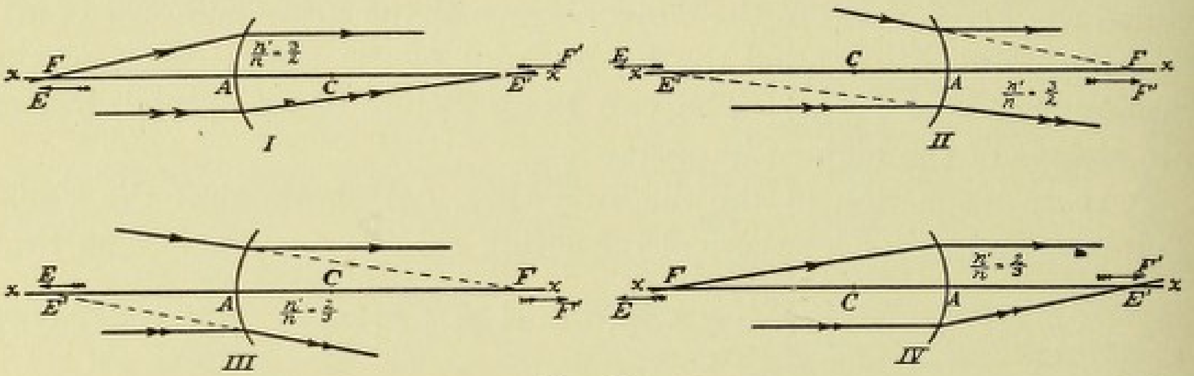

FIG. 61.

REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE. Construction of the Focal Points F and E'. In I and II the rays are refracted from air to glass. In III and IV the rays are refracted from glass to air. In I and III spherical refracting surface is convex. In II and IV spherical refracting surface is concave. In I and IV incident rays parallel to the axis are converged to a real focus at E'; whereas in II and III E' is a virtual focus.

ART. 36. REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE. EXTRA-AXIAL CONJUGATE POINTS. CONJUGATE PLANES. THE FOCAL PLANES AND THE FOCAL LENGTHS.

121. To an Object-Point Q lying not on the axis, but very near to it, evidently there will correspond an Image-Point Q' lying on the straight line joining Q with the centre C of the spherical refracting surface, the position of which is determined by the equation

$$(CUQQ') = n'/n,$$

where U designates the point where the self-corresponding ray QQ' meets the spherical surface. Employing here exactly the same reasoning as was used in § 114 in the similar case of Reflexion at a Spherical Mirror, we may copy *verbatim* the results which were obtained there, merely changing the words "mirror", "reflexion", etc., to adapt the statements to the case of refraction at a spherical surface. Thus:

(1) *The image of a plane object perpendicular to the axis of a spherical refracting surface is likewise a plane perpendicular to the axis;* (2) *A straight line drawn through the centre of the spherical refracting surface will intersect a pair of such conjugate planes in a pair of conjugate points;* and (3) *To a homocentric bundle of incident paraxial rays proceeding from a point Q in a plane perpendicular to the axis of the spherical refracting surface there corresponds a homocentric bundle of refracted rays with its vertex Q' lying in the conjugate Image-Plane.*

122. The Construction of the Image-Point Q' Corresponding to the Extra-Axial Object-Point Q may be performed also by a process precisely similar to that used in § 115. Thus, in the diagrams (Figs. 62 and 63), which are drawn according to the plan explained in § 113, the points designated by the letters A and C represent the vertex and centre, respectively, of the spherical refracting surface. In Fig. 62 the surface is convex, and in Fig. 63 it is concave. If the positions of the Focal Points F and E' are not assigned, they can be determined directly by the relations given in formulæ (68) and (69). Both of the diagrams show the case when n' is greater than n.

The incident ray proceeding from the point Q towards the centre C will meet the spherical surface normally and will continue its route into the second medium without change of direction. Thus, as was stated also above, the corresponding point Q' must lie on the straight line joining Q with C. To the incident ray QV proceeding from the Object-Point Q parallel to the axis and meeting the straight line Ay in the point V there corresponds a refracted ray which passes (really or virtually) through the secondary Focal Point E'. Thus, the Image-

Point Q' will be at the point of intersection of the straight lines QC and VE'.

The intersection of any pair of refracted rays emanating originally from the Object-Point Q will determine the position of the Image-Point Q'. Thus, for example, instead of one of those used above, we might have employed the ray which proceeding from the Object-Point Q towards the primary Focal Point F and meeting the straight line Ay in the point designated in the diagrams by W is refracted parallel to the axis of the spherical surface.

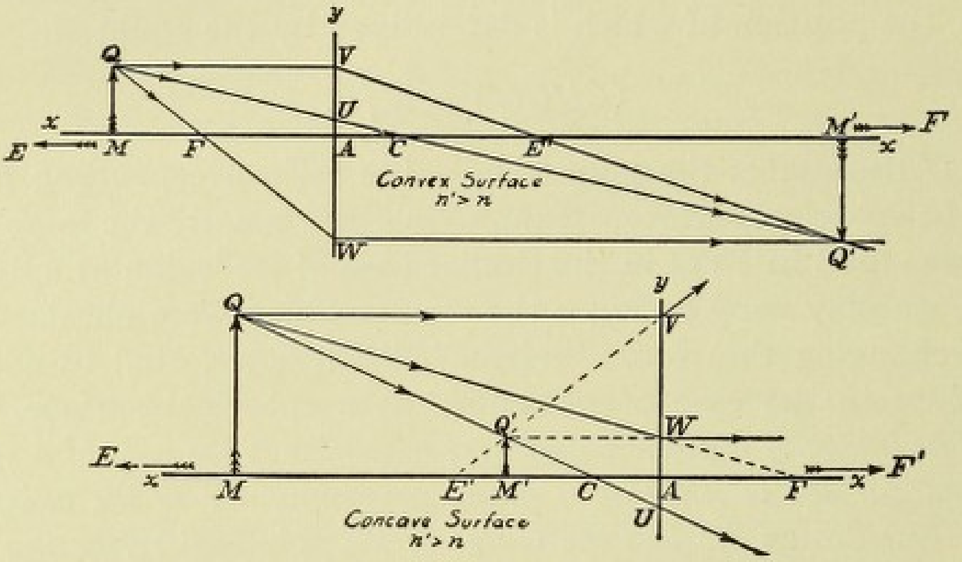

FIG. 62 and FIG. 63.

REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE. Construction of Image-Point Q' corresponding to extra-axial Object-Point Q. The points A and C designate the positions of the vertex and centre of the spherical refracting surface, and F and E' designate the positions of the Focal Points. In Fig. 62 the surface is convex, in Fig. 63 it is concave; for both diagrams $n' > n$. In Fig. 62 $M'Q'$ is a real, inverted image of MQ; whereas in Fig. 63 the image is virtual and erect.

If M, M' designate the feet of the perpendiculars let fall from Q, Q', respectively, on the axis, then $M'Q'$ will be the image, by paraxial rays, of the infinitely small straight line MQ. In Fig. 62 this image is real and inverted, whereas in Fig. 63 it is virtual and erect.

123. The Focal Planes of a Spherical Refracting Surface. If the Object-Point Q is the infinitely distant point of the straight line QC (Fig. 64), it will be a point of the infinitely distant plane of the Object-Space to which is conjugate a plane perpendicular to the axis at the Focal Point E' of the Image-Space. This plane is called the Focal Plane of the Image-Space or the Secondary Focal Plane. Its trace in the plane of the paper (which shows a meridian section of the spherical surface) is the straight line e' which we may call the secondary Focal Line. Thus, we can say:

To a bundle of parallel incident paraxial rays there corresponds a *homocentric bundle of refracted rays with its vertex lying in the secondary focal plane of the spherical refracting surface.*

Similarly, the plane perpendicular to the axis at the primary Focal Point F is called the Focal Plane of the Object-Space or the Primary Focal Plane, and its trace in the plane of the paper (Fig. 64) is the straight line f, which we may call the Primary Focal Line in the plane of this meridian section. The Image-Plane conjugate to the Primary Focal Plane is the infinitely distant plane of the Image-Space; and, hence, if the Object-Point Q lies in the Primary Focal Plane, the corresponding Image-Point Q' will be the infinitely distant point of the straight line QC. Thus:

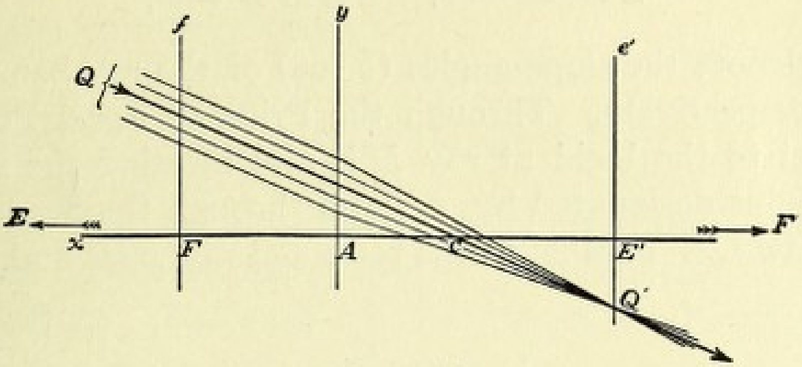

FIG. 64 (*a*).

REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE. Incident Parallel Rays intersect after refraction in a point of the focal plane of the Image-Space, the trace of which in the plane of the paper is the focal line e'.

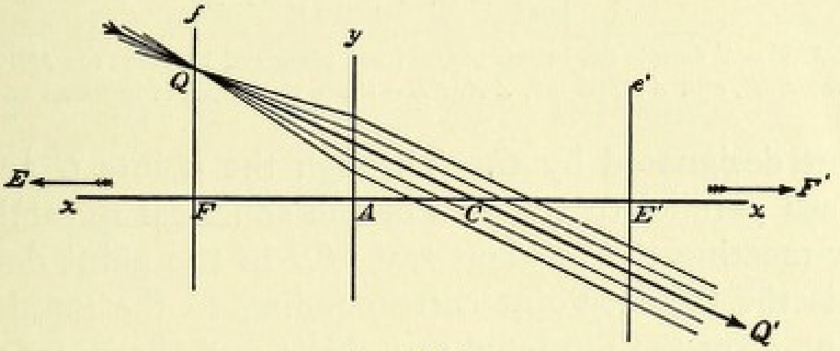

FIG. 64 (*b*).

REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE. Incident Rays emanating from a point of the Focal Plane of the Object-Space (the trace of which in the plane of the paper is the Focal Line f) are made parallel by refraction.

To a homocentric bundle of incident paraxial rays, with its vertex lying in the Primary Focal Plane of the spherical refracting surface, there corresponds a bundle of parallel refracted rays.

124. The Focal Lengths f and e' of a Spherical Refracting Surface. In Fig. 65 the points designated by M, M' are the points where a paraxial ray crosses the axis, before and after refraction, respectively, at a spherical surface, and the point B is the incidence-point of this ray. The vertex of the spherical surface is at the point marked A, and the Focal Points are at F and E'. Let

$$\angle AMB = \theta, \quad \angle AM'B = \theta',$$

where θ, θ' denote the slope-angles (§ 108) of the ray before and after refraction, respectively. Through the Primary Focal Point F draw FK' parallel to the incident ray MB and meeting the straight line Ay in the point designated by K', and through the Secondary Focal Point E' draw GE' parallel to the refracted ray BM' and meeting Ay in the point designated by G. Through the points G and K' draw straight lines parallel to the axis of the spherical refracting surface; the former meeting the incident ray MB in the point designated by R, which is the Object-Point corresponding to the infinitely distant Image-Point R' of the refracted ray BM'; and the latter meeting the refracted ray BM' in the point S', which is the Image-Point corresponding to the infinitely distant Object-Point S of the incident ray MB. The point R will lie in the Primary Focal Plane, and the point S' will lie in the Secondary Focal Plane. Let us put

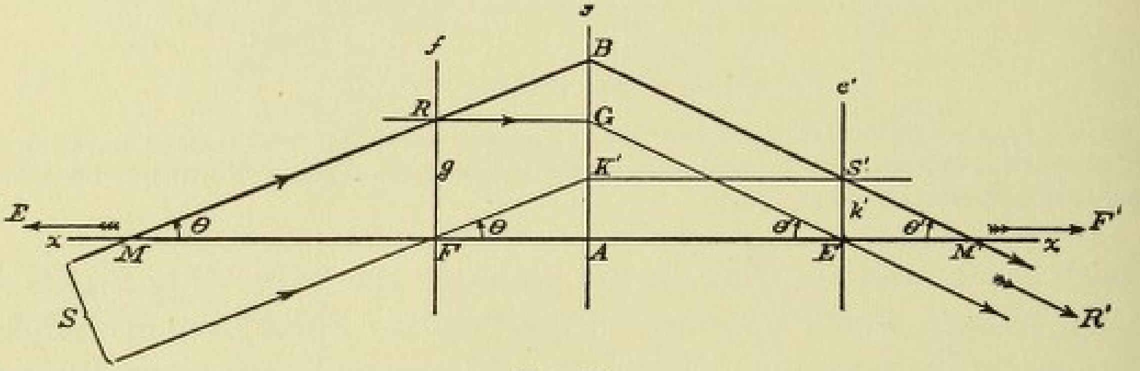

FIG. 65.

REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE. The focal lengths of the spherical refracting surface are:

$$f = \frac{k'}{\tan\theta} = FA, \quad e' = \frac{g}{\tan\theta'} = E'A,$$

where $FR = g$, $E'S' = k'$ denote the intercepts on focal planes of incident ray MB and corresponding refracted ray BM', and $\angle AMB = \theta$, $\angle AM'B = \theta'$ are the slopes of incident and refracted rays.

$$FR = AG = g, \quad E'S' = AK' = k'.$$

Evidently, we have then the following relations:

$$\frac{g}{\tan\theta'} = E'A, \quad \frac{k'}{\tan\theta} = FA;$$

so that whatever be the slopes of the incident and refracted rays, the intercepts g and k' will always be such that the above ratios have constant values. If we denote these constant values by f and e', that is, if we put

$$FA = f, \quad E'A = e',$$

these equations can be written:

$$\frac{k'}{\tan\theta} = f, \quad \frac{g}{\tan\theta'} = e'. \tag{70}$$

The constants denoted here by the symbols f and e' are called the *Primary and Secondary Focal Lengths*, respectively, of the spherical refracting surface. The proper definitions of the Focal Lengths (see § 178) are given by formulæ (70); thus:

The Primary Focal Length (f) *is equal to the quotient of the distance from the optical axis of the point where a refracted ray crosses the Secondary Focal Plane by the tangent of the slope-angle of the corresponding incident ray;* and similarly:

The Secondary Focal Length (e') *is equal to the quotient of the distance from the optical axis of the point where an incident ray crosses the Primary Focal Plane by the tangent of the slope-angle of the corresponding refracted ray.*

In the special case where the optical system consists of a single spherical refracting surface, the Focal Lengths may also be defined as follows:

The Focal Lengths of a Spherical Refracting Surface are equal to the abscissæ of the vertex A with respect to each of the Focal Points; that is, $f = FA$, $e' = E'A$, as above stated.

The Focal Lengths f and e' of a spherical refracting surface may easily be expressed in terms of the radius $r = AC$. Thus, in Fig. 60, from the two pairs of similar triangles AFY_f, $G'Y_fG$ and $E'AY_e$, $G'Y_fY_e$ we obtain the following proportions:

$$FA : Y_fG' = AY_f : G'G, \quad E'A : G'Y_f = AY_e : Y_fY_e;$$

and since

$$YG' = AC = r, \quad AY = G'G = CG - CG', \quad CG : CG' = n' : n,$$

we derive immediately the following formulæ for the magnitudes of the Focal Lengths in terms of the radius r:

$$f = \frac{n}{n' - n}\,r, \quad e' = -\frac{n'}{n' - n}\,r; \tag{71}$$

whence also we find:

$$n'f + ne' = 0, \tag{72}$$

which is equivalent to formula (69); and also:

$$f + e' + r = 0; \tag{73}$$

which is equivalent to formula (68).

ART. 37. THE IMAGE-EQUATIONS IN THE CASE OF THE REFRACTION OF PARAXIAL RAYS AT A SPHERICAL SURFACE.

125. The Abscissa-Equation in Terms of the Constants n, n' and r. If the vertex A of the spherical refracting surface is taken as origin of a system of rectangular axes whose x-axis is the optical axis determined by the centre C and the vertex A, the co-ordinates of an Object-Point Q may be denoted by u, y and of the corresponding Image-Point Q' by u', y'; thus:

$$AM = u, \quad AM' = u', \quad MQ = y, \quad M'Q' = y'.$$

The problem is to determine u', y' in terms of u, y.

Since

$$CM = CA + AM = u - r, \quad CM' = CA + AM' = u' - r,$$

equation (66) may be written in the following form:

$$\frac{u-r}{u'-r} : \frac{u}{u'} = \frac{n'}{n};$$

or, finally:

$$\frac{n'}{u'} - \frac{n}{u} = \frac{n'-n}{r}. \tag{74}$$

To every value of u comprised between $u = -\infty$ and $u = +\infty$, we obtain by this equation a corresponding value of the abscissa u'; thus, *to every axial Object-Point M there corresponds one, and only one, axial Image-Point M'*. This linear equation connecting the abscissæ of conjugate axial points in the case of the refraction of paraxial rays at a spherical surface is one of the most important formulæ of Geometrical Optics. It is entirely independent of the special law of refraction known as SNELL's Law; for if the angles of incidence and refraction α, α' are connected by any equation of the form $f(\alpha, \alpha') = 0$, wherein it is assumed that the angles denoted by α, α' are small, it is easy to show that the limiting value of the ratio α/α' will be a constant which may be denoted by n'/n; in which case we shall derive formula (74) as the most general expression of the relation between conjugate points of any paraxial ray which passes through the centre C of the spherical surface. In a supplement to this chapter it will be shown that this equation is the analytical expression of *Central Collineation in a Plane.*

126. The so-called Zero-Invariant. If according to the convenient method of notation, introduced by ABBE, we denote the difference of the values of an expression before and after refraction at a spherical surface by the symbol Δ written before the expression, formula (74) may be written also in the following abbreviated form:

$$\Delta\frac{n}{u} = \frac{1}{r}\Delta n. \tag{75}$$

The magnitude

$$J = n\left(\frac{1}{r} - \frac{1}{u}\right) = n'\left(\frac{1}{r} - \frac{1}{u'}\right), \tag{76}$$

which has the same value before and after refraction at the spherical surface, is called the "*Zero-Invariant*" or the invariant in the case of the refraction of paraxial rays at a spherical surface. This magnitude denoted here by J plays an important part in the Theory of Spherical Aberrations, and the following formulæ, all easily derived from (76), will be found useful in the investigations of that theory. For example, we obtain:

$$\Delta\frac{1}{u} = -J\,\Delta\frac{1}{n}; \tag{77}$$

and, also:

$$\Delta\frac{1}{nu} = \frac{1}{r}\Delta\frac{1}{n} - J\,\Delta\frac{1}{n^2}. \tag{78}$$

Again, we find:

$$\Delta\frac{1}{u^2} = J^2\Delta\frac{1}{n^2} - \frac{2J}{r}\Delta\frac{1}{n}. \tag{79}$$

Combining formulæ (77) and (79), we obtain:

$$\Delta\frac{1}{u^2} = J^2\Delta\frac{1}{n^2} + \frac{2}{r}\Delta\frac{1}{u}; \tag{80}$$

and combining formulæ (78) and (79):

$$J\Delta\frac{1}{nu} = -\frac{J}{r}\Delta\frac{1}{n} - \Delta\frac{1}{u^2}. \tag{81}$$

Moreover, if θ, θ' denote the slopes of a ray before and after refraction at a spherical surface, and if α, α' denote the angles of incidence and refraction, and, finally, if φ denotes the central angle ($\varphi = \angle BCA$, Fig. 51), then, as in formula (60):

$$\alpha - \theta = \alpha' - \theta' = \varphi.$$

In the case of Paraxial Rays where these angles are all so small that we may neglect powers above the first, we have (see § 108):

$$\theta = -\frac{h}{u}, \quad \theta' = -\frac{h}{u'}, \quad \varphi = \frac{h}{r}, \tag{82}$$

where $h = DB$ (Figs. 58 and 59) denotes the incidence-height of the ray. From these relations we obtain easily:

$$\alpha = \frac{hJ}{n}, \quad \alpha' = \frac{hJ}{n'}, \tag{83}$$

127. The Lateral Magnification. The ratio $Y = y'/y$ is called the Lateral Magnification of the spherical refracting surface with respect to the axial Object-Point M. Referring to Figs. 62 and 63, we see that we have the proportion:

$$M'Q' : MQ = CM' : CM,$$

and, consequently:

$$\frac{y'}{y} = \frac{u' - r}{u - r}.$$

This equation, together with formula (74), enables us to write the transformation-formulæ between Object-Space and Image-Space as follows:

$$u' = \frac{n'ru}{(n' - n)\,u + nr}, \quad y' = \frac{nry}{(n' - n)\,u + nr}; \tag{84}$$

whereby, being given the co-ordinates u, y of the Object-Point Q, we can find the co-ordinates u', y' of the corresponding Image-Point Q'.

The formula for the Lateral Magnification Y may also be written as follows:

$$Y = \frac{y'}{y} = \frac{nu'}{n'u}; \tag{85}$$

whence we see that the Lateral Magnification Y is a function of the abscissa u, and that it is independent of the absolute magnitude of the ordinate y. For a given pair of conjugate planes at right angles to the axis of a spherical refracting surface, the ratio denoted by Y is constant, but it is different for different pairs of conjugate planes.

128. The Image-Equations in Terms of the Focal Lengths f, e'. If, by means of formulæ (72) and (73), we eliminate n, n' and r from the formulæ (84), the Image-Equations for the Refraction of Paraxial Rays at a Spherical Surface may be obtained also in the following forms:

$$\frac{f}{u}+\frac{e'}{u'}=-1, \quad \frac{y'}{y}=\frac{f}{f+u}; \tag{86}$$

wherein the constants which determine the spherical refracting surface are the two focal lengths f and e'.

If, instead of taking the vertex A as the origin of abscissæ, both in the Object-Space and in the Image-Space, we take the two Focal Points F and E' as origins for the Object-Space and Image-Space, respectively, we may put:

$$FM = x, \quad E'M' = x';$$

so that the co-ordinates of the conjugate points Q, Q' referred to axes with origins at F, E' will be x, y and x', y', respectively. Evidently,

$$u = AM = AF + FM = x - f, \quad u' = AM' = AE' + E'M' = x' - e';$$

and substituting these values in place of u and u' in equations (86), we obtain the Image-Equations in their simplest forms, as follows:

$$xx' = fe', \quad Y = \frac{y'}{y} = \frac{f}{x} = \frac{x'}{e'}. \tag{87}$$

129. The case of the Reflexion of Paraxial Rays at a Spherical Mirror, which was treated at length in Arts. 33 and 34, may be regarded as a special case of the Refraction of Paraxial Rays at a Spherical Surface. Thus, according to the general principle explained in § 26, we have merely to put $n' = -n$ in the formulæ of Arts. 35–37 in order to derive at once the corresponding formulæ of Reflexion. Thus, for example, if we put $n' = -n$ in formulæ (71), we obtain $f = e' = r/2$; which shows that the two Focal Points F, E' coincide in the case of a spherical mirror (§ 112).

Another interesting special case that may be remarked here also is obtained by putting $r = \infty$; in which case we shall obtain the formulæ for the *Refraction of Paraxial Rays at a Plane Surface;* thus we find:

$$u' = \frac{n'}{n}u, \quad Y = \frac{y'}{y} = 1, \quad f = e' = \infty,$$

which will be recognized as the same as the results obtained in § 53.

The last of these equations is of special interest, for it shows that the Focal Points F and E' of a refracting plane are themselves the infinitely distant points of the two ranges of conjugate axial points. Hence, to a bundle of parallel incident rays refracted at a plane there corresponds a bundle of parallel refracted rays. Any optical system which treats parallel incident rays in this way is called a *Telescopic System*—a name which is derived from the fact that the Focal Points of a telescope are both at infinity.

III. SUPPLEMENT: CONTAINING CERTAIN SIMPLE APPLICATIONS OF THE METHODS OF PROJECTIVE GEOMETRY.

ART. 38. CENTRAL COLLINEATION OF TWO PLANE-FIELDS.

130. In the investigation of the refraction (or reflexion) of paraxial rays at a spherical surface, we have seen that *the imagery is ideal;* so long at least as the rays of light are supposed to be monochromatic, so that the refractive indices n, n' have fixed values. Thus, to a homocentric bundle of Object-Rays there corresponds always a homocentric bundle of Image-Rays, and to each point of the Object-Space, within the region of the paraxial rays, there corresponds one, and only one, point of the Image-Space. This unique point-to-point correspondence by means of rectilinear rays between the Object-Space and the Image-Space, which is the fundamental and essential requirement of Optical Imagery, is called in the modern geometry "*Collineation*". Thus,

Two space-systems Σ and Σ' are said to be "collinear" with each other, if to every point P of Σ there corresponds one, and only one, point P' of Σ', and to every straight line of Σ which goes through P there corresponds one straight line of Σ' which goes through P'.

In the theory of optics these two spaces Σ and Σ' are designated as the Object-Space and the Image-Space. They are not to be thought of as separate and distinct parts of space; they penetrate and include one another, so that a point or a ray may be regarded as belonging to either of the two space-systems.

Inasmuch as the problem of the refraction of paraxial rays affords a simple and at the same time a very useful application of the elegant methods of the modern geometry, it is proposed to give here a special investigation of it from this point of view; especially, too, because this study will prove a good introduction to the general theory of Optical Imagery which is treated at length in Chapter VII.

Since the optical axis of the spherical surface is an axis of symmetry for both the Object-Space and the Image-Space, it will suffice, as we have seen, to investigate the imagery in any meridian plane of the spherical surface; that is, in any plane containing the optical axis. In this plane in space we have two *collinear plane-fields*, one belonging to the Object-Space and one belonging to the Image-Space, which correspond with each other point by point and ray by ray. The totality of all the points and straight lines situated in an infinitely extended plane is what is here meant by the term "plane-field".

The distinguishing characteristics of the kind of Collineation which we have in the case of the Refraction of Paraxial Rays at a Spherical Surface may be said to be two in number, although, indeed, one is a consequence of the other. These characteristics are contained in the following statements:

(1) If Q, Q' are a pair of corresponding, or conjugate, points the straight line QQ' passes through the centre C of the spherical refracting surface; or, *the straight lines joining pairs of conjugate points all intersect in one point* (C).

(2) Since an incident ray and its corresponding refracted ray meet in the spherical refracting surface, and, moreover, since we are concerned here only with paraxial rays, which, therefore, meet the spherical surface at points infinitely near to its vertex A, so that the straight line (y) in the meridian plane which is tangent to the spherical surface at A may be regarded as the section of the surface made by this plane (see § 113); it follows, therefore, that *any pair of corresponding rays of the two collinear plane-fields will meet in this straight line* (y).

When two collinear plane-fields are so situated relative to each other that they have in common *a self-corresponding range of points*, we have the special case of the "*Central Collineation*" of two plane-fields. The straight line (y) which corresponds with itself point by point is called the "*Axis of Collineation*". The point C through which every straight line joining a pair of corresponding points passes is called the "*Centre of Collineation*". This point C is a "double point" or self-corresponding point of the two collinear plane-fields. Hence, *every straight line drawn through C contains two double points*, viz., the Centre of Collineation itself and the point where the straight line intersects the Axis of Collineation.

131. Projective Relation of Two Collinear Plane-Fields. If P, Q, R, S (Fig. 66) are a range of four points lying on a straight line s of one of the plane-fields, the points P', Q', R', S' conjugate to P, Q, R, S, respectively, will be ranged along the corresponding straight line s' of the collinear plane-field, and it is easy to show that we have the following relation:

$$(PQRS) = (P'Q'R'S');$$

that is, *two collinear plane-fields are in "projective" relation to each other.*

The proof of this is especially simple when we have Central Collineation of the two plane-fields; for then every straight line joining a pair of corresponding points passes through the centre of collineation C, and hence the two plane-fields in this case will be in perspective; whence it follows that any two corresponding straight lines s, s' will intersect a pencil of rays with its vertex at C in two projective ranges of points.

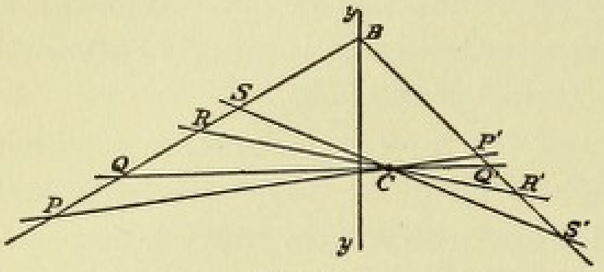

Fig. 66.

Central Collineation of Two Plane-Fields. The centre of collineation (C) and the axis of collineation (y) are the centre and axis of perspective; so that if s, s' are a pair of corresponding rays,

$$(PQRS) = (P'Q'R'S').$$

In case, however, the ray s itself passes through the centre C, so that s, s' are, therefore, a pair of self-corresponding rays (Fig. 67), the above proof of the projective relation of s, s' will not be applicable. In such a case we may proceed as follows:

Through the point C draw any other straight line, and take on it a point O. Connect O by straight lines with the Object-Points P, Q, R, S ranged along the straight line s. The straight lines joining the corresponding Image-Points P', Q', R', S' ranged along the straight line s' with the points where the straight lines PO, QO, RO, SO, respectively, intersect the axis of collineation (y) will all pass through the point O' conjugate to O, which is a point of the straight line joining O and C. Since, therefore, the Object-Ray OC and the corresponding Image-Ray $O'C$ coincide in the straight line joining O and O', the pencils of rays O, O' are in perspective with each other; so that *for conjugate points P, Q, R, S and P', Q', R', S' of a central or self-corresponding ray s (or s') we have also the projective relation*, characterized by the equation:

$$(PQRS) = (P'Q'R'S').$$

The self-corresponding ray at right angles to the axis of collineation (y) coincides with the optical axis of the system. This ray will be designated as the ray x of the Object-Space and the ray x' of the Image-Space. And the point A where it crosses the axis of collineation will be selected, in the special case of Central Collineation, as the most convenient point for the origin of a system of rectangular co-ordinates, the axes whereof are the optical axis and the axis of collineation.

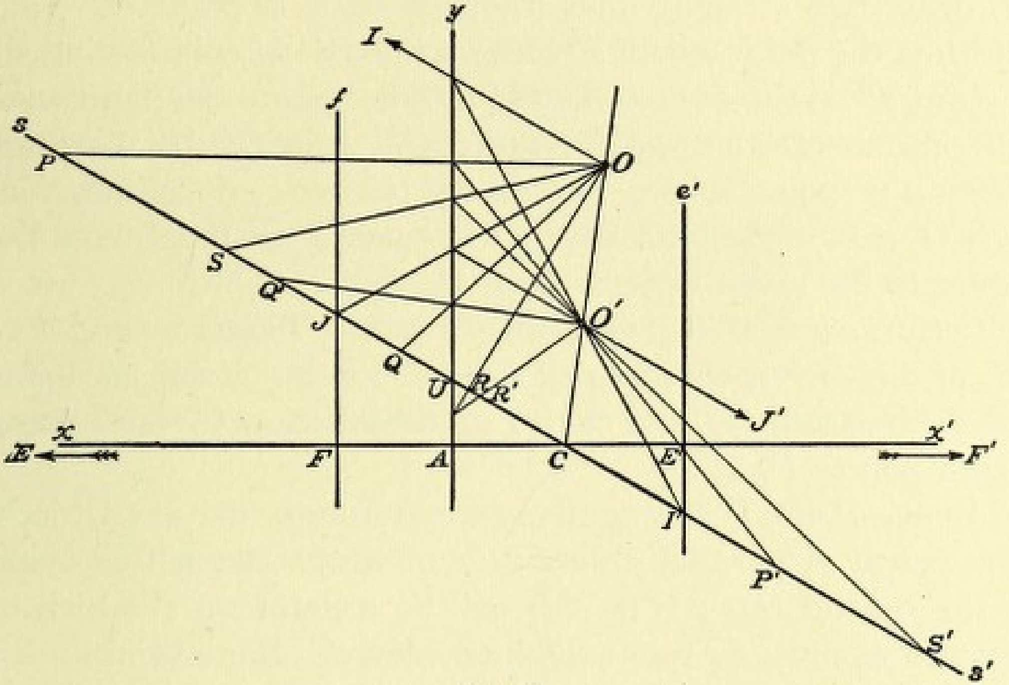

FIG. 67.

CENTRAL COLLINEATION OF TWO PLANE-FIELDS. Construction of Conjugate Points of a self-corresponding or central ray s (or s'). The centre of collineation is at C; the axis of collineation, or the perspective-axis, is the straight line designated by y. O, O' are a pair of conjugate points on any straight line passing through C. The points P, Q, R, S of the range of Object-Points s are projected by a pencil of rays from O, and the conjugate points P', Q', R', S' of the range of image-points s' are projected by a pencil of rays from O' which is in perspective with the pencil O. The points I', J are points of s', s conjugate to the infinitely distant points I, J' of s, s', respectively. The self-corresponding ray x (or x') which meets the axis of collineation at A at right angles is the optical axis; and the straight lines parallel to the axis of collineation which are drawn through J and I' and which meet the optical axis at right angles at F and E', respectively, are the two focal lines.

132. Geometrical Constructions.

(1) *Given the axis of collineation* (y) *and the centre of collineation* (C), *together with the positions of two conjugate points* P, P': *it is required to construct the Image-Point* Q' *of a given Object-Point* Q.

Through the two given Object-Points P, Q draw the straight line s meeting the axis of collineation in the double point B. Suppose (i) that *the straight line s joining* P, Q *does not pass through the centre* C, as in Fig. 66. This is the general case. The straight line s' corresponding to s will connect B with the given point P', and this straight line must also pass through the point Q' conjugate to Q. But Q' must also lie on the self-corresponding ray which goes through Q and the centre C; and hence the Image-Point Q' will be uniquely determined by the intersection of the straight lines BP' and QC. Again suppose (ii) that *the straight line s joining* P, Q *passes through the centre* C, as in Fig. 67; so that s (or s') is a self-corresponding ray.

This is a special case of great importance. In this case the above construction fails, and we may proceed, therefore, as follows: From P and C draw two straight lines intersecting in a point O. Join by a straight line the point where PO meets the axis of collineation with the given point P' conjugate to P, and let O' designate the point where this straight line meets the straight line CO. Join QO by a straight line and from the point where QO meets the axis of collineation draw through O' a straight line, which will meet s' in the Image-Point Q' conjugate to the given Object-Point Q.

(2) *Construction of the so-called "Flucht" Points J and I' of any central, or self-corresponding, ray s (or s')*; being given, as before, the axis of collineation (y), the centre of collineation (C) and the pair of conjugate points P, P'.

The Image-Point I' conjugate to the infinitely distant Object-Point I of the pencil of parallel object-rays of which the self-corresponding ray is the central ray s (Fig. 67) will be a point on s' which may be constructed exactly as was explained above. For example, knowing the positions of P, P', we can locate the positions of a pair of conjugate points O, O', as was done above. A straight line drawn through O parallel to s will go through the infinitely distant point I of s. The straight line joining the point where OI meets the axis of collineation with the point O' will intersect s' in the Image-Point I' conjugate to the infinitely distant Object-Point I. German writers on geometry call this point I' the "*Flucht*" *Point* of the ray s'.

Similarly, the "Flucht" Point J of the Object-Ray s is that point of this range which corresponds with the infinitely distant point J' of the Image-Ray s'. It may be constructed in a way precisely analogous to the construction of I' above, in the manner indicated in the diagram.

The "Flucht" Points J and I' are, in general, actual, or finite, points of the projective ranges of points s and s', respectively. In particular, the "Flucht" Points, designated by F and E', of the self-corresponding ray x, x', which coincides with the optical axis, are the points called the Focal Points of the optical system (§ 120).

(3) *Given the axis of collineation* (y), *together with the positions of the "Flucht" Points, J and I', of any central ray s, s', to construct the Image-Point P' corresponding to a given Object-Point P of s.*

Take any point O (Fig. 67), and through O draw the straight line OI parallel to s; and draw the straight line joining with I' the point where OI meets the axis of collineation. Draw the straight lines JO, PO, and from the point where JO meets the axis of collineation draw a straight line parallel to s' meeting in O' the straight line drawn through

I'. The straight line which joins with O' the point where PO meets the axis of collineation will meet s' in the Image-Point P' conjugate to the Object-Point P.

In particular, knowing the positions of the two Focal Points F and E' on the optical axis, and knowing also the position of the axis of collineation, we may, as above, *construct any pair of conjugate axial points* M, M'.

(4) *Given the axis of collineation* (y) *and the centre of collineation* (C), *together with the positions of two conjugate points* P, P' *it is required to construct the image-ray* v' *corresponding to a given object-ray* v.

Let H (Fig. 68) designate the double point where the given object-ray meets the axis of collineation. Through the given Object-Point P draw any ray s meeting the given ray v in a point Q and the axis of collineation in a point B. The point of intersection of QC and BP' will determine the Image-Point Q' conjugate to the Object-Point Q; and hence the straight line HQ' will be the image-ray v' conjugate to the given object-ray v.

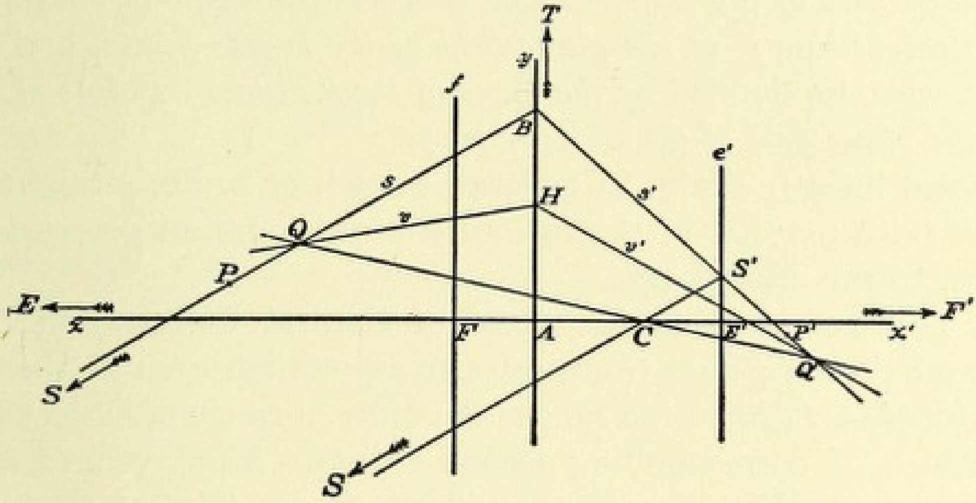

FIG. 68.

CENTRAL COLLINEATION OF TWO PLANE-FIELDS. Construction of Image-Ray v' conjugate to given Object-Ray v; also, construction of the "Flucht" Lines or Focal Lines f, e'.

(5) *If the given object-ray in* (4) *is the infinitely distant straight line* e *of the Object-Plane, we can construct the conjugate straight line* e' *of the Image-Plane, as follows:*

The point of intersection of the infinitely distant straight line e of the Object-Plane with the axis of collineation (y) is the infinitely distant point T (Fig. 68) of y; and hence e' will be parallel to y. Any ray s drawn through the given Object-Point P will meet the infinitely distant straight line e of the Object-Plane in the infinitely distant point S of s. If the object-ray s meets y in B, the corresponding image-ray s' will be the straight line BP', and a straight line drawn through the centre C parallel to s will determine by its intersection with s' the Image-Point S' conjugate to the infinitely distant Object-Point S. The straight line drawn through S' parallel to y will, therefore, be the image-ray e' conjugate to the infinitely distant object-ray e.

This straight line e' which is conjugate to the infinitely distant straight line e of the Object-Plane is called in Optics the *Focal Line* of the Plane-Field of the Image-Space (see § 123). Since e' passes through the point S', which is the "Flucht" Point of *any* ray of the plane-field of the Image-Space, it follows that *the Focal Line e' is the locus of the "Flucht" Points of all the image-rays in this plane-field.*

In a precisely similar way, we can *construct also the straight line f in the plane-field of the Object-Space which is conjugate to the infinitely distant straight line f' of the plane-field of the Image-Space*, and which may, likewise, be defined as *the locus of the "Flucht" Points of all the rays in the plane-field of the Object-Space.*

The focal lines f, e' are, in general, actual, or finite, straight lines. They are both parallel to the axis of collineation, and perpendicular, therefore, to the optical axis.

133. The Invariant in the Case of Central Collineation. Since all the rays of the pencil C are self-corresponding, each of these rays is the base of two projective ranges of points, a range of Object-Points and a range of corresponding Image-Points. Moreover, *to each of these self-corresponding rays belongs a pair of double, or self-corresponding, points* (§ 130); one of these double points being the centre of collineation itself and the other the point where the ray crosses the axis of collineation.

Similarly, each point on the axis of collineation is the common vertex of two projective pencils of rays, viz., a pencil of object-rays and a pencil of corresponding image-rays; and *each pair of such pencils of corresponding rays contains two self-corresponding rays*, of which the axis of collineation itself is one, and the ray joining the common vertex of the two pencils with the centre of collineation is the other.

Let P, P' (Fig. 69) and Q, Q' be two pairs of conjugate points of the self-corresponding ray s, s', and let U designate the double point where this ray crosses the axis of collineation (y). Since the ray s, s' is the common base of two projective ranges of points, the double ratio of the four Object-Points C, U, P, Q on s is equal to the double ratio of the four corresponding Image-Points C, U, P', Q' on s'; that is,

$$(CUPQ) = (CUP'Q');$$

whence it follows immediately that we have also:

$$(CUPP') = (CUQQ');$$

and, consequently, *the double ratio of any pair of conjugate points P, P' with the two self-corresponding points C, U of the two projective ranges of points which have the common base PP' has a constant value, which is independent of the positions of the conjugate points P, P'.*

Let M, M' be any other pair of conjugate points not situated on the straight line PP'; for example, it will be perfectly general if we

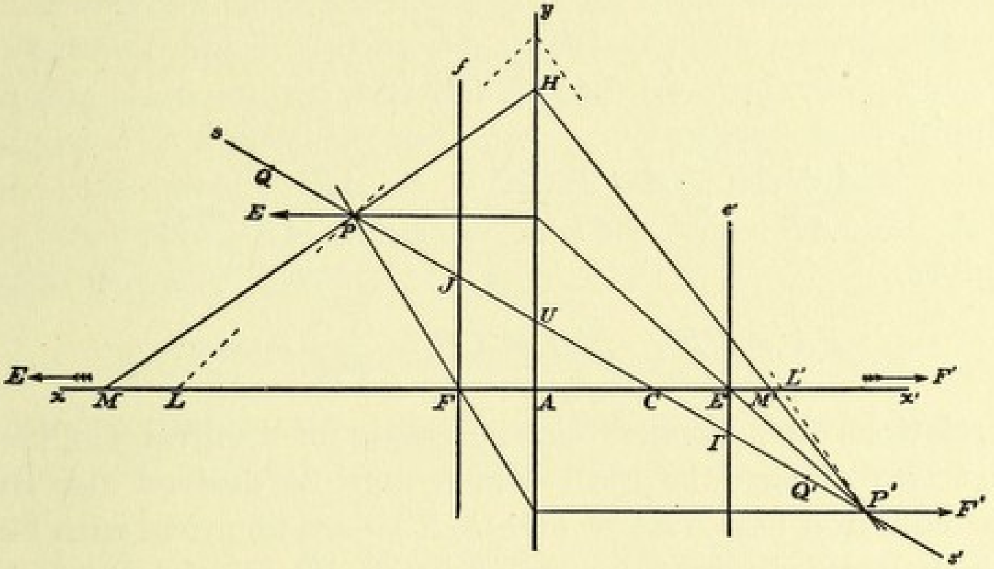

FIG. 69.

CENTRAL COLLINEATION OF TWO PLANE-FIELDS.

$(CUPP') = (CUQQ') = (CAMM') = (CALL') = (CAEE') = (CAFF')$.

take the points M, M' on the optical axis x, x' which crosses the axis of collineation (y) at the point A. Let the two corresponding rays MP, $M'P'$ intersect in a point H on the axis of collineation. It is obvious immediately that the two ranges of points C, U, P, P' and C, A, M, M' are in perspective, since they are both sections of the pencil of rays which has its vertex at H. Hence, the double ratios of each of these ranges of four points are equal; and if we denote the value of this double ratio by the symbol c, we have the following remarkable relations:

$$\left.\begin{aligned} c = (CUPP') &= (CUQQ') = \text{etc.},\\ = (CAMM') &= (CALL') = \text{etc.},\\ = (CAFF') &= CF : AF,\\ = (CAEE') &= AE' : CE'; \end{aligned}\right\} \qquad (88)$$

where, as heretofore, F and E' designate the positions of the two Focal Points, and F' and E designate the infinitely distant points of x' and x, respectively.

The most striking characteristic of the Central Collineation of two plane-fields consists, therefore, in the fact which we have here discovered, that it has an *invariant:*

The Double Ratio of any pair of conjugate points of a self-corresponding ray and the two double points of the ray has the same value for all such rays.

The value of this invariant, as above stated, is:

$$c = \frac{CF}{AF} = \frac{AE'}{CE'};$$

accordingly,

$$\frac{CA + AF}{AF} = \frac{AC + CE'}{CE'} = \frac{AE'}{CA + AE'} = c,$$

which gives:

$$FA = CE', \quad E'A = CF, \quad \frac{E'A}{FA} = -c. \tag{89}$$

These relations are likewise characteristic of Central Collineation. The first two of formulæ (89)—which may be derived also directly from the equation $(CFAE) = (CF'AE')$—are identical with formulæ (67) which were obtained for the special case of the Refraction of Paraxial Rays at a Spherical Surface; whereas the third equation corresponds with the relation given in formula (69).

134. The Characteristic Equation of Central Collineation. In particular, if M, M' designate the positions of any two conjugate points of the optical axis, the relation

$$(CAMM') = c$$

may be written in the following form:

$$\frac{c}{u'} - \frac{1}{u} = \frac{c-1}{r}; \tag{90}$$

where the symbols u, u' and r denote the abscissæ, with respect to the point A as origin, of the points M, M' and C, respectively; thus,

$$u = AM, \quad u' = AM', \quad r = AC.$$

This equation, which expresses for the case of any Central Collineation the relation between the abscissæ of conjugate axial points, is a perfectly general expression of the one-to-one correspondence of two projective ranges of points lying upon the same straight line. *The cases which occur in Optics* are comparatively restricted; we shall proceed to examine them.

If the sign of the invariant (c) *is positive*, the conjugate points M, M' are not "separated", in the geometrical sense, by the axis of collineation (y) and the centre of collineation (C). That is, for $c > 0$, the points M and M' are either both situated between C and A, or neither of them lies between C and A. In other words, the points C, A, M, M' are what is called a "hyperbolic throw", $(CAMM') > 0$. This case occurs always when the rays are refracted from one medium to another; so that in Optics *a positive value of c indicates Refraction;* whereas, on the contrary, *whenever the light-rays are reflected at a mirror, the imagery is of a kind that corresponds to a negative value of c* ($c < 0$); in which case one of the points M or M' will lie between C and A, but not the other point. In this latter case the points C, A, M, M' are an "elliptical throw", $(CAMM') < 0$.

Case I. Refraction of Paraxial Rays; $c > 0$.

A. Suppose, first, that $r = AC$ *is not equal to zero;* that is, that the centre of collineation (C) does not lie on the axis of collineation (y).

This is the case of *the Refraction of Paraxial Rays at a Spherical Surface,* which has been specially treated in this chapter. The invariant c in formula (90) is identical in value with the relative index of refraction (n'/n) from the first medium to the second medium, while the other constant r denotes here the radius of the spherical surface, as will be seen by comparing formula (90) with formula (74). The points A and C are, therefore, identical with the vertex and centre, respectively, of the spherical refracting surface.

Several special cases included under this head may be briefly noticed:

(1) If $c = +1$ (the value of r, as above specified, being different from zero), the relative index of refraction is equal to unity ($n' = n$). In this case equation (90) gives $u' = u$; and, hence, Object-Space and Image-Space coincide point by point; in fact, the two spaces are identical. When $n' = n$, there is no optical difference between the first medium and the second medium.

(2) *The case when $r = \infty$.* An infinite value of r in this case means merely that the centre C is at an infinite distance away in the direction of a line at right angles to the axis of collineation (y); so that now

the refracting surface is a *plane surface.* Formula (90) becomes now:

$$u' = \frac{n'}{n} u,$$

which is the abscissa-relation for the case of *the Refraction of Paraxial Rays at a Plane* (§ 53 and § 129). Since the centre of collineation (C) is at an infinite distance in a direction perpendicular to the refracting plane, the trace of which in the plane of the diagram (Fig. 70) is the axis of collineation (y), all straight lines joining pairs of conjugate points are parallel to the abscissa-axis. In this case the infinitely distant straight lines of the two collinear plane-fields must pass through the infinitely distant double point C; and, therefore, the two infinitely distant straight lines must be a pair of self-corresponding rays, and, accordingly, the five points designated by C, E, E', F, F' must all be coincident. In the modern geometry two collinear plane-fields are said to be in *affinity* with each other when their infinitely distant straight lines are conjugate to each other. Hence in this case the two focal lines f and e' are coincident with the infinitely distant straight lines e and f', respectively. In Optics this type of imagery is called *telescopic* (§ 129).

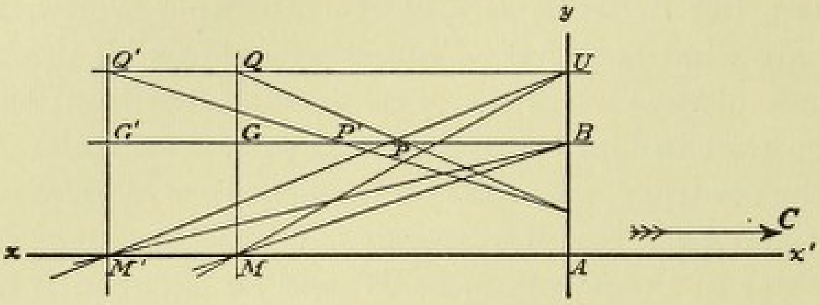

FIG. 70.

CENTRAL COLLINEATION OF TWO PLANE-FIELDS FOR THE CASE WHEN $c > 0$ AND $r = \infty$. The diagram shows the case when $n = c > 1$. This case ($c > 0$, $r = \infty$) is the case of the Refraction of Paraxial Rays at a Plane Surface. The double point C is the infinitely distant point of the optical axis xx', and the two Focal Points F, E' both coincide with C.

$$AM = u, \quad AM' = u', \quad MQ = y = M'Q' = y'.$$

B. A case of Central Collineation which is of much importance in Optics is the case *when the invariant* $c = +1$ *and the other constant* $r = 0$. If $r = 0$, *the centre of collineation* (C) *is situated on the axis of collineation* (y), so that the two double points A and C of the optical axis coincide. In this case we find:

$$FA = AE', \quad \text{or} \quad f + e' = 0;$$

where $f = FA$, $e' = E'A$ denote the focal lengths of the optical system. This type of imagery is characterized, therefore, by the fact that *the two focal points are equidistant from the axis of collineation, and on opposite sides thereof*. In the following chapter it will be seen that this is the imagery obtained by the *refraction of paraxial rays through an Infinitely Thin Lens*, or through any number of such lenses in successive contact with each other.

In this special case, the expression on the right-hand side of formula (90) becomes illusory. This leads us to remark here that we can obtain the abscissa-relation of Central Collineation in another form, which is characteristic not only of Central Collineation, but of the collinear relation in general. Thus, since

$$(MAFE) = (M'AF'E'),$$

we derive the equation:

$$xx' = fe',$$

where $x = FM$, $x' = E'M'$ denote the abscissæ of M, M' referred to the Focal Points F, E', respectively, as origins. In the special case here under consideration for which we have $e' = -f$, this formula takes the form:

$$xx' = -f^2.$$

Case II. Reflexion of Paraxial Rays; $c < 0$.

The only negative value of c that has any practical significance in Optics is the value $c = -1$. For this value of c we have:

$$(CAMM') = -1;$$

so that *each pair of conjugate points is harmonically separated by the centre* (C) *and the axis of collineation* (y). This formula will be recognized immediately as the formula for the *Reflexion of Paraxial Rays at a Spherical Mirror* (§ 111).

Since $(CAMM') = -1 = (CAM'M)$, the two ranges of points lying upon any central ray are in "involutory position"; so that, if, for example, a point M' of one range x' is conjugate to a point M of the conlocal range x, the same point M regarded now as a point of x' will be conjugate to M' regarded as a point of x (see § 110). We find here:

$$FA = E'A, \quad \text{or} \quad f - e' = 0;$$

so that, as was pointed out in § 112, the Focal Points F, E' of a spherical mirror are coincident.

Finally, if $r = \infty$, we have, for $c = -1$, $u' = -u$, which (see § 50) is the formula for *Reflexion at a Plane Mirror*.[1]

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[1] In connection with Art. 38, see J. P. C. SOUTHALL: The geometrical theory of optical imagery: *Astrophys. Journ.*, xxiv. (1906), 156–184.

CHAPTER VI.

REFRACTION OF PARAXIAL RAYS THROUGH A THIN LENS, OR THROUGH A SYSTEM OF THIN LENSES.

ART. 39. REFRACTION OF PARAXIAL RAYS THROUGH A CENTERED SYSTEM OF SPHERICAL SURFACES.

135. Centered System of Spherical Surfaces. Nearly all optical instruments consist of a combination of transparent isotropic media, each separated from the next by a spherical (or plane) surface; the centres of these surfaces lying all on one and the same straight line, called the "optical axis" of the centered system of spherical surfaces, which is an axis of symmetry. The spherical surface which the rays encounter first is called the first surface of the system; in our diagrams, in which the light is represented as being propagated from left to right, the first surface will be the one farthest to the left. The two media separated by this surface will be called the first and second media, respectively, in the sense in which the light travels. If the number of spherical surfaces is m, the number of media will be $m + 1$, the $(m + 1)$th medium being the last medium into which the rays emerge after refraction (or reflexion) at the mth surface. The absolute index of refraction of the first medium will be denoted by n_1 $(= n'_0)$; and, generally, the index of refraction of the kth medium (where k denotes any positive integer from 0 to m) will be denoted by n'_{k-1}. Thus, the index of refraction of the last medium will be denoted by n'_m. The centre of the kth spherical surface will be designated by C_k, and the point where the optical axis meets this surface, called the vertex of the surface, will be designated by A_k. The centered system of spherical surfaces is completely determined provided we know the index of refraction of each of the successive media and the positions on the optical axis of the centres and vertices of the spherical surfaces.

136. To a homocentric bundle of incident paraxial rays there corresponds a homocentric bundle of rays refracted at the first surface. The image-point or vertex of this bundle of refracted rays may be real or virtual; but in either case it is to be regarded as lying in the second medium, even though the actual position of this point in space may lie in a region which is occupied by the material of some one of the other media (see § 10). This bundle of rays refracted at the first surface will be a homocentric bundle of paraxial rays incident on the second surface, to which, therefore, there corresponds a homocentric bundle of rays refracted at this latter surface, with its image-point lying in the third medium. Proceeding thus from surface to surface, remaining always a bundle of homocentric rays, and producing a point-image in each successive medium of the series, the rays emerge finally into the last medium and form there a point-image, which, with respect to the entire centered system of spherical surfaces, is the point conjugate to the Object-Point in the first medium from which the rays originally came. Thus, precisely as in the case of the refraction of paraxial rays at a single spherical surface, we have also for the refraction of such rays through a centered system of spherical surfaces strict collinear correspondence between Object-Space and Image-Space.

The accompanying figure (Fig. 71) represents a centered system of three spherical refracting surfaces; the sections of the surfaces made by a plane containing the optical axis (the plane of the diagram) being shown by the tangent-lines y_1, y_2, etc., in accordance with the graphical method explained in § 113. Consider a ray M_1B_1 lying in the plane of the diagram which crosses the optical axis at the point designated by M_1 and meets the first spherical surface at the incidence-point B_1. After refraction at this surface this ray crosses the axis in the second medium at the point designated by M_1', which is, therefore, the axial image-point in this medium conjugate to the Object-Point M_1. Incident at B_2 on the second surface, the ray is again refracted, and again crosses the axis at a point M_2' which is the image-point in the third

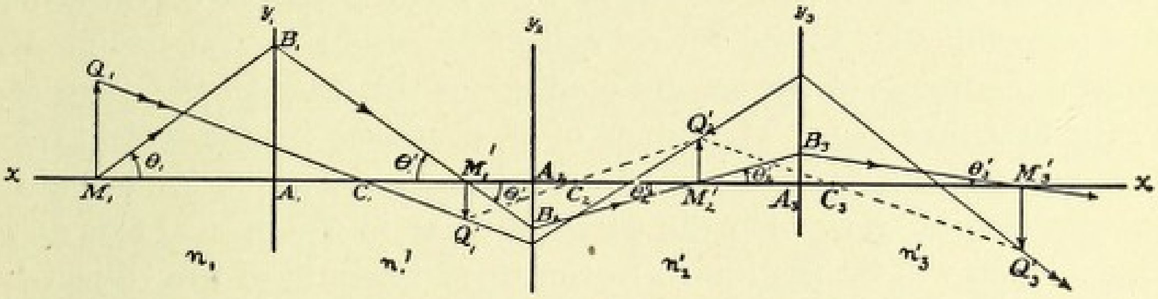

FIG. 71.

IMAGERY BY REFRACTION OF PARAXIAL RAYS THROUGH A CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES. In the diagram, the spherical surfaces are represented by the straight lines y_1, y_2, etc.; in the figure all the surfaces are represented as convex, with no two centres C_1, C_2, etc., in the same medium. Moreover, each image is represented as a real image formed between the centre of one surface and the vertex of the next following. The diagram is thus drawn merely for the sake of simplicity

$$A_1M_1 = u_1,\quad A_1M_1' = u_1',\quad A_1C_1 = r_1,\quad A_1A_2 = d_1,\quad A_2M_1' = u_2,\quad A_2M_2' = u_2',$$
$$A_2C_2 = r_2,\quad A_2A_3 = d_2,\quad A_kM_{k-1}' = u_k,\quad A_kM_k' = u_k',\quad A_kC_k = r_k,$$
$$A_{k-1}A_k = d_{k-1},\quad M_1Q_1 = y_1,\quad M_1'Q_1' = y_1',\quad M_2'Q_2' = y_2',\quad M_k'Q_k' = y_k'.$$

medium conjugate to the axial Object-Point M_1 in the first medium. Any one of these image-points may be real or virtual, depending on circumstances. If the number of spherical surfaces is m, the point M'_m where the ray crosses the axis after refraction at the last surface will be the image-point which, with respect to the entire system of surfaces, is conjugate to the axial Object-Point M_1.

The diagram shows also the path of a ray which, emanating from an Object-Point Q_1 near the optical axis, but not on it, traverses the centered system of spherical surfaces. The actual ray whose path is drawn is the ray which in the first medium is directed from Q_1 towards the centre C_1 of the first surface, and which, meeting this surface normally, proceeds into the second medium without change of direction; so that the point Q'_1 in the second medium which is conjugate to Q_1 must lie, therefore, on the straight line connecting Q_1 and C_1. If the extra-axial Object-Point Q_1 is a point on the straight line perpendicular to the optical axis at M_1, the point Q'_1 will lie on the straight line perpendicular to the optical axis at M'_1, and the straight line $M'_1Q'_1$ will be the image in the second medium of the short Object-Line M_1Q_1 in the first medium. The image of Q'_1 produced by the second refraction will be at a point Q'_2 in the third medium, which is the point of intersection of Q'_1C_2 with the perpendicular to the optical axis at M'_2; thus, $M'_2Q'_2$ will be the image in the third medium of the Object-Line M_1Q_1. The point Q'_m in the last medium will be the Image-Point, with respect to the entire system, of the extra-axial Object-Point Q_1, and $M'_mQ'_m$ will be the image, produced by the refraction of paraxial rays through a centered system of m spherical refracting surfaces, of a small Object-Line M_1Q_1 in the first medium perpendicular to the optical axis.

Thus, exactly as in the case of a single spherical refracting surface, *any plane of the Object-Space perpendicular to the optical axis of a centered system of spherical refracting surfaces will be imaged by paraxial rays by a plane of the Image-Space also perpendicular to the optical axis.*

137. The abscissæ, with respect to the vertex A_k of the kth surface, of the points M'_{k-1}, M'_k where a paraxial ray crosses the optical axis before and after refraction at this surface will be denoted by u_k, u'_k, respectively; thus,

$$A_kM'_{k-1} = u_k, \quad A_kM'_k = u'_k,$$

where k denotes any integer from 1 to m. For $k = 1$, we have $A_1M_1 = u_1$, since we write here M_1 instead of M'_0. The radius of the kth surface is denoted by r_k, and is defined as the abscissa, with respect to the vertex A_k, of the centre C_k. Moreover, the abscissa of the vertex A_{k+1} of the $(k + 1)$th surface with respect to the vertex A_k of the kth surface, called the *thickness* of the $(k + 1)$th medium, is denoted by d_k. Thus,

$$A_k C_k = r_k, \quad A_k A_{k+1} = d_k.$$

Thus, for the kth Spherical Refracting Surface, we have, according to formula (76):

$$J_k = n'_k \left(\frac{1}{r_k} - \frac{1}{u'_k} \right) = n'_{k-1} \left(\frac{1}{r_k} - \frac{1}{u_k} \right); \tag{91}$$

wherein, since

$$A_{k-1}M'_{k-1} + M'_{k-1}A_k = A_{k-1}A_k,$$

the value of u_k is determined by:

$$u_k = u'_{k-1} - d_{k-1}. \tag{92}$$

In these formulæ (91) and (92) we must give k in succession all integral values from $k = 1$ to $k = m$, where m is the total number of spherical surfaces (Note that $d_0 = 0$). Thus, provided we know the magnitudes denoted here by n, r and d, that is, provided we are given the centered system of spherical surfaces, we can, by means of these recurrent formulæ, eliminate in order the magnitudes denoted by u, and thus obtain the final value u'_m corresponding to a given value of u_1; that is, determine the position of the Image-Point M'_m corresponding to a given position of the axial Object-Point M_1.

The *Focal Point* E' of the Image-Space of a centered system of spherical refracting surfaces is the point where a paraxial ray, which in the first medium is parallel to the optical axis, crosses this axis after refraction at the last, or mth, surface. If in the above equations we put $u_1 = \infty$, then $u'_m = A_m E'$ will be the abscissa of the point E' with respect to the vertex A_m of the last spherical surface. We shall have $(2m - 1)$ equations with $(2m - 1)$ unknown magnitudes, viz., $u_2, u_3, \cdots u_m$ and $u'_1, u'_2, \cdots u'_m$. Accordingly, by successive substitutions we can find u'_m. Similarly, the *Focal Point* F of the Object-Space is the point where a ray crosses the optical axis in the first medium which emerges in the last medium parallel to the optical axis. In order to locate this point F, we must put $u'_m = \infty$ and find the value of the abscissa $u_1 = A_1 F$ of the Focal Point F with respect to the vertex A_1 of the first spherical surface.

138. The Lateral Magnification Y. Putting $M'_k Q'_k = y'_k$, and making use of formula (85), we obtain the following equations:

$$\frac{y'_1}{y_1} = \frac{n_1 u'_1}{n'_1 u_1}, \quad \frac{y'_2}{y'_1} = \frac{n'_1 u'_2}{n'_2 u_2}, \quad \cdots, \quad \frac{y'_k}{y'_{k-1}} = \frac{n'_{k-1} u'_k}{n'_k u_k}.$$

Multiplying these equations together, we obtain:

$$\frac{y'_k}{y_1} = \frac{n_1}{n'_k} \frac{u'_1 \cdot u'_2 \cdots u'_k}{u_1 \cdot u_2 \cdots u_k}.$$

The ratio

$$\frac{M'_m Q'_m}{M_1 Q_1} = \frac{y'_m}{y_1} = Y$$

is called the *Lateral Magnification* of the centered system of spherical surfaces with respect to the axial Object-Point M_1. Thus, according to the formula above, we have:

$$Y = \frac{y'_m}{y_1} = \frac{n_1}{n'_m} \prod_{k=1}^{k=m} \frac{u'_k}{u_k} \tag{93}$$

where the symbol Π means the continued product of the terms obtained by giving k in succession all integral values from $k = 1$ to $k = m$. Y is evidently a function of u_1.

139. The Principal Points of a Centered System of Spherical Surfaces. The pair of conjugate planes perpendicular to the optical axis for which the Lateral Magnification has the special value $Y = +1$, so that for this pair of planes object and image are equal both as to magnitude and sign, were called by GAUSS[1] the *Principal Planes* of the optical system; and the two conjugate axial points, designated here by the letters A, A', where the Principal Planes were cut by the optical axis, were called similarly the *Principal Points*. Putting $Y = +1$ in formula (93), we have:

$$\frac{n_1}{n'_m} \prod_{k=1}^{k=m} \frac{u'_k}{u_k} = 1,$$

which, together with the equations (91) and (92), gives us $2m$ equations with $2m$ unknown quantities, whereby we can determine the abscissæ $u_1 = A_1 A$ and $u'_m = A_m A'$, and thus ascertain the positions of the Principal Points A, A'.

The earlier writers on Geometrical Optics proceeded by computing the values of u, u' from surface to surface. MOEBIUS and, especially, GAUSS strove to derive general formulæ for finding the position of the image-point conjugate to a given object-point, without involving the tedious process of tracing the path of the ray from surface to surface. It was GAUSS who introduced the notion of the so-called *Cardinal Points* of the optical system. These are certain distinguished pairs of conjugate axial points, the most important of which are the Principal Points A, A', which are briefly referred to here. We may remark that GAUSS's method marked a great advance in the science of Geometrical Optics, and led to very simple and elegant formulæ. More recently, ABBE (as we shall see in the following chapter), without employing the Cardinal Points at all, has obtained the so-called "image-equations" by a still simpler method depending only on the characteristics of the Focal Points of the optical system. ABBE's theory of optical imagery will be explained at length in the following chapter; where will be found also a more extended reference to the Cardinal Points of the system (§ 180).

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[1] C. F. GAUSS: *Dioptrische Untersuchungen* (Goettingen, 1841), p. 13.

The formulæ which have been obtained will be applied in this chapter to the problem of the refraction of paraxial rays through an infinitely thin lens.

ART. 40. TYPES OF LENSES; OPTICAL CENTRE OF LENS.

140. A centered system of two spherical refracting surfaces constitutes what is known in Optics as a *Lens*. In practice the Lens is usually surrounded by the same medium on both sides, and we shall assume in this chapter that such is the case. We may denote the indices of refraction of the two media by the symbols n and n'; thus,

$$n = n_1 = n_2', \quad n' = n_1'.$$

Since $m = 2$, we obtain from equations (91) and (92) the following formulæ for the refraction of paraxial rays through a Lens surrounded by the same medium on both sides:

$$\left.\begin{aligned} &\frac{n'}{u_1'} - \frac{n}{u_1} = \frac{n' - n}{r_1}, \\ &\qquad\qquad\qquad\qquad u_2 = u_1' - d, \\ &\frac{n}{u_2'} - \frac{n'}{u_2} = \frac{n - n'}{r_2}, \end{aligned}\right\} \tag{94}$$

where here we use d instead of d_1 to denote the thickness A_1A_2 of the Lens. Thus, if we know the radii r_1, r_2 of the two surfaces of the Lens and the index of refraction of the Lens-medium relative to the surrounding medium (n'/n), together with the thickness d of the Lens, we can find the position of the Image-Point M_2' conjugate to the axial

Object-Point M_1. The positions of the Focal Points F and E' may be determined by putting, first, $u_2' = \infty$ and solving for $u_1 = A_1F$, and, second, $u_1 = \infty$ and solving for $u_2' = A_2E'$.

The Lateral Magnification with respect to the Object-Point M_1 is obtained at once by putting $m = 2$ in formula (93); thus, we have:

$$Y = \frac{y_2'}{y_1} = \frac{u_1' \cdot u_2'}{u_1' \cdot u_2}. \tag{95}$$

141. Lenses may be conveniently divided into two main classes, as follows:

(1) *Lenses which are thickest along the optical axis.* In this group are included, therefore, such forms of lenses as are shown in the figure (Fig. 72), viz., the Bi-convex Lens, the Plano-convex Lens and the Convexo-concave (or Concavo-convex) Lens with a shallow concavity (the so-called "Positive Meniscus").

(2) *Lenses which are thinnest along the optical axis.* To this group belong the Lenses shown in Fig. 73, viz.: the Bi-concave Lens, the Plano-concave Lens and the Concavo-convex (or Convexo-concave) Lens with a deep concavity (the so-called "Negative Meniscus").

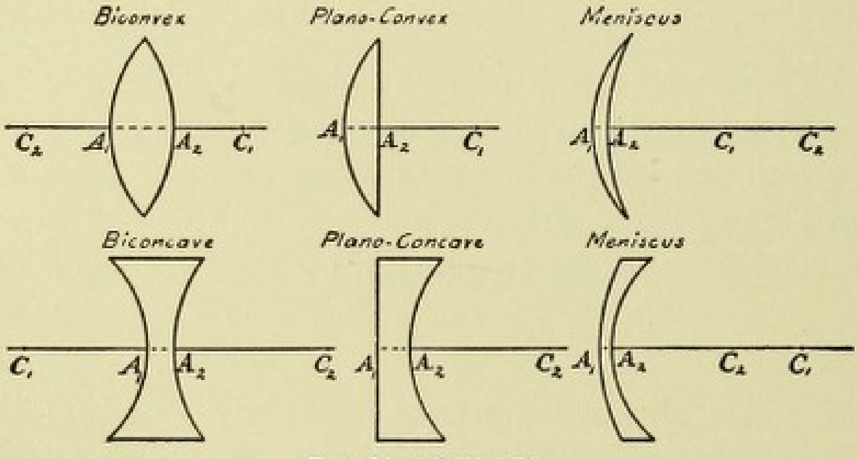

FIG. 72 and FIG. 73.

TYPES OF LENSES. In Fig. 72 the lenses are "convergent" or positive; in Fig. 73 the lenses are "divergent" or negative; assuming that the lenses are glass lenses surrounded by air and not too thick.

$A_1A_2 = d, \quad A_1C_1 = r_1, \quad A_2C_2 = r_2.$

A bundle of incident parallel paraxial rays falling on a Lens of the first group—supposed to be a moderately thin glass lens surrounded by air—will be converged to a real focus on the far side of the Lens; whereas, under the same circumstances, a beam of parallel rays will be made divergent by passing through a Lens of the second group. On account of this characteristic treatment of incident parallel rays, the Lenses of the first group are sometimes called "Convergent"

Lenses, and those of the second group are called "Divergent" Lenses. But this property depends essentially on the thickness of the Lens and on the relative index of refraction.

142. Optical Centre of Lens. Any ray, whether paraxial or not, which leaves the Lens (supposed to be surrounded by the same medium on both sides) in a direction parallel to that of the corresponding incident ray, will have passed, within the Lens, (either really or virtually) through a remarkable point on the optical axis called the *Optical Centre* of the Lens. In order to prove this statement, and at the same time to determine the position of this point, let us draw through the centres C_1, C_2 of the two Lens-surfaces any two parallel radii C_1B_1, C_2B_2 (Fig. 74): then the point O where the straight line B_1B_2 crosses the optical axis is a fixed point. For in the similar triangles OC_1B_1 and OC_2B_2 we have:

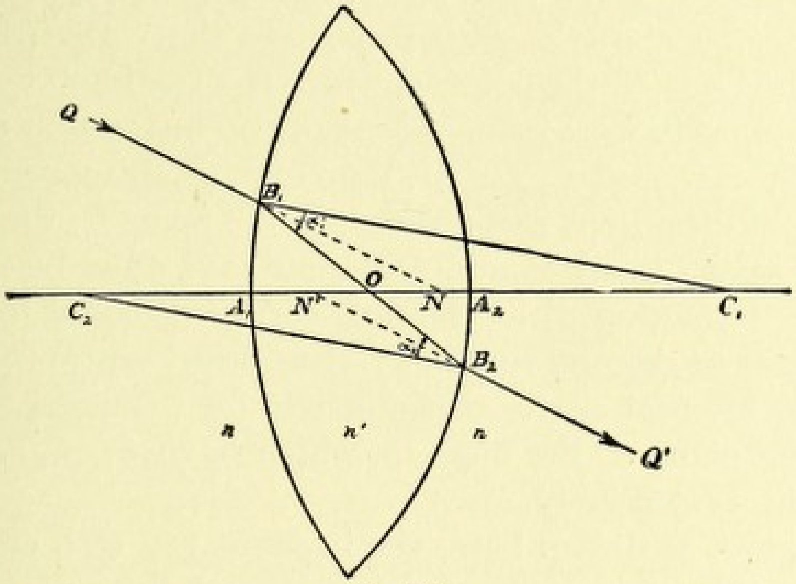

FIG. 74.

OPTICAL CENTRE OF LENS AT THE POINT MARKED O. Any ray passing through O emerges from the lens in a direction parallel to the direction of the incident ray; the lens being surrounded on both sides by the same medium.

$$A_1C_1 = r_1,\quad A_2C_2 = r_2,\quad A_1A_2 = d,\quad \angle C_1B_1O = \alpha_1' = \angle C_2B_2O = \alpha_2.$$

$$\frac{C_1O}{C_2O} = \frac{C_1B_1}{C_2B_2},$$

or

$$\frac{C_1O}{C_2O} = \frac{C_1A_1}{C_2A_2},$$

or

$$\frac{C_1A_1 + A_1O}{C_2A_2 + A_2O} = \frac{C_1A_1}{C_2A_2},$$

and, hence:

$$\frac{A_1O}{A_2O} = \frac{A_1C_1}{A_2C_2} = \frac{r_1}{r_2}.$$

And, since

$$A_2O = A_2A_1 + A_1O = A_1O - A_1A_2 = A_1O - d,$$

where $d = A_1A_2$ denotes the thickness of the Lens, we obtain finally:

$$A_1O = \frac{r_1}{r_1 - r_2}d. \tag{96}$$

Thus, for a Lens of given form and thickness, this equation enables us to determine the abscissa, with respect to the vertex A_1 of the first surface of the Lens, of the point O, which is a fixed point on the optical axis, since its position is independent of the inclination of the pair of parallel radii C_1B_1 and C_2B_2. If, therefore, B_1B_2 represents the path of a ray within the Lens going through this point O, the directions of the corresponding incident and emergent rays must be parallel, since the angle of refraction α_1' at the first surface is equal to the angle of incidence α_2 at the second surface. The optical centre O will be recognized as the internal centre of similitude (or perspective) of the two circles in the plane of the diagram which have C_1, C_2 as centres and r_1, r_2 as radii, respectively.

In the figure, as drawn here, the incident ray QB_1 crosses the axis virtually at the point designated by N, and the emergent ray B_2Q' which is parallel to QB_1 crosses the axis virtually at the point designated by N'. If the ray is a paraxial ray, the points N, N' will be a pair of axial conjugate points—the so-called "Nodal Points" of the Lens.

In case one of the surfaces of the Lens is plane, the optical centre O will coincide with the vertex of the curved surface, as is evident from formula (96). When the curvatures of the two surfaces of the Lens have the same sign, as is the case with either the positive or negative meniscus, the optical centre does not lie within the Lens at all.

ART. 41. FORMULÆ FOR THE REFRACTION OF PARAXIAL RAYS THROUGH AN INFINITELY THIN LENS.

143. When the Lens is so thin that we may neglect its thickness (d) in comparison with the other linear magnitudes which are measured along the optical axis, we have the case of an Infinitely Thin Lens. In comparison with the other dimensions the thickness of the Lens is often quite small, but an Infinitely Thin Lens is, of course, unrealizable, so that such a Lens is sometimes called an "ideal Lens". If we put $A_1A_2 = d = 0$, this is equivalent to regarding the vertices A_1, A_2 as coincident, and the Lens-surfaces as, therefore, in contact with each other. The approximate formulæ that are obtained under these circumstances are often of very great utility, especially in the preliminary design of an optical instrument; and in many cases such formulæ are quite sufficient to enable us to form a proper idea of the behaviour and general characteristics of a real Lens of not too great thickness.

144. Conjugate Axial Points in the case of the Refraction of Paraxial Rays through an Infinitely Thin Lens. In accordance with the graphical mode of representation explained in § 113, an infinitely thin lens may be represented in a diagram by a straight line perpendicular to the optical axis. The point A (Fig. 75) where this straight line crosses the axis is not only the common vertex of the two spherical surfaces, but it is also the position of the optical centre of the Lens; for, according to formula (96), when $d = 0$, the optical centre coincides with the common vertex of the Lens-surfaces. The form of the Lens is shown in the figure by the positions of the centres C_1, C_2 of the two spherical surfaces. If in the second of formulæ (94) we put $d = 0$, we have $u_2 = u'_1$. Imposing this condition, and adding the two other equations, and at the same time writing here u and u' in place of u_1 and u'_2, respectively, we obtain the useful abscissa-relation for the refraction of paraxial rays through an infinitely thin Lens in the following form:

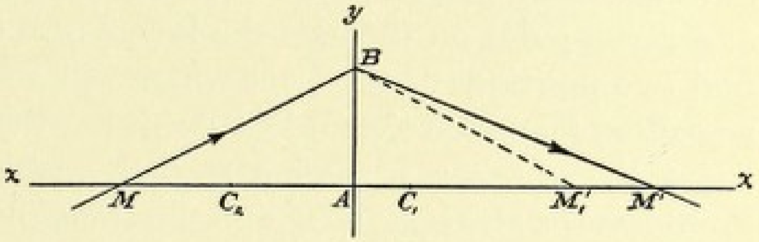

FIG. 75.

REFRACTION OF PARAXIAL RAYS THROUGH INFINITELY THIN LENS. M, M' are a pair of conjugate axial points. The points designated in the diagram by M, M', C_1, C_2 and A may be ranged along the optical axis in any order whatever, depending on the form and optical properties of the lens and on the direction of the incident ray MB. The lens represented in the diagram is a Biconvex Lens, the lens-medium being more highly refracting than the surrounding medium.

$AC_1 = r_1$, $AC_2 = r_2$, $AM = u$, $AM' = u'$.

$$\frac{1}{u'} - \frac{1}{u} = \frac{n' - n}{n}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{97}$$

The expression on the right-hand side of this equation, involving only the Lens-constants, r_1, r_2 and n'/n, has for a given Lens a perfectly definite value. If we denote this constant by $1/f$, so that

$$\frac{1}{f} = \frac{n' - n}{n}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{98}$$

the formula above may be written as follows:

$$\frac{1}{u'} - \frac{1}{u} = \frac{1}{f}. \tag{99}$$

Thus, having determined by means of formula (98) the value of the magnitude denoted by f, or else being given its value directly, we can ascertain the position of the Image-Point M' corresponding to a given axial Object-Point M; that is, knowing u, we can find u', and *vice versa*.

It may be remarked that equation (99) is symmetrical with respect to u and $-u'$; that is, if $-u$ be written in place of u' and $-u'$ in place of u, the equation will not be altered. Hence, if the Object-Point M is situated on the axis at the point $(u, 0)$ and the Image-Point M' at the point $(u', 0)$, and if the Object-Point is then supposed to be transferred to a new position $(-u', 0)$, the new Image-Point will have the position $(-u, 0)$. Or, if we adjust the Lens so as to produce at a given point on the axis the image of a fixed Object-Point, we can find two positions of the Lens which will accomplish the purpose, viz., a position for which the Object-Point has the abscissa u and the Image-Point the abscissa u' and a second position for which the Object-Point has the abscissa $-u'$ and the Image-Point has the abscissa $-u$.

145. The Focal Points of an Infinitely Thin Lens. Putting $u = \infty$ in formula (99), we obtain:

$$AE' = f,$$

where E' designates the position on the optical axis of the Secondary Focal Point of the Infinitely Thin Lens. Similarly, putting $u' = \infty$, we find:

$$AF = -f,$$

where F designates the position of the Primary Focal Point of the Lens. Thus, *the two Focal Points F and E' of an Infinitely Thin Lens are equidistant from the Lens, and on opposite sides of it.*

The imagery of an Infinitely Thin Lens is completely determined so soon as we know the positions of the three points A, F and E'; and, since the point A lies midway between the Focal Points F and E', Lenses may also be divided into two classes, as follows:

(1) Lenses in which the points F, A, E' are ranged along the optical axis in the order named in the sense in which the light is propagated (therefore, in our diagrams from left to right); so that for Lenses of this type incident rays which proceed parallel to the axis will be converged to a real focus at the point E' beyond the Lens, as shown in

the first diagram of Fig. 76: and, hence, such Lenses are called *Convergent Lenses*. They are also called *Positive Lenses*, because $FA = f$ is positive, if we take the direction along which the light is propagated as the positive direction of the ray. Assuming that $n' > n$ (as, for

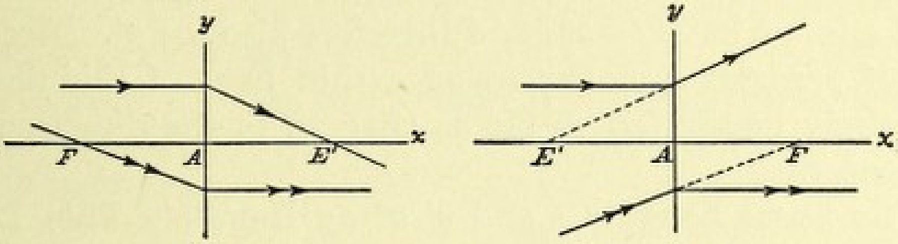

FIG. 76.

Diagram on Left represents a Convergent Lens ($f > 0$); diagram on Right represents a Divergent Lens ($f < 0$).

$$f = FA,\quad e' = E'A,\quad f = -e'.$$

example, in the case of a glass lens in air), the sign of f, according to formula (98), is the same as the sign of $(1/r_1 - 1/r_2)$. In the Biconvex Lens ($r_1 > 0$, $r_2 < 0$), the Plano-convex Lens ($r_1 = \infty$, $r_2 < 0$, or $r_1 > 0$, $r_2 = \infty$) and the Positive Meniscus ($r_2 > r_1 > 0$)—that is, for all Lenses which are thicker in the middle than towards the edges—the sign of $(1/r_1 - 1/r_2)$ is positive, and, therefore, $f > 0$; and, hence, as already stated (§ 141), such lenses (provided $n' > n$) are convergent.

(2) Lenses in which the order of the above-named points is E', A, F. For lenses of this class incident rays which proceed parallel to the axis are made divergent by passing through the Lens, and emerge as if they had come from a virtual focus at the Secondary Focal Point E', lying in front of the Lens, as shown in the second diagram of Fig. 76. Accordingly, such Lenses are called *Divergent* or *Negative Lenses*, since here $FA = f$ is negative. In case $n' > n$, the sign of f, as above stated, agrees with the sign of $(1/r_1 - 1/r_2)$. In the Biconcave Lens ($r_1 < 0, r_2 > 0$), the Plano-concave Lens ($r_1 = \infty, r_2 > 0$ or $r_1 < 0$, $r_2 = \infty$) and the Negative Meniscus ($r_2 < r_1 < 0$)—that is, for all Lenses which are thinner in the middle than they are at the edges—the sign of $(1/r_1 - 1/r_2)$ is negative; and, hence (provided $n' > n$), such Lenses are divergent.

Several special forms of Lenses may be mentioned here, viz.:

The Equiconvex and the Equiconcave Lens, for which $r_2 = -r_1$, for which we have, therefore, $f = nr_1/2(n' - n)$. In the case of the Equiconvex Lens $r_1 > 0$, and, therefore (assuming $n' > n$), we have $f > 0$; whereas for the Equiconcave Lens $r_1 < 0$, and, therefore, $f < 0$.

The Plano-convex and the Plano-concave Lens: Assuming that the first surface of the Lens is the plane surface, we have here $r_1 = \infty$;

so that $f = -nr_2/(n'-n)$. The sign of f depends therefore on the sign of r_2.

An interesting limiting case is that of an Ideal Meniscus in which $r_1 = r_2$. Such a Lens is neither thicker nor thinner in the middle than it is at the edges, and, therefore, is neither convergent nor divergent. For this Lens we have $f=\infty$, and therefore also $u=u'$. Accordingly a bundle of paraxial rays traversing a thin Lens of this description will be entirely unaffected so far as changes in the directions of the rays are concerned.

146. The Focal Lengths f and e' of an Infinitely Thin Lens. If the Focal Lengths of the Lens are defined exactly in the same way as the Focal Lengths of a single spherical refracting surface were defined in § 124, we shall find that the Focal Lengths of an Infinitely Thin Lens are also equal to FA and $E'A$; that is, they are equal to the abscissæ, with respect to the Focal Points F and E', of the common vertex A of the two Lens-surfaces. If, therefore, we denote the Focal Lengths here also by f and e', we have:

$$f = FA, \quad e' = E'A;$$

so that the constant f introduced above and defined by formula (98), which we saw was equal to FA, is in fact the Primary Focal Length of the Lens. Obviously, we have the following relations:

$$f = -e' = \frac{nr_1r_2}{(n'-n)(r_2-r_1)}. \tag{100}$$

Thus, the focal lengths of an Infinitely Thin Lens are equal in magnitude, but opposite in sign. If we reverse a thin Lens, so that the first surface of the Lens is the surface which was formerly the second surface, we do not alter the Focal Lengths, and hence the character of the Lens will not be altered; for r_1 becomes $-r_2$, and the formula above is not altered.

The reciprocal of the Focal Length is called the *Power* or *Strength* of the Lens. If we put $1/f = \varphi$, and if also the curvatures of the two surfaces of the Lens are denoted by c and c', that is, $c = 1/r_1$, $c' = 1/r_2$, formula (98) may be written as follows:

$$\varphi = \frac{n'-n}{n}(c-c'); \tag{101}$$

so that *the Power of an Infinitely Thin Lens is proportional to the difference of the curvatures of the two surfaces of the Lens.*

Since the Focal Lengths of an Infinitely Thin Lens are equal to the distances of the Lens from the two Focal Points, the theory of the refraction of paraxial rays through such a Lens is very similar to that of the refraction of paraxial rays at a spherical surface; only, in the case of the Lens the theory is simpler, because the Focal Points are equidistant from the Lens.

147. Putting $u_2 = u'_1$ in formula (95), and writing u, u' in place of u_1, u'_2, respectively, we obtain for the **Lateral Magnification of an Infinitely Thin Lens:**

$$Y = \frac{y'}{y} = \frac{u'}{u}; \tag{102}$$

that is, *the ratio of the linear dimensions of the Object and Image is equal to the ratio of the distances of the Object and Image from the Lens.*

If x, x' denote the abscissæ, with respect to the Focal Points F, E', of the conjugate axial points M, M', respectively, that is, if

$$FM = x, \quad E'M' = x',$$

then

$$\begin{aligned} u &= AM = AF + FM = x - f, \\ u' &= AM' = AE' + E'M' = x' - e'; \end{aligned}$$

and substituting these values in formulæ (99) and (102), we obtain the so-called "Image-Equations" of an Infinitely Thin Lens in the following simple and convenient forms:

$$xx' = -f^2, \quad \frac{y'}{y} = \frac{f}{x}. \tag{103}$$

The abscissa-equation is the same as the characteristic equation of the Central Collineation of two plane-fields for the case when the invariant $c = +1$ (see § 134). It may be derived at once from the projective relation:

$$(MAFE) = (M'AF'E'),$$

where E and F' are the infinitely distant points of the two corresponding ranges of Object-Points and Image-Points, respectively, lying upon the optical axis of the Lens.

148. Construction of the Image Formed by the Refraction of Paraxial Rays through an Infinitely Thin Lens. In the diagrams (Figs. 77 and 78) MQ represents a very short Object-Line perpendicular to the optical axis at the axial Object-Point M. The Infinitely Thin Lens is itself represented by the straight line y perpendicular to the optical axis at the point designated by A. Fig. 77 shows the case of a Convergent Lens, and Fig. 78 shows the case of a Divergent Lens. Since the point A where the optical axis meets the Infinitely Thin Lens is also the optical centre of the Lens (§144), any ray directed towards A will emerge from the Lens without change of direction, and, hence, the straight line joining any pair of conjugate points Q, Q' will go through this point A. Thus, we see that the Object-Space and Image-Space of an Infinitely Thin Lens are in perspective relation to each other with respect to the point A as centre of perspective. This is obvious also from formula (102). As was remarked above (§ 146), the imagery in the case of an Infinitely Thin Lens is quite similar to that of a single spherical refracting surface, where the centre of the surface is the centre of perspective of the Object-Space and Image-Space.

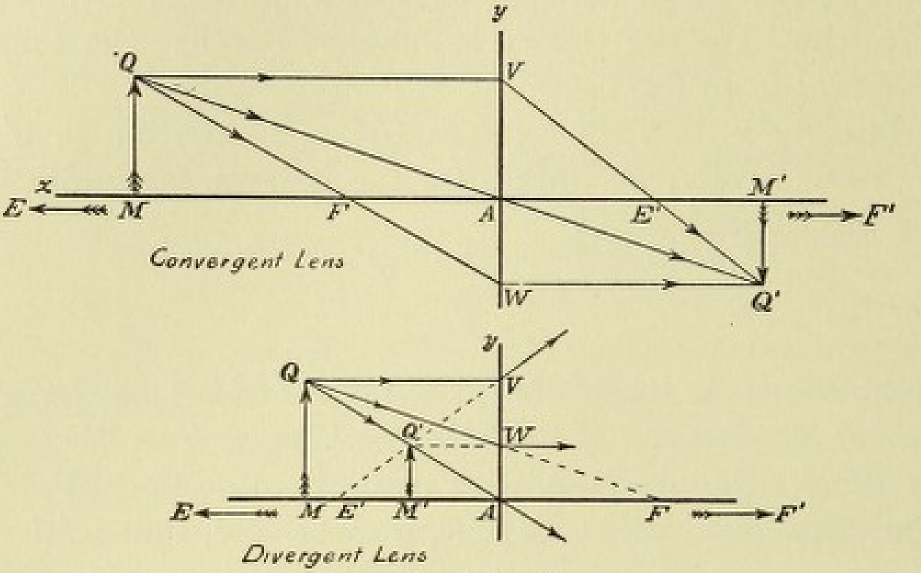

FIG. 77 and FIG. 78.

REFRACTION OF PARAXIAL RAYS THROUGH AN INFINITELY THIN LENS. Construction of Image.

$AM = u$, $AM' = u'$, $FA = AE' = f$, $MQ = y$, $M'Q' = y'$.

Knowing the positions of the axial points A, F and E' of an Infinitely Thin Lens, we may easily construct the Image $M'Q'$ conjugate to MQ. All that we have to do is to locate the position of the point Q', and then draw $M'Q'$ perpendicular to the optical axis at M'. The point of intersection of any pair of emergent rays emanating originally from the Object-Point Q will suffice to determine the corresponding Image-Point Q'. In the diagrams (which need no farther explanation) three such rays are shown, any two of which are sufficient.

The imagery in the case of the Refraction of Paraxial Rays through an Infinitely Thin Lens is exhibited in the two diagrams, Figs. 79 and 80, the first of which shows the case of a Convergent Lens and the second the case of a Divergent Lens. The numerals 1, 2, 3, etc., designate various successive positions of an Object-Point, which, starting at an infinite distance in front of the Lens, is supposed to travel towards the Lens along a straight line parallel to the optical axis. The corresponding positions of the Image-Point on the straight line connecting the point *V* with the Secondary Focal Point *E′* are designated in the diagram by the same numerals with primes. Thus the straight lines 11′, 22′, etc., connecting each pair of conjugate points, will, if they are drawn, all pass through the perspective-centre *A*. In both types of Lens the Object-Point and Image-Point coincide with each other at the point *V* on the Lens itself, and hence the two Principal Points (§ 139) of an Infinitely Thin Lens coincide with each other at the point *A*. If the Object-Point lies beyond the Lens (that is, to the right of the Lens in the diagrams), it is a virtual Object-Point.

So long as the Object is in front of the Primary Focal Plane of a Convergent Lens (Fig. 79), we have a real, inverted Image lying on

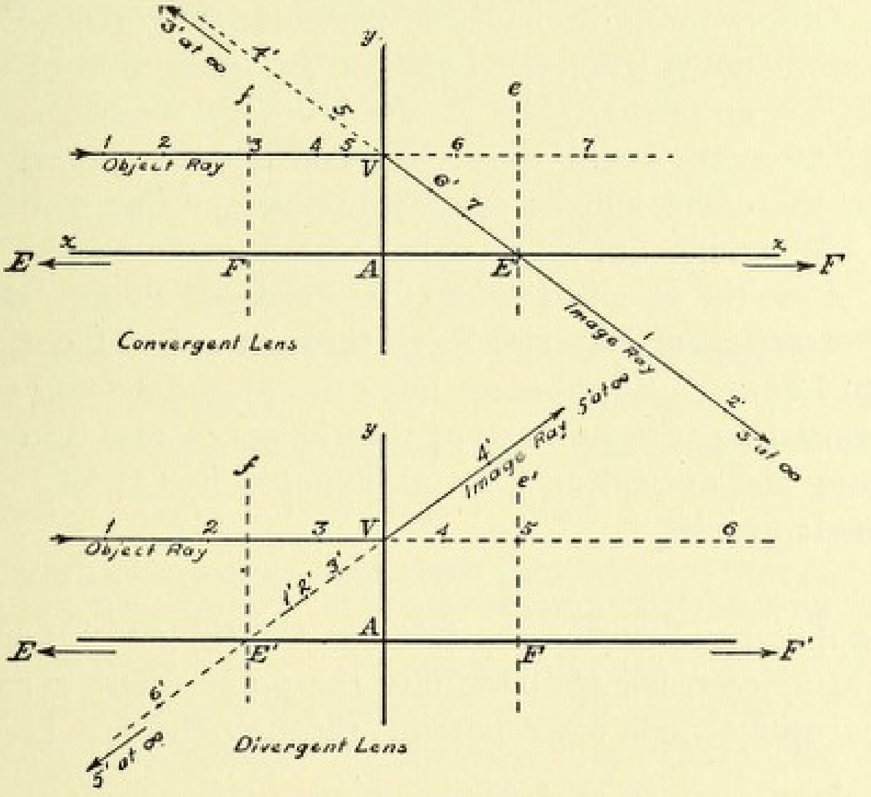

FIG. 79 and FIG. 80.

REFRACTION OF PARAXIAL RAYS THROUGH INFINITELY THIN LENS. IMAGERY OF IDEAL LENS. The numerals 1, 2, 3, etc., show a number of selected positions of an object-point supposed to move from left to right along a straight line parallel to the optical axis. The numerals with primes show the corresponding positions of the image-point on the straight line *E′ V*.

the far side of the Secondary Focal Plane; and when the Object is in the Primary Focal Plane, the Image is at infinity. If the Object lies between the Primary Focal Plane of a Convergent Lens and the Lens itself, the Image is virtual, erect and magnified. The Image of a virtual Object formed by a Convergent Lens is real, erect and diminished, and lies between the Lens and the Secondary Focal Plane.

In a Divergent Lens (Fig. 80) the Image of a real Object is always virtual, erect and diminished, and lies between the Secondary Focal Plane and the Lens. A Divergent Lens, however, will produce a real image of a virtual Object which is placed between the Lens and the Primary Focal Plane; but if the virtual Object lies beyond the Primary Focal Plane, the Image produced by a Divergent Lens will be virtual and inverted and will lie in front of the Secondary Focal Plane.

When $u = -u'$ (that is, when $AM = M'A$), formula (102) shows that Image and Object are equal in size but opposite in sign ($y'/y = -1$). Putting $u' = -u$ in formula (99), we find $u = -2f = 2AF$; so that for this special position of Object and Image, the Primary Focal Point F is midway between the points designated by A and M. When the Object-Point M has this position, the Image-Point M' is at the same distance from the Lens on the other side of it, and the Image $M'Q'$ is equal to QM. If the point M moves nearer to the Lens, $M'Q'$ becomes larger than QM, and if the point M moves the other way, the effect will be exactly opposite. The conjugate axial points M, M' which are so situated with respect to the Lens that $AM = M'A$ are the so-called "Negative Principal Points" of the Lens.

149. Refraction of Paraxial Rays through a Combination of Infinitely Thin Lenses. Suppose we have a centered system of spherical surfaces consisting of a number of thin lenses, whose optical centres, ranged along the optical axis, are designated by A_1, A_2, etc., in the order named; and let

$$d_k = A_k A_{k+1}$$

denote the distance between the kth and the $(k+1)$th Lenses of the system. Moreover, let M'_k designate the point where a paraxial ray crosses the optical axis after passing through the kth Lens, and let

$$u_k = A_k M'_{k-1}, \quad u'_k = A_k M'_k$$

denote the abscissæ, with respect to A_k, of the points M'_{k-1}, M'_k where the ray crosses the optical axis before and after passing through the kth Lens. Evidently, we shall have then the following equations:

$$u'_{k-1} = u_k + d_{k-1}, \quad \frac{1}{u'_k} - \frac{1}{u_k} = \frac{1}{f_k}; \tag{104}$$

where f_k denotes the Primary Focal Length of the kth Lens. If the number of Lenses is m, we must give k in succession all integral values from $k=1$ to $k=m$; noting also that $d_0=0$. Instead of u'_0 we write $u_1=A_1M_1$, where M_1 designates the position of the axial Object-Point. By means of these equations, we can determine the abscissa $u'_m=A_mM'_m$ of the Image-Point M'_m corresponding to the axial Object-Point M_1, provided we know the positions and Focal Lengths of the Lenses.

The Lateral Magnification of a combination of m Infinitely Thin Lenses is given by the formula:

$$Y=\frac{y'_m}{y_1}=\frac{u'_1\cdot u'_2\cdots u'_m}{u_1\cdot u_2\cdots u_m}. \tag{105}$$

In the special case *when the Infinitely Thin Lenses are all in contact*, so that we have

$$d_1=d_2=\cdots=d_k=0,$$

we obtain:

$$u'_{k-1}=u_k;$$

and, hence, by addition of all the equations of the type

$$1/u'_k-1/u_k=1/f_k,$$

we derive the following formula:

$$\frac{1}{u'_m}-\frac{1}{u_1}=\sum_{k=1}^{k=m}\frac{1}{f_k}=\frac{1}{f}. \tag{106}$$

The combination of Thin Lenses in Contact is therefore seen to be equivalent to a Single Thin Lens of Focal Length f such that $1/f$ is equal to the sum of the reciprocals of the Focal Lengths of the separate Lenses.

ART. 42. COTES'S FORMULA FOR THE "APPARENT DISTANCE" OF AN OBJECT VIEWED THROUGH ANY NUMBER OF THIN LENSES

150. More than twenty years ago, Lord RAYLEIGH[1] directed attention to the almost forgotten work on Optics published in 1783 by ROBERT SMITH,[2] professor of Astronomy in Cambridge University

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[1] Lord RAYLEIGH: Notes, chiefly Historical, on some Fundamental Propositions in Optics: *Phil. Mag.*, (5), xxi. (1886), 466–476.

[2] ROBERT SMITH: *A Compleat System of Opticks in four books, viz. A Popular, a Mathematical, a Mechanical, and a Philosophical Treatise. To which are added Remarks upon the Whole.* Cambridge, 1738. This work is in two large octavo volumes.

A German translation, edited with Notes and Additions by A. G. KAESTNER, was published by RICHTER in Altenburg in 1755. The title of this translation is: *Vollstaendiger Lehrbegriff der Optik nach Herrn* ROBERT SMITHS *Englischen mit Aenderungen und Zusaetzen ausgearbeitet von* A. G. KAESTNER.

In 1757 at Avignon, a French translation by LE PERE PEZENAS was published with the and afterwards Master of Trinity College. SMITH was the literary executor of his cousin ROGER COTES, the first Plumian Professor of Astronomy in Cambridge University and the editor of the second edition of NEWTON'S *Principia*, who died in 1716, *æt.* 34, leaving unfinished a series of elaborate researches in Optics. "Had COTES lived, we might have known something!" is a speech attributed to NEWTON, with whom he was closely associated. Chapter V of Book II of SMITH's treatise is founded on a "noble and beautiful theorem", said to have been the last piece of work of COTES's life. The theorem is remarkable as being, perhaps, the earliest generalization in Optics, and it was used by SMITH in a masterly fashion in deriving a number of very important corollaries, which, as RAYLEIGH observes, were afterwards "rediscovered in a somewhat different form by LAGRANGE, KIRCHHOFF, and VON HELMHOLTZ". For these reasons, and, also, on account of its elegant form and intrinsic value, COTES's Theorem should not be omitted from a treatise on Optics—especially too as it is now very hard to obtain a copy of SMITH's Optics. The theorem is given by P. CULMANN in his article on *Die Realisierung der optischen Abbildung*, published in the first volume of *Die Theorie der optischen Instrumente*, edited by M. VON ROHR (Berlin, 1904).

151. COTES used the term "*Apparent Distance*" to mean the distance at which the Object would have to be placed so as to appear to the naked eye of the same angular magnitude as its Image appeared when viewed through the optical system. Although this meaning has never come into general use, the term will be employed in this sense in the following exposition of COTES's Theorem.

In Fig. 81 let MQ represent an Object-Line perpendicular to the optical axis of a system of Infinitely Thin Lenses whose optical centres are at the points designated by A_1, A_2, etc. In the diagram the system is represented as consisting of only three lenses. Let the broken line $QP_1P_2P_3L'_3$ represent the path of the ray proceeding from the end-point Q of the Object and entering the eye supposed to be placed on the axis at the point L'_3. The points designated by P_1, P_2, P_3 are the points where this ray meets the lenses A_1, A_2, A_3, respectively. The point L_1 is the point where the incident ray QP_1 crosses the optical axis, and the point L'_3 is the point where the ray crosses the axis after

title: *Cours complet d'optique, traduit de l'anglois de* ROBERT SMITH, *contenant la théorie, la pratique et les usages de cette science. Avec des additions considérables sur toutes les nouvelles découvertes qu'on a faites en cette manière depuis la publication de l'ouvrage anglois.*

RAYLEIGH mentions another French translation also published in 1767, at Brest, by DUVAL LEROY.

passing through all three of the lenses. In the diagram the intersection at L_1 is represented as virtual, and that at L_3' is real. In the same way L_1' and L_2' designate the positions of the points where the ray crosses the axis after passing through the first and second lenses, respectively: both of these intersections, as shown in the diagram, are virtual. From the points Q, P_1 and P_2 draw straight lines parallel to the optical axis and produce them until they meet the straight line determined by the emergent ray P_3L_3', and from each of these points of intersection let fall perpendiculars on the axis at the points designated by K, J and H, respectively.

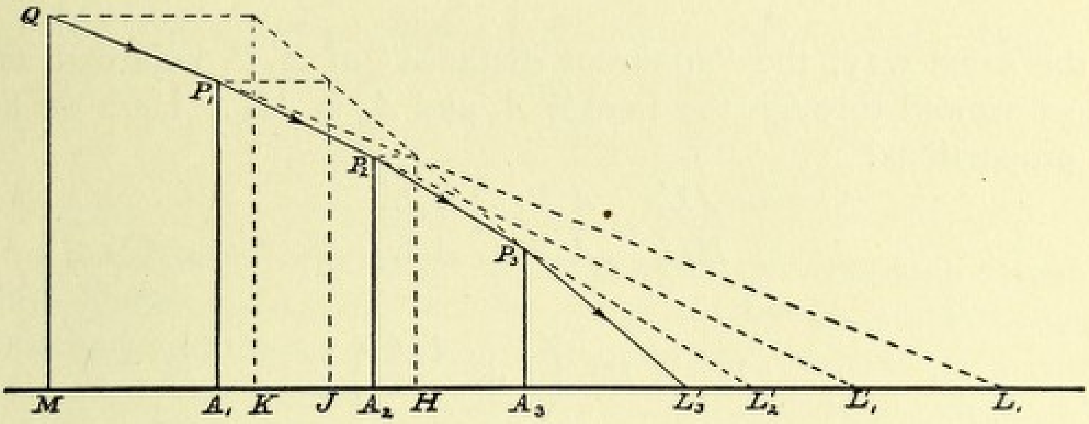

FIG. 81.

FIGURE USED IN DEDUCING COTES'S THEOREM. A_1P_1, A_2P_2, A_3P_3 represent three infinitely thin lenses. In the diagram these lenses are all represented as concave or divergent lenses. MQ is object-line perpendicular to optical axis of system of lenses. $QP_1P_2P_3L_3'$ is the outermost ray proceeding from the end-point Q of the object and traversing all the lenses. The eye is supposed to be placed on the axis at L_3'. KL_3' is the "apparent distance" of the object from the eye.

The "apparent distance" of A_2P_2, regarded as an object viewed through the Lens A_3 is HL_3'; and from the similar right triangles of the figure we obtain the following proportions:

$$\frac{HL_3'}{A_3L_3'} = \frac{A_2P_2}{A_3P_3} = \frac{A_2L_2'}{A_3L_2'};$$

and consequently:

$$HL_3' = A_3L_3'\left(1 + \frac{A_2A_3}{A_3L_2'}\right).$$

Since L_2', L_3' are a pair of conjugate axial points with respect to the Lens A_3, we have, according to formula (99):

$$\frac{1}{A_3L_2'} = \frac{1}{A_3L_3'} - \frac{1}{f_3};$$

where f_3 denotes the Focal Length of the Lens A_3. Introducing this value of $1/A_3L'_2$ in the above, we obtain:

$$HL'_3 = A_3L'_3 - \frac{A_2A_3 \cdot A_3L'_3}{f_3}.$$

In the same way, the "apparent distance" of A_1P_1 regarded as an Object viewed through the Lenses A_2 and A_3 is JL'_3. Here we have the proportions:

$$\frac{JL'_3}{HL'_3} = \frac{A_1P_1}{A_2P_2} = \frac{A_1L'_1}{A_2L'_1},$$

or

$$JL'_3 = HL'_3\left(1 + \frac{A_1A_2}{A_2L'_1}\right).$$

In the same way, also, we have here:

$$\frac{1}{A_2L'_1} = \frac{1}{A_2L'_2} - \frac{1}{f_2} = \frac{1}{HL'_3} \cdot \frac{A_3L'_3}{A_3L'_2} - \frac{1}{f_3} = \frac{1}{HL'_3}\left(1 - \frac{A_3L'_3}{f_3}\right) - \frac{1}{f_2};$$

and hence:

$$JL'_3 = A_1L'_3 - \frac{A_1A_2 \cdot A_2L'_3}{f_2} - \frac{A_1A_3 \cdot A_3L'_3}{f_3} + \frac{A_1A_2 \cdot A_2A_3 \cdot A_3L'_3}{f_2f_3}.$$

Again, the "apparent distance" of the Object-Line MQ viewed through the three Lenses A_1, A_2 and A_3 is KL'_3; and, as before:

$$\frac{KL'_3}{JL'_3} = \frac{MQ}{A_1P_1} = \frac{ML_1}{A_1L_1},$$

or

$$KL'_3 = JL'_3\left(1 + \frac{MA_1}{A_1L_1}\right).$$

Also,

$$\begin{aligned} \frac{1}{A_1L_1} &= \frac{1}{A_1L'_1} - \frac{1}{f_1} = \frac{HL'_3}{JL'_3}\left(\frac{1}{A_2L'_2} - \frac{1}{f_2}\right) - \frac{1}{f_3} \\ &= \frac{1}{JL'_3}\left(\frac{A_3L'_3}{A_3L'_2} - \frac{HL'_3}{f_2}\right) - \frac{1}{f_1} = \frac{1}{JL'_3}\left(1 - \frac{A_3L'_3}{f_3} - \frac{HL'_3}{f_2}\right) - \frac{1}{f_1}, \end{aligned}$$

so that

$$KL'_3 = JL'_3 + MA_1\left(1 - \frac{A_3L'_3}{f_3} - \frac{HL'_3}{f_2} - \frac{JL'_3}{f_1}\right);$$

and, finally we obtain the following formula for the "apparent distance" of the Object viewed through the Lenses A_1, A_2 and A_3:

$$KL'_3 = ML'_3 - \frac{MA_1 \cdot A_1L'_3}{f_1} - \frac{MA_2 \cdot A_2L'_3}{f_2} - \frac{MA_3 \cdot A_3L'_3}{f_3} + \frac{MA_1 \cdot A_1A_2 \cdot A_2L'_3}{f_1f_2} + \frac{MA_1 \cdot A_1A_3 \cdot A_3L'_3}{f_1f_3} + \frac{MA_2 \cdot A_2A_3 \cdot A_3L'_3}{f_2f_3} - \frac{MA_1 \cdot A_1A_2 \cdot A_2A_3 \cdot A_3L'_3}{f_1f_2f_3}. \quad (107)$$

This is COTES's Formula for the case when the system is composed of three lenses A_1, A_2, A_3; but the law of the formation of the terms is apparent, and the formula can be immediately written for a system of any number of Lenses. Thus, if we observe that the piece of the optical axis included between the Object at M and the eye at L'_3 may be considered as divided by the Lenses at A_1, A_2, A_3 into two, three and four segments in the following ways:

$$\begin{aligned} ML'_3 &= MA_1 + A_1L'_3 = MA_2 + A_2L'_3 = MA_3 + A_3L'_3 \\ &= MA_1 + A_1A_2 + A_2L'_3 = MA_1 + A_1A_3 + A_3L'_3 \\ &= MA_2 + A_2A_3 + A_3L'_3 = MA_1 + A_1A_2 + A_2A_3 + A_3L'_3, \end{aligned}$$

it will be seen that the members of each of these groups when multiplied together form the products which are the numerators of the fractions on the right-hand side of equation (107), while the denominators are the products of the Focal Lengths of the Lenses which occur in the numerators; the signs of the fractions being positive or negative according as the number of factors in the denominator is even or odd. The "apparent distance" is equal to the real distance added to the algebraic sum of the set of fractions whose numerators and denominators are formed according to the rule just explained.

A general proof of COTES's Theorem was given by LAGRANGE,[1] who was evidently acquainted with SMITH's work on Optics, as he refers to it in his paper.

152. Lord RAYLEIGH in the article above-mentioned (§ 150) quotes at length several of the corollaries which SMITH derives from COTES's Theorem, the first of which is as follows:

"While the glasses are fixt, if the eye and object be supposed to change places, the apparent distance, magnitude and situation of the object will be the same as before. For the interval ML'_3 being the same, and being divided by the same glasses into the same parts, will give the same theorem for the apparent distance as before."

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[1] J. L. DE LAGRANGE: Sur la théorie des lunettes: *Mémoires de l'Acad. de Berlin* (1780), 162–180.

Thus, in Fig. 81, if we suppose that the axial point of the Object is at L_3' and that the centre of the pupil of the eye is on the axis at the point designated by M, then A_3P_3 will be proportional to the breadth at the object-glass A_3 of the bundle of incident rays from the axial Object-Point L_3', and MQ will be proportional to the breadth of the corresponding bundle of emergent rays where they enter the eye at M, and from the figure we have evidently:

$$\frac{MQ}{A_3P_3}=\frac{KL_3'}{A_3L_3'};$$

whence is derived SMITH's Second Corollary, which he states as follows:

"When an object MQ is seen through any number of glasses, the breadth of the principal pencil where it falls on the eye at L_3', is to its breadth at the object-glass A_1, as the apparent distance of the object, to its real distance from the object-glass; and consequently in Telescopes, as the true magnitude of the object, to the apparent."

This very striking result can be put in a different form. Thus, from the figure, we obtain:

$$\frac{MQ}{A_3P_3}=\frac{MQ}{A_1P_1}\cdot\frac{A_1P_1}{A_2P_2}\cdot\frac{A_2P_2}{A_3P_3}=\frac{ML_1}{A_1L_1}\cdot\frac{A_1L_1'}{A_2L_1'}\cdot\frac{A_2L_2'}{A_3L_2'};$$

and therefore:

$$\frac{KL_3'}{ML_1}=\frac{A_1L_1'\cdot A_2L_2'\cdot A_3L_3'}{A_1L_1\cdot A_2L_1'\cdot A_3L_2'}.$$

The expression on the right-hand side of this equation, according to formula (105), is the value of the Lateral Magnification y_3'/y_1 at the conjugate axial points L_1, L_3', so that we have:

$$\frac{KL_3'}{ML_1}=\frac{y_3'}{y_1}.$$

Moreover,

$$\frac{KL_3'}{ML_1}=\frac{\tan\angle A_1L_1P_1}{\tan\angle A_3L_3'P_3}=\frac{\tan\theta_1}{\tan\theta_3'};$$

and hence we derive the formula:

$$y_3'\cdot\tan\theta_3'=y_1\cdot\tan\theta_1;$$

or, if the system consists of m Lenses:

$$y'_m \cdot \tan \theta'_m = y_1 \cdot \tan \theta_1. \qquad (108)$$

This formula, which was given by LAGRANGE[1] more than fifty years after the publication of SMITH's Optics, is a particular case of the general formula usually known in Optics as the HELMHOLTZ *Equation* (see § 194).

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[1] J. L. DE LAGRANGE: Sur une loi générale d'optique: *Mémoires de l'Acad. de Berlin*, 1803.

CHAPTER VII.

THE GEOMETRICAL THEORY OF OPTICAL IMAGERY.

I. INTRODUCTION.

ART. 43. ABBE'S THEORY OF OPTICAL IMAGERY.

153. The function of an optical instrument is to produce an image of an external object. Each point of the object is the base (or vertex) of a bundle of rays, of which, in general, only a part is utilized in the formation of the image. These object-rays which are affected by the instrument are called the "incident" rays. Within the apparatus these rays undergo a series of refractions (or reflexions) at the plane or curved boundary-surfaces of suitably disposed optical media; and, thus modified, they "emerge" into the last medium and form there a more or less perfect image of the object, which may be "real" or "virtual", etc.; the nature of the image in the several respects of position, dimensions, orientation, etc., depending primarily on the peculiarity and design of the instrument itself. Proceeding from any point P of the Object, a bundle of incident rays "enters" the optical instrument, and emerging therefrom, a portion of these rays at least, if not all of them, will intersect ("really" or "virtually") in the corresponding, or "conjugate", point P' of the Image. In the case of an ideal, or geometrically perfect, image, *all* of the emergent rays corresponding to the rays of the bundle of incident rays P will intersect in the Image-Point P'; so that a homocentric bundle of object-rays will be (as the German writers say) "imaged" (*abgebildet*) by a homocentric bundle of image-rays.

154. Until comparatively recent times the method of investigation of the relations between image and object in Optics was to advance by a process of mathematical induction from simple special cases to more complex general cases of homocentric imagery. This method was used with conspicuous success by ROGER COTES (§ 150), first Plumian Professor of Astronomy in Cambridge University, whose brilliant and original contributions to optical science were cut short by his untimely death (1716) at the age of thirty-four years. The same method was employed also by C. F. GAUSS in his famous *Dioptrische Untersuchungen* (Goettingen, 1841), who developed completely the theory of the refraction of paraxial rays through a centered system of infinitely thin lenses. By substituting in place of the original data, such as the radii, refractive indices, etc., certain constants of a much more general kind, GAUSS obtained remarkably simple formulæ, which marked a great advance in optical theory and added a new encouragement to such investigations. But even GAUSS, with his extraordinary insight and rare gift of analysis, seems not to have discerned that the general laws of optical imagery are independent of all special assumptions as to the particular mode of producing the image.

MOEBIUS,[1] indeed, came nearer to the real and essential idea of an optical image when he pointed out that the unique connection between Object-Point and Image-Point in the case of the refraction of paraxial rays at a spherical surface is equivalent to the expression of the relation of Collinear Correspondence between Object-Space and Image-Space; and that if this is true in the case of a single spherical refracting surface, it must be true also for the relation between object and image in the refraction of paraxial rays through a centered system of spherical refracting surfaces; and, hence, finally, that all the formulæ showing the relation between object and image in such a case as this were deducible from the theory of Collinear Correspondence. This presentment was quickly seized by other investigators (as F. LIPPICH,[2] A. BECK[3] and H. HANKEL[4]) who, following the lead of MOEBIUS, and, like him, employing the methods of projective geometry, extended this idea of optical imagery to less simple cases. Thus, for example, F. LIPPICH[5] showed that there is also collinear correspondence of object and image in the case of infinitely narrow bundles of rays incident on a spherical refracting surface at finite slopes. Yet neither MOEBIUS himself nor any of his followers in this mode of treating the matter was able to discard entirely the idea that some kind of Dioptric action was essential for the production of an optical image. At least not one of them stated distinctly that a purely geometrical assumption was all that was necessary, viz., that *an optical image is produced by rays.*

155. A remarkable paper "On the General Laws of Optical Instruments" was contributed in 1858 by JAMES CLERK MAXWELL to *The*

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[1] A. F. MOEBIUS: Entwickelung der Lehre von dioptrischen Bildern mit Huelfe der Collineations-Verwandschaft: *Leipziger Berichte*, vii. (1855), 8–32.

[2] F. LIPPICH: Fundamentalpunkte eines Systemes centrirter brechender Kugelflaechen: *Mittheilungen des naturwissenschaftlichen Vereines für Steiermark*, ii. (1871), 429–459.

[3] A. BECK: Die Fundamentaleigenschaften der Linsensysteme in geometrischer Darstellung: *Zft. f. Math. u. Phys.*, xviii. (1873), 588–600.

[4] H. HANKEL: *Die Elemente der projektivischen Geometrie in synthetischer Behandlung* (Leipzig, 1875).

[5] F. LIPPICH: Ueber Brechung und Reflexion unendlich duenner Strahlensysteme an Kugelflaechen: *Wiener Denkschr.*, xxxviii. (1878), 163–192.

Quarterly Journal of Pure and Applied Mathematics, ii., 233–246; it is reprinted in the collection of MAXWELL's *Scientific Papers*, vol. i., 271–285. In an introduction to this article, MAXWELL describes the undertaking as follows:

"The investigations which I now offer are intended to show how simple and how general the theory of optical instruments may be rendered, by considering the optical effects of the entire instrument, without examining the mechanism by which these effects are obtained. I have thus established a theory of 'perfect instruments', geometrically complete in itself, although I have also shown that no instrument depending on refraction and reflexion (except the plane mirror) can be optically perfect."

A "perfect instrument" is one which is free from "certain defects incident to optical instruments"; thus, according to MAXWELL, "a perfect instrument must fulfil three conditions:

"I. Every ray of the pencil, proceeding from a single point of the object, must, after passing through the instrument, converge to, or diverge from, a single point of the image. The corresponding defect when the emergent rays have not a common focus, has been appropriately called (by Dr. WHEWELL) *Astigmatism*.

"II. If the object is a plane surface, perpendicular to the axis of the instrument, the image of any point of it must lie in a plane perpendicular to the axis. When the points of the image lie in a curved surface, it is said to have the defect of *curvature*.

"III. The image of an object on this plane must be similar to the object, whether its linear dimensions be altered or not; when the image is not similar to the object, it is said to be *distorted*."

Assuming that the image is free from these three defects, and is therefore a "perfect image", MAXWELL derives formulæ for the relative positions and magnitudes of the object and image which are precisely equivalent to the formulæ obtained by GAUSS; but the difference consists in the fact that, whereas GAUSS's investigations are based on certain physical assumptions not only in regard to the Law of Refraction of light-rays, but also as to a centered system of spherical surfaces and paraxial rays, the *modus operandi* is left out of consideration entirely by MAXWELL, who shows that an optical image, however it may be produced, provided it is free from the geometrical "defects" above enumerated, must have certain perfectly definite geometrical relations with the object. This very important idea seems to have been clearly perceived and distinctly stated by MAXWELL first of all.

156. The most notable contribution in recent years to the literature of Geometrical Optics is Dr. S. CZAPSKI's *Theorie der optischen Instrumente nach ABBE*, the first edition of which was published in Breslau in 1893. In this brilliant work, recognized immediately as an epoch-making book, was set forth for the first time a complete and masterly exposition of the remarkable theories of Professor ABBE, of Jena.

ABBE, without a knowledge of the investigations of MOEBIUS and MAXWELL, discerned even more clearly than they that the physical agency or mechanism which was employed in the actual formation of an optical image was in no wise involved in the geometrical theory of optical imagery; so that without any special assumptions whatever as to the construction or constitution of the optical apparatus, and even without reference to the physical laws of reflexion and refraction, he deduced in the simplest and most direct way all the laws concerning the relative positions, dimensions, etc. of the object and image.

Thus, the fundamental and essential characteristic of optical imagery is a point-to-point correspondence, by means of rectilinear rays, between object and image; and from this one assumption—at once the most natural and the most obvious—ABBE, in his celebrated university lectures, used to deduce the general laws of optical images.

The advantage of this is that in investigating an actual image produced by an optical instrument it will be possible to separate what in the laws of this image depends on the general fundamental laws of optical imagery and what is due to the particular mode of producing the image. Moreover, although to-day a certain optical instrument may be a mechanical impossibility, it is possible to say whether such a system is theoretically practicable; so that the geometrical theory will point the way of future inventions.

In the modern geometry this unique point-to-point correspondence by means of rectilinear rays between image and object is called "*Collineation*"—a term introduced by MOEBIUS in his great work entitled *Der barycentrische Calcul* (Leipzig, 1827).

II. THE THEORY OF COLLINEATION, WITH SPECIAL REFERENCE TO ITS APPLICATIONS TO GEOMETRICAL OPTICS.

ART. 44. TWO COLLINEAR PLANE-FIELDS.

157. Definitions. In this treatment we shall employ the beautiful and appropriate methods of projective geometry. As some readers may not be entirely familiar with the terms here employed, a brief introduction may be required.

The totality of points and straight lines which are contained in a plane is called a "*plane-field*", and the plane is then said to be the "base "of this system of points and lines. All the points lying on a straight line of the field, taken together, form a "*range of points*", the straight line itself being called the "base" of the point-range. A straight line considered as a whole (that is, without reference to the points which lie on it) is called a "*ray*". All the straight lines of a plane-field which go through one point form a "*pencil of rays*", and the common point of intersection of these rays may be regarded as the "base" of the pencil.

Two plane-fields π and π' are said to be collinear, if to every point P of π there corresponds one point P' of π', and to every straight line p of π which goes through P there corresponds a straight line p' of π' which goes through P'.

The totality of rays which go through a single point O in space is called a "*bundle of rays*", so that a bundle of rays consists of an infinite number of pencils of rays. We speak also of a "*bundle of planes*", meaning thereby the totality of planes which pass through one point O. In either case the point O which is common to all the elements of the bundle is the "base" of the bundle. A "*sheaf of planes*" is the term applied to the totality of planes which all have one common line of intersection: thus, in a bundle of planes are comprised an infinite number of sheaves of planes. The common line of intersection is the "base" of the sheaf of planes.

A plane-field π and a bundle of rays O' are said to be collinear with each other if to every point P of π there corresponds a ray p' of O', and to every straight line l of π that goes through P there corresponds a plane λ' of the bundle O' that contains the straight line p'.

And, again:

Two bundles of rays O and O' are said to be collinear with each other, if to each ray p of O there corresponds a ray p' of O', and to each plane λ of O that contains p there corresponds a plane λ' of O' that contains p'.

158. Projective Relation of Two Collinear Plane-Fields.

Two collinear plane-fields π and π' are also called "projective", because to each harmonic range of four points of π there corresponds a harmonic range of four points of π'.

Thus, if P, Q, R, S (Fig. 82) are a harmonic range of four points of the plane-field π, and if P', Q', R', S' are the four corresponding points of the collinear plane-field π', in the first place, since the points P, Q, R, S all lie upon a straight line s, the points P', Q', R', S' must all likewise lie upon a straight line s' which is conjugate to s. Let

$ABCD$ be any quadrangle of the plane-field π, such that the two opposite sides AB and CD intersect in the point P, and the other two opposite sides AD and BC intersect in the point Q, while the two diagonals BD and AC go through the points R and S, respectively. To this quadrangle of π there will correspond a certain quadrangle $A'B'C'D'$ of π', such that the two opposite sides $A'B'$ and $C'D'$ intersect in the point P', the other two opposite sides $A'D'$ and $B'C'$ intersect in the point Q', and the two diagonals $B'D'$ and $A'C'$ go through the points R' and S', respectively. Accordingly, the points P', Q', R', S' are also a harmonic range of points; and this is the condition that the two plane-fields π and π' shall be *projective*.

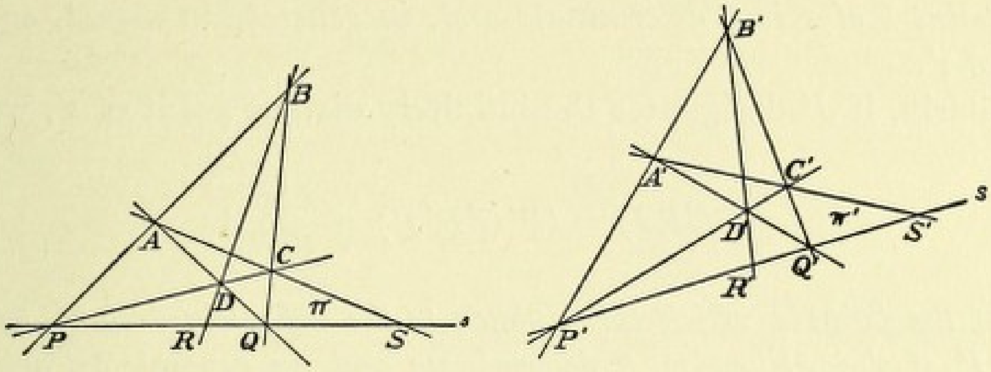

FIG. 82

PROJECTIVE RELATION OF TWO COLLINEAR PLANE-FIELDS.

By a similar method we can show also that two collinear bundles of rays or a bundle of rays and a plane-field in collinear relation are projective to each other.

159. The so-called "Flucht" Points of Conjugate Rays. Let s and s' denote two conjugate rays of the collinear plane-fields π and π'. Since, as has just been shown, the point-ranges s, s' are projective, it follows that *the Double Ratio* ($PQRS$) *of any four points* P, Q, R, S *of* s *is equal to the Double Ratio* ($P'Q'R'S'$) *of the four corresponding points* P', Q', R', S' *of* s'. That is,

$$\frac{PR}{QR} : \frac{PS}{QS} = \frac{P'R'}{Q'R'} : \frac{P'S'}{Q'S'}.$$

If we suppose that P, Q and R are three fixed points of s and that S is a variable point, the Double Ratio ($PQRS$) will vary in value as the point S moves along s; and if the point S moves away to an infinite distance until it coincides with the infinitely distant point[1]

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[1] Every actual straight line contains one (and only one) infinitely distant (or ideal) point, and all rays having in common the same infinitely distant point are parallel.

I of s, we shall have:

$$(PQRI) = (P'Q'R'I') = \frac{PR}{QR},$$

where I' designates the point on s' which corresponds to the infinitely distant point I of s. Since P, Q and R are three actual, or finite, points of s, no pair of which are supposed to be coincident, the value of the ratio $PR : QR$ is finite; and hence *the point I' conjugate to the ideal point I of s is a determinate and, in general, an actual, or finite, point of s'.*

Similarly, if J' designates the infinitely distant point of s', we shall have:

$$(PQRJ) = (P'Q'R'J') = \frac{P'R'}{Q'R'};$$

so that *the point J which corresponds to the infinitely distant, or ideal, point J' of s' is, likewise, a determinate and, in general, an actual, or finite, point of s.*

In general, therefore, the points J and I', corresponding to the infinitely distant points J' and I of s' and s, respectively, are actual, or finite, points having perfectly determinate positions on s and s', respectively. In the German treatises the points J and I' are called the "*Flucht*" *Points* of the two projective point-ranges s and s'.

It will be remarked that we are careful to say that the so-called "Flucht" Points are "in general" actual, or finite, points; for in one special case, viz., when

$$(PQRJ) = (P'Q'R'I') = \frac{PR}{QR} = \frac{P'R'}{Q'R'},$$

the "Flucht" Point I' will coincide with the infinitely distant point J' of s', and the "Flucht" Point J will, likewise, coincide with the infinitely distant point I of s; and *in this particular case the infinitely distant points I and J' of the projective point-ranges s and s' will also be a pair of conjugate points.*

160. The so-called "Flucht" Lines (or Focal Lines) of Conjugate Planes. In the plane-field π consider now a quadrangle $ABCD$ (Fig. 83) such that the two pairs of opposite sides form two pairs of parallel straight lines. The two parallel sides AB and CD intersect in the infinitely distant point P, and, similarly, the other two parallel sides AD and BC intersect in the infinitely distant point Q; so that if R and S designate the infinitely distant points of the two diagonals BD and AC, respectively, the four points P, Q, R, S are a harmonic range of points of the infinitely distant, or ideal, straight line i of the plane π.[1]

In the collinear plane-field π' the ray $A'B'$ conjugate to AB will go through the point P' conjugate to the infinitely distant point P of the ray AB; so that the point P' is therefore the "Flucht" Point of the ray $A'B'$. Obviously, the point P' is also the "Flucht" Point of the ray $C'D'$ conjugate to CD. Precisely in the same way, the point Q', conjugate to the infinitely distant point Q of the parallel rays AD and BC, is the common "Flucht" Point of each of the rays $A'D'$ and $B'C'$ conjugate to the rays AD and BC, respectively. Let R' and S' designate the positions of the "Flucht" Points of the rays $B'D'$ (conjugate to BD) and $A'C'$ (conjugate to AC), respectively.

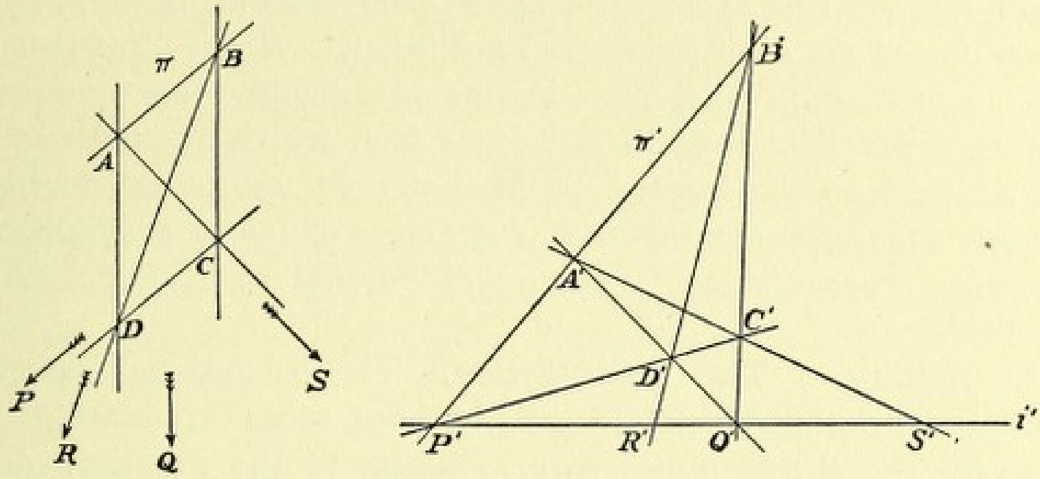

FIG. 83.

THE "FLUCHT" LINE i' OF THE PLANE-FIELD π' CORRESPONDING TO THE INFINITELY DISTANT STRAIGHT LINE OF THE COLLINEAR PLANE-FIELD π.

Since (§ 158) the points P', Q', R', S' are a harmonic range of points, they all lie on a certain definite straight line i' of the plane-field π'; and *this straight line i', which is conjugate to the infinitely distant straight line i of the plane-field π, is the locus of the "Flucht" Points of all the rays of the plane-field π' collinear with π.*

Similarly, *there is a certain straight line j of the plane-field π, conjugate to the infinitely distant straight line j' of the collinear plane-field π', which is the locus of the "Flucht" Points of all the rays of π.*

German writers call these two straight lines j and i' the "*Flucht*"

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[1] All the infinitely distant points of a plane are assumed to lie in an infinitely distant, or ideal, straight line. The ideal line of a plane must be a straight line, because every actual straight line of the plane meets it in only one point — the infinitely distant point of that line; whereas a curved line may have in common with a straight line more than one point. Just as a pencil of parallel rays determines one infinitely distant point common to all the rays, so a sheaf of parallel planes determines one infinitely distant straight line common to all the planes.

Lines (or "*Gegenaxen*") of the two projective plane-fields. We shall designate them hereafter, from the stand-point of Optics, as the *Focal Lines* of the two conjugate planes π and π'.

If two plane-fields are collinear, then, in general (that is, except in one particular case considered in § 161 below), *to the infinitely distant, or ideal, straight line of one field there corresponds an actual, or finite, straight line of the other field, the so-called "Focal Line" of that field.*

To a pencil of parallel rays in one plane-field there corresponds therefore a pencil of rays in the other field which all intersect in a point situated on the Focal Line of that field; or, as we might say, *the Focal Line of one plane-field is the locus of the bases of pencils of rays which are conjugate to pencils of parallel rays of the collinear plane-field.*

161. Affinity of Two Plane-Fields. The exceptional case mentioned above cannot be passed over without some explanation. The points P', Q', R', S' of π', corresponding to the infinitely distant points P, Q, R, S of π are, in general (as was stated), actual, or finite, points, and determine, therefore, an actual, or finite, straight line i'; except in the one particular case when the quadrangle $A'B'C'D'$, as well as the quadrangle $ABCD$, has each pair of its opposite sides parallel. In this special case the points P', Q', R', S' will be ranged along the infinitely distant straight line j' of the plane-field π', and the Focal Line i' will therefore coincide with the infinitely distant straight line j'.

This special case, in which the two Focal Lines j and i' are also the infinitely distant straight lines i and j' of the collinear plane-fields π and π', respectively, is the so-called case of "Affinity" of the two plane-fields.

This extremely important special case will be met with again. Here we merely call attention to it.

ART. 45. TWO COLLINEAR SPACE-SYSTEMS.

162. *Two Space-Systems Σ and Σ' are said to be collinear with each other if to every point P of Σ there corresponds one (and only one) point P' of Σ', and to every straight line p of Σ, which goes through P, there corresponds one straight line p' of Σ' which goes through P'.*

It is not necessary to think of Σ and Σ' as two separate and distinct regions of space; they are to be regarded rather as completely interpenetrating one another, so that any point, ray or plane in space may be considered, according to the point of view, one time as belonging to the system Σ and another time as belonging to the system Σ'. In fact, when we say that two space-systems are collinear with each other, we mean that the whole of space is in collinear relation with itself; or in the language of the modern geometry, the whole of space is the common "base" of the two space-systems Σ, Σ'.

In the geometrical theory of Optics the two space-systems Σ, Σ' are distinguished as the *Object-Space* and the *Image-Space*, respectively; and the points, rays and planes of space, according as they are regarded as belonging to the one or the other of these two space-systems, are called Object-Points, Object-Rays and Object-Planes or Image-Points, Image-Rays and Image-Planes. Since the relation between the Object-Space and the Image-Space is perfectly reciprocal, there is no essential difference between them; whence is deduced at once the theorem known as the *Principle of the Reversibility of the Light-Path* (§ 18).

A direct consequence of the unique point-to-point and ray-to-ray correspondence between Object-Space and Image-Space is plane-to-plane correspondence; so that to every plane π of the Object-Space there corresponds a definite plane π' of the Image-Space, and *vice versa*. Thus, using the language of the modern geometry, we may say:

In two Collinear Space-Systems to every plane-field there corresponds a collinear plane-field; to every bundle of rays or planes, a collinear bundle of rays or planes; and to every point-range, a projective point-range.

163. Two Space-Systems Σ and Σ' may be placed in collinear correspondence with each other by taking any two bundles of rays A and B of Σ and associating them with any two bundles of rays A' and B' of Σ' in such fashion that the rays AB, $A'B'$ common to the two pairs of bundles are corresponding rays, and the sheaf of planes AB of Σ corresponds with the sheaf of planes $A'B'$ of Σ'. For if this correspondence is established, and if P designates a point of Σ, the pair of rays AP, BP determine a certain plane η of the sheaf of planes AB to which corresponds in Σ' a plane η' of the sheaf of planes $A'B'$, and corresponding to the rays AP, BP of Σ, which intersect in P, there will be two rays $A'P'$, $B'P'$ of Σ', which determine by their intersection the point P' of Σ' corresponding to any point P of Σ. Moreover, corresponding to any ray s of Σ projected from A and B by the planes As and Bs, respectively, there will be a ray s' of Σ' which is determined by the intersection of the two planes $A's'$ and $B's'$ corresponding to the planes As and Bs, respectively. And, finally, if π denotes any plane-field of Σ whereby the two bundles of rays A and B are in perspective with each other, with the ray AB

common to the two bundles, the two corresponding bundles of rays, A' and B', being also in perspective relation with each other, with the ray $A'B'$ in common, will, accordingly, determine a plane-field π' of Σ' collinear with the plane-field π of Σ. Therefore, the two Space-Systems Σ and Σ' are placed in complete collinear correspondence.

From this we derive immediately the following rule:

If we take any five points of one Space-System, no four of which lie in one plane, and associate them as corresponding with five such points of the other Space-System, the two Space-Systems will be completely collinear to each other.

Thus, suppose we take five points A, B, C, D, E of Σ, no four of which lie in one plane, and associate them with five such points A', B', C', D', E' of Σ', then the two bundles of rays AB, AC, AD, AE and BA, BC, BD, BE of Σ correspond to the two bundles of rays $A'B'$, $A'C'$, $A'D'$, $A'E'$ and $B'A'$, $B'C'$, $B'D'$, $B'E'$, respectively, of Σ', and to the sheaf of planes ABC, ABD, ABE of Σ corresponds the sheaf of planes $A'B'C'$, $A'B'D'$, $A'B'E'$ of Σ'; and we see, accordingly, that the rule given above is equivalent to the method which we gave first.

Since each point of a space-pentagon may have a 3-fold infinitude of positions, it is obvious that two Space-Systems may have a 15-fold infinitude of collineations.

164. The so-called "Flucht" Planes, or Focal Planes, of Two Collinear Space-Systems.

In two collinear Space-Systems Σ and Σ' let π and π' designate two corresponding plane-fields, wherein $KLMN$ and $K'L'M'N'$ are two corresponding quadrangles. The quadrangle $KLMN$ determines a harmonic range of four points P, Q, R, S which all lie on a straight line s of the plane-field π; and, similarly, the quadrangle $K'L'M'N'$ determines also a harmonic range of four points P', Q', R', S', which are conjugate to P, Q, R, S, respectively, and which all lie on a straight line s' of π' which is conjugate to s. From a point A of Σ, lying outside the plane-field π, this field is projected by a bundle of rays or planes, and from the corresponding point A' of Σ' the plane-field π' will be projected by a bundle of rays or planes which is projective with the bundle A; so that, for example, the four rays AP, AQ, AR, AS and the four corresponding rays $A'P'$, $A'Q'$, $A'R'$, $A'S'$ form two harmonic pencils of rays. The complete quadrangles $KLMN$ and $K'L'M'N'$ are projected from A and A', respectively, by two complete four-edges.

Now suppose that the two pairs of opposite sides KL, MN and LM, NK of the quadrangle $KLMN$ are two pairs of parallel straight lines, so that the four points P, Q, R, S are a harmonic range of points all lying on the infinitely distant straight line i of the plane-field π. In the collinear plane-field π' the two pairs of opposite sides of the quadrangle $K'L'M'N'$ will, in general, not be pairs of parallel straight lines, so that the harmonic range of points P', Q', R', S' conjugate to the infinitely distant points P, Q, R, S will, in general, determine a finite straight line i', the so-called "Flucht" Line, or Focal Line (§ 160) of the plane-field π' corresponding to the infinitely distant straight line i of the plane-field π.

To the plane Ai parallel to the plane-field π corresponds the plane $A'i'$ of Σ', which, in general, will not be parallel to the plane-field π'. If now the point A is itself an infinitely distant point of the Space-System Σ, the corresponding point A' will be the common "Flucht" Point of all the rays of the bundle conjugate to the bundle of parallel rays of Σ whose direction is determined by the infinitely distant point A; and, in general, A' will be a determinate and actual, or finite, point of the Space-System Σ'. In this case the plane Ai will be the infinitely distant plane[1] ϵ of Σ, and the corresponding plane $A'i'$ is the so-called "*Flucht*" *Plane* ϵ' of Σ'. It contains the "Flucht" Lines of all the planes and the "Flucht" Points of all the rays of Σ'.

Similarly, there is a certain plane φ of Σ, conjugate to the infinitely distant plane φ' of Σ', in which are contained the "Flucht" Lines of all the planes and the "Flucht" Points of all the rays of Σ.

These two planes φ and ϵ' are the so-called "Flucht" Planes of the two Space-Systems Σ and Σ', respectively. In the geometrical theory of optical imagery they play a very important part, and are called the *Focal Planes* of the Object-Space and Image-Space. Hence:

If we have two collinear Space-Systems Σ *and* Σ', *which, in the language of Geometrical Optics, we shall call the "Object-Space" and the "Image-Space", respectively, then* (except in the so-called case of Telescopic Imagery, referred to below) *to the infinitely distant* (*or ideal*) *plane of one system there will correspond a finite* (*or actual*) *plane, the so-called "Flucht" Plane or Focal Plane, of the other system.*

Thus, to a bundle of parallel rays in one space there will correspond a bundle of rays in the other space which all intersect in a point of the Focal Plane of that space.

165. Affinity-Relation between Object-Space and Image-Space. In the exceptional case when the quadrangle $K'L'M'N'$, as well as the quadrangle $KLMN$, has each pair of its opposite sides parallel, so that the points P', Q', R', S', corresponding to the infinitely distant points P, Q, R, S of the plane-field π, are themselves also infinitely distant points lying on the infinitely distant straight line j' of the plane-field π', the plane $A'j'$ conjugate to the plane Ai is parallel to the plane π'. And if also the two corresponding points A, A' are infinitely distant points of the Space-Systems Σ, Σ', respectively, then *the two Focal Planes* φ *and* ϵ' *are also the infinitely distant planes* ϵ *and* φ' *of the Object-Space and the Image-Space, respectively.* This case, which actually occurs in certain optical systems, is called in geometry the case of "*Affinity*" of the two Space-Systems. In Optics it is the important case known as "*Telescopic Imagery*".

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[1] The infinitely distant points and lines of space are assumed to lie in an infinitely distant or ideal surface, which, since it is intersected by every actual straight line in only one point and by every actual plane in a straight line, must be a plane surface—*the infinitely distant plane* of space.

ART. 46. GEOMETRICAL CHARACTERISTICS OF OBJECT-SPACE AND IMAGE-SPACE.

166. Conjugate Planes. The two Focal Planes φ and ϵ' of the Object-Space and Image-Space, respectively, not only from the optical but from the geometrical stand-point as well, are the most distinguished planes of the Space-Systems to which they belong. The exceptional case of *Telescopic Imagery*, alluded to in § 165, in which the Focal Planes φ and ϵ' are themselves the infinitely distant planes of the Space-Systems Σ and Σ', respectively, will be treated specially and in detail in a separate division of this chapter. Therefore entirely excluding this case for the present, and *assuming that the Focal Planes* φ *and* ϵ' *are finite, or actual, planes*, we proceed to enumerate the most striking general characteristics of the collinear correspondence of two Space-Systems Σ and Σ'.

1. *In general, to a sheaf of parallel planes of one of the two Space-Systems there will correspond a sheaf of non-parallel planes of the other Space-System.*

The axis of the sheaf of parallel planes is an infinitely distant straight line of the Space-System Σ to which this sheaf is supposed to belong; and the axis of the conjugate sheaf of planes will be, therefore, a straight line lying in the Focal Plane of the other Space-System Σ'. Generally speaking, this straight line will be a finite, or actual, line of Σ', and such a line can only be the base of a sheaf of non-parallel planes.

However, there is one very important exception to the above statement, viz.:

2. *The two sheaves of parallel planes to which the Focal Planes themselves belong are conjugate sheaves.*

The infinitely distant straight lines of the Focal Planes φ and ϵ' are the bases or axes of these two sheaves of parallel planes, and since these infinitely distant straight lines are the lines of intersection of the Focal Planes with the infinitely distant planes of their respective Space-Systems, they are a pair of infinitely distant conjugate straight lines, in fact (the case of Telescopic Imagery being excluded) the only such pair of conjugate lines. Hence, the two sheaves of parallel planes which have these two infinitely distant conjugate straight lines as axes are conjugate sheaves of planes.

Two conjugate planes σ *and* σ' *which are parallel to the Focal Planes* φ *and* ϵ', *respectively, are in the relation of "Affinity" to each other,* because their infinitely distant straight lines are a pair of conjugate straight lines. Since, therefore, to each infinitely distant point of one such plane there corresponds also an infinitely distant point of the plane in "affinity" with it, it follows that:

Parallel straight lines of the plane σ *correspond to parallel straight lines of the plane* σ'; so that a parallelogram in the Object-Plane σ will be "imaged" by a parallelogram in the Image-Plane σ'.

Moreover:

Any range of points r of the Object-Space, which is parallel to the Focal Plane φ, *will be "imaged" in a "projectively similar"*[1] *range of points r′ of the Image-Space, which is likewise parallel to the Focal Plane* ϵ'.

167. The Focal Points and the Principal Axes of the Object-Space and the Image-Space.

3. *To a bundle of parallel rays in the Object-Space will correspond, in general, a bundle of non-parallel rays in the Image-Space, the vertex of which lies in the Focal Plane of that space; and vice versa.*

The particular point of the Focal Plane which will be the vertex of the bundle of non-parallel rays will depend on the direction of the bundle of parallel rays. If, for example, the bundle of parallel rays in one space meets the Focal Plane of that space *at right angles*, the vertex of the corresponding bundle of rays in the other space will determine a certain definite point in the Focal Plane of that space, viz., the so-called *Focal Point* of that space. The *Focal Point of the Object-Space*, designated by F, is the vertex of the bundle of object-rays to which corresponds a bundle of parallel image-rays which cross the

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[1] The peculiarity of "*projectively similar*" *ranges of points* is that the lengths of corresponding segments of them are in a constant ratio to each other. Thus, for example, if r, r' are two projective ranges of points whose infinitely distant points W, W' correspond to each other, and if A, A'; B, B'; C, C' are any three pairs of conjugate finite points of r, r', then, since $(ABCW) = (A'B'C'W')$, we have immediately:

$$AC : BC = A'C' : B'C', \quad \text{or} \quad A'C' : AC = B'C' : BC.$$

Focal Plane of the Image-Space at right angles; and, similarly, the *Focal Point of the Image-Space*, designated by E', is the vertex of the bundle of image-rays to which corresponds a bundle of parallel object-rays which cross the Focal Plane of the Object-Space at right angles.

The two straight lines drawn through the Focal Points F and E' perpendicular to the Focal Planes φ and ϵ' are called the *Principal Axes* of the Object-Space and the Image-Space, respectively. This pair of straight lines will be designated as the axes of x and x'. Since the ray x' passes through the Focal Point E' (which corresponds to the infinitely distant point E of x) and also through the infinitely distant point F' (which corresponds to the Focal Point F likewise situated on x), it follows that *the Principal Axes x and x' are a pair of conjugate straight lines* and, in fact, *this is the only pair of conjugate rays which are at right angles to the Focal Planes.*

168. Axes of Co-ordinates. An immediate consequence of the fact that x and x' are a pair of conjugate rays is the following:

To the sheaf of planes in the Object-Space which has for its axis the x-axis corresponds the sheaf of planes in the Image-Space which has for its axis the x'-axis.

Of these two projective sheaves of so-called "*Meridian Planes*", there is, according to an elementary law of projective geometry, one pair of Meridian Object-Planes at right angles to each other to which corresponds a pair of Meridian Image-Planes which are also at right angles to each other. In each space this particular pair of Meridian Planes at right angles to each other, together with a third plane perpendicular to the Principal Axis, and, therefore, perpendicular to each of the two Meridian Planes, will determine by their intersections a set of three mutually perpendicular straight lines. Hereafter, when we come to derive the Image-Equations, we shall find it convenient to select these two sets of straight lines as the axes of two systems of rectangular co-ordinates, one in the Object-Space and the other in the Image-Space. One of these straight lines is, of course, the Principal Axis x or x' of the Space-System. But, whereas x, x' will always be a pair of conjugate straight lines, the other two pairs of straight lines, designated after the manner of Analytic Geometry, as the y-axis and z-axis in the Object-Space and the y'-axis and z'-axis in the Image-Space, may, or may not, be pairs of conjugate straight lines. This will depend on whether the yz-plane and the $y'z'$-plane are a pair of conjugate planes.[1]

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[1] Strict consistency in the matter of notation, which is eminently desirable, especially in Geometrical Optics, cannot, however, always be observed without sacrificing something

ART. 47. METRIC RELATIONS.

169. Relation between Conjugate Abscissæ. Let s, s' be a pair of conjugate rays of the two collinear Space-Systems Σ and Σ', and let J and I' designate the points where these rays cross the Focal Planes φ and ϵ', respectively. Moreover, let I and J' designate the infinitely distant points of s and s' conjugate to I' and J, respectively. Finally, if P, P' and Q, Q' are two other pairs of conjugate points of s and s', we shall have:

$$(PQJI) = (P'Q'J'I'),$$

or

$$\frac{PJ}{QJ}:\frac{PI}{QI} = \frac{P'J'}{Q'J'}:\frac{P'I'}{Q'I'};$$

which, since I and J' are the ideal points of s and s', reduces to the following:

$$\frac{PJ}{QJ} = \frac{Q'I'}{P'I'}.$$

This equation may be written:

$$JP \cdot I'P' = JQ \cdot I'Q' = \text{a constant.}$$

Stated in words this *Characteristic Metric Relation of Optical Imagery* may be expressed as follows:

The product of the "abscissæ" of two conjugate points, P and P', with respect to the so-called "Flucht" Points, J and I', of two conjugate rays s and s' which go through P and P', respectively, is constant.

In this statement the term "abscissa" is employed (for lack of a better word) to describe the position of a point on a ray with respect to the "Flucht" Point of the ray as origin. Thus, for example, the "abscissa" of the point P of the ray s is JP, which means the segment of the ray included between J and P, and *reckoned from J to P*, that is, reckoned always in the sense indicated by the order in which the letters are written. (See Appendix, Art. 4.)

The product of the "abscissæ" of pairs of conjugate points of any one pair of conjugate rays s, s' is constant, but the magnitude of

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of greater importance. Thus, according to the system of notation employed in this chapter and very generally throughout this book, the designation "$y'z'$-plane" would naturally imply a plane in the Image-Space conjugate to the yz-plane in the Object-Space. But even when these two co-ordinate planes are not conjugate, we shall continue to designate the plane in the Image-Space as the $y'z'$-plane rather than complicate and, perhaps, confuse things by introducing a pair of entirely new letters. As a matter of fact, except in the important case when these planes yz, $y'z'$ are the two Focal Planes ϕ, ε', they are generally a pair of conjugate planes.

this constant, will in general, be different for different pairs of conjugate rays. In particular, if the two conjugate rays are the Principal Axes x, x' themselves, we shall have for a pair of conjugate axial points M, M':

$$FM \cdot E'M' = \text{a constant};$$

so that, if we put

$$x = FM, \quad x' = E'M',$$

we obtain the so-called *Abscissa-Equation:*

$$xx' = a, \tag{109}$$

where a denotes the value of the constant for the conjugate rays x, x'.

170. The Lateral Magnifications. In the special case when the object-ray r was parallel to the Focal Plane φ of the Object-Space, we saw (§ 166) that the corresponding image-ray r' was likewise parallel to the Focal Plane ϵ' of the Image-Space; so that, because the "Flucht" Points of these rays coincide with their infinitely distant points, the so-called "Abscissa" Relation obtained in § 169 is of no value when applied to such a pair of conjugate rays. But the fact, that to a range of Object-Points r which is parallel to the Focal Plane φ corresponds a "projectively similar" range of Image-Points r' which is parallel to the Focal Plane ϵ', leads also to a very important metrical relation concerning the rays r, r'; so that if A, A'; B, B'; C, C' are any three pairs of conjugate points of r, r', we shall have:

$$\frac{A'C'}{AC} = \frac{B'C'}{BC}.$$

This is called the *Magnification-Ratio* for the two conjugate rays r, r'.

Let σ, σ' be two conjugate planes parallel to the Focal Planes φ and ϵ' and containing the pair of conjugate rays r, r', respectively. The plane-fields σ, σ', as has been explained (§ 166), are in the relation of "affinity" to each other; so that to a pencil of parallel rays of σ corresponds a pencil of parallel rays of σ'; and, hence, *for all point-ranges of σ which are parallel to r the Magnification-Ratio has the same value.*

In § 168 it was explained that there was one pair of Meridian Planes in the Object-Space at right angles to each other to which corresponded in the Image-Space a pair of Meridian Planes also at right angles to each other. The intersection of the plane σ with this pair of Meridian Planes in the Object-Space will determine a pair of straight lines of σ at right angles to each other which are parallel to the co-ordinate axes y and z; and, similarly, there will be determined in the Image-Space in the same way a pair of perpendicular straight lines of σ' conjugate to those of σ which are parallel to the co-ordinate axes y' and z' of the Image-Space. If, therefore, y, z and y', z' denote the co-ordinates of a pair of conjugate points of σ and σ', so that y, y' and z, z' in this sense are used to denote the lengths of corresponding segments of point-ranges of σ, σ' which are parallel to the axes y, y' and the axes z, z', respectively, the Magnification-Ratios for rays of σ and σ' parallel to these axes will be y'/y and z'/z.

These ratios y'/y and z'/z are called the *Lateral Magnifications* for the pair of conjugate planes σ and σ'.

For a given Object-Plane σ parallel to the Focal Plane φ, the two Lateral Magnifications have perfectly definite values. Thus, for example, the value of y'/y for the plane σ may be denoted by the symbol Y, so that

$$Y = \frac{y'}{y},$$

which states that the value of Y is independent of the actual magnitudes of y and y'.

As origins of the two systems of rectangular co-ordinates of the Object-Space Σ and the Image-Space Σ' let us select the two Focal Points F and E' (Fig. 84), respectively. The Principal Axes will be the axes of x and x'. Let it be observed that the Focal Planes which are the planes yz and $y'z'$ are not conjugate planes, as the notation would imply.

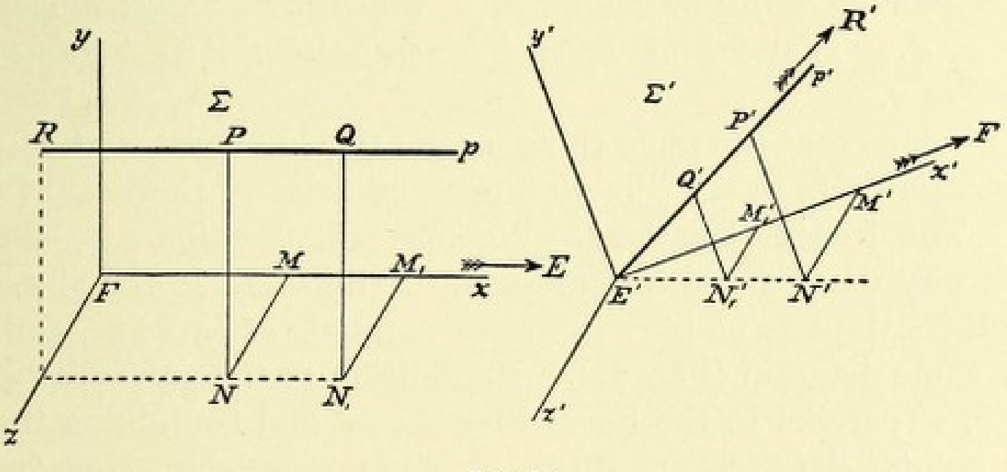

FIG. 84.

COLLINEATION OF TWO SPACE-SYSTEMS Σ, Σ': showing how the Lateral Magnification of Conjugate Planes parallel to the Focal Planes depends on the distances from the Focal Planes. The ranges p, p', as drawn in this figure, are "oppositely projective"—a case that does not actually occur in optical imagery; but that fact is immaterial so far as the question under consideration is concerned.

Take any point P of the Object-Space, whose co-ordinates are $x = FM$, $z = MN$, $y = NP$, and through P draw the ray p parallel to the x-axis. Let P' designate the point in the Image-Space conjugate to P, and let $x' = E'M'$, $z' = M'N'$, $y' = N'P'$ denote the co-ordinates of P'. Corresponding to the object-ray p going through the Object-Point P, we shall have an image-ray p' which connects the Image-Point P' with the Focal Point E'. The pair of conjugate planes perpendicular to the Principal Axes x, x' at the points M, M' will be designated by σ, σ', and the value of the Lateral Magnification for this pair of planes and for rays which are parallel to y and y' will be denoted by Y; so that $Y = y'/y$.

If the Object-Point P is supposed to move along a straight line parallel to the Focal Line y, it is obvious that the Image-Point P' must also traverse a straight line parallel to the Focal Line y' in such fashion that the Lateral Magnification $y'/y = N'P'/NP = Y$ shall be constant.

Again, if the Object-Point P is supposed to move along the ray p which is parallel to the x-axis, the Image-Point P' will travel along the conjugate ray p' which connects P' with the Focal Point E'; so that as the ordinate $y = NP$ remains constant as to both magnitude and sign, its image $y' = N'P'$ assumes all values from $-\infty$ to $+\infty$.

Thus, it appears that *the Lateral Magnification Y has different values for each pair of conjugate planes σ, σ' which are parallel to the Focal Planes φ, ϵ'.* That is, *the Lateral Magnification Y is a function of the abscissa x.*

It is obvious that the same thing is true also in regard to the Lateral Magnification z'/z in the direction perpendicular to the Focal Line y.

171. The Image-Equations. We proceed, therefore, to ascertain in what way the Lateral Magnification Y depends on the abscissa x. We shall continue to employ the same symbols as in § 170, and shall use the same diagram (Fig. 84). In addition to the pair of conjugate planes σ, σ' parallel to the Focal Planes φ, ϵ' and containing the conjugate points P (x, y, z), $P'(x', y', z')$, respectively, consider also another pair of such planes σ_1, σ_1' perpendicular to the Principal Axes x, x' at the points M_1, M_1', respectively. And let Y, Y_1 denote the values of the Lateral Magnification for these two pairs of conjugate planes σ, σ' and σ_1, σ_1', respectively. Let the object-ray p parallel to the x-axis cross the plane σ_1 at the point Q whose co-ordinates are $FM_1 = x_1$, $M_1N_1 = z_1 = z$, $N_1Q = y_1 = y$. Similarly, in the Image-Space let the ray p' conjugate to the object-ray p meet the plane σ_1' in the point Q' whose co-ordinates are $E'M_1' = x_1'$, $M_1'N_1' = z_1'$, $N_1'Q' = y_1'$.

If the point R of the Focal Plane φ is the "Flucht" Point of the object-ray p, then, by the abscissa-relation of § 169, we have:

$$RP \cdot E'P' = RQ \cdot E'Q',$$

and since $RP = FM = x$, $RQ = FM_1 = x_1$, and since from the figure we have also:

$$\frac{E'Q'}{E'P'} = \frac{N_1'Q'}{N'P'} = \frac{y_1'}{y'},$$

we may write the relation above as follows:

$$\frac{x}{x_1} = \frac{y_1'}{y'}.$$

Now $y_1' = Y_1 \cdot y_1 = Y_1 \cdot y$, and $Y = y'/y$; accordingly, we obtain finally:

$$\frac{Y}{Y_1} = \frac{x_1}{x};$$

that is,

$$Y \cdot x = \text{a constant} = b \text{ (say)}.$$

Thus, we see that *the Lateral Magnification Y is inversely proportional to the abscissa x.*

By precisely the same process we should find that the Lateral Magnification z'/z is also inversely proportional to the abscissa x.

Accordingly, we are able now to express the co-ordinates x', y', z' of any point P' of the Image-Space in terms of the co-ordinates x, y, z of the corresponding point P of the Object-Space. Thus, taking the Focal Points F and E' as the origins of the two systems of rectangular co-ordinates, and therefore using equation (109) together with the results which we have just obtained, we can write the *Image-Equations* as follows:

$$x' = \frac{a}{x}, \quad y' = \frac{by}{x}, \quad z' = \frac{cz}{x}, \tag{110}$$

from which we infer that the most general case of optical imagery, as defined by these equations, involves at least three constants a, b and c.[1]

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[1] CZAPSKI, in his celebrated book, derives the Image-Equations entirely by the methods of Analytic Geometry. Taking as the basis of his mathematical investigation the plane-to-plane correspondence which is characteristic of the collinear relation of the Space-

III. COLLINEAR OPTICAL SYSTEMS.

ART. 48. CHARACTERISTICS OF OPTICAL IMAGERY.

172. Signs of the Image-Constants a, b and c. Up to this point we have developed the theory of Optical Imagery from the standpoint of pure geometry, and on this account, while keeping steadily in view the application to the theory of optical instruments, we have purposely avoided introducing in this general treatment any of the physical properties of optical rays whereby the problem would become

Systems Σ and Σ' and denoting the co-ordinates of any point P of Σ, with respect to an arbitrary system of rectangular axes in Σ, by x, y, z, and the co-ordinates of the conjugate point P', also with respect to an arbitrary system of rectangular axes in Σ', by x', y', z', CZAPSKI shows that the following equations, involving 15 independent constants (cf. end of § 163), are the analytical expression of collinear correspondence between Σ and Σ':

$$x' = \frac{a_1x + b_1y + c_1z + d_1}{a_4x + b_4y + c_4z + d_4},$$

$$y' = \frac{a_2x + b_2y + c_2z + d_2}{a_4x + b_4y + c_4z + d_4},$$

$$z' = \frac{a_3x + b_3y + c_3z + d_3}{a_4x + b_4y + c_4z + d_4}.$$

From this system of equations we may obtain, in general, also a second system which may be written as follows:

$$x = \frac{\alpha_1x' + \alpha_2y' + \alpha_3z' + \alpha_4}{\delta_1x' + \delta_2y' + \delta_3z' + \delta_4},$$

$$y = \frac{\beta_1x' + \beta_2y' + \beta_3z' + \beta_4}{\delta_1x' + \delta_2y' + \delta_3z' + \delta_4}.$$

$$z = \frac{\gamma_1x' + \gamma_2y' + \gamma_3z' + \gamma_4}{\delta_1x' + \delta_2y' + \delta_3z' + \delta_4}.$$

In each of these two sets of equations it will be remarked that the right-hand members are fractions with linear numerators and denominators, and that the denominators of the fractions are identical for all three equations in each group. It is obvious that

$$a_4x + b_4y + c_4z + d_4 = 0,$$
$$\delta_1x' + \delta_2y' + \delta_3z' + \delta_4 = 0$$

are the equations of the "Flucht" Planes or Focal Planes ϕ, ε' of the two Space-Systems Σ, Σ', respectively.

Having thus obtained the equations above, CZAPSKI proceeds to show how by a suitable choice of axes of co-ordinates the equations may be reduced finally to the simpler forms given in equations (110), where, instead of as many as 15 independent constants in the case of arbitrary systems of co-ordinates, the number of independent constants is only 3. See CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1894), pages 27–33.

See also E. WANDERSLEB: Die geometrische Theorie der optischen Abbildung nach E. ABBE: Chapter III of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904) edited by M. VON ROHR.

Also: JAMES P. C. SOUTHALL: The Geometrical Theory of Optical Imagery: *Astrophys. Journ.*, xxiv. (1906), 156–184.

more or less specialized. But having obtained the Image-Equations (§ 171), we shall find it convenient now to call attention to the manifest singularity which distinguishes optical rays from the rays of ordinary geometry. Along each optical ray there is one direction, viz., the direction which the light follows, which is the obvious, or natural, direction of the ray. First of all, therefore, we may take advantage of this property by agreeing to define *the positive direction of an optical ray as that direction along the ray which the light takes.*

In the case, therefore, of two conjugate ranges of points s, s', there are two possibilities. Thus, for example, if $P, Q, R, \cdots$ is a series of points of s which are traversed by the light in the order named, the series of conjugate points $P', Q', R', \cdots$ lying on s' will be traversed either in the same or in the reverse order. In the former case (when the direction of the ray s' is therefore the same as the direction $P'Q'$), we shall call s and s' a pair of *directly projective* ranges of points (Fig. 85); and in the latter case (when the direction of the ray s' is opposite to that of $P'Q'$), we shall call s and s' a pair of *oppositely projective* ranges of points (Fig. 86). Obviously, *in optical imagery we can have only directly projective ranges of points*, and, consequently, so far as our purposes are concerned, we may leave out of account altogether oppositely projective ranges.

If, therefore P, P' are a pair of conjugate points of the directly projective ranges of points s, s' (Fig. 85), and if J, I and I', J' designate the "Flucht" Points and the Infinitely Distant Points of s and s', respectively, then, as the point P is supposed to travel along s from J to I in the direction of the ray s, P' will travel along s' from

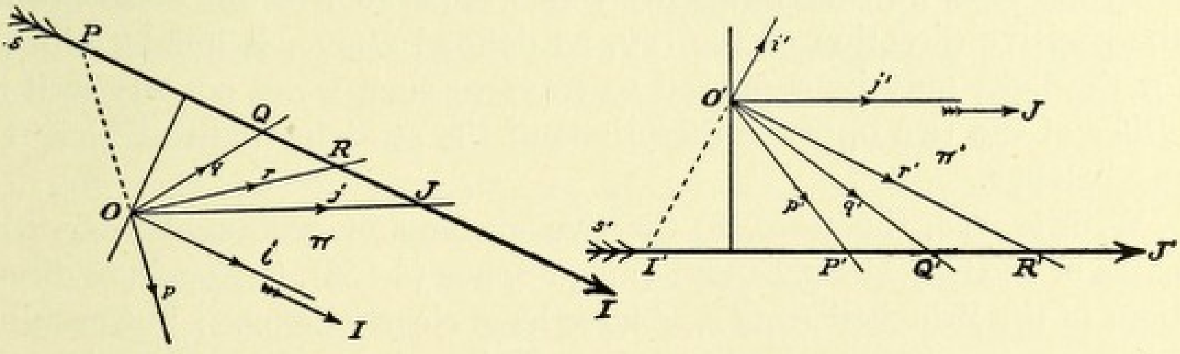

FIG. 85.

DIRECTLY PROJECTIVE RANGES OF POINTS; SUCH AS WE HAVE ALWAYS IN OPTICAL IMAGERY. O, O' are a pair of conjugate points, from which the point-ranges $P, Q, R, \cdots$ and $P', Q', R', \cdots$ lying on the straight lines s, s', respectively, are projected. The points $P, Q, R, \cdots$ and $P', Q', R', \cdots$ are traversed by the light in the order in which the points are named. J and I' are the "Flucht" Points and I and J' the infinitely distant points of s, s', respectively. (However, the rays in the diagram which are designated as j and i' do not here correspond to the "Flucht" Lines of the Plane-Fields π and π'.)

J' to I' in the positive direction of the ray s'; so that, supposing, for example, as is represented in the figure, that the point P lies on the negative side of the "Flucht" Point J ($JP < 0$), the point P' will lie on the positive side of the "Flucht" Point I' ($I'P' > 0$), and *vice versa.* Hence, in the case of two directly, or, as we might say, "optically", projective point-ranges, the "abscissæ" (§ 169) JP, $I'P'$ always have opposite signs. Accordingly, recalling the Abscissa-Relation derived in § 169, we may say:

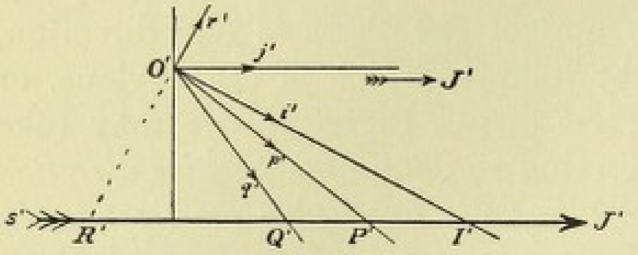

FIG. 86.

The range P', Q', R', $\cdots$ of points lying on s' is oppositely projective with the range P, Q, R, $\cdots$ lying on s (see left-hand side of Fig. 85). This case cannot occur in optical imagery.

In the case of Optical Imagery, the product $JP \cdot I'P' =$ a constant *is always negative.*

The value of this constant for the two projective point-ranges lying along the Principal Axes x, x' was denoted by a; hence, *provided the positive directions of the axes of x, x' are defined as the directions which light pursues along these rays, the Image-Constant a is negative in all cases of optical imagery;* that is,

$$a < 0.$$

In the case, however, of a ray which is parallel to the Focal Plane, the positive direction of the ray, as defined above, is indeterminate, for the light may be supposed to traverse such a ray equally well in either of the two opposite directions of the straight line to which the ray belongs.

With regard, therefore, to the two systems of rectangular co-ordinates of the Object-Space and Image-Space (§ 168), the positive directions of the Principal Axes x, x' have been clearly defined; but nothing whatever has been done towards choosing the positive directions of the secondary axes y, z in the Object-Space and y', z' in the Image-Space. So far as our previous investigation goes, the positive direction of each one of these axes is entirely arbitrary; and, accordingly, the signs of the two constants b and c which enter into the Image-Equations (110) may be positive or negative and like or unlike, depending only on the choice of the positive directions of the axes of y, z and of y', z'.

It makes no difference which directions we choose as the positive directions of the axes of y, z in the Object-Space; but, having chosen these, let us contrive so that the positive directions of the axes of y', z' in the Image-Space shall be thereby determined. Accordingly, we have merely to make, for example, the following agreement:

The positive directions of the axes of y *and* y' *are to be chosen relative to each other in such manner that the constant* b *shall be a positive number.*

And in the same way, *the positive directions of the axes of* z *and* z' *are to be chosen with respect to each other so that the constant* c *shall be a positive number.*

Thus,

$$b > 0, \quad c > 0.$$

Hence, assuming the positive directions of the secondary axes to be determined according to these considerations, it follows that *the Lateral Magnifications* y'/y *and* z'/z *always have the same sign, viz., the sign of the abscissa* x.

The signs of the three constants a, b and c which enter into the Image-Equations are dependent, therefore, only on the choice of the positive directions of the axes of co-ordinates. If these directions are defined as above, then the signs of these constants are as follows:

$$a < 0, \quad b > 0, \quad c > 0.$$

173. So long as we do not assume any definite position-relation between the Object-Space and the Image-Space, we shall define the positive directions of the axes of co-ordinates in this way, so that a is negative and b and c are positive. For the entirely general case this is the best choice to make. But when the optical system consists of a centered system of spherical refracting surfaces, as is usually the practical case, *the corresponding axes of the two systems of co-ordinates are parallel*, and then it will generally be more convenient to define the positive directions in such a way that *the positive directions of corresponding, or parallel, axes will be the same.* If this method is used, the signs of the Image-Constants may be different, in some cases, from the signs which they have above. It is important to bear this in mind, as the student may be puzzled when he finds that the signs of the Image-Constants are sometimes different from the signs as given above; merely because the positive directions of the axes of co-ordinates have been determined by different considerations (see § 176).

174. Symmetry around the Principal Axes. In the most general case of optical imagery, defined by equations (110), which involve at least as many as three constants a, b and c, the imagery is not symmetrical with respect to the Principal Axes of the Object-Space and Image-Space; that is, in general, the two Magnification-Ratios y'/y and z'/z have different values corresponding to the same value of x. However, in most actual optical systems, in fact almost without exception, *the Principal Axes are axes of symmetry;* and, since we are concerned primarily with the applications of these laws to the theory of optical instruments, it will be assumed hereafter that this is the case. Thus, we shall put

$$c = b;$$

in which case the *Image-Equations* become:

$$x' = \frac{a}{x}, \quad y' = \frac{by}{x}, \quad z' = \frac{bz}{x}; \tag{111}$$

so that the character of the imagery will be defined now by the two constants a and b. The Principal Axes being axes of symmetry, every pair of Meridian Planes of the Object-Space which are at right angles to each other has a corresponding pair of Meridian Planes of the Image-Space also at right angles to each other; so that the choice of the axes of y and z is now indeterminate.

Moreover, in the case of symmetry with respect to the Principal Axes of x, x', when we have $c = b$, the collinear plane-fields σ, σ' parallel to the Focal Planes φ, ϵ', respectively, are not only in affinity with each other (§ 166), but they are also *similar;* so that if A, B, C are three points of σ not in the same straight line, and A', B', C' the three corresponding points of σ', then

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC};$$

and, consequently, corresponding angles of two similar plane-fields are equal.

The simplest, and at the same time the most perfect, kind of optical image would be one which was geometrically exactly similar to the object; so that it would always be possible to conceive the object and image oriented with respect to one another in such fashion that all corresponding lines were parallel. This case of complete geometrical similarity between an object and its optical image may not, in general, be realized by any optical apparatus, although it is possible in special cases. But if the object is a plane figure lying in a plane parallel to the Focal Plane of the Object-Space, the image will be a completely similar figure lying in a plane parallel to the Focal Plane of the Image-Space—assuming that we have collinear correspondence between Object-Space and Image-Space, and that the Principal Axes are axes of symmetry.

175. The Different Types of Optical Imagery.[1] In collinear bundles of rays there are always two corresponding rectangular three-edges. Thus, at any point O of the Object-Space, let OA, OB, OC be three rays mutually at right angles to each other, to which there correspond three rays $O'A'$, $O'B'$, $O'C'$ meeting at the conjugate point O' of the Image-Space, and also mutually at right angles to each other. Let us suppose that the three edges OA, OB, OC of the octant O-ABC form a *canonical* or *right-screw system of axes* (so that, if a right-screw was turned in the direction from OB to OC, the point of the screw would advance along OA). Two cases may occur, as follows: (1) The system $O'A'$, $O'B'$, $O'C'$ may also be a right-screw system; or (2) The system $O'A'$, $O'B'$, $O'C'$ may be a *left-screw* (or *acanonical*) system of axes.

In the first case, we can see how it might be possible, by placing the points O and O' together, to fit one of the octants into the other in such fashion that the directions of the three pairs of corresponding edges of the two conjugate octants OA, $O'A'$; OB, $O'B'$; OC, $O'C'$ agree with one another; so that except for the fact that the pairs of corresponding points A, A'; B, B'; C, C' will not, in general, be superposed on each other, we should have "*congruence*" of the two rectangular corners O-ABC and O'-$A'B'C'$.

In the latter case, when one system is canonical and the other acanonical, no such "congruence" would be possible. Thus, for example, if in this case we place the points O and O' in coincidence with each other, and if we orient the two octants relative to each other so that the directions of two pairs of conjugate edges, say, OB, $O'B'$ and OC, $O'C'$ are the same, the directions of the third pair of edges OA, $O'A'$ will be exactly opposite to each other. Instead, therefore, of a so-called "congruence" of the two conjugate octants O-ABC and O'-$A'B'C'$, such as was possible in the first case, we shall have here a certain "*symmetry*" of the two octants; although here again we are employing a term in a sense somewhat different from the precise meaning attached to it in geometry. Strictly speaking, both "congruence" and "symmetry" involve the idea of the *equality* of corresponding line-segments, which is by no means necessarily implied in the employment of these terms in the present connection.

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[1] The following discussion is based on the admirable treatment of this matter in E. WANDERSLEB's article on "Die geometrische Theorie der optischen Abbildung nach E. ABBE", which is Chapter III of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by Dr. M. VON ROHR. See pages 92, foll.

By virtue of the Principle of Continuity, it is obvious that if we have "congruence", or "symmetry", between one pair of conjugate octants, we shall have "congruence", or "symmetry", between all pairs of conjugate octants. It is true that possibly at the Focal Planes (which are sometimes called the "Discontinuity Planes") of the two Space-Systems, the imagery might change from "congruence" to "symmetry", or *vice versa*. That this change does not occur in crossing from one side of the Focal Plane to the other, we shall now proceed to show.

Let us assume that the two pairs of conjugate points O, O' and A, A' are situated on the Principal Axes x, x'; so that OB, $O'B'$ and OC, $O'C'$ are parallel to the co-ordinate axes y, y' and z, z', respectively. Moreover, let us assume also that the points A, B, C are all infinitely near to O; and, consequently, the points A', B', C' will also be infinitely near to O'. Hence, if

$$x = FO, \quad x' = E'O'$$

denote the abscissæ of the points O, O' with respect to the Focal Points F, E', respectively, as origins, we shall have:

$$\begin{aligned} OA &= dx, & OB &= dy, & OC &= dz; \\ O'A' &= dx', & O'B' &= dy', & O'C' &= dz'. \end{aligned}$$

And, finally, let us suppose that the directions OA, OB, OC agree with the positive directions of the axes of x, y, z, respectively (no matter how these directions may have been defined), so that the magnitudes denoted here by dx, dy, dz are all positive. If we write the Image-Equations (110) in the differential form, as follows:

$$dx' = -\frac{a}{x^2}dx, \quad dy' = \frac{b}{x}dy, \quad dz' = \frac{c}{x}dz,$$

we see that, as the point O is supposed to cross the Focal Plane at F, whereas the abscissa x changes its sign, the sign of dx' remains the same; but the signs of dy' and dz' both change with change of the sign of x. Consequently, the octant O'-$A'B'C'$ remains of the same type, so that if it was "congruent" (or "symmetric") with the octant O-ABC when the point O was on one side of the Focal Plane, it will remain "congruent" (or "symmetric") with it when the point O is taken on the other side of the Focal Plane.

Accordingly, from this purely geometrical standpoint, and entirely without reference to the actual signs of the Image-Constants a, b, c, it appears that there are these two essentially different types of optical imagery, which may be conveniently distinguished as follows:

1. *Right-Screw Imagery*—the case when the two conjugate octants are "congruent"; in this case the image of a right-screw will be a right-screw, but, in general, distorted; and

2. *Left-Screw Imagery*—the case when the two conjugate octants are "symmetric"; in this case the image of a right-screw will be a left-screw, although, in general, distorted.

These two types of imagery may be exhibited by diagrams (Figs. 87, 88 and 89) as follows:

In the Object-Space parallel to the x-axis draw two pairs of rays, viz., two rays b, b (Fig. 87) in the xy-plane at equal distances from, and on opposite sides of, the x-axis; and, similarly, two rays c, c drawn in the same way in the xz-plane. In the Image-Space (Figs. 88 and 89), corresponding to the two pairs of object-rays b, b and c, c, we have two pairs of image-rays b', b' and c', c' all passing through the Focal Point E'. The pair of rays b', b' will lie in the $x'y'$-plane, and the pair of rays c', c' will lie in the $x'z'$-plane, and the x'-axis will bisect the angles at E' between each of the pairs of rays b', b' and c', c'. In Fig. 87 O_1, O_2, etc. represent a series of equidistant Object-Points ranged along the x-axis. Through each one of these points draw a pair of lines parallel to the axes of y and z, and consider the segments of these lines comprised between b, b and c, c. The image of one of these rectangular crosses made by such a pair of line-segments will be a rectangular cross with its arms parallel to the

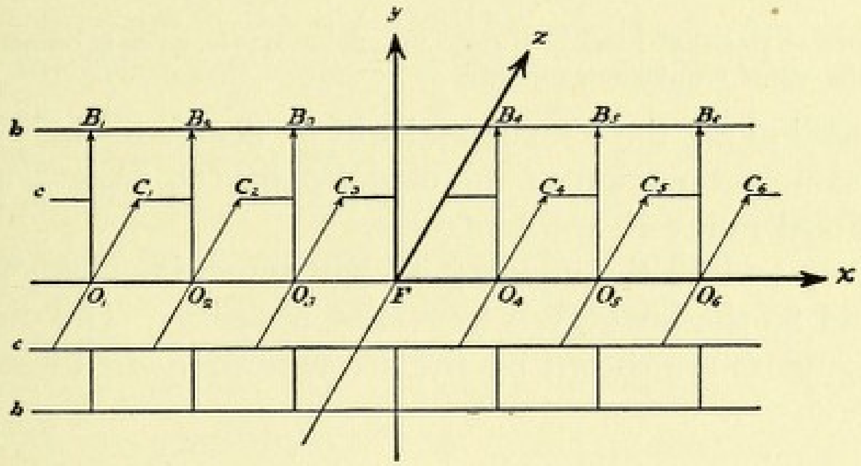

FIG. 87.

TYPES OF OPTICAL IMAGERY: OBJECT-SPACE. This figure shows a series of equidistant axial Object-Points O_1, O_2, etc. and two series of equidistant Object-Points B_1, B_2, etc. and C_1, C_2, etc., lying in the planes xy and xz, respectively, on the straight lines b and c parallel to the Principal Axis (x) of the Object-Space.

axes of y' and z'; the end-points of these arms lying in the pairs of rays b', b' and c', c', as shown in Figs. 88 and 89. The points O'_1, O'_2, etc., corresponding to the axial Object-Points O_1, O_2, etc. (Fig. 87),

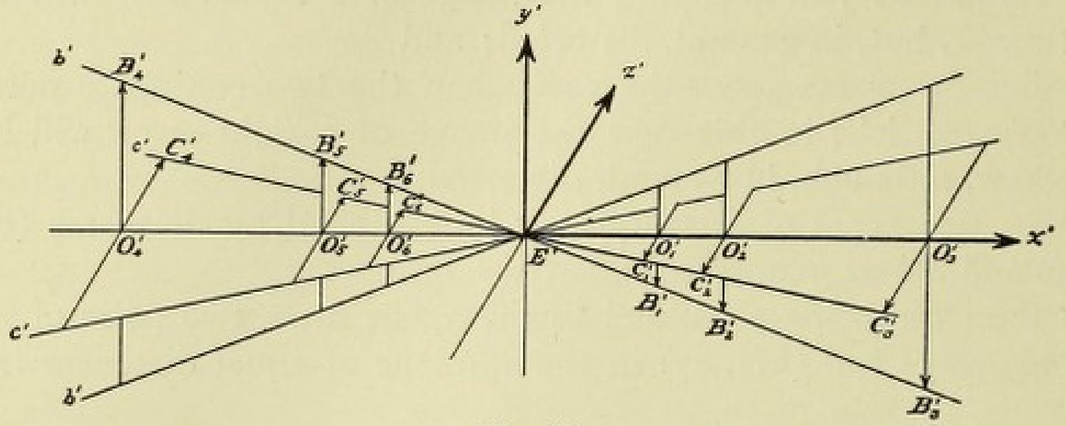

FIG. 88.
TYPES OF OPTICAL IMAGERY: IMAGE-SPACE. This figure is to be taken in connection with Fig. 87. It shows the case of Right-Screw Imagery.

will be ranged along the x'-axis, and will lie nearer to the Focal Point E' in the same proportion as the object-points O_1, O_2, etc. are farther from the Focal Point F, and *vice versa*.

In Figs. 87 and 88 the imagery is right-screw imagery; whereas in Figs. 87 and 89 the imagery is left-screw imagery. The directions of the line-segments are shown by the arrow-heads. In these diagrams

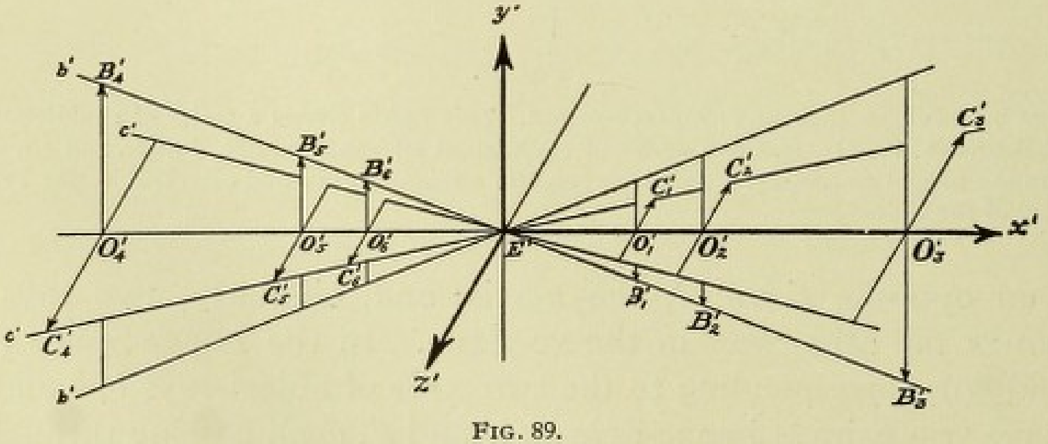

FIG. 89.
TYPES OF OPTICAL IMAGERY: IMAGE-SPACE. This figure is to be taken in connection with Fig. 87. It shows the case of Left-Screw Imagery.

the positive directions of the axes are chosen so that the signs of the Image-Constants (§ 172) are given by the following relations:

$$a < 0, \quad b > 0, \quad c > 0;$$

and, consequently, corresponding to positive values of the co-ordinates x, y, z, we shall have x' negative and y' and z' positive; whereas if y and z are both positive, but x negative, x' will be positive, and y' and z' both negative.

One of the most obvious and characteristic features of optical imagery is the symmetry of the imagery with respect to the two Focal Planes. Each of the two space-regions is divided by its Focal Plane into two equal halves, and to each half of the Object-Space corresponds one of the two halves of the Image-Space.

176. In order not to affect the generality of our results, up to this point we have purposely nowhere assumed any *definite position-relation between the Object-Space and the Image-Space.* As a matter of fact, however, practically all optical instruments consist of a *centered system of spherical refracting* (*or reflecting*) *surfaces*, so that the system is perfectly symmetrical with respect to the *optical axis* (§ 135), or straight line along which lie the centres of the spherical surfaces. In such a system the Principal Axes x, x' of the Object-Space and the Image-Space are both coincident with the optical axis. A ray lying in a Meridian Plane of the Object-Space must in its transit through the system continue to lie always in this same plane in space, so that a Meridian Object-Plane and its conjugate Image-Plane are the same plane in space. Thus, for example, the two Meridian Planes of the system of co-ordinates of the Object-Space, viz., the planes xy and xz, are coincident with the planes $x'y'$ and $x'z'$, respectively, of the Image-Space. Hence, *the axes of y and z in the Object-Space are parallel to the axes of y' and z', respectively, in the Image-Space.* This being the case, it is usually found convenient to *select the positive directions of the axes of x', y', z' so that these directions shall be the same as the positive directions of the axes x, y, z, respectively.* Thus, while we shall always select the positive direction of the x-axis as the direction taken by the *incident* light along that line (§ 172), the positive direction of the x'-axis may, or may not, be the direction pursued by the light along it. And, therefore, the constant a may in a case of this kind be either positive or negative, depending on which direction of the x'-axis is the positive direction (see § 173).

In an optical system composed of a centered system of spherical surfaces, it is important to emphasize the fact that *the positive direction along the optical axis is always the direction of the incident light;* so that, for example, if one of the spherical surfaces is a reflecting surface whereby the original direction of the light along the optical axis is reversed, notwithstanding, we must continue to reckon as positive that direction which was originally the positive direction; and all axial line-segments, irrespective of any subsequent change of the direction of the light, are to be reckoned as positive or negative according as they have the same direction as, or the opposite direction to, the incident axial ray (see §§ 26 and 108).

So also in regard to the other Image-Constant $b = c$: since here we do not, as in the general case, choose the positive directions of the secondary axes of y' and z' so that $b = c$ shall be positive, the Image-Constant $b = c$ may, therefore, be positive or negative. In brief, in these special circumstances, the two systems of axes are chosen with respect to each other so that a mere displacement along the optical axis of the origin of co-ordinates of the Image-Space is all that is needed in order to bring the axes of co-ordinates of the Image-Space into coincidence with the axes of co-ordinates of the Object-Space.

If we write again the Image-Equations in their differential forms, viz.:

$$dx' = -\frac{a}{x^2}dx, \quad dy' = \frac{b}{x}dy, \quad dz' = \frac{b}{x}dz,$$

and assume always that dx, dy, dz are positive, we may consider the following cases:

I. As to the sign of the constant a:

(1) If $a < 0$, then whatever may be the sign of the abscissa x, the sign of dx' must be positive. The signs of dy' and dz' are always either both positive or both negative, depending on the sign of x. Consequently, the two conjugate octants which have dx, dy, dz and dx', dy', dz' as corresponding edges are "congruent", and, hence, when $a < 0$, we have *Right-Screw Imagery* (§ 175).

(2) When $a > 0$, the sign of dx' must be negative for both positive and negative values of x; whereas, as before, the signs of dy' and dz' are either both positive or both negative, depending on the sign of x; so that the two conjugate octants which have dx, dy, dz and dx', dy', dz' as corresponding edges are "symmetric" (§ 175). Hence, when $a > 0$, we have *Left-Screw Imagery.*

II. As to the sign of the constant $b = c$:

(1) When $b > 0$, the signs of dy' and dz' are the same as that of x. Accordingly, for positive values of x, we have *erect* images, and for negative values of x, we have *inverted* images. An optical system of this kind is called a *convergent system.*

(2) When $b < 0$, the signs of dy' and dz' are opposite to that of x; so that the positive half of the Object-Space is portrayed by *inverted* images, whereas the other half (the negative half) is portrayed by *erect* images. This case is, accordingly, precisely opposite to the one above, and a system of this kind is called *divergent.*

These results may be summarized as follows:

A centered system of spherical refracting (or reflecting) surfaces is convergent or divergent according as the Image-Constant $b >$ *or* < 0; *and the Imagery is Right-Screw or Left-Screw Imagery according as the other Image-Constant* $a <$ *or* > 0.

ART. 49. THE FOCAL LENGTHS, MAGNIFICATION-RATIOS, CARDINAL POINTS, ETC.

177. Analytical Investigation of the Relation between a Pair of Conjugate Rays. Let the Focal Point F (Fig. 90) be the origin of the system of rectangular co-ordinates of the Object-Space, the Principal Axis of the Object-Space being the x-axis, and the Focal Plane φ being the yz-plane. Similarly, in the Image-Space (not represented in the figure) the Focal Point E' is the origin of a system of rectangular axes, the Principal Axis of the Image-Space being the x'-axis, and the Focal Plane ϵ' being the $y'z'$-plane. Consider a pair of conjugate rays, an Object-Ray (s), which crosses the co-ordinate-planes xy, xz, yz at the points designated in the figure by the letters P, Q, R, respectively, and the corresponding Image-Ray (s'), which crosses the co-ordinate-planes $x'y'$, $x'z'$, $y'z'$ of the Image-Space at the points N', O', S', respectively. Draw RU perpendicular to the y-axis and $S'V'$ perpendicular to the y'-axis; then the straight lines PU and $N'V'$ meeting the x-axis in the point L and the x'-axis in the point

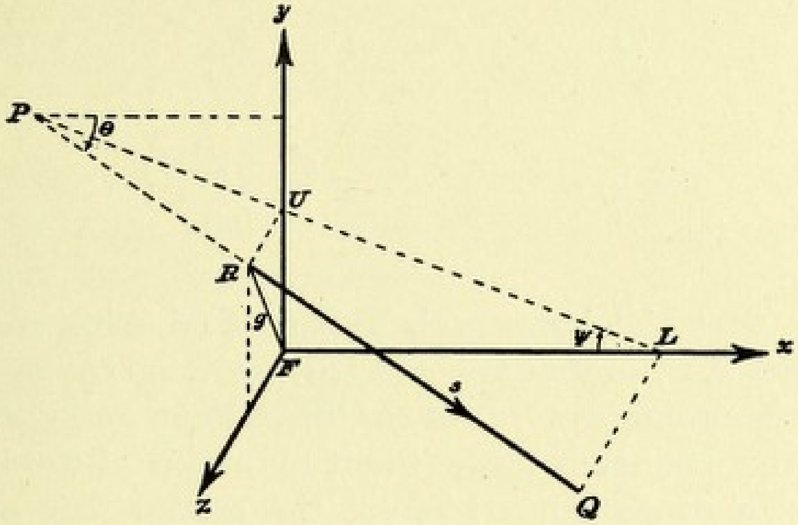

FIG. 90.

RELATION OF OBJECT-RAY AND CONJUGATE IMAGE-RAY. The figure shows only the object-ray; a similar diagram, with letters suitably changed, may be imagined for the image-ray. PQ (or s) represents an object-ray which crosses the Focal Plane yz at the point designated by R. $FR = g$, this distance being reckoned positive or negative according as R is above or below the xz-plane. The angle θ may have any value between $\pi/2$ and $-\pi/2$; the sign of this angle being always the same as that of the angle $\psi = \angle FLU$, where PU is the projection of PQ on the xy-plane.

M', will be the projections on the xy-plane and the $x'y'$-plane of the conjugate rays s, s', respectively.

The co-ordinates of the points R and S' where the object-ray (s) and the image-ray (s') cross the Focal Planes φ and ϵ', respectively, will be:

$$(0, FU, UR) \quad \text{and} \quad (0, E'V', V'S')\ ;$$

and, hence, if l, m, n and l', m', n' are the direction-cosines of the straight lines s, s', respectively, the Cartesian equations of these straight lines will be:

$$\frac{x}{l} = \frac{y - FU}{m} = \frac{z - UR}{n}\,,$$

$$\frac{x'}{l'} = \frac{y' - E'V'}{m'} = \frac{z' - V'S'}{n'}\,,$$

respectively.

Assuming that the imagery is symmetrical with respect to the Principal Axes x, x', so that $b = c$ (§ 174), we can express the relations between the co-ordinates x, y, z of an Object-Point and the co-ordinates x', y', z' of the conjugate Image-Point by means of the Image-Equations (111); in consequence whereof the second pair of the above equations may be written as follows:

$$y = \frac{E'V'}{b}x + \frac{a}{b}\,\frac{m'}{l'}\,, \quad z = \frac{V'S'}{b}x + \frac{a}{b}\,\frac{n'}{l'}\,.$$

Comparing this pair of equations with the first pair above, we obtain immediately the following relations for the co-ordinates of the two points R and S' in the Focal Planes yz and $y'z'$, respectively:

$$FU = \frac{a}{b}\,\frac{m'}{l'}\,, \quad UR = \frac{a}{b}\,\frac{n'}{l'}\ ;$$

$$E'V' = b\,\frac{m}{l}\,, \quad V'S' = b\,\frac{n}{l}\,.$$

Let us denote the focal distances of the points R, S' where s, s' cross the Focal Planes φ, ϵ' by g, k', respectively; that is, $FR = g$, $E'S' = k'$; and, moreover, let θ, θ' denote the angles of inclination to the axes x, x' of the conjugate rays s, s', respectively. Squaring and adding the two equations in the top line, and doing the same for the two equations in the lower line, and introducing the symbols which we have just defined, at the same time remarking that we have also:

$$l^2 + m^2 + n^2 = l'^2 + m'^2 + n'^2 = 1$$

and

$$l = \cos\theta, \quad l' = \cos\theta',$$

we obtain immediately the following results:

$$g^2 = \frac{a^2}{b^2}\tan^2\theta', \quad k'^2 = b^2\tan^2\theta.$$

Thus, we find:

$$g = \pm\frac{a}{b}\cdot\tan\theta', \quad k' = \pm\, b\cdot\tan\theta.$$

In order to avoid ambiguity of signs in this pair of equations, it is necessary to define more precisely the linear magnitudes g and k' and the angular magnitudes θ and θ'.

1. As to the signs of the linear magnitudes g and k': The focal distances g and k' are to be reckoned positive or negative according as their projections FU and $E'V'$ on the y-axis and y'-axis, respectively, are positive or negative. Thus, according as the point U lies on the positive or negative half of the y-axis, the sign of g will be plus or minus; and, according as the point V' lies on the positive or negative half of the y'-axis, the sign of k' will be plus or minus.

2. As to the angular magnitudes θ and θ': If through the point P where the object-ray meets the xy-plane a straight line is drawn parallel to the x-axis, in the same direction as the positive direction of the x-axis, the angle θ is the acute angle through which this straight line has to be turned about P in order to bring it into coincidence with the straight line PQ. The sign of this angle may be positive or negative, its value being comprised between $\theta = \pi/2$ and $\theta = -\pi/2$. The sign of the angle θ can always be ascertained by the following rule: If $\psi = \angle FLP$ denotes the acute angle through which the x-axis must be revolved about the point L in order to make it coincide in position with the projection PL of the object-ray PQ on the xy-plane, and if the sign of the angle ψ is determined by the relation

$$\tan\psi = -\frac{FU}{FL},$$

let us agree that *the signs of the angles here denoted by θ and ψ shall always be the same.* Thus, for example, in the figure, as it is drawn, both FU and FL are positive, since their directions are the same as the positive directions of the axes of y and x, respectively; and hence the angle θ in the figure is negative.

The angle θ' in the Image-Space is defined in an entirely similar way.

If the pair of conjugate rays lie in a pair of conjugate Meridian Planes, we shall find, on investigation, that it will not be necessary to extract a square-root, as it was in the general case above, and that, with the above definitions of the magnitudes denoted by g, k', θ, θ', the positive sign in the two formulæ is alone admissible. Thus, the ambiguity disappears, and we must write:

$$g = \frac{a}{b} \cdot \tan\theta', \quad k' = b \cdot \tan\theta. \tag{112}$$

From these formulæ we derive the following:

To object-rays, whose inclinations (θ) *to the Principal Axis* (x) *are all equal, correspond image-rays which cross the Focal Plane* (ϵ') *of the Image-Space at equal distances* (k') *from the Focal Point* E'; *and, similarly, to object-rays, which cross the Focal Plane* (φ) *of the Object-Space at equal distances* (g) *from the Focal Point* F, *correspond image-rays whose inclinations* (θ') *to the Principal Axis* (x') *are all equal.*

We had already perceived (§ 167) that to a bundle of *parallel* rays of one Space-System, say, Σ, there corresponds a bundle of non-parallel rays of the other Space-System, the vertex of which lies in the Focal Plane of Σ'. We see now that this fact is merely a particular case of a more general law of optical imagery, as given in the above statement. The absolute value of the focal distance of the point R or S', where the object-ray or image-ray crosses the Focal Plane φ or ϵ', depends only on the magnitude of the inclination θ' or θ of the conjugate ray to the x'- or x-axis, respectively.

178. The Focal Lengths f and e'. Equations (112) obtained in the last section, which may be written:

$$\frac{g}{\tan\theta'} = \frac{a}{b}, \quad \frac{k'}{\tan\theta} = b,$$

afford us a new way of defining the Image-Constants a and b. Thus, the constant b may be defined as the ratio of the Focal distance k' of the point where an image-ray crosses the Focal Plane of the Image-Space to the tangent of the angle of inclination θ of the corresponding object-ray to the Principal Axis x of the Object-Space; and, similarly, the magnitude a/b may be defined as the ratio of the Focal distance g of the point where an object-ray crosses the Focal Plane of the Object-Space to the tangent of the angle of inclination θ' of the corresponding image-ray to the Principal Axis x' of the Image-Space. From the equations above, as well as from the Image-Equations themselves (§ 174), it is apparent that the dimensions of the Image-Constants a and b are different; thus, whereas b denotes a length, a denotes an area. For this and other reasons it is convenient to introduce at this point a new pair of symbols f and e' instead of a and b, and to write:

$$f = b, \quad e' = \frac{a}{b}. \tag{113}$$

Thus, the Image-Constants denoted by f and e' will be defined by the following formulæ:

$$f = \frac{k'}{\tan\theta}, \quad e' = \frac{g}{\tan\theta'}. \tag{114}$$

The constants f and e' are called the *Focal Lengths* of the optical system. According to Abbe, the proper definitions of these characteristic constants of the optical system are given only by formulæ (114). So soon as the magnitudes denoted by f and e' are ascertained, the optical system may be regarded as completely determined.

The definition of the Focal Lengths of a system of lenses, as given by Gauss,[1] is essentially the same as Abbe's definition by means of the above equations; thus:

The Focal Length of the Object-Space (denoted here by f) *is equal to the ratio of the linear magnitude of an image formed in the Focal Plane of the Image-Space to the apparent* (*or angular*) *magnitude of the corresponding infinitely distant object;* and

The Focal Length of the Image-Space (denoted by e') *is equal to the ratio of the linear magnitude of an object lying in the Focal Plane of the Object-Space to the apparent magnitude of its infinitely distant image.*

Introducing the Focal Lengths f and e', we may now write the Image-Equations (111) as follows:

$$xx' = fe', \quad \frac{y'}{y} = \frac{z'}{z} = \frac{f}{x} = \frac{x'}{e'}. \tag{115}$$

Provided we adhere to the choice of the positive directions of the axes of co-ordinates which was made in § 172 (where we had $a < 0$,

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[1] See S. Czapski: *Theorie der optischen Instrumente nach* Abbe (Breslau, 1893), p. 40.

$b > 0$), we shall have always:

$$f > 0, \quad e' < 0.$$

179. The Magnification-Ratios and their Relations to one another.

1. *The Lateral Magnification* Y. This, as has been already defined (§ 170), is the ratio of conjugate line-segments lying in planes at right angles to the Principal Axes. Thus,

$$Y = \frac{y'}{y} = \frac{f}{x} = \frac{x'}{e'}; \tag{116}$$

whence we see that the Lateral Magnification Y may have any value from $-\infty$ to $+\infty$, depending on the value of the abscissa x.

2. *The Axial or Depth-Magnification* X. By differentiating the abscissa-equation

$$xx' = fe',$$

we obtain for the ratio of infinitely small conjugate line-segments dx, dx' of the Principal Axes:

$$X = \frac{dx'}{dx} = -\frac{fe'}{x^2} = -\frac{x'^2}{fe'}. \tag{117}$$

This ratio, denoted by X, is called the *Axial* or *Depth-Magnification*. It is inversely proportional to the square of the abscissa x. If we choose the positive directions of the axes of co-ordinates so that $f > 0$, $e' < 0$ (see § 178), then X will be necessarily positive, and may have any value comprised between 0 and $+\infty$.

Comparing formulæ (116) and (117), we obtain the following relation between the Axial Magnification (X) and the Lateral Magnification (Y):

$$\frac{X}{Y^2} = -\frac{e'}{f}; \tag{118}$$

and, hence, we can say: *At each point the Axial Magnification is proportional to the square of the Lateral Magnification.*

3. *The Angular Magnification* Z. Let M, M' (Fig. 91) designate the positions of two axial conjugate points, whose abscissæ with respect to the Focal Points F, E' are denoted by x, x', respectively; so that

$$FM = x, \quad E'M' = x'.$$

Let the straight line MR represent an object-ray crossing the Focal Plane of the Object-Space at the point R and making with the Principal Axis x of the Object-Space an angle $xMR = \theta$. Let S' designate the point where the conjugate ray $S'M'$ crosses the Focal Plane of the Image-Space, and let $\theta' = \angle E'M'S'$ denote the inclination of this image-ray to the Principal Axis x' of the Image-Space. Putting

$$FR = g, \quad E'S' = k',$$

we have, in accordance with our agreement in § 177 concerning the signs of the angles θ, θ':

$$\tan\theta = -\frac{g}{x}, \quad \tan\theta' = -\frac{k'}{x'}.$$

The Focal Lengths f and e', by definition, are given by the formulæ:

$$f = \frac{k'}{\tan\theta}, \quad e' = \frac{g}{\tan\theta'}.$$

And, hence, if Z denotes the ratio of the tangents of the angles of inclination to the Principal Axes of a pair of conjugate rays in any two conjugate Meridian Planes, we have:

$$Z = \frac{\tan\theta'}{\tan\theta} = -\frac{x}{e'} = -\frac{f}{x'}; \tag{119}$$

whence it will be seen that Z is independent of the values of θ, θ' themselves; so that for a given value of x, the ratio denoted by Z has a constant value. Thus, *for all rays which pass through the axial point M, the ratio* $\tan\theta' : \tan\theta$ *is constant.*

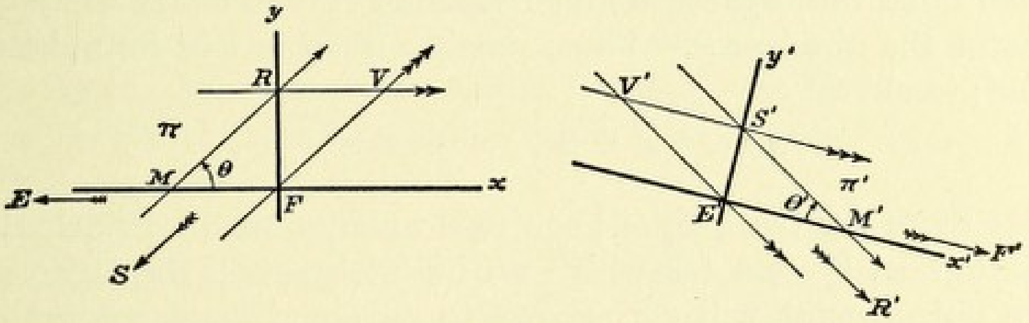

FIG. 91.

ANGULAR MAGNIFICATION.
$FM = x$, $E'M' = x'$, $FR = g$, $E'S' = k'$;
$\tan\theta = -FR/FM$, $\tan\theta' = -E'S'/E'M'$, $Z = \tan\theta'/\tan\theta$,
where Z denotes the angular magnification for the conjugate axial points M, M'.

This ratio denoted by Z is called the *Angular Magnification*, or the "*Convergence-Ratio*", and is an important magnitude in the theory of optical instruments.

Comparing the values of the Magnification-Ratios X, Y and Z, as given by formulæ (116), (117) and (119), we have the following relations between them:

$$\frac{X}{Y^2} = -\frac{e'}{f}, \quad Y \cdot Z = -\frac{f}{e'}, \quad X \cdot Z^2 = -\frac{f}{e'}, \quad \frac{Y}{X \cdot Z} = 1. \qquad (120)$$

180. The Cardinal Points of an Optical System. As we see from formulæ (116) and (119), the Magnification-Ratios Y and Z may have any values comprised between $-\infty$ and $+\infty$, depending on the value of the abscissa x; whereas the Depth-Magnification X, as is shown by formulæ (117), may have any value between 0 and $+\infty$; since we assume in this discussion that the positive directions of the axes of x, x' are so chosen that the Focal Lengths f and e' have always opposite signs. Each of these ratios is a function of the abscissa x, so that by assigning any particular value to one of these ratios, we shall thereby determine at least one pair of conjugate axial points. Those pairs of conjugate axial points for which one or other of the magnitudes denoted by X, Y, Z has the absolute value unity are all of more or less interest, and certain of them are especially distinguished in the theory of optics. They may be enumerated in the following order:

1. The two pairs of conjugate axial points for which the Depth-Magnification X has the value $+1$; for, since X is a function of x^2, we shall obtain always for a given value of X two equal and opposite values of the abscissa x. Thus, putting $X = +1$ in formulæ (117), we find:

$$x = \pm \sqrt{-fe'}, \quad x' = \mp \sqrt{-fe'};$$

so that there are two pairs of conjugate points on the Principal Axes of the optical system for which an infinitely small displacement dx of the object-point will correspond to an equal displacement dx' of the image-point. Moreover, it will be remarked that the Focal Points F and E' are midway between the two axial object-points and the two axial image-points, respectively. However, these two pairs of axial conjugate points are of slight importance, and need not detain us any longer, except merely to add that the Lateral and Angular Magnifications at these points are equal. Thus, we have:

$$Y = Z = \pm \sqrt{-f/e'}.$$

2. The most important and the most celebrated of all these pairs

of conjugate axial points is the pair named by Gauss[1] the *Principal Points* (see § 139) of the optical system, which in our diagrams will be designated by the letters A and A' (Fig. 92). The Principal Points are defined by the value $Y = +1$. Putting $Y = y'/y = +1$ in the equations (116), we obtain for the abscissæ of the Principal Points:

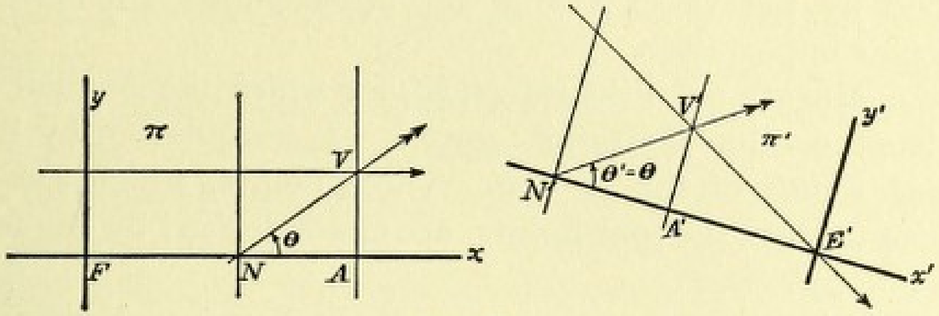

Fig. 92.

Cardinal Points of Optical System. Focal Points F, E'; Principal Points A, A'; Nodal Points N, N'.

$FA = N'E' = f$; $E'A' = NF = e'$; $NA = N'A'$; $AV = A'V'$; $\angle ANV = \theta = \angle A'N'V' = \theta'$.

$$x = FA = f, \quad x' = E'A' = e'; \tag{121}$$

and, hence, *The Focal Lengths f, e' of an optical system may also be defined as the abscissæ, with respect to the Focal Points F, E', of the Principal Points A, A', respectively.*

If the positive directions of the Principal Axes are determined by the directions pursued by the light in traversing these lines, then, as has been repeatedly stated, f will be positive and e' negative (see § 178); hence, the Primary Principal Point A will lie always on the positive half of the x-axis, and the Secondary Principal Point A' will lie on the negative half of the x'-axis.

The pair of conjugate planes at right angles to the Principal Axes at the Principal Points A, A' were likewise named by Gauss the *Principal Planes* of the system. These planes are characterized by the fact that to any point V in the Principal Plane of the Object-Space there corresponds a point V' in the Principal Plane of the Image-Space, such that $AV = A'V'$; so that an object lying in the Primary Principal Plane will be portrayed by an image lying in the Secondary Principal Plane, which is equal to the object in every particular.

We may remark also that at the Principal Points $x = f$, $x' = e'$ we have also:

$$X = \frac{1}{Z} = -\frac{e'}{f}.$$

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[1] C. F. Gauss: *Dioptrische Untersuchungen* (Goettingen, 1841), § 7.

The points A, A' are sometimes called also the Positive Principal Points in order to distinguish them from another pair of axial conjugate points called by TOEPLER[1] the Negative Principal Points, which are defined by the value $Y = -1$. These points are, however, of no particular importance.

3. The conjugate axial points N, N', for which the Angular Magnification has the value $Z = \tan\theta' : \tan\theta = +1$, were named by LISTING[2] the *Nodal Points* of the system. These points, which are next in importance to the Principal Points, are characterized by the following property:

To an object-ray crossing the x-axis at the Primary Nodal Point N at an inclination θ there corresponds an image-ray crossing the x′-axis at the Secondary Nodal Point N′ at an inclination θ′ = θ.

Putting $Z = +1$ in formulæ (119), we find for the abscissæ, with respect to the Focal Points F, E', of the Nodal Points N, N':

$$x = FN = -e', \quad x' = E'N' = -f;$$

or (Fig. 92):

$$FA = N'E' = f; \quad E'A' = NF = e'. \tag{122}$$

Moreover, since

$$AN = AF + FN = -(f + e'), \quad A'N' = A'E' + E'N' = -(f + e'),$$

we have:

$$AN = A'N'. \tag{123}$$

Hence, *the two Nodal Points are equidistant from the Principal Points;* and, since the abscissæ of N, N', with respect to A, A', respectively, have the same sign, the Nodal Points lie always either both to the right or both to the left of the corresponding Principal Points. And if $AN = 0$, then $A'N' = 0$ also.

For $Z = +1$, we have:

$$X = Y = -\frac{f}{e'}.$$

The planes perpendicular to the Principal Axes at the points N, N' are called the *Nodal Planes* of the system. TOEPLER likewise distinguished a pair of *Negative Nodal Points* defined by $Z = -1$.

These distinguished pairs of conjugate axial points are called the

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[1] A. TOEPLER: Bemerkungen ueber die Anzahl der Fundamentalpuncte eines beliebigen Systems von centrirten brechenden Kugelflaechen: POGG. *Ann.*, cxlii. (1871), 232–251.

[2] J. B. LISTING: Beitrag zur physiologischen Optik: *Goettinger Studien*, 1845. See also article by LISTING on the Dioptrics of the Eye, published in R. WAGNER's *Handwoerterbuch d. Physiologie* (Braunschweig, 1853), Bd. iv., p. 451.

Cardinal Points of the optical system; and some writers include also under this designation the Focal Points F, E'. Knowing the positions of one pair of the Cardinal Points, and knowing also the Focal Lengths of the optical system, we can determine completely the character of the imagery.[1]

181. The Image-Equations referred to a Pair of Conjugate Axial Points. It will be convenient sometimes, and always in the case of Telescopic Imagery (Art. 50), to select as origins of the two systems of co-ordinates some other pair of axial points besides the Focal Points which have been used hitherto for this purpose. Thus, suppose we take two conjugate axial points O, O' as origins, and let the co-ordinates of the conjugate points Q, Q' with respect to O, O' be denoted as follows:

$$OM = \xi, \quad MQ = y, \quad O'M' = \xi', \quad M'Q' = y';$$

where M, M' are the feet of the perpendiculars let fall from Q, Q' on the axes of x, x', respectively. Moreover, let

$$FM = x, \quad E'M' = x', \quad FO = x_0, \quad E'O' = x_0';$$

so that

$$xx' = x_0x_0' = fe'.$$

Now

$$x = x_0 + \xi, \quad x' = x_0' + \xi',$$

and, therefore:

$$(x_0 + \xi)(x_0' + \xi') = x_0x_0';$$

which may be written:

$$\frac{x_0}{\xi} + \frac{x_0'}{\xi'} + 1 = 0; \tag{124}$$

which is the required relation between the abscissæ ξ and ξ'. The constants x_0, x_0' which occur in this formula are the distances of the origins O, O' from the Focal Points F, E', respectively.

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[1] In connection with this subject, the following writers may be consulted (in addition to those already named):

C. G. NEUMANN: *Die Haupt- und Brenn-Punkte eines Linsen-Systems. Elementare Darstellung der durch* GAUSS *begruendeten Theorie* (Leipzig, 1866).

J. A. GRUNERT: Ueber merkwuerdige Puncte der Spiegel- und Linsen-Systeme: *Grun. Arch. f. Math. Phys.*, xlvii. (1867), 84–105.

F. LIPPICH: Fundamentalpunkte eines Systemes centrirter brechender Kugelflaechen: *Mitt. des naturw. Ver. f. Steiermark*, ii. (1871), 429–459.

L. MATTHIESSEN: *Grundriss der Dioptrik geschichteter Linsensysteme. Mathematische Einleitung in die Dioptrik des menschlichen Auges* (Leipzig, 1877). Also, Ueber eine Methode zur Berechnung der sechs Cardinalpuncte eines centrierten Systems sphaerischer Linsen: *Zft. f. Math. u. Phys.*, xxiii. (1878), 187–191. Also, Bestimmung der Cardinalpuncte eines dioptrisch-katoptrischen Systems centrirter sphaerischer Flaechen, mittels Kettenbruchdeterminanten dargestellt: *Zft. f. Math. u. Phys.*, xxxii. (1887), 170–175.

The above is only a partial list of the writers on this subject.

For the Lateral Magnification at M, M' we obtain:

$$Y = \frac{y'}{y} = \frac{f}{x_0 + \xi} = \frac{x_0' + \xi'}{e'}\,. \tag{125}$$

The Angular Magnification, in terms of the abscissæ ξ, ξ', is given as follows:

$$Z = \frac{\tan\theta'}{\tan\theta} = -\frac{x_0 + \xi}{e'} = -\frac{f}{x_0' + \xi'}\,. \tag{126}$$

If Y_0 denotes the Lateral Magnification at the points O, O', then $Y_0 = f/x_0 = x_0'/e'$. Hence, if we choose, we may eliminate the constants x_0, x_0' in the above equations by putting:

$$x_0 = \frac{f}{Y_0}, \quad x_0' = e'Y_0\,;$$

thus,

$$\left.\begin{aligned} &\frac{f}{Y_0\xi} + \frac{e'Y_0}{\xi'} + 1 = 0, \\ Y &= \frac{fY_0}{f + Y_0\xi} = \frac{e'Y_0 + \xi'}{e'} = -\frac{f}{e'}\frac{\xi'}{\xi}\frac{1}{Y_0}\,, \\ Z &= -\frac{f + Y_0\xi}{e'Y_0} = -\frac{f}{e'Y_0 + \xi'} = \frac{\xi}{\xi'}Y_0\,. \end{aligned}\right\} \tag{127}$$

In particular, if the origins of the two systems of co-ordinates are the pair of conjugate axial points A, A' called the Principal Points (§ 180), and if for this special case we denote the abscissæ of the points Q, Q' by u, u', so that

$$AM = u, \quad A'M' = u',$$

then, writing u, u' in place of ξ, ξ', respectively, in formulæ (127), and putting $Y_0 = 1$, we obtain the *Image-Equations referred to the Principal Points as origins*, as follows:

$$\left.\begin{aligned} &\frac{f}{u} + \frac{e'}{u'} + 1 = 0, \\ Y &= \frac{y'}{y} = \frac{f}{f + u} = \frac{e' + u'}{e'} = -\frac{fu'}{e'u}\,, \\ Z &= \frac{\tan\theta'}{\tan\theta} = -\frac{f + u}{e'} = -\frac{f}{e' + u'} = \frac{u}{u'}\,. \end{aligned}\right\} \tag{128}$$

182. Geometrical Constructions of Conjugate Points of an Optical System.

1. *Construction of Conjugate Axial Points.* The equation

$$\frac{f}{u}+\frac{e'}{u'}+1=0$$

suggests a simple method of construction of the pair of conjugate axial points M, M', provided we know the positions and directions of the Principal Axes x, x', the positions of the two Principal Points A, A', and the magnitudes of the Focal Lengths f, e' of the Optical System. For since $f = FA$, $e' = E'A'$, the equation above may be written:

$$\frac{AF}{u}+\frac{A'E'}{u'}=1;$$

and, hence, if we suppose the two Principal Points A, A' (Fig. 93) are placed in coincidence with each other so that the positive directions of the Principal Axes x, x' make with each other at A (or A') any angle xAx' different from zero, and if through the Focal Points F and E' we draw straight lines parallel to x' and x, respectively, intersecting each other at a point O, then any straight line drawn through O will make on the axes x, x' intercepts $AM = u$, $A'M' = u'$, respectively, which will satisfy the above equation. In fact the point O is the centre of perspective of the two point-ranges x, x'.

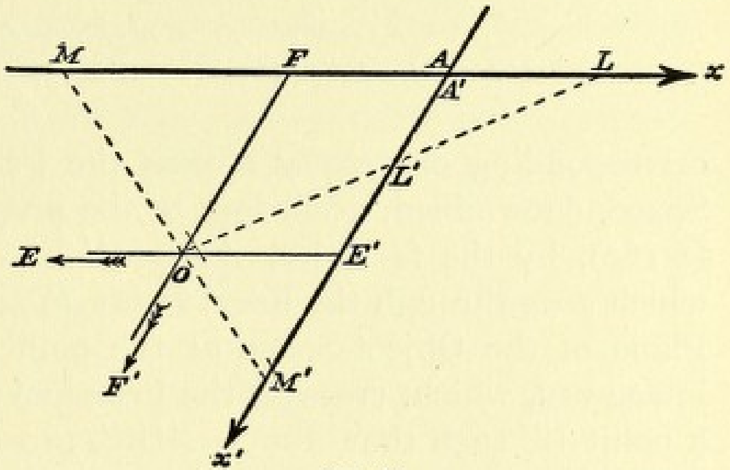

FIG. 93.

CONSTRUCTION OF CONJUGATE AXIAL POINTS M, M' (OR L, L') OF AN OPTICAL SYSTEM.

$FA=f$; $E'A'=e'$; $AM=u$; $A'M'=u'$.

2. *Construction of Conjugate Points Q, Q' not on the Principal Axes.* Suppose that the optical system is given by assigning the positions and directions of the Principal Axes x, x' (Fig. 94), the positions of the two Focal Points F, E' and the Focal Lengths f, e'. The Principal Points A, A' may be located at once, since $FA=f$, $E'A'=e'$ (§ 180); and the planes through these points perpendicular to the Principal Axes are the Principal Planes. The point Q' is the vertex

of the bundle of image-rays corresponding to the bundle of object-rays whose vertex is the Object-Point Q. If, therefore, we can determine two of the rays of the bundle of image-rays, they will suffice to determine by their point of intersection the Image-Point Q'. Thus, for example, to the object-ray $Q\,V$ which is parallel to the x-axis there corresponds an image-ray which goes through the Focal Point E'; and moreover, this image-ray will cross the Principal Plane of the Image-Space at a point V' conjugate to the point V where the corresponding object-ray crosses the Principal Plane of the Object-Space, determined, according to the property of the Principal Planes (§ 180), by the fact that $A\,V = A'\,V'$. Again, the object-ray QW, which goes through the Focal Point F, and which meets the Principal Plane of the Object-Space at the point W, must correspond to an image-ray, which, crossing the Principal Plane of the Image-Space at a point W' such that $A\,W = A'W'$, proceeds parallel to the Principal Axis x'; and the intersection of this ray with the other image-ray $V'E'$ will determine the Image-Point Q' conjugate to the Object-Point Q.

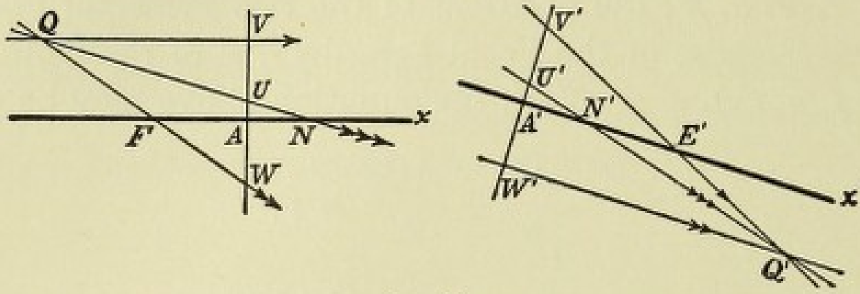

FIG. 94.

CONSTRUCTION OF CONJUGATE POINTS Q, Q' OF AN OPTICAL SYSTEM. F and E' are the Focal Points; A and A' are the Principal Points; and N and N' are the Nodal Points.

$$FA = N'E' = f;\quad E'A' = NF = e';\quad AN = A'N';$$
$$\angle ANU = \angle A'N'U';\quad AV = A'V';\quad AU = A'U';\quad AW = A'W'.$$

If we know the positions of the two Focal Points F, E', and if we know also the Focal Lengths f, e', we may locate the Nodal Points N, N' (§ 180). To an object-ray $Q\,N$ meeting the Principal Plane of the Object-Space at a point U there corresponds an image-ray, which, crossing the Principal Plane of the Image-Space at a point U' such that $A\,U = A'\,U'$, has the same inclination to the x'-axis as the object-ray $Q\,N$ has to the x-axis; that is, $\angle A\,N\,U = \angle A'\,N'\,U'$. Thus, the point Q' may be determined as the point of intersection of any pair of the three image-rays $V'Q'$, $U'Q'$ and $W'Q'$.

ART. 50. TELESCOPIC IMAGERY.

183. The Image-Equations in the Case of Telescopic Imagery. In the special and singular case of Telescopic Imagery (§ 165), the Image-Equations (110) referred to the Focal Points F, E' are not applicable, because in this case the Focal Planes φ and ϵ' are no longer actual, or finite, planes, but they are the infinitely distant planes ϵ and φ' of the Object-Space Σ and the Image-Space Σ', respectively. The infinitely distant planes are not only the Focal Planes, but they are also a pair of conjugate planes; so that, if we were consistent in our notation, and if we designated the Focal Plane of the Object-Space by φ, in the case of Telescopic Imagery we should designate the Focal Plane of the Image-Space by φ'. In the language of geometry the two Space-Systems Σ, Σ' are said to be in "*affinity*" with each other. Each pair of conjugate plane-fields π, π' of Σ, Σ' are also in affinity with each other, because to every infinitely distant straight line of Σ there corresponds an infinitely distant straight line of Σ'. Thus, also, each pair of conjugate point-ranges of Σ and Σ' are "projectively similar", so that corresponding segments of them are in a constant ratio to each other (§ 166). Hence, the image in Σ' of a parallelogram of Σ will likewise be a parallelogram; and so, also, the image of a parallelepiped will be a parallelepiped. To a bundle of parallel object-rays will correspond a bundle of parallel image-rays.

Since corresponding point-ranges are "projectively similar", we can say:

In the case of Telescopic Imagery, the Magnification-Ratio has the same value for all parallel rays.

This fundamental characteristic of Telescopic Imagery will enable us to deduce the Image-Equations immediately. Thus, selecting as origins of the two systems of co-ordinates any pair of conjugate points O, O', let us take as axes of x, y, z any three straight lines meeting in O, and let the three straight lines conjugate to x, y, z which meet in O' be selected as axes of x', y', z'. If the magnification-ratios for the three bundles of parallel object-rays to which the axes of x, y, z belong are denoted by the constants p, q, r, we may write the *Image-Equations for the case of Telescopic Imagery*, as follows:

$$x' = px, \quad y' = qy, \quad z' = rz. \tag{129}$$

If we select a set of *rectangular axes* in the Object-Space, the axes of x', y', z' in the Image-Space will in general be oblique. But in the two projective bundles of rays O, O', there is always one set of mutually perpendicular rays of O to which corresponds also a system of three mutually perpendicular rays of O'; so that if we choose these two particular sets of corresponding rays as axes of co-ordinates of the two Space-Systems, the equations above will be the general Image-Equations, referred to rectangular axes, for the case of Telescopic Imagery.

It will be remarked that in the general case of Telescopic Imagery the Image-Equations involve at least three independent constants. A striking difference between Telescopic Imagery and Optical Imagery in general is to be noted in the fact that in the former there are no Principal Axes; so that it is merely a matter of preference which of the axes of co-ordinates is selected as the axis of x.

However, if (as is practically nearly always the case) the Imagery is *symmetrical* with respect to one pair of the conjugate axes of the two systems of rectangular co-ordinates, it is usual to select these as the axes of x and x'. In this case putting $r = q$, we may write the Image-Equations as follows:

$$x' = px, \quad y' = qy, \quad z' = qz. \tag{130}$$

184. Characteristics of Telescopic Imagery. In the case of Telescopic Imagery, both the Lateral Magnification Y and the Depth-Magnification X are constant. Thus:

$$Y = \frac{y'}{y} = q = Y_0, \quad X = \frac{dx'}{dx} = \frac{x'}{x} = p = X_0. \tag{131}$$

In regard to the Focal Lengths f and e', defined as in § 178, it is obvious that we have here:

$$f = e' = \infty.$$

But by formula (118) we have $X/Y^2 = -e'/f$; hence, here:

$$\frac{e'}{f} = -\frac{p}{q^2}. \tag{132}$$

Accordingly, we may say that the characteristic of Telescopic Imagery is that, *whereas the Focal Lengths f and e' are infinite, the ratio of the Focal Lengths is finite.*

Introducing this finite ratio as one of the image-constants and the Lateral Magnification $Y = q = Y_0$ as the other constant, we may write the Image-Equations for the case of Telescopic Imagery as follows:

$$\frac{x'}{x} = -Y_0\frac{e'}{f}, \quad \frac{y'}{y} = Y_0. \tag{133}$$

The Angular Magnification Z is given by the formula:

$$Z = \frac{\tan\theta'}{\tan\theta} = \frac{Y}{X}.$$

Hence, for the case of Telescopic Imagery:

$$Z = \frac{q}{p} = -\frac{f}{e'}\frac{1}{Y_0}. \tag{134}$$

Thus, the Angular Magnification in Telescopic Imagery is constant also. It may be remarked that the Angular Magnification is an especially important magnitude in this kind of imagery; for when we are considering the infinitely distant image of an infinitely distant object, the Angular Magnification is the only kind of magnification that conveys any meaning. If in the Image-Equations we introduce $Z = q/p = Z_0$ and the ratio $e'/f = -p/q^2$ as the two image-constants, these equations may also be expressed as follows:

$$\frac{x'}{x} = -\frac{f}{e'}\frac{1}{Z_0^2}, \quad \frac{y'}{y} = -\frac{f}{e'}\frac{1}{Z_0}. \tag{135}$$

ART. 51. COMBINATION OF TWO OPTICAL SYSTEMS.

185. The Problem in General. A series of Optical Systems may be so arranged one after the other that the Image-Space of one system is at the same time the Object-Space of the following system, and so on. The resultant effect of all these successive imageries will be an imagery which may be regarded as due to a single optical system which by itself would produce the same effect. An optical instrument, whose function is to produce an image of an external object, is, in fact, nearly always a compound system or combination of simpler systems. Provided we know the Focal Lengths and the positions of the Focal Points and of the Principal Axes of each of the component systems, it is always possible to ascertain the Focal Lengths and the positions of the Focal Points of the compound system. In case the system is composed of spherical refracting (or reflecting) surfaces with their centres ranged along a straight line, the Principal Axes of each of the elements of the system will coincide with the "optical axis" (§ 135). This is usually the case in an actual optical instrument, and the problem is greatly simplified by this condition. However, following CZAPSKI,[1] and supposing at first that we have *only two component systems*, we shall consider here a rather more general case than the one above-mentioned. Thus, the only restriction which we shall make is the following:

The Principal Axis (x_1') *of the Image-Space of System* (*I*) *shall be also the Principal Axis* (x_2) *of the Object-Space of System* (*II*).

(Since this condition is usually satisfied in the case of imagery by means of narrow bundles of rays inclined to the Principal Axes at finite angles, the results which we shall obtain here will be directly applicable also to this case, as we shall have occasion of seeing in a subsequent chapter. See § 248.)

Let F_1, E_1' and F_2, E_2' (Fig. 95) designate the positions of the Focal Points of the systems (I) and (II), respectively; and let f_1, e_1' and f_2, e_2' denote the Focal Lengths of the two systems. Since we have assumed that the Principal Axis (x_1') of the Image-Space of (I) is coincident with the Principal Axis (x_2) of the Object-Space of (II), the Focal Planes at E_1' and F_2 will be parallel. The relative position of the two component systems may be assigned by giving the distance of the point F_2 from the point E_1', that is, the abscissa of F_2 with respect to E_1'. This magnitude, usually denoted by writers on Optics by the symbol Δ, that is,

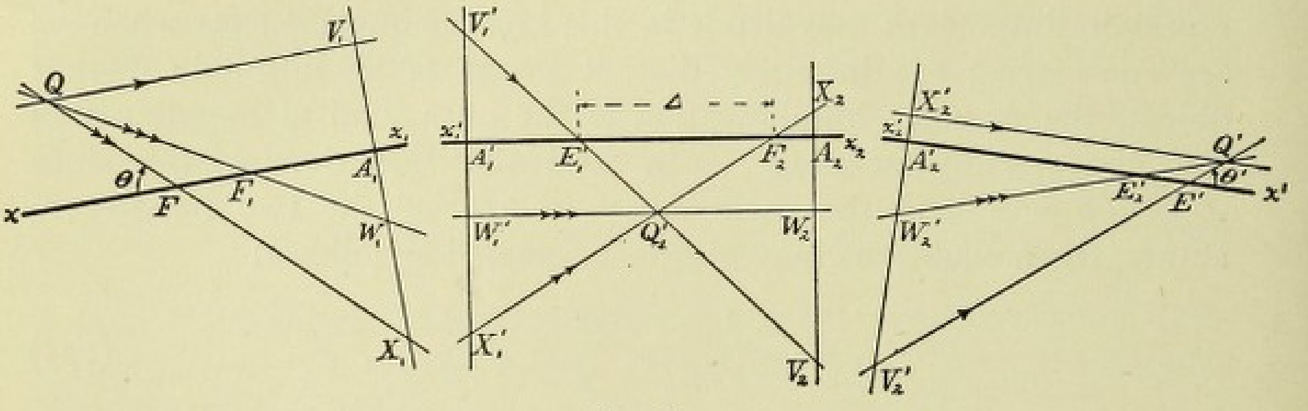

FIG. 95.

COMBINATION OF TWO OPTICAL SYSTEMS. F_1 and E_1' and F_2, E_2' mark the positions of the Focal Points, and A_1, A_1' and A_2, A_2' the positions of the Principal Points of systems (I) and (II), respectively; and F, E' mark the positions of the Focal Points of the compound system (I + II). $f_1 = F_1A_1$, $e_1' = E_1'A_1'$, and $f_2 = F_2A_2$, $e_2' = E_2'A_2'$ denote the Focal Lengths of systems (I) and (II), respectively; whereas f, e' denote the Focal Lengths of the compound system. The "interval" between the two systems (I) and (II) is $E_1'F_2 = \Delta$.

$$\Delta = E_1'F_2,$$

and reckoned positive or negative according as the direction from E_1'

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[1] S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), pages 46, foll.

to F_2 is the same as or opposite to that along which the light travels, is called *the interval* between the two systems.

The Focal Points of the compound system will be denoted by the letters F and E' (without any subscripts); and, similarly, the Focal Lengths of the compound system will be denoted by the symbols f and e'.

The problem is, therefore, with the data above-mentioned defining the component systems, to determine the Focal Points and the Focal Lengths of the compound system.

In the first place, it is obvious that the Focal Planes at the points F and F_1 are parallel, as is also the case with the Focal Planes at the points E' and E'_2. For in the system (I), to the sheaf of planes which are parallel to the Focal Plane at F_1 there corresponds a sheaf of planes which are parallel to the Focal Plane at E'_1 (§ 166), and which are therefore, by hypothesis, parallel likewise to the Focal Plane at F_2; and to this sheaf of planes in the Object-Space of system (II) there corresponds a sheaf of planes which are parallel to the Focal Plane at E'_2. Hence, to planes which are parallel to the Focal Plane at F_1 there correspond planes which are parallel to the Focal Plane at E'_2. But we have seen that in an optical system, in general, the only two sheaves of parallel planes which are conjugate sheaves of planes are the sheaves to which the Focal Planes belong. Hence, the Focal Planes at F and E' are parallel to the Focal Planes at F_1 and E'_2, respectively.

Moreover, the Principal Axis (x_1) of the Object-Space of system (I) is also the Principal Axis (x) of the Object-Space of the compound system; and, similarly, the Principal Axis (x'_2) of the Image-Space of system (II) is also the Principal Axis (x') of the Image-Space of the compound system. For since x_1 and x' are obviously a pair of conjugate rays with respect to the compound system, and since the Principal Axes of an optical system have been defined (§ 167) as that pair of conjugate rays which meet at right angles the Focal Planes of the system, it follows that x_1 and x'_2 must coincide with x and x', respectively.

We proceed, in the next place, to ascertain *the positions of the Focal Points F, E' of the compound system.* Consider, for example, an object-ray proceeding in the direction QV_1 parallel to the Principal Axis (x) of the Object-Space. Emerging from the first system, this ray will go through the Focal Point E'_1 of the Image-Space of this system, and, traversing the second system, will finally cross the Principal Axis (x') of the Image-Space of the compound system at the point E'; so that, with respect to system (II), the points E'_1 and E' are a pair of conjugate points, and therefore, according to the first of formulæ (115):

$$F_2E'_1 \cdot E'_2E' = f_2e'_2\,;$$

that is,

$$E'_2E' = -\frac{f_2e'_2}{\Delta}\,.$$

Since we know the position of the point E'_2, this equation enables us to determine that of the Focal Point E' of the Image-Space of the compound system.

Again, consider an object-ray QF going through the Focal Point F of the Object-Space of the compound system. This ray, after traversing the entire system, must emerge parallel to the Principal Axis (x') of the Image-Space of the compound system. And, therefore, it must have passed through the Focal Point F_2 of the Object-Space of system (II), and, hence, the points F and F_2 must be a pair of conjugate axial points with respect to system (I). Accordingly, in the same way as above:

$$F_1F \cdot E'_1F_2 = f_1e'_1;$$

that is,

$$F_1F = \frac{f_1e'_1}{\Delta}\,;$$

whereby the position of the Focal Point F of the Object-Space of the compound system can be located with respect to the position of the given point F_1.

Thus, having located the positions of the Focal Points F, E' of the compound system, we have next *to determine the Focal Lengths* f, e'. Recalling the definitions of the Focal Lengths as given by formulæ (114), viz.:

$$f = \frac{k'}{\tan\theta}, \quad e' = \frac{g}{\tan\theta'}\,;$$

let us consider again the two rays which we have already employed.

The ray QV_1, which proceeds parallel to the Principal Axis (x) of the Object-Space of the compound system, crosses the Focal Plane φ at the height $g = A_1V_1$, and after traversing the entire system, emerges so as to cross the Principal Axis (x') of the Image-Space at the Focal Point E' at an angle $\theta' = \angle A'_2E'V'_2$, so that $e' = g/\tan\theta'$. This ray crosses the Principal Axis (x'_1) of the Image-Space of system (I) at the point E'_1 (Fig. 95), so that

$$g = e'_1 \cdot \tan\angle A'_1E'_1V'_1.$$

And since E'_1 and E' are a pair of conjugate axial points with respect to system (II), the ratio $\tan\theta' : \tan\angle A'_1E'_1V'_1$ is the value of the Angular Magnification (Z_2) of system (II) for this pair of conjugate points. Applying, therefore, formula (119), we obtain:

$$\frac{\tan\theta'}{\tan\angle A'_1E'_1V'_1} = -\frac{F_2E'_1}{e'_2} = \frac{\Delta}{e'_2}.$$

Thus, the formula $e' = g/\tan\theta'$ becomes:

$$e' = \frac{e'_1e'_2}{\Delta},$$

whereby the magnitude of the Focal Length e' of the Image-Space of the compound system is determined in terms of the known magnitudes e'_1, e'_2 and Δ.

Similarly, if $\angle A_1FX_1 = \theta$ is the inclination to the x-axis of the object-ray QX_1, which goes through the Focal Point F of the Object-Space, and which, after traversing the entire system, emerges in the direction X'_2Q' parallel to the x'-axis, and if k' denotes the height at which the emergent ray crosses the Focal Plane ϵ' of the Image-Space, then $f = k'/\tan\theta$. This ray crossed the Principal Axis (x_2) of the Object-Space of system (II) at the point F_2, so that

$$k' = f_2 \cdot \tan\angle A_2F_2X_2.$$

The points F and F_2 are a pair of conjugate axial points with respect to system (I), and the value of the Angular Magnification (Z_1) of this system for this pair of conjugate points is given by the ratio

$$\tan\angle A_2F_2X_2 : \tan\theta.$$

Thus, by formula (119), we obtain:

$$\frac{\tan\angle A_2F_2X_2}{\tan\theta} = -\frac{f_1}{E'_1F_2} = -\frac{f_1}{\Delta}.$$

Accordingly, we find:

$$f = -\frac{f_1f_2}{\Delta};$$

whereby the magnitude of the Focal Length f of the Object-Space of the compound system can be determined in terms of the known constants f_1, f_2 and Δ.

The formulæ for the Focal Lengths f and e' may be obtained also by considering the ray, which, proceeding from the Object-Point Q, goes through the Focal Point F_1 of the Object-Space of system (I), and which, therefore, emerges from system (II) so as to cross the x'-axis at the Focal Point E' of the Image-Space of system (II). Thus, from Fig. 95, we have:

$$f_1 = \frac{A'_1 W'_1}{\tan \angle A_1 F_1 W_1}, \quad e'_2 = \frac{A_2 W_2}{\tan \angle A'_2 E'_2 W'_2}.$$

Now $A'_1 W'_1 = A_2 W_2$, and therefore:

$$\frac{\tan \angle A'_2 E'_2 W'_2}{\tan \angle A_1 F_1 W_1} = \frac{f_1}{e'_2}.$$

But since F_1 and E'_2 are conjugate axial points with respect to the compound system, we have by formula (119):

$$\frac{\tan \angle A'_2 E'_2 W'_2}{\tan \angle A_1 F_1 W_1} = -\frac{f}{E'E'_2} = -\frac{FF_1}{e'}.$$

Hence:

$$\frac{f_1}{e'_2} = -\frac{f}{E'E'_2} = -\frac{FF_1}{e'};$$

and, therefore, as before

$$f = -\frac{f_1 f_2}{\Delta}, \quad e' = \frac{e'_1 e'_2}{\Delta}.$$

The formulæ derived above may be collected as follows:

$$\left.\begin{array}{l}
\textit{Positions of the Focal Points } F,\ E'\textit{:} \\
\qquad F_1F = \dfrac{f_1 e'_1}{\Delta}, \quad E'_2E' = -\dfrac{f_2 e'_2}{\Delta}; \\
\textit{Magnitudes of the Focal Lengths } f,\ e'\textit{:} \\
\qquad f = -\dfrac{f_1 f_2}{\Delta}, \quad e' = \dfrac{e'_1 e'_2}{\Delta}.
\end{array}\right\} \qquad (136)$$

The influence of the interval Δ between the two systems, which forms the denominator of the right-hand member of each of these formulæ, is at once apparent. Two given systems (I) and (II) may be combined in an infinite number of ways by merely altering the interval Δ either as to its magnitude or as to its sign or as to both. So long as this magnitude Δ is different from zero, and none of the Focal Points of the component systems are situated at infinity, we shall have a compound system with finite Focal Lengths f, e'.

186. Special Cases of the Combination of Two Optical Systems. We may consider several special cases of the combination of two optical systems as follows:

1. *The Case when the "Interval" is zero* ($\Delta = 0$), *the Focal Lengths* f_1, e'_1 *and* f_2, e'_2 *of the component systems* (*I*) *and* (*II*) *being all finite.*

In this case the compound system will be *telescopic*, since according to equations (136), we have here $f = e' = \infty$, whereas the ratio $f/e' = -f_1 f_2/e'_1 e'_2$ is finite by hypothesis (see § 184).

In a telescopic system the three magnification-ratios X, Y and Z, as we saw in § 184, are all constant; let us denote their values here by the special symbols X_0, Y_0 and Z_0, respectively. So soon as we know the values of any two of these magnitudes, the system will be completely determined. In fact, since we know already the value of the finite ratio of the Focal Lengths of the Telescopic System, it will be sufficient if we know also only one of the magnitudes denoted by X_0, Y_0, Z_0.

The diagram (Fig. 96) represents the case of the combination of two non-telescopic systems into a telescopic system, which is the case now under consideration. The letters in this figure have the same meanings as they have in the preceding figure, so that they do not need to be explained again.

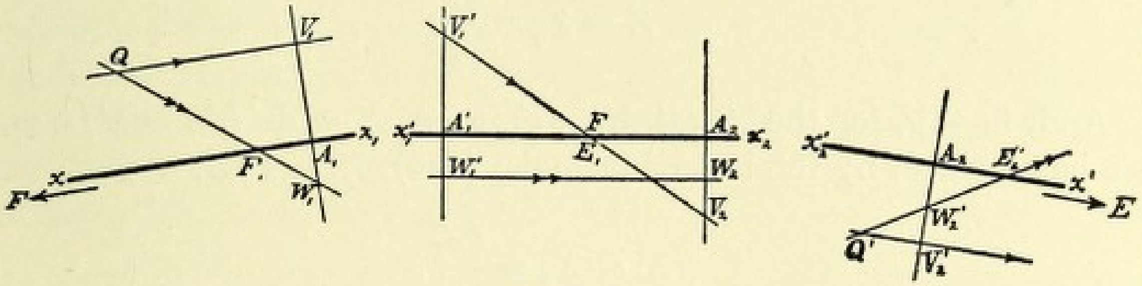

FIG. 96.

TELESCOPIC SYSTEM RESULTING FROM THE COMBINATION OF TWO NON-TELESCOPIC SYSTEMS PLACED TOGETHER SO THAT $\Delta = 0$.

The *Lateral Magnification* is evidently:

$$Y = \frac{y'}{y} = \frac{A'_2 V'_2}{A_1 V_1} = Y_0.$$

If we wish to obtain the value of the magnitude Y_0 in terms of the Focal Lengths of the component systems, we have from the definitions of the Focal Lengths (§ 178):

$$e'_1 = \frac{A_1 V_1}{\tan \angle A'_1 E'_1 V'_1}, \quad f_2 = \frac{A'_2 V'_2}{\tan \angle A_2 F_2 V_2};$$

and since $\Delta = 0$, so that the points E'_1 and F_2 are coincident, it follows that

$$\frac{A'_2V'_2}{A_1V_1} = \frac{f_2}{e'_1};$$

and, hence:

$$Y = Y_0 = \frac{f_2}{e'_1}.$$

Similarly, the *Angular Magnification* $Z = Z_0$, in terms of the Focal Lengths of the component systems, may be obtained as follows: Consider the axial points F_1 and E'_2, which, with respect to the compound system, are a pair of conjugate points; evidently, we have:

$$Z = Z_0 = \frac{\tan \angle A'_2E'_2W'_2}{\tan \angle A_1F_1W_1},$$

and since

$$\tan \angle A'_2E'_2W'_2 = \frac{A_2W_2}{e'_2}, \quad \tan \angle A_1F_1W_1 = \frac{A'_1W'_1}{f_1}, \quad A_2W_2 = A'_1W'_1,$$

we find:

$$Z = Z_0 = \frac{f_1}{e'_2}.$$

And, finally, for the *Axial Magnification*, $X = dx'/dx = x'/x = X_0$, we have, since, by the last of formulæ (120), $X = Y/Z$,

$$X = X_0 = \frac{f_2e'_2}{f_1e'_1}.$$

2. *Combination of Telescopic System (I) with Non-Telescopic, or Finite, System (II).*

Let the Telescopic System (I) be given by the values of the constant axial and lateral magnification-ratios X_1, Y_1, respectively, and by the positions of the conjugate axial points M_1, M'_1 (Fig. 97). Here, as in the preceding case, we assume that the Principal Axis (x'_1) of the Image-Space of system (I) and the Principal Axis (x_2) of the Object-Space of system (II) are coincident. If F_2 designates the position of the Focal Point of the Object-Space of system (II), the relative positions of the two systems may be assigned by the value of the abscissa of the point F_2 with respect to the point M'_1. Let us denote this abscissa by the symbol a, so that $M'_1F_2 = a$.

A ray proceeding parallel to the Principal Axis (x_1) of the Object-Space of system (I) will also be parallel to the Principal Axis (x'_1) of the Image-Space of this system, and, emerging finally from system

(II), will cross the Principal Axis (x') of the Image-Space of the compound system at the Focal Point E_2' of the Image-Space of system (II), which is likewise also the Focal Point E' of the Image-Space of the compound system. And since the position of E_2' is given, we know, therefore, the position of E'.

The position of the other Focal Point F of the compound system will be determined if we ascertain its position with respect to the given axial Object-Point M_1. An image-ray which emerges from the compound system in a direction parallel to the x'-axis corresponds to an object-ray which crosses the x-axis at the required point F, and which must also have passed through the Focal Point F_2 of the Object-Space of system (II). Hence, the points F and F_2 must be a pair of conjugate axial points with respect to the telescopic system (I). And since, by the second of equations (131),

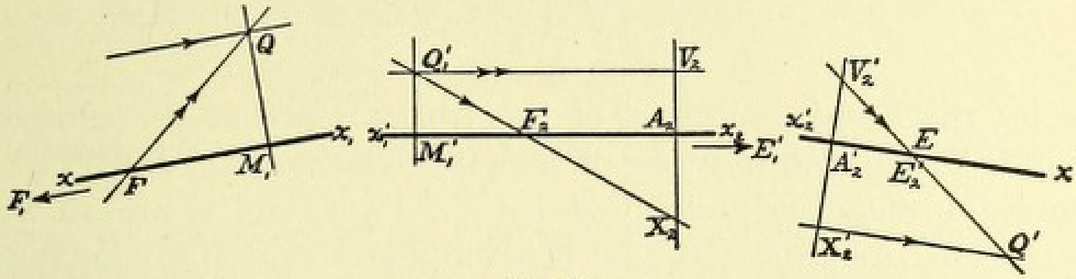

FIG. 97.

COMBINATION OF A TELESCOPIC WITH A NON-TELESCOPIC SYSTEM INTO A NON-TELESCOPIC COMPOUND SYSTEM.

$$X_1 = \frac{M_1'F_2}{M_1F} = \frac{a}{M_1F},$$

we obtain immediately:

$$M_1F = \frac{a}{X_1};$$

whereby the position of the Focal Point F is ascertained.

It only remains therefore to determine the magnitudes of the Focal Lengths f and e'. From the figure we obtain:

$$A_2'X_2' = f_2 \cdot \tan \angle A_2F_2X_2,$$

$$f = \frac{A_2'X_2'}{\tan \angle M_1FQ} = \frac{f_2 \cdot \tan \angle A_2F_2X_2}{\tan \angle M_1FQ} = f_2 \cdot Z_1;$$

where Z_1 denotes the constant value of the Angular Magnification of the telescopic system (I). Since $Z_1 = Y_1/X_1$, we have, therefore:

$$f = \frac{f_2 \cdot Y_1}{X_1};$$

where f is determined in terms of the known constants f_2, X_1 and Y_1.

Again,

$$M_1Q = \frac{M'_1Q'}{Y_1} = \frac{A_2V_2}{Y_1} = \frac{e'_2 \cdot \tan \angle A'_2E'_2V'_2}{Y_1},$$

$$e' = \frac{M_1Q}{\tan \angle A'_2E'_2V'_2};$$

and, hence:

$$e' = \frac{e'_2}{Y_1};$$

whereby the Focal Length e' of the Image-Space of the compound system is determined in terms of the given constants e'_2 and Y_1.

Since the magnitudes denoted by X_1, Y_1, f_2 and e'_2 are all finite, it is evident from the formulæ here obtained that the combination of a telescopic system with a non-telescopic system is a non-telescopic system. If the system (II) were the telescopic system, the procedure would be entirely similar to that given above.

3. *Both Systems Telescopic.*

Let us suppose that the two component telescopic systems are given by the values of their constant Axial and Lateral Magnification-Ratios X_1, Y_1, and X_2, Y_2; and that the relative position of the two systems is given by the positions of a pair of conjugate points L, L'_1 (Fig. 98) of system (I) and the positions of a pair of conjugate axial points M_2, M' of system (II). Let us write $a = L'_1M_2$.

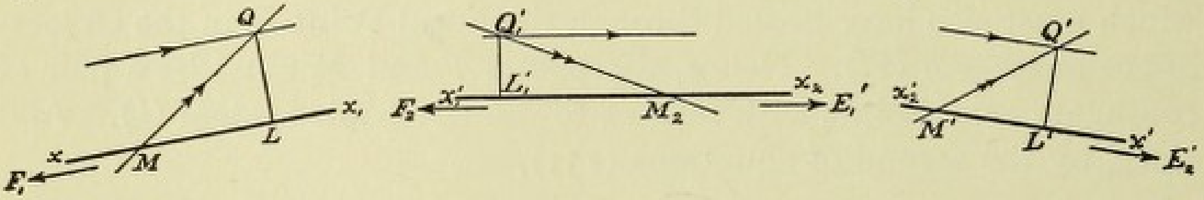

FIG. 98.

COMBINATION OF TWO TELESCOPIC SYSTEMS INTO A TELESCOPIC SYSTEM.

To begin with, it is obvious that the compound system is also telescopic. Thus, the Lateral Magnification of the compound system is:

$$Y = \frac{L'Q'}{LQ} = \frac{L'Q'}{L'_1Q'_1} \cdot \frac{L'_1Q'_1}{LQ} = Y_1 \cdot Y_2 = \text{a constant};$$

and the Axial Magnification of the compound system is:

$$X = \frac{L'M'}{LM} = \frac{L'M'}{L'_1M_2} \cdot \frac{L'_1M_2}{LM} = X_1 \cdot X_2 = \text{a constant}.$$

Here the letters M and L' designate the positions of the points, which, with respect to the compound system, are conjugate to the given points M' and L, respectively. The positions of the points M and L' may be determined as follows:

Since $L'_1M_2 : LM = X_1$, and $L'_1M = a$, we find:

$$LM = \frac{a}{X_1};$$

whereby the position of the point M is determined relative to that of the given point L. Again, since $M'L' : M_2L'_1 = X_2$, we have:

$$M'L' = -a \cdot X_2;$$

whereby the point L' may be located with respect to the given point M'.

ART. 52. GENERAL FORMULÆ FOR THE DETERMINATION OF THE FOCAL POINTS AND FOCAL LENGTHS OF A COMPOUND OPTICAL SYSTEM.

187. We shall suppose that the compound system consists of m component systems, and we shall assume that *the Principal Axis* (x'_k) *of the Image-Space of the kth system is likewise the Principal Axis* (x_{k+1}) *of the Object-Space of the* ($k + 1$)*th system.* In this statement the symbol k denotes an integer, which is supposed to have in succession every value from $k = 1$ to $k = m - 1$.

In the diagram (Fig. 99), F_k and E'_k designate the Focal Points of the kth system, and A_k and A'_k designate the Principal Points of this system. To an axial Object-Point M_1 lying on the Principal Axis (x_1) of the Object-Space of the first system there corresponds an axial Image-Point M'_m lying on the Principal Axis (x'_m) of the Image-Space of the last, or mth, system. A ray proceeding originally from the point M_1 will cross the Principal Axis (x_k) of the Object-Space of the kth system at the point M'_{k-1} and the Principal Axis (x'_k) of the

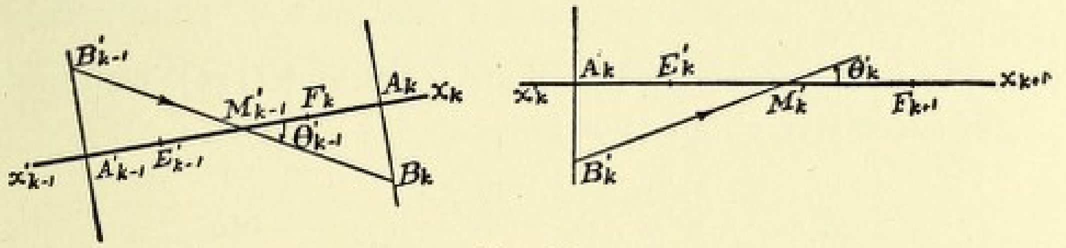

FIG. 99.

DETERMINATION OF FOCAL POINTS (F, E') AND FOCAL LENGTHS (f, e') OF A COMPOUND OPTICAL SYSTEM. The diagram shows the Principal Axes (x_k, x_k'), the Principal Points (A_k, A_k') and the Focal Points (F_k, E_k') of the kth member of the system, and the path of a ray traversing this component.

$$F_kA_k = f_k, \quad E_k'A_k' = e_k', \quad E_k'F_{k+1} = \Delta_k.$$

$$F_kM_{k-1}' = x_k, \quad E_k'M_k' = x_k', \quad A_kB_k = h_k = A_k'B_k' = h_k'.$$

$$\angle A_{k-1}'M_{k-1}'B_{k-1}' = \theta_{k-1}', \quad \angle A_k'M_k'B_k' = \theta_k'.$$

Image-Space of this system at the point M'_k. Moreover, let B_k and B'_k designate the points where this ray crosses the Principal Planes of the kth system, and let us write

$$A_kB_k = h_k, \quad A'_kB'_k = h'_k\,;$$

and, also:

$$F_kM'_{k-1} = x_k, \quad E'_kM'_k = x'_k.$$

The slope of the ray at M'_{k-1} is:

$$\angle A'_{k-1}M'_{k-1}B'_{k-1} = \theta'_{k-1} = \angle A_kM'_{k-1}B_k.$$

The Focal Lengths of the kth system will be denoted by f_k, e'_k; thus,

$$F_kA_k = f_k, \quad E'_kA'_k = e'_k;$$

and, finally, the interval between the kth and the $(k + 1)$th systems will be denoted by

$$\Delta_k = E'_kF_{k+1}.$$

Evidently, we have the following two systems of equations:

$$x_kx'_k = f_ke'_k, \quad (k = 1, 2, \cdots, m), \tag{137}$$

and

$$x'_k = x_{k+1} + \Delta_k, \quad (k = 1, 2, \cdots, m - 1). \tag{138}$$

1. *Determination of the Positions of the Focal Points F, E′ of the Compound System.*

From the two systems of equations (137) and (138), we obtain by process of successive elimination:

$$x_1 = \frac{f_1e'_1}{\Delta_1 + \dfrac{f_2e'_2}{x'_2}} = \frac{f_1e'_1}{\Delta_1 + \dfrac{f_2e'_2}{\Delta_2 + \dfrac{f_3e'_3}{x'_3}}} = \text{etc.}$$

Now if the Object-Point M_1 coincides with the Focal Point F of the Object-Space of the compound system, the Image-Point M'_m will be the infinitely distant point of the Principal Axis (x') of the Image-Space of the compound system; and in this case:

$$x_1 = F_1F, \quad x'_m = \infty.$$

Introducing these values, we obtain the abscissa of the Focal Point F of the compound system in the form of a continued fraction as follows:

$$F_1F = \cfrac{f_1e'_1}{\Delta_1 + \cfrac{f_2e'_2}{\Delta_2 + \cdots \vdots \, \Delta_{m-2} + \cfrac{f_{m-1}e'_{m-1}}{\Delta_{m-1}}}}. \tag{139}$$

On the other hand, if we suppose that the Image-Point M'_m coincides with the Focal Point E' of the Image-Space of the compound system, the Object-Point M_1 will be the infinitely distant point of the Principal Axis (x) of the Object-Space of the compound system; so that in this case we shall have:

$$x_1 = \infty, \quad x'_m = E'_mE';$$

and, by a process precisely analogous to the above, we obtain:

$$E'_mE' = -\cfrac{f_me'_m}{\Delta_{m-1} + \cfrac{f_{m-1}e'_{m-1}}{\Delta_{m-2} + \cdots \vdots \, \Delta_2 + \cfrac{f_2e'_2}{\Delta_1}}}; \tag{140}$$

whereby the abscissa of the Focal Point E' of the compound system with respect to the Focal Point E'_m of the mth system is expressed also in the form of a continued fraction.

The continued fractions which form the right-hand sides of equations (139) and (140) may be expressed in the form of determinants. Thus, writing

$$F_1F = \frac{A}{B} \quad \text{and} \quad E'_mE' = \frac{A'}{B'},$$

we have for the numerators A, A' and the denominators B, B' the following determinant-arrays:

$$A = \begin{vmatrix} f_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -e'_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_1 & \Delta_2 & -e'_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & f_3 & \Delta_3 & -e'_4 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & -e'_{m-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & f_{m-1} & \Delta_{m-1} \end{vmatrix},$$

$$A' = \begin{vmatrix} f_m & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -e'_m & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_{m-1} & \Delta_{m-2} & -e'_{m-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & f_{m-2} & \Delta_{m-3} & -e'_{m-3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & -e'_2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & f_2 & \Delta_1 \end{vmatrix},$$

$$B = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \Delta_1 & -e'_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_2 & \Delta_2 & -e'_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & f_3 & \Delta_3 & -e'_4 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & -e'_{m-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & f_{m-1} & \Delta_{m-1} \end{vmatrix},$$

$$B' = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \Delta_{m-1} & -e'_{m-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_{m-1} & \Delta_{m-2} & -e'_{m-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & f_{m-2} & \Delta_{m-3} & -e'_{m-3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & -e'_2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & f_2 & \Delta_1 \end{vmatrix}.$$

A mere inspection of the last two arrays will show that the denominators B and B' are equal.

2. *Determination of the Focal Lengths f, e' of the Compound System.*

In order to determine the Focal Length e' of the Image-Space of the Compound System, let us consider an object-ray parallel to the Principal Axis (x) of the Object-Space, to which corresponds, therefore, an image-ray which goes through the Focal Point E' of the Image-Space. Accordingly, for this ray:

$$x_1 = \infty, \quad x'_m = E'_m E'.$$

By the definition of the Focal Length e', we have:

$$e' = \frac{h_1}{\tan \theta'_m},$$

where $\theta'_m = \angle A'_m E' B'_m$.

This equation may be written:

$$e' = \frac{h_1}{\tan \theta'_1} \cdot \frac{\tan \theta'_1}{\tan \theta'_2} \cdot \frac{\tan \theta'_2}{\tan \theta'_3} \cdots \frac{\tan \theta'_{m-1}}{\tan \theta'_m};$$

and since $e'_1 = h_1/\tan \theta'_1$, and since, also, by formula (119):

$$\frac{\tan \theta'_k}{\tan \theta'_{k-1}} = -\frac{x_k}{e'_k},$$

we have:

$$e' = (-1)^{m-1} \frac{e'_1 \cdot e'_2 \cdots e'_m}{x_2 \cdot x_3 \cdots x_m} \cdot (x_1 = \infty). \tag{141}$$

Putting $x_1 = \infty$, we obtain from the two systems of equations (137) and (138):

$$\begin{aligned} x_2 \cdot x_3 \cdots x_m &= (-1)^{m-1} \Delta_1 \left(\Delta_2 + \frac{f_2 e'_2}{\Delta_1}\right)\left(\Delta_3 + \cfrac{f_3 e'_3}{\Delta_2 + \cfrac{f_2 e'_2}{\Delta_1}}\right) \cdots \\ &= (-1)^{m-1} R_1 \cdot R_2 \cdots R_{m-1}, \end{aligned}$$

where R_k is used to denote the kth term of this product of continued fractions. Writing

$$R_k = \frac{P_k}{Q_k},$$

we may express each of these continued fractions as the quotient of two determinants as follows:

$$P_k = \begin{vmatrix} \Delta_k & -e'_k & 0 & 0 & 0 & \cdots & 0 & 0 \\ f_k & \Delta_{k-1} & -e'_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_{k-1} & \Delta_{k-2} & -e'_{k-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & f_{k-2} & \Delta_{k-3} & -e'_{k-3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & 0 & -e'_2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & f_2 & \Delta_1 \end{vmatrix},$$

$$Q_k = \begin{vmatrix} \Delta_{k-1} & -e'_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-1} & \Delta_{k-2} & -e'_{k-2} & 0 & \cdots & 0 & 0 \\ 0 & f_{k-2} & \Delta_{k-3} & -e'_{k-3} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & -e'_2 \\ 0 & 0 & 0 & 0 & \cdots & f_2 & \Delta_1 \end{vmatrix}.$$

A mere inspection of these two determinants shows that we have:

$$Q_{k+1} = P_k;$$

accordingly,

$$x_2 \cdot x_3 \cdots x_m = (-1)^{m-1} \frac{P_{m-1}}{Q_1} = (-1)^{m-1} P_{m-1},$$

since we must put $Q_1 = 1$. Thus, we find:

$$e' = \frac{e'_1 \cdot e'_2 \cdots e'_m}{P_{m-1}}. \tag{142}$$

Again, in order to determine the Focal Length f of the Object-Space of the compound system, consider an image-ray parallel to the Principal Axis (x') of the Image-Space; to which corresponds an object-ray which goes through the Focal Point F of the Object-Space. Accordingly,

$$x_1 = F_1 F, \quad x'_m = \infty.$$

By the definition of the Focal Length f, we have:

$$f = \frac{h'_m}{\tan \theta'_1},$$

where $\theta'_1 = \angle A_1 F B_1$. This equation may be written:

$$f = \frac{\tan \theta'_1}{\tan \theta_1} \cdot \frac{\tan \theta'_2}{\tan \theta'_1} \cdots \frac{\tan \theta'_{m-1}}{\tan \theta'_{m-2}} \cdot \frac{h'_m}{\tan \theta'_{m-1}};$$

and since

$$\frac{h'_m}{\tan \theta'_{m-1}} = f_m,$$

and since, also, by formula (119), we have:

$$\frac{\tan \theta'_k}{\tan \theta'_{k-1}} = -\frac{f_k}{x'_k},$$

we obtain here:

$$f = (-1)^{m-1}\frac{f_1 \cdot f_2 \cdots f_m}{x'_1 \cdot x'_2 \cdots x'_{m-1}} \cdot (x'_m = \infty). \qquad (143)$$

Putting $x'_m = \infty$, we derive from the two systems of equations (137) and (138):

$$x'_1 \cdot x'_2 \cdots x'_{m-1} = \left(\Delta_1 + \cfrac{f_2 e'_2}{\Delta_2 + \cfrac{f_3 e'_3}{\begin{matrix}\vdots \\ \Delta_{m-2} + \cfrac{f_{m-1} e'_{m-1}}{\Delta_{m-1}}\end{matrix}}}\right) \cdots \left(\Delta_{m-2} + \frac{f_{m-1} e'_{m-1}}{\Delta_{m-1}}\right)\Delta_{m-1}$$

$$= R'_1 \cdot R'_2 \cdots R'_{m-1},$$

where R'_k is used to denote the kth term of this product of continued fractions. Writing

$$R'_k = \frac{P'_k}{Q'_k},$$

we may express each of these continued fractions as the quotient of two determinants as follows:

$$P'_k = \begin{vmatrix} \Delta_k & -e'_{k+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ f_{k+1} & \Delta_{k+1} & -e'_{k+2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_{k+2} & \Delta_{k+2} & -e'_{k+3} & 0 & \cdots & 0 & 0 \\ 0 & 0 & f_{k+3} & \Delta_{k+3} & -e'_{k+4} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & 0 & -e'_{m-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & f_{m-1} & \Delta_{m-1} \end{vmatrix}$$

$$Q'_k = \begin{vmatrix} \Delta_{k+1} & -e'_{k+2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ f_{k+2} & \Delta_{k+2} & -e'_{k+3} & 0 & 0 & \cdots & 0 & 0 \\ 0 & f_{k+3} & \Delta_{k+3} & -e'_{k+4} & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & 0 & -e'_{m-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & f_{m-1} & \Delta_{m-1} \end{vmatrix}.$$

Here also it is evident that:

$$P'_k = Q'_{k-1};$$

accordingly,

$$x'_1 \cdot x'_2 \cdots x'_{m-1} = \frac{P'_1}{Q'_{m-1}} = P'_1,$$

since we must put $Q'_{m-1} = 1$. Thus, we find:

$$f = (-1)^{m-1} \frac{f_1 \cdot f_2 \cdots f_m}{P'_1}. \tag{144}$$

Comparing the two determinant-arrays denoted by P_k and P'_k, we see that P_{m-1} and P'_1 are equal; if, therefore, we write:

$$P_{m-1} = P'_1 = D,$$

formulæ (142) and (144) may be written:

$$e' = \frac{e'_1 \cdot e'_2 \cdots e'_m}{D}, \quad f = (-1)^{m-1} \frac{f_1 \cdot f_2 \cdots f_m}{D}. \tag{145}$$

Consequently, also:

$$\frac{f}{e'} = (-1)^{m-1} \frac{f_1 \cdot f_2 \cdots f_m}{e'_1 \cdot e'_2 \cdots e'_m}. \tag{146}$$

If, therefore, we know the determination-constants and the relative positions of the members of the compound system, the formulæ which we have here obtained will enable us to determine the Focal Points and the Focal Lengths of the compound system.[1]

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[1] In regard to the literature dealing with the subject of Art. 51, the following is a partial list of the writers:

With reference to the matters treated in §§ 185 and 186, consult: S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), pages 46–51. Also, with reference to § 187, see CZAPSKI, pages 51–53. See also E. WANDERSLEB: Die geometrische Theorie der optischen Abbildung nach E. ABBE, Chapter III of Vol. I of *Die Theorie der optischen Instrumente* (Berlin, 1904), edited by M. VON ROHR; pages 112–121. Also, Dr. J. CLASSEN's *Mathematische Optik* (Leipzig, 1901), Art. 46.

With special reference to § 187:

A. F. MOEBIUS: Beitraege zu der Lehre von der Kettenbruechen, nebst einem Anhang dioptrischen Inhalts: CRELLES *Journ.*, vi. (1830), 215–243.

F. W. BESSEL: Ueber die Grundformeln der Dioptrik: *Astr. Nach.*, xviii. (1841), No. 415, pages 97–108.

F. LIPPICH: Fundamentalpuncte eines Systemes centrirter brechender Kugelflaechen: *Mitt. naturw. Ver. Steiermark*, ii. (1871), 429–459.

S. GUENTHER: *Darstellung der Naeherungswerte von Kettenbruechen in independenter Form:* Habil.-Schr. Erlangen, 1873.

O. ROETHIG: *Die Probleme der Brechung und Reflexion* (Leipzig, 1876).

F. MONOYER: Théorie générale des systèmes dioptriques centrés: Paris, *Soc. Phys. Séances*, 1883, 148–174.

L. MATTHIESSEN: Allgemeine Formeln zur Bestimmung der Cardinalpunkte eines brechenden Systems centrierter sphaerischer Flaechen; mittels Kettenbruchdeterminanten dargestellt: *Zft. f. Math. u. Phys.*, xxix. (1884), 343–350.

CHAPTER VIII.

IDEAL IMAGERY BY PARAXIAL RAYS. LENSES AND LENS-SYSTEMS.

ART. 53. INTRODUCTION.

188. The geometrical theory of optical imagery, as developed in the preceding chapter, is entirely independent of the physical laws of Optics. The fundamental and single assumption on which the theory rests is that of point-to-point correspondence, by means of rectilinear rays, between Object-Space and Image-Space. With regard to the angular apertures of the bundles of rays employed in the production of the image, as also with regard to the dimensions of the object to be portrayed, absolutely no conditions were imposed. In that chapter we were not at all concerned with the mechanism whereby an image may be realized; we merely assumed that such imagery was possible and investigated the laws thereof. Whatever practical difficulties may lie in the way of realizing the geometrical condition of collinear correspondence, we have not yet encountered them, as we shall have to do hereafter.

The investigation of the Refraction of Paraxial Rays of monochromatic light through a centered system of spherical refracting surfaces had prepared the way for the geometrical theory of optical imagery; for in this special, and, to be sure, more or less impractical, case we saw that there was strict collinear correspondence between Object-Space and Image-Space. Hence, here at any rate, the formulæ of the preceding chapter are immediately applicable. The theory of the refraction of paraxial rays through a centered system of lenses was first fully worked out by GAUSS[1]; and, hence, the imagery which we have under these circumstances is frequently called "GAUSSian Imagery".

The determination-data of a centered system of spherical surfaces are usually the refractive indices of the successive isotropic media, the radii of the spherical surfaces, and the distances between the consecutive vertices. If we introduce these constants into the general formulæ of the preceding chapter, we shall obtain not only all the results which for the case of Paraxial Rays we have previously obtained by independent methods, but also a number of new and useful formulæ, particularly in regard to the Focal Lengths and the Magnification-Ratios of the optical system.

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[1] C. F. GAUSS: *Dioptrische Untersuchungen* (Goettingen, 1841). See also paper by F. W. BESSEL, entitled "Ueber die Grundformeln der Dioptrik" (*Astr. Nach.*, xviii., 1841, No. 415, pages 97–108).

189. In the case of a *Single Spherical Surface*, of radius r, separating two media of refractive indices n, n', we found (§ 124) that the focal lengths f, e' were given by the following formulæ:

$$f = \frac{n}{n' - n} r, \quad e' = -\frac{n'}{n' - n} r, \quad \frac{f}{e'} = -\frac{n}{n'}. \tag{147}$$

If the vertex of the surface is at the point designated by A, and if the positions of the focal points are designated by F, E', then $f = FA$, $e' = E'A$; and, consequently (§180), the Principal Points coincide with each other at the vertex A. The Nodal Points, evidently, coincide at the centre C.

If, therefore, M, M' designate the positions of a pair of conjugate axial points of the Spherical Surface, and if, according to our previous system of notation, we put $AM = u$, $AM' = u'$, we obtain at once from formulæ (128):

$$\left.\begin{aligned} \frac{f}{u} + \frac{e'}{u'} + 1 &= 0, \\ Y = \frac{y'}{y} = -\frac{fu'}{e'u} &= \frac{nu'}{n'u}, \\ Z = \frac{\tan\theta'}{\tan\theta} &= \frac{u}{u'}. \end{aligned}\right\} \tag{148}$$

The first two of these formulæ will be recognized as identical with formulæ (85) and (86).

ART. 54. THE FOCAL LENGTHS OF A CENTERED SYSTEM OF SPHERICAL SURFACES.

190. In § 137 we showed how to determine the positions of the Focal Points F and E' of a centered system of spherical refracting surfaces. Thus, using the same system of notation as was employed there, we obtained two sets of equations of the following types:

$$\left.\begin{aligned} u_k &= u'_{k-1} - d_{k-1}, \\ n'_k(1/r_k - 1/u'_k) &= n'_{k-1}(1/r_k - 1/u_k); \end{aligned}\right\} \tag{149}$$

wherein we must give k in succession all integral values from $k = 1$ to $k = m$; noting also that $d_0 = 0$. The diagram (Fig. 100) shows the path $B_k B_{k+1}$ of a ray between the kth and the $(k + 1)$th surfaces of the centered system of spherical surfaces; and we have:

$$A_kM'_{k-1} = u_k, \quad A_kM'_k = u'_k, \quad A_kC_k = r_k, \quad A_kA_{k+1} = d_k.$$

Also, in accordance with our previous notation, let us write:

$$A_kB_k = h_k, \quad \angle\, A_kM'_kB_k = \theta'_k.$$

Here A_k and C_k designate the vertex and centre of the kth spherical surface, and M'_k designates the point where the paraxial ray crosses the axis after refraction at the kth surface.

In order to determine the *Focal Length* e' of the Image-Space of the system of m spherical surfaces, in terms of the magnitudes denoted by r, n and d, let us consider a ray which in the Object-Space is parallel to the optical axis ($u_1 = \infty$), and which, therefore, in the Image-Space crosses the optical axis at the Focal Point $E'(u'_m = A_mE')$. By the definition of the Focal Length e' given in § 178, we have:

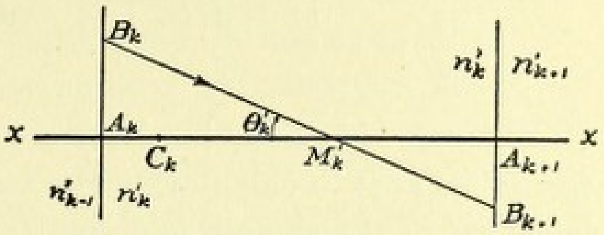

FIG. 100.

PATH OF A PARAXIAL RAY BETWEEN kth AND $(k+1)$th SURFACES OF A CENTERED SYSTEM OF SPHERICAL SURFACES.

$$A_kM_k' = u_k', \quad A_{k+1}M_k' = u_{k+1}, \quad A_kA_{k+1} = d_k,$$
$$A_kC_k = r_k, \quad A_kB_k = h_k, \quad A_{k+1}B_{k+1} = h_{k+1},$$
$$\angle\, A_kM_k'B_k = \theta_k'.$$

$$e' = \frac{h_1}{\tan\theta'_m},$$

where $\theta'_m = \angle A_mE'B_m$. Thus, we may write:

$$e' = \frac{h_1}{h_2}\cdot\frac{h_2}{h_3}\cdots\frac{h_{m-1}}{h_m}\cdot\frac{h_m}{\tan\theta'_m}.$$

By the diagram we have obviously:

$$\frac{h_k}{h_{k+1}} = \frac{u'_k}{u_{k+1}}, \quad \tan\theta'_k = -\frac{h_k}{u'_k};$$

and, accordingly, we derive the following formula:

$$e' = E'A_m\cdot\frac{u'_1\cdot u'_2\cdots u'_{m-1}}{u_2\cdot u_3\cdots u_m}, \tag{150}$$

where the magnitudes u_k, u'_k can be determined in terms of the known constants by means of the $(2m-1)$ equations (149).

191. Formulæ (149) and (150) may be written in the following forms, adapted to the logarithmic computation of the path of a paraxial ray through a centered system of spherical surfaces:

$$\left.\begin{aligned}\frac{1}{u'_k}&=\frac{1}{u_k}\cdot\frac{n'_{k-1}}{n'_k}+\frac{1}{r_k}\cdot\frac{n'_k-n'_{k-1}}{n'_k},\\ \frac{1}{u_{k+1}}&=\frac{1}{u'_k}\cdot\frac{1}{1-d_k/u'_k},\\ \frac{1}{e'}&=-\frac{(1-d_1/u'_1)\cdot(1-d_2/u'_2)\cdots(1-d_{m-1}/u'_{m-1})}{u'_m}.\end{aligned}\right\}\qquad(151)$$

In Chapter X there is given a numerical illustration of the use of formulæ (151) in calculating the value of $u'_m = A'_m E'$. When the focal length e' of the system is given, the last of these formulæ gives a very convenient way of checking the logarithmic computation.

192. The *Focal Length* f of the Object-Space may be determined in an analogous way by considering a second ray for which $u_1 = A_1 F$, $u'_m = \infty$. Thus, we shall obtain a similar formula for f, as follows:

$$f = FA_1\cdot\frac{u_2\cdot u_3\cdots u_m}{u'_1\cdot u'_2\cdots u'_{m-1}};\qquad(152)$$

in which, however, the magnitudes denoted by u_k, u'_k will, of course, not have the same values as they have in formula (150), because here they have reference to a different ray.

193. Ratio of the Focal Lengths f and e'.

The system of spherical surfaces may be regarded as a compound system formed by the combination of m spherical surfaces; thus, if f_k, e'_k denote the Focal Lengths of the kth spherical surface, we must have, according to the third of formulæ (147):

$$\frac{f_k}{e'_k} = -\frac{n'_{k-1}}{n'_k};$$

hence, applying formula (146) at the end of Chapter VII, we obtain directly:

$$\frac{f}{e'} = -\frac{n_1}{n'_1}\cdot\frac{n'_1}{n'_2}\cdots\frac{n'_{m-1}}{n'_m}.$$

In case, therefore, all of the spherical surfaces are refracting surfaces, we obtain the following formula:

$$\frac{f}{e'} = -\frac{n_1}{n'_m};\qquad(153)$$

but if the system consists of both refracting and reflecting surfaces, and *if the number of the reflecting surfaces is odd*, the formula will be:

$$\frac{f}{e'} = + \frac{n_1}{n'_m}. \tag{153a}$$

In connection with these formulæ, it is well to remind the reader again, as was stated in § 176, that *in the case of a centered system of spherical surfaces, the positive direction along the optical axis is determined by the direction of the incident axial ray;* and no matter if the direction of this ray should be reversed by one or more reflexions, the positive direction of the optical axis remains unchanged. Thus, the Focal Lengths, Radii, etc., are to be reckoned positive or negative, according as they are measured in the same direction as, or in the opposite direction to, the direction of the incident axial ray (see § 26).

The useful result, which we have just found, may be stated as follows: *In any centered system of spherical surfaces, the absolute value of the ratio of the two focal lengths is equal to the ratio of the indices of refraction of the first and last media.* This ratio is negative, except in the case when an odd number of the m spherical surfaces are reflecting. In this exceptional case the ratio f/e' is positive.

In particular, if the media of the Object-Space and Image-Space are identical in substance, that is, if $n_1 = n'_m$, the absolute values of the focal lengths are equal. In this case, which is so often realized in optical instruments, if we suppose that all the surfaces are refracting, or that an even number of them are reflecting, we shall have $f = -e'$, and, therefore, the nodal points N, N' will coincide with the principal points A, A', respectively; for, according to § 180, we have:

$$FN = A'E' = -e' = f = FA, \quad \text{and} \quad E'N' = AF = -f = e' = E'A'.$$

ART. 55. SEVERAL IMPORTANT FORMULÆ FOR THE CASE OF THE REFRACTION OF PARAXIAL RAYS THROUGH A CENTERED SYSTEM OF SPHERICAL SURFACES.

194. Robert Smith's Law.

According to the second of formulæ (120), we found that in an optical system the product of the Lateral Magnification $Y = y'/y$ and the Angular Magnification $Z = \tan\theta'/\tan\theta$ is constant; that is,

$$Y \cdot Z = -f/e',$$

or

$$\frac{y' \cdot \tan\theta'}{y \cdot \tan\theta} = -\frac{f}{e'}.$$

If the indices of refraction of the Object-Space and Image-Space are denoted by n and n', respectively, formulæ (153) and (153a) may be written as follows:

$$\frac{f}{e'} = \mp \frac{n}{n'};$$

and combining this with the preceding equation, we obtain one of the most important relations of Geometrical Optics, as follows:

$$n'y'\tan\theta' = \pm\, ny\tan\theta, \tag{154}$$

or

$$n'y'\,\theta' = \pm\, ny\,\theta, \tag{154a}$$

where the lower, or negative, sign applies only in case we have an odd number of reflexions.

In most German books on Optics this formula is called the "LAGRANGE-HELMHOLTZ" *Equation.* A very interesting account of the history of this celebrated law of Optics is given in a note at the end of P. CULMANN's article on "Die Realisierung der optischen Abbildung", which is Chapter IV of the first volume of *Die Theorie der optischen Instrumente*, edited by M. VON ROHR (Berlin, 1904). CULMANN concludes that the formula should be called the "HELMHOLTZ Equation", inasmuch as HELMHOLTZ[1] gave the equation in the form in which it is now used. He suggests also that it might be called the "SMITH-HELMHOLTZ" *Equation.* HELMHOLTZ himself attributed the law to LAGRANGE, who published, in 1803, a special case of the law.[2] But Lord RAYLEIGH[3] has pointed out that ROBERT SMITH,[4] with whose work LAGRANGE was undoubtedly acquainted, had enunciated the law for the special case of a system of infinitely thin lenses as early as 1738, deducing it from COTES's Theorem (see Art. 42, especially § 152). SMITH treats the whole subject in a very masterful way, and recognizes fully the importance and the consequences of the law, although it is true he did not establish it for the most general case. The formula, as applied to a centered system of spherical surfaces, was given by LUDWIG SEIDEL, in 1856, in his paper "Zur Dioptrik," etc., published in Vol. xliii., No. 1027, of *Astronomische Nachrichten* (pages 290–303): see formulæ (8) in SEIDEL's paper.

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[1] H. HELMHOLTZ: *Handbuch der physiologischen Optik* (1867), p. 50.

[2] J. L. DE LAGRANGE: Sur une loi générale d'optique: *Mémoires de l'Académie de Berlin* (1803), 3–12.

[3] J. W. STRUTT, Lord RAYLEIGH: Notes, chiefly Historical, on some Fundamental Propositions in Optics: *Phil. Mag.* (5), xxi. (1886), pp. 466–476.

[4] ROBERT SMITH: *A Compleat System of Opticks* (Cambridge, 1738); Book II, Chap. V.

195. Formulæ of L. Seidel. The following formulæ, due to L. Seidel,[1] will be frequently employed in the Theory of Spherical Aberration.

Let M_1, $\boldsymbol{M}_1$ designate the positions of two points on the optical axis of a centered system of spherical refracting surfaces, and let $A_1M_1 = u_1$, $A_1\boldsymbol{M}_1 = \boldsymbol{u}_1$, where A_1 designates the vertex of the first surface. Consider two Paraxial Rays, which, before refraction at the first surface, cross the optical axis at M_1, $\boldsymbol{M}_1$, and which, before refraction at the kth surface, cross the optical axis at the points designated by M'_{k-1}, $\boldsymbol{M}'_{k-1}$, and which are incident on the kth surface at points designated by B_k, $\boldsymbol{B}_k$, respectively. In agreement with our previous notation, we shall write:

$$A_kM'_{k-1} = u_k, \quad A_kM'_k = u'_k, \quad A_kB_k = h_k, \quad \angle A_kM'_kB_k = \theta'_k;$$
$$A_k\boldsymbol{M}'_{k-1} = \boldsymbol{u}_k, \quad A_k\boldsymbol{M}'_k = \boldsymbol{u}'_k, \quad A_k\boldsymbol{B}_k = \boldsymbol{h}_k, \quad \angle A_k\boldsymbol{M}'_k\boldsymbol{B}_k = \boldsymbol{\theta}'_k.$$

Since, by formulæ (149),

$$n'_k\left(\frac{1}{r_k} - \frac{1}{u'_k}\right) = n'_{k-1}\left(\frac{1}{r_k} - \frac{1}{u_k}\right),$$
$$n'_k\left(\frac{1}{r_k} - \frac{1}{\boldsymbol{u}'_k}\right) = n'_{k-1}\left(\frac{1}{r_k} - \frac{1}{\boldsymbol{u}_k}\right),$$

we obtain:

$$n'_k\left(\frac{1}{\boldsymbol{u}'_k} - \frac{1}{u'_k}\right) = n'_{k-1}\left(\frac{1}{\boldsymbol{u}_k} - \frac{1}{u_k}\right).$$

Also, since

$$u'_{k-1} - u_k = \boldsymbol{u}'_{k-1} - \boldsymbol{u}_k = d_{k-1},$$

and

$$\frac{h_{k-1}}{u'_{k-1}} = \frac{h_k}{u_k} = -\theta'_{k-1}, \quad \frac{\boldsymbol{h}_{k-1}}{\boldsymbol{u}'_{k-1}} = \frac{\boldsymbol{h}_k}{\boldsymbol{u}_k} = -\boldsymbol{\theta}'_{k-1},$$

we find:

$$h_k\boldsymbol{h}_k\left(\frac{1}{\boldsymbol{u}_k} - \frac{1}{u_k}\right) = h_{k-1}\boldsymbol{h}_{k-1}\left(\frac{1}{\boldsymbol{u}'_{k-1}} - \frac{1}{u'_{k-1}}\right);$$

and combining the two equations thus obtained, we derive the first of Seidel's formulæ, as follows:

$$\begin{aligned} n'_kh_k\boldsymbol{h}_k\left(\frac{1}{\boldsymbol{u}'_k} - \frac{1}{u'_k}\right) &= n'_{k-1}h_k\boldsymbol{h}_k\left(\frac{1}{\boldsymbol{u}_k} - \frac{1}{u_k}\right) \\ &= n'_{k-1}h_{k-1}\boldsymbol{h}_{k-1}\left(\frac{1}{\boldsymbol{u}'_{k-1}} - \frac{1}{u'_{k-1}}\right) \end{aligned}$$

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[1] L. Seidel: Zur Dioptrik: *Astr. Nachr.*, xxxvii. (1853), No. 871, pages 105–120.

$$
\begin{aligned}
&= n'_{k-2} h_{k-1} \boldsymbol{h}_{k-1} \left(\frac{1}{\boldsymbol{u}_{k-1}} - \frac{1}{u_{k-1}} \right) \\
&= \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&= n_1 h_1 \boldsymbol{h}_1 \left(\frac{1}{\boldsymbol{u}_1} - \frac{1}{u_1} \right).
\end{aligned}
$$

If we introduce here the so-called "zero-invariant" (see §126), and write

$$n(1/r - 1/u) = n'(1/r - 1/u') = J,$$

$$n(1/r - 1/\boldsymbol{u}) = n'(1/r - 1/\boldsymbol{u}') = \boldsymbol{J},$$

the formulæ above may be written as follows:

$$
\begin{aligned}
h_k \boldsymbol{h}_k (J_k - \boldsymbol{J}_k) = h_{k-1} \boldsymbol{h}_{k-1} (J_{k-1} - \boldsymbol{J}_{k-1}) = \cdots \\
= h_1 \boldsymbol{h}_1 (J_1 - \boldsymbol{J}_1). \qquad (155)
\end{aligned}
$$

Moreover, as can be seen easily from Fig. 100, we have:

$$\frac{h_{k-1} - h_k}{h_{k-1}} = \frac{d_{k-1}}{u'_{k-1}}, \quad \frac{\boldsymbol{h}_{k-1} - \boldsymbol{h}_k}{\boldsymbol{h}_{k-1}} = \frac{d_{k-1}}{\boldsymbol{u}'_{k-1}};$$

and, hence, combining these equations, and using formula (155), we obtain:

$$\frac{h_k}{h_{k-1}} - \frac{\boldsymbol{h}_k}{\boldsymbol{h}_{k-1}} = (J_{k-1} - \boldsymbol{J}_{k-1}) \frac{d_{k-1}}{n'_{k-1}} = h_1 \boldsymbol{h}_1 (J_1 - \boldsymbol{J}_1) \frac{d_{k-1}}{n'_{k-1} \cdot h_{k-1} \cdot \boldsymbol{h}_{k-1}},$$

or

$$\frac{\boldsymbol{h}_{k-1}}{h_{k-1}} - \frac{\boldsymbol{h}_k}{h_k} = h_1 \boldsymbol{h}_1 (J_1 - \boldsymbol{J}_1) \frac{d_{k-1}}{n'_{k-1} \cdot h_k \cdot h_{k-1}}.$$

Giving k in succession the values 2, 3, $\cdots$ k, and adding all the equations thus obtained, we derive the second of SEIDEL's Formulæ, as follows:

$$\frac{\boldsymbol{h}_1}{h_1} - \frac{\boldsymbol{h}_k}{h_k} = h_1 \boldsymbol{h}_1 (J_1 - \boldsymbol{J}_1) \sum_{k=2}^{k=k} \frac{d_{k-1}}{n'_{k-1} \cdot h_k \cdot h_{k-1}}. \qquad (156)$$

Moreover, since

$$h_k \boldsymbol{h}_k (J_k - \boldsymbol{J}_k) = h_1 \boldsymbol{h}_1 (J_1 - \boldsymbol{J}_1),$$

and

$$J_k - \boldsymbol{J}_k = n'_k \left(\frac{1}{\boldsymbol{u}'_k} - \frac{1}{u'_k} \right),$$

we can write:

$$h_1 \boldsymbol{h}_k \left(\frac{1}{\boldsymbol{u}'_k} - \frac{1}{u'_k} \right) = \frac{h_1^2 \boldsymbol{h}_1}{n'_k h_k} (J_1 - \boldsymbol{J}_1),$$

or

$$\frac{h_1\boldsymbol{h}_k}{\boldsymbol{u}'_k} - \frac{\boldsymbol{h}_1 h_k}{u'_k} = \frac{h_1^2\boldsymbol{h}_1}{n'_k h_k}(J_1 - \boldsymbol{J}_1) - \frac{h_1\boldsymbol{h}_1}{u'_k}\left(\frac{h_k}{h_1} - \frac{\boldsymbol{h}_k}{\boldsymbol{h}_1}\right),$$

and, hence, by (156), we derive also the following:

$$\frac{h_1\boldsymbol{h}_k}{\boldsymbol{u}'_k} - \frac{\boldsymbol{h}_1 h_k}{u'_k} = h_1^2\boldsymbol{h}_1(J_1 - \boldsymbol{J}_1)\left\{\frac{1}{n'_k h_k} - \frac{h_k}{u'_k}\sum_{k=2}^{k=k}\frac{d_{k-1}}{n'_{k-1}h_{k-1}h_k}\right\}; \qquad (157)$$

which is the third of SEIDEL's Formulæ.

The expression

$$\sum_{k=2}^{k=k}\frac{d_{k-1}}{n'_{k-1}\cdot h_{k-1}\cdot h_k},$$

which occurs frequently, especially in SEIDEL's optical formulæ, may be transformed as follows:

$$\begin{aligned}
\sum_{k=2}^{k=k}\frac{d_{k-1}}{n'_{k-1}\cdot h_{k-1}\cdot h_k} &= \sum_{k=2}^{k=k}\frac{u_k}{n'_{k-1}\cdot h_k}\left(\frac{1}{h_k} - \frac{1}{h_{k-1}}\right) = \sum_{k=2}^{k=k}\frac{1}{n'_{k-1}}\left(\frac{u_k}{h_k^2} - \frac{u'_{k-1}}{h_{k-1}^2}\right)\\
&= \frac{1}{h_1^2}\left(\frac{u_1}{n_1} - \frac{u'_1}{n'_1}\right) + \frac{1}{h_2^2}\left(\frac{u_2}{n'_1} - \frac{u'_2}{n'_2}\right) + \cdots + \frac{1}{h_k^2}\left(\frac{u_k}{n'_{k-1}} - \frac{u'_k}{n'_k}\right) + \frac{u'_k}{n'_k h_k^2} - \frac{u'_1}{n_1 h_1^2}\\
&= \frac{u'_k}{n'_k h_k^2} - \frac{u_1}{n_1 h_1^2} + \sum_{k=1}^{k=k}\frac{1}{h_k^2}\left(\frac{u_k}{n'_{k-1}} - \frac{u'_k}{n'_k}\right)\\
&= \frac{u'_k}{n'_k h_k^2} - \frac{u_1}{n_1 h_1^2} + \sum_{k=1}^{k=k}\frac{u_k u'_k}{h_k^2}\left(\frac{1}{n'_{k-1}u'_k} - \frac{1}{n'_k u_k}\right);
\end{aligned}$$

and, finally:

$$\sum_{k=2}^{k=k}\frac{d_{k-1}}{n'_{k-1}\cdot h_{k-1}\cdot h_k} = \frac{u'_k}{n'_k h_k^2} - \frac{u_1}{n_1 h_1^2} - \sum_{k=1}^{k=k}\frac{u_k u'_k}{h_k^2}\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k; \qquad (158)$$

where Δ is the symbol of operation introduced in § 126; so that here

$$\Delta(1/n)_k = 1/n'_k - 1/n'_{k-1}.$$

196. In order to illustrate some of the uses of these formulæ, we shall apply them, as SEIDEL does, to determine the positions of the Focal Points F, E' and of the Principal Points A, A' and the magnitudes of the Focal Lengths $f = FA$, $e' = E'A'$ of a centered system of m spherical refracting surfaces.

We shall suppose that we know completely the path of a certain paraxial ray through the system, for example, the ray which in the object-space crosses the optical axis at the point designated by M_1, and which meets the first spherical surface at the point designated by

B_1. Thus, instead of the usual determination-constants of the optical system, viz., the radii (r) and the axial thicknesses (d), we shall suppose that we have given here the elements of this ray, so that the system is determined by the intercepts (u) of the ray on the optical axis and by the incidence-heights (h); as is perfectly possible, since the former magnitudes can be expressed in terms of the latter, as follows:

$$r_k = \frac{\Delta\left(\frac{1}{n}\right)_k}{\frac{1}{n'_k u_k} - \frac{1}{n'_{k-1} u'_k}}, \quad d_k = u'_k - u_{k+1}.$$

In deriving the following formulæ, we shall, chiefly for the sake of brevity in writing, introduce a constant of the optical system, denoted by the symbol F, and defined by the expression:

$$\frac{1}{F} = \frac{h_1 h_m}{u_1 u'_m}\left\{\frac{u_1}{n_1 h_1^2} - \frac{u'_m}{n'_m h_m^2} + \sum_{k=2}^{k=m} \frac{d_{k-1}}{n'_{k-1} \cdot h_{k-1} \cdot h_k}\right\}, \tag{159}$$

which may be written also as follows:

$$\frac{1}{F} = -\frac{h_1 h_m}{u_1 u'_m} \sum_{k=1}^{k=m} \frac{u_k u'_k}{h_k^2} \cdot \frac{1}{r_k} \cdot \Delta\left(\frac{1}{n}\right)_k, \tag{159a}$$

as is evident from the transformation given in formula (158).

The positions of the Focal Points F, E' can be found by means of formula (157). Thus, if in this formula we put, 1st, $\boldsymbol{u}_1 = A_1 F$, $\boldsymbol{u}'_m = \infty$, and, 2nd, $\boldsymbol{u}_1 = \infty$, $\boldsymbol{u}'_m = A_m E'$, we obtain

$$\left.\begin{aligned} A_1 F &= u_1\left(1 - \frac{1}{n_1} \frac{h_m}{h_1} \frac{F}{u'_m}\right), \\ A'_m E' &= u'_m\left(1 + \frac{1}{n'_m} \frac{h_1}{h_m} \frac{F}{u_1}\right). \end{aligned}\right\} \tag{160}$$

In order to determine the positions of the Principal Points A, A' of the centered system of spherical surfaces, we recall (§139) that this pair of conjugate points is characterized by the condition:

$$\frac{n_1}{n'_m} \prod_{k=1}^{k=m} \frac{\boldsymbol{u}'_k}{\boldsymbol{u}_k} = 1,$$

where $\boldsymbol{u}_1 = A_1 A$, $\boldsymbol{u}'_m = A_m A'$; which, since

$$\prod_{k=1}^{k=m} \frac{\boldsymbol{u}'_k}{\boldsymbol{u}_k} = \frac{\boldsymbol{h}_1}{\boldsymbol{h}_m} \cdot \frac{\boldsymbol{u}'_m}{\boldsymbol{u}_1},$$

may be expressed by the following relation:

$$\frac{n_1\boldsymbol{h}_1}{\boldsymbol{u}_1}=\frac{n'_m\boldsymbol{h}_m}{\boldsymbol{u}'_m}.$$

In formula (157), therefore, we must, 1st, substitute $n_1\boldsymbol{h}_1/n'_m\boldsymbol{u}_1$ for $\boldsymbol{h}_m/\boldsymbol{u}'_m$, solve for $\boldsymbol{u}_1$, and put $\boldsymbol{u}_1=A_1A$; and, 2nd, substitute $n'_m\boldsymbol{h}_m/n_1\boldsymbol{u}'_m$ for $\boldsymbol{h}_1/\boldsymbol{u}_1$, eliminate $(J_1-\boldsymbol{J}_1)$ by (155), solve for $\boldsymbol{u}'_m$, and, finally, put $\boldsymbol{u}'_m=A_mA'$. Performing both of these operations, and making use of formula (159) each time, we obtain the following formulæ:

$$\left.\begin{aligned} A_1A &= u_1 - F\left(\frac{1}{n_1}\cdot\frac{h_m}{h_1}\cdot\frac{u_1}{u'_m}-\frac{1}{n'_m}\right),\\ A'_mA' &= u'_m - F\left(\frac{1}{n_1}-\frac{1}{n'_m}\cdot\frac{h_1}{h_m}\cdot\frac{u'_m}{u_1}\right). \end{aligned}\right\} \qquad (161)$$

Having determined the positions of the Focal Points and of the Principal Points, we can find at once the magnitudes of the Focal Lengths f, e'; for

$$\begin{aligned} f &= FA = FA_1 + A_1A,\\ e' &= E'A' = E'A_m + A_mA', \end{aligned}$$

and, hence, by formulæ (160) and (161):

$$f = F/n'_m, \quad e' = -F/n_1. \qquad (162)$$

If the given ray, to which the symbols u, h refer, is incident on the first surface of the system in a direction parallel to the optical axis, we must put $u_1=\infty$ in the above formulæ, whereby we obtain for the Focal Lengths:

$$f=\frac{n_1}{n'_m}\cdot\frac{u'_1\cdot u'_2\cdots u'_m}{u_2\cdot u_3\cdots u_m}, \quad e' = -\frac{u'_1\cdot u'_2\cdots u'_m}{u_2\cdot u_3\cdots u_m}; \qquad (163)$$

in agreement with the results as expressed by formulæ (150) and (153).

LENSES AND LENS-SYSTEMS.

ART. 56. THICK LENSES.

197. A centered system of two spherical refracting surfaces ($m=2$) is called a Lens. In the following discussion of Lenses it will be assumed that the media of the incident and emergent rays are identical in substance, and the symbol

$$n = n'_1/n_1 = n'_1/n'_2$$

will be employed to denote the relative index of refraction of the medium of the lens-substance and the surrounding medium. A Lens is usually described by assigning: (1) The magnitude of the relative index of refraction (n); (2) The positions A_1, A_2 of the vertices of the two spherical surfaces, whereby the optical axis of the lens may be directly determined, both as to its position and as to its direction, by drawing the straight line from A_1 to A_2. The distance

$$A_1A_2 = d,$$

called the "thickness" of the Lens, is reckoned always in the positive direction of the optical axis, so that d is essentially a positive magnitude; and (3) The magnitudes and signs of the radii $r_1 = A_1C_1$, $r_2 = A_2C_2$. Employing the same letters and notation as are used in the preceding portion of this chapter, we shall regard a Lens as a compound system consisting of two single spherical refracting surfaces, whose focal lengths, in terms of the above data, will be expressed as follows:

$$\left.\begin{aligned} F_1A_1 &= f_1 = \frac{r_1}{n-1}, & E_1'A_1 &= e_1' = -\frac{nr_1}{n-1}, \\ F_2A_2 &= f_2 = -\frac{nr_2}{n-1}, & E_2'A_2 &= e_2' = \frac{r_2}{n-1}. \end{aligned}\right\} \tag{164}$$

In order to determine the positions of the Focal Points F, E' and the magnitudes of the Focal Lengths f, e' of the Lens, we shall employ formulæ (136), in which the determination-data are the Focal Lengths f_1, e_1' and f_2, e_2' of the two partial systems and the "interval" Δ between them. This latter magnitude is defined as follows:

$$\Delta = E_1'F_2 = E_1'A_1 + A_1A_2 + A_2F_2,$$

that is,

$$\Delta = e_1' - f_2 + d; \tag{165}$$

and, accordingly, expressed in terms of the original data, this magnitude is as follows:

$$\Delta = \frac{n(r_2 - r_1) + d(n-1)}{n-1} = \frac{N}{(n-1)^2}, \tag{166}$$

where

$$N = (n-1)\{n(r_2 - r_1) + d(n-1)\} \tag{167}$$

denotes a constant of the Lens.

198. (1) *The abscissæ of the Focal Points F, E' of the Lens with respect to F_1, E_2', respectively:*

These are obtained immediately from the first two of formulæ (136) as follows:

$$F_1F = -\frac{nr_1^2}{N}, \quad E_2'E' = -\frac{nr_2^2}{N}. \tag{168}$$

(2) *The Focal Lengths f, e' of the Lens:*

Likewise, from the last two of formulæ (136), we obtain:

$$FA = f = \frac{nr_1r_2}{N} = -e' = A'E'; \tag{169}$$

where A, A' designate the positions on the optical axis of the two Principal Points of the Lens. We see that *in every Lens, surrounded by the same medium on both sides, the Focal Lengths f, e' are equal in magnitude, but opposite in sign.* (This is true of an optical system consisting of any number of spherical surfaces, provided $n_1 = n_m'$; see § 193.)

Hence, also, the Nodal Points N, N' of the Lens coincide with the Principal Points A, A', respectively; which is characteristic likewise of any optical system for which the media of the incident and emergent rays are identical (§ 193).

(3) *Abscissæ of the Focal Points F, E' of the Lens referred to the vertices A_1, A_2, respectively:*

Since

$$A_1F = A_1F_1 + F_1F = -f_1 + F_1F,$$
$$A_2E' = A_2E_2' + E_2'E' = -e_2' + E_2'E',$$

we derive, by means of formulæ (164) and (168), the following formulæ for locating the positions of F and E':

$$A_1F = -\frac{Nr_1 + n(n-1)r_1^2}{N(n-1)}, \quad A_2E' = -\frac{Nr_2 - n(n-1)r_2^2}{N(n-1)}. \tag{170}$$

(4) *Abscissæ of the Principal Points A, A' of the Lens, reckoned from the vertices A_1, A_2, respectively:*

Since

$$A_1A = A_1F + FA, \quad A_2A' = A_2E' + E'A',$$

formulæ (169) and (170), together with formula (165), give the following formulæ for determining the positions of A, A':

$$A_1A = -\frac{n-1}{N}r_1d, \quad A_2A' = -\frac{n-1}{N}r_2d, \quad \frac{A_1A}{A_2A'} = \frac{r_1}{r_2}; \tag{171}$$

whence we see that the abscissæ of the Principal Points are in the same ratio to each other as the radii of the surfaces of the Lens.

The distance AA' between the two Principal Points may be expressed as follows:

$$AA' = AA_1 + A_1A_2 + A_2A',$$

whence we obtain:

$$AA' = \frac{(n-1)(r_1 - r_2) + N}{N} d. \tag{172}$$

199. Character of the Different Forms of Lenses.

An inspection of formulæ (167) and (169) will show that the sign of the Focal Length f, which determines the character of the Lens, depends not only on the magnitudes and signs of the radii r_1, r_2, but also on the thickness d of the Lens. It will depend also on whether n is greater or less than unity, but in the following discussion *it will be assumed that the Lens is of the type of a glass lens in air*, that is,

$$n - 1 > 0.$$

Evidently, with this assumption, the magnitude denoted by N will be greater than, equal to, or less than, zero, according as

$$d \gtreqless \frac{n(r_1 - r_2)}{n-1}. \tag{173}$$

Now d itself is always positive; and hence in any form of Lens, for which $r_1 - r_2$ is negative, the two lower signs in formula (173) cannot possibly occur, so that for any Lens of such form, the sign of N will necessarily be negative.

How the sign of the Focal Length f of the Lens depends on the magnitude of the thickness d, will be apparent in the following classification of the different forms of Lenses.

(1) *Biconvex Lens* ($r_1 > 0$, $r_2 < 0$). In a Biconvex Lens the radii r_1, r_2 have opposite signs, and hence the Focal Length

$$f = \frac{nr_1r_2}{N}$$

of a Biconvex Lens is positive, so long as the thickness

$$d < \frac{n(r_1 - r_2)}{n-1};$$

so that a Biconvex Lens whose thickness d does not exceed this limiting value is a convergent Lens. This is the usual character of a Biconvex glass Lens in air, an example of which is shown in Fig. 101, where $r_1 = +10$, $r_2 = -15$ and $d = +3$. For comparatively small values of d, as here, the Principal Points A, A' are situated within the Lens itself. If the Lens is made thicker, the two Principal Points will approach nearer to each other, until when d attains the value $d = r_1 - r_2$, so that the two surfaces of the Lens have a common centre, the Principal Points A, A' coincide with each other at this common centre. Fig. 102 shows a Biconvex Lens with concentric surfaces; such a Lens, made of glass and surrounded by air, will be convergent. Here, likewise, belongs the Spherical Lens, characterized by the value $d = r_1 - r_2 = 2r_1$. The Spherical Lens is also to be regarded as a particular case of the Equi-Biconvex Lens ($r_1 = -r_2$, $r_1 > 0$). A Spherical Lens is shown in Fig. 103.

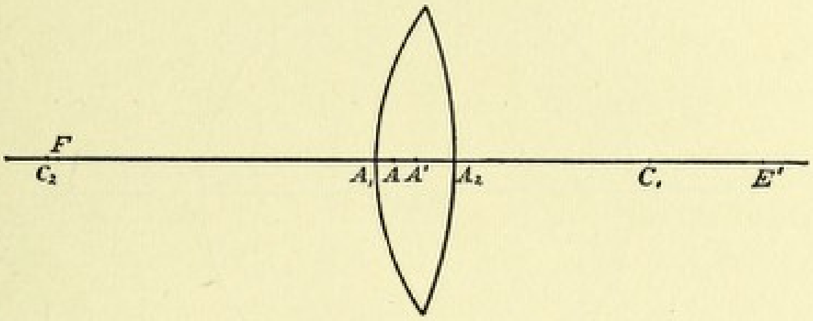

FIG. 101.

CONVERGENT BICONVEX GLASS LENS IN AIR.

$n = 3/2$; $r_1 = A_1C_1 = +10$; $r_2 = A_2C_2 = -15$; $d = A_1A_2 = +3$; $A_1F = -11.66$; $A_2E' = +11.25$; $A_1A = +0.833$; $A_2A' = -1.25$; $f = FA = -e' = A'E' = +12.5$.

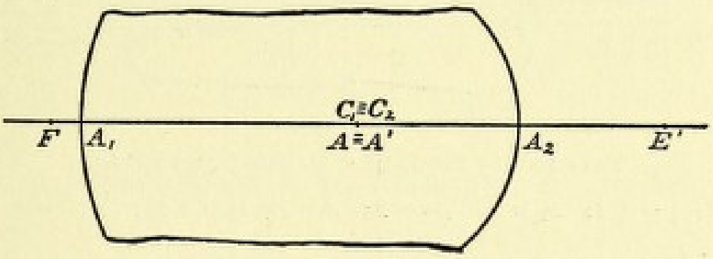

FIG. 102.

CONVERGENT BICONVEX GLASS LENS IN AIR: SPECIAL CASE—TWO SURFACES CONCENTRIC.

$n = 3/2$; $r_1 = +5$; $r_2 = -3$; $d = +8$; $A_1F = -0.625$; $A_2E' = +2.625$; $A_1A = A_1C_1 = A_1C_2 = +5$; $A_2A' = A_2C_2 = A_2C_1 = -3$; $f = FA = -e' = A'E' = +5.625$.

If we suppose the thickness of the Biconvex Lens to be greater than $(r_1 - r_2)$, the Lens continues at first to be convergent, but the Primary Principal Point A will lie now beyond (or to the right of) the Secondary Principal Point A'; and as the thickness d is increased, these points separate farther and farther from each other, so that at length we shall find the Secondary Principal Point A' in front of the

Lens and the Primary Principal Point A beyond the Lens, the Lens still being convergent. And when d attains the value

$$d = n(r_1 - r_2)/(n - 1),$$

the Biconvex Lens becomes a Telescopic Optical System, with its Focal Planes, and its Principal Planes also, at infinity. The case of a Telescopic Biconvex glass Lens in air is shown in Fig. 104; for which the determination-constants have the following values: $r_1 = +3$, $r_2 = -2$, $d = +15$. The Optical Centre O of this Lens coincides with the Focal Point E'_1 of the first surface of the Lens and with the Focal Point F_2 of the second surface of the Lens.

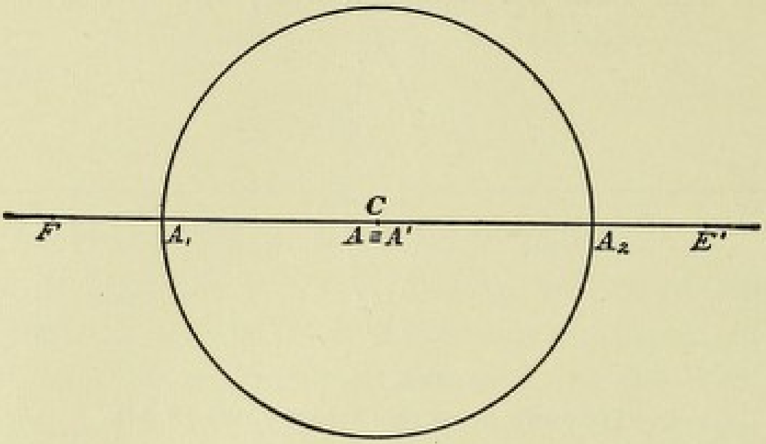

FIG. 103.

CONVERGENT EQUIBICONVEX GLASS LENS IN AIR: SPECIAL CASE—SPHERICAL LENS.

$n = 3/2$; $r_1 = A_1C_1 = -r_2 = CA_2 = +3$; $d = A_1A_2 = +6$; $A_1F = E'A_2 = -1.5$; $A_1A = A_1C = A'A_2 = CA_2 = +3$; $f = FA = -e' = A'E' = +4.5$.

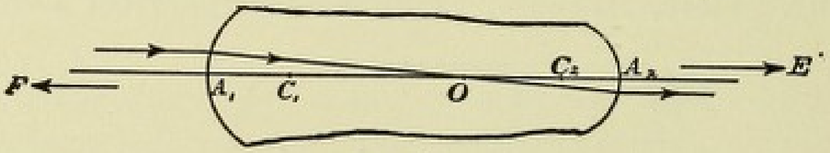

FIG. 104.

TELESCOPIC BICONVEX GLASS LENS IN AIR.

$n = 3/2$; $r_1 = A_1C_1 = +3$; $r_2 = A_2C_2 = -2$; $d = A_1A_2 = +15$; $A_1O = +9$; $f = -e' = \infty$.

And, finally, in case $d > n(r_1 - r_2)/(n - 1)$, a Biconvex glass Lens in air will be divergent ($f < 0$). But no matter how great the thickness d becomes, we shall find that the Focal Point F of a Biconvex Lens lies always in front of the Lens.

(2) *Biconcave Lens* ($r_2 > 0 > r_1$). Here also, as in the case of the Biconvex Lens, the radii of the two surfaces have opposite signs, r_1 being negative and r_2 being positive. Hence, assuming that n is greater than unity, we find in the case of a Biconcave Lens that the constant N is always positive, and, therefore, f is negative; so that

a Biconcave glass Lens in air is always a divergent Lens. The Principal Points A, A' of a Biconcave Lens lie always in the interior of the Lens, the Primary Principal Point A being situated in front of the Secondary Principal Point A'. Fig. 105 shows a Biconcave glass Lens in air, for which the constants have the following values: $r_1 = -10$, $r_2 = +15$, $d = +3$.

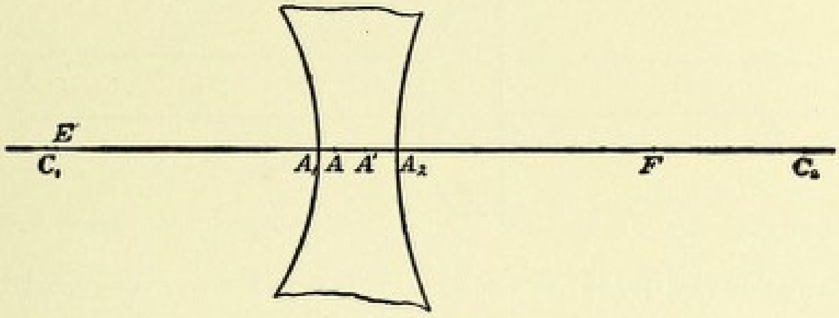

Fig. 105.

Divergent Biconcave Glass Lens in Air.

$n = 3/2$; $r_1 = A_1C_1 = -10$; $r_2 = A_2C_2 = +15$; $d = A_1A_2 = +3$; $A_1F = +12.3$; $A_2E' = -12.7$; $A_1A = +0.77$; $A_2A' = -1.154$; $f = FA = -e' = A'E' = -11.54$.

In an Equi-Biconcave Lens we have $r_2 = -r_1$, $r_2 > 0$.

(3) *Lens with One Surface Plane.* In this case, therefore, one of the radii r_1, r_2 is infinite.

If the first surface is the plane surface, then $r_1 = \infty$, and we find:

$$f = -e' = -\frac{r_2}{n-1},$$

so that the character of the Lens depends on the sign of the curvature of the curved surface. Thus, for example, in a *Plano-Convex Lens* ($r_1 = \infty$, $r_2 < 0$), f is positive, and the Lens is a convergent Lens. On the contrary, in a *Plano-Concave Lens* ($r_1 = \infty$, $r_2 > 0$), f is negative, and the Lens is a divergent Lens. (In these statements it is assumed, as always in this discussion, that $n > 1$).

For $r_1 = \infty$, we find also:

$$A_1F = \frac{nr_2 + (n-1)d}{n(n-1)}, \quad A_2E' = -\frac{r_2}{n-1},$$

$$A_1A = d/n, \quad A_2A' = 0.$$

When one surface of the Lens is plane, one of the Principal Points will coincide with the vertex of the curved surface.

The diagrams (Figs. 106 and 107) show the cases of a Plano-Convex Lens and of a Plano-Concave Lens ($n = 3/2$). The two Lenses are represented as having the same thicknesses, and the absolute value of the radius of the curved surface is the same for both Lenses. In the figures the first surface is the plane surface ($r_1 = \infty$); but if the light is supposed to go from right to left, so that the curved surface is the first surface, the figures, except for certain obvious changes in the letters, will be correct.

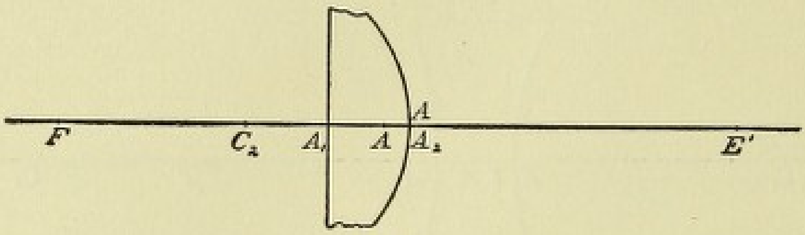

FIG. 106.

PLANO-CONVEX GLASS LENS IN AIR. This Lens is always Convergent.

$n = 3/2$; $r_1 = A_1C_1 = \infty$; $r_2 = A_2C_2 = -6$; $d = A_1A_2 = +3$; $A_1F = -10$; $A_2E' = +12$; $A_1A = +2$; $A_2A' = 0$; $f = FA = -e' = A'E' = +12$.

(4) *Concavo-Convex, or Convexo-Concave, Lens.* In a Lens of this form the two radii r_1, r_2 have the same sign, and, hence, the sign of the Focal Length $f = nr_1r_2/N$ will be the same as the sign of the constant N. If, therefore, N is positive, the Lens will be convergent; if N is negative, the Lens will be divergent; and if $N = 0$, the Lens will be telescopic. According to (173), the sign of N will depend on the value of d.

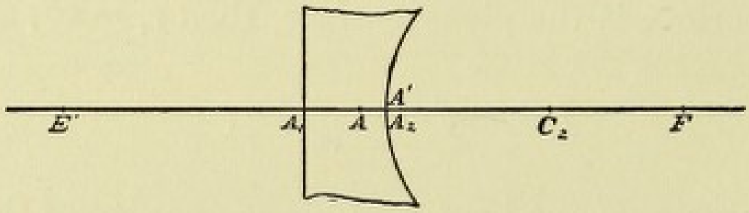

FIG. 107.

PLANO-CONCAVE GLASS LENS IN AIR. This Lens is always Divergent.

$n = 3/2$; $r_1 = A_1C_1 = \infty$; $r_2 = A_2C_2 = +6$; $d = A_1A_2 = +3$; $A_1F = +14$; $A_2E' = -12$; $A_1A = +2$; $A_2A' = 0$; $f = FA = -e' = A'E' = -12$.

Let us suppose that both radii are positive, so that the first surface of the Lens is convex ($r_1 > 0$) and the second surface is concave ($r_2 > 0$). It will be necessary to consider only this case, since we have merely to suppose that the direction of the light is reversed in order to obtain the opposite case.

Accordingly, assuming that both radii are positive, we have to consider the following three cases of the Convexo-Concave Lens:

(*a*) *Positive Meniscus* ($r_2 > r_1 > 0$).

Since in this case ($r_1 - r_2$) is negative, and since d is always positive, it follows that $d > n(r_1 - r_2)/(n - 1)$, and, therefore, N is positive. Hence, a Lens of this form, called a "Positive Meniscus", is always a convergent Lens (Fig. 108). It will be remarked that the Primary Principal Point A lies to the left of the vertex A_1, and the Secondary Principal Point A' lies to the left of the vertex A_2; and that the line-segment AA' is always positive.

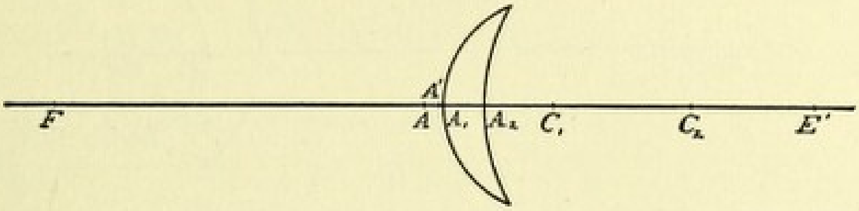

FIG. 108.

POSITIVE MENISCUS (GLASS LENS IN AIR). This Lens is always Convergent.

$n = 3/2$; $r_1 = A_1C_1 = +6$; $r_2 = A_2C_2 = +12$; $d = A_1A_2 = +2$; $A_1F = -22.8$; $A_2E' = +19.2$; $A_1A = -1.2$; $A_2A' = -2.4$; $f = FA = -e' = A'E' = +21.6$.

(*b*) The case when $r_1 > r_2 > 0$. In this case $(r_1 - r_2)$ is positive, so that the Lens may be divergent, convergent or telescopic, depending on the value of the ratio $d/(r_1 - r_2)$.

The most common case under this head is that for which

$$d < n(r_1 - r_2)/(n - 1).$$

When this is the case, we have a divergent Lens, called a "*Negative Meniscus*" (Fig. 109). The Principal Points A, A' lie beyond (that is, to the right of) the vertices A_1, A_2, respectively. If $d = r_1 - r_2$, the two Principal Points coincide at a point which is also the common centre of the two surfaces of the Lens (Fig. 110). An infinitely thin Lens of this kind is not divergent, but telescopic.

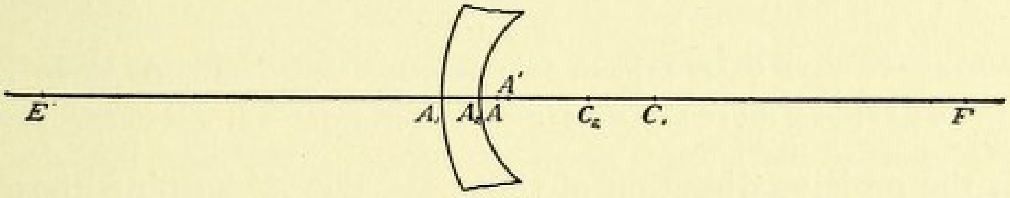

FIG. 109.

NEGATIVE MENISCUS (GLASS LENS IN AIR).

$n = 3/2$; $r_1 = A_1C_1 = +12$; $r_2 = A_2C_2 = +6$; $d = A_1A_2 = +2$; $A_1F = +30$; $A_2E' = -25.5$; $A_1A = +3$; $A_2A' = +1.5$; $f = FA = -e' = A'E' = -27$.

Again, if when $r_1 > r_2 > 0$, we have also

$$d = \frac{n(r_1 - r_2)}{n - 1},$$

the Lens will be of the kind represented in Fig. 111, where the constants have the following values: $n = 3/2$, $r_1 = +12$, $r_2 = +6$, $d = +18$;

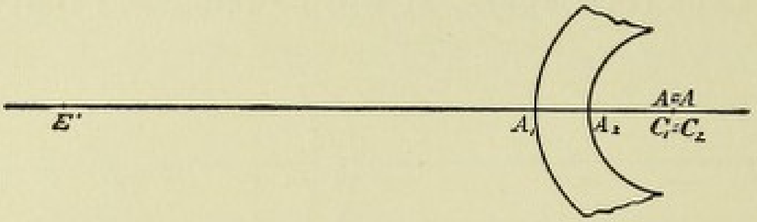

FIG. 110.

CONVEXO-CONCAVE GLASS LENS IN AIR: SPECIAL CASE OF NEGATIVE MENISCUS. Two surfaces of Lens have the same centre: Lens is Divergent. (Focal Point F not shown in the diagram; it lies far to the right.)

$n = 3/2$; $r_1 = A_1C_1 = A_1C_2 = A_1A = A_1A' = +5$; $r_2 = A_2C_2 = A_2C_1 = A_2A = A_2A' = +3$; $d = A_1A_2 = r_1 - r_2 = +2$; $A_1F = +27.5$; $A_2E' = -19.5$; $f = FA = -e' = A'E' = -22.5$.

whence we find $f = -e' = \infty$. This type of Lens may, therefore, be called a "Telescopic Meniscus".

As d increases from the value $d = r_1 - r_2$ to the value

$$d = n(r_1 - r_2)/(n - 1),$$

the Principal Points, which, as we saw above, were coincident, separate farther and farther from each other, both moving along the optical axis in the positive direction of that axis, but A' keeping ahead of A until they both arrive together at infinity.

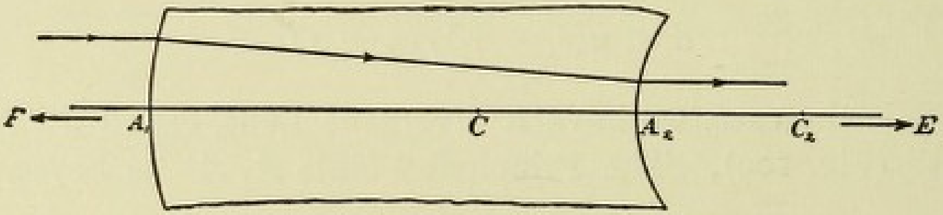

FIG. 111.

CONVEXO-CONCAVE GLASS LENS IN AIR: SPECIAL CASE—TELESCOPIC MENISCUS.

$n = 3/2$; $r_1 = A_1C_1 = C_1C_2 = +12$; $r_2 = A_2C_2 = C_1A_2 = +6$; $f = -e' = \infty$.

And, finally, if, when $r_1 > r_2 > 0$, we have also

$$d > n(r_1 - r_2)/(n - 1),$$

as in the case of the Lens represented in Fig. 112, where the constants

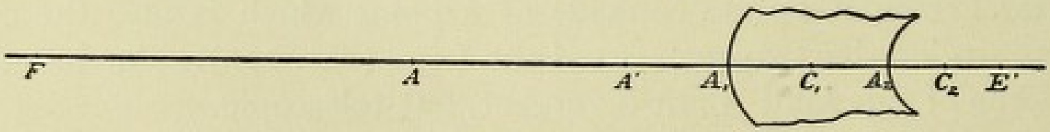

FIG. 112.

CONVERGENT MENISCUS ($r_1 > r_2 > 0$): GLASS LENS IN AIR.

$n = 3/2$; $r_1 = A_1C_1 = +3$; $r_2 = A_2C_2 = +2$; $d = A_1A_2 = +6$; $A_1F = -24$; $A_2E' = +4$; $A_1A = -12$; $A_2A' = -8$; $f = FA = -e' = A'E' = +12$.

have the following values: $n = 3/2$, $r_1 = +3$, $r_2 = +2$, $d = +6$, the Lens will again be a convergent Lens, and now the Principal Points A, A' will lie in front of the vertices A_1, A_2, respectively, and A will lie in front of A'.

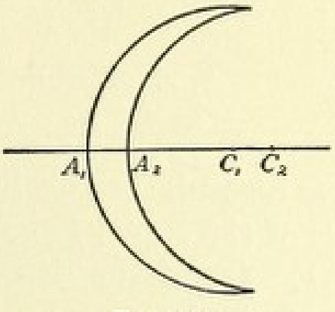

FIG. 113.

CONVERGENT CONVEXO-CONCAVE GLASS LENS IN AIR: Special Case—Two surfaces have equal curvature: Called "Lens of Zero Curvature."

$$r = r_1 = r_2 > 0;\ f = -e' = \frac{6r^2}{d};$$
$$d = A_1A_2 = C_1C_2.$$

(*c*) The last case to be considered is the case when $r_1 = r_2 > 0$. In this form of Lens, sometimes called "Lens of Zero-Curvature", the curvatures of the two surfaces are equal; and since r_1, r_2 have the same sign, and N is positive, the Focal Length f is positive, so that this Lens is always convergent. The diagram (Fig. 113) represents the case of a Lens of Zero-Curvature, determined by the values: $n = 3/2, f = -e' = 6r_1^2/d$. Obviously, in this Lens we have

$$A_1A_2 = C_1C_2.$$

In the limiting case when $d = 0$, this Lens will be an infinitely thin telescopic Lens.

ART. 57. THIN LENSES.

200. Practically speaking, the thickness of a Lens is almost always small in comparison with the other linear constants of the Lens. And (except in the case of the so-called "Lens of Zero-Curvature", for which $r_1 = r_2$) the term $(n-1)d$ which occurs in the expression of the constant N is generally quite small in comparison with the other term $n(r_2 - r_1)$. The value of N may be written:

$$N = n(n-1)(r_2 - r_1)\left\{1 + \frac{(n-1)d}{n(r_2 - r_1)}\right\};$$

and, hence, *if we neglect terms involving powers of d higher than the first,* we have:

$$\frac{1}{N} = \frac{1}{n(n-1)(r_2 - r_1)}\left\{1 - \frac{(n-1)d}{n(r_2 - r_1)}\right\}.$$

Substituting this value of $1/N$ in formulæ (169) and (171), we obtain the following *approximate formulæ of Thin Lenses:*

$$\left.\begin{aligned} f = -e' &= \frac{r_1r_2}{(n-1)(r_2-r_1)}\left\{1-\frac{(n-1)d}{n(r_2-r_1)}\right\}, \\ A_1A = -\frac{r_1d}{n(r_2-r_1)}&, \quad A_2A' = -\frac{r_2d}{n(r_2-r_1)}. \end{aligned}\right\} \tag{174}$$

These formulæ are useful for approximate determinations of the magnitudes of the Focal Lengths and of the positions of the Principal Points.

201. Infinitely Thin Lenses. If the Lens is *infinitely thin*, the formulæ above may be still further simplified by putting $d = 0$. Thus, we obtain:

$$\left.\begin{aligned} d = A_1A_2 = 0, \quad f = -e' = \frac{r_1r_2}{(n-1)(r_2-r_1)}, \\ A_1F = -\frac{r_1r_2}{(n-1)(r_2-r_1)} = E'A_2, \quad A_1A = A_2A' = 0. \end{aligned}\right\} \tag{175}$$

These formulæ, which are identical with those formerly obtained in Chapter VI, need no further remark here.

ART. 58. LENS-SYSTEMS.

202. Consider a compound system consisting of two Lenses with their optical axes in the same straight line, and let A_1, A_1' and F_1, E_1' designate the positions of the Principal Points and of the Focal Points, respectively, of the first Lens; similarly, let A_2, A_2' and F_2, E_2' designate the positions of the Principal Points and of the Focal Points, respectively, of the second Lens. Thus,

$$F_1A_1 = f_1 = -e_1' = A_1'E_1', \quad F_2A_2 = f_2 = -e_2' = A_2'E_2';$$

where f_1, e_1' and f_2, e_2' denote the Focal Lengths of the two Lenses. Moreover, here let us put

$$A_1'A_2 = d, \quad E_1'F_2 = \Delta.$$

Then, since

$$\Delta = E_1'A_1' + A_1'A_2 + A_2F_2,$$

we have:

$$\Delta = -(f_1 + f_2 - d).$$

Finally, let F, E' and A, A' designate the positions of the Focal Points and of the Principal Points, respectively, of the compound system of the two Lenses, and let f, e' denote the Focal Lengths of the compound system. Then, by processes entirely similar to those employed above in Art. 56, we derive the following system of formulæ for an *Optical*

System composed of two Lenses:

$$\left.\begin{aligned} &f = -e' = \frac{f_1 f_2}{f_1 + f_2 - d}, \\ &F_1 F = \frac{f_1^2}{f_1 + f_2 - d}, \qquad E_2' E' = -\frac{f_2^2}{f_1 + f_2 - d}, \\ &A_1 F = -\frac{f_1(f_2 - d)}{f_1 + f_2 - d}, \quad A_2' E' = \frac{f_2(f_1 - d)}{f_1 + f_2 - d}, \\ &A_1 A = \frac{f_1 d}{f_1 + f_2 - d}, \qquad A_2' A' = -\frac{f_2 d}{f_1 + f_2 - d}. \end{aligned}\right\} \qquad (176)$$

Thus, being given the two Lenses and their positions relative to each other, we can, by means of the above formulæ, determine completely the compound system.

203. If, instead of two Lenses, we had *Two Systems of Lenses*, the formulæ (176) can be employed to determine the compound system, provided the letters with the subscript 1 and the letters with the subscript 2 be understood as applying to the first and second systems of Lenses, respectively.

204. A case of considerable interest is an *Optical System composed of Two Infinitely Thin Lenses*. Since the two Principal Points of an infinitely thin Lens coincide at the optical centre of the Lens, the letters A_1 and A_2', as employed in formulæ (176), will designate for this case the positions of the optical centres of the two Lenses, and, therefore,

$$d = A_1' A_2 = A_1 A_2 = A_1 A_2'$$

denotes now the *distance* of the second Lens from the first. If the two infinitely thin Lenses are in contact ($d = 0$), we find:

$$1/f = 1/f_1 + 1/f_2,$$

in agreement with the general formula (106).

Assuming, for the sake of simplicity, that the optical system consists of two infinitely thin Lenses, we may discuss formulæ (176) briefly, as follows:

(*a*) *Suppose that both Lenses are convergent* ($f_1 > 0$, $f_2 > 0$.) If the two Lenses are in contact ($d = 0$), we have:

$$f = \frac{f_1 f_2}{f_1 + f_2},$$

and, consequently, $f > 0$. But this is the smallest positive value of f;

so that as we increase the distance d between the two Lenses, the resulting system is less and less convergent; until, when $d = f_1 + f_2$, we have $f = \infty$, in which case the compound system is telescopic. If we continue to separate the Lenses still farther, we have at first a feebly divergent system; but the divergence increases as d is made greater and greater.

(*b*) In case *both Lenses are divergent* ($f_1 < 0, f_2 < 0$), we have always $f < 0$, so that the compound system will be divergent. The divergence will be greatest when the two Lenses are in contact, and will decrease as the Lenses are separated farther and farther.

(*c*) Finally, suppose that *one of the Lenses is convergent, and the other divergent.* For example, let us assume that $f_1 > 0$ and $f_2 < 0$. In this case the compound system will be divergent, if $d < (f_1 + f_2)$; convergent, if $d > (f_1 + f_2)$; and telescopic, if $d = f_1 + f_2$. Since f_1, f_2 have opposite signs, there are two cases here to be considered, as follows:

1st, The case when $(f_1 + f_2) < 0$: that is, the absolute value of the Focal Length of the convergent Lens is less than that of the divergent Lens. Since d is essentially positive, the only possibility here is $d > (f_1 + f_2)$, and hence this system will also be convergent. The greatest value of f is obtained by placing the two Lenses in contact ($d = 0$); and as the Lenses are separated farther and farther apart, the convergence increases.

2nd, The case when $(f_1 + f_2) > 0$: that is, the absolute value of f_1 is greater than that of f_2. When the two Lenses are in contact ($d = 0$), the system is divergent and f has its least negative value. As d increases, the absolute value of f increases, its sign remaining negative; until, when $d = f_1 + f_2$, f is infinite, and the system is a telescopic system. For values of d greater than $(f_1 + f_2)$, the sign of f will be positive, and the system will be convergent, the convergence increasing with continued increase of d.

CHAPTER IX.

EXACT METHODS OF TRACING THE PATH OF A RAY REFRACTED AT A SPHERICAL SURFACE.

ART. 59. INTRODUCTION.

205. In the preceding chapter we have seen how an ideal image is produced by a centered system of spherical surfaces so long as the rays concerned are the so-called "Paraxial Rays" which are all contained within the infinitely narrow cylindrical region immediately surrounding the optical axis of the system. In this case to a homocentric bundle of incident rays corresponds a homocentric bundle of emergent rays.

But, according to the Wave-Theory of Light, in order to have an optical imagery, a mere homocentric convergence of the rays is not sufficient. This theory requires not only that the wave-front after the light has traversed the optical system shall be spherical, so that the rays of light proceeding originally from a point shall meet again in a point, but that the effective portion of the wave-surface shall be as great as possible in comparison with its radius, which means that the effective rays shall constitute a *wide-angle* bundle of rays (see § 45). Only when this last condition is complied with will the resultant effect of the spherical wave be reduced approximately to a point at the centre, so that there will be point-to-point correspondence between object and image.

Moreover, there is also still another practical reason why we find it necessary to use wide-angle bundles of rays in the production of an image. For if the wide-angle bundle of rays is a condition of a distinct, clear-cut image, it is equally essential for the production of a bright image, since the light-intensity will evidently be greater in proportion as the effective portion of the wave-surface is larger.

Both theoretically and practically, therefore, we require to have an optical system which will, if possible, converge to a point a wide-angle homocentric bundle of incident rays, so that not merely those rays which we call Paraxial Rays but those rays which have finite inclinations to the optical axis will be converged again to one and the same image-point. Generally speaking, this requirement is found to be impossible of fulfilment. Indeed, there may be said to be only one actual optical system which perfectly satisfies the condition of collinear correspondence, viz., the *Plane Mirror;* which, inasmuch as it produces only a virtual image without magnification, hardly deserves to be ranked as an "optical instrument" at all. The "*Pin-Hole Camera*" is no exception to this statement, because only when the aperture through which the rays enter the apparatus is a mathematical point will there be strict point-to-point correspondence of object and image —even then assuming that there were no exceptions to the Law of the Rectilinear Propagation of Light such as we encounter in Physical Optics.

Instead of the ideal case of collinear correspondence of Object-Space and Image-Space, the theory of optical instruments is complicated by numerous practical and, for the most part, irreconcilable difficulties, due chiefly to the so-called "*aberrations*"—some of which are aberrations of *sphericity*, while others are *chromatic* aberrations—and due also, in a less degree, to the assumptions at the foundation of Geometrical Optics, which, as we have pointed out, are not entirely in accordance with the facts of Physical Optics. It is not our purpose, however, to enter into a discussion of these questions here, as they will be extensively treated in subsequent chapters of this treatise. In this chapter we propose to investigate the path of a ray which makes a finite angle with the axis.

ART. 60. GEOMETRICAL METHOD OF INVESTIGATING THE PATH OF A RAY REFRACTED AT A SPHERICAL SURFACE.

206. Construction of the Refracted Ray.

In § 29, we showed how to construct the path of a ray refracted at a surface of any form, and that method is, of course, applicable to the refraction of a ray at a spherical surface. The following elegant and useful construction of the path of a ray refracted at a spherical surface was first given by THOMAS YOUNG in his lectures on Natural Philosophy.[1] WEIERSTRASS,[2] in 1858, and LIPPICH,[3] in 1877, gave the same construction, each entirely independently.

Let C (Figs. 114 and 115) designate the position of the centre, and let r denote the radius, of the spherical refracting surface μ, and let

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[1] *A course of lectures on Natural Philosophy and the Mechanical Arts,* by THOMAS YOUNG, M.D., London, 1807 (two volumes); II, p. 73, Art. 425.

[2] See article by K. SCHELLBACH entitled " Der Gang der Lichtstrahlen in einer Glaskugel ": *Zft. phys. chem. Unt.*, 1889, II, 135.

[3] F. LIPPICH: Ueber Brechung und Reflexion unendlich duenner Strahlensysteme an Kugelflaechen: *Denkschriften der kaiserl. Akad. der Wissenschaften zu Wien,* XXXVII (1878), pp. 163–192. See also a paper published by F. KESSLER in *Wied. Ann. Phys.*, XV (1882).

QB represent the path of the incident ray meeting μ at the point B; and let n, n' denote the absolute indices of refraction of the first and second medium, respectively. Concentric with the spherical refracting surface μ, and with radii equal to $n'r/n$ and nr/n', describe two spherical surfaces τ, τ', respectively. Let Z designate (as shown in the diagram) the point where QB, produced if necessary, meets the auxiliary spherical surface τ. Join Z by a straight line with the centre C, and let Z' designate the point where this straight line intersects the other auxiliary spherical surface τ'. The path BR' of the refracted ray is determined by the straight line which joins B and Z'. In making this construction, we must be careful to select for the point Z that one of the two possible points of intersection of the incident ray QB with the spherical surface τ which will make the piece of the incident ray which lies in the first medium and the piece of the refracted ray which lies in the second medium fall on opposite sides of the incidence-normal CB, in accordance with the Law of Refraction.

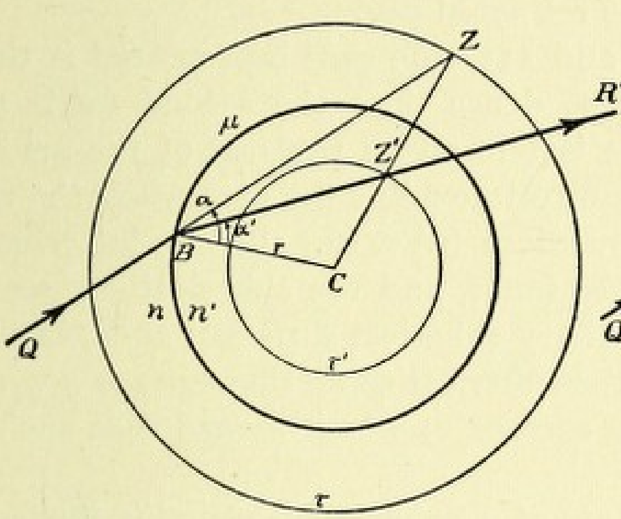

Fig. 114.

Young's Construction of the Path of a Ray Refracted at a Spherical Surface. The figure shows the case when the surface is convex and the second medium more highly refracting than the first ($n' > n$). If the letters Q and R', Z and Z' and τ and τ' are interchanged, and if the arrow-heads are reversed, the same diagram will show Young's Construction for the case when the ray is refracted at a concave spherical surface into an optically less dense medium.

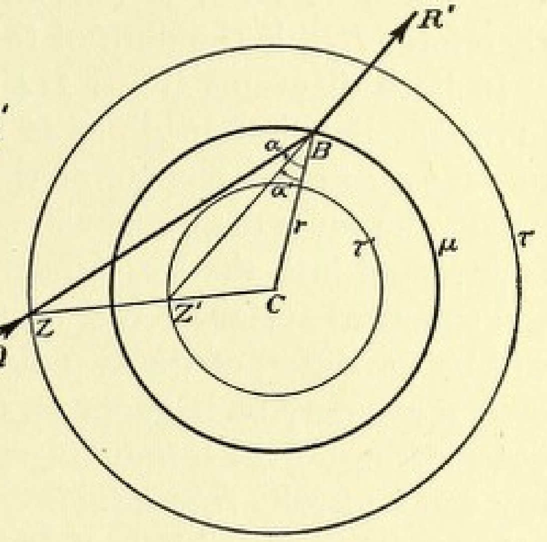

Fig. 115.

Young's Construction of the Path of a Ray Refracted at a Spherical Surface. The figure shows the case when the surface is concave, and the second medium more highly refracting than the first ($n' > n$). If the letters Q and R', Z and Z' and τ and τ' are interchanged, and if the arrow-heads are reversed, the same diagram will show Young's Construction for the case when the ray is refracted at a convex surface into an optically less dense medium.

The proof of the construction is very simple. Since

$$CZ : CB = CB : CZ' = n' : n,$$

the triangles CBZ, CBZ' are similar, and, hence, $\angle CBZ = \angle BZ'C$. But in the triangle CBZ,

$$\frac{\sin \angle CBZ}{\sin \angle BZC} = \frac{CZ}{CB} = \frac{n'}{n},$$

and, since by the Law of Refraction

$$\sin \alpha / \sin \alpha' = n'/n,$$

where $\alpha = \angle CBZ$, it follows that $\angle BZC = \angle CBZ' = \alpha'$ and, therefore, BR' is the path of the refracted ray.

In both diagrams (Figs. 114 and 115) the case represented is that in which the first medium is less dense than the second ($n' > n$); but by a suitable change of the letters and a reversal of the arrow-heads, the same diagrams will suffice to exhibit the case when the ray is refracted into the less dense medium ($n' < n$). In this latter case the spherical surface τ will be the inner, and the spherical surface τ' will be the outer, of the two auxiliary spherical surfaces; thus, in this case, a ray may be incident on the spherical refracting surface μ without meeting at all the auxiliary surface τ; which means that such a ray will be *totally reflected*.

207. "Aplanatic" Pair of Points of a Spherical Refracting Surface. The first point to be remarked in connection with YOUNG's Construction is the extraordinary property of every pair of such points as Z, Z'. Any straight line drawn through the centre C of the spherical refracting surface will determine by its intersections with the auxiliary spherical surfaces τ, τ' a pair of points Z, Z', at distances from C equal to $n'r/n$, nr/n', respectively, characterized by the property that to a homocentric bundle of incident rays Z corresponds a homocentric bundle of refracted rays Z'. Moreover, this property is entirely independent of the angular opening of the bundle of incident rays, and is true, therefore, of a bundle of rays of finite aperture.

The pair of conjugate points Z, Z', which lie on the axis of the spherical refracting surface (Fig. 116), and which are situated as above described, are called the "*aplanatic*" *pair of points* of the spherical refracting surface; with respect to these points the spherical refracting surface is an "aberrationless" surface.

Since rays which are directed towards the centre C enter the second medium without being changed in their directions, the point C may also be regarded as a pair of coincident conjugate points (§ 44) which possess a property similar therefore to that of the aplanatic points. Moreover, each point on the surface of the refracting sphere is a "self-corresponding", or "double", point. But the only pair of such points that are separated is the pair Z, Z'.

Since $\angle BZ'C = \alpha$, $\angle BZC = \alpha'$, it follows that the angles of inclination to the axis of the incident and refracted rays are equal to the angles of refraction and incidence, respectively; so that the aplanatic points are likewise characterized by the fact that *the sines of the angles of inclination to the axis of any pair of conjugate rays crossing the axis at Z and at Z' have a constant ratio.* Another way of remarking this characteristic property of the aplanatic pair of points of a spherical refracting surface is by the relation:

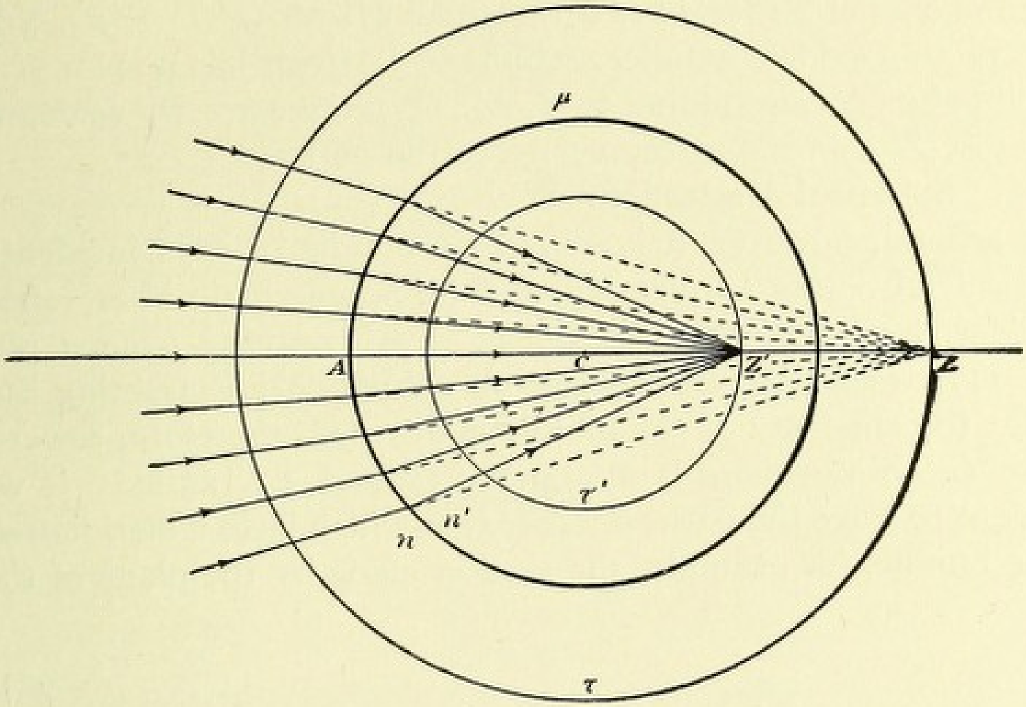

Fig. 116.

So-called Aplanatic (or Aberrationless) Points of a Refracting Sphere.

$$BZ/BZ' = n'/n.$$

Moreover, since

$$CZ \cdot CZ' = r^2,$$

the geometer will recognize that Z, Z' are the so-called "inverse" points with respect to the spherical surface of radius r, which are harmonically separated by the end-points of the diameter on which they lie.

The points Z, Z' lie always on the same side of the centre C of the spherical refracting surface, so that whereas the rays will pass "really" through one of these points, the corresponding rays will pass "virtually" through the other point. Thus, one of the spherical surfaces τ, τ' is the virtual image of the other.

The circle of contact, in which the tangent-cone, drawn from that one of the points Z, Z' which lies outside the spherical refracting surface, touches this surface, divides it into two portions, and the incidence-point B lies always on the greater of these two portions of the surface.

In the case of *Reflexion at a Spherical Mirror* ($n' = -n$), YOUNG's Construction evidently fails. A Spherical Mirror has no pair of "aplanatic" points corresponding to Z, Z'; or, more correctly speaking, the points Z, Z' coincide at the vertex of the mirror.

208. Spherical Aberration.

In general, however, a homocentric bundle of rays incident on a spherical refracting surface will not be homocentric after refraction. Consider, for example, a bundle of rays diverging from a point L (Fig. 117), and incident directly on a Spherical Refracting Surface, so that the chief ray of the bundle is directed, therefore, towards the centre C. Since there is symmetry around LC as axis, it will be sufficient to trace the paths of those rays which lie in a meridian section of the bundle, for example, the section made by the plane of the diagram; for it is obvious that the entire bundle of rays may be regarded as generated by the rotation of this meridian pencil around the chief ray LC as axis.

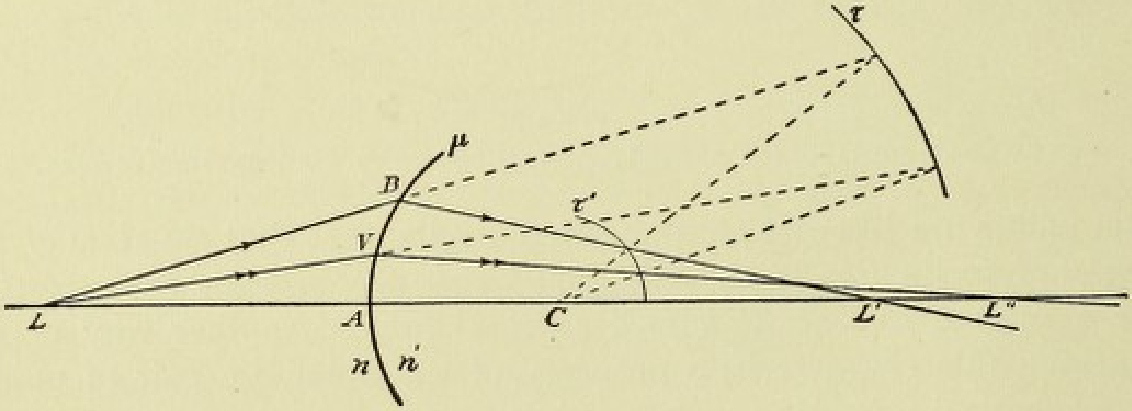

FIG. 117.

SPHERICAL ABERRATION. Whereas the incident rays all cross the axis of the spherical surface at one point L, the corresponding refracted rays cross the axis, in general, at different points L' L'', etc.

If LB is an incident ray, and BL' the corresponding refracted ray meeting the straight line LC in L', and if LBL' is revolved around LC as axis, to the incident rays lying on the surface of the right-circular cone CLB will correspond a system of refracted rays lying on the surface of the right-circular cone $CL'B$.

The position of the point L' can be seen to depend, in general, on the slope of the incident ray LB, so that different rays of the pencil of incident rays L will determine different positions of the point L'. Accordingly, whereas all the rays of the bundle of incident rays L will be grouped in cones, which have a common vertex at L, the corresponding refracted rays will be grouped in cones, which, while they have all a common axis LC, will, in general, have different vertices L'. This variation of the position of the point L' corresponding to a fixed position of the point L is called *Spherical Aberration* (see § 260): which will be treated at length in a special chapter devoted to that subject.

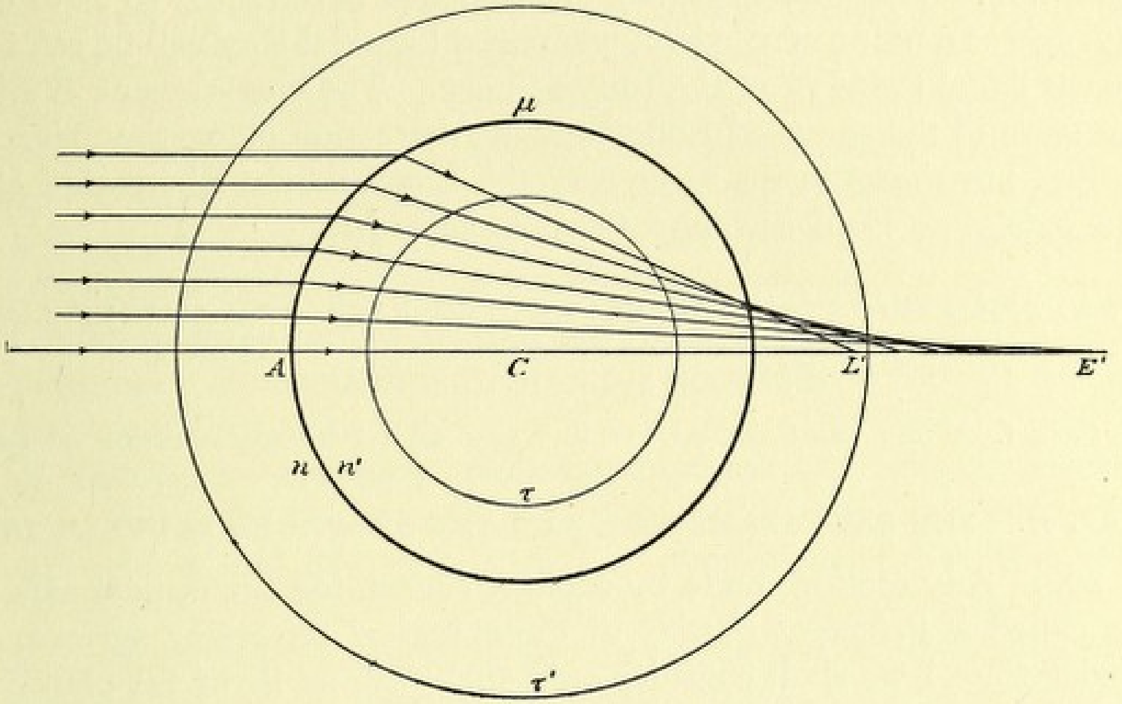

Fig. 118.

Spherical Aberration. Case when a Pencil of Parallel Incident Rays is Refracted at a Spherical Surface.

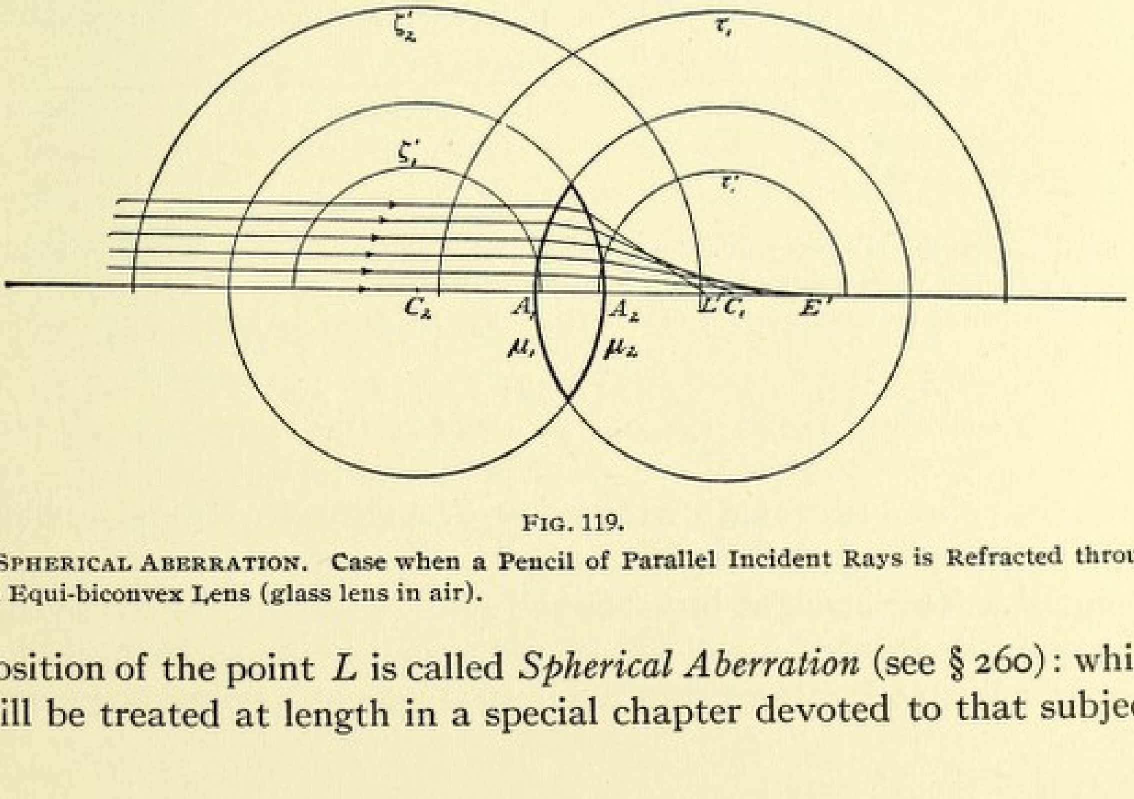

Fig. 119.

Spherical Aberration. Case when a Pencil of Parallel Incident Rays is Refracted through an Equi-biconvex Lens (glass lens in air).

The diagram (Fig. 118) shows the case of a meridian pencil of incident rays parallel to the axis of the spherical refracting surface. The paths of the refracted rays have been traced by YOUNG's Construction. The outermost ray of the pencil is refracted so as to cross the axis at a point marked L', whereas a Paraxial Ray will be refracted to the Focal Point E' of the Image-Space. The line-segment $E'L'$ is a measure of the so-called Longitudinal Aberration along the axis.

Fig. 119 shows in the same way the Longitudinal Aberration along the axis of an Equi-Biconvex Glass Lens in Air.

TRIGONOMETRIC COMPUTATION OF THE PATH OF A RAY OF FINITE INCLINATION TO THE AXIS, REFRACTED AT A SINGLE SPHERICAL SURFACE.

CASE I. WHEN THE PATH OF THE RAY LIES IN A PRINCIPAL SECTION OF THE SPHERICAL REFRACTING SURFACE.

ART. 61. THE RAY-PARAMETERS, AND THE RELATIONS BETWEEN THEM.

209. Any section made by a plane containing the optical axis will be called a *Principal Section* of the spherical refracting surface, and under Case I we shall consider only such rays as lie in the plane of a principal section.

In the diagram (Fig. 120) the plane of the paper represents a principal section of the spherical refracting surface, the centre of which is at the point designated by the letter C. Through C draw a straight line in the plane of the paper meeting the spherical surface in the point A. This straight line we shall take as the optical axis, so that the point designated by A will be the vertex of the surface. Let the straight line QB, intersecting the optical axis at a point L, represent

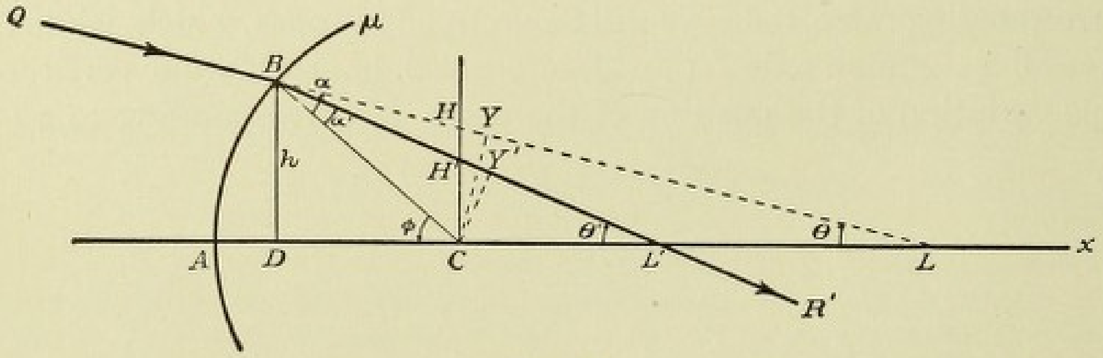

FIG. 120.

TRIGONOMETRIC CALCULATION OF THE PATH OF A RAY REFRACTED AT A SPHERICAL SURFACE: CASE WHEN THE RAY-PATH LIES IN THE PLANE OF A PRINCIPAL SECTION. The straight line QB shows the path of the incident ray, and the straight line BR' shows the path of the corresponding refracted ray.

$AC = r$, $AL = v$, $AL' = v'$, $CL = c$, $CL' = c'$, $DB = h$, $\angle CBL = \alpha$, $\angle CBL' = \alpha'$,
$\angle BCA = \phi$, $\angle ALB = \theta$, $\angle AL'B = \theta'$, $BL = l$, $BL' = l'$, $CH = b$, $CH' = b'$.

the path of a ray of light incident on the spherical refracting surface at the point B, and draw the radius BC.

We shall employ here pretty nearly the same letters and symbols as were used in Chapter V, with such changes, however, as will be necessary in order to distinguish the present case from that of a Paraxial Ray. Moreover, as we shall also have to introduce symbols for several new magnitudes, and as the relations derived below will be frequently referred to in the course of this work, it will be well to define clearly the precise meaning that is to be attached to each of the symbols employed; which we therefore proceed to do.

1. *Notation of the Linear Magnitudes.*

The abscissa of the centre C, with respect to the vertex A, will be denoted by r; thus, $AC = r$.

The abscissæ, with respect to the vertex A, of the points designated by L, L', where the ray crosses the axis, really or virtually, before and after refraction, will be denoted by v, v', respectively; thus, $AL = v$, $AL' = v'$.

The abscissæ, with respect to the centre, of the points L, L' will be denoted by c, c', respectively; thus, $CL = c$, $CL' = c'$; and, consequently,

$$c = v - r, \quad c' = v' - r.$$

In regard to the signs of these abscissæ, they are to be reckoned positive or negative according as they are measured in the positive or negative direction of the axis: the positive direction of the axis being determined by the direction of the incident axial ray (§193).

The "*ray-lengths*" are the segments of the incident and refracted rays, measured from the point of incidence B to the points L, L' where the incident and refracted rays, respectively, cross the axis. These magnitudes will be denoted by the symbols l, l'; thus, $BL=l$, $BL'=l'$. These magnitudes are to be reckoned positive or negative according as they are measured along the ray in the same direction as the light goes or in the opposite direction.

If D designates the foot of the perpendicular let fall from the incidence-point B on the axis, the magnitude $DB = h$ is called the "incidence-height" of the ray, and is reckoned positive or negative according as the point B lies above or below the axis.

The perpendicular to the optical axis erected at the centre C of the spherical and refracting surface will be called the "central perpendicular", and the intercepts CH, CH' of the incident and refracted rays on the central perpendicular will be denoted by the symbols b, b';

thus, $CH = b$, $CH' = b'$. This intercept b is to be reckoned positive or negative according as the point H lies above or below the optical axis. And a perfectly similar rule obtains with regard to the sign of b'.

2. *Notation of the Angular Magnitudes.*

The angles of incidence and refraction are denoted by the symbols α and α'; thus, in the diagram, $\angle CBL = \alpha$, $\angle CBL' = \alpha'$. These angles are the acute angles through which the radius CB must be rotated around the point B in order to come into coincidence with the straight lines to which the incident and refracted rays belong.

The "*Slope*" of the incident ray, or its inclination to the axis, is denoted by the symbol θ; and, similarly, the symbol θ' is used to denote the "slope" of the corresponding refracted ray. Thus,

$$\angle ALB = \theta, \quad \angle AL'B = \theta'.$$

These are the acute angles through which the axis must be turned around the points L, L', in order to be brought into coincidence with the straight lines to which the incident and refracted rays, respectively, belong. Moreover, since

$$\tan\theta = -h/DL, \quad \tan\theta' = -h/DL',$$

the signs of θ and θ' are the same as the signs of $-h/v$ and $-h/v'$, respectively.

The acute angle through which the radius CB drawn to the incidence-point B has to be turned around C in order for it to coincide with CA will be denoted by the symbol φ; thus, $\angle BCA = \varphi$.

210. We proceed now to remark a number of useful relations between the magnitudes denoted by the symbols v, l, b, h, α, θ and φ.

The position of a straight line is determined so soon as we know the positions of two points on the line or the positions of one point together with the direction of the line. The equation of a straight line lying in a given plane—for example, the plane of a Principal Section of the spherical surface—involves at most two arbitrary constants or *parameters;* and to each set of values of any such pair of parameters there corresponds a perfectly definite straight line of the given plane.

Thus, for example, the position (but not the direction) of the incident ray LB lying in the plane of the Principal Section will be completely determined provided we know the values, say, of the parameters v, θ, which are called by some writers the "ray-co-ordinates". Instead of using v, θ, we might also define the position of the ray by means of various other pairs of the magnitudes denoted by the symbols v, l, b, h, α, θ and φ. L. SEIDEL, for example, uses in his system of optical formulæ ray-parameters that are equivalent to h, θ.

The relations between these magnitudes are obtained easily by an inspection of the triangle LBC. Evidently,

$$\alpha = \theta + \varphi. \tag{177}$$

This formula exhibits the connection between the angular magnitudes.

By the so-called Law of Sines, we derive from the triangle LBC the following formulæ:

$$\left.\begin{aligned} l\cdot\sin\theta &= -\,r\cdot\sin\varphi, \\ -r\cdot\sin\alpha &= (v-r)\sin\theta, \\ (v-r)\sin\varphi &= l\cdot\sin\alpha; \end{aligned}\right\} \tag{178}$$

and by the so-called Law of Cosines:

$$\left.\begin{aligned} l^2 &= (v-r)^2 + r^2 + 2r(v-r)\cos\varphi, \\ (v-r)^2 &= r^2 + l^2 - 2rl\cdot\cos\alpha, \\ r^2 &= (v-r)^2 + l^2 - 2l(v-r)\cos\theta. \end{aligned}\right\} \tag{179}$$

Finally, by projecting two of the sides of the triangle LBC on the third side, we obtain:

$$\left.\begin{aligned} r &= l\cdot\cos\alpha - (v-r)\cos\varphi, \\ v-r &= l\cdot\cos\theta - r\cdot\cos\varphi, \\ l &= r\cdot\cos\alpha + (v-r)\cos\theta. \end{aligned}\right\} \tag{180}$$

Also, in the right triangles CBD, LBD, we have:

$$\sin\varphi = h/r, \tag{181}$$

and

$$\sin\theta = -\,h/l. \tag{182}$$

Finally, if Y designates the foot of the perpendicular let fall from the centre C on the straight line BH, we have evidently:

$$CY = r\cdot\sin\alpha = b\cdot\cos\theta. \tag{183}$$

By priming the magnitudes denoted by v, l, b, α and θ in the above formulæ (177), (178), (179), (180), (181), (182) and (183), we shall obtain the corresponding relations for the refracted ray BL'.

ART. 62. TRIGONOMETRIC COMPUTATION OF THE PATH OF THE REFRACTED RAY.

211. The problem is as follows: Given the spherical refracting surface and the values of the indices of refraction n, n' of the two media separated by it, and the position of the incident ray, to determine the position of the corresponding refracted ray. In other words, being given the constants denoted by n, n' and r, and the co-ordinates v, θ of the incident ray, we are required to find the co-ordinates v', θ' of the refracted ray.

By the Law of Refraction:

$$n \cdot \sin \alpha = n' \cdot \sin \alpha'.$$

Moreover, since

$$\alpha = \theta + \varphi, \quad \alpha' = \theta' + \varphi,$$

we have the invariant-relation:

$$\alpha - \theta = \alpha' - \theta'.$$

By means of these equations and the second of equations (178) above, we obtain easily the following system of equations for calculating the values of v', θ':

$$\left.\begin{array}{ll} \sin \alpha = (1 - v/r) \sin \theta, & \sin \alpha' = n \cdot \sin \alpha / n', \\ \theta' = \theta + \alpha' - \alpha, & v' = r(1 - \sin \alpha' / \sin \theta'). \end{array}\right\} \quad (184)$$

We may also remark here a number of other useful relations between the parameters of the incident and refracted rays. For example, since the incidence-height has the same value for both rays, we have the following invariant relation:

$$l \cdot \sin \theta = l' \cdot \sin \theta'; \quad (185)$$

and, since

$$\sin \alpha / \sin \theta = - c/r, \quad \sin \alpha' / \sin \theta' = - c'/r,$$

and, therefore,

$$\sin \theta' / \sin \theta = nc/n'c',$$

we obtain from (185):

$$nc/l = n'c'/l',$$

which may also be written:

$$n \frac{v - r}{l} = n' \frac{v' - r}{l'}.$$

This formula is, in fact, a mere transformation of the Optical Invariant (§ 25)

$$K = n \cdot \sin \alpha = n' \cdot \sin \alpha'$$

for the special case of Refraction at a Spherical Surface. The magnitude

$$I = \frac{n(v-r)}{lr} = \frac{n'(v'-r)}{l'r}, \tag{186}$$

or

$$I = \frac{K}{r\cdot\sin\varphi},$$

which remains unchanged as the ray is refracted from one medium into the next, and which may be called the "invariant of refraction at a spherical surface", plays an important part in ABBE's Theory of Spherical Aberration.

Note 1. In the special case of *Reflexion at a Spherical Mirror*, we have only to put $n' = -n$ in the above formulæ (see § 26). For example, putting $n' = -n$ in formulæ (184), we obtain:

$$\left.\begin{aligned} \sin\alpha &= \sin\theta(1 - v/r),\\ \alpha' &= -\alpha,\\ \theta' &= \theta - 2\alpha,\\ v' &= r\left(1 + \frac{\sin\alpha}{\sin(\theta - 2\alpha)}\right) \end{aligned}\right\} \text{Reflexion at Spherical Mirror.}$$

Note 2. The following formula, adapted to logarithmic computation, is convenient as a "check" formula in calculating the magnitude v':

$$\begin{aligned} v' &= AL' = AC + CD + DL'\\ &= r - r\cdot\cos\varphi - h\cdot\cot\theta'\\ &= 2r\cdot\sin^2\varphi/2 - r\cdot\sin\varphi\cdot\cos\theta'/\sin\theta'\\ &= 2r\left(\sin\frac{\varphi}{2} - \cos\frac{\varphi}{2}\cdot\frac{\cos\theta'}{\sin\theta'}\right)\sin\frac{\varphi}{2}; \end{aligned}$$

so that, finally, we may write:

$$v' = -\frac{2r\cdot\sin\frac{\varphi}{2}\cdot\cos\left(\theta' + \frac{\varphi}{2}\right)}{\sin\theta'} = -\frac{2r\cdot\sin\frac{\varphi}{2}\cdot\cos\frac{\alpha'+\theta'}{2}}{\sin\theta'};$$

and, similarly:

$$v = -\frac{2r\cdot\sin\frac{\varphi}{2}\cdot\cos\left(\theta + \frac{\varphi}{2}\right)}{\sin\theta} = -\frac{2r\cdot\sin\frac{\varphi}{2}\cdot\cos\frac{\alpha+\theta}{2}}{\sin\theta}.$$

Dividing one of these formulæ by the other, we obtain:

$$\frac{v'}{v}=\frac{\sin\theta\cdot\cos\left(\theta'+\frac{\varphi}{2}\right)}{\sin\theta'\cdot\cos\left(\theta+\frac{\varphi}{2}\right)}.$$

In the special case of *Refraction at a Plane Surface*, putting $r = \infty$, we have $\varphi = 0$, and the above formula becomes:

$$v'\cdot\tan\theta' = v\cdot\tan\theta, \text{ (Refraction at Plane Surface),}$$

which may also be easily derived directly (§ 52).

Note 3. *Spherical Aberration.* The co-ordinates v', θ' of the refracted ray BR' can be found, as we have shown, in terms of the co-ordinates v, θ of the incident ray. If in the formula

$$n(v - r)/l = n'(v' - r)/l'$$

we substitute for l, l' their values as given by the first of formulæ (179), it is obvious that v' will thus be expressed as a function of n, n', r, v and φ. The magnitudes n, n' and r are constants, so that v' is, in fact, a function of the variables v and φ; and, therefore, if v is kept constant, it is obvious that we shall, generally, obtain different values of v' by merely changing the value of φ. This is the analytical statement of the fact of *Spherical Aberration* mentioned in § 208.

The positions on the axis of the so-called "*aplanatic*" pair of points Z, Z' (§ 207) can be found easily by means of the formulæ obtained above. The condition that the abscissa v' corresponding to a certain fixed value of v shall be independent of the angle φ must be imposed upon the equations. Since

$$nc/l = n'c'/l',$$

and

$$l^2 = c^2 + r^2 + 2rc\cdot\cos\varphi,$$

$$l'^2 = c'^2 + r^2 + 2rc'\cdot\cos\varphi,$$

we obtain:

$$(n'^2 - n^2)c^2c'^2 + (n'^2c'^2 - n^2c^2)r^2 + 2rcc'(n'^2c' - n^2c)\cos\varphi = 0.$$

If, for a given value of c, the value of c' derived from this equation is to be independent of φ, we must have:

$$c' = \frac{n^2}{n'^2}c,$$

which shows that for this particular pair of values c, c' must have the same sign; that is, the points Z, Z' must lie on the same side of the centre C. If in the equation above we substitute this special value of c', we obtain

$$c^2 = \frac{n'^2}{n^2} r^2.$$

This equation gives two values of c, of which only the value

$$c = +\frac{n'}{n} r$$

is admissible here where we have to do with optical rays as distinguished from mere geometrical rays. (The value $c = -n'r/n$ corresponds to the other intersection of the ray with the auxiliary spherical surface τ in Figs. 114 and 115.) Thus, we find:

$$v = r + n'r/n, \quad v' = r + nr/n'$$

for the abscissæ AZ, AZ' of the pair of aplanatic points of a spherical refracting surface; in agreement with the results of § 207.

A characteristic property of the aplanatic points of a single spherical refracting surface, which was also remarked in § 207, may be stated as follows: If θ, θ' denote the slopes of the incident and refracted rays BZ, BZ', then

$$\sin\theta/\sin\theta' = n/n';$$

that is, the ratio of the sines of the "slope"-angles is independent of the magnitude of the angle of incidence, and constant, therefore, for all pairs of corresponding incident and refracted rays. If Y denotes the value of the Lateral Magnification by means of Paraxial Rays for the pair of conjugate points Z, Z', we shall find that:

$$Y = n^2/n'^2,$$

and, hence, the relation obtained above may be written:

$$\sin\theta/\sin\theta' = n'Y/n.$$

Expressed in this form, this relation, which we have obtained for the aplanatic points of a single spherical refracting surface, represents a very important general law of Optics known as the *Sine-Condition* (Art. 86), which will be fully considered in a subsequent chapter.

Note 4. If the position of the ray is defined by means of its slope-angle θ and its intercept b on the central perpendicular, then by formula (183):

$$r\cdot\sin\alpha = b\cdot\cos\theta, \quad r\cdot\sin\alpha' = b'\cdot\cos\theta';$$

and, hence, since

$$n\cdot\sin\alpha = n'\cdot\sin\alpha',$$

we obtain the following invariant-relation:

$$n\cdot b\cdot\cos\theta = n'\cdot b'\cdot\cos\theta'.$$

Thus, being given the parameters b, θ of the incident ray, we can find the parameters b', θ' of the refracted ray by means of the following system of equations:

$$\left.\begin{aligned} \sin\alpha &= \frac{b\cdot\cos\theta}{r}, & \sin\alpha' &= \frac{n\cdot\sin\alpha}{n'}, \\ \theta' &= \theta - \alpha + \alpha', & b' &= \frac{n}{n'}\,\frac{\cos\theta}{\cos\theta'}\,b. \end{aligned}\right\} \tag{187}$$

Since $CH = LC\cdot\tan\theta$, the connection between the intercept $AL = v$ on the optical axis and the intercept $CH = b$ on the central perpendicular is given by the formula:

$$b = (r - v)\tan\theta.$$

ART. 63. FORMULÆ FOR FINDING THE POINT OF INTERSECTION AND THE INCLINATION TO EACH OTHER OF A PAIR OF REFRACTED RAYS LYING IN THE PLANE OF A PRINCIPAL SECTION OF THE SPHERICAL REFRACTING SURFACE.

212. Let one of the incident rays, distinguished as the *chief* of the two, cross the optical axis at the point $\boldsymbol{L}$ (Fig. 121); which, when the ray is the chief of a bundle of incident rays, will coincide with the position of the centre of the "stop", or circular diaphragm, which is used to limit the bundle of object-rays that are permitted to pass through the optical system; and let the incidence-point of the chief ray be designated by the letter $\boldsymbol{B}$. The other ray (which we may call the "secondary"ray) crosses the optical axis at the point L, and meets the spherical surface at the point B. The positions of both of these rays are supposed to be known, so that we may consider that we know their "slopes",

$$\theta = \angle ALB, \quad \boldsymbol{\theta} = \angle A\boldsymbol{LB},$$

and their intercepts on the optical axis,

$$v = AL, \quad \boldsymbol{v} = A\boldsymbol{L};$$

so that we also know (or can find) the angles of incidence,

$$\alpha = \angle CBT, \quad \boldsymbol{\alpha} = \angle C\mathbf{B}T,$$

the point of intersection of the secondary ray with the chief ray being designated in the diagram by the letter T. The magnitudes

$$\mathbf{B}T = t, \quad \angle \mathbf{B}TB = \lambda$$

may be regarded as the co-ordinates of the secondary ray with respect to the chief ray. This intercept t on the chief ray is measured always from the incidence-point $\mathbf{B}$ as origin, and is to be reckoned positive or negative according as the light travels along the straight line $\mathbf{B}T$ in the direction from $\mathbf{B}$ towards T or in the opposite direction. The "aperture-angle" λ is defined as the acute angle through which the chief ray $\mathbf{B}T$ must be turned around the point T in order to bring it into coincidence with the secondary ray BT.

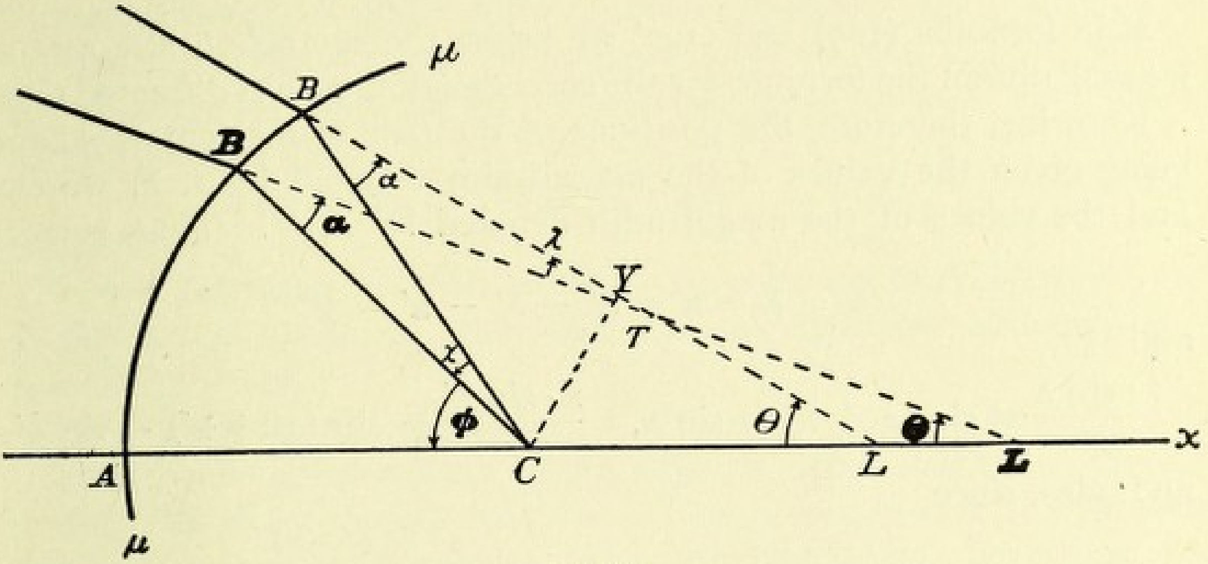

FIG. 121.

Figure represents a pair of rays, lying in the plane of a principal section of a spherical refracting surface, and incident on this surface at the points designated by $\mathbf{B}$ and B. These rays cross the optical axis at the points designated by $\mathbf{L}$ and L, and intersect each other at the point designated by T. The refracted rays are not shown.

$$A\mathbf{L} = \mathbf{v}, \quad AL = v, \quad AC = r, \quad \mathbf{B}T = t, \quad \angle BCA = \phi, \quad \angle \mathbf{B}CA = \boldsymbol{\phi},$$
$$\angle BC\mathbf{B} = \chi, \quad \angle ALB = \theta, \quad \angle A\mathbf{L}\mathbf{B} = \boldsymbol{\theta}, \quad \angle \mathbf{B}TB = \lambda.$$

Putting

$$\angle BCA = \varphi, \quad \angle \mathbf{B}CA = \boldsymbol{\phi},$$

we have:

$$\varphi = \boldsymbol{\phi} + \chi, \tag{188}$$

where $\chi = \angle BC\mathbf{B}$ denotes the increase of the central angle $\boldsymbol{\phi}$.

From the figure we have evidently:

$$\alpha - \lambda = \boldsymbol{\alpha} + \chi. \tag{189}$$

From the centre C draw CY perpendicular to the straight line BT at Y. The orthogonal projection of the radius CB on the straight line CY is equal to the sum of the orthogonal projections on CY of the line-segments $\boldsymbol{CB}$ and $\boldsymbol{BT}$; and, since these projections are equal to $r \cdot \sin\alpha$, $r \cdot \sin(\mathbf{a} + \lambda)$ and $-t \cdot \sin\lambda$, respectively, we obtain the relation:

$$r \cdot \sin\alpha = r \cdot \sin(\mathbf{a} + \lambda) - t \cdot \sin\lambda,$$

or

$$\frac{t \cdot \sin\lambda}{r} = \sin(\mathbf{a} + \lambda) - \sin\alpha. \tag{190}$$

If in formulæ (189) and (190) we prime the symbols t, λ, $\mathbf{a}$ and α, we shall obtain the formulæ for the corresponding pair of refracted rays.

Knowing, therefore, the positions of the pair of incident rays, and being given the values of the magnitudes denoted by t, λ, we can find the values of the magnitudes denoted by t', λ'. Thus, since

$$\alpha - \lambda - \mathbf{a} = \alpha' - \lambda' - \mathbf{a}' = \chi,$$

and

$$\frac{t \cdot \sin\lambda}{r} = \sin(\mathbf{a} + \lambda) - \sin\alpha, \quad \frac{t' \cdot \sin\lambda'}{r} = \sin(\mathbf{a}' + \lambda') - \sin\alpha',$$

and, also, since

$$\frac{\sin(\mathbf{a}' + \lambda') - \sin\alpha'}{\sin(\mathbf{a} + \lambda) - \sin\alpha} = \frac{\cos\dfrac{\alpha' + \mathbf{a}' + \lambda'}{2}}{\cos\dfrac{\alpha + \mathbf{a} + \lambda}{2}},$$

we derive the following formulæ:

$$\left.\begin{aligned} \lambda' &= \lambda + (\mathbf{a} - \mathbf{a}') - (\alpha - \alpha'), \\ \frac{t'}{t} &= \frac{\sin\lambda}{\sin\lambda'} \frac{\cos\dfrac{\alpha' + \mathbf{a}' + \lambda'}{2}}{\cos\dfrac{\alpha + \mathbf{a} + \lambda}{2}}. \end{aligned}\right\} \tag{191}$$

CASE II. WHEN THE PATH OF THE RAY DOES NOT LIE IN A PRINCIPAL SECTION OF THE SPHERICAL REFRACTING SURFACE.

ART. 64. PARAMETERS OF OBLIQUE RAY.

213. The equation of a straight line in space involves as many as four arbitrary constants, and the forms of the refraction-formulæ which we shall obtain will depend on how these ray-parameters are chosen.

Let us take the centre C of the spherical refracting surface as the origin of a system of rectangular co-ordinates. Naturally, also, we shall take the optical axis itself as the x-axis. The plane of a principal section of the spherical surface may be conveniently selected as the xy-plane; nor will it at all affect the generality of the following treatment if for this plane we take that meridian section of the spherical surface which contains also the object-point. The plane of the principal section, which is perpendicular to the xy-plane, will then be the xz-plane, and a transversal plane at right angles to the optical axis will be the yz-plane. For convenience, we may suppose that the axis of y is vertical, and that the axes of x and z are horizontal.

The letters G, H and I will be used to designate the points where the incident ray, prolonged if necessary, crosses the xy-, yz- and xz-planes, respectively; and the rectangular co-ordinates of these points will be denoted by

$$x_g, y_g, 0; \quad 0, y_h, z_h, \quad \text{and} \quad x_i, 0, z_i,$$

respectively.

In the following we shall explain the methods of A. Kerber and L. Seidel of calculating the path of a ray refracted obliquely at a spherical surface.

214. Method of A. Kerber.

In the calculation-system of A. Kerber,[1] the position of the ray is determined by the co-ordinates of the points G and I, where the ray crosses the vertical plane of the principal section (xy-plane) and the horizontal meridian plane (xz-plane). In the figure (Fig. 122) the spherical triangle AA_gA_i represents a piece of the spherical refracting surface. The point A, where the optical axis crosses this surface, is the vertex of the surface; AA_gC is the plane of the principal section, and AA_iC is the meridian section perpendicular to the principal section. Let

$$\angle ACA_g = \varphi_g, \quad \angle ACA_i = \varphi_i.$$

These angles are precisely defined by the following relations:

$$\tan\varphi_g = -\frac{y_g}{x_g}, \quad \tan\varphi_i = -\frac{z_i}{x_i}. \tag{192}$$

Also, regarding the radius A_gC as a secondary axis of the spherical surface, let us denote the abscissa of the point G, with respect to A_g as origin, by v_g; and, similarly, regarding the radius A_iC as another secondary axis, we shall denote the abscissa of the point I, with respect to A_i as origin, by v_i; thus, $v_g = A_gG$, $v_i = A_iI$. From the figure,

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[1] A. Kerber: *Beitraege zur Dioptrik*, Heft II (Leipzig, Gustav Fock, 1896), pages 5–8.

we obtain: $$x_g = CG \cdot \cos\varphi_g, \quad x_i = CI \cdot \cos\varphi_i,$$
and, since,
$$CG = v_g - r, \quad CI = v_i - r,$$
we have:
$$v_g - r = \frac{x_g}{\cos\varphi_g}, \quad v_i - r = \frac{x_i}{\cos\varphi_i}. \tag{193}$$

The projection of the incident ray in the plane of the principal section (xy-plane) makes with the optical axis an angle ϵ, and with the ray itself an angle δ; these angles being exactly defined by the following formulæ:

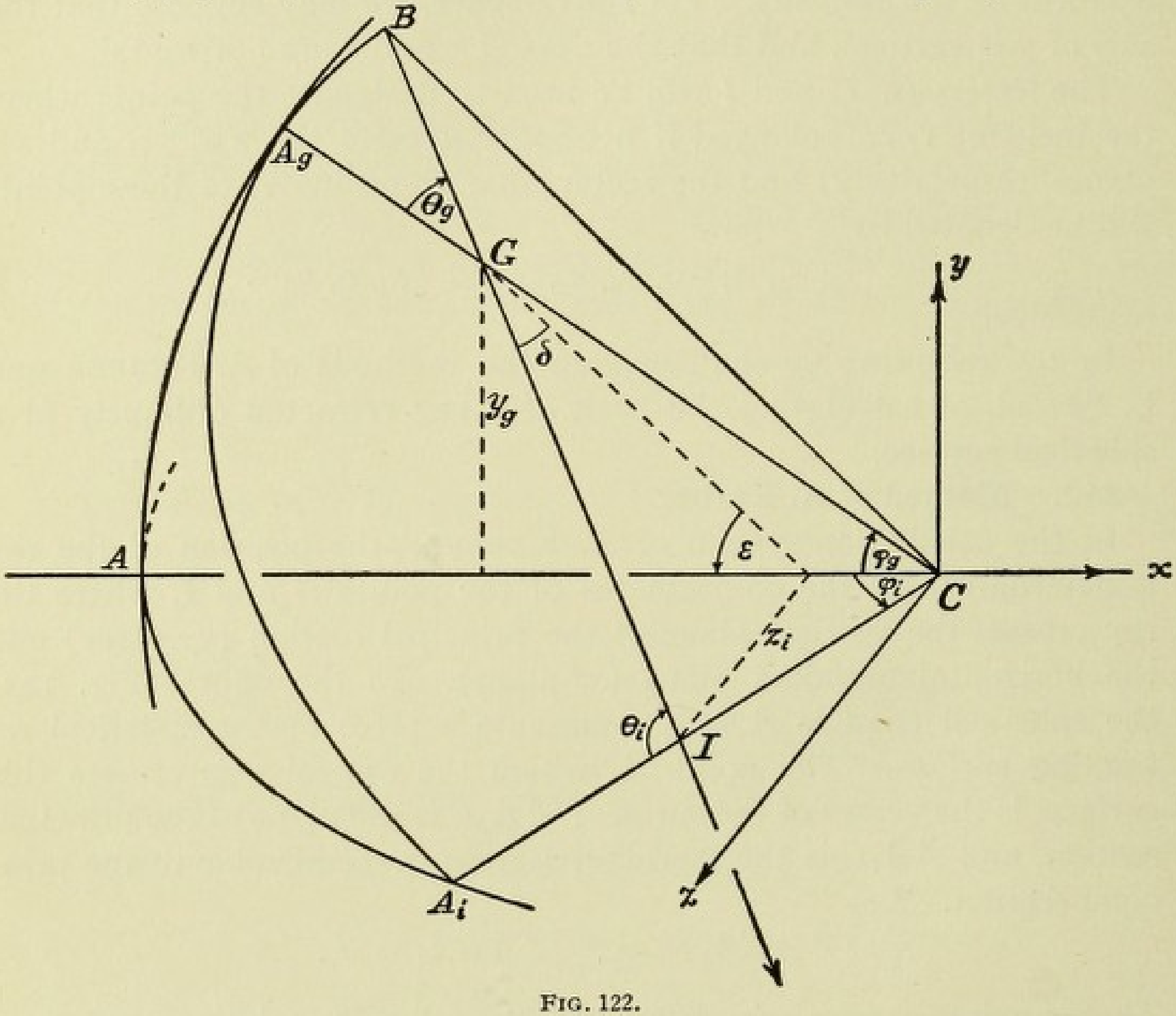

FIG. 122.

KERBER'S METHOD OF DEALING WITH THE OBLIQUE RAY. The plane of the paper (xy-plane) represents a principal section of a spherical refracting surface, centre at C, and optical axis coinciding with the x-axis of co-ordinates. A ray, whose path does not lie in the plane of the principal section, is incident on the spherical surface at the point B. This ray crosses the xy-plane at the point designated by G and the xz-plane at the point designated by I. The spherical triangle AA_gA_i is formed by the intersections of the vertical xy-plane, the horizontal xz-plane and the plane of incidence with the spherical refracting surface.

$\angle ACA_g = \phi_g$, $\angle ACA_i = \phi_i$, $\angle A_gGB = \theta_g$, $\angle A_iIB = \theta_i$, $\angle CBG = \alpha$, $A_gG = v_g$, $A_iI = v_i$.

$$\tan\epsilon = \frac{y_g}{x_i - x_g}, \quad \tan\delta = \frac{z_i \cdot \cos\epsilon}{x_g - x_i}. \tag{194}$$

Moreover, let B designate the point where the ray meets the spherical refracting surface, and let us put

$$\angle A_g GB = \theta_g, \quad \angle A_i IB = \theta_i.$$

The angle θ_g may be determined from the following relation:

$$\cos\theta_g = \cos(\epsilon - \varphi_g)\cdot\cos\delta, \tag{195}$$

which may easily be derived from the figure; and the angle θ_i may be determined in terms of θ_g by means of the formula:

$$\sin\theta_i = \frac{v_g - r}{v_i - r}\sin\theta_g, \tag{196}$$

which may also be derived without difficulty.

The plane $A_g A_i C$ contains the incident ray GI and the incidence-normal BC, so that this plane is the plane of incidence. The radii $A_g C$, $A_i C$ both lie in this plane, as do also the line-segments denoted by v_g, v_i and the angles denoted by θ_g, θ_i; and, consequently, regarding $A_g C$ and $A_i C$ each as axes of the spherical surface, we have evidently the following relations exactly similar to the relations expressed by equation (177) and the second of equations (178):

$$\alpha = \theta_g + \angle BCA_g = \theta_i + \angle BCA_i, \tag{197}$$

and

$$-r\cdot\sin\alpha = (v_g - r)\sin\theta_g = (v_i - r)\sin\theta_i, \tag{198}$$

where α denotes the angle of incidence.

If in the figure the letters G and I are primed, the diagram will answer to show the corresponding case of a ray refracted at a spherical surface, and by priming all the symbols x, y, z, v, θ, α, ϵ and δ in the formulæ (192) to (198) above, we shall obtain the corresponding relations between the parameters of the refracted ray.

215. Method of L. Seidel.

Instead of determining the position of the ray by its points of intersection with two selected planes, L SEIDEL[1] makes use of only one such point, and, in place of the co-ordinates of a second point, employs two angular parameters to define the direction of the ray. The point of the ray which he selects is the point designated by H (Fig. 123) where the ray crosses the transversal (or *yz*-) plane. Moreover, instead of using the rectangular co-ordinates (y_h, z_h) of this point, he introduces a system of polar co-ordinates (p, π) in the *yz*-plane. Employing other symbols than those used by SEIDEL himself, we shall write:

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[1] L. SEIDEL: Trigonometrische Formeln für den allgemeinsten Fall der Brechung des Lichtes an centrirten sphaerischen Flaechen: *Sitzungsber. der math.-phys. Cl. der kgl. bayr. Akad. der Wissenschaften*, vom 10. Nov. 1866. Reprinted in Beilage III of STEINHEIL & VOIT'S *Handbuch der angewandten Optik*, Bd. I (Leipzig, B. G. TEUBNER, 1891), pages 257–270.

$$p = CH, \quad \pi = \angle HCy,$$

which magnitudes are connected with the rectangular co-ordinates of H by the following relations:

$$y_h = p \cdot \cos \pi, \quad z_h = p \cdot \sin \pi. \tag{199}$$

Both the radius-vector p and the polar angle π are to be considered as always positive in sign. The angle π, which may thus have any value comprised between 0° and 360°, may be defined as the angle through which CH has to be turned about C, always in the sense of positive rotation, in order that it may come into coincidence with the positive direction of the y-axis.

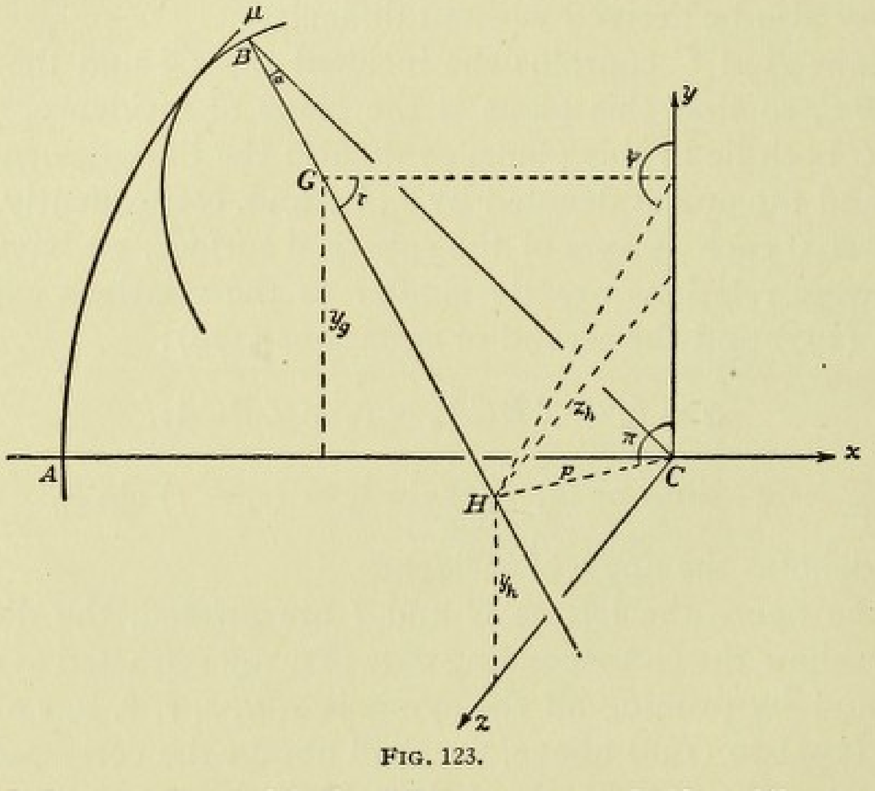

FIG. 123.

METHOD OF L. SEIDEL. The straight line BH represents a ray incident obliquely at the point B on a spherical refracting surface, whose centre is at the point designated by C. The optical axis coincides with the x-axis of co-ordinates, and the plane of the paper is the plane of a principal section (xy-plane); $A\mu$ being the section of the spherical surface made by this plane. CB is the incidence-normal, and ACB is the plane of incidence. The ray BH crosses the xy-plane at the point designated by G, and the yz-plane at the point designated by H. The polar co-ordinates of the point H are $p = CH$, $\pi = \angle HCy$. The angle at B is the angle of incidence α. The acute angle made by the ray with the x-axis is the angle denoted by τ; and the angle made by the projection of the ray on the yz-plane with the positive direction of the y-axis is the angle denoted by ψ.

One of the two angular magnitudes that define the direction of the ray is the acute angle (τ) between the direction of the ray and the positive direction of the x-axis; this angle being reckoned always as positive.

The other angular magnitude selected for this purpose by L. SEIDEL is the angle (ψ) made with the positive direction of the y-axis by the projection of the ray on the transversal (or yz-) plane. This angle, likewise, is always reckoned as positive, but it may have any value comprised between 0° and 360°.

If the direction-cosines of the straight line HI are denoted by α, β, γ, then

$$\frac{\alpha}{x_i} = -\frac{\beta}{y_h} = \frac{\gamma}{z_i - z_h},$$

and, since

$$\tan\psi = \gamma/\beta$$

(as may be easily verified), we obtain:

$$\tan\psi = \frac{z_h - z_i}{y_h}, \tag{200}$$

whereby the angle ψ is precisely defined.

Moreover, since

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \quad \text{and} \quad \alpha = \cos\tau,$$

we find (taking the minus sign, which is in agreement with the definitions above):

$$\beta = -\sin\tau\cdot\cos\psi;$$

and, hence:

$$\tan\tau = \frac{y_h}{x_i\cdot\cos\psi}, \tag{201}$$

or

$$\tan\tau = \frac{\sqrt{y_h^2 + (z_h - z_i)^2}}{x_i}, \tag{201a}$$

which is, therefore, the definition-equation of the angle τ.

An auxiliary angle (μ) is also employed in the calculation-scheme of L. SEIDEL. Let B designate the point where the ray meets the spherical surface; in the triangle BHC, the angle at H, but not necessarily the interior angle, is the angle denoted by μ. This angle, which is also reckoned as positive, may have any value comprised between 0° and 180°, and is defined exactly by the following formula:

$$\cos\mu = -\sin\tau\cdot\cos(\psi - \pi), \tag{202}$$

a relation which may easily be verified from the above definitions of the angles denoted by π, τ and ψ.

From the triangle BHC we derive also the following formula connecting the angle of incidence α at B and the auxiliary angle μ at H:

$$r \cdot \sin \alpha = p \cdot \sin \mu; \tag{203}$$

wherein it should be noted that, according to this formula, since by definition both p and $\sin \mu$ are positive magnitudes, and the angle α is an acute angle, *the sign of the angle α must be reckoned always as the same as the sign of the radius r.*[1]

The point where the refracted ray crosses the transversal yz-plane is designated, similarly, by H'; and if the symbols x, y, z, p, π, τ, ψ, μ and α in formulæ (199) to (203) above are primed, we shall obtain at once the relations between the corresponding magnitudes which relate to the refracted ray.

ART. 65. TRIGONOMETRIC COMPUTATION OF PATH OF RAY REFRACTED OBLIQUELY AT A SPHERICAL SURFACE.

216. The Refraction-Formulæ of A. Kerber.

The problem is as follows: Being given the rectangular co-ordinates (x_g, y_g) and (x_i, z_i) of the points G and I where the incident ray crosses the xy- and xz-planes, respectively, to determine the co-ordinates (x'_g, y'_g) and (x'_i, z'_i) of the corresponding points G' and I' where the refracted ray crosses these same planes.

By the Law of Refraction, we have:

$$n \cdot \sin \alpha = n' \cdot \sin \alpha';$$

and, moreover, since

$$\alpha = \theta_g + \angle BCA_g = \theta_i + \angle BCA_i, \quad \alpha' = \theta'_g + \angle BCA_g = \theta'_i + \angle BCA_i,$$

we have:

$$\alpha - \theta_g = \alpha' - \theta'_g, \quad \alpha - \theta_i = \alpha' - \theta'_i.$$

By means of these formulæ and the formulæ (192) to (198), we obtain A. Kerber's[2] *System of Refraction-Formulæ*, as follows:

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[1] This is practically equivalent to the method used by B. Wanach in a paper entitled *Ueber L. v. Seidel's Formeln zur Durchrechnung von Strahlen durch ein zentriertes Linsensystem, nebst Anwendung auf photographische Objective*, published in *Zeitschrift für Instrumentenkunde*, XX. (1900), pp. 162–171. In Seidel's formulæ, as originally published, the symbol R is used to denote the absolute value of the radius of the refracting surface, so that Seidel has to employ the double sign in order to include the cases of both convex and concave surfaces. Seidel adopted this method by preference, as being, in his opinion, practically the most convenient.

[2] A. Kerber: *Beitraege zur Dioptrik*. Zweites Heft. (Leipzig, Gustav Fock, 1896), pages 5–8.

$$\left.\begin{aligned}
&\tan\varphi_g = -y_g/x_g, \quad \tan\varphi_i = -z_i/x_i\,;\\
&\tan\epsilon = \frac{y_g}{x_i - x_g}; \qquad \tan\delta = \frac{z_i\cdot\cos\epsilon}{x_g - x_i};\\
&r - v_g = -\frac{x_g}{\cos\varphi_g}, \quad r - v_i = -\frac{x_i}{\cos\varphi_i};\\
&\cos\theta_g = \cos(\epsilon - \varphi_g)\cdot\cos\delta, \quad \sin\theta_i = \frac{r - v_g}{r - v_i}\sin\theta_g;\\
&\sin\alpha = \frac{r - v_g}{r}\sin\theta_g; \qquad \sin\alpha' = \frac{n}{n'}\sin\alpha;\\
&\theta'_g = \theta_g - \alpha + \alpha', \quad \theta'_i = \theta_i - \alpha + \alpha';\\
&r - v'_g = \frac{r\cdot\sin\alpha'}{\sin\theta'_g}, \quad r - v'_i = \frac{r\cdot\sin\alpha'}{\sin\theta'_i};\\
&x'_g = -(r - v'_g)\cdot\cos\varphi_g, \quad x'_i = -(r - v'_i)\cdot\cos\varphi_i;\\
&y'_g = -x'_g\cdot\tan\varphi_g, \quad z'_i = -x'_i\cdot\tan\varphi_i.
\end{aligned}\right\} \tag{204}$$

217. In the special case of a *Plane Refracting Surface*, the centre C is the infinitely distant point of the optical axis, and, hence, *the origin of co-ordinates will have to be shifted from C to the point A* where the optical axis meets the refracting plane, which is effected very simply by writing $x - r$ in place of x. If we do this, and then put $r = \infty$, KERBER's Formulæ for a Plane Refracting Surface will be found. as follows:

$$\left.\begin{aligned}
&\varphi_g = \varphi_i = 0;\\
&\tan\epsilon = \frac{y_g}{x_i - x_g}, \quad \tan\delta = \frac{z_i\cdot\cos\epsilon}{x_g - x_i};\\
&v_g = x_g, \quad v_i = x_i;\\
&\cos\theta_g = \cos\epsilon\cdot\cos\delta;\\
&\theta_i = \theta_g, \quad \theta'_g = \theta'_i;\\
&\sin\theta'_g = \frac{n}{n'}\sin\theta_g;\\
&v'_g = v_g\frac{\tan\theta_g}{\tan\theta'_g}, \quad v'_i = v_i\frac{\tan\theta_i}{\tan\theta'_i};\\
&x'_g = v'_g; \quad x'_i = v'_i;\\
&y'_g = y_g; \quad z'_i = z_i.
\end{aligned}\right\} \tag{205}$$

218. In case the angle θ_g is very small, the determination of this angle by means of the formula

$$\cos\theta_g = \cos(\epsilon - \varphi_g)\cdot\cos\delta$$

is not satisfactory, and a greater numerical accuracy will be possible by determining, first, the value of the angle β between the plane of incidence and the vertical plane of the Principal Section by means of the following formula:[1]

$$\tan\beta = \frac{\tan\delta}{\sin(\epsilon - \varphi_g)}; \tag{206}$$

whence we can find afterwards:

$$\sin\theta_g = -\frac{\sin\delta}{\sin\beta}. \tag{207}$$

In connection with KERBER's Refraction-Formulæ, the following suggestion, also due to Messrs. KOENIG and VON ROHR,[2] is worthy of remark:

By taking as the ray-parameters the co-ordinates x_g, y_g and the angular magnitudes denoted by δ and ϵ, the calculation of all of the magnitudes denoted above by symbols with the subscript i can be entirely avoided. Since, by (207), we have:

$$\sin\beta = -\frac{\sin\delta}{\sin\theta_g} = -\frac{\sin\delta'}{\sin\theta'_g},$$

we obtain:

$$\sin\delta' = \frac{\sin\theta'_g}{\sin\theta_g}\sin\delta, \tag{208}$$

whereby we can determine the angle δ'; and the value of the angle ϵ' may be found by the formula:

$$\cos(\epsilon' - \varphi_g) = \frac{\cos\theta'_g}{\cos\delta'},$$

or by the formula:

$$\sin(\epsilon' - \varphi_g) = \frac{\tan\delta'}{\tan\beta}.$$

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[1] This suggestion is found in *Die Theorie der optischen Instrumente* (Berlin, JULIUS SPRINGER, 1904), Bd. I, II Kapitel, "Die Durchrechnungsformeln": von A. KOENIG und M. VON ROHR, p. 65.

[2] Same reference as preceding.

219. The Refraction-Formulæ of L. Seidel.[1]

Here the problem is as follows: Being given the angular magnitudes (τ, ψ), which define the direction of the incident ray, and the polar co-ordinates (p, π) of the point H where this ray crosses the yz-plane, to find the corresponding parameters (τ', ψ') and (p', π') of the refracted ray.

Since the plane of the triangle BHC contains the incident ray BH and the incidence-normal BC, this is the plane of incidence, which likewise, therefore, contains the refracted ray BH'. That is, the two planes BHC and $BH'C$ coincide, and, consequently, their lines of intersection with the yz-plane coincide also. Hence, the three points C, H and H' all lie on one and the same straight line; accordingly, the radii vectores CH, CH' have the same (or opposite) directions, so that the polar angles π, π' are either equal or differ by 180°. In the case of a refracting surface, we shall have:

$$\pi' = \pi;$$

and for a reflecting surface:

$$\pi' = 180^\circ + \pi.$$

By formula (203), we have:

$$r \cdot \sin \alpha = p \cdot \sin \mu, \quad r \cdot \sin \alpha' = p' \cdot \sin \mu',$$

where μ, μ' are the two auxiliary angles at the vertices H, H' of the triangles BHC, $BH'C$; and hence, by the Law of Refraction, we derive the invariant relation:

$$n \cdot p \cdot \sin \mu = n' \cdot p' \cdot \sin \mu'. \tag{209}$$

Moreover, since the angle at C is common to these two triangles, we obtain also another invariant relation as follows:

$$\mu + \alpha = \mu' + \alpha'. \tag{210}$$

By means of the above formulæ, the position of the point H' may be determined.

Still another invariant relation, depending on the fact that the plane of incidence and the plane determined by the optical axis and the radius vector CH coincide with the plane of refraction and the plane determined by the optical axis and the radius vector CH', respectively, which may also be easily derived, is the following:

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[1] L. v. SEIDEL: Trigonometrische Formeln für den allgemeinsten Fall der Brechung des Lichtes an centrierten sphaerischen Flaechen: *Sitzungsber. der math.-phys. Cl. der kgl. bayr. Akad. der Wissenschaften*, vom 10. Nov. 1866. Reprinted in Beilage III of Bd. I of STEINHEIL & VOIT's *Handbuch der angewandten Optik* (Leipzig, B. G. TEUBNER, 1891).

$$\frac{\sin \tau' \cdot \sin (\psi' - \pi)}{\sin \mu'} = \frac{\sin \tau \cdot \sin (\psi - \pi)}{\sin \mu}. \tag{211}$$

Moreover, the following relation, also, is obvious:

$$\frac{\cos \tau'}{\sin \mu'} = \frac{\cos \tau}{\sin \mu};$$

but this formula, which is convenient by reason of its simplicity, is not a very practical formula for numerical calculation in case the angles τ, τ' are small, as, in fact, they usually are. But if we combine this formula with (211), we obtain:

$$\tan \tau' \cdot \sin (\psi' - \pi) = \tan \tau \cdot \sin (\psi - \pi); \tag{212}$$

whereby we can find the tangent of the angle τ'.

Finally, arranging the above formulæ in the order in which they are used, we have L. SEIDEL's *Calculation-Scheme* for determining the refracted ray corresponding to a ray incident obliquely on a spherical refracting surface, as follows:

(1) *Determination of the Position of H′ by means of its Polar Co-ordinates* (p', π'):

$$\left.\begin{aligned}
\cos \mu &= -\sin \tau \cdot \cos (\psi - \pi), \\
\sin \alpha &= \frac{p \cdot \sin \mu}{r}, \\
\sin \alpha' &= \frac{n \cdot \sin \alpha}{n'}, \\
\mu' &= \mu + \alpha - \alpha', \\
p' &= r \cdot \frac{\sin \alpha'}{\sin \mu'} = p \cdot \frac{n}{n'} \frac{\sin \mu}{\sin \mu'}, \\
\pi' &= \pi.
\end{aligned}\right\} \tag{213}$$

Note.—As we shall have to calculate below the quotient $\sin \mu' / \sin \mu$, it is worth while to compute the value of p' by means of each of the two formulæ above; as this will afford us some way of checking the values obtained for the angles α, α'.

(2) *Determination of the Direction* (τ', ψ') *of the Refracted Ray:*

$$\left.\begin{aligned}\sin\tau'\cdot\sin(\psi'-\pi) &= \frac{\sin\mu'}{\sin\mu}\cdot\sin\tau\cdot\sin(\psi-\pi),\\ \tan(\pi-\psi') &= \frac{\sin\tau'\cdot\sin(\psi'-\pi)}{\cos\mu'},\\ \tan\tau' &= \tan\tau\frac{\sin(\psi-\pi)}{\sin(\psi'-\pi)}.\end{aligned}\right\} \qquad (214)$$

Note.—The second of these formulæ is obtained by combining formula (212) with the formula:

$$\cos\mu' = -\sin\tau'\cdot\cos(\psi'-\pi);$$

and it enables us to find the magnitude of the angle ψ'.

220. In the special case of a *Plane Refracting Surface*, for which the centre C is the infinitely distant point of the optical axis, the plane surface must be taken for the yz-plane, and hence the three points B, H and H' coincide. Accordingly, for this special case we have:

$$p' = p, \quad \pi' = \pi.$$

And since the incidence-normal is parallel to the optical axis, we have also $\alpha = \tau$, $\alpha' = \tau'$; and, therefore,

$$\sin\tau' = \frac{n}{n'}\sin\tau$$

is the equation for determining the magnitude of the angle τ'. Moreover, since both the incident and refracted rays lie in the plane of incidence, containing the incidence-normal, which here is parallel to the x-axis, the projections of these rays on the yz-plane must coincide with each other; and, therefore,

$$\psi' = \psi.$$

By means of the above equations, we can find the four parameters p', π', τ' and ψ' of a ray refracted at a Plane Surface.

CHAPTER X.

TRIGONOMETRIC FORMULÆ FOR CALCULATING THE PATH OF A RAY REFRACTED THROUGH A CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES.

Case I. When the Ray Lies in the Plane of a Principal Section.

ART. 66. CALCULATION-SCHEME FOR THE PATH OF A RAY LYING IN THE PLANE OF A PRINCIPAL SECTION OF A CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES.

221. In order to compute the path of a ray, which undergoes successive refractions (or reflexions) at a series of centered spherical surfaces, whose optical axis lies in the plane of incidence of the first surface, it is sufficient to obtain the set of formulæ for one of the surfaces, say, the *k*th; for the calculation will consist merely in the repeated employment of this set of formulæ for each of the surfaces in succession. We shall require also a so-called "transformation-formula", which will enable us to pass from one surface to the next.

The points where the ray crosses the optical axis, before and after refraction at the *k*th surface, will be designated by L'_{k-1}, L'_k, respectively, and the abscissæ of these points, with respect to the vertex A_k (Fig. 124) of the *k*th surface, will be denoted by v_k, v'_k; thus,

$$A_kL'_{k-1} = v_k; \quad A_kL'_k = v'_k.$$

The "transformation-formula", by which we transform from the origin of abscissæ A_k of the *k*th surface to the origin A_{k+1} of the next surface is:

$$d_k = v'_k - v_{k+1};$$

where $d_k = A_kA_{k+1}$ denotes the so-called "thickness" of the medium which lies between the *k*th and $(k + 1)$th spherical surfaces, and whose absolute index of refraction is denoted by n'_k.

The radius of the *k*th surface will be denoted by r_k $(= A_kC_k)$; and the angles of incidence and refraction at the *k*th surface will be denoted by α_k, α'_k. The "slope"-angles of the ray before and after refraction at the *k*th surface will be denoted by θ'_{k-1}, θ'_k, respectively; thus,

$$\angle A_{k-1}L'_{k-1}B_{k-1} = \theta'_{k-1}, \quad \angle A_kL'_kB_k = \theta'_k.$$

The following system of formulæ (see § 211) may now be written:

$$\left.\begin{aligned} \sin\alpha_k &= \left(1-\frac{v_k}{r_k}\right)\cdot\sin\theta'_{k-1}\\ \sin\alpha'_k &= \frac{n'_{k-1}}{n'_k}\sin\alpha_k,\\ \theta'_k &= \theta'_{k-1}+\alpha'_k-\alpha_k,\\ v'_k &= r_k\cdot\left(1-\frac{\sin\alpha'_k}{\sin\theta'_k}\right),\\ v_{k+1} &= v'_k-d_k. \end{aligned}\right\} \qquad (215)$$

In these formulæ we must give k in succession all integral values from $k = 1$ to $k = m$, where m denotes the total number of spherical surfaces. If we know the values of the constants n'_{k-1}, n'_k and r_k for the kth refracting surface, and if we have determined the ray-co-ordinates v_k, θ'_{k-1} of the ray incident on this surface, the first four of the formulæ (215) above enable us to find the ray-co-ordinates v'_k, θ'_k of the ray after refraction at the kth surface; whereas the last of these formulæ enables us to pass to the next surface, provided we know the axial "thickness" d_k between the kth and the $(k+1)$th surfaces. Thus, having found the magnitude v_{k+1}, we can proceed to make the same calculation for the $(k+1)$th surface, and so on, until we obtain, finally, the co-ordinates of the emergent ray, viz., $v'_m = A_mL'_m$, $\theta'_m = \angle A_mL'_mB_m$. An actual numerical example, illustrating the calculation-process by means of formulæ (215), is given in Art. 67.

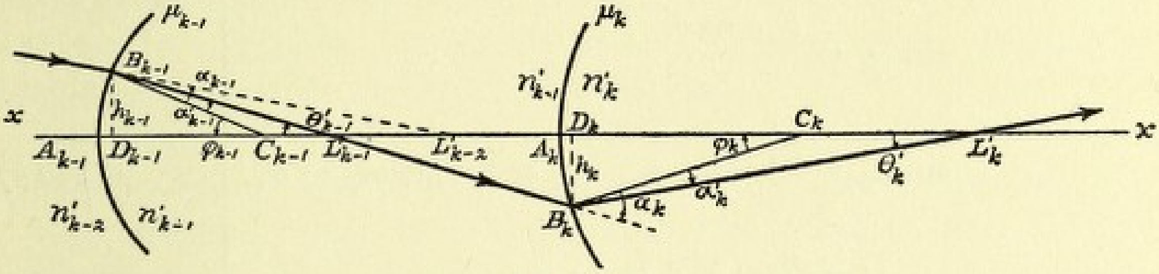

FIG. 124.

PATH OF A RAY IN A PRINCIPAL SECTION OF A CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES.

$A_kL'_{k-1} = v_k$, $A_kL'_k = v'_k$, $A_kC_k = r_k$, $D_kB_k = h_k$, $A_{k-1}A_k = d_{k-1}$, $B_{k-1}B_k = \delta_{k-1}$,
$B_kL'_{k-1} = l_k$, $B_kL'_k = l'_k$, $\angle A_{k-1}L'_{k-1}B_{k-1} = \theta'_{k-1}$, $\angle A_kL'_kB_k = \theta'_k$, $\angle B_kC_kA_k = \phi_k$.

222. We have also a number of other relations, which are often very useful and convenient. Thus, if the symbols φ_k, h_k, l_k, l'_k have the following significations:

$$\varphi_k = \angle B_k C_k A_k,\ h_k = D_k B_k,\ l_k = B_k L'_{k-1},\ l'_k = B_k L'_k,$$

where the letters designate the points shown in the diagram (Fig. 124), we have immediately, in connection with formulæ (215):

$$\left.\begin{aligned} & h_k = r_k \cdot \sin(\alpha_k - \theta'_{k-1}), \\ & l_k = -\,h_k/\sin\theta'_{k-1},\quad l'_k = -\,h_k/\sin\theta'_k, \\ & \varphi_k = \alpha_k - \theta'_{k-1} = \alpha'_k - \theta'_k. \end{aligned}\right\} \tag{216}$$

223. If the position of the ray is defined by its "slope" (θ'_{k-1}) and its intercept b_k ($= C_k H_k$) on the "central perpendicular", we obtain (see § 211, Note 4) the following calculation-scheme:

$$\left.\begin{aligned} \sin\alpha_k &= \frac{b_k \cdot \cos\theta'_{k-1}}{r_k} \\ \sin\alpha'_k &= \frac{n'_{k-1}}{n'_k}\sin\alpha_k,\quad \theta'_k = \theta'_{k-1} + \alpha'_k - \alpha_k, \\ b'_k &= \frac{n'_{k-1}}{n'_k}\,\frac{\cos\theta'_{k-1}}{\cos\theta'_k}\,b_k; \end{aligned}\right\} \tag{217}$$

together with the following "transformation-formula", for passing from the kth to the $(k+1)$th surface:

$$b_{k+1} = b'_k + a_k \cdot \tan\theta'_k; \tag{218}$$

where

$$a_k = d_k + r_{k+1} - r_k \tag{219}$$

denotes the abscissa of the centre C_{k+1} with respect to the centre C_k; that is, $a_k = C_k C_{k+1}$.

The relation between the intercepts b_k and v_k is given by the following formula:

$$b_k = (r_k - v_k)\tan\theta'_{k-1}. \tag{220}$$

ART. 67. NUMERICAL ILLUSTRATION.

224. By means of the formulæ (151), we can find the position of the image-point M'_m, which corresponds by Paraxial Rays with the axial object-point M_1 (or L_1), and by means of formulæ (215) above we can determine the position on the axis of the point L'_m where the extreme outside ray, or so-called "edge-ray", of the bundle of rays crosses the optical axis after emerging from the centered system of m spherical refracting surfaces: and thus we can compute the longitudinal aberration along the axis:

$$M'_m L'_m = v'_m - u'_m.$$

In practice this is found to be a very useful way of computing the magnitude of this aberration, especially in the case of optical systems of comparatively wide apertures, to which the theory of aberrations of the first order does not apply very well. By repeated trials in this fashion, it is possible, also, to discover how the thicknesses and radii will have to be altered so that, for example, the edge-ray will emerge so as to cross the optical axis at a point L'_m, which coincides, very nearly at least, with the so-called "GAUSSian" image-point M'_m; in which case for this pair of rays (that is, for a paraxial ray and the edge-ray), we shall have $v'_m - u'_m = 0$, approximately. In the design of optical instruments this calculation-process is found to be extremely serviceable. In order to exhibit the use of the formulæ, we shall give here a rather simple numerical illustration.

For this purpose, we shall select an example given in TAYLOR'S *System of Applied Optics* (London, 1906), page 101, as follows:

The optical system is a large Telescope Object-Glass, of 12-in. aperture ($h_1 = 6$ in.), consisting of a biconvex crown-glass lens and a biconcave flint-glass lens, with the following radii and thicknesses (all measured in inches):

$$r_1 = +59.8;\quad d_1 = +1;\quad r_2 = -90.15;\quad d_2 = 0.013;$$
$$r_3 = -84.7;\quad d_3 = +1;\quad \text{and}\quad r_4 = +410.$$

The values of the refractive indices, for rays corresponding to the FRAUNHOFER-Line C, are as follows:

$$n_1 = n'_2 = n'_4 = 1;\quad n'_1 = 1.5146;\quad n'_3 = 1.6121.$$

The incident rays are parallel to the optical axis, so that

$$u_1 = v_1 = \infty, \text{ and } \theta_1 = 0.$$

According to the first of formulæ (216), we have, therefore, in such a case as this:

$$\sin\alpha_1 = \frac{h_1}{r_1},\quad (v_1 = \infty), \tag{221}$$

which is the formula we must employ here in order to determine the value of α_1.

The calculation will be divided into two parts, as follows:

(1) The calculation of the Path of a Paraxial Ray, by means of formulæ (151); and

(2) The trigonometric calculation of the Path of the Edge-Ray by means of formulæ (215) above, together also with formula (221) above.

The sign + or − written after a logarithm indicates the sign of the number to which the logarithm belongs. Each vertical column contains the calculation for one surface: accordingly, in the present example, where there are four refracting surfaces, each table will contain four such columns.

For the Edge-Ray: $h_1 = 6$ inches, $v_1 = \infty$ and $\theta_1 = 0$: hence, according to formula (221) above, we have:

$$\begin{aligned} \lg h_1 &= 0.7781513 + \\ \operatorname{clg} r_1 &= \underline{8.2232988 +} \\ \lg \sin \alpha_1 &= 9.0014501 + \end{aligned}$$

This forms the starting point for the calculation of this ray.

The two parts of the calculation follow.

I. *Paraxial Ray:* $u_1 = \infty$.

Formulæ:

$$\frac{1}{u'_k} = \frac{1}{u_k} \cdot \frac{n'_{k-1}}{n'_k} + \frac{1}{r_k} \cdot \frac{n'_k - n'_{k-1}}{n'_k},$$

$$\frac{1}{u_{k+1}} = \frac{1}{u'_k} \cdot \frac{1}{1 - d_k/u'_k}.$$

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$\operatorname{clg} u_k$		7.7569451+	8.1573196+	7.6481546+
$\lg (n'_{k-1}/n'_k)$		0.1802980+	9.7926080+	0.2073920+
$\lg (n'_{k-1}/u_k n'_k)$		7.9372431+	7.9499276+	7.8555466+
$\operatorname{clg} r_k$	8.2232988+	8.0450343−	8.0721166−	7.3872161+
$\lg (n'_k - n'_{k-1})$	9.7114698+	9.7114698−	9.7868224+	9.7868224−
$\operatorname{clg} n'_k$	9.8197020+	0.0000000	9.7926080+	0.0000000
$\lg \frac{n'_k - n'_{k-1}}{r_k n'_k}$	7.7544706+	7.7565041+	7.6515470−	7.1740385−
$n'_{k-1}/u_k n'_k$	0.0000000	+0.0086545	+0.0089110	+0.0071705
$(n'_k - n'_{k-1})/r_k n'_k$	+0.0056816	+0.0057083	−0.0044828	−0.0014929
$1/u'_k$	+0.0056816	+0.0143628	+0.0044282	+0.0056776
$\operatorname{clg} u'_k$	7.7544706+	8.1572385+	7.6462272+	7.7541648+
$\lg d_k$	0.0000000	8.1139434+	0.0000000	
$\lg d_k/u'_k$	7.7544706+	6.2711819+	7.6462272+	
$1 - d_k/u'_k$	+0.9943184	+0.9998133	+0.9955718	
$\operatorname{clg} u'_k$	7.7544706+	8.1572385+	7.6462272+	
$\operatorname{clg} (1 - d_k/u'_k)$	0.0024745+	0.0000811+	0.0019274+	
$\lg 1/u_{k+1}$	7.7569451+	8.1573196+	7.6481546+	

$u'_4 = + \underline{\underline{176.13077}}$ in.

Formula for the Focal Length e':

$$1/e' = -(1 - d_1/u'_1)(1 - d_2/u'_2)(1 - d_3/u'_3)(1/u'_4).$$

$$\begin{aligned} \lg(1 - d_1/u'_1) &= 9.9975255+ \\ \lg(1 - d_2/u'_2) &= 9.9999189+ \\ \lg(1 - d_3/u'_3) &= 9.9980726+ \\ \operatorname{clg} u'_4 &= \underline{7.7541648}+ \\ \operatorname{clg} e' &= \underline{7.7496818}- \\ e' &= -177.9583 \text{ inches.} \end{aligned}$$

II. *Edge-Ray:* See formulæ (215) of this Chapter.

	$k=1$	$k=2$	$k=3$	$k=4$
$\lg(1 - v_k/r_k)$		0.4679039+	0.2585662+	9.6547511+
$\lg\sin\theta'_{k-1}$		8.5340672−	8.9356090−	8.4225418−
$\lg\sin\alpha_k$	9.0014501+	9.0019711−	9.1941752−	8.0772929−
$\lg n'_{k-1}/n'_k$	9.8197020+	0.1802980+	9.7926080+	0.2073920+
$\lg\sin\alpha'_k$	8.8211521+	9.1822691−	8.9867832−	8.2846849−
θ'_{k-1}	0° 0′ 0″	−1°57′36″.3	−4°56′46″.3	−1°30′57″.8
$-\alpha_k$	−5°45′30″.3	+5°45′55″.3	+8°59′48″.2	+0°41′ 4″.5
$\theta'_{k-1} - \alpha_k$	−5°45′30″.3	+3°48′19″.0	+4° 3′ 1″.9	−0°49′53″.3
α'_k	+3°47′54″.0	−8°45′ 5″.3	−5°33′59″.7	−1° 6′13″.2
θ'_k	−1°57′36″.3	−4°56′46″.3	−1°30′57″.8	−1°56′ 6″.5
$\lg\sin\alpha'_k$	8.8211521+	9.1822691−	8.9867832−	8.2846849−
$\operatorname{clg}\sin\theta'_k$	1.4659328−	1.0643910−	1.5774582−	1.4715561−
$\lg(\sin\alpha'_k/\sin\theta'_k)$	0.2870849−	0.2466601+	0.5642414+	9.7562410+
$\sin\alpha'_k/\sin\theta'_k$	−1.9368	+1.764655	+3.666414	+0.570481
$1 - \sin\alpha'_k/\sin\theta'_k$	+2.9368	−0.764655	−2.666414	+0.429519
$\lg\left(1 - \dfrac{\sin\alpha'_k}{\sin\theta'_k}\right)$	0.4678744+	9.8834656−	0.4259276−	9.6330430+
$\lg r_k$	1.7767012+	1.9549657−	1.9278834−	2.6127839+
$\lg v'_k$	2.2445756+	1.8384313+	2.3538110+	2.2458269+
v'_k	+175.6206	+68.93365	+225.8452	$\underline{\underline{+176.1273}}$
$-d_k$	− 1.0000	− 0.013	− 1.0000	
v_{k+1}	+174.6206	+68.92065	+224.8452	
$\lg v_{k+1}$	2.2420955+	1.8383494+	2.3518836+	
$\operatorname{clg} r_{k+1}$	8.0450343−	8.0721166−	7.3872161+	
$\lg v_{k+1}/r_{k+1}$	0.2871298−	9.9104660−	9.7390997+	
v_{k+1}/r_{k+1}	−1.9370	−0.813703	+0.548403	
$1 - v_{k+1}/r_{k+1}$	+2.9370	+1.813703	+0.451597	

$$M'_4L'_4 = v'_4 - u'_4 = -\underline{\underline{0.0035 \text{ inches.}}}$$

Case II. When the Path of the Ray does not Lie in the Plane of a Principal Section of the Centered System of Spherical Refracting Surfaces.

ART. 68. TRIGONOMETRIC FORMULÆ OF A. KERBER FOR CALCULATING THE PATH OF AN OBLIQUE RAY THROUGH A CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES.

225. In the calculation-scheme of A. Kerber[1] (see §§ 214 and 216) the parameters of the ray before refraction at the kth surface of the system of spherical refracting surfaces are the co-ordinates $(x_{g,k}, y_{g,k})$ and $(x_{i,k}, z_{i,k})$ of the points G_k and I_k where the ray crosses the two meridian co-ordinate planes, viz., the xy-plane and the xz-plane, respectively; and, similarly, the parameters of the ray after refraction at this surface are the co-ordinates $(x'_{g,k}, y'_{g,k})$ and $(x'_{i,k}, z'_{i,k})$ of the points G'_k (or G_{k+1}) and I'_k (or I_{k+1}) where the refracted ray crosses the xy- and xz-planes, respectively. In order to obtain the refraction-formulæ for the kth surface, we have merely to affix to the symbols in formulæ (204) the k-subscript to indicate that the formulæ are to be applied to the kth refracting surface.

It will also be necessary to obtain a system of "*Transformation-Formulæ*", whereby, having ascertained the values of the co-ordinates $(x'_{g,k}, y'_{g,k})$ and $(x'_{i,k}, z'_{i,k})$ of the points G'_k (or G_{k+1}) and I'_k (or I_{k+1}), referred to the centre C_k of the kth surface as origin, we can compute the values of the co-ordinates $(x_{g,k+1}, y_{g,k+1})$ and $(x_{i,k+1}, z_{i,k+1})$ of these same points referred to the centre C_{k+1} of the $(k+1)$th surface as origin. This shifting of the origin along the x-axis will affect only the x-co-ordinates. Thus, evidently, we shall have:

$$x_{g,k+1} = x'_{g,k} - a_k, \quad x_{i,k+1} = x'_{i,k} - a_k,$$

where

$$a_k = C_k C_{k+1} = d_k + r_{k+1} - r_k. \tag{222}$$

Accordingly, in the *Calculation-Scheme* of A. Kerber, we have the following system of formulæ:

(1) *Refraction-Formulæ for Finding the Values of the Parameters* $x'_{g,k}$, $y'_{g,k}$, $x'_{i,k}$ *and* $z'_{i,k}$ *of the Ray After Refraction at the kth surface:*

$$\left.\begin{aligned} \tan\varphi_{g,k} = -\,y_{g,k}/x_{g,k}, \quad \tan\varphi_{i,k} &= -\,z_{i,k}/x_{i,k}; \\ \tan\epsilon'_{k-1} &= \frac{y_{g,k}}{x_{i,k} - x_{g,k}}; \\ \tan\delta'_{k-1} &= \frac{z_{i,k}\cdot\cos\epsilon'_{k-1}}{x_{g,k} - x_{i,k}}; \end{aligned}\right\} \tag{223}$$

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[1] A. Kerber: *Beitraege zur Dioptrik*, Heft II (Leipzig, Gustav Fock, 1896), pages 5–8.

$$\left.\begin{aligned}
r_k - v_{g,k} = -\frac{x_{g,k}}{\cos\varphi_{g,k}}, \quad r_k - v_{i,k} = -\frac{x_{i,k}}{\cos\varphi_{i,k}};\\
\cos\theta'_{g,k-1} = \cos(\epsilon'_{k-1} - \varphi_{g,k})\cdot\cos\delta'_{k-1};\\
\sin\theta'_{i,k-1} = \frac{r_k - v_{g,k}}{r_k - v_{i,k}}\sin\theta'_{g,k-1};\\
\sin\alpha_k = \frac{r_k - v_{g,k}}{r_k}\sin\theta'_{k-1};\\
\sin\alpha'_k = \frac{n'_{k-1}}{n'_k}\sin\alpha_k;\\
\theta'_{g,k} = \theta'_{g,k-1} - \alpha_k + \alpha'_k, \quad \theta'_{i,k} = \theta'_{i,k-1} - \alpha_k + \alpha'_k;\\
r_k - v'_{g,k} = \frac{r_k\cdot\sin\alpha'_k}{\sin\theta'_{g,k}}, \quad r_k - v'_{i,k} = \frac{r_k\cdot\sin\alpha'_k}{\sin\theta'_{i,k}};\\
x'_{g,k} = -(r_k - v'_{g,k})\cdot\cos\varphi_{g,k}, \quad x'_{i,k} = -(r_k - v'_{i,k})\cdot\cos\varphi_{i,k};\\
y'_{g,k} = -x'_{g,k}\cdot\tan\varphi_{g,k}, \quad z'_{i,k} = -x'_{i,k}\cdot\tan\varphi_{i,k}.
\end{aligned}\right\}\text{(223, continued)}$$

(2) *Transformation-Formulæ for Determining the Parameters* $x_{g,k+1}$, $y_{g,k+1}$, $x_{i,k+1}$ *and* $z_{i,k+1}$ *of the Ray Before Refraction at the* $(k+1)$*th surface:*

$$\left.\begin{aligned}
x_{g,k+1} = x'_{g,k} + r_k - r_{k+1} - d_k, \quad x_{i,k+1} = x'_{i,k} + r_k - r_{k+1} - d_k;\\
y_{g,k+1} = y'_{g,k}, \quad z_{i,k+1} = z'_{i,k}.
\end{aligned}\right\}\quad(224)$$

226. The Initial Values.

The position of the ray incident on the first surface of the centered system of spherical refracting surfaces will be defined generally by giving the co-ordinates of the object-point P_1, whence the ray emanates, and the co-ordinates of the point $\boldsymbol{P}_1$, where the ray crosses the plane of the so-called "Entrance-Pupil" (see § 257 and § 361). Usually, it will be possible to select as the plane of the principal section (xy-plane) the meridian plane of the optical system which contains the object-point P_1, so that this point will, therefore, coincide with the point designated by G_1. If M_1 designates the foot of the perpendicular let fall from P_1 on the optical axis, and if we put

$$u_1 = A_1M_1, \quad \eta_1 = M_1P_1,$$

the co-ordinates of the point P_1, referred to a system of rectangular axes with origin at C_1, will be:

$$x_{g,1} = C_1M_1 = u_1 - r_1, \quad y_{g,1} = M_1P_1 = \eta_1, \quad z_{g,1} = 0.$$

In every actual optical instrument the angular opening of the bundle of "effective" rays, which, emanating from the object-point P_1, traverse the system of lenses, is limited in some way, usually by a "stop", consisting of a plane screen perpendicular to the optical axis with a circular opening in it, whose centre (called the "stop-centre") lies on the optical axis of the instrument. Even when no screen of this description is employed, the cone of effective rays will be determined by the rim of one of the glasses — in some instances, also, by the iris of the eye of the observer. The "stop" is not always situated in front of the entire system of lenses; it may lie between one pair of them, or it may even be placed beyond them all. Let us take the most general case and assume that the "stop" is situated between, say, the bth and the $(b+1)$th surfaces of the system of m spherical surfaces, and let us designate the position of the stop-centre by $\boldsymbol{M}'_b$. This point $\boldsymbol{M}'_b$ will be the image, formed by Paraxial Rays, after having traversed the first b surfaces of the system, of a certain axial object-point $\boldsymbol{M}_1$; which latter point is the centre of the so-called "Entrance-Pupil". The transversal plane $\boldsymbol{\sigma}_1$ perpendicular to the optical axis at $\boldsymbol{M}_1$ (which in any given optical system will always be a perfectly definite plane) is the Plane of the Entrance-Pupil. And the point where an object-ray, emanating from the object-point P_1, crosses this plane will be designated here by $\boldsymbol{P}_1$, as has been stated above. Moreover, we shall put $A_1\boldsymbol{M}_1 = \boldsymbol{u}_1$, and shall denote the co-ordinates of $\boldsymbol{P}_1$, referred to rectangular axes with C_1 as origin, as follows:

$$\boldsymbol{u}_1 - r_1,\ \boldsymbol{\eta}_1,\ \boldsymbol{\zeta}_1.$$

As has been remarked, the position of the object-ray is usually given by assigning the values of the magnitudes denoted here by the symbols u_1, η_1 and $\boldsymbol{\eta}_1$, $\boldsymbol{\zeta}_1$. By drawing a simple diagram, the reader will easily perceive that, if K designates the projection of the point I_1 on the xy-plane ($C_1K = x_{i,1}$, $KI_1 = z_{i,1}$), we have the following relations:

$$\frac{\eta_1}{\boldsymbol{\eta}_1} = \frac{M_1K}{\boldsymbol{M}_1K}, \quad \frac{KI_1}{\boldsymbol{\zeta}_1} = \frac{M_1K}{M_1\boldsymbol{M}_1};$$

whence, since

$$M_1K = M_1A_1 + A_1C_1 + C_1K = x_{i,1} + r_1 - u_1,$$
$$\boldsymbol{M}_1K = \boldsymbol{M}_1A_1 + A_1C_1 + C_1K = x_{i,1} + r_1 - \boldsymbol{u}_1,$$

and

$$M_1\boldsymbol{M}_1 = M_1A_1 + A_1\boldsymbol{M}_1 = \boldsymbol{u}_1 - u_1,$$

we obtain:

$$\frac{\eta_1}{\boldsymbol{\eta}_1} = \frac{x_{i,1} + r_1 - u_1}{x_{i,1} + r_1 - \boldsymbol{u}_1}, \quad \frac{z_{i,1}}{\boldsymbol{\zeta}_1} = \frac{x_{i,1} + r_1 - u_1}{\boldsymbol{u}_1 - u_1}.$$

Thus, we obtain the initial values $x_{i,1}$, $z_{i,1}$ as follows:

$$\left.\begin{aligned} x_{i,1} &= \frac{\eta_1 \boldsymbol{u}_1 - \boldsymbol{\eta}_1 u_1}{\eta_1 - \boldsymbol{\eta}_1} - r_1, \\ z_{i,1} &= \frac{x_{i,1} + r_1 - u_1}{\boldsymbol{u}_1 - u_1} \boldsymbol{\zeta}_1 = \frac{\eta_1}{\eta_1 - \boldsymbol{\eta}_1} \boldsymbol{\zeta}_1. \end{aligned}\right\} \qquad (225)$$

In case *the object-point* P_1 *is infinitely distant*, the object-rays will constitute a bundle of parallel rays; and, since, in general, η_1, as well as u_1, will be infinite, the value of $x_{i,1}$, as given by the first of formulæ (225), will be illusory. Under these circumstances, we shall require to know the direction of the object-ray, and, since all the object-rays proceeding from one and the same point of the object are parallel, it will be sufficient if we are given the slope-angle $\boldsymbol{\theta}_1$ of that one of the bundle of object-rays which crosses the optical axis at the centre $\boldsymbol{M}_1$ of the Entrance-Pupil.[1] Now, evidently,

$$\tan \boldsymbol{\theta}_1 = \frac{\eta_1}{\boldsymbol{M}_1 M_1} = \frac{\eta_1}{u_1 - \boldsymbol{u}_1};$$

if, therefore, in the expression for $x_{i,1}$ given in (225), we substitute the value of the ratio u_1/η_1, as obtained from this last equation, and then put $u_1 = \eta_1 = \infty$, we shall derive the first of the two following formulæ:

$$\left.\begin{aligned} x_{i,1} &= \boldsymbol{u}_1 - \boldsymbol{\eta}_1 \cdot \cot \boldsymbol{\theta}_1 - r_1; \\ z_{i,1} &= \boldsymbol{\zeta}_1. \end{aligned}\right\} \qquad (u_1 = \infty) \quad (226)$$

The latter formula is obvious immediately from the second of formulæ (225).

ART. 69. THE TRIGONOMETRIC FORMULÆ OF L. SEIDEL FOR CALCULATING THE PATH OF AN OBLIQUE RAY THROUGH A CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES.

227. Employing here the same notation as was used in §§ 215 and 219, where the calculation-scheme of L. Seidel for the case of the refraction of an oblique ray at a single spherical surface was given, we shall designate the points where the ray crosses the kth transversal (or yz-) plane, before and after refraction at the kth surface, by H_k, H'_k, respectively; and shall denote the rectangular co-ordinates of these points by $(0, y_{h,k}, z_{h,k})$, $(0, y'_{h,k}, z'_{h,k})$, and their polar co-ordinates by (p_k, π_k), (p'_k, π'_k), respectively: the relations of these two sets of co-ordinates being defined as follows:

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[1] This will not be the *Chief Ray* of the bundle, unless the stop-centre coincides with the centre of the Entrance-Pupil; or unless, with respect to these two points, the spherical aberration of that part of the optical system which precedes the stop-centre has been abolished.

$$\left.\begin{aligned} y_{h,k} &= p_k\cdot\cos\pi_k, \quad z_{h,k} = p_k\cdot\sin\pi_k, \\ y'_{h,k} &= p'_k\cdot\cos\pi'_k, \quad z'_{h,k} = p'_k\cdot\sin\pi'_k, \end{aligned}\right\} \tag{227}$$

where

$$\pi'_k = \pi_k \tag{228}$$

(or, in case the kth surface is a reflecting surface, $\pi'_k = \pi_k + 180°$).

The directions of the ray, before and after refraction at the kth surface, are defined by two pairs of angular magnitudes denoted by τ_k, ψ_k and τ'_k, ψ'_k, respectively. Since the direction of the ray after refraction at the kth surface is identical with its direction before refraction at the $(k+1)$th surface, we have:

$$\tau'_k = \tau_{k+1}, \quad \psi'_k = \psi_{k+1}; \tag{229}$$

which are, therefore, the "Transformation-Formulæ" for SEIDEL's Direction-Parameters. These Direction-Parameters are defined, precisely as in § 215, by the following formulæ:

$$\left.\begin{aligned} \tan\psi'_k &= \frac{z'_{h,k} - z'_{i,k}}{y'_{h,k}} = \tan\psi_{k+1} = \frac{z_{h,k+1} - z_{i,k+1}}{y_{h,k+1}}; \\ \tan\tau'_k &= \frac{y'_{h,k}}{x'_{i,k}\cdot\cos\psi'_k} = \tan\tau_{k+1} = \frac{y_{h,k+1}}{x_{i,k+1}\cdot\cos\psi_{k+1}}. \end{aligned}\right\} \tag{230}$$

It remains to obtain L. SEIDEL's Formulæ for the transformation from the parameters π_k, p'_k to the parameters π_{k+1}, p_{k+1}; which we proceed to do.

If the Direction-Cosines of the ray after refraction at the kth surface are denoted by α, β, γ, then, precisely as in § 215, we have:

$$\frac{\beta}{\alpha} = -\tan\tau'_k\cdot\cos\psi'_k, \quad \frac{\gamma}{\alpha} = -\tan\tau'_k\cdot\sin\psi'_k;$$

and since this ray goes through the two points H'_k and H_{k+1}, whose rectangular co-ordinates, referred to the centre of the kth surface as origin are:

$$(0,\ p'_k\cdot\cos\pi_k,\ p'_k\cdot\sin\pi_k) \quad \text{and} \quad (a_k,\ p_{k+1}\cdot\cos\pi_{k+1},\ p_{k+1}\cdot\sin\pi_{k+1}),$$

respectively, we have:

$$\frac{a_k}{\alpha}=\frac{p_{k+1}\cdot\cos\pi_{k+1}-p'_k\cdot\cos\pi_k}{\beta}=\frac{p_{k+1}\cdot\sin\pi_{k+1}-p'_k\cdot\sin\pi_k}{\gamma}.$$

Eliminating α, β, γ from these two sets of equations, we obtain:

$$\begin{aligned}p_{k+1}\cdot\cos\pi_{k+1}-p'_k\cdot\cos\pi_k&=-a_k\cdot\tan\tau'_k\cdot\cos\psi'_k,\\ p_{k+1}\cdot\sin\pi_{k+1}-p'_k\cdot\sin\pi_k&=-a_k\cdot\tan\tau'_k\cdot\sin\psi'_k.\end{aligned}$$

Combining these equations, we obtain easily the Transformation-Formulæ of L. SEIDEL,[1] as follows:

$$\left.\begin{aligned}p_{k+1}\cdot\sin(\psi'_k-\pi_{k+1})&=p'_k\cdot\sin(\psi'_k-\pi_k),\\ p_{k+1}\cdot\cos(\psi'_k-\pi_{k+1})&=p'_k\cdot\cos(\psi'_k-\pi_k)-a_k\cdot\tan\tau'_k.\end{aligned}\right\}\qquad(231)$$

Accordingly, in the *Calculation-Scheme* of L. SEIDEL for the refraction of an oblique ray through a centered system of spherical surfaces we have the following formulæ (see formulæ (213) and (214)):

(1) *Determination of the Position of the Point H'_k by means of its Polar Co-ordinates* (p'_k, π_k):

$$\left.\begin{aligned}\cos\mu_k&=-\sin\tau'_{k-1}\cdot\cos(\psi'_{k-1}-\pi_k);\\ \sin\alpha_k&=p_k\cdot\sin\mu_k/r_k;\\ \sin\alpha'_k&=\frac{n'_{k-1}}{n'_k}\sin\alpha_k;\\ \mu'_k&=\mu_k+\alpha_k-\alpha'_k;\\ p'_k&=r_k\cdot\frac{\sin\alpha'_k}{\sin\alpha_k}=p_k\frac{n'_{k-1}}{n'_k}\frac{\sin\mu_k}{\sin\mu'_k}.\end{aligned}\right\}\qquad(232)$$

(2) *Determination of the Direction* (τ'_k, ψ'_k) *of the Refracted Ray:*

$$\left.\begin{aligned}\sin\tau'_k\cdot\sin(\psi'_k-\pi_k)&=\frac{\sin\mu'_k}{\sin\mu_k}\cdot\sin\tau'_{k-1}\cdot\sin(\psi'_{k-1}-\pi_k);\\ \tan(\pi_k-\psi'_k)&=\frac{\sin\tau'_k\cdot\sin(\psi'_k-\pi_k)}{\cos\mu'_k};\\ \tan\tau'_k&=\tan\tau'_{k-1}\cdot\frac{\sin(\psi'_{k-1}-\pi_k)}{\sin(\psi'_k-\pi_k)}.\end{aligned}\right\}\qquad(233)$$

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[1] L. SEIDEL: Trigonometrische Formeln für den allgemeinsten Fall der Brechung des Lichtes an centrierten sphaerischen Flaechen: *Sitzungsber. der math.-phys. Cl. der kgl. bayr. Akad. der Wissenschaften*, vom 10. Nov. 1866. Reprinted in Beilage III of STEINHEIL & VOIT'S *Handbuch der angewandten Optik*, Bd. I (Leipzig, B. G. TEUBNER, 1891), pages 257–270.

(3) *Transformation-Formulæ for finding the parameters* p_{k+1}, π_{k+1} *of the Ray Before Refraction at the* $(k+1)$*th Surface:*

$$\left.\begin{aligned} a_k &= d_k - r_k + r_{k+1}; \\ p_{k+1}\cdot\sin(\psi'_k - \pi_{k+1}) &= p'_k\cdot\sin(\psi'_k - \pi_k); \\ p_{k+1}\cdot\cos(\psi'_k - \pi_{k+1}) &= p'_k\cdot\cos(\psi'_k - \pi_k) - a_k\cdot\tan\tau'_k. \end{aligned}\right\} \quad (234)$$

228. Seidel's "Control" Formulæ.

In order to check the numerical work from time to time, and thereby to avoid the disagreeable necessity, in case of arithmetical errors, of having to repeat sometimes a very considerable portion of the calculation, L. SEIDEL has proposed, in connection with the above formulæ, several so-called "*Control*" *Formulæ*, the first of which is as follows:

$$\frac{\sin\mu_k\cdot\sin\mu'_k}{\sin(\psi'_{k-1} - \pi_k)} = \frac{\sin\mu'_k\cdot\sin\tau'_{k-1}}{\sin(\psi'_k - \pi_k)} = \frac{\sin(\alpha_k - \alpha'_k)}{\sin(\psi'_{k-1} - \psi'_k)}. \quad (235)$$

The equality of the two expressions on the left follows from the first of formulæ (232); and the equality between each of these and the third expression can be deduced easily from the first of formulæ (232) and the first of formulæ (233). Accordingly, this "control" formula (235) serves to test only the accuracy of computations by these formulæ from which it is derived.

The values of the sines of the angles of incidence and refraction are checked, along with the value of p', by the double calculation of this latter magnitude by means of the two expressions for p' in formulæ (232). But as it is possible that, even though we have found the correct value of the sine of an angle, an error may be introduced in determining the value of the corresponding angle itself, or that a mistake may be made in obtaining the difference $\alpha - \alpha'$, thereby involving also a mistake in the value obtained for the angle μ', and as the "control" formula (235) would not enable us to detect an error of any of these kinds, SEIDEL suggests also a second "control" formula, as follows:

$$\frac{\sin(\alpha_k + \alpha'_k)\cdot\sin(\alpha_k - \alpha'_k)}{\sin\alpha_k\cdot\sin\alpha'_k} = \frac{n'_k}{n'_{k-1}} - \frac{n'_{k-1}}{n'_k}; \quad (236)$$

which is a simple consequence of the Law of Refraction. The magnitude on the right is constant for all rays of the same wave-length refracted between the same two media; so that in case the calculation has to be made for a number of such rays (as usually happens in such calculations), it will not be necessary to calculate at all the value of the left-hand side of the equation, but it will be sufficient merely to see that the values of the expressions on the right are the same for all the rays. Moreover, in the usual case of an optical system consisting of a series of glass lenses, each surrounded by air, where, therefore, the ray proceeding from a medium (n) into a medium (n'), emerges again into the medium (n), the values of the constant on the right-hand side of (236) for two successive refracting surfaces will be equal in magnitude, but opposite in sign; and in such a case it will merely be necessary to calculate the values of the expression on the left-hand side for each surface, and see that the condition above-mentioned is fulfilled.

Finally, a third "control" formula, deduced from the two transformation-formulæ (231), is as follows:

$$\frac{p_{k+1}}{\sin(\psi'_k - \pi_k)} = \frac{a_k \cdot \tan \tau'_k}{\sin(\pi_k - \pi_{k+1})} = \frac{p'_k}{\sin(\psi'_k - \pi_{k+1})}. \qquad (237)$$

In Steinheil & Voit's *Handbuch der angewandten Optik*, I. Bd. (Leipzig, B. G. Teubner, 1891), the reader will find numerous complete calculations by means of the trigonometric formulæ of L. Seidel.

229. The Initial Values.

The position of the object-ray will usually be defined by the position of the object-point $P_1(u_1 - r_1, \eta_1, 0)$ and the position of the point $\boldsymbol{P}_1(\boldsymbol{u}_1 - r_1, \boldsymbol{\eta}_1, \boldsymbol{\zeta}_1)$ where the ray crosses the plane of the Entrance-Pupil (see § 226). This ray crosses the first transversal (or yz-) plane at $H_1(0, y_{h,1}, z_{h,1})$ and the horizontal xz-plane at $I_1(x_{i,1}, 0, z_{i,1})$. The positions on the x-axis of the points designated below by C_1, M_1, $\boldsymbol{M}_1$ and K are defined as follows:

$$r_1 = A_1C_1, \quad u_1 = A_1M_1, \quad \boldsymbol{u}_1 = A_1\boldsymbol{M}_1, \quad x_{i,1} = C_1K.$$

By drawing a figure, the following relations will be immediately obvious:

$$\frac{y_{h,1}}{\boldsymbol{\eta}_1} = \frac{C_1K}{\boldsymbol{M}_1K}, \quad \frac{z_{h,1}}{\boldsymbol{\zeta}_1} = \frac{M_1C_1}{M_1\boldsymbol{M}_1}.$$

Here

$$\boldsymbol{M}_1K = \boldsymbol{M}_1A_1 + A_1C_1 + C_1K = x_{i,1} + r_1 - \boldsymbol{u}_1,$$
$$M_1C_1 = M_1A_1 + A_1C_1 = r_1 - u_1,$$

and

$$M_1\boldsymbol{M}_1 = M_1A_1 + A_1\boldsymbol{M}_1 = \boldsymbol{u}_1 - u_1;$$

and if for $x_{i,1}$ we substitute its value as given by the first of formulæ

(225), we obtain:

$$y_{h,1} = \frac{\boldsymbol{\eta}_1(\boldsymbol{u}_1 - r_1) - \eta_1(u_1 - r_1)}{\boldsymbol{u}_1 - u_1}, \quad z_{h,1} = -\boldsymbol{\zeta}_1 \frac{u_1 - r_1}{\boldsymbol{u}_1 - u_1}. \tag{238}$$

By means of formulæ (238), together with (225), we can determine now the magnitudes of the direction-parameters (τ_1, ψ_1) of the object-ray; for according to the definition-formulæ of these angles we have:

$$\tan\psi_1 = \frac{z_{h,1} - z_{i,1}}{y_{h,1}}, \quad \tan\tau_1 = \frac{\sqrt{y_{h,1}^2 + (z_{h,1} - z_{i,1})^2}}{x_{i,1}},$$

and, consequently:

$$\tan\psi_1 = \frac{\boldsymbol{\zeta}_1}{\boldsymbol{\eta}_1 - \eta_1}, \quad \tan\tau_1 = \frac{\sqrt{(\boldsymbol{\eta}_1 - \eta_1)^2 + \boldsymbol{\zeta}_1^2}}{\boldsymbol{u}_1 - u_1}. \tag{239}$$

The initial values p_1, π_1 of the other two SEIDEL-parameters may be determined by the equations:

$$\left.\begin{aligned} p_1 \cdot \sin(\psi_1 - \pi_1) &= \pm\, \eta_1 \cdot \sin\psi_1, \\ p_1 \cdot \cos(\psi_1 - \pi_1) &= \pm\, \eta_1 \cdot \cos\psi_1 - (r_1 - u_1) \cdot \tan\tau_1, \end{aligned}\right\} \tag{240}$$

wherein the upper sign must be used in case the object-point lies above the optical axis, and the lower sign in the opposite case.

In the special case *when the object-point P_1 is the infinitely distant point of the object-ray*, then, in general, both η_1 and u_1 will be infinite. In this case, instead of being given the co-ordinates u_1, η_1, we shall be given the direction of the ray—which will usually be done by assigning the value of the slope-angle $\boldsymbol{\theta}_1$ of that one of the bundle of parallel object-rays which crosses the optical axis at the centre $\boldsymbol{M}_1$ of the Entrance-Pupil, and which, therefore, crosses the first central transversal plane at a point whose distance from the optical axis is:

$$(r_1 - \boldsymbol{u}_1) \cdot \tan\boldsymbol{\theta}_1.$$

If $\boldsymbol{\eta}_1$, $\boldsymbol{\zeta}_1$ denote the co-ordinates of the point where the general object-ray lying outside the plane of the principal section crosses the plane of the Entrance-Pupil, we shall have in this case the following formulæ for determining the parameters p_1, π_1:

$$\left.\begin{aligned} y_{h,1} &= p_1 \cdot \cos\pi_1 = \boldsymbol{\eta}_1 + (r_1 - \boldsymbol{u}_1) \cdot \tan\boldsymbol{\theta}_1, \\ z_{h,1} &= p_1 \cdot \sin\pi_1 = \boldsymbol{\zeta}_1 \end{aligned}\right\}, \quad (u_1 = \eta_1 = \infty). \tag{241}$$

Evidently, also, for the case of an infinitely distant object-point, we have $\psi_1 = 0°$ or $180°$ and $\tau_1 = \pm\,\boldsymbol{\theta}_1$.

§ 229a. Trigonometric Formulæ of M. Lange for Calculating the Path of an Oblique Ray through a Centered System of Spherical Refracting Surfaces.

A very convenient set of formulæ for the trigonometric calculation of the path of a ray through a centered system of spherical refracting surfaces has been published recently by MAX LANGE.[1] In these formulæ (which are easily derived) the symbols have been changed to accord with the plan of notation of this book.

Let x_k, y_k, z_k denote the rectangular co-ordinates, with respect to the centre C_{k-1} of the $(k-1)$th surface as origin, of the point B_k where the ray is incident on the kth surface; and let the direction-cosines of the straight line $B_{k-1}B_k$ be each multiplied by the refractive index (n_k) of the medium comprised between the $(k-1)$th and kth surfaces, and the products denoted by the symbols λ_k, μ_k, ν_k. Moreover, put

$$n_k \cdot B_{k-1}B_k = p_k;$$

that is, let p_k denote the so-called "optical length" (§ 38) of the ray-path between the $(k-1)$th and kth surfaces. And, finally, put

$$B_kC_{k+1} = q_k, \quad \angle\, C_{k+1}B_kB_{k+1} = \omega_k.$$

Here it is important to note particularly that *the magnitude denoted by q_k is to be regarded as essentially positive.* In this connection we remark also that in the following formulæ *the radius r_k is to be reckoned always as positive.* The angles of incidence and refraction are denoted by α_k, α_k'; but with regard to the sign of $\cos\alpha_k$, the following rule is to be borne in mind: *The* $\cos\alpha_k$ *is to be reckoned positive or negative, according as the kth surface is concave or convex, respectively.*

The symbol a_k is employed as formerly in this chapter; thus,

$$a_k = C_kC_{k+1}.$$

Moreover, the symbol V_k is used as an abbreviation for the following expression:

$$\frac{n_{k+1}\cos\alpha_k' - n_k\cos\alpha_k}{r_k} = V_k.$$

Introducing these symbols, we have, according to LANGE, the following system of formulæ:

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[1] MAX LANGE: *Vereinfachte Formeln für die trigonometrische Durchrechnung optischer Systeme* (Inaugural-Dissertation zur Erlangung der Doktorwuerde der philosophischen Fakultaet der Universitaet Rostock): Leipzig, 1909. In this elegant paper Dr. LANGE gives a complete system of formulæ for the calculation of centered optical systems, with actual calculations of a telescope-objective.

$$\lambda_{k+1} = \lambda_k + V_k x_k, \quad \mu_{k+1} = \mu_k + V_k y_k, \quad \nu_{k+1} = \nu_k + V_k z_k;$$

$$q_k = \sqrt{(r_k - a_{k+1})^2 + 2a_{k+1}(r_k - x_k)};$$

$$\cos\omega_k = \frac{1}{q_k}\left(r_k \cos\alpha_k' - \frac{a_{k+1}\lambda_{k+1}}{n_{k+1}}\right);$$

$$\sin\alpha_{k+1} = \frac{q_k \sin\omega_k}{r_{k+1}}, \quad \sin\alpha'_{k+1} = \frac{n_{k+1}\sin\alpha_{k+1}}{n_{k+2}};$$

$$p_{k+1} = n_{k+1}(r_{k+1}\cos\alpha_{k+1} - q_k\cos\omega_k);$$

$$x_{k+1} = x_k + \frac{p_{k+1}\lambda_{k+1}}{n^2_{k+1}} - a_{k+1}, \quad y_{k+1} = y_k + \frac{p_{k+1}\mu_{k+1}}{n^2_{k+1}},$$

$$z_{k+1} = z_k + \frac{p_{k+1}\nu_{k+1}}{n^2_{k+1}}.$$

For the initial values, we have:

$$a_1 = 0, \quad \cos\alpha_1 = \frac{x_1\lambda_1 + y_1\mu_1 + z_1\nu_1}{r_1}.$$

The following "control" formulæ are useful as a check on the calculation:

$$\lambda_k^2 + \mu_k^2 + \nu_k^2 = n_k^2,$$

$$x_k^2 + y_k^2 + z_k^2 = r_k^2.$$

NOTE CONCERNING NEW SYSTEM OF FORMULÆ OF A. KERBER.

While the second edition of this book was passing through the press, a new and ingenious scheme for the calculation of the path of a ray through a centered system of spherical refracting surfaces has been proposed by Dr. A. KERBER,[1] which seems to possess certain advantages, especially inasmuch as the tedious trigonometric evaluation of angles is almost entirely avoided. It is a source of regret that it is not possible to give these formulæ here, but it may be remarked that the method consists in determining the co-ordinates of the incidence-points B_k and is based on a certain simple invariant relation, which is derived at once from the following obvious formulæ (wherein the letters and symbols have the same meanings as in § 214):

$$\frac{BG}{ny_g} = \frac{\sin\angle BCA_g}{n\cdot\sin\alpha\cdot\sin\phi_g}, \quad \frac{BG'}{n'y'_g} = \frac{\sin\angle BCA_g}{n'\cdot\sin\alpha'\cdot\sin\phi_g},$$

whence, since $n\cdot\sin\alpha = n'\cdot\sin\alpha'$, we obtain the fundamental relation above mentioned, viz.:

$$\frac{BG}{ny_g} = \frac{BG'}{n'y'_g}$$

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[1] A. KERBER: Neue Durchrechnungsformeln für windschiefe Strahlen: *Zft. f. Instrumentenk.*, xxiii (1913), 75–84.

CHAPTER XI.

GENERAL CASE OF THE REFRACTION OF AN INFINITELY NARROW BUNDLE OF RAYS THROUGH AN OPTICAL SYSTEM. ASTIGMATISM.

ART. 70. GENERAL CHARACTERISTICS OF A NARROW BUNDLE OF RAYS REFRACTED AT A SPHERICAL SURFACE.

230. Meridian and Sagittal Rays.

To an infinitely narrow homocentric bundle of incident rays refracted (or reflected) at a spherical surface there corresponds, in general, an *astigmatic* bundle of refracted (or reflected) rays, which, provided we *neglect magnitudes of the second order of smallness*, is characterized by the following properties:

The chief ray u' of the bundle of refracted rays is that one of the refracted rays which corresponds to the chief ray u of the bundle of incident rays. All the refracted rays meet two infinitely short straight lines, the so-called *Image-Lines* (§ 47), which lie in two perpendicular planes both containing the refracted chief ray u', and which are perpendicular to u'. These two planes are the planes of Principal Curvature of the element of the refracted wave-surface at any point P' of the refracted chief ray u', which pierces the surface-element at P' normally, and their traces on the element of wave-surface at P' are two elements of arc intersecting at right angles at P'. The two pencils of rays of the bundle of refracted rays which lie in the planes of Principal Curvature have their vertices on the refracted chief ray u' at the centres of curvature S' and $\overline{S}'$. Thus, to an object-point S lying on the incident chief ray u, which is the vertex of an infinitely narrow homocentric bundle of incident rays, correspond two image-points S', $\overline{S}'$ lying on the refracted chief ray u', which we shall call the Primary and Secondary Image-Points, respectively. The two Image-Lines are perpendicular to the refracted chief ray u' at these Image-Points. Thus, the I. Image-Line is perpendicular to the refracted chief ray at S', and lies in the plane of Principal Curvature of the refracted wave-surface for which the II. Image-Point $\overline{S}'$ is the centre of curvature; and, similarly, the II. Image-Line is perpendicular at $\overline{S}'$ to the chief refracted ray u', and lies in the plane of Principal Curvature of the refracted wave-surface for which the I. Image-Point S' is the centre of curvature.

When the plane determined by the chief rays u, u', which we shall call the Plane of Incidence, is at the same time a plane of Principal Curvature of the refracted wave-surface, one of the image-lines will lie in this plane, and the other will lie in a plane perpendicular to the plane of incidence.

The special problem which we have to consider presents a comparatively simple case; for, since the refracting surface is spherical, the two systems of incident and refracted rays are symmetrical about an axis. Thus, if C designates the centre of the spherical refracting surface, not only this surface but the incident and refracted wave-surfaces as well are surfaces of revolution around the straight line SC as axis. The plane of incidence uC, containing the common axis of these three surfaces of revolution, is a meridian plane of each one of these surfaces, and is, therefore, also a plane of Principal Curvature; so that one of the Image-Lines will lie in the plane of incidence, and the other will lie in the plane perpendicular to the plane of incidence which contains the refracted chief ray u'. According to the usage of most writers on Optics, we shall designate the latter as the I. Image-Line and the former as the II. Image-Line.[1] The II. Image-Line is perpendicular to the chief refracted ray u' at the point $\overline{S}'$ where this ray crosses the axis of symmetry SC.

Thus, in the case of an infinitely narrow homocentric bundle of incident rays refracted at a spherical surface, the directions of the Image-Lines of the astigmatic bundle of refracted rays will depend only on the position and direction of the chief refracted ray u'; so that to a range of object-points lying on a given incident chief ray u there will correspond a series of parallel I. Image-Lines and a series of parallel II. Image-Lines.

The planes of Principal Curvature of the wave-surface determine two principal sections of the infinitely narrow bundle of rays. The plane of incidence uC, which in the case of a spherical refracting surface coincides with one of these planes, cuts the infinitely narrow homocentric bundle of incident rays and the corresponding astigmatic bundle of refracted rays in a pencil of incident rays with its vertex at the Object-Point S and in a pencil of refracted rays with its vertex at the I. Image-Point S'. These are the so-called *Meridian* Rays; since the plane of incidence uC is at the same time a meridian plane of the spherical refracting surface.

If the incident chief ray u is supposed to be revolved about SC

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[1] Some writers, however, for example, LIPPICH, use the contrary method of designating these lines.

as axis through an infinitely small angle to one side and the other of its actual position, it will coincide in succession with all the rays which lie on the surface of a right circular cone of which SC is the axis and the straight line SB (where B designates the point where the chief ray meets the refracting surface) is an element. The corresponding refracted rays will likewise lie on the surface of a right circular cone generated by the revolution of the chief refracted ray $B\bar{S}'$ about the same line as axis. Provided we neglect infinitely small magnitudes of the second order, this group of incident rays may be regarded as lying in a plane $\bar{\pi}$ which contains the incident chief ray and is perpendicular to the plane of incidence uC (or π); and, similarly, the corresponding refracted rays may also be regarded as lying in a plane $\bar{\pi}'$ which contains the chief refracted ray u' and is likewise perpendicular to the plane uC. These planes are evidently tangent to the conical surfaces generated by the revolution of u, u' around SC as axis. Following the usage of most modern writers, we shall call the incident and refracted rays lying in the planes $\bar{\pi}$, $\bar{\pi}'$, respectively, the *Sagittal* Rays.[1]

231. Different Degrees of Convergence of the Meridian and Sagittal Rays.

The diagram (Fig. 125) shows a meridian section of the spherical refracting surface μ containing the chief incident ray u and the chief refracted ray u', the plane of the diagram being, therefore, the Plane of Incidence. The numerals 1 and 2 in the figure are used to designate two points of the meridian section of the spherical surface both very close to the incidence-point B of the chief incident ray and lying

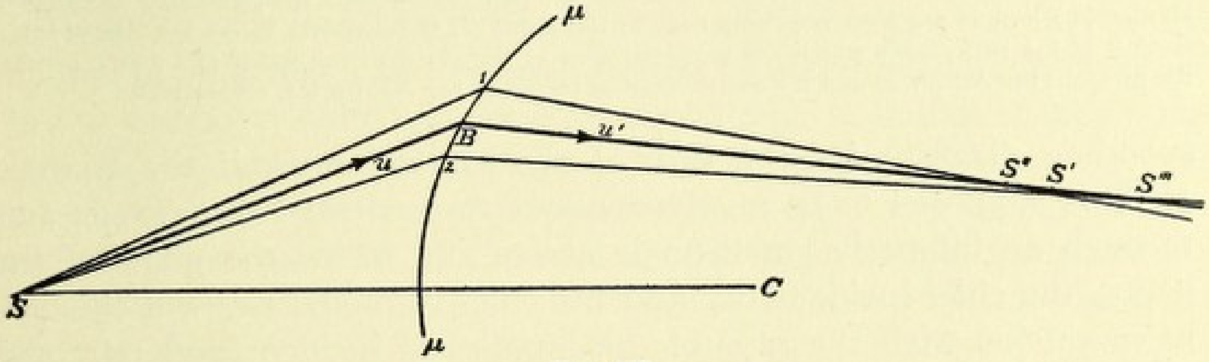

Fig. 125.

Convergence of Meridian Rays after Refraction at a Spherical Surface. All the lines in the figure lie in the plane of a meridian section of the refracting sphere.

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[1] "Sagittal" is a term borrowed from Anatomy. Many writers use the antonym "tangential" instead of "meridian". On the other hand, some writers, who use the term "meridian", prefer to be more consistent and use therefore the word "equatorial" instead of "sagittal".

on opposite sides of this point. Thus, $S1$, $S2$ belong to the pencil of meridian incident rays. After refraction, these rays will intersect the chief refracted ray u' in the points designated in the figure by S'', S''', which, while they are infinitely close together, cannot, in general, be regarded as coincident unless we neglect infinitesimals of the first order. In fact, the position of the I. Image-Point S' depends on the arc $B1$, so that for different rays of the meridian pencil we shall obtain values of BS' which differ from each other by magnitudes of the same order of smallness as the arc $B1$. Hence, the convergence of the refracted rays in the meridian section is said to be a "convergence of the first order".

The convergence of the refracted rays in the sagittal section is of a higher order than the first. Thus, in Fig. 126, which is the corresponding diagram for the case of the sagittal rays, if the triangle $SB\overline{S}'$ is supposed to be revolved about the central line $SC\overline{S}'$ as axis through an infinitely small angle above and below the plane of the paper, the chief incident ray and the chief refracted ray will coincide in succession with the rays of the bundles of incident and refracted rays, respectively, which lie on the conical surfaces generated by this revolution. These refracted rays all intersect exactly at the point $\overline{S}'$ and these rays are very nearly identical with the sagittal rays themselves. In fact, it is easy to see that the convergence of the sagittal rays at $\overline{S}'$ is a "convergence of the second order", and is optically more effective than that of the meridian rays.

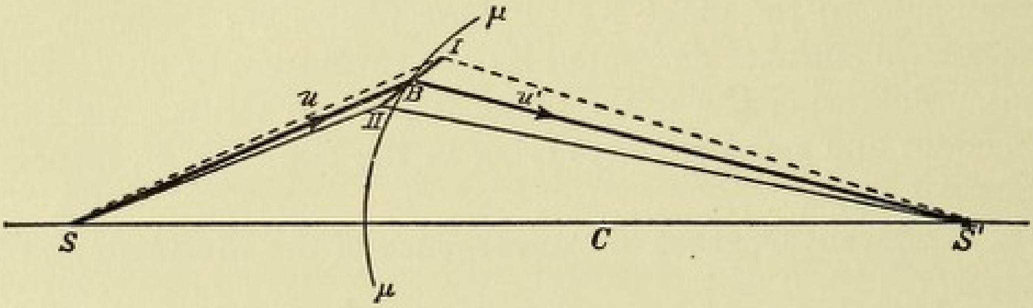

FIG. 126.

CONVERGENCE OF SAGITTAL RAYS AFTER REFRACTION AT A SPHERICAL SURFACE. The plane of the paper represents a meridian section of the refracting sphere. The points designated in the diagram by the letters S, $\overline{S}'$, B and C lie in this plane. The points designated by the Roman Numerals I and II are both infinitely near to the point B; these points lie on the line of intersection of the two planes which are perpendicular to the plane of the paper and which contain the incident ray SB (or u) and the corresponding refracted ray $B\overline{S}'$ (or u'), respectively.

232. The Image-Lines.

The astigmatic bundle of rays may be regarded as composed either of pencils of meridian rays whose chief rays all meet in the II. Image-Point $\overline{S}'$, or as pencils of sagittal rays whose chief rays all meet in the

I. Image-Point S'. The vertices of the meridian pencils form the I. Image-Line, and the vertices of the sagittal pencils form the II. Image-Line.

An incident ray proceeding from S, lying in the plane of the meridian section, and meeting the spherical refracting surface at a point 1 a little above B (Fig. 125) will be refracted so as to intersect the central line SC at a point slightly to the left of $\overline{S}'$ (Fig. 126); and this point will be the point of convergence of all the refracted rays which correspond to incident rays lying on the conical surface generated by the revolution about SC as axis of the incident ray $S1$. And, similarly, if the point of incidence lies in the meridian section at a point 2 slightly below B, the rays incident on the spherical surface at points in the arc of the circle described by 2 when the figure is revolved about SC as axis will be refracted so as to cross the central line at a point a little to the right of $\overline{S}'$. Thus, all the rays of the infinitely narrow astigmatic bundle of refracted rays will cross the central line SC within an infinitely short piece of it lying on either side of the II. Image-Point $\overline{S}'$. This line-element may be regarded, and, indeed, from a purely geometrical point of view, should be regarded, as in reality the II. Image-Line.[1] However, this line is not perpendicular to the chief refracted ray u', and it is more convenient and quite permissible to consider both of the Image-Lines, according to Sturm's definition, as perpendicular to the chief ray of the astigmatic bundle (see § 49). In fact, as Czapski[2] and others have pointed out, a section of the bundle of rays made by a plane through $\overline{S}'$ perpendicular to the chief ray u' differs very little from a straight line; the actual shape of the section is a curve with two loops, not unlike a slender figure 8. It is easy to see that this is so; for whereas the rays of the sagittal section proper all intersect in $\overline{S}'$, the rays of the other so-called sagittal sections intersect in points which lie on the axis to one side and the other of $\overline{S}'$, and, hence, the rays of each of these latter pencils will meet the plane, which is drawn perpendicular to u' at $\overline{S}'$, either before or after they meet each other at the vertex of the pencil on the central line SC, according as this vertex lies to the one side or the other of the II. Image-Point $\overline{S}'$. Thus, we see that the section of the bundle made by this plane opens out on each side of $\overline{S}'$. Moreover, it can very easily be shown that the width of this section is a magnitude of the second order of smallness, and hence the section itself may be regarded as a straight line, since we are neglecting infinitesimals of the second order. As CZAPSKI says, the two 8-shaped sections, which we have at both the I. Image-Point S' and the II. Image-Point $\overline{S}'$, with the axes of the 8's at right angles to each other, are as near an approach to what may be called the Image-Lines of the astigmatic bundle of rays as any other pair of lines.

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[1] See, particularly, L. Matthiessen: Ueber die Form der unendlich duennen astigmatischen Strahlenbuendel und ueber die Kummer'schen Modelle: *Sitzungber. der math.-phys. Cl. der koenigl. bayer. Akad. der Wissenschaften zu Muenchen*, xiii. (1883), 83.

[2] S. Czapski: Zur Frage nach der Richtung der Brennlinien in unendlich duennen optischen Bueschеln: Wied. *Ann.*, xliii. (1891), 332–337.

ART. 71. THE MERIDIAN RAYS.

233. Relation between the Object-Point S and the I. Image-Point S'. Let the chief ray u of an infinitely narrow homocentric bundle of incident rays proceeding from an Object-Point S meet the spherical refracting surface μ in the point B (Fig. 127), and let the refracted chief ray u' corresponding to u be constructed as in YOUNG's Construction (§ 206) by means of the concentric spherical surfaces τ, τ' described around C as centre with radii equal to $n'r/n$, nr/n', respectively, where C designates the centre of the spherical refracting surface, and r denotes its radius, and n, n' denote the absolute indices of refraction of the first and second medium, respectively. Let G designate a point of the spherical refracting surface in the plane of incidence uC and infinitely near to B, so that SG will represent a secondary ray of the pencil of incident meridian rays. This ray will meet the auxiliary spherical surface τ in a point N infinitely near to the point Z where the chief incident ray u meets this surface, and the refracted ray corresponding to the incident ray SG will meet the spherical surface τ' in a point N' infinitely near to the point Z' where the chief refracted ray u' meets this surface. The point of intersection of this refracted ray with the chief refracted ray will determine the I. Image-Point S', which is the vertex of the pencil of meridian refracted rays.

The relation between the Object-Point S and its I. Image-Point S' may be found in various ways, either analytically or geometrically. A very elegant geometrical method, involving however certain kinematical notions which appear to be a little foreign in a treatise on Optics, is given by L. BURMESTER in his interesting paper, "Homocentrische Brechung des Lichtes durch die Linse" (*Zs. f. Math. u. Phys.*, xl., 1895, 321). The method which is given below is in some ways very similar to that used by F. KESSLER in a paper entitled "Beitraege zur graphischen Dioptrik" (*Zs. f. Math. u. Phys.*, xxix., 1884, 65–74).

On BC (Fig. 127) as diameter, describe a semi-circle meeting the chief incident ray u in a point Y and the chief refracted ray u' in a point Y', and let us imagine that straight lines are drawn connecting the points Y, Y' with each of the points G, Z and Z'. Since the angles are infinitely small, we can write the following proportions:

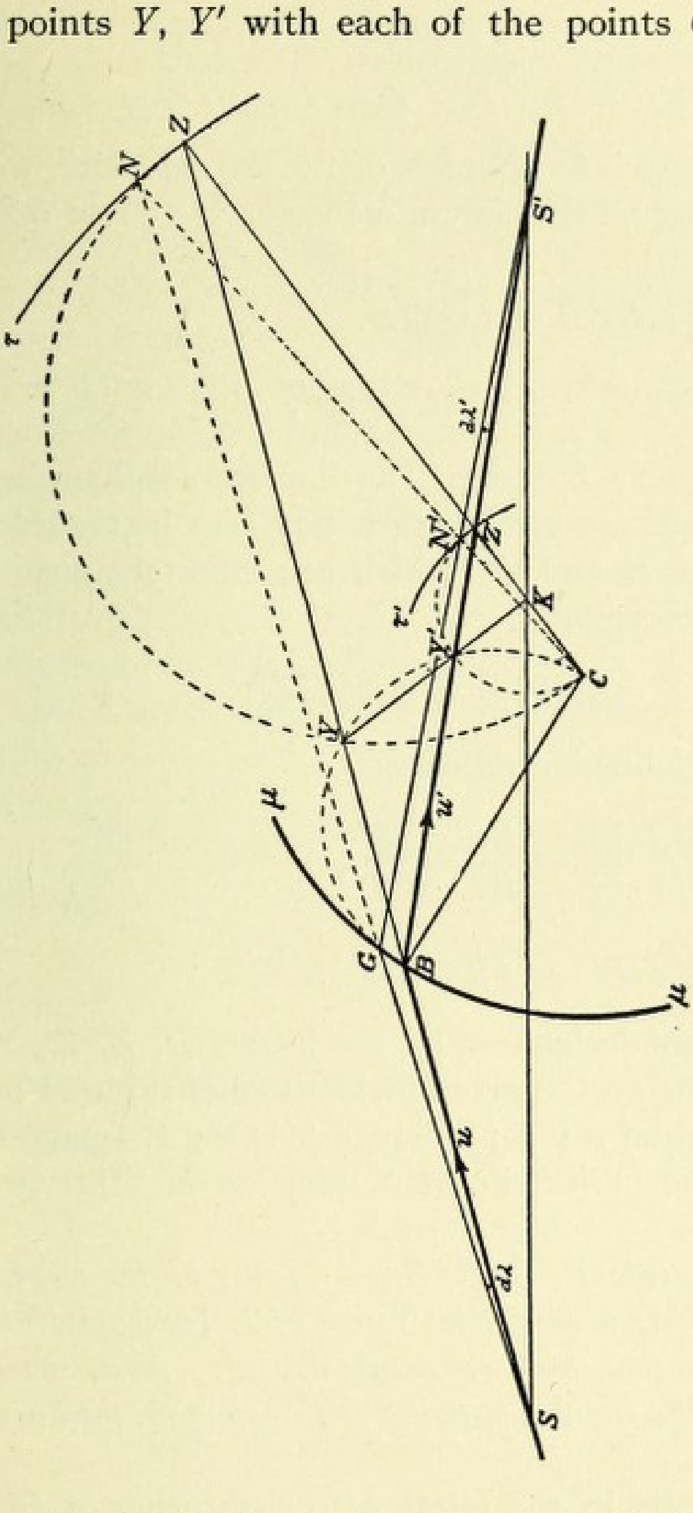

Fig. 127.

Construction of the I. Image-Point of Astigmatic Bundle of Refracted Rays corresponding to an Infinitely Narrow, Homocentric Bundle of Rays Incident on a Spherical Refracting Surface.

The plane of the paper represents a meridian section of the spherical refracting surface $\mu\mu$, whose centre is at the point C. S designates the position on the chief incident ray u of the Object-Point, and S' the position on the corresponding refracted ray u' of the I. Image-Point. τ and τ' are spherical surfaces concentric with the spherical refracting surface, of radii equal to $n'r/n$ and nr/n', respectively. The point designated by K is the Centre of Perspective of the range of Object-Points lying on the chief incident ray u and the range of I. Image-Points lying on the corresponding refracted ray u'.

$BS = s$, $BS' = s'$, $\angle BSG = d\lambda$, $\angle BS'G = d\lambda'$.

$$\frac{\angle GYB}{\angle GSB} = \frac{BS}{BY}, \quad \frac{\angle GY'B}{\angle GS'B} = \frac{BS'}{BY'},$$
$$\frac{\angle GSB}{\angle NYZ} = \frac{YZ}{SZ}; \quad \frac{\angle GS'B}{\angle N'Y'Z'} = \frac{Y'Z'}{S'Z'}.$$

Therefore, multiplying each of the two upper equations by the one directly below it, we obtain:

$$\frac{\angle GYB}{\angle NYZ}=\frac{BS\cdot YZ}{BY\cdot SZ},\quad \frac{\angle GY'B}{\angle N'Y'Z'}=\frac{BS'\cdot Y'Z'}{BY'\cdot S'Z'}.$$

Since we are neglecting here infinitesimals of the second order, we can regard the point G as lying on the circumference of the circle $BYY'C$, and therefore we can write:

$$\angle GYB = \angle GY'B.$$

Moreover, since

$$\angle BYC = \angle BY'C = 90°,$$

the semi-circles described on CZ and CZ' as diameters will go through Y and Y', respectively. These semi-circles may also be regarded as going through the points N and N' which are infinitely near to Z and Z', respectively. Accordingly,

$$\angle NYZ = \angle NCZ = \angle N'Y'Z';$$

and, thus, we obtain the following relation:

$$\frac{BS\cdot YZ}{BY\cdot SZ}=\frac{BS'\cdot Y'Z'}{BY'\cdot S'Z'},$$

or

$$(BZSY) = (BZ'S'Y').$$

In this equation the points designated by the letters B, Z, Z', Y and Y' are all fixed points lying on the given incident chief ray u or on the corresponding refracted chief ray u'; whereas S' is the I. Image-Point on u' corresponding to an Object-Point S lying on u. Interpreting the equation, we can say:

To a range of Object-Points P, Q, R, S, $\cdots$ lying on the chief incident ray u, there corresponds a projective range of I. Image-Points P', Q', R', S', $\cdots$ lying on the chief refracted ray u'. And, moreover, since the two ranges have the incidence-point B in common, they are also in perspective.

That the points Y, Y' are in the relation to each other of Object-Point and I. Image-Point is evident not only from the above equation, but geometrically also; for if we imagine an infinitely narrow pencil of meridian incident rays with its vertex at Y, these rays will meet the spherical refracting surface at points infinitely near to B which may all be regarded as lying on the circumference of the semi-circle BYC; so that for all rays converging to Y, the angles of incidence, being all subtended by the arc CY, will all be equal, and hence, the angles of refraction also must all be equal, and be angles in the circumference standing on the arc CY'.

234. The **Centre of Perspective** K is determined by the intersection of the straight lines YY', ZZ'. The existence of this point seems to have been recognized first by THOMAS YOUNG.[1] The point K was afterwards found again, independently, by CORNU[2] in 1863 and by LIPPICH[3] in 1878.

Since $\angle CBZ' = \angle CYY'$, both being inscribed angles standing on the same arc CY', it follows that YY' is perpendicular to CZZ' at K. Thus, we have the following simple *Construction of the I. Image-Point* S' corresponding to an Object-Point S on the chief incident ray u:

Having constructed the refracted chief ray u' corresponding to the chief incident ray u, draw CY perpendicular to u at Y and YK perpendicular to CZ at K; the straight line connecting S with K will intersect the chief refracted ray u' in the I. Image-Point S'.

The position of the Centre of Perspective K may also be computed as follows:

Since

$$CY = r\cdot\sin\alpha, \quad CK = CY\cdot\sin\alpha' = r\cdot\sin\alpha\cdot\sin\alpha',$$

we obtain:

$$CK = \frac{nr\cdot\sin^2\alpha}{n'} = \frac{n'r\cdot\sin^2\alpha'}{n}. \tag{242}$$

If we draw the straight line BK (Fig. 127), then

$$\begin{aligned}\angle CBK &= \angle CBZ' - \angle KBZ' = \alpha' - \angle KBZ' \\ &= \alpha' - (\angle BKC - \angle BZ'C) = \alpha + \alpha' - \angle BKC;\end{aligned}$$

thus, in the triangle BKC we obtain:

$$\frac{BC}{CK} = \frac{\sin\angle BKC}{\sin\angle CBK} = \frac{\sin\angle BKC}{\sin(\alpha + \alpha' - \angle BKC)}.$$

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[1] THOMAS YOUNG: On the Mechanism of the Human Eye: *Phil. Trans.*, 1801, xcii. p. 23. This paper is reprinted in *The Works of* THOMAS YOUNG, in three volumes, edited by GEO. PEACOCK, D.D. (London, JOHN MURRAY, 1855); Vol. I, pages 12–63. See "Prop. IV." on p. 16.

[2] A. CORNU: Caustiques — Centre de Jonction: *Nouv. Ann. de Math.*, 1863, (2), ii., 311–317. See also A. CORNU: Construction géométrique des deux images d'un point lumineux produit par réfraction oblique sur une surface spherique; *Journ. de physique*, Ser. III, x. (1901), 607.

[3] F. LIPPICH: Ueber Brechung und Reflexion unendlich duenner Strahlensysteme an Kugelflaechen: *Wiener Denkschr.*, 1878, xxxviii., 163–192.

And since

$$\frac{YC}{BC} = \sin\alpha, \quad \frac{CK}{YC} = \sin\alpha',$$

we have also:

$$\frac{BC}{CK} = \frac{1}{\sin\alpha\cdot\sin\alpha'};$$

so that

$$\frac{1}{\sin\alpha\cdot\sin\alpha'} = \frac{\sin\angle BKC}{\sin(\alpha+\alpha'-\angle BKC)};$$

whence we find:

$$\tan\angle BKC = \tan\alpha + \tan\alpha'. \qquad (243)$$

Commenting on this result, we observe that $\angle BKC$, and, hence, also $\angle CBK = \alpha + \alpha' - \angle BKC$, is independent of the radius of the spherical refracting surface, so that the values of these angles will depend only on the angle of incidence α and on the indices of refraction n, n'.

Indeed, it is obvious also from the geometrical relations in the diagram, that if, keeping the incidence-angle α unchanged, we suppose the radius of the refracting sphere to be variable, although the actual distances from B of both C and K will vary, the directions of the straight lines BC and BK will remain unaltered. Perhaps, the easiest way of seeing this is by drawing through any point on BC a straight line parallel to CZ, and constructing a point on this line exactly in the same way as the point K was constructed on CZ. It will be seen that the point thus determined will lie always on the straight line BK.

235. The Focal Points J and I' of the Meridian Rays.

If the Object-Point S is the infinitely distant point I of the chief incident ray u, the meridian incident rays will be a pencil of parallel rays to which will correspond a pencil of meridian refracted rays meeting the chief refracted ray u' in the "Flucht" Point I' of the range of I. Image-Points. And, on the other hand, if the I. Image-Point S' is the infinitely distant point J' of the chief refracted ray u', the meridian incident rays will intersect in the "Flucht" Point J of the range of I. Object-Points lying on the chief incident ray u. The "Flucht" Points J and I', or, as we shall now call them, the Primary and Secondary *Focal Points of the Meridian Rays*, may be located by drawing through K (Fig. 128) straight lines parallel to u' and u meeting u and u' in the points J and I', respectively.

According to a well-known law of projective ranges of points, we have evidently:

$$JB\cdot I'B = JS\cdot I'S' = (JB + BS)\,(I'B + BS');$$

and if we put

$$BS = s, \quad BS' = s',$$

we obtain the following equation:

$$\frac{BJ}{s} + \frac{BI'}{s'} = 1, \tag{244}$$

which, as the reader will remark, is completely analogous to the relation

$$AF/u + AE'/u' = 1, \quad \text{or} \quad f/u + e'/u' = -1,$$

which we found for the case of an infinitely narrow bundle of normally incident rays refracted at a spherical surface; see formulæ (148).

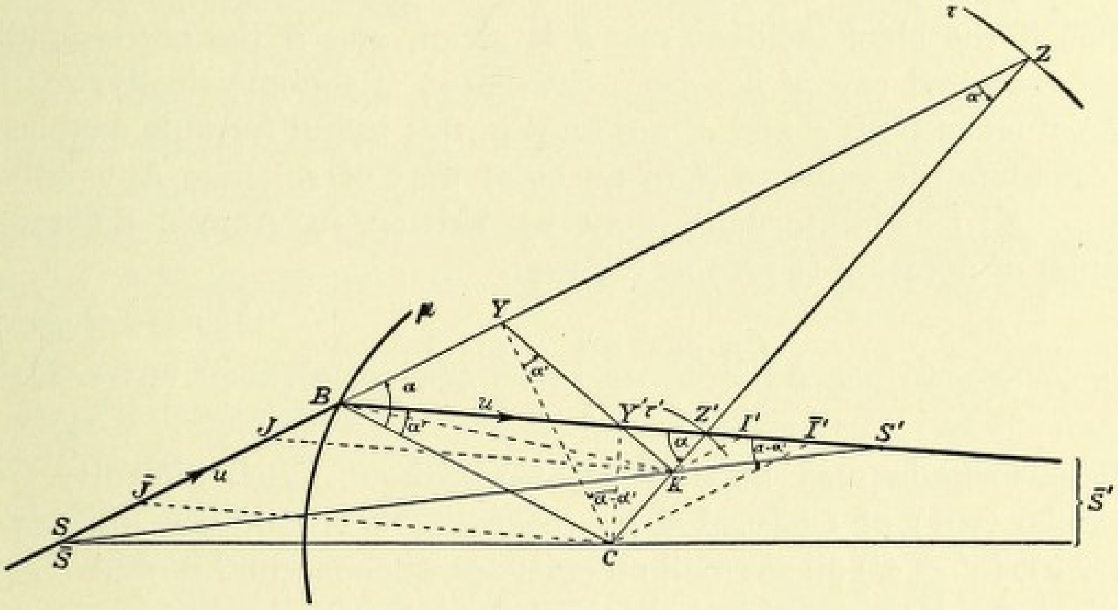

FIG. 128.

REFRACTION AT A SPHERICAL SURFACE OF AN INFINITELY NARROW BUNDLE OF RAYS. Perspective Relations of the Range of Object-Points lying on the chief incident ray u and the Ranges of I. and II. Image-Points lying on the corresponding refracted ray u'.

The positions of the Focal Points J, I' may be calculated as follows: From the figure (Fig. 128), we obtain:

$$BJ = I'K = -KZ' \cdot \frac{\sin \alpha}{\sin(\alpha - \alpha')} = -CZ' \cdot \frac{\sin \alpha \cdot \cos^2 \alpha}{\sin(\alpha - \alpha')},$$

$$BI' = JK = KZ \cdot \frac{\sin \alpha'}{\sin(\alpha - \alpha')} = CZ \cdot \frac{\sin \alpha' \cdot \cos^2 \alpha'}{\sin(\alpha - \alpha')};$$

and since

$$CZ = n'r/n, \quad CZ' = nr/n',$$

the above formulæ may be written:

$$\left.\begin{aligned} BJ &= -\frac{r\cdot\sin\alpha'\cdot\cos^2\alpha}{\sin(\alpha-\alpha')} = -\frac{nr\cdot\cos^2\alpha}{n'\cdot\cos\alpha'-n\cdot\cos\alpha},\\ BI' &= \frac{r\cdot\sin\alpha\cdot\cos^2\alpha'}{\sin(\alpha-\alpha')} = \frac{n'r\cdot\cos^2\alpha'}{n'\cdot\cos\alpha'-n\cdot\cos\alpha}. \end{aligned}\right\} \qquad (245)$$

236. Formula for Calculating the Position of the I. Image-Point S' corresponding to an Object-Point S on a given incident chief ray u.

If in formula (244) above we substitute the values of BJ and BI', as given by formulæ (245), we obtain the following relation:

$$\frac{n'\cdot\cos^2\alpha'}{s'} - \frac{n\cdot\cos^2\alpha}{s} = \frac{n'\cdot\cos\alpha' - n\cdot\cos\alpha}{r}. \qquad (246)$$

Thus, if the chief incident ray u is given, and if the corresponding chief refracted ray u' has been calculated trigonometrically, so that the values of both α and α' are known, this useful formula enables us to calculate the value of s' in terms of that of s. (See Appendix to Chap. XI.) The formula may be written in ABBE's differential system of notation (§ 126) as follows:

$$\Delta\left(\frac{n\cdot\cos^2\alpha}{s}\right) = \frac{1}{r}\,\Delta(n\cdot\cos\alpha). \qquad (246a)$$

This formula may be derived also without much difficulty from formulæ (191) of Chap. IX by regarding λ, λ' and the differences $(\alpha - \mathfrak{a})$, $(\alpha' - \mathfrak{a}')$ all as small magnitudes whose second powers may be neglected. RAYLEIGH[1] has obtained the formula also in a very simple way by the use of the Principle of the Shortest Light-Path (§ 38).

237. Convergence-Ratio of Meridian Rays.

If (Fig. 127) we put

$$\angle BSG = d\lambda, \quad \angle BS'G = d\lambda',$$

then

$$Z_u = \frac{d\lambda'}{d\lambda}$$

will denote the convergence-ratio of the pencil of meridian rays. Now we saw above that

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[1] J. W. STRUTT, Lord RAYLEIGH: Investigations in Optics with special reference to the Spectroscope: *Phil. Mag.*, (5), ix. (1880), 40–55.

$$\angle BSG = \frac{BY}{BS}\cdot\angle BYG,\quad \angle BS'G = \frac{BY'}{BS'}\cdot\angle BY'G,\quad \angle BYG = \angle BY'G;$$

and, accordingly:

$$\frac{\angle BS'G}{\angle BSG} = \frac{BY'}{BY}\cdot\frac{BS}{BS'}.$$

Hence, since

$$BY = 2r\cdot\cos\alpha,\quad BY' = 2r\cdot\cos\alpha',$$

we obtain:

$$Z_u = \frac{d\lambda'}{d\lambda} = \frac{s\cdot\cos\alpha'}{s'\cdot\cos\alpha}. \tag{247}$$

ART. 72. THE SAGITTAL RAYS.

238. Relation between the Object-Point $\overline{S}$ and the II. Image-Point $\overline{S}'$.

Let $\overline{S}$ designate the vertex of the pencil of sagittal incident rays. If the bundle of incident rays is homocentric, $\overline{S}$ will coincide with S. We have seen that the vertex $\overline{S}'$ of the pencil of sagittal refracted rays is the point of intersection of the chief refracted ray u' with the central line $\overline{S}C$; and, hence, without further study, we may make the following statement:

The range of Object-Points $\overline{P}$, $\overline{Q}$, $\overline{R}$, $\overline{S}$, $\cdots$ *lying on the incident chief ray* u *is in perspective with the corresponding range of II. Image-Points* $\overline{P}'$, $\overline{Q}'$, $\overline{R}'$, $\overline{S}'$, $\cdots$ *lying on the chief refracted ray* u'*; the centre* C *of the spherical refracting surface being the Centre of Perspective of these two corresponding ranges, since the straight lines* $\overline{P}\overline{P}'$, $\overline{Q}\overline{Q}'$, $\overline{R}\overline{R}'$, $\overline{S}\overline{S}'$, $\cdots$ *all pass through the centre* C.

239. The Focal Points $\bar{J}$, $\bar{I}'$ of the Sagittal Rays.

If the Object-Point $\overline{S}$ is the infinitely distant point $\bar{I}$ (or I) of the chief incident ray u, the sagittal incident rays will be a pencil of parallel rays to which will correspond a pencil of sagittal refracted rays with its vertex at the "Flucht" Point $\bar{I}'$ of the range of II. Image-Points lying on the chief refracted ray u'; and, similarly, if the II. Image-Point $\overline{S}'$ coincides with the infinitely distant point $\bar{J}'$ (or J) of the chief refracted ray u', the sagittal refracted rays will be a pencil of parallel rays to which will correspond a pencil of sagittal incident rays with its vertex at the "Flucht" Point $\bar{J}$ of the range of II. Object-Points lying on the chief incident ray u.

The positions of the "Flucht" Points $\bar{J}$ and $\bar{I}'$, or, as we shall now call them, the Primary and Secondary *Focal Points of the Sagittal Rays*, may be found by drawing through C straight lines parallel to u' and u which will meet u and u' in the points $\bar{J}$ and $\bar{I}'$, respectively.

Since the ranges $\bar{P}$, $\bar{Q}$, $\bar{R}$, $\bar{S}$, $\cdots$ and $\bar{P}'$, $\bar{Q}'$, $\bar{R}'$, $\bar{S}'$, $\cdots$ are projective, and since the point B is a double-point of the two ranges, we have the following relation:

$$\bar{J}B \cdot \bar{I}'B = \bar{J}\bar{S} \cdot \bar{I}'\bar{S}' = (\bar{J}B + B\bar{S})(\bar{I}'B + B\bar{S}');$$

and, hence, if we put

$$B\bar{S} = \bar{s}, \quad B\bar{S}' = \bar{s}',$$

we derive an equation exactly analogous to formula (244) which was obtained for the meridian rays, viz.:

$$\frac{B\bar{J}}{\bar{s}} + \frac{B\bar{I}'}{\bar{s}'} = 1. \tag{248}$$

The positions of the Focal Points $\bar{J}$, $\bar{I}'$ may be calculated as follows: Since

$$B\bar{J} = \bar{I}'C, \quad B\bar{I}' = \bar{J}C,$$

we obtain directly (Fig. 128):

$$\left.\begin{aligned} B\bar{J} &= -\frac{r \cdot \sin \alpha'}{\sin (\alpha - \alpha')} = -\frac{nr}{n' \cdot \cos \alpha' - n \cdot \cos \alpha}, \\ B\bar{I}' &= \frac{r \cdot \sin \alpha}{\sin (\alpha - \alpha')} = \frac{n'r}{n' \cdot \cos \alpha' - n \cdot \cos \alpha}. \end{aligned}\right\} \tag{249}$$

240. Formula for Calculating the Position of the II. Image-Point $\bar{S}'$ corresponding to an Object-Point $\bar{S}$ on a given chief incident ray u.

Substituting in formula (248) the values of $B\bar{J}$ and $B\bar{I}'$, as given by formula (249), we obtain the following formula for determining $\bar{s}'$ in terms of $\bar{s}$:

$$\frac{n'}{\bar{s}'} - \frac{n}{\bar{s}} = \frac{n' \cdot \cos \alpha' - n \cdot \cos \alpha}{r}; \tag{250}$$

or, in Abbe's method of writing:

$$\Delta\left(\frac{n}{\bar{s}}\right) = \frac{1}{r}\Delta(n \cdot \cos \alpha). \tag{250a}$$

This formula, which enables us, for given values of r, α and α', to calculate the position of the II. Image-Point $\bar{S}'$ on u' corresponding to a given Object-Point $\bar{S}$ on u, remains true even if we retain infinitesimals of the second order; whereas the corresponding formula (246), which we obtained for the meridian rays, is only correct so long as we neglect infinitesimals of the second order. If we retained infinitesimals of the second order, the formula which would be obtained for the meridian rays would give values of s' which depend on the inclinations of the secondary rays to the chief ray of the pencil of meridian rays: that is, the values of s' would differ from each other by infinitesimals of the first order. Thus, as has been stated above (§ 231), the convergence of the meridian rays at the I. Image-Point is a "convergence of the first order", whereas the convergence of the sagittal rays at the II. Image-Point is a "convergence of the second order".

241. Convergence-Ratio of the Sagittal Rays.

Let $d\bar{\lambda}$, $d\bar{\lambda}'$ denote the angular apertures of the pencils of sagittal incident and refracted rays. Obviously, we have the following relation:

$$\bar{Z}_u = \frac{d\bar{\lambda}'}{d\bar{\lambda}} = \frac{\bar{s}}{\bar{s}'}; \qquad (251)$$

where $\bar{Z}_u$ denotes the Convergence-Ratio of the Sagittal Rays.

ART. 73. THE ASTIGMATIC DIFFERENCE, AND THE MEASURE OF THE ASTIGMATISM.

242. If the bundle of incident rays is homocentric, the Object-Points S and $\bar{S}$ on the chief incident ray u are coincident, and in this case, therefore, we shall have $s = \bar{s}$. Thus,

To a range of Object-Points P, Q, R, S, $\cdots$ lying on the chief incident ray u there corresponds a projective range of I. Image-Points P', Q', R', S', $\cdots$ and a projective range of II. Image-Points $\bar{P}'$, $\bar{Q}'$, $\bar{R}'$, $\bar{S}'$, $\cdots$, both lying on the chief refracted ray u'; and, hence, also, the two ranges of Image-Points are projective with each other.

The points designated in the figures by the letters B and Z' are the double-points of these two projective ranges of Image-Points. At the incidence-point B the Object-Point and its two Image-Points coincide. The point Z', as we saw, is the vertex of the bundle of refracted rays corresponding to a homocentric bundle of incident rays—which need not be an infinitely narrow bundle—with its vertex at the Object-Point Z where the chief incident ray meets the auxiliary spherical surface τ (see § 207).

In the case of an infinitely narrow homocentric bundle of Object-Rays proceeding from the Object-Point S and undergoing refraction at a spherical surface, the astigmatic difference is the segment

$$\bar{S}'S' = s' - \bar{s}'$$

of the chief refracted ray u' comprised between the II. Image-Point and the I. Image-Point. The astigmatic difference may also be defined as the central projection from the homocentric object-point S of the line-segment CK on the chief refracted ray u'.

However, as the *Measure of the Astigmatism* of the astigmatic bundle of refracted rays, it is found more convenient to take, not the length of the segment of the chief refracted ray comprised between the two Image-Points, but the value of the expression

$$\frac{1}{n'}\left(\frac{1}{s'}-\frac{1}{\bar{s}'}\right).$$

Thus, let us suppose that the bundle of incident rays is not homocentric, but that, in consequence, perhaps, of previous refractions, it also is an astigmatic bundle. In this general case the magnitudes denoted by s and $\bar{s}$ will not be equal. Combining equations (246) and (250), and using the relations

$$\cos^2\alpha = 1 - \sin^2\alpha, \quad \cos^2\alpha' = 1 - \sin^2\alpha',$$

we obtain:

$$n'\left(\frac{1}{s'}-\frac{1}{\bar{s}'}\right) - n\left(\frac{1}{s}-\frac{1}{\bar{s}}\right) = \frac{n'\cdot\sin^2\alpha'}{s'} - \frac{n\cdot\sin^2\alpha}{s}; \qquad (252)$$

which equation, by introducing the so-called "Optical Invariant"

$$K = n\cdot\sin\alpha = n'\cdot\sin\alpha',$$

may be written in Abbe's method of notation as follows:

$$\Delta\left\{n\left(\frac{1}{s}-\frac{1}{\bar{s}}\right)\right\} = K^2\cdot\Delta\left(\frac{1}{ns}\right). \qquad (252a)$$

This formula gives the expression for the measure of the *Change of Astigmatism* produced by the refraction of the bundle of rays at the spherical surface.

In particular, we may observe that in the special case when we have $n's' = ns$, the change of astigmatism is equal to zero. Thus, for example, if the bundle of incident rays is homocentric ($\bar{s} = s$), the bundle of refracted rays will be homocentric also ($\bar{s}' = s'$), provided $n's' = ns$. We have seen that Z' is the homocentric Image-Point of the Object-Point Z; and, therefore, we must have $n'\cdot BZ' = n\cdot BZ$ (see § 207). Thus, for the pair of points Z, Z' we obtain from equation (246):

$$\left.\begin{aligned} BZ &= r(\cos\alpha + n'\cdot\cos\alpha'/n), \\ BZ' &= r(n\cdot\cos\alpha/n' + \cos\alpha'). \end{aligned}\right\} \qquad (253)$$

The points Z, Z' are the so-called *aplanatic points* of the spherical refracting surface.

ART. 74. HISTORICAL NOTE, CONCERNING ASTIGMATISM.

243. The theory of Astigmatism, at least in its beginnings and early development, is due almost entirely to British men of science.

The earliest investigations along this line, of which we have any record, are to be found in the optical writings of the distinguished mathematician ISAAC BARROW, professor of Geometry in the University of Cambridge (1663–1669) and the preceptor of NEWTON, who succeeded to his chair in the university, and who aided BARROW in preparing for publication his *Lectiones Opticæ* (London, 1674). In this excellent and interesting work, BARROW investigates very skilfully the paths of rays lying in the meridian plane of a spherical refracting surface, and shows how to construct the I. Image-Point.

But the real discoverer of Astigmatism was Sir ISAAC NEWTON himself, who in his *Lectiones Opticæ Annis* 1669, 1670, 1671 (London, 1728) deals with the problem of the refraction of a narrow bundle of rays at both plane and spherical surfaces, and who not only recognizes the existence of the two Image-Points, but seeks also to determine what intermediate point is selected by the eye as the place of the image.

The next most important advances in this study were made by ROBERT SMITH, who investigated very thoroughly the properties of caustics both by reflexion and by refraction at spherical surfaces, and who showed clearly the relations between the Object-Point and the I. Image-Point, not merely for the case of refraction or reflexion at a single spherical surface, but for the general case of refraction through a centered system of spherical surfaces. See especially Chapter IX of Book 2 of SMITH's *Compleat System of Opticks* (Cambridge, 1738). Thus, in Sec. 423 (Vol. i., p. 165) SMITH finds that

$$JS \cdot I'S' = JB \cdot I'B,$$

where the letters here used refer to Fig. 128. This result is a direct consequence of the fact that if P', Q', R', S', etc., lying on the chief refracted ray u', are the I. Image-Points of P, Q, R, S, etc., respectively, lying on the chief incident ray u, these two ranges of points, as we found in § 233, are *projective* with each other, so that we have:

$$(PQRS) = (P'Q'R'S').$$

For example, according to this relation, we have (Fig. 128):

$$JS \cdot I'S' = JY \cdot I'Y' = JZ \cdot I'Z' = JB \cdot I'B = \text{a constant}.$$

For the *Construction of the Focal Points J, I'*, SMITH gives (Sec. 419, Vol. i., p. 164) the following convenient method: From the centre C (Fig. 128) of the spherical refracting surface draw CY, CY' perpendicular at Y, Y' to the chief incident and refracted rays u, u', respectively; and draw the radius CB to the point of incidence B. From Y, Y' drop perpendiculars on CB, and through the foot of the perpendicular let fall from Y draw a straight line parallel to u', and through the foot of the perpendicular let fall from Y' draw a straight line parallel to u. These straight lines will intersect u and u' in the required points J and I', respectively.

Among the most important contributions to this subject are those of THOMAS YOUNG, who recognized clearly and distinctly the value of NEWTON's discovery of Astigmatism. In YOUNG's celebrated paper "On the mechanism of the eye" (*Phil. Trans.*, 1801, cii., 23–88; reprinted in the *Miscellaneous Works of the late* THOMAS YOUNG, London, 1855), he gives the formula, obtained first by L'HOSPITAL ("*Analyse des infiniment petits*", Second Edition, Paris, 1716), for calculating the intercept s' on the chief refracted ray of the meridian rays, and shows how to find the positions of the Focal Points I', $\bar{I}'$ of both the meridian and the sagittal rays. Moreover, YOUNG perceived the perspective centres K and C of the rays of the meridian and sagittal sections of the narrow bundle of rays, and also discussed very thoroughly the astigmatism of the eye. In his *Lectures on Natural Philosophy* (London, 1807), YOUNG gives, likewise, the formula for the intercept $\bar{s}'$ of the sagittal refracted rays and applies all these various formulæ to a number of important special cases. He seems also to have been the first to recognize the existence of "image-lines". Moreover, YOUNG was cognizant of the so-called "aplanatic" points of a refracting sphere.

The contributions of AIRY[1] and of CODDINGTON[2] to the theory of astigmatism deserve also to be ranked among the most important.

For a complete and very learned account of the theory of Astigmatism from the earliest times to the present, the reader is referred to the historical note, "Ueber den Astigmatismus", at the end of P. CULMANN's article entitled "Die Realisierung der optischen Abbildung", which is Chapter IV of *Die Theorie der optischen Instrumente*, edited by M. VON ROHR (Berlin, 1904).

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[1] G. B. AIRY: On a peculiar defect in the eye and mode of correcting it: *Camb. Phil. Trans.* (1827), ii., 227–252. Also: On the spherical aberration of the eye-pieces of telescopes (Cambridge, 1827); and in *Camb. Phil. Trans.*, iii. (1830), 1–64.

[2] H. CODDINGTON: *A Treatise on the Reflexion and Refraction of Light:* London, 1829.

ART. 75. INQUIRY AS TO THE NATURE AND POSITION OF THE IMAGE OF AN EXTENDED OBJECT FORMED BY NARROW ASTIGMATIC BUNDLES OF RAYS.

244. It appears, therefore, that, in general, when an infinitely narrow homocentric bundle of rays is refracted at a spherical surface, the bundle of refracted rays will not be homocentric, but will be astigmatic; so that to an Object-Point there corresponds, not a single Image-Point, such as we have in the case of ideal imagery, but a pair of infinitely short Image-Lines at right angles to each other and lying in different planes. An eye placed on the chief refracted ray may accommodate itself to regard either of these two Image-Lines as the image of the Object-Point whence the rays emanate.

If, instead of one single Object-Point, we have an aggregation of such points forming an extended object, each of these points being the vertex of an infinitely narrow bundle of incident rays whose chief rays (we may suppose) all meet the spherical refracting surface at the same point B, the image of the object will be more or less blurred and distorted. Thus, if the eye is accommodated to view the primary Image-Lines, the dimensions of the object parallel to these lines will be exaggerated in the image, whereas when the eye is focussed on the other set of Image-Lines, there will be a similar exaggeration parallel to these lines, so that in either case the quality of the image will be defective. Thus, as a rule, we do not obtain either faithful or distinct images by means of astigmatic bundles of rays. It is assumed by most writers that on the whole the best image in such a case will be obtained by accommodating the eye to view neither of the two sets of Image-Lines of the astigmatic bundle of rays, but a place lying somewhere between these, the place of the so-called "*Circle of Least Confusion*". In fact, it is said, the eye unconsciously selects these sections of the astigmatic bundles of rays.[1] Corresponding to each point of the object, the eye will thus see a small area, so that according to this view of the matter, the image of an object, as HEATH expresses it, "is taken to be the aggregation of the overlapping 'Circles of least confusion'." In general, this is no doubt a correct explanation, but in some special cases a more perfect and satisfactory image may be obtained by viewing the Image-Lines directly.

CZAPSKI[2] considers, for example, the case of an infinitely short

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[1] See, for example, HEATH'S *Geometrical Optics* (Cambridge, 1887), Art. 145. Also, O. LUMMER'S work on Optics, published as Vol. II of the Ninth Edition of MUELLER-POUILLET'S *Lehrbuch der Physik*, Art. 183. — The designation of this section of the bundle of rays as the place of "least confusion" is rather misleading, as the *definition* is better at either of the two Image-Lines.

[2] S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), S. 76. See.

Object-Line a perpendicular, say, at S to the incident ray SB (or u) which we may consider here as a "mean" incident chief ray. From each point of the Object-Line a there proceeds a bundle of object-rays which all meet the spherical refracting surface in points closely adjacent to the incidence-point B of the "mean" incident chief ray. We shall regard as the chief rays of all these bundles of rays those rays which meet the refracting surface at the point B. Thus, as CZAPSKI says, the conditions are very nearly the same for all these bundles of object-rays, and, therefore, we shall have very nearly the same phenomena. It is true that the chief rays of the bundles will meet the spherical refracting surface at slightly different angles of incidence, and, consequently, the astigmatic differences of the corresponding bundles of refracted rays will be also slightly different; but the I. Image-Lines will all be very nearly parallel, and the same is true also of the II. Image-Lines. Moreover, the lengths of the lines in each group will not be very different from each other.

The image of the Object-Line a will appear, therefore, as the assemblage of the I. Image-Lines a' (Fig. 129) or of the II. Image-Lines $\bar{a}'$ (Fig. 129), according as the eye is focussed to view one or the other of these aggregations. In either case, the image will evidently be an unsatisfactory representation of the object.

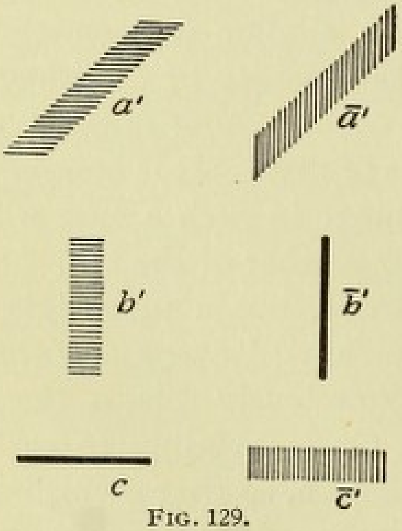

FIG. 129.

IMAGES OF A SMALL OBJECT-LINE PERPENDICULAR TO THE "MEAN" CHIEF INCIDENT RAY, as produced by means of Narrow Astigmatic Bundles of Refracted Rays.

But if the Object-Line is a short line b lying in the plane of incidence of the "mean" chief ray (Meridian Plane), and perpendicular to this ray, we have a special case that is worth considering. Now all the chief rays of the bundles of object-rays will lie in the Meridian Plane, so that the I. Image-Points S' and the II. Image-Points $\bar{S}'$ corresponding to the points of the Object-Line b all lie in the plane of incidence. The I. Image-Lines are perpendicular to the plane of incidence at the I. Image-Points, so that the image produced by this assemblage of Image-Lines will have the form of a rectangle b' (Fig. 129) perpendicular to the plane of incidence. The II. Image-Lines, on the other hand, lie in the plane of incidence, being

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also, P. CULMANN's article "Die Realisierung der optischen Abbildung", which forms Chapter IV of *Theorie der optischen Instrumente*, edited by M. VON ROHR (Bd. I, Berlin, 1904), S. 167. Also, *Theorie und Geschichte des Photographischen Objektivs*, by M. VON ROHR (Berlin, 1899), 42, 43.

spread over a small area in this plane, so that if a screen were placed at right angles to the "mean" refracted chief ray at $\overline{S}'$, it would intersect the bundles of refracted rays in a line. Thus, an eye placed on the "mean" refracted chief ray and accommodated to the II. Image-Point $\overline{S}'$ would view there a linear image $\bar{b}'$ (Fig. 129) of the linear object b.

An entirely analogous case is presented when the Object-Line is a small line c lying in the plane containing the "mean" incident chief ray and perpendicular to the plane of incidence. An eye placed on the "mean" refracted chief ray and accommodated for the I. Image-Point S' would see there an image of c in the form of a straight line c' (Fig. 129) parallel to c itself; whereas if the eye were focussed on the II. Image-Point $\overline{S}'$, the image of c will be found to be a rectangular figure $\bar{c}'$ (Fig. 129) perpendicular at $\overline{S}'$ to the plane of incidence.[1]

ART. 76. COLLINEAR RELATIONS IN THE CASE OF THE REFRACTION OF A NARROW BUNDLE OF RAYS AT A SPHERICAL SURFACE.

245. The Principal Axes of the Two Pairs of Collinear Plane Systems.

To the chief ray u of an infinitely narrow homocentric bundle of incident rays which meets the spherical refracting surface at the incidence-point B corresponds the chief refracted ray u' of the astigmatic bundle of refracted rays. Both the incident meridian rays and the refracted meridian rays proceed in the plane uu', which may, therefore, be designated as the plane π or π'. Similarly, the planes of the incident sagittal rays and the refracted sagittal rays may be designated by the symbols $\bar{\pi}$, $\bar{\pi}'$, respectively.

Consider, first, a point V lying in the plane π of the meridian rays and very near to the chief incident ray u; and let us suppose that V itself, regarded as an Object-Point, is the vertex of a narrow bundle of incident rays all meeting the spherical surface at points nearly adjacent to the incidence-point B. The incident ray VB (or v) lying in the plane π may be treated as the chief ray of this bundle. The angle at B between the rays u, v being an infinitesimal angle of the first order, so likewise is the angle between the corresponding refracted rays u', v'; and, since v lies in the plane uC or π, v' will lie in the plane $u'C$ or π', which is coincident with π; and to the pencil of incident rays proceeding from V and lying in the plane π will correspond a pencil of refracted rays lying in the plane π' and converging to the I. Image-Point V' on v'. Thus, if we utilize only such rays as before and after refraction proceed infinitely near to u and u', respectively, the plane-fields π, π' will be characterized by the property that to a homocentric pencil of rays of π there corresponds by refraction a homocentric pencil of rays of π'.

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[1] For a very clear and interesting treatment of the images formed by astigmatic bundles of rays see L. Matthiessen: Ueber die Form der unendlich duennen astigmatischen Strahlenbuendel und ueber die Kummer'schen Modelle: *Sitzungber. der math.-phys. Cl. der k. bayer. Akad. der Wiss. zu Muenchen*, xiii. (1883), 35–51.

In the next place, let us consider a point $\overline{W}$ lying in the plane of the sagittal section and also infinitely near to the chief ray u of the bundle of incident rays. Regarding $\overline{W}$ as an Object-Point, we shall suppose that it is the vertex of a narrow bundle of incident rays whose chief ray $\overline{w}$ meets the spherical refracting surface also at the point B so that the angles between the two incident chief rays u and $\overline{w}$ and between the corresponding refracted chief rays u' and $\overline{w}'$ are both infinitesimal angles of the first order. If we use YOUNG's Construction (§ 206) for drawing the refracted ray $\overline{w}'$, it will be obvious that, if we neglect infinitesimals of the second order, $\overline{w}'$ will lie in the plane $\overline{\pi}'$; and, with the same degree of exactness, all incident rays proceeding from $\overline{W}$ and lying in the plane $\overline{\pi}$ will, after refraction, lie in the plane $\overline{\pi}'$, and will converge to the II. Image-Point $\overline{W}'$ on $\overline{w}'$ corresponding to the Object-Point $\overline{W}$ on $\overline{w}$.

Thus, within the infinitely narrow region surrounding the so-called "mean" incident chief ray u in the Object-Space and the corresponding refracted chief ray u' in the Image-Space, we have a *collinear* relation between the plane-fields π, π' and also between the plane-fields $\overline{\pi}$, $\overline{\pi}'$; because to every incident ray in π (or $\overline{\pi}$) there corresponds a refracted ray in π' (or $\overline{\pi}'$), and to every Object-Point of π (or $\overline{\pi}$) there corresponds a I. (or II.) Image-Point of π' (or $\overline{\pi}'$).

It may be remarked also that the plane-fields π, π' have in common the range of points which lie in the plane of incidence π along the tangent to the spherical refracting surface at the incidence-point B; whereas the plane-fields $\overline{\pi}$, $\overline{\pi}'$ have in common the range of points which lie in the line of intersection of these planes.

These results, which appear to have been first obtained by LIPPICH,[1] may, accordingly, be stated as follows:

(1) The plane-fields π, π' lying in the plane of incidence are in perspective with each other; and

(2) The plane-fields $\overline{\pi}$, $\overline{\pi}'$, which are both perpendicular to the plane uu', and which contain u, u', respectively, are likewise in perspective with each other.

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[1] F. LIPPICH: Ueber Brechung und Reflexion unendlich duenner Strahlensysteme an Kugelflaechen: *Denkschriften der kaiserl. Akad. der Wissenschaften zu Wien*, xxxviii. (1878), 163–192.

In order to get a proper idea of the imagery which we obtain by means of the meridian rays, suppose we consider an infinitely short object-line SV lying in the plane of incidence SBC and perpendicular at S to the "mean" incident chief ray SB (or u). To the narrow pencil of meridian object-rays with its vertex at V will correspond a pencil of meridian image-rays with its vertex at the I. Image-Point V'' on the refracted ray BV'' corresponding to the incident chief ray VB; and if S' is the I. Image-Point on u' corresponding to the Object-Point S on u, the infinitely short line $S'V''$, which is the image of the object-line SV will, in general, not be perpendicular to the "mean" refracted chief ray u' (or BS'). Accordingly, let us draw $S'V'$ perpendicular to BS' at S' and meeting BV'' in the point V'. Now the distance between the two points V' and V'' and also the angular aperture of the pencil of image-rays V'' are infinitesimals of the first order, and, therefore, the piece of $S'V'$ intercepted between the two extreme rays of this pencil will be an infinitesimal of the second order, and, consequently, may be treated as a mere point, since here we neglect infinitesimals of order higher than the first. Thus, according to ABBE, we may regard the point V' as the vertex of the pencil of image-rays corresponding to object-rays proceeding from V and $S'V'$, therefore, as the image of SV. In brief, provided we neglect infinitesimals of the second order, we have a right to say that the image, by means of meridian rays, of an infinitely short object-line perpendicular to the "mean" incident chief ray is an infinitely short line perpendicular to the "mean" refracted chief ray.[1]

If $d\lambda$ denotes the inclination to the chief ray u of a secondary ray of the pencil of meridian object-rays whose vertex is at S, and if $d\lambda'$ denotes the inclination to the chief refracted ray u' of the corresponding refracted secondary ray, it is a very simple matter to show that (always neglecting infinitesimals of the second order) we have for the meridian rays the following relation:

$$n \cdot SV \cdot d\lambda = n' \cdot S'V' \cdot d\lambda';$$

which will be recognized as perfectly analogous to the Law of ROBERT SMITH for the refraction of paraxial rays, the so-called LAGRANGE-HELMHOLTZ Formula, § 194.

And, finally, if we consider in the same way the imagery in the planes $\bar{\pi}$, $\bar{\pi}'$, it will be obvious, on mere grounds of symmetry, that the image, by means of the sagittal rays, of an infinitely short object-line $S\overline{W}$ lying in the plane $\bar{\pi}$ and perpendicular at S to the "mean" incident chief ray u will be an infinitely short line $\overline{S}'\overline{W}'$ in the plane $\bar{\pi}'$ and perpendicular to the refracted chief ray u' at the II. Image-Point $\overline{S}'$ corresponding to the object-point S; provided that here also we neglect infinitesimals of the second order.

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[1] See CZAPSKI'S *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), S. 78. Also, P. CULMANN'S "Die Realisierung der optischen Abbildung", which forms Chapter IV of *Die Theorie der optischen Instrumente*, edited by M. VON ROHR (Bd. I, Berlin, 1904), S. 171.

Thus, according to ABBE, the "mean" incident chief ray u and the corresponding refracted ray u' are to be regarded as the *Principal Axes* of the narrow collinear plane-fields π, π' and also of the narrow collinear plane-fields $\bar{\pi}$, $\bar{\pi}'$, since in both cases to an object-line perpendicular to u there corresponds, as we have seen, an image-line perpendicular to u'. This was not the case in LIPPICH's mode of treating this matter, but it will be found to simplify the problem very greatly to be able to consider the chief rays u, u' as the Principal Axes of the two pairs of collinear plane systems.

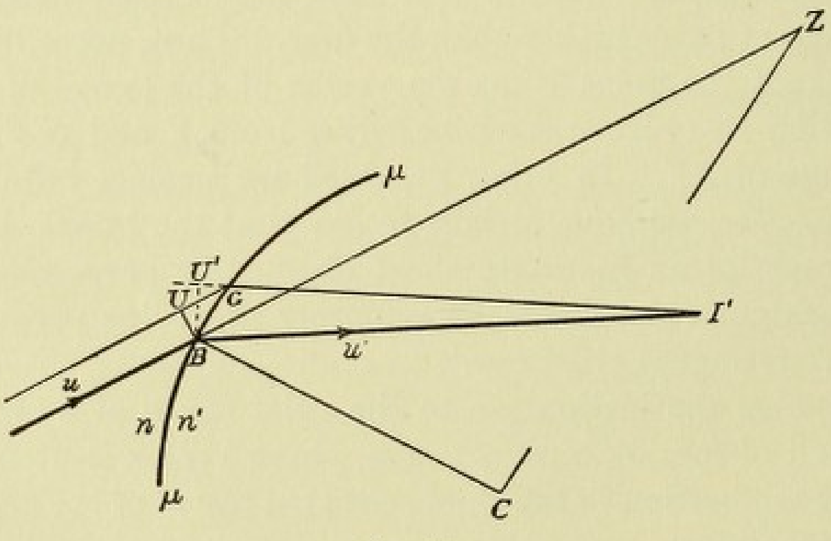

FIG. 130.

FIGURE FOR FINDING THE SECONDARY FOCAL LENGTH (e_u') OF THE SYSTEM OF MERIDIAN RAYS.

246. Having determined the Principal Axes, we can now proceed to obtain the formulæ for calculating **The Focal Lengths** of the two plane systems of rays; the Focal Lengths being defined as in § 178.

For example, Fig. 130 represents the case of a narrow pencil of parallel meridian incident rays to which corresponds a pencil of refracted meridian rays with its vertex at the Focal Point I'. The incidence-points of the chief ray u and a secondary ray of the pencil of incident rays are designated in the diagram by the letters B and G, respectively; the corresponding refracted rays are BI' (or u') and GI'.

At B erect BU, BU' perpendicular to u, u' and meeting the secondary incident ray and the secondary refracted ray in the points U, U', respectively. If, then, we put $d\lambda' = \angle BI'G$, and if e'_u denotes the Secondary Focal Length of the system of meridian rays for which u and u' are the chief incident and refracted rays, according to the definition referred-to above, we shall have:

$$e'_u = \frac{BU}{d\lambda'}.$$

Similarly, in the case of a pencil of parallel meridian refracted rays emanating before refraction from the Focal Point J on the chief incident ray u, the Primary Focal Length f_u will be given by the formula:

$$f_u = \frac{BU'}{d\lambda},$$

where $d\lambda = \angle BJG$.

If α, α' denote the angles of incidence and refraction of the chief ray, we have evidently the following relations:

$$BU = BG\cdot\cos\alpha, \qquad BU' = BG\cdot\cos\alpha',$$
$$d\lambda = -\frac{BU}{BJ} = -\frac{BG\cdot\cos\alpha}{BJ}, \quad d\lambda' = -\frac{BU'}{BI'} = -\frac{BG\cdot\cos\alpha'}{BI'};$$

whence, therefore, we obtain:

$$f_u = JB\cdot\frac{\cos\alpha'}{\cos\alpha}, \quad e'_u = I'B\cdot\frac{\cos\alpha}{\cos\alpha'}.$$

Thus, we see that the Focal Lengths f_u and e'_u are not equal to the segments JB and $I'B$ on u and u' comprised between the Focal Points J and I', respectively, and the incidence-point B, as, having in mind the special case of normally incident rays, where we have $f = FA$, $e' = E'A$, we might have expected.

Substituting for JB and $I'B$ their values as derived from formulæ (245), we obtain finally:

$$f_u = \frac{nr\cdot\cos\alpha\cdot\cos\alpha'}{n'\cdot\cos\alpha' - n\cdot\cos\alpha}, \quad e'_u = -\frac{n'r\cdot\cos\alpha\cdot\cos\alpha'}{n'\cdot\cos\alpha' - n\cdot\cos\alpha}. \quad (254)$$

By a process exactly similar to the above, we shall obtain, even more simply, for the Focal Lengths $\bar{f}_u$, $\bar{e}'_u$ of the system of sagittal rays, for which u and u' are the chief incident and refracted rays, expressions as follows:

$$\bar{f}_u = \bar{J}B, \quad \bar{e}'_u = \bar{I}'B,$$

so that in the case of the sagittal rays the distances of the incidence-point B from the Focal Points $\bar{J}$ and $\bar{I}'$ are equal to the Focal Lengths. Thus, employing formulæ (249), we obtain:

$$\bar{f}_u = \frac{nr}{n'\cdot\cos\alpha' - n\cdot\cos\alpha}, \quad \bar{e}'_u = -\frac{n'r}{n'\cdot\cos\alpha' - n\cdot\cos\alpha}. \tag{255}$$

For given values of the constants n, n' and r, the Focal Lengths f_u, e'_u and $\bar{f}_u$, $\bar{e}'_u$, as we see from formulæ (254) and (255), depend only on the angle of incidence (that is, therefore, on the slope and position) of the chief incident ray u. In the special case when the chief incident ray meets the spherical surface normally at the vertex A, by putting $\alpha = 0$, in the formulæ (254) and (255), and writing F in place of J or $\bar{J}$ and E' in place of I' or $\bar{I}'$ and A in place of B, we obtain:

$$\text{For } \alpha = 0: \quad \begin{aligned} f_u &= \bar{f}_u = f = FA = nr/(n' - n), \\ e' &= e'_u = \bar{e}'_u = E'A = -n'r/(n' - n); \end{aligned}$$

as in formulæ (147) of Chapter VIII.

Moreover, we find also:

$$f_u/e'_u = \bar{f}_u/\bar{e}'_u = -n/n', \tag{256}$$

which corresponds with the relation already found in Chapter VIII, viz., $f/e' = -n/n'$.

The magnification-ratios for the meridian and sagittal rays may be derived without difficulty by means of the formulæ given in Chap. VII, § 179.

ART. 77. REFRACTION OF NARROW BUNDLE OF RAYS THROUGH A CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES.

247. Formulæ for Calculating the Astigmatism of the Bundle of Emergent Rays.

We shall consider here only the simple case when the chief incident ray u_1 lies in a plane which contains the optical axis of the centered system of spherical surfaces. Thus, all the meridian sections of the astigmatic bundles of rays arising by refraction at the successive surfaces will lie in this plane.

Let B_k designate the point where the chief ray meets the kth spherical refracting surface, and let

$$\delta_k = B_k B_{k+1}$$

denote the length of the path of the chief ray comprised between the incidence-point B_k at the kth surface and the incidence-point B_{k+1} at the $(k+1)$th surface. Moreover, let S'_k, $\overline{S}'_k$ designate the positions on the chief ray of the I. and II. Image-Points, respectively, after the refraction of the ray at the *k*th surface. We shall employ also the following symbols:

$$B_kS'_{k-1} = s_k, \quad B_kS'_k = s'_k,$$
$$B_k\overline{S}'_{k-1} = \bar{s}_k, \quad B_k\overline{S}'_k = \bar{s}'_k.$$

The relations between these intercepts on the chief ray before and after refraction at the *k*th surface are given, for the meridian rays, by formula (246) and, for the sagittal rays, by formula (250) of this Chapter. Thus if r_k $(= A_kC_k)$ denotes the radius of the *k*th spherical surface, and if α_k, α'_k denote the angles of incidence and refraction of the chief ray at this surface, we shall have:

$$\frac{n'_k \cdot \cos^2\alpha'}{s'_k} - \frac{n'_{k-1} \cdot \cos^2\alpha_k}{s_k} = \frac{n'_k}{\bar{s}'_k} - \frac{n'_{k-1}}{\bar{s}_k} = V_k, \tag{257}$$

where, by way of abbreviation, we have put:

$$\frac{n'_k \cdot \cos\alpha'_k - n'_{k-1} \cdot \cos\alpha_k}{r_k} = V_k; \tag{258}$$

this magnitude being called sometimes the "astigmatic constant" of the *k*th spherical surface for the ray incident on it at the angle α_k.

For the Logarithmic Computation of the positions on the emergent chief ray of the I. and II. Image-Points S'_m and $\overline{S}'_m$ corresponding to an Object-Point S_1 on the chief incident ray, it will be necessary, in the first place, to determine, by means of the system of formulæ (215) of Chapter X, the path of the chief ray through the centered system of *m* spherical refracting surfaces, whereby we shall obtain the values of the angles of incidence α, α' at each surface in succession. We may then proceed to employ the following system of formulæ, which are written in a form adapted to logarithmic work:

$$\left.\begin{array}{l}
V_k = \dfrac{n'_{k-1} \cdot \sin(\alpha_k - \alpha'_k)}{r_k \cdot \sin\alpha'_k}. \\[2ex]
\text{I. } \textit{Meridian Rays:} \\[1ex]
\dfrac{1}{s'_k} = \dfrac{n'_{k-1} \cdot \cos^2\alpha_k}{n'_k \cdot \cos^2\alpha'_k} \cdot \dfrac{1}{s_k} + \dfrac{V_k}{n'_k \cdot \cos^2\alpha'_k}; \\[2ex]
s_{k+1} = s'_k - \delta_k. \\[2ex]
\text{II. } \textit{Sagittal Rays:} \\[1ex]
\dfrac{1}{\bar{s}'_k} = \dfrac{n'_{k-1}}{n'_k} \cdot \dfrac{1}{s_k} + \dfrac{V_k}{n'_k}; \\[2ex]
\bar{s}_{k+1} = \bar{s}'_k - \delta_k.
\end{array}\right\} \tag{259}$$

In these formulæ k must receive in succession all integral values from $k = 1$ to $k = m$ ($\delta_m = 0$). Accordingly, if we are given the values of the constants of the optical system, that is, the magnitudes denoted by n, r and d, and if we are also given the ray-co-ordinates (v_1, θ_1) of the chief ray incident on the first spherical surface, so that we have the data for determining, by means of formulæ (215) of Chapter X, the magnitudes denoted by α, α' and δ; and, if finally, we are given the positions on the chief incident ray of the I. Object-Point S_1 and of the II. Object-Point $\overline{S}_1$, that is, if we are given the values of the intercepts $s_1 (= B_1S_1)$ and $\bar{s}_1$ $(= B_1\overline{S}_1)$; we can, by successive substitutions in formulæ (259), obtain the values of the magnitudes $\bar{s}'_m$ $(= B_m\overline{S}'_m)$ and s'_m $(= B_mS'_m)$, and thus determine the positions on the emergent chief ray of the I. and II. Image-Points S'_m and $\overline{S}'_m$, and the magnitude of the Astigmatic Difference $\overline{S}'_mS'_m = s'_m - \bar{s}'_m$. The calculation, to be sure, is quite long and tedious, especially if the system consists of as many as four or five refracting surfaces; but there is no shorter process of solving the required problem.[1]

The condition that the Astigmatic Difference of the bundle of emergent rays shall vanish is $\overline{S}'_mS'_m = 0$, or $s'_m = \bar{s}'_m$. If the Optical System consists of a single Lens ($m = 2$), it is not difficult to show that this condition leads to a quadratic equation for determining $s_1 (= \bar{s}_1)$. The problem of the Homocentric Refraction of Light-Rays through a Lens has been beautifully and completely investigated by L. BURMESTER.[2] By a simple process of geometrical reasoning, he shows that when an infinitely narrow bundle of rays is refracted through a Lens, there are two object-points (which may be real or imaginary, and which may be coincident) lying on the chief object-ray, to each of which there corresponds on the chief image-ray a "Homocentric" Image-Point. Moreover, the same reasoning can be extended immediately to show that the same thing is true also in the case of a centered system of any number of spherical refracting surfaces. BURMESTER shows also how to construct the two object-points and the corresponding "Homocentric" Image-Points in the case of a Lens, and discusses a number of interesting special cases.

248. Collinear Relations.

Within the infinitely narrow region surrounding the chief ray before and after refraction at the kth spherical surface, we have a collinear relation between the plane-systems π'_{k-1} and π'_k, which lie in the plane of the meridian section of the centered system of spherical refracting surfaces; and, likewise, a collinear relation between the plane-systems $\bar{\pi}'_{k-1}$ and $\bar{\pi}'_k$, which lie in the planes of the sagittal sections of the astigmatic bundles of rays before and after refraction at the kth surface. In Art. 76 we saw that the chief rays before and after refraction at this surface were to be regarded as the Principal Axes of each of these two pairs of collinear plane systems. And since the chief ray after refraction at the kth surface is identical with the chief ray before refraction at the $(k + 1)$th surface, the following is the state of things which we have here:

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[1] See A. GLEICHEN: *Lehrbuch der geometrischen Optik* (Leipzig und Berlin, B. G. TEUBNER, 1902), pages 441–467, for the complete calculation of the "Astigmatische Bildpunkte" of P. GOERZ's Double Anastigmatic Photographic Objective.

[2] L. BURMESTER: Homocentrische Brechung des Lichtes durch die Linse: *Zft. f. Math. u. Phys.*, xl. (1895), 321.

The Principal Axis of the Image-Space of the kth surface is at the same time the Principal Axis of the Object-Space of the $(k + 1)$th surface; and it will be recalled that this is precisely the one condition that was assumed in Chapter VII, Art. 52, in deriving the formulæ for finding the determining-constants of a compound system due to the combination of any number of given simpler systems. Thus, if we know the positions of the Focal Points J_k, I'_k and $\bar{J}_k$, $\bar{I}'_k$ and the magnitudes of the Focal Lengths $f_{u,k}$, $e'_{u,k}$ and $\bar{f}_{u,k}$, $\bar{e}'_{u,k}$ for the Meridian and Sagittal Rays, respectively, for each one of the m spherical surfaces of the centered system, we can employ straightway the formulæ referred-to above, in order to determine the positions of the Focal Points J, I' and $\bar{J}$, $\bar{I}'$ and the magnitudes of the Focal Lengths f_u, e'_u and $\bar{f}_u$, $\bar{e}'_u$ of the entire compound system.

Obviously, we may also employ here exactly the same method as was used in Chapter VIII, Art. 54, for finding the Focal Lengths of a centered system of spherical refracting surfaces for the case of Paraxial Rays. Thus, for the Sagittal Rays we should find without difficulty:

$$\bar{e}'_u = \bar{I}'B_m \cdot \frac{\bar{s}'_1 \cdot \bar{s}'_2 \cdots \bar{s}'_{m-1}}{\bar{s}_2 \cdot \bar{s}_3 \cdots \bar{s}_m}, \qquad (\bar{s}_1 = \infty). \qquad (260)$$

For the case of the Meridian Rays, since (Fig. 130)

$$\frac{B_k U'_k}{B_k U_k} = \frac{\cos\alpha'_k}{\cos\alpha_k},$$

we should find, in the same way, the following formula:

$$e'_u = I'B_m \cdot \frac{\cos\alpha_1 \cdot \cos\alpha_2 \cdots \cos\alpha_m}{\cos\alpha'_1 \cdot \cos\alpha'_2 \cdots \cos\alpha'_m} \cdot \frac{s'_1 \cdot s'_2 \cdots s'_{m-1}}{s_2 \cdot s_3 \cdots s_m}, \quad (s_1 = \infty). \qquad (261)$$

Thus, having found by means of formulæ (260) and (261) the magnitudes of the two Secondary Focal Lengths e'_u and $\bar{e}'_u$, the magnitudes of the Primary Focal Lengths f_u and $\bar{f}_u$ can be calculated from the following relations:

$$\frac{f_u}{e'_u} = \frac{\bar{f}_u}{\bar{e}'_u} = -\frac{n_1}{n'_m}. \tag{262}$$

ART. 78. SPECIAL CASES.

249. The Special Case of the Refraction of a Narrow Bundle of Rays at a Plane Surface.

When we are given a chief ray u incident at a certain point B of a spherical refracting surface, we have seen how we can construct the corresponding refracted ray u' (Chapter IX, § 206) and determine the position of a certain fixed point K (§ 234), which is the centre of perspective of the range of Object-Points lying on u and the corresponding range of I. Image-Points lying on u', just as the centre C of the sphere is also the centre of perspective of the range of Object-Points lying on u and the range of II. Image-Points lying on u'. We saw also that when the radius of the spherical surface varies, these points C and K do not remain fixed, but *move along two fixed straight lines*. In particular, if the radius of the refracting surface becomes infinite, so that this surface is, therefore, a *Plane Surface*, the points C and K will be the infinitely distant points of the two fixed straight lines. And, hence, in the case of a Plane Refracting Surface, as was shown in Chapter III, Art. 20, the straight lines joining the Object-Points lying on the chief incident ray u with their corresponding II. Image-Points lying on u' will all be parallel to the fixed straight line BC normal to the refracting plane; and, similarly, the straight lines joining the Object-Points lying on u with their corresponding I. Image-Points lying on u' will all be parallel to the other fixed straight line BK. In this special case, therefore, the range of Object-Points on u and the two ranges of I. and II. Image-Points on u' are *three similar ranges of points*.[1]

The refracted ray u' corresponding to a given ray u incident on a plane refracting surface $\mu\mu$ at the point B (Fig. 131) may be constructed by using YOUNG's Construction, as follows:

On the incidence-normal take a point O, and with this point as centre and with radii equal to $n' \cdot OB/n$ and $n \cdot OB/n'$ describe in the plane of incidence the arcs of two concentric circles c_1, c_2. If Z_1 designates the point of intersection of the incident ray u with the arc c_1, and if Z_2 designates the point of intersection of the straight line OZ_1 with the arc c_2, the straight line BZ_2 will be the path of the refracted ray u'.

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[1] See F. LIPPICH: Ueber Brechung und Reflexion unendlich duenner Strahlensysteme an Kugelflaechen: *Denkschr. der kaiserl. Akad. der Wissenschaften zu Wien*, xxxviii. (1878), 163–192.

Also, F. KESSLER: Beitraege zur graphischen Dioptrik: *Zft. f. Math. u. Phys.*, xxix. (1884), 65–74.

The normal to the plane refracting surface gives the direction of the infinitely distant point C. The direction of the infinitely distant point K is found by drawing OY perpendicular to BZ_1 and YH perpendicular to OZ_1. Then the point K will be the infinitely distant point of the straight line BH.

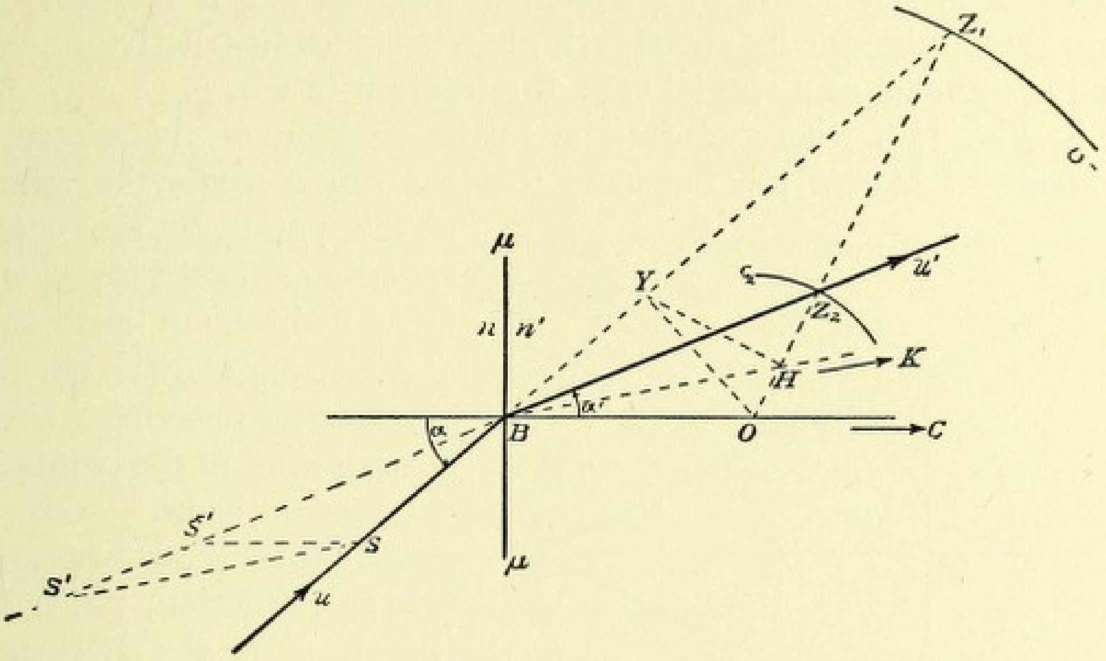

Fig. 131.

Refraction of Infinitely Narrow Bundle of Rays at a Plane Surface. Construction of Chief Refracted Ray u' Corresponding to Chief Incident Ray u; and Construction of I. and II. Image-Points S' and $\overline{S}'$ corresponding to a given Object Point S on u. Centres of Perspective C and K both at infinity. Plane Surface is regarded as a Spherical Surface with Infinite Radius.

The I. Image-Point S' and the II. Image-Point $\overline{S}'$ corresponding to an Object-Point S on the chief incident ray u are found by drawing through S straight lines parallel to BK and BC, which will meet the chief refracted ray u' in the required points S' and $\overline{S}'$ respectively. The Focal Points of the Meridian and Sagittal Rays coincide with the infinitely distant points of the chief incident and refracted rays.

By putting $r = \infty$ in the formulæ of Arts. 71 and 72 of this chapter, we shall derive immediately the same formulæ as were obtained in Chapter III, Art. 19.

250. Reflexion at a Spherical Mirror Treated as a Special Case of Refraction at a Spherical Surface.

In the case of Reflexion ($n'/n = -1$), we cannot use Young's

Construction for constructing the reflected ray u' corresponding to a ray u incident on a spherical mirror, for the obvious reason that the auxiliary spherical surfaces τ and τ', and with them the Aplanatic Points Z, Z', used in this construction (§§ 206, 207), have here no meaning. Except, however, such properties as depend on these particular features, we have in the case of Reflexion at a Spherical Mirror relations corresponding precisely to those which we found in the investigation of Refraction at a Spherical Surface. It is very easy to obtain these relations independently, but it is also instructive to consider the problem as a special case of refraction (§ 26).

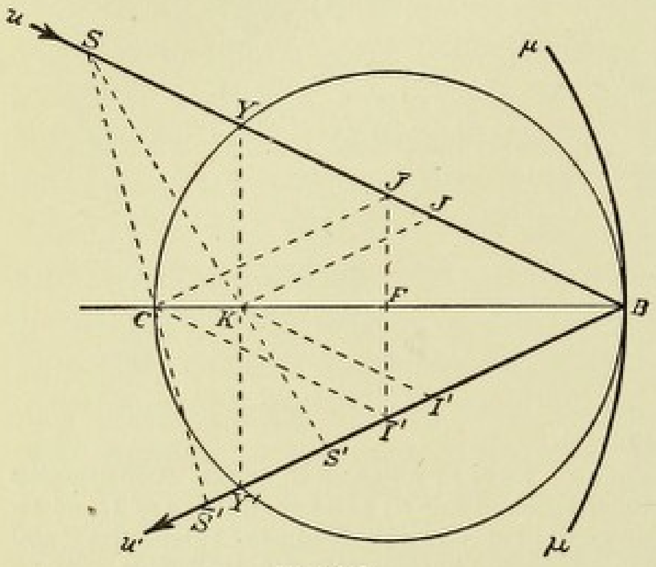

Fig. 132.

Reflexion of Infinitely Narrow Bundle of Rays at a Spherical Mirror. u, u' Chief Incident and Reflected Rays, respectively.

$\angle CBS = \alpha = \angle S'BC, \quad BS = s, \quad BS' = s', \quad B\bar{S}' = \bar{s}'.$

If in Fig. 132, where C designates the position of the centre of the Spherical Mirror $\mu\mu$, the chief incident ray u meets the mirror at the point B, the corresponding reflected ray u' will have a direction such that $\angle CBu = \angle u'BC$. On CB as diameter describe a circle cutting u, u' in the points Y, Y', respectively. Obviously, exactly as was the case in refraction, the point Y' on u' is the I. Image-Point of the Object-Point Y on u (§ 233); and, hence, the centre of perspective K (§ 234) of the range of Object-Points on u and the range of corresponding I. Image-Points on u' will lie on the straight line YY'. The actual position of K is found by drawing CK perpendicular to YY'; thus, K is seen to be the point of intersection of the straight lines YY' and CB.

The I. and II. Image-Points S' and $\bar{S}'$ on the chief reflected ray u' corresponding to an Object-Point S on the chief incident ray u are determined by drawing from S straight lines through K and C; the intersections of SK and SC with u' will determine the points S' and $\bar{S}'$, respectively. Straight lines drawn through K and C parallel to the ray u will determine by their intersections with the reflected ray u' the Focal Points I' and $\bar{I}'$, respectively. Similarly, the Focal Points J and $\bar{J}$ on u are found by drawing through K and C, respectively, straight lines parallel to u'.

The Metric Relations which we have for the case of the Reflexion of a narrow bundle of rays at a Spherical Mirror can be derived from the corresponding Refraction-Formulæ, which have been obtained in this chapter, by merely putting $n' = -n$ and $\alpha' = -\alpha$. However, in the formulæ derived in this way, the reader should bear in mind, that, according to the convention we made in § 26, the *positive direction* of any straight line is the direction along that line which light would pursue if the line were the path of an *incident ray*, and, accordingly, the positive direction along a reflected ray is the direction exactly opposite to that in which the reflected light is propagated along it. Failure to note this point has been a source of frequent confusion with writers on Optics.

We derive, therefore, the following set of *Formulæ for the Reflexion of a Narrow Bundle of Rays at a Spherical Mirror:*

$$\left.\begin{aligned}
&\text{I. \textit{Meridian Rays:}}\\
&\qquad CK = r\cdot\sin^2\alpha, \quad \angle BKC = 0;\\
&\qquad f_u = e'_u = JB = I'B = -\frac{r\cdot\cos\alpha}{2};\\
&\qquad \frac{1}{s}+\frac{1}{s'} = \frac{2}{r\cdot\cos\alpha}; \quad Z_u = \frac{s}{s'}.\\
&\text{II. \textit{Sagittal Rays:}}\\
&\qquad \bar{f}_u = \bar{e}'_u = \bar{J}B = \bar{I}'B = -\frac{r}{2\cos\alpha};\\
&\qquad \frac{1}{\bar{s}}+\frac{1}{\bar{s}'} = \frac{2\cos\alpha}{r}; \quad \bar{Z}_u = \frac{\bar{s}}{\bar{s}'}.
\end{aligned}\right\} \tag{263}$$

251. Astigmatism of an Infinitely Thin Lens.

Provided we assume that the length of the path of the chief ray within the Lens is negligible (which may sometimes be a rather big assumption, even though the Lens is infinitely thin), and accordingly put $B_1B_2 = 0$, we shall have:

$$s'_1 = s_2, \quad \bar{s}'_1 = \bar{s}_2;$$

and since here there is no possibility of confusion, we shall find it convenient to write: $s = s_1$, $s' = s'_2$ and $\bar{s} = \bar{s}_1$, $\bar{s}' = \bar{s}'_2$. Moreover, since the Lens is supposed to be surrounded by the same medium on both sides, we may also write: $n_1 = n'_2 = n$, $n'_1 = n'$. Thus, for the case of an *Infinitely Thin Lens* ($m = 2$), formulæ (259) give the following relations:

$$\left.\begin{array}{l}
V_1 = \dfrac{n' \cdot \cos \alpha_1' - n \cdot \cos \alpha_1}{r_1}, \quad V_2 = \dfrac{n \cdot \cos \alpha_2' - n' \cdot \cos \alpha_2}{r_2}; \\
\text{I. } \textit{Meridian Rays:} \\
\dfrac{1}{s'} = \dfrac{\cos^2 \alpha_1 \cdot \cos^2 \alpha_2}{\cos^2 \alpha_1' \cdot \cos^2 \alpha_2'} \cdot \dfrac{1}{s} + \dfrac{\cos^2 \alpha_2}{n \cdot \cos^2 \alpha_2'} \left(\dfrac{V_1}{\cos^2 \alpha_1'} + \dfrac{V_2}{\cos^2 \alpha_2} \right); \\
\text{II. } \textit{Sagittal Rays:} \\
\dfrac{1}{\bar{s}'} = \dfrac{1}{\bar{s}} + \dfrac{1}{n} (V_1 + V_2).
\end{array}\right\} \quad (264)$$

The conditions that to an Object-Point Σ lying on the chief object-ray u there shall correspond a "*Homocentric*" Image-Point Σ' lying on the chief image-ray u' are $s = \bar{s} = B\Sigma$, $s' = \bar{s}' = B\Sigma'$, whence we find:

$$B\Sigma = \frac{n(\cos^2 \alpha_1 \cdot \cos^2 \alpha_2 - \cos^2 \alpha_1' \cdot \cos^2 \alpha_2')}{V_1(\cos^2 \alpha_1' \cdot \cos^2 \alpha_2' - \cos^2 \alpha_2) - V_2 \cdot \cos^2 \alpha_1' \cdot \sin^2 \alpha_2'}. \quad (265)$$

In general, therefore, on every incident chief ray u there is one such Object-Point Σ to which on the corresponding emergent chief ray u' there corresponds a "Homocentric" Image-Point Σ'.

A case of both theoretical and practical interest occurs *when the chief ray goes through the Optical Centre of the Infinitely Thin Lens* (which is easily contrived by placing a screen with a small circular opening right in front of the Lens). In this case the paths of the incident and emergent chief rays are along the same straight line, and, accordingly, we have:

$$\alpha_1 = \alpha_2'; \quad \text{and also} \quad \alpha_1' = \alpha_2;$$

and, therefore,

$$r_1 V_1 + r_2 V_2 = 0.$$

Introducing these values in formulæ (264) above, we obtain for this special case:

Formulæ for Calculating the Astigmatism of an Infinitely Thin Lens for the case when the Chief Ray goes through the Optical Centre:

$$\left.\begin{array}{l}
\text{I. } \textit{Meridian Rays:} \\
\dfrac{1}{s'} - \dfrac{1}{s} = \dfrac{V_1}{n \cdot \cos^2 \alpha_1} \cdot \dfrac{r_2 - r_1}{r_2}; \\
\text{II. } \textit{Sagittal Rays:} \\
\dfrac{1}{\bar{s}'} - \dfrac{1}{\bar{s}} = \dfrac{V_1}{n} \cdot \dfrac{r_2 - r_1}{r_2}.
\end{array}\right\} \quad (266)$$

The positions of the Secondary Focal Points I' and $\bar{I}'$ of the systems of Meridian and Sagittal Rays of the astigmatic bundle of emergent rays may be found by putting $s = \bar{s} = \infty$ in the above formulæ. Thus, if A designates the position on the axis of the Optical Centre of the Thin Lens, we have

$$AI' = \frac{nr_2}{V_1(r_2 - r_1)}\cos^2\alpha_1, \quad A\bar{I}' = \frac{nr_2}{V_1(r_2 - r_1)}. \qquad (267)$$

If in formulæ (261) and (262), we introduce the special conditions which we have in the present case, viz.: $m = 2$, $s_1' = s_2$, $\alpha_1 = \alpha_2'$, $\alpha_1' = \alpha_2$ and $n_1 = n_2'$, we find for the Focal Lengths of the system of Meridian Rays:

$$f_u = -e_u' = AI'.$$

In the same way, formulæ (260) and (262), give for the Focal Lengths of the system of Sagittal Rays:

$$\bar{f}_u = -\bar{e}_u' = A\bar{I}'.$$

Accordingly, in the special case which we have here the Focal Lengths of both systems of rays are equal to the distances of the Focal Points from the incidence-point A. Thus, we have:

$$f_u = \bar{f}_u \cdot \cos^2\alpha_1 = -e_u' = -\bar{e}_u' \cdot \cos^2\alpha_1 = \frac{nr_2}{V_1(r_2 - r_1)} \cdot \cos^2\alpha_1; \qquad (268)$$

so that now formulæ (266) may be put in the following forms:

$$\left.\begin{array}{ll} \text{I. } \textit{Meridian Rays:} & 1/s' - 1/s = 1/f_u; \\ \text{II. } \textit{Sagittal Rays:} & 1/\bar{s}' - 1/\bar{s} = 1/\bar{f}_u. \end{array}\right\} \qquad (269)$$

These equations, as will be immediately recognized, have precisely the same form as the formula for the Refraction of Paraxial Rays through an Infinitely Thin Lens, formula (99) of Chap. VI. The Focal Lengths f_u and $\bar{f}_u$ are both functions of the slope-angle α_1 of the chief ray, and for the value $\alpha_1 = 0$ we obtain:

$$(\alpha_1 = 0), \quad f_u = \bar{f}_u = f = nr_1r_2/(n' - n)(r_2 - r_1).$$

When the chief ray goes through the Optical Centre of the Infinitely Thin Lens, the Astigmatic Difference vanishes, in general, only for the case when the Object-Point is in contact with the Lens; but for $\alpha_1 = 0$ it vanishes for all positions of the Object-Point.[1]

Another interesting special case which has been investigated by H. HARTING[2] is the case when the chief ray crosses the optical axis at the common vertex of a System of Thin Lenses in Contact.

APPENDIX TO CHAPTER XI.

I. Concerning the Focal Points of the Meridian and Sagittal Rays (see §§ 235 & 239).

1. If, instead of reckoning the distances of the object-points and image-points from the double point B of the two projective point-ranges lying on the chief incident and refracted rays u, u', we take as origins a pair of conjugate points $\mathbf{S}$, $\mathbf{S}'$, and put $\mathbf{S}S = p$, $\mathbf{S}'S' = p'$, we shall obtain the relation:

$$\frac{\mathbf{S}J}{p} + \frac{\mathbf{S}'I'}{p'} = 1, \tag{244a}$$

analogous to formula (244). This result is derived immediately by writing:

$$J\mathbf{S} \cdot I'\mathbf{S}' = JS \cdot I'S' = (J\mathbf{S} + p)(I'\mathbf{S}' + p').$$

In the same way, generalizing formula (248) for the sagittal rays, we shall find:

$$\frac{\bar{\mathbf{S}}\bar{J}}{\bar{p}} + \frac{\bar{\mathbf{S}}'\bar{I}'}{\bar{p}'} = 1. \tag{248a}$$

2. The following formulæ (which will be needed in the derivation of the formulæ for the Magnification-Ratios) may also be derived from the projective-relations of the I. and II. object-points and image-points lying on the incident and refracted chief rays:

$$\left.\begin{aligned} \frac{\mathbf{S}'S'}{\mathbf{S}'B} : \frac{\mathbf{S}S}{\mathbf{S}B} &= \frac{JB}{JS} = \frac{I'S'}{I'B}; \\ \frac{\bar{\mathbf{S}}'\bar{S}'}{\bar{\mathbf{S}}'B} : \frac{\bar{\mathbf{S}}\bar{S}}{\bar{\mathbf{S}}B} &= \frac{\bar{J}B}{\bar{J}\bar{S}} = \frac{\bar{I}'\bar{S}'}{\bar{I}'B}. \end{aligned}\right\} \tag{i}$$

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[1] See *Die Theorie der optischen Instrumente*, edited by M. VON ROHR (Bd. I, Berlin, 1904); IV. Kapitel, "Die Realisierung der optischen Abbildung", von P. CULMANN, S. 179.

[2] H. HARTING: Einige Bemerkungen zu dem Aufsatze des Hrn. B. WANACH: Ueber L. v. SEIDELS Formeln zur Durchrechnung von Strahlen u. s. w.: *Zft. f. Instr.*, xx. (1900), 234–237. See also *Die Theorie der optischen Instrumente*, edited by M. VON ROHR (Bd. I, Berlin, 1904); V. Kapitel, " Die Theorie der sphaerischen Aberrationen", S. 254.

3. In this connection we take occasion also to call attention to a recent contribution to this subject by von Höegh[1] concerning the position of the focal point $\bar{I}'$. If in Fig. 128 a line is drawn through $\bar{I}'$ parallel to $Z'C$ and meeting the normal to the spherical refracting surface at the incidence-point of the chief ray in a point which we may designate here by the letter N, then

$$N\bar{I}' = B\bar{I}'\cdot\frac{\sin\alpha'}{\sin(\alpha+\alpha')},$$

and hence by the second of formulæ (249):

$$N\bar{I}' = \frac{r\sin\alpha\sin\alpha'}{\sin(\alpha-\alpha')\sin(\alpha+\alpha')}.$$

Now it may easily be shown that

$$\frac{\sin(\alpha-\alpha')\sin(\alpha+\alpha')}{\sin\alpha\sin\alpha'} = \frac{n'}{n} - \frac{n}{n'},$$

as is given in formula (236), § 228. Hence, we find:

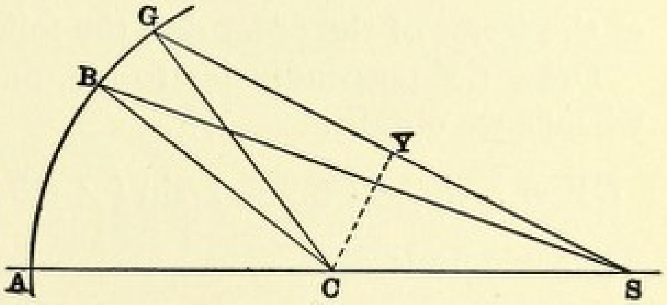
$$N\bar{I}' = \frac{nn'}{n'^2-n^2}r = \text{a constant.}$$

Accordingly, for a given incidence-point B, the focal point $\bar{I}'$ is at a constant distance from a fixed point N lying on the normal at B to the spherical refracting surface. In other words, for all chief rays incident at B, the locus of the focal point $\bar{I}'$ is a circle around N as centre with radius equal to $nn'r/(n'^2-n^2)$. The point N is situated at a distance from B such that

$$BN = \frac{n'}{n}N\bar{I}' = \frac{n'^2}{n'^2-n^2}r.$$

II. Concerning the Formulæ for Calculating the Position of the I. Image-Point (see § 236).

1. Formula (246) of § 236 may be derived very simply as follows:



In the adjoining diagram (where the letters have their usual meanings), the angles are denoted as follows:

$$\angle\, CBS = \alpha,\quad \angle\, CGS = \alpha + d\alpha,\quad \angle\, ASB = \theta,\quad \angle\, BSG = d\theta,$$
$$\angle\, BCA = \phi,\quad \angle\, GCB = d\phi.$$

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[1] E. von Höegh: Dioptrische Untersuchungen: C. P. Goerz *Festschrift* (Berlin-Friedenau, 1911), 171–175.

Moreover,

$$AC = r, \quad BS = s.$$

Since

$$n \sin \alpha = n' \sin \alpha',$$

we have:

$$n \cos \alpha \cdot d\alpha = n' \cos \alpha' \cdot d\alpha'.$$

Now

$$d\alpha = d\theta + d\phi$$

and

$$d\theta = -\frac{\text{arc } GB \cdot \cos \alpha}{s}, \quad d\phi = \frac{\text{arc } GB}{r};$$

hence,

$$d\alpha = \text{arc } GB \left(\frac{1}{r} - \frac{\cos \alpha}{s} \right),$$

and, similarly:

$$d\alpha' = \text{arc } GB \left(\frac{1}{r} - \frac{\cos \alpha'}{s'} \right).$$

Accordingly, we derive:

$$n \cos \alpha \left(\frac{\cos \alpha}{s} - \frac{1}{r} \right) = n' \cos \alpha' \left(\frac{\cos \alpha'}{s'} - \frac{1}{r} \right), \qquad (246b)$$

which is equivalent to formula (246). Using ABBE's difference-notation, we may also write this formula as follows:

$$\Delta \left\{ n \cos \alpha \left(\frac{\cos \alpha}{s} - \frac{1}{r} \right) \right\} = 0. \qquad (246c)$$

2. If in the development of this important formula we take account of the terms of the order $d\phi^2$, the following method may be employed:

Draw CY perpendicular to GS, and project CB and BS on to CY, whence we obtain:

$$CY = r \cdot \sin \angle CGS = r \cdot \sin (\angle CBS + \angle BSG) + BS \cdot \sin \angle GSB.$$

Thus, we find:

$$r \cdot \sin (\alpha + d\alpha) = r \cdot \sin (\alpha + d\theta) - s \cdot \sin d\theta,$$

or, since $d\alpha - d\theta = d\phi$,

$$s \cdot d\theta = \left\{ \frac{r}{2} \sin \alpha (2d\theta + d\phi) - r \cos \alpha \right\} d\phi.$$

Hence,

$$d\theta = \frac{-r \cos \alpha + \frac{1}{2} r \sin \alpha \cdot d\phi}{s - r \sin \alpha \cdot d\phi} d\phi;$$

whence, after actually performing the division and neglecting powers of $d\phi$ higher than the second, we obtain:

$$d\theta = -\frac{r\cos\alpha}{s}d\phi + \left(\frac{1}{2}\frac{r\sin\alpha}{s} - \frac{r^2\sin\alpha\cos\alpha}{s^2}\right)d\phi^2.$$

Now let us write by way of abbreviation

$$K = n\sin\alpha = n'\sin\alpha',$$

so that

$$dK = n\{\sin(\alpha + d\alpha) - \sin\alpha\} = 2n\cos(\alpha + \tfrac{1}{2}d\alpha)\sin\tfrac{1}{2}d\alpha,$$

which, after discarding powers of $d\alpha$ higher than the second, may be written:

$$dK = n\cos\alpha\cdot d\alpha - \tfrac{1}{2}K\cdot d\alpha^2$$

or, since $d\alpha = d\theta + d\phi$:

$$dK = n\cos\alpha\cdot(d\theta + d\phi) - \tfrac{1}{2}K(d\theta + d\phi)^2.$$

From the expression for $d\theta$ derived above, we obtain:

$$d\theta + d\phi = \left(1 - \frac{r\cos\alpha}{s}\right)d\phi + \left(\frac{1}{2}\frac{r\sin\alpha}{s} - \frac{r^2\sin\alpha\cos\alpha}{s^2}\right)d\phi^2,$$

$$(d\theta + d\phi)^2 = \left(1 - \frac{r\cos\alpha}{s}\right)^2 d\phi^2;$$

and introducing these values in the expression for dK, we can write:

$$\frac{dK}{d\phi} = n\cos\alpha\left(1 - \frac{r\cos\alpha}{s}\right) + \frac{3}{2}K\frac{r\cos\alpha}{s}\left(1 - \frac{r\cos\alpha}{s}\right)d\phi - \frac{K}{2}d\phi.$$

And, similarly, for the refracted pencil:

$$\frac{dK}{d\phi} = n'\cos\alpha'\left(1 - \frac{r\cos\alpha'}{s'}\right) + \frac{3}{2}K\frac{r\cos\alpha'}{s'}\left(1 - \frac{r\cos\alpha'}{s'}\right)d\phi - \frac{K}{2}d\phi.$$

Equating these two expressions, we find:

$$\Delta\left\{n\cos\alpha\left(\frac{1}{r} - \frac{\cos\alpha}{s}\right)\right\} = \frac{3}{2}Krd\phi\cdot\Delta\left\{\frac{\cos\alpha}{s}\left(\frac{1}{r} - \frac{\cos\alpha}{s}\right)\right\},$$

which is the required formula, wherein there remains now a term involving $d\phi$. If we neglect this term, the equation immediately reduces to the formula (246*c*) above.

III. Concerning the Formula for the Calculation of the Position of the II. Image-Point (see § 240).

Let E designate a point on the spherical refracting surface which is infinitely near to the incidence-point B of the incident chief ray u and which lies in the plane of the sagittal section of the bundle of incident rays belonging to the object-point $\bar{S}$. To the incident ray $\bar{S}E$ there corresponds a refracted ray lying in the plane determined by $\bar{S}$, E and C which will meet the chief refracted ray u' in the point $\bar{S}'$ where this ray intersects the straight line $\bar{S}C$. This point $\bar{S}'$ is the II. image-point corresponding to the object-point $\bar{S}$. If L marks the point where the spherical surface is crossed by the straight line $\bar{S}C$, and if we put

$$B\bar{S} = \bar{s}, \quad \angle BCL = \epsilon,$$

we obtain from the triangle $CB\bar{S}$:

$$\frac{1}{\bar{s}} = \frac{1}{r}\,\frac{\sin(\epsilon - \alpha)}{\sin\epsilon};$$

which may be written as follows:

$$n\left(\frac{\cos\alpha}{r} - \frac{1}{\bar{s}}\right) = \frac{n\sin\alpha\cot\epsilon}{r}.$$

Similarly, if $B\bar{S}' = \bar{s}'$, we find:

$$n'\left(\frac{\cos\alpha'}{r} - \frac{1}{\bar{s}'}\right) = \frac{n'\sin\alpha'\cot\epsilon}{r};$$

and, hence, by the law of refraction:

$$n'\left(\frac{\cos\alpha'}{r} - \frac{1}{\bar{s}'}\right) = n\left(\frac{\cos\alpha}{r} - \frac{1}{\bar{s}}\right); \qquad (250b)$$

or

$$\Delta\left\{n\left(\frac{\cos\alpha}{r} - \frac{1}{\bar{s}}\right)\right\} = 0. \qquad (250c)$$

If, as in formula (258) of § 247, we abbreviate by writing:

$$\frac{n'\cos\alpha' - n\cos\alpha}{r} = V \qquad (258\ bis)$$

formulæ (246*a*) and (250*a*) may be written:

$$\Delta\frac{n\cos^2\alpha}{s} = \Delta\frac{n}{\bar{s}} = V. \qquad \text{(ii)}$$

IV. Collinear Relations in the Case of a Narrow Bundle of Rays Refracted at a Spherical Surface, or through a Centered System of Spherical Surfaces (see Arts. 76 and 77).

1. *Magnification-Ratios* i, $\bar{i}$.

On the incident chief ray uB belonging to the object-point S, let $\boldsymbol{S}$ designate the position of the I. object-point corresponding to the stop-centre. Through S draw a straight line perpendicular to the ray u in its meridian plane, and on this perpendicular take a point V infinitely near to S. The chief ray belonging to the I. object-point V will also go through $\boldsymbol{S}$, and will be incident on the spherical surface at a point very near to B which may be designated by D. Draw BN parallel to SV meeting the straight line DV in the point N. From this construction we obtain:

$$\frac{SV}{BN}=\frac{\boldsymbol{S}S}{\boldsymbol{S}B}.$$

If the angle of incidence of the ray u is denoted by α, we may put $BN = BD\cdot\cos\alpha$, and accordingly:

$$SV=\frac{\boldsymbol{S}S}{\boldsymbol{S}B}\cdot BD\cdot\cos\alpha.$$

Similarly, performing the analogous construction for the refracted chief ray u', and using the same letters and symbols with accents, we find the corresponding relation:

$$S'V'=\frac{\boldsymbol{S}'S'}{\boldsymbol{S}'B}\cdot BD\cdot\cos\alpha'.$$

Thus, we obtain:

$$\frac{S'V'}{SV}=\left(\frac{\boldsymbol{S}'S'}{B\boldsymbol{S}'}:\frac{\boldsymbol{S}S}{B\boldsymbol{S}}\right)\frac{\cos\alpha'}{\cos\alpha}.$$

Again, on the incident chief ray uB belonging to the object-point $\bar{S}$, let $\bar{\boldsymbol{S}}$ designate the position of the II. object-point corresponding to the stop-centre, and through $\bar{S}$ draw a straight line perpendicular to u in the plane of the sagittal section, and on this straight line take a point $\bar{W}$ infinitely near to $\bar{S}$. The chief ray belonging to $\bar{W}$ will also go through $\bar{\boldsymbol{S}}$ and will be incident on the spherical surface at a point very near to B which may be designated by E. Evidently,

$$\frac{\bar{S}\bar{W}}{BE}=\frac{\bar{\boldsymbol{S}}\bar{S}}{\bar{\boldsymbol{S}}B};$$

and, similarly, for the refracted ray:

$$\frac{\bar{S}'\bar{W}'}{BE} = \frac{\bar{\mathbf{S}}'\bar{S}'}{\bar{\mathbf{S}}'B}.$$

Hence,

$$\frac{\bar{S}'\bar{W}'}{\bar{S}\bar{W}} = \frac{\bar{\mathbf{S}}'\bar{S}'}{B\bar{\mathbf{S}}'} : \frac{\bar{\mathbf{S}}\bar{S}}{B\bar{\mathbf{S}}}.$$

The ratios

$$i = \frac{S'V'}{SV}, \quad \bar{i} = \frac{\bar{S}'\bar{W}'}{\bar{S}\bar{W}}$$

are called the *Magnification-Ratios* in the meridian and sagittal planes with respect to the object-points S and $\bar{S}$, respectively. Referring to formulæ (i), we obtain:

$$i = \frac{JB}{JS}\frac{\cos\alpha'}{\cos\alpha}, \quad \bar{i} = \frac{\bar{J}B}{\bar{J}\bar{S}}. \tag{iii}$$

If the magnification-ratios with respect to the incidence-point B are denoted by j and $\bar{j}$, then:

$$j = \frac{\cos\alpha'}{\cos\alpha}, \quad \bar{j} = 1; \tag{iv}$$

and, hence:

$$i = j \cdot \frac{JB}{JS}, \quad \bar{i} = \bar{j} \cdot \frac{\bar{J}B}{\bar{J}\bar{S}}. \tag{v}$$

Introducing the focal lengths, we have according to § 246,

$$JB = \frac{\cos\alpha}{\cos\alpha'} \cdot f_u, \quad \bar{J}B = \bar{f}_u.$$

Thus, we find from (iv):

$$j = f_u/JB, \quad \bar{j} = \bar{f}_u/\bar{J}B = 1; \tag{vi}$$

and since

$$JS \cdot I'S' = f_u e_u', \quad \bar{J}\bar{S} \cdot \bar{I}'\bar{S}' = \bar{f}_u \bar{e}_u',$$

we obtain from (v):

$$i = \frac{f_u}{JS} = \frac{I'S'}{e_u'}, \quad \bar{i} = \frac{\bar{f}_u}{\bar{J}\bar{S}} = \frac{\bar{I}'\bar{S}'}{\bar{e}_u'}. \tag{vii}$$

Indeed, since according to the view of ABBE (§ 245), the chief incident and refracted rays u, u' are to be regarded as the *principal axes* of the narrow collinear plane-fields π, π' (and likewise also of the narrow collinear plane-fields $\bar{\pi}$, $\bar{\pi}'$), the formulæ which we have just derived may be obtained directly from formulæ (116), page 234.

2. *The Image-Equations Referred to a Pair of Conjugate Points* $\boldsymbol{S}, \boldsymbol{S}'$ (or $\overline{\boldsymbol{S}}, \overline{\boldsymbol{S}}'$).

If in formula (244*a*), page 366, we put

$$J\boldsymbol{S} = \frac{1}{\boldsymbol{i}} f_u, \quad I'\boldsymbol{S}' = \boldsymbol{i} e_u' = -\boldsymbol{i} \frac{n'}{n} f_u,$$

where $\boldsymbol{i}$ denotes the magnification-ratio with respect to the point $\boldsymbol{S}$ in the meridian plane of the incident chief ray u; and if, also, we note that

$$JS = J\boldsymbol{S} + p = -\frac{p}{p'} I'\boldsymbol{S}' = \frac{n'}{n} \frac{p}{p'} f_u \boldsymbol{i};$$

we obtain easily the equations of the imagery in the meridian section, referred to the pair of conjugate points $\boldsymbol{S}$, $\boldsymbol{S}'$, in the following forms:

$$\boldsymbol{i}^2 \frac{n'}{p'} = \frac{n}{p} + \boldsymbol{i} \frac{n}{f_u}, \quad i = \frac{1}{\boldsymbol{i}} \cdot \frac{n}{n'} \cdot \frac{p'}{p}. \tag{viii}$$

Writing $\bar{p}$, $\bar{p}'$, $\bar{f}_u$, $\bar{\boldsymbol{i}}$ in place of p, p', f_u, $\boldsymbol{i}$, respectively, in the above formulæ, we obtain the image-equations for the sagittal section referred to the pair of conjugate points $\overline{\boldsymbol{S}}$, $\overline{\boldsymbol{S}}'$.

These equations might also have been obtained directly by combining the relation given in formula (256) with the first two of equations (127), page 240.

3. *Reduced Abscissæ and Reduced Convergences. The Dioptry.*

The abscissa of a point S with respect to a fixed point O lying on the ray u which passes through S, and in the same medium (n) as S, is a term which will be employed to denote the segment OS of the ray. Following GULLSTRAND,[1] we shall also employ the expression "*reduced abscissa*" to denote the quotient OS/n obtained by dividing the abscissa by the index of refraction; but the reader must be advised that in other parts of optics the expression "reduced length" is used in a different sense from that given above (*cf.* § 38), so that there is a possibility of confusion if this is not borne in mind. The reciprocal of the reduced abscissa, viz., n/OS, is called the "*reduced convergence*" with respect to the point O. Thus, for example,

$$P = \frac{n}{p}, \quad P' = \frac{n'}{p'}, \tag{ix}$$

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[1] See, for example, A. GULLSTRAND's article on "Die optische Abbildung" in the appendix to the third edition of H. VON HELMHOLTZ's *Handbuch der physiologischen Optik*, Bd. I (Hamburg u. Leipzig, 1909), page 239.

are the reduced convergences with respect to the conjugate points $\boldsymbol{S}, \boldsymbol{S}'$. If, in the same fashion, we write

$$D = \frac{n}{f_u}, \quad \overline{D} = \frac{n}{\bar{f}_u}, \tag{x}$$

the image-equations, referred to the conjugate points $\boldsymbol{S}, \boldsymbol{S}'$ (and $\overline{\boldsymbol{S}}, \overline{\boldsymbol{S}}'$), will have the following forms:

$$\left.\begin{aligned} \boldsymbol{i}^2P' &= P + \boldsymbol{i}D, \quad i = \frac{1}{\boldsymbol{i}}\frac{P}{P'}; \\ \bar{\boldsymbol{i}}^2\overline{P}' &= \overline{P} + \bar{\boldsymbol{i}}\overline{D}, \quad \bar{i} = \frac{1}{\bar{\boldsymbol{i}}}\frac{\overline{P}}{\overline{P}'}. \end{aligned}\right\} \tag{xi}$$

The magnitudes denoted by D and $\overline{D}$ are called the "*refractivities*" or "*the refracting powers*" of the optical system with respect to the given chief ray u.

The unit of measurement of the reduced convergences and refractivities may be chosen arbitrarily. But since, as GULLSTRAND[1] remarks, the *Dioptry* is employed in Ophthalmology as the unit of refractivity of lenses, and is defined as the refractivity of a lens surrounded by air of focal length one metre, it is advisable to adopt this unit generally. Accordingly, in order to express a convergence P in dioptries, the abscissa p must be expressed in metres, and then $P = n/p$ will be the value of the convergence in dioptries. Thus, if $p = 0.15$ m., and if $n = 1.5$, then $P = 10$ dioptries. Similarly, if the focal lengths f_u, $\bar{f}_u$ are expressed in metres, the refractivities D, $\overline{D}$, as defined by formulæ (x), will be given in dioptries.

4. *Optical System Composed of a Series of Spherical Refracting Surfaces with their Centres Ranged Along a Straight Line.*

Using the same notation as in the preceding, we can write for the kth surface:

$$\frac{n_k}{p_k} = P_k, \quad \frac{n_{k+1}}{p_k'} = P_k', \quad \frac{n_k}{f_{u,k}} = D_k,$$

and, hence, if $\boldsymbol{i}_k$ denotes the magnification-ratio in the meridian plane of the ray in question, for the refraction at the kth surface, with respect to the points $\boldsymbol{S}_{k-1}'$, $\boldsymbol{S}_k'$, we obtain:

$$\boldsymbol{i}_k^2P_k' = P_k + \boldsymbol{i}_kD_k;$$

and employing this recurrent equation in connection with the geometrical relation

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[1] See, for example, 3d edition of HELMHOLTZ's *Handbuch der physiolog. Optik*, Bd. I, 246.

$$P_k' = P_{k+1},$$

we obtain for a system of m spherical surfaces:

$$(\boldsymbol{i}_1\cdot\boldsymbol{i}_2\cdots\boldsymbol{i}_m)^2P_m' = P_1 + \left\{\boldsymbol{i}_1D_1 + \frac{\boldsymbol{i}_1^2\cdot\boldsymbol{i}_2^2}{\boldsymbol{i}_2}D_2 + \cdots + \frac{\boldsymbol{i}_1^2\cdot\boldsymbol{i}_2^2\cdots\boldsymbol{i}_m^2}{\boldsymbol{i}_m}D_m\right\}.$$

If we use the following abbreviations, viz.,

$$\boldsymbol{i} = \prod_{k=1}^{k=m}\boldsymbol{i}_k,\quad \boldsymbol{i}D = \sum_{k=1}^{k=m}\left\{\frac{D_k}{\boldsymbol{i}_k}\prod_{k=1}^{k=k}\boldsymbol{i}_k^2\right\}, \tag{xii}$$

the equation above may be simplified as follows:

$$\boldsymbol{i}^2P_m' = P_1 + \boldsymbol{i}D. \tag{xiii}$$

Moreover, if i_k denotes the magnification-ratio with respect to the points S_{k-1}', S_k' and the refraction at the kth surface, then

$$i_k = \frac{1}{\boldsymbol{i}_k}\frac{P_k}{P_k'},$$

and hence:

$$\prod_{k=1}^{k=m} i_k = \frac{P_1}{P_k'}\prod_{k=1}^{k=m}\frac{1}{\boldsymbol{i}_k};$$

so that if we put

$$\prod_{k=1}^{k=m} i_k = i,$$

we obtain:

$$i = \frac{1}{\boldsymbol{i}}\cdot\frac{P_1}{P_m'}. \tag{xiv}$$

Similarly, for the imagery by means of the sagittal rays, we obtain the analogous formulæ:

$$\bar{\boldsymbol{i}}^2\bar{P}_m' = \bar{P}_1 + \overline{\boldsymbol{i}D},\quad \bar{i} = \frac{1}{\bar{\boldsymbol{i}}}\cdot\frac{\bar{P}_1}{\bar{P}_m'}. \tag{xv}$$

The *positions of the Focal Points* J, I', lying on the chief ray u in the object-space (n_1) and on the correspondong chief ray u' in the image-space ($n_m' = n_{m+1}$), respectively, may be found from formula (xiii) by putting, first, $P_m' = 0$, and, second, $P_1 = 0$; thus, we obtain:

$$\frac{\boldsymbol{S}_1J}{n_1} = -\frac{1}{\boldsymbol{i}D},\quad \frac{\boldsymbol{S}_m'I'}{n_m'} = \frac{\boldsymbol{i}}{D}; \tag{xvi}$$

whereby the positions of the focal points are defined with respect to the fixed points $\boldsymbol{S}_1$, $\boldsymbol{S}_m'$.

Analogous formulæ define the positions of the Focal Points $\bar{J}$, $\bar{I}'$ with respect to the fixed points $\bar{\boldsymbol{S}}_1$, $\bar{\boldsymbol{S}}_m'$, respectively.

If the *Principal Points*, lying on the incident and emergent chief rays u, u', for the case of the imagery in the meridian plane, are designated by the letters H, H', the positions of these points may be found by putting the following values in formulæ (xiii) and (xiv):

$$i = 1, \quad P_1 = \frac{n_1}{S_1H}, \quad P_m' = \frac{n_m'}{S_m'H'};$$

thus, we obtain:

$$\frac{S_1H}{n_1} = \frac{i-1}{i} \cdot \frac{1}{D}, \quad \frac{S_m'H'}{n_m'} = (i-1)\frac{1}{D}. \tag{xvii}$$

Accordingly, combining formulæ (xvi) and (xvii), we find:

$$JH = \frac{n_1}{D}, \quad I'H' = -\frac{n_m'}{D}; \tag{xviii}$$

and if we put

$$f_u = \frac{n_1}{D}, \quad e_u' = -\frac{n_m'}{D}, \tag{xix}$$

then

$$JH = f_u, \quad I'H' = e_u'. \tag{xx}$$

The magnitudes denoted here by f_u, e_u' are the *Focal Lengths* in the object-space and image-space, respectively, for the imagery in the meridian plane of the chief ray. Precisely analogous formulæ will be found for the imagery by means of the sagittal rays.

5. *Imagery along the Axial Ray x, x' (Optical Axis).*

In the special case when the chief ray meets the spherical refracting surface normally, we must put $\alpha = \alpha' = 0$ in formulæ (ii) to (xi); and then we find that *the two imageries in the meridian and sagittal planes are identical.* If the normally incident ray coincides with the axis of symmetry (x, x') of the optical system, we employ the letters and symbols which in our notation are characteristic of the optical axis. Thus, for example, the principal points which lie on the axial ray and which are always meant when we speak simply of "the principal points of the optical system" are designated by A, A'. If the optical system consists of *a single spherical surface*, these points A, A' will be found to coincide with each other at the vertex A where the axial ray meets the surface of the sphere. Moreover, putting $\alpha = \alpha' = 0$ in formula (258 *bis*), and writing V_0 in place of V, we have:

$$V_0 = \frac{n' - n}{r}; \tag{xxi}$$

and hence if the focal points on the axial ray, which are distinguished as *the focal points of the optical system*, are designated by F, E', their positions with respect to the vertex A of the spherical surface will be defined by the following equations:

$$AF = -\frac{n}{V_0}, \quad AE' = \frac{n'}{V_0}; \tag{xxii}$$

as may be verified by putting $\alpha = \alpha' = 0$ either in formulæ (245) or in formulæ (249).

The focal lengths in the case of the axial ray are called *the principal focal lengths* or simply *the focal lengths of the optical system*, and they are denoted by the symbols f, e'. Specializing formulæ (254) or formulæ (255), we obtain for a single spherical refracting surface:

$$f = \frac{n}{V_0}, \quad e' = -\frac{n'}{V_0}, \quad n'f + ne' = 0. \tag{xxiii}$$

These formulæ may be derived also from (xxii) from the fact that the focal lengths of a single spherical surface may be defined as the abscissæ of the vertex A with respect to the focal points F, E'; that is,

$$f = FA, \quad e' = E'A.$$

If the optical system consists of *a series of spherical surfaces with their centres ranged along a straight line* (optical axis), and if the indices of refraction of the first and the last medium are denoted by n, n' ($n = n_1$, $n' = n_m'$), and if in the case of the axial ray we write the special symbol F in place of D, then, according to formulæ (xviii), (xix) and (xx), we have:

$$FA = \frac{n}{F} = f, \quad E'A' = -\frac{n'}{F} = e'. \tag{xxiv}$$

The magnitude denoted here by F is called the *refracting power of the optical system* or *the refractivity*, and it may be defined as *the quotient obtained by dividing the value of the refractive index* (n) *in the object-space by the focal length* f; thus,

$$F = \frac{n}{f} = -\frac{n'}{e'}. \tag{xxv}$$

The stop-centre designated by O will determine (cf. § 361) an axial object-point $\boldsymbol{M}$ and an axial image-point $\boldsymbol{M}'$ which are conjugate axial points with respect to the optical system. These points are the centres of the so-called *pupils*, $\boldsymbol{M}$ being the centre of the *entrance-pupil* and $\boldsymbol{M}'$ the centre of the *exit-pupil*. Let M, M' designate the positions of another pair of axial points which are conjugate to each other with respect to the optical system, and let us put

$$\boldsymbol{M}M = \xi, \quad \boldsymbol{M}'M' = \xi'.$$

Moreover, let us use the symbols Y, $\boldsymbol{Y}$ to denote the magnification-ratios with respect to the two pairs of conjugate axial points M, M' and $\boldsymbol{M}$, $\boldsymbol{M}'$, respectively. If now in formulæ (xv) we put

$$i = Y, \quad \boldsymbol{i} = \boldsymbol{Y}, \quad D = F, \quad P_1 = \frac{n}{\xi}, \quad P_m' = \frac{n'}{\xi},$$

we derive *the image-equations for the axial ray, referred to the pupil-centres* $\boldsymbol{M}$, $\boldsymbol{M}'$, viz.,

$$\boldsymbol{Y}^2\frac{n'}{\xi'} = \frac{n}{\xi} + \boldsymbol{Y}F, \quad Y = \frac{n}{n'}\,\frac{1}{\boldsymbol{Y}}\,\frac{\xi'}{\xi}. \tag{xxvi}$$

If the principal points A, A' are selected as the origins of abscissæ measured along the axial ray, and if we put

$$AM = u, \quad A'M' = u',$$

and if the principal-point convergences are denoted by U, U', that is, if

$$\frac{n}{u} = U, \quad \frac{n'}{u'} = U', \tag{xxvii}$$

the image-equations for the axial ray, referred to the principal points A, A', will have the following forms:

$$U' = U + F, \quad YU' = U. \tag{xxviii}$$

The convergences may be reckoned from any pair of fixed points on the incident and refracted rays: for example, in the case of the axial ray, from the focal points F, E'. Thus, if we put

$$FM = x, \quad E'M' = x', \quad L = \frac{n}{x}, \quad L' = \frac{n'}{x'},$$

we shall obtain *the image-equations for the axial ray, referred to the focal points* F, E', as follows:

$$L \cdot L' = -F^2, \quad Y = \frac{L}{F}. \tag{xxix}$$

6. *Combination of Two Optical Systems.* It is assumed that each of the two systems is composed of a centered system of spherical surfaces, and that their optical axes are coincident. The two systems will be distinguished in the order in which the light traverses them as

systems I. and II. The refractive indices of the first and last media of system I. will be denoted by n and n'', and the refractive indices of the first and last media of system II. will be denoted by n'' and n', respectively. We shall consider the case of the imagery in the meridian plane of a ray emanating from an object-point S_1 lying in the medium n; the same method of procedure can be applied to the imagery by means of the sagittal rays.

Let S_1' designate the position of the image-point in the medium n'' which is common to the two systems, and let S_2' designate the position of the image-point in the last medium n'. Moreover, let the principal points of systems I. and II., with respect to the given ray, be designated by H_{I}, H_{I}' and H_{II}, H_{II}', respectively. And, finally, let

$$T_1 = \frac{n}{H_{\text{I}}S_1}, \quad T_1' = \frac{n''}{H_{\text{I}}'S_1'}, \quad T_2 = \frac{n''}{H_{\text{II}}S_1'}, \quad T_2' = \frac{n'}{H_{\text{II}}S_2'}.$$

Then, by analogy with the first of formulæ (xxviii), we have:

$$T_1' = T_1 + D_1, \quad T_2' = T_2 + D_2.$$

Moreover, if the reduced distance of H_{II} from H_{I}' is denoted by c_a, that is, if

$$H_{\text{I}}'H_{\text{II}} = n'' \cdot c_a,$$

we have the geometrical equation:

$$T_2 = \frac{T_1'}{1 - c_a T_1'}.$$

From these equations we find:

$$T_2' = \frac{(1 - c_a D_2)T_1 + D}{1 - c_a(T_1 + D_1)}, \quad T_1 = \frac{(1 - c_a D_1)T_2' - D}{1 - c_a(T_2' - D_2)}, \qquad \text{(xxx)}$$

where

$$D = D_1 + D_2 - c_a D_1 D_2. \qquad \text{(xxxi)}$$

If the focal points of the imagery, with respect to the compound system, in the meridian plane of the ray in question, are designated by J and I', the positions of these points may be determined from formulæ (xxxi); for if $T_2' = 0$, then $T_1 = \frac{n}{H_{\text{I}}J}$, and if $T_1 = 0$, then $T_2' = \frac{n'}{H_{\text{II}}I'}$; and if we substitute these values in the above formulæ, we obtain:

$$\frac{H_{\text{I}}J}{n} = -\frac{1 - c_a D_2}{D}, \quad \frac{H_{\text{II}}'I'}{n'} = \frac{1 - c_a D_1}{D}. \qquad \text{(xxxii)}$$

The principal points H, H' of the imagery, with respect to the compound system, in the meridian plane of the ray in question, will be determined by imposing the condition:

$$i = i_1 \cdot i_2 = 1;$$

and, since by analogy with the second of formulæ (xxviii), the magnitudes i_1, i_2 may be defined by the relations:

$$i_1 T_1' = T_1, \quad i_2 T_2' = T_2,$$

the condition above may be expressed as follows:

$$T_1 \cdot T_2 = T_1' \cdot T_2'.$$

Introducing this relation, and at the same time putting

$$T_1 = \frac{n}{H_{\mathrm{I}}H}, \quad T_2' = \frac{n'}{H_{\mathrm{II}}'H'},$$

we obtain the following formulæ for finding the positions of the principal points designated by H, H':

$$\frac{H_{\mathrm{I}}H}{n} = \frac{c_a D_2}{D}, \quad \frac{H_{\mathrm{II}}'H'}{n'} = -\frac{c_a D_1}{D}. \qquad \text{(xxxiii)}$$

Finally, if

$$f_a = JH, \quad e_a' = I'H'$$

denote the focal lengths of the imagery, with respect to the compound system, in the meridian plane of the given ray, we find from formulæ (xxxii) and (xxxiii):

$$D = \frac{n}{f_a} = -\frac{n'}{e_a'}. \qquad \text{(xxxiv)}$$

For the **Axial Ray,** in place of the letters S, H, J and I', we shall employ the letters M, A, F and E', respectively; and in place of the symbols T, c_a, D, i, f_a and e_a', we shall employ the symbols U, c, F, Y, f and e', respectively. Hence, in the case of the Axial Ray, the formulæ above have the following forms:

$$\left.\begin{aligned}
&\frac{n}{f} = -\frac{n'}{e'} = F, \quad F = F_1 + F_2 - cF_1F_2;\\
&\frac{A_{\mathrm{I}}F}{n} = -\frac{1 - cF_2}{F}, \quad \frac{A_{\mathrm{II}}'E'}{n'} = \frac{1 - cF_1}{F};\\
&\frac{A_{\mathrm{I}}A}{n} = \frac{cF_2}{F}, \quad \frac{A_{\mathrm{II}}'A'}{n'} = -\frac{cF_1}{F}.
\end{aligned}\right\} \qquad \text{(xxxv)}$$

CHAPTER XII.

THE THEORY OF SPHERICAL ABERRATIONS.

I. INTRODUCTION.

ART. 79. PRACTICAL IMAGES.

252. The requirements of a good image are (1) that it shall be sharp or *distinct*, corresponding, therefore, to the object point by point, (2) that it shall be *accurate*, that is, completely similar to the object, and thus faithfully reproducing it, and (3) that it shall be *bright*. This last condition necessarily implies the use of wide-angle bundles of rays, because obviously the light-intensity at any point will be greater in proportion to the number of rays that unite at that point. On the other hand, the first two requirements, which are both purely geometrical, will, in general, be fulfilled by an optical system only in the special and unrealizable case when the bundles of rays concerned in the production of the image are infinitely narrow. Thus, in the theory of the Imagery by means of Paraxial Rays, which was developed according to general laws first by GAUSS, and which has, therefore, been appropriately called "GAUSSian Imagery" (§ 188), we have seen that for an optical system of centered refracting (or reflecting) spherical surfaces a distinct and accurate image was formed only when the rays concerned were all comprised within an indefinitely narrow cylindrical space immediately surrounding the optical axis; this region being more explicitly defined by the condition that a "paraxial" ray is one for which both the angle of incidence α and the central angle φ were so small that all powers of these angles higher than the first could be neglected (§ 109).

In general, even with infinitely narrow bundles of rays, stigmatic imagery, except in the case of normally incident rays just mentioned, is possible only for certain special positions of the object-point.

It goes without saying that from the standpoint of the optician the formation of images under such impracticable restrictions is almost without interest. Without dwelling on the obvious objection that such images would be of infinitesimal dimensions (as would be likewise true of the objects to be depicted), we encounter a still greater difficulty in the fact that Physical Optics—which in all optical questions is the court of last resort—pronounces that these images are not true images at all. For according to the Wave-Theory of Light, a mere homocentric convergence of the image-rays is not of itself sufficient for the formation of a distinct optical image. If the wave-surface in the Image-Space is spherical—so that the image-rays all meet in one point, viz., at the centre of the spherical wave-surface—instead of an image-point, we shall obtain a resultant effect (in the plane perpendicular to the optical axis through the centre of the spherical wave-surface) consisting of a central luminous disc surrounded by alternate dark and diminishingly bright rings. The brightness of this disc fades from the centre out towards the circumference. The greater the extent of the effective portion of the spherical wave-surface as compared with its radius, that is, the wider the angle of the homocentric bundle of image-rays, the smaller will be the diameter of the diffraction-disc, which therefore tends to be reduced more and more nearly to a point as the angular aperture of the bundle of image-rays is increased. From the point of view of Physical Optics, as LUMMER[1] remarks, this is the only sense in which the term "point-image" can have any meaning. Thus, both for a clear and distinct image as well as for a bright image, theory insists that wide-angle bundles of rays must be employed (see § 45).

On the other hand, from the geometrical standpoint the fundamental requirement of optical imagery is the convergence of the rays to one point; and, in general, this requirement in the case of bundles of finite aperture is impossible.

Consequently, actual optical images, which are necessarily formed by bundles of rays of finite aperture, are, in general, more or less faulty. These faults—which are called *aberrations*—may sometimes escape unnoticed merely because the eye which views the image cannot or does not distinguish the defects which it contains. But to the practical optician who strives to obtain an image as nearly perfect as possible it is of the highest importance to ascertain the nature of these various so-called aberrations, to distinguish them the one from the other, and to perceive clearly what factors contribute to produce them in each instance, so that in the design of an optical instrument he may contrive to reduce, perhaps to abolish entirely, at any rate those aberrations which for the particular type of instrument are to be regarded as the most objectionable. Along these lines, and especially since the rise of Photography, wonderful progress has been achieved in the design and construction of optical instruments.

The plan that is employed is to combine optical systems in such a way that, although each single refracting or reflecting surface gives by itself a point-to-point imagery only within the narrow region to which the paraxial rays are confined, in the compound system these limitations are very considerably extended in one direction or another or perhaps in several directions simultaneously. The duty of refracting the rays so that they will emerge finally in suitable directions is not assigned to a single surface, but is distributed over a number of separate surfaces. By suitable combinations, it has been found possible in this way to construct systems which by means of wide-angle bundles of rays will give a true image of an axial object-point or of a small surface-element placed at right angles to the optical axis. The objective of a microscope, for example, is a system of this kind. In the eye-piece, or ocular, on the other hand, we have an illustration of a system which by means of relatively narrow bundles of rays produces the image of a large object. Thus, in the compound microscope the duty of the objective is to produce an image of a small object by means of wide-angle bundles of rays, whereas the duty of the ocular is, by means of narrow bundles, to spread over the large field of vision the image produced by the objective. In the case of the photographic objective, we must have both wide aperture and extensive field of vision, and in order to meet both of these requirements at once, something else has to be sacrificed, and, accordingly, we are obliged to be content with a less distinct image than we require in the case of the objective of a microscope.

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[1] See MUELLER-POUILLET's *Lehrbuch der Physik* (neunte Auflage), Bd. II, 447.

Of course, it would be idle for the optician to seek to produce an image which is free from faults that could not be detected by the eye if they were present. The resolving-power of the human eye is comparatively poor (*cf.* § 377). Thus, for example, details in the object which are separated by an angular distance, say, of one minute will not be recognized by the eye as separate and distinct. Accordingly, the practical image need be perfect only to the degree that in it those elements of the object which are to be preceived as separate must be presented to the eye at a visual angle of not less than one minute of arc.

ART. 80. THE SO-CALLED SEIDEL IMAGERY.

253. The theory developed by GAUSS[1] in his *Dioptrischen Untersuchungen* proceeds on the assumption that the central angle φ is so small that the second and higher powers thereof are negligible. The theory is applicable, therefore, only to optical systems of narrow aperture and of small visual field, since both the incidence-points of the rays and the object-points whence they emanate must all lie very close to the optical axis of the centered system of spherical surfaces. The investigations of Euler,[1] Schleiermacher,[2] Seidel[3] and others were first directed towards taking account of the aberrations due to increase of the aperture of the system; but, later, with the rise of Photography and the development of the Photographic Objective, it became necessary to take into consideration not only a wider aperture but a greater field of vision, in order to portray objects which were at some distance from the optical axis. The complete theory of spherical aberrations was worked out by J. Petzval[4] and L. Seidel,[5] and in the following sections of this article the methods of these two investigators form the basis of the mode of treatment.

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[1] C. F. GAUSS: *Dioptrische Untersuchungen* (Goettingen, 1841).

254. Order of the Image, according to J. Petzval. Taking the optical axis of the centered system of spherical surfaces as the x-axis of a system of rectangular co-ordinates, let us denote the co-ordinates of an object-point P by ξ, η, ζ. The transversal plane σ, which passes through P and is perpendicular to the optical axis, will be called the *Object-Plane*. Let $\boldsymbol{P}$ ($\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$) designate the position of the point where the rectilinear path of an object-ray, proceeding from the object-point P, crosses a second fixed transversal plane $\boldsymbol{\sigma}$ parallel to the object-plane σ. The position of this object-ray will be completely determined by the four parameters η, ζ, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$. In the image-space let σ', $\boldsymbol{\sigma}'$ designate a pair of fixed transversal planes perpendicular to the optical axis; we shall call the plane σ' the *Image-Plane*. Let P', $\boldsymbol{P}'$ designate the positions of the points where the rectilinear path of the image-ray, corresponding to the object-ray $P\boldsymbol{P}$, crosses the planes σ', $\boldsymbol{\sigma}'$, respectively, and let the rectangular co-ordinates of P', $\boldsymbol{P}'$ be denoted by (ξ', η', ζ') and by ($\boldsymbol{\xi}'$, $\boldsymbol{\eta}'$, $\boldsymbol{\zeta}'$), respectively. The position of the image-ray will, therefore, be defined by the four parameters η', ζ', $\boldsymbol{\eta}'$, $\boldsymbol{\zeta}'$.

Since to every object-ray there corresponds one, and only one, image-ray, it is obvious that each of the four parameters of the image-ray must be a definite function of the four parameters of the object-ray, so that we may write:

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[1] L. Euler: *Dioptricæ pars prima* (Petersburg, Akad. Wiss., 1769); *pars secunda* (*ibid.*, 1770); *pars tertia* (*ibid.*, 1771).

[2] L. Schleiermacher: *Ueber den Gebrauch der analytischen Optik bei Construction optischer Werkzeuge* (Pogg. *Ann.*, 1828, xiv.); also, Analytische Optik (Baumgartners und von Ettingshausens *Zft. f. Phys. u. Math.*, 1831, ix., 1–35; 161–178; 454–474; 1832, x., 171–200; 329–357).

[3] L. Seidel: Zur Theorie der Fernrohr Objective: *Astr. Nachr.*, 1853, xxxv. No. 836, 301–316.

[4] Joseph Petzval: *Bericht ueber die Ergebnisse einiger dioptrischer Untersuchungen* (Pesth, 1843). See also: Bericht ueber optische Untersuchungen. *Sitzungsber. der math.-naturwiss. Cl. der kaiserl. Akad. der Wissenschaften*, Wien, xxvi. (1857), 50–75, 92–105, 129–145. (See especially page 95, in regard to the "order" of the image.)

[5] L. Seidel: Zur Dioptrik. Ueber die Entwicklung der Glieder 3ter Ordnung, welche den Weg eines ausserhalb der Ebene der Axe gelegenen Lichtstrahles durch ein System brechender Medien bestimmen: *Astr. Nachr.*, 1856, xliii., No. 1027, 289–304; No. 1028, 305–320; No. 1029, 321–332.

$$\eta' = f_1(\eta, \zeta, \boldsymbol{\eta}, \boldsymbol{\zeta}), \quad \zeta' = f_2(\eta, \zeta, \boldsymbol{\eta}, \boldsymbol{\zeta}),$$
$$\boldsymbol{\eta}' = f_3(\eta, \zeta, \boldsymbol{\eta}, \boldsymbol{\zeta}), \quad \boldsymbol{\zeta}' = f_4(\eta, \zeta, \boldsymbol{\eta}, \boldsymbol{\zeta});$$

where the functions f_1, f_2, f_3, f_4 can be deduced by the laws of refraction.

Moreover, taking account of the symmetry with respect to the optical axis, we observe that if the signs of the parameters η, ζ, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ are all reversed, the signs of the parameters η', ζ', $\boldsymbol{\eta}'$, $\boldsymbol{\zeta}'$ will all likewise be reversed; and, consequently, if each of the functions above is developed in a series of ascending powers and products of η, ζ, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$, each of these series can contain only the terms of the odd degrees. And, hence, if the parameters of the ray are regarded as magnitudes of the first order of smallness, these series-developments will contain only terms of the odd orders of smallness.

Now, if for *all* rays proceeding from the object-point P we obtain exactly the same values of the co-ordinates η', ζ', we shall obtain at P' a perfect image of the object-point P. In general, however, this will not be the case, and for a second object-ray coming from P, whose parameters are, say, η, ζ, $\boldsymbol{\eta} + \delta\boldsymbol{\eta}$, $\boldsymbol{\zeta} + \delta\boldsymbol{\zeta}$, we shall obtain a new set of values $\eta' + \delta\eta'$, $\zeta' + \delta\zeta'$, $\boldsymbol{\eta}' + \delta\boldsymbol{\eta}'$, $\boldsymbol{\zeta}' + \delta\boldsymbol{\zeta}'$ for all four of the parameters of the corresponding image-ray. Obviously, in the series-developments the differences $\delta\eta'$, $\delta\zeta'$ will contain also only the terms of the odd degrees. If, as compared with the magnitudes η, ζ, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$, these differences $\delta\eta'$, $\delta\zeta'$ are, say, of the $(2k + 1)$th order of smallness, then, according to J. PETZVAL, the spot of light formed around P' by the totality of all such points as P' is to be considered as an "*image*" *of the* $(2k + 1)$*th order* in the image-plane σ' corresponding to the object-point P. *The higher the order of the image, the more nearly perfect it will be.* An image of the 3rd order is one in which there are uncorrected faults of the 3rd order.

255. Parameters of Object-Ray and Image-Ray, according to L. Seidel. A complete development of the theory of Spherical Aberrations was first published by L. SEIDEL, who extended GAUSS's theory so as to take account of magnitudes of the 3rd order of smallness, neglecting therefore the terms of the 5th and higher orders. Thus, *in the so-called* SEIDEL *Imagery, the image is of the fifth order.*

The comparative simplicity and elegance of SEIDEL's methods are due to his choice of the four parameters which define the rectilinear path of the ray, viz., the two pairs of rectangular co-ordinates (η, ζ)

and $(\boldsymbol{\eta}, \boldsymbol{\zeta})$ of the points P, $\boldsymbol{P}$ where the ray crosses the two fixed transversal planes σ, $\boldsymbol{\sigma}$. In order to make this clear, let us suppose now that $P\boldsymbol{P}$ represents the path, not of the object-ray itself, as formerly, but of this ray before refraction at, say, the kth surface of the optical system, and, in the same way, let $P'\boldsymbol{P}'$ represent the path of the ray after refraction at this surface. The actual locations of the four transversal planes σ, σ' and $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ have not been specified; and, accordingly, we may establish an arbitrary connection between, say, σ and σ', on the one hand, and between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$, on the other hand. If, for example, M, M' designate the points where the optical axis meets the planes σ, σ', respectively, these points may be selected with reference to each other so that, in the sense of GAUSS's Theory, M, M' are a pair of conjugate axial points with respect to the spherical refracting surface which is here under consideration. And the same relation can be established between the pair of points $\boldsymbol{M}$, $\boldsymbol{M}'$ where the optical axis crosses the transversal planes $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$, respectively. Thus, by GAUSS's Theory, the transversal planes σ, σ' and $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ will be two pairs of conjugate planes with respect to the spherical surface in question. If A designates the vertex and C the centre of this spherical surface, and if we put:

$$AC = r, \quad AM = u, \quad AM' = u', \quad A\boldsymbol{M} = \boldsymbol{u}, \quad A\boldsymbol{M}' = \boldsymbol{u}',$$

the relations between M and M' and between $\boldsymbol{M}$ and $\boldsymbol{M}'$ will be expressed as follows (see § 126):

$$\left.\begin{aligned} n\left(\frac{1}{r} - \frac{1}{u}\right) &= n'\left(\frac{1}{r} - \frac{1}{u'}\right) = J, \\ n\left(\frac{1}{r} - \frac{1}{\boldsymbol{u}}\right) &= n'\left(\frac{1}{r} - \frac{1}{\boldsymbol{u}'}\right) = \boldsymbol{J}. \end{aligned}\right\} \tag{270}$$

The co-ordinates of the four points P, $\boldsymbol{P}$, P', $\boldsymbol{P}'$ may now be expressed as follows:

$$\left.\begin{array}{llll} \eta = y + \delta y, & \zeta = z + \delta z, & \boldsymbol{\eta} = \boldsymbol{y} + \delta \boldsymbol{y}, & \boldsymbol{\zeta} = \boldsymbol{z} + \delta \boldsymbol{z}; \\ \eta' = y' + \delta y', & \zeta' = z' + \delta z', & \boldsymbol{\eta}' = \boldsymbol{y}' + \delta \boldsymbol{y}', & \boldsymbol{\zeta}' = \boldsymbol{z}' + \delta \boldsymbol{z}'; \end{array}\right\} \tag{271}$$

wherein the first term on the right-hand side of each of these equations denotes the *approximate* (or "GAUSSian") value of the parameter obtained by neglecting the terms of the 3rd order, and the second term denotes the *correction of the 3rd order*, which, being added to the principal, or approximate, value, gives a value which will be exact except for residual errors of the 5th and higher orders. Evidently, the points $Q(y, z)$ and $Q'(y', z')$, lying in the planes σ, σ' and not far from the points P, P', respectively, are a pair of conjugate points according to Gauss's Theory; and the same thing is true also of the pair of points $\boldsymbol{Q}(\boldsymbol{y}, \boldsymbol{z})$ and $\boldsymbol{Q}'(\boldsymbol{y}', \boldsymbol{z}')$, which lie in the transversal planes $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ and not far from the points $\boldsymbol{P}(\boldsymbol{\eta}, \boldsymbol{\zeta})$, $\boldsymbol{P}'(\boldsymbol{\eta}', \boldsymbol{\zeta}')$, respectively.

256. The Correction-Terms or Aberrations of the 3rd Order. Thus, Seidel employs *two independent systems of transversal planes* perpendicular to the optical axis of the centered system of spherical surfaces, so that for each medium traversed by the ray there is one plane of each system. The position of the object-ray before refraction at the first spherical surface is given by assigning the co-ordinates (η_1, ζ_1), $(\boldsymbol{\eta}_1, \boldsymbol{\zeta}_1)$ of the points P_1, $\boldsymbol{P}_1$ where the ray crosses two arbitrary transversal planes σ_1, $\boldsymbol{\sigma}_1$.

For the plane σ_1 naturally we shall select the transversal plane which contains the object-point $P_1(\eta_1, \zeta_1)$; this is the so-called Object-Plane mentioned above (§ 254). Moreover, without affecting at all the generality of the discussion, we may select the xy-plane of the system of rectangular co-ordinates so that the object-point P_1 lies in this plane, in which case we shall have $\zeta_1 = 0$. Since the bundle of object-rays is homocentric, the point $Q_1(y_1, z_1)$ will coincide with P_1; that is, $\delta y_1 = 0$, $\delta z_1 = 0$.

The plane σ'_k is the transversal plane, which, according to Gauss's theory, is conjugate, with respect to the first k spherical surfaces of the optical system, to the Object-Plane σ_1. After refraction at the kth surface, the ray (prolonged either forwards or backwards, if necessary) will cross the plane σ'_k at the point $P'_k(\eta'_k, \zeta'_k)$. If m denotes the total number of spherical surfaces, the corresponding image-ray, emerging from the optical system, will cross the Image-Plane σ'_m at the point P'_m, whose co-ordinates are:

$$\eta'_m = y'_m + \delta y'_m, \quad \zeta'_m = z'_m + \delta z'_m,$$

where y'_m, z'_m denote the co-ordinates of the point Q'_m which, by Gauss's theory, is the image of the object-point P_1 (or Q_1). The magnitudes y'_m, z'_m can be determined by the approximate formulæ of Gauss. Obviously, the point Q'_m will lie in the meridian plane through the point Q_1, and since Q_1 is coincident with P_1, if the meridian plane containing the object-point is taken as the xy-plane, we must have:

$$\zeta_1 = z_1 = z'_m = 0;$$

and, hence, $\zeta'_m = \delta z'_m$.

The magnitudes denoted by $\delta y'_m$, $\delta z'_m$ are the correction-terms, or *aberrations of the 3rd order*, which measure the errors of the image. By some writers $\delta y'_m$, $\delta z'_m$ are called the "*Tangential*" and "*Sagittal*" *aberrations*, respectively, in the GAUSSian Image-Plane σ'_m. We may also call them the *y-aberration* and the *z-aberration* in this plane.

257. Planes of the Pupils of the Optical System. So far as the meanings of the magnitudes $\delta y'_m$, $\delta z'_m$ are concerned, it is a matter of no consequence what plane $\boldsymbol{\sigma}_1$ is selected for the initial plane of the other system (or $\boldsymbol{\sigma}$-system) of transversal planes. In all optical instruments the aperture of the cone of effective rays is limited by certain diaphragms or circular openings, called "stops", which are placed with their planes perpendicular to the optical axis and with their centres on this axis. Even in case there is no such artificial diaphragm, the rims or fastenings of the lenses themselves will act as such, so that of all the rays emitted from an object-point only a certain limited number succeed in making their way through the entire apparatus. When there are several diaphragms, the effective stop is that one which permits the fewest rays to pass. This stop may be situated, according to circumstances, in front of the entire system or somewhere within the system or even beyond the entire system. It is found to be most convenient to select the plane $\boldsymbol{\sigma}_1$ so that one of the planes of this series of transversal planes shall coincide with the plane of the effective stop. In the most general case, when the stop is situated within the optical system, say, between the kth and the $(k+1)$th spherical surfaces, the plane of the stop will be the transversal plane $\boldsymbol{\sigma}'_k$, and the stop-centre will be at the point $\boldsymbol{M}'_k$ where the optical axis crosses this plane. The axial point $\boldsymbol{M}_1$, whose image produced by the refraction of paraxial rays through the first k spherical surfaces of the optical system is $\boldsymbol{M}'_k$, will determine, therefore, the position of the initial transversal plane $\boldsymbol{\sigma}_1$. All the object-rays cross the plane $\boldsymbol{\sigma}_1$ at points lying within the space which would be covered by a thin circular disc placed with its centre on the optical axis at $\boldsymbol{M}_1$ and perpendicular to the optical axis and of such dimensions that, with respect to the first k surfaces of the optical system, its GAUSSian image at $\boldsymbol{M}'_k$ exactly coincided with the effective stop there. This circle around $\boldsymbol{M}_1$ in the plane $\boldsymbol{\sigma}_1$ has been well called by ABBE the *Entrance-Pupil* (see § 361) of the optical system; and we shall, therefore, speak of the initial plane $\boldsymbol{\sigma}_1$ as the "Plane of the Entrance-Pupil". Similarly, all the image-rays will cross the last transversal plane $\boldsymbol{\sigma}'_m$ in points contained within a circle around $\boldsymbol{M}'_m$, which, with respect to the entire system, is the image, by GAUSS's theory, of the Entrance-Pupil. This circle is called the

Exit-Pupil, and the plane σ'_m is called the "Plane of the Exit-Pupil".

Obviously, by this method of selecting the initial plane σ_1 we have the advantage of knowing the greatest possible values which the coordinates $\boldsymbol{y}_1$, $\boldsymbol{z}_1$ can have in the case of a given optical system, and, since the values of $\delta y'_m$, $\delta z'_m$ depend also on the values of $\boldsymbol{y}_1$, $\boldsymbol{z}_1$, this knowledge will be of service in considering the relative importance of the various terms in the series-developments.

258. Chief Ray of Bundle. Of all the rays proceeding from the object-point P_1 there is one, which, lying in the meridian plane through P_1, will, in traversing the medium in which the stop is situated, go through the centre of the stop. This ray, distinguished as the *chief ray* of the bundle, will, in general, cross the optical axis for the first time at a point $\boldsymbol{L}_1$ not very far from the centre $\boldsymbol{M}_1$ of the Entrance-Pupil. The slope of the chief ray of the bundle of object-rays emanating from P_1 is

$$\angle M_1\boldsymbol{L}_1P_1 = \theta_1,$$

more exactly defined by the following equation:

$$\tan\theta_1 = \frac{M_1P_1}{\boldsymbol{L}_1M_1} = \frac{\eta_1}{u_1 - \boldsymbol{v}_1}, \tag{272}$$

where $\boldsymbol{v}_1 = A_1\boldsymbol{L}_1$ denotes the abscissa, with respect to the vertex A_1 of the first surface, of the point $\boldsymbol{L}_1$. Of course, under certain circumstances the point $\boldsymbol{L}_1$ may coincide with $\boldsymbol{M}_1$, as is often the case.

259. Relative Importance of the Terms of the Series-Developments of the Aberrations of the 3rd Order. The maximum value of y_1 will be fixed by the limits of the required field of vision, and in the same way the maximum values of $\boldsymbol{y}_1$, $\boldsymbol{z}_1$ will depend on the size of the aperture of the optical system. Thus, for example, in the case of an optical system of relatively small field of vision, and, on the other hand, of relatively large aperture, the most important terms in the series-developments of the aberrations $\delta y'_m$, $\delta z'_m$ will be the terms which do not contain y_1 at all, that is, the terms $\boldsymbol{y}_1^3$, $\boldsymbol{y}_1^2\boldsymbol{z}_1$, $\boldsymbol{y}_1\boldsymbol{z}_1^2$ and $\boldsymbol{z}_1^3$. Next in importance will be the terms which contain the first power of y_1, viz., $y_1\boldsymbol{y}_1^2$, $y_1\boldsymbol{y}_1\boldsymbol{z}_1$ and $y_1\boldsymbol{z}_1^2$; and then the terms which contain the second power of y_1, viz., $y_1^2\boldsymbol{y}_1$ and $y_1^2\boldsymbol{z}_1$; and, finally, least important of all for this particular case, the term which contains y_1^3. In the developments of the aberrations of the 3rd order, where y_1, $z_1 = 0$, $\boldsymbol{y}_1$ and $\boldsymbol{z}_1$ denote the approximate values of the parameters of the object-ray, the ten terms above-mentioned are all that can occur.

The expressions for the aberrations $\delta y'_m$, $\delta z'_m$, which are developed by SEIDEL (see Art. 102), enable us to compute the resultant defects of the 3rd order of the image of an object-point, and by specializing these general formulæ (as SEIDEL himself does), we can ascertain the nature of the various component defects which go to make up this resultant. However, in order to obtain a clear comprehension of these errors, it is best to follow the plan adopted by KOENIG and VON ROHR[1] in their admirable and exhaustive treatise on the Theory of Spherical Aberrations, and thus, first, to develop separately the formulæ for each one of these special aberrations, and afterwards to give, at the end of the chapter, SEIDEL's general theory (Arts. 102, foll.). Accordingly, this method will be pursued here also.

II. The Spherical Aberration in the Case when the Object-Point Lies on the Optical Axis.

ART. 81. CHARACTER OF A BUNDLE OF REFRACTED RAYS EMANATING ORIGINALLY FROM A POINT ON THE OPTICAL AXIS.

260. Longitudinal Aberration, or Aberration along the Optical Axis. The simplest case of all is the case when the object-point lies on the optical axis of the centered system of spherical surfaces, so that the point designated by P_1 coincides with M_1, that is, $y_1 = 0$. When the bundle of image-rays is symmetrical about an axis, as is the case when the rays emanate originally from a point on the optical axis, one of the caustic surfaces (§ 46) is a surface of revolution around the axis and is touched by each ray of the bundle; whereas the other caustic surface, in this particular instance, collapses into the segment of the axis comprised between the point M' (Fig. 133) where the paraxial rays cross the axis and the point L' where the outermost rays of the bundle meet the axis.[2] All the rays of the bundle will intersect the axis at points which are comprised between the two extreme points M' and L'. This axial line-segment $M'L'$ is called the *Longitudinal Aberration* of the outermost ray. Let A designate the vertex of the spherical surface, and let us put $AM' = u'$, $AL' = v'$. If the Spherical Aberration along the axis, or the Longitudinal Aberration $M'L'$ is denoted by $\delta u'$, we shall have:

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[1] A. KOENIG und M. VON ROHR: Die Theorie der sphaerischen Aberrationen; being Chapter V (pages 208–338) of *Die Theorie der optischen Instrumente*, Bd. I, edited by M. VON ROHR (Berlin, 1904). This treatise of Messrs. KOENIG and VON ROHR has been of inestimable service to the author in the preparation of the present chapter of this work.

[2] All the letters in the figure should, as a matter of fact, be written with the subscript *m*, to indicate that the letters relate to the rays after refraction at the last, or *m*th, surface of the system. But in all such cases as the one here considered the surface-numerals written as subscripts may be conveniently omitted where only one of the surfaces of the system is being treated, since there is no risk of confusion.

$$\delta u' = v' - u'.$$

Now if $\theta' = \angle AL'B$ denotes the slope of the ray which, in the plane of the diagram, crosses the optical axis at the point designated by L', it is evident that $\delta u'$ is a function of this angle θ'; and, moreover, it is also evident that if the function $\delta u'$ is developed in a series of ascending powers of θ', only the even powers will occur, because for a ray lying in the same meridian plane and symmetrically situated on the other side of the optical axis, so that its slope-angle is equal to $-\theta'$, we shall obtain the same value of the function $\delta u'$. If, therefore, θ' may be regarded as a magnitude of the 1st order of smallness, we can write:

$$\delta u' = v' - u' = a'\theta'^2, \tag{273}$$

since all the succeeding terms of the series, involving magnitudes of the orders of smallness higher than the 3rd, are, by the limitations of this investigation, to be neglected.

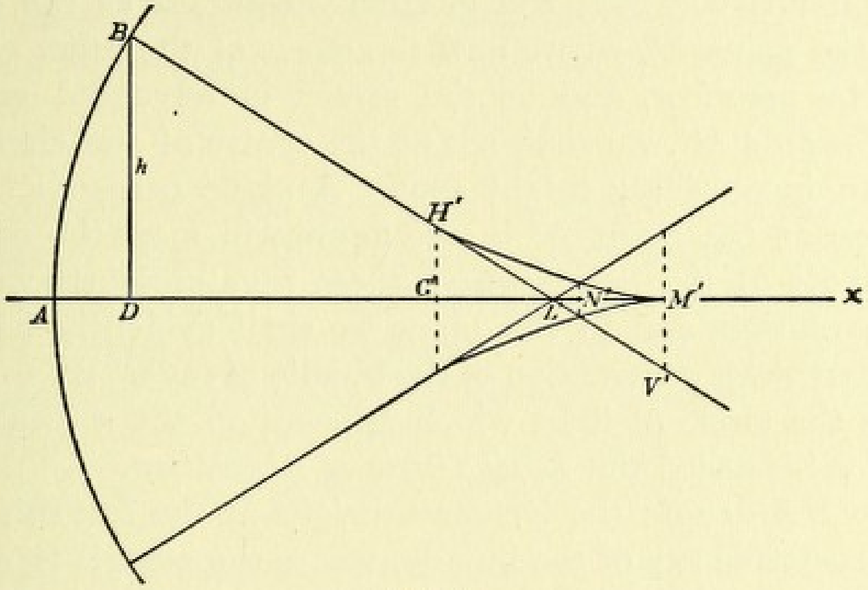

FIG. 133.

CHARACTER OF BUNDLE OF RAYS SYMMETRICALLY SITUATED WITH RESPECT TO THE OPTICAL AXIS.

$AM' = u'$, $AL' = v'$, $M'L' = \delta u'$, $M'V' = \delta y'$, $DB = h$, $\angle AL'B = \theta'$.

The co-efficient a' is entirely characteristic both of the magnitude and of the nature, or sign, of the aberration $\delta u'$, since, for a given value of θ', we can determine $\delta u'$, so soon as we have ascertained also the value of a'. Thus, if $a' = 0$, we have $v' = u'$, in which case we say that the system is "*spherically corrected*" for this ray. According as the sign of the co-efficient a' is positive or negative, the optical system is said to be "*spherically over-corrected*" or "*spherically under-corrected*" for the particular ray in question.

261. Least Circle of Aberration. If the rays of the bundle are received on a plane screen placed at right angles to the optical axis, and if this screen is gradually translated along the axis from the position $G'H'$ in the figure towards the point M', we shall see on the screen at first a circular patch of light surrounded on its outer edge by a brighter ring, which will gradually contract as the screen approaches the point L', where now the effect of the other caustic will begin to be manifest, and as the screen is advanced still farther from L' towards M', we shall see at the centre of the circular patch of light an increasingly bright spot. A plane perpendicular to the optical axis at the point N' in the figure will meet the outside rays of the bundle at the points where these rays cross the caustic surface of revolution, and in this plane we shall evidently have, therefore, the narrowest contraction of the bundle of rays. By some writers on Optics the circle of light which appears on the screen when it is placed at N' is called the *Least Circle of Aberration.*

Let $h = DB$ denote the incidence-height at the last spherical surface of the extreme ray of the bundle, whose slope-angle is

$$\theta' = \angle AL'B;$$

and let L'' (not shown in the figure) mark the position of the point between L' and M', where some other ray of the pencil of image-rays lying in the plane of the figure crosses the optical axis. The slope-angle of this ray may be denoted by θ''. Finally, let i denote the ordinate of the point of intersection of this general ray with the outermost ray. The least value i_0 of i will be the radius of the Least Circle of Aberration. Evidently,

$$L''L' = i(\cot\theta'' - \cot\theta');$$

moreover, if a' denotes the so-called aberration-co-efficient (§ 260):

$$L''L' = a'(\theta'^2 - \theta''^2):$$

so that we obtain the following relation:

$$i(\cot\theta'' - \cot\theta') = a'(\theta'^2 - \theta''^2).$$

Differentiating this equation with respect to θ'', and putting $\partial i/\partial\theta'' = 0$,

we obtain:

$$2\theta''\cdot\sin^2\theta''(\cot\theta'' - \cot\theta') = \theta'^2 - \theta''^2;$$

and expanding the trigonometric functions in series,[1] and neglecting terms involving powers and products of θ', θ'' higher than the 3rd, we find:

$$\theta''(2\theta'' - \theta') = \theta'^2,$$

which is satisfied by the values $\theta'' = \theta'$ and $\theta'' = -\theta'/2$. The first of these values corresponds to the maximum value of i represented in the diagram by the ordinate $G'H'$; whereas the second value

$$\theta'' = -\theta'/2$$

gives the slope of the ray for which $i = i_0$ is a minimum. The incidence-height of this ray is, therefore, approximately half that of the outside ray, but opposite in sign. If this value of θ'' is substituted in the above equation connecting i and θ'', we shall find (neglecting, as before, powers of θ' above the 3rd) for the radius of the Least Circle of Aberration:

$$i_0 = -a'\theta'^3/4.$$

The position on the axis of the point N' can be determined from the fact that $N'L'$ must be equal to $-i_0\cdot\cot\theta'$; hence, to the same degree of approximation, we find:

$$N'L' = a'\theta'^2/4 = M'L'/4.$$

Accordingly, the distance of the Least Circle of Aberration from the GAUSSian Image-Point M' is equal, approximately, to three-fourths of the Longitudinal Aberration of the extreme outside ray.

262. The so-called Lateral Aberration. Exactly what point on the axis is to be regarded as the image of the axial object-point M in such a case as that which we are here discussing is a question that cannot be decided by merely theoretical considerations; especially, too, as there is some diversity of opinion on the subject. In order to be answered, the matter, as CZAPSKI observes, needs to be considered rather from the point of view of Physical Optics than from that of Geometrical Optics. Most optical writers are agreed, however, that the place probably selected by the eye as most nearly reproducing the axial object-point is the place of the least circle of aberration. This

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[1] The development of the cotangent in series is as follows:

$$\cot x = 1/x - x/3 - x^3/45 - 2x^5/945 - \cdots.$$

circle is pierced by all the rays of the bundle, and the radius of it might be considered as the measure of the spherical aberration. Instead of this magnitude, ABBE employs the radius of the circle in the GAUSSian Image-Plane σ' (§ 254), inside of which all the rays of the bundle cross this plane. This radius

$$M'V' = \delta y'$$

is called the *Lateral Aberration* of the extreme ray, and its magnitude is equal to $M'L'/N'L'$ times the radius of the least circle of aberration: that is,

$$\delta y' = -\theta' \cdot \delta u' = -a' \cdot \theta'^3. \tag{274}$$

Thus, we see also that, whereas the Longitudinal Aberration $\delta u'$ is of the 2nd order of smallness, the Lateral Aberration $\delta y'$ is of the 3rd order.

ART. 82. DEVELOPMENT OF THE FORMULA FOR THE SPHERICAL ABERRATION OF A DIRECT BUNDLE OF RAYS.

263. Since the bundle of rays emanating originally from a point on the optical axis of the centered sytem of spherical surfaces is symmetrical with respect to this axis, it will be sufficient to investigate the rays in any meridian plane. Consider, therefore, any ray of the bundle, and let the meridian plane containing this ray be the xy-plane of the system of co-ordinates. Hence, for this ray not only do we have $y_1 = z_1 = 0$ (as is the case for all the rays of the bundle), but also $\mathbf{z}_1 = 0$; so that the only term in the series-development of the Lateral Aberration $\delta y'_m$ will be the $\mathbf{y}_1^3$-term (see § 259).

Discarding for the present the subscript-notation, let us designate by L, L' the points where the path of this ray crosses the optical axis before and after refraction at the kth spherical surface. Employing here the same letters and symbols as in § 209, viz.:

$$r = AC, \quad v = AL, \quad v' = AL', \quad \angle BCA = \varphi, \quad \angle ALB = \theta,$$
$$\angle AL'B = \theta',$$

denoting also the angles of incidence and refraction by α, α', respectively, and the indices of refraction by n, n', we may write the fundamental formulæ for the refraction of the ray at the spherical surface in question, as follows (see § 210):

$$\left.\begin{aligned} r \cdot \sin\alpha &= -(v - r) \cdot \sin\theta, \\ r \cdot \sin\alpha' &= -(v' - r) \cdot \sin\theta', \\ n' \cdot \sin\alpha' &= n \cdot \sin\alpha, \\ \alpha - \theta &= \alpha' - \theta' = \varphi. \end{aligned}\right\} \tag{275}$$

If, also, M, M' designate the points where the path of a paraxial ray crosses the optical axis, before and after refraction, respectively, at the kth spherical surface, and if

$$AM = u, \quad AM' = u',$$

then (§ 126)

$$n\left(\frac{1}{r} - \frac{1}{u}\right) = n'\left(\frac{1}{r} - \frac{1}{u'}\right) = J.$$

Moreover,

$$ML = \delta u = v - u, \quad M'L' = \delta u' = v' - u'$$

will denote the magnitudes of the Longitudinal Aberration of the ray before and after refraction at the spherical surface.

If we neglect all magnitudes higher than those of the second order, the approximate values of the slope-angles θ, θ', expressed in terms of the central angle φ, are $\theta = - r\varphi/u$, $\theta' = - r\varphi/u'$. But if, as we propose to do here, we take account also of the terms of the 3rd order, the expressions for θ, θ' must evidently have the following forms:

$$\left.\begin{aligned} \theta &= -\frac{r}{u}\varphi + A\varphi^3, \\ \theta' &= -\frac{r}{u'}\varphi + A'\varphi^3, \end{aligned}\right\} \qquad (276)$$

wherein the co-efficients A, A' are undetermined. Moreover, since

$$\alpha = \theta + \varphi, \quad \alpha' = \theta' + \varphi,$$

we may expand α, α' likewise in a series of odd powers of φ, as follows:

$$\left.\begin{aligned} \alpha &= \frac{Jr}{n}\varphi + A\varphi^3, \\ \alpha' &= \frac{Jr}{n'}\varphi + A'\varphi^3, \end{aligned}\right\} \qquad (277)$$

where the co-efficients A, A' have the same meanings as in formulæ (276).

If x denotes a small magnitude of the 1st order, and if we take account of terms as far as x^3, then

$$\sin x = x - x^3/6;$$

and, hence, employing formulæ (276) and (277), we have here the following series-developments for the sines of the angles α, α', θ, θ':

$$\left.\begin{aligned}\sin\alpha &= \frac{Jr}{n}\varphi - \left(\frac{1}{6}\frac{J^3r^3}{n^3} - A\right)\varphi^3,\\ \sin\alpha' &= \frac{Jr}{n'}\varphi - \left(\frac{1}{6}\frac{J^3r^3}{n'^3} - A'\right)\varphi^3,\\ \sin\theta &= -\frac{r}{u}\varphi + \left(\frac{1}{6}\frac{r^3}{u^3} + A\right)\varphi^3,\\ \sin\theta' &= -\frac{r}{u'}\varphi + \left(\frac{1}{6}\frac{r^3}{u'^3} + A'\right)\varphi^3.\end{aligned}\right\} \qquad (278)$$

Substituting in the first three of equations (275) these values of the sines of the angles α, θ, etc., putting

$$v = u + \delta u, \quad v' = u' + \delta u',$$

and neglecting all magnitudes of orders higher than the 3rd, we obtain, after some reductions:

$$\begin{gathered}\frac{\delta u}{u} = -\frac{\varphi^2}{6}\left\{\frac{Jr^2}{n}\left(\frac{J}{n} - \frac{1}{u}\right) - 6A\frac{u}{r}\right\},\\ \frac{\delta u'}{u'} = -\frac{\varphi^2}{6}\left\{\frac{Jr^2}{n'}\left(\frac{J}{n'} - \frac{1}{u'}\right) - 6A'\frac{u'}{r}\right\},\\ 6(n'A' - nA) = J^3r^3\left(\frac{1}{n'^2} - \frac{1}{n^2}\right).\end{gathered}$$

If, now, we multiply the first of these equations by n/u and the second by n'/u', and then subtract the first from the second, we obtain:

$$n'\frac{\delta u'}{u'^2} - n\frac{\delta u}{u^2} = -\frac{r^2\varphi^2}{6}J\left\{J\left(\frac{1}{n'u'} - \frac{1}{nu}\right) - \left(\frac{1}{u'^2} - \frac{1}{u^2}\right) - \frac{6}{Jr^3}(n'A' - nA)\right\};$$

and, accordingly, by means of the third of the equations above, we can eliminate at the same time both of the unknown co-efficients A, A'. Thus, employing ABBE's convenient Difference-Notation, whereby the difference $q' - q$ between the values q, q' of a magnitude before and after refraction is denoted by Δq, we derive the following equation:

$$\Delta\frac{n\cdot\delta u}{u^2} = -\frac{r^2\varphi^2}{6}J\left(J\cdot\Delta\frac{1}{nu} - \Delta\frac{1}{u^2} - J^2\cdot\Delta\frac{1}{n^2}\right),$$

which, since (§126)

$$J\cdot\Delta\frac{1}{n^2}=\frac{1}{r}\Delta\frac{1}{n}-\Delta\frac{1}{nu},$$

and

$$\Delta\frac{1}{u^2}=-J\left(\Delta\frac{1}{nu}+\frac{1}{r}\Delta\frac{1}{n}\right),$$

may be still further simplified as follows:

$$\Delta\frac{n\cdot\delta u}{u^2}=-\tfrac{1}{2}r^2\varphi^2J^2\cdot\Delta\frac{1}{nu}. \qquad (279)$$

Thus, provided we know the Longitudinal Aberration δu of the given ray before refraction at the spherical surface in question, we may, by means of formula (279), compute the magnitude $\delta u'$ of the Longitudinal Aberration after refraction.

264. From the incidence-point B draw BD perpendicular to the optical axis at B, and put $DB = h$, so that h denotes the ordinate of the incidence-point B, that is, the incidence-height of the ray of slope θ. Then, since

$$h = r\cdot\sin\varphi,$$

to the degree of approximation required in this investigation, we may write:

$$h = r\varphi - r\frac{\varphi^3}{6}; \qquad (280)$$

and, consequently, in formula (279) we can put $r^2\varphi^2 = h^2$. If we do this, and if now at the same time we attach to the symbols the surface-number in the form of a subscript, noting also that the point L_k where the ray crosses the axis before refraction at the kth surface is identical with the point L'_{k-1} where the ray crosses the axis after refraction at the $(k - 1)$th surface, so that

$$\delta u_k = \delta u'_{k-1};$$

we may write the formula for the kth surface as follows:

$$\frac{n'_k\cdot\delta u'_k}{u'^2_k}-\frac{n'_{k-1}\cdot\delta u'_{k-1}}{u^2_k}=-\tfrac{1}{2}h^2_kJ^2_k\left(\frac{1}{n'_ku'_k}-\frac{1}{n'_{k-1}u_k}\right), \qquad (281)$$

or in Abbe's abbreviated notation:

$$\Delta\left(\frac{n\cdot\delta u}{u^2}\right)_k=-\tfrac{1}{2}h^2_kJ^2_k\cdot\Delta\left(\frac{1}{nu}\right)_k. \qquad (281a)$$

265. By means of this recurrent formula (281), we can obtain finally the value of the Longitudinal Aberration $\delta u'_m$ of the ray after refraction at the last surface of the centered system of m spherical surfaces. How this is done, we proceed now to show.

By combining formulæ (276) and (280), we find:

$$h = -u\theta + \left(Au - \frac{r}{6}\right)\varphi^3 = -u'\theta' + \left(A'u' - \frac{r}{6}\right)\varphi^3, \quad (282)$$

and, hence, again introducing the subscripts, and remarking that the angles denoted by θ_k and θ'_{k-1} are identical, we may write, taking account of the terms of the 3rd order:

$$\frac{h_k^2}{u_k^2} = \frac{h_{k-1}^2}{u'^2_{k-1}}. \quad (283)$$

If, therefore, we multiply both sides of equation (281) by h_k^2, and at the same time use the relation (283), we shall derive the following formula:

$$\frac{n'_k \cdot h_k^2 \cdot \delta u'_k}{u'^2_k} - \frac{n'_{k-1} \cdot h_{k-1}^2 \cdot \delta u'_{k-1}}{u'^2_{k-1}} = -\tfrac{1}{2} h_k^4 J_k^2 \cdot \Delta\left(\frac{1}{nu}\right)_k.$$

If, now, in this formula we give k in succession the values 1, 2, $\cdots$ m, and add together the equations thus obtained, and note also that, since the bundle of object-rays is supposed to be homocentric, we must put $\delta u_1 = 0$, we obtain finally:

$$\delta u'_m = -\frac{u'^2_m}{2n'_m}\,\frac{h_1^4}{h_m^2}\sum_{k=1}^{k=m}\left(\frac{h_k}{h_1}\right)^4 J_k^2 \cdot \Delta\left(\frac{1}{nu}\right)_k. \quad (284)$$

In this formula we need to know, in addition to the constants which determine the optical system (refractive indices, radii, thicknesses, etc.), only the position on the axis of the object-point M_1 and the incidence-height h_1 of the object-ray; for then we can compute the values of all the other magnitudes that occur on the right-hand side of the equation. Practically, the formula is very convenient, because it exhibits the effect on the Longitudinal Aberration $\delta u'_m$ which is produced at each refraction. For a given axial object-point, it will always be theoretically possible, by employing a sufficient number of surfaces, to contrive so that the aberration $\delta u'_m = 0$; the condition whereof is:

$$\sum_{k=1}^{k=m} h_k^4 J_k^2 \cdot \Delta\left(\frac{1}{nu}\right)_k = 0. \quad (285)$$

It must be remembered, however, that the accuracy of this *formula for the abolition of the spherical aberration along the axis* depends on the magnitude of the aperture of the bundle of rays; for it has been assumed throughout that we can safely afford to neglect the powers of the slope-angle θ higher than the 3rd. Thus, for example, in the case of the objective of a telescope, the aperture of which, although by no means negligible, is relatively small, the formula will usually give a very high approximation. On the other hand, in the calculation of a photographic objective the formula would generally not be very accurate. In the objective of a microscope the magnitude of the angle θ is often equal to nearly 90°, and the approximate formulæ here derived are not applicable to wide-angle systems at all.

266. Abbe's Measure of the "Indistinctness" of the Image. By means of formulæ (274) and (282), we find for the *Lateral Aberration:*

$$\delta y'_m = \frac{h_m}{u'_m}\,\delta u'_m;$$

and hence:

$$\delta y'_m = -\frac{u'_m}{2n'_m}\,\frac{h_1^4}{h_m}\sum_{k=1}^{k=m}\left(\frac{h_k}{h_1}\right)^4 J_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k. \tag{286}$$

If $e_{1,k}$ denotes the length of the object-line perpendicular to the optical axis at M_1, whose Gaussian image at M'_k is equal to the value of the Lateral Aberration $\delta y'_k$ after refraction at the kth surface, it is evident that details in an object at M_1 which are separated by an interval greater than $e_{1,k}$ will, on account of the spherical aberration, not appear separated in the image formed after refraction at the kth surface. Thus, according to ABBE, the magnitude denoted by $e_{1,k}$, measured at the object, affords a convenient *measure of the lack of detail, or "indistinctness", of the image.*

The approximate value of the slope-angle θ'_k is:

$$\theta'_k = -\frac{h_k}{u'_k} = -\frac{h_{k+1}}{u_{k+1}};$$

and, hence by the Law of ROBERT SMITH (§ 194):

$$\frac{n_1 h_1 e_{1,k}}{u_1} = \frac{n'_k h_k\cdot\delta y'_k}{u'_k},$$

that is,

$$e_{1,k} = \frac{h_k}{h_1}\,\frac{n'_k}{n_1}\,\frac{u_1}{u'_k}\,\delta y'_k. \tag{287}$$

Thus, from formula (286) we obtain:

$$e_{1,m} = -\frac{1}{2n_1} h_1^3 u_1 \sum_{k=1}^{k=m} \left(\frac{h_k}{h_1}\right)^4 J_k^2 \cdot \Delta \left(\frac{1}{nu}\right)_k ; \tag{288}$$

which shows that the "indistinctness" is proportional to the cube of the aperture h_1 of the bundle of object-rays.

In case the object-point M_1 is very far away, it will be convenient to determine the angle $\epsilon_{1,m}$ subtended at the vertex A_1 of the first surface by the linear magnitude $e_{1,m}$; thus, since

$$\epsilon_{1,m} = e_{1,m}/u_1,$$

we have:

$$\epsilon_{1,m} = -\frac{h_1^3}{2n_1} \sum_{k=1}^{k=m} \left(\frac{h_k}{h_1}\right)^4 J_k^2 \cdot \Delta \left(\frac{1}{nu}\right)_k ;$$

and, hence, the angular value of the lack of detail in the image, on account of spherical aberration, is proportional to the cube of the linear aperture h_1 of the bundle of object-rays. For example, in the case of the objective of a telescope, it is proportional to the cube of the diameter of the objective.

ART. 83. SPHERICAL ABERRATION OF DIRECT BUNDLE OF RAYS IN SPECIAL CASES.

267. Case of a Single Spherical Refracting Surface.

If the optical system consists of a single spherical surface ($m = 1$), we have for the Longitudinal Aberration of the bundle of image-rays corresponding to a bundle of object-rays proceeding from the axial point M ($u = AM$):

$$\delta u' = -\frac{J^2 h^2 u'^2}{2n'} \left(\frac{1}{n'u'} - \frac{1}{nu}\right);$$

and if we substitute for J its value, viz.:

$$J = \frac{n'(u' - r)}{ru'},$$

we obtain:

$$\delta u' = -\frac{h^2 (u' - r)^2 (nu - n'u')}{2nr^2 uu'} . \tag{289}$$

In each of the following three cases the Longitudinal Aberration will be equal to zero:

(1) When $u = u' = 0$, in which case object-point M and image-point M' coincide at the vertex A of the sphere;

(2) When $u = u' = r$, in which case object-point M and image-point M' coincide at the centre C of the sphere; and

(3) When $nu = n'u'$, in which case:

$$u = (n + n')r/n, \quad u' = (n + n')r/n',$$

and the points M, M' coincide with the aplanatic points Z, Z', respectively (§ 207, § 211, Note 3). This latter case is the only one that may be said to have any practical importance.

For all other positions of the object-point M the point L' will not coincide with M'. The sign of $\delta u'$ will depend on the sign of the factor

$$\frac{1}{n'u'} - \frac{1}{nu} = \frac{n' - n}{n'^2}\left(\frac{1}{r} - \frac{n' + n}{n} \cdot \frac{1}{u}\right),$$

and for any given spherical surface may be positive or negative depending on the sign of u. If the object-point M is at an infinite distance, the sign of $\delta u'$ will depend on that of $(n' - n)/r$. If this expression is positive, the refracting surface will be a convergent surface, the sign of $\delta u'$ will be negative, and the spherical surface will be "spherically under-corrected" (§ 260).

268. Case of an Infinitely Thin Lens.

When the optical system consists of two spherical surfaces, we must put $m = 2$ in formula (284). Assuming that the Lens is surrounded by the same medium on both sides, we may conveniently write:

$$n = n'_1/n_1 = n'_1/n'_2,$$

so that in the following discussion n will be used to denote the relative index of refraction for the two media concerned. Moreover, in the case of an Infinitely Thin Lens, we have:

$$u'_1 = u_2,$$

and we may therefore afford to dispense with the subscripts in the symbols u_1 and u'_2, and write these u and u', respectively. Likewise, we shall write: $h = h_1 = h_2$. Under these circumstances, we obtain by the general formula (284) the following expression for the *Longitudinal Aberration of an Infinitely Thin Lens:*

$$\delta u' = -\frac{h^2 u'^2}{2}\left\{\left(\frac{1}{r_1} - \frac{1}{u}\right)^2\left(\frac{1}{nu'_1} - \frac{1}{u}\right) - \left(\frac{1}{r_2} - \frac{1}{u'}\right)^2\left(\frac{1}{nu'_1} - \frac{1}{u'}\right)\right\}. \quad (290)$$

For the case of an Infinitely Thin Lens we shall employ a *special notation*, as follows: Thus, let $x = 1/u$, $x' = 1/u'$ denote the reciprocals of the intercepts on the axis of the paraxial object-rays and image-rays, respectively, and let $c = 1/r_1$, $c' = 1/r_2$ denote the curvatures of the bounding surfaces of the Lens. Finally, let $\varphi = 1/f$ denote here the reciprocal of the primary focal length of the Lens. With this system of symbols the formulæ of Chapter VI, Art. 41, for the *Refraction of Paraxial Rays through an Infinitely Thin Lens* will have the following forms:

$$\left.\begin{aligned} \varphi &= (n-1)(c-c'), \\ x' &= x + \varphi, \\ \frac{1}{u_1'} &= \frac{x + (n-1)c}{n}. \end{aligned}\right\} \qquad (291)$$

Employing these relations, we can eliminate from formulæ (290) the magnitudes denoted by u_1', x' and c', and thus we shall obtain:

$$\delta u' = -\frac{h^2 u'^2 \varphi}{2}\left[\left(\frac{n}{n-1}\right)^2 \varphi^2 + \left(\frac{1}{n-1}\right)\left\{nx - (2n+1)(c-x)\right\}\varphi + \frac{1}{n}(c-x)\left\{(n+2)(c-x) - 2nx\right\}\right],$$

or

$$\delta u' = -\frac{h^2 u'^2 \varphi}{2}\left\{\left(\frac{n}{n-1}\right)^2 \varphi^2 + \left(\frac{3n+1}{n-1}x - \frac{2n+1}{n-1}c\right)\varphi + \frac{3n+2}{n}x^2 - \frac{4(n+1)}{n}cx + \frac{n+2}{n}c^2\right\}. \qquad (292)$$

If the object-rays are parallel to the axis ($x = 0$, $x' = \varphi$), the image-point M' coincides with the secondary focal point E', and for this special case we obtain:

$$E'L' = -\frac{h^2\varphi}{2(n-1)}\left\{\frac{(n-1)(n+2)}{n}\frac{c^2}{\varphi^2} - (2n+1)\frac{c}{\varphi} + \frac{n^2}{n-1}\right\}. \qquad (293)$$

In the case of a Thin Lens of semi-diameter h and focal length f, whose thickness is greatest along the optical axis, one can easily see from the geometrical properties of the circle that the thickness of the Lens is very nearly equal to

$$\frac{h^2}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{h^2}{2}(c - c') = \frac{h^2\varphi}{2(n-1)};$$

and thus for a Lens of this character the expression within the large brackets of formula (293) is the factor by which the thickness of the Lens has to be multiplied in order to obtain the spherical aberration along the axis for an infinitely distant axial object-point. If the Lens is a convergent glass Lens in air ($n = 3/2$), its thickness is very nearly equal to h^2/f.

By way of illustration, let us compute by formula (293) the Longitudinal Aberration for Lenses of special forms; thus, we shall find:

(1) In case the *first surface of the Lens is plane* ($c = 0$):

$$E'L' = -\left(\frac{n}{n-1}\right)^2 \cdot \frac{h^2}{2f}; \quad \text{for} \quad n = 3/2, \quad E'L' = -\frac{9}{2}\frac{h^2}{f}.$$

(2) In case the *second surface of the Lens is plane* ($c = \varphi/(n-1)$):

$$E'L' = -\frac{n^3 - 2n^2 + 2}{n(n-1)^2} \cdot \frac{h^2}{2f}; \quad \text{for} \quad n = 3/2, \quad E'L' = -\frac{7}{6}\frac{h^2}{f}.$$

(3) In case of an *Equi-Biconvex Lens* ($c = -c' = \varphi/2(n-1)$):

$$E'L' = -\frac{4n^3 - 4n^2 - n + 2}{8n(n-1)^2} \cdot \frac{h^2}{f}; \quad \text{for} \quad n = 3/2, \quad E'L' = -\frac{5}{3}\frac{h^2}{f}.$$

Assuming, therefore, that the focal length f of each of these Lenses has the same numerical value, we see that the Longitudinal Aberration is greatest in the Lens with its plane side turned towards, and least in the Lens with its plane side turned away from, the object-rays.

269. The next question to be investigated is, *What are the conditions that the Longitudinal Aberration of a Thin Lens shall vanish?*

If, for brevity, the expression within the large brackets in formula (292) is put equal to Z, we may write the formula for the Longitudinal Aberration of a Thin Lens as follows:

$$\delta u' = -\frac{h^2 u'^2 \varphi}{2} Z. \tag{294}$$

If $\varphi = 0$ (that is, $f = \infty$), we shall have $\delta u' = 0$. In this case $u = u'$, $r_1 = r_2$, so that the two surfaces of the Infinitely Thin Lens are parallel. This case has evidently no practical interest. It remains, therefore, to investigate the cases when the function Z vanishes.

We shall assume that we have given a Lens of a definite focal length, and that the position on the axis of the object-point M is also given; and, under these circumstances, we are required to determine the form of the Lens in order that the Longitudinal Aberration shall be zero; that is, we must ascertain the curvatures c, c' of the two surfaces of the Lens. Since $c' = c + \varphi$, and since the value of φ is supposed to be prescribed, the problem, in reality, consists merely in finding the curvature c of the first surface. This process of varying the curvatures of the surfaces without altering the focal length is called "*bending*" *the Lens.*

Accordingly, regarding c as the independent variable, and treating both x and φ as constants, we shall write the function Z in the following form:

$$Z = \frac{n+2}{n}c^2 - \left(\frac{4(n+1)}{n}x + \frac{2n+1}{n-1}\varphi\right)c + \frac{3n+2}{n}x^2 + \frac{3n+1}{n-1}x\varphi + \left(\frac{n}{n-1}\right)^2\varphi^2. \quad (295)$$

For $Z = 0$, we obtain two values of c, as follows:

$$c = \frac{4(n^2-1)x + n(2n+1)\varphi \pm n\sqrt{4(n-1)^2x(x+\varphi) - (4n-1)\varphi^2}}{2(n-1)(n+2)},$$

and if these values of c are to be real, the expression under the radical must be positive; that is, for real values of c, we must have:

$$4x(x+\varphi) - \frac{4n-1}{(n-1)^2}\varphi^2 > 0;$$

or, since $x + \varphi = x'$,

$$4xx' - \frac{4n-1}{(n-1)^2}(x'-x)^2 > 0.$$

Accordingly, we see that a necessary condition that the aberration shall vanish is that x and x' shall have the same sign; which means that the object-point and image-point must lie both on the same side of the Lens. In the practical and more important case when the image is a real image, it is impossible to abolish the Longitudinal Aberration of an Infinitely Thin Lens.

The condition above may also be put in the following form:

$$\frac{\{n-1+\sqrt{n(n+2)}\}^2}{4n-1} > \frac{x'}{x} > \frac{\{n-1-\sqrt{n(n+2)}\}^2}{4n-1}.$$

Thus, with a glass Lens in air ($n = 3/2$), it is possible to abolish the Longitudinal Aberration only in case the ratio x'/x, or u'/u, is comprised between the values $(11 - \sqrt{21})/10$ and $(11 + \sqrt{21})/10$, that is, between the values 0.642 and 1.558.

In exactly the same way, by considering Z as a function of x, and treating c and φ as constants, we shall find that in order for Z to vanish for real values of x, the Infinitely Thin Lens must have a form such that

$$\frac{c^2}{\varphi^2} - \frac{1}{n-1}\frac{c}{\varphi} - \frac{(n+1)(3n-1)}{4(n-1)^2} > 0;$$

that is, the ratio c/φ must be comprised between the values

$$\frac{1-\sqrt{n(3n+2)}}{2(n-1)} \quad \text{and} \quad \frac{1+\sqrt{n(3n+2)}}{2(n-1)}.$$

For example, for $n = 3/2$, the value of c/φ must lie between $(2-\sqrt{39})/2$ and $(2+\sqrt{39})/2$ that is, between -2.1225 and $+4.1225$.

Practically speaking, these results are without value.

270. Since, therefore, it is practically not feasible to abolish entirely the Longitudinal Aberration in the case of an Infinitely Thin Lens, let us seek now to find the condition that *the Aberration shall be a minimum.*

Equation (295), in which c and Z are to be considered as the variables, evidently represents a Parabola with its axis parallel to the Z-axis of co-ordinates and with its vertex at the point:

$$\left.\begin{aligned} Z_0 &= \frac{n(4n-1)}{4(n-1)^2(n+2)}\varphi^2 - \frac{n}{n+2}x(x+\varphi), \\ c_0 &= \frac{2(n+1)}{n+2}x + \frac{n(2n+1)}{2(n-1)(n+2)}\varphi; \end{aligned}\right\} \tag{296}$$

and it is obvious that for a Lens of given "power" (φ) and for a given position (x) of the object-point M on the axis, the minimum value of the function Z will be $Z = Z_0$.

So long as $xx' = x(x+\varphi)$ is not positive, the value of Z_0 cannot be equal to zero; if $xx' < 0$, then $Z_0 > 0$. That is, for a real image-point M' on the other side of the Lens from the object-point M, the minimum value of Z is positive. In case $xx' > 0$, Z_0 will, in general, be negative, and in special cases it may be equal to zero, in agreement with the results found in the preceding discussion. We need consider only the case when $Z_0 > 0$. The minimum value of the Longitudinal Aberration is:

$$\delta u'_0 = -\frac{h^2 u'^2 \varphi}{2} Z_0.$$

For an infinitely distant object-point ($x = 0$, $x' = \varphi$), the curvatures of the Lens-surfaces for minimum aberration are:

$$c_0 = \frac{n(2n+1)}{2(n-1)(n+2)}\varphi, \quad c'_0 = \frac{2n^2-n-4}{2(n-1)(n+2)}\varphi; \quad (x=0);$$

and the minimum aberration is:

$$(E'L')_0 = -\frac{n(4n-1)}{8(n-1)^2(n+2)}h^2\varphi.$$

For $n = 3/2$, we find: $c_0 = 12\varphi/7$, $(E'L')_0 = -15h^2\varphi/14$; and for $n = 2$ (diamond), $c_0 = 5\varphi/4$, $(E'L')_0 = -7h^2\varphi/16$. The minimum value of the Longitudinal Aberration of a Diamond Lens is very much less than that of a Glass Lens of equal focal length. And, generally, for values of n greater than unity, it is easy to show that the minimum value of the aberration decreases with increase of n.

When $x = 0$, we have:

$$c_0 : c'_0 = \frac{n(2n+1)}{2n^2-n-4};$$

which, for $n = 3/2$, gives $c_0/c'_0 = -6$. Thus, with an infinitely distant object-point a biconvex glass Lens has the least Longitudinal Aberration, viz., $-15h^2\varphi/14$, when the curvature of its first surface is six times as great as the curvature of its farther surface.

271. Case of a System of Two or More Thin Lenses.

If the optical system consists of a system of m Infinitely Thin Lenses, with the centres of their surfaces ranged along one and the same straight line, we can determine the Longitudinal Aberration by means of the formula (284). We shall employ here a notation entirely similar to that used above in the case of a single Lens (§ 268); but it should be noted also that in the following formulæ the subscript attached to a symbol will indicate, not, as usually, the ordinal number of the spherical refracting surface, but the ordinal number of the Lens to which the symbol has reference. The bundle of object-rays is supposed to emanate from an object-point M_1 on the optical axis, and the point where the paraxial image-rays cross the axis will be designated here by M'_m, and, similarly, the point where the outermost ray of the bundle of image-rays crosses the axis will be designated by L'_m. For the Longitudinal Aberration of the system of m Lenses, we obtain:

$$M'_m L'_m = -\frac{u'^2_m}{2h^2_m}\sum_{k=1}^{k=m} h^4_k \varphi_k Z_k, \qquad (297)$$

where

$$Z_k = \left(\frac{n_k}{n_k - 1}\right)^2 \varphi_k^2 + \left(\frac{3n_k + 1}{n_k - 1} x_k - \frac{2n_k + 1}{n_k - 1} c_k\right) \varphi_k$$
$$+ \frac{3n_k + 2}{n_k} x_k^2 - \frac{4(n_k + 1)}{n_k} c_k x_k + \frac{n_k + 2}{n_k} c_k^2. \quad (298)$$

In this formula n_k denotes the relative index of refraction from air into the medium of the kth Lens; c_k and φ_k denote the reciprocals of the radius of the first surface and the primary focal length, respectively, of this lens; x_k denotes the reciprocal of the intercept $A_k M_k$, where M_k designates the point where paraxial rays cross the axis before entering the kth Lens; h_k denotes the incidence-height of the outermost ray at the kth Lens; and, finally, $u'_m = A_m M'_m$ is the intercept of the paraxial image-rays.

If the distances that separate the Lenses are all negligible, so that we have a *System of m Thin Lenses in Contact*, the formula becomes:

$$M'_m L'_m = -\frac{u'^2_m h_1^2}{2} \sum_{k=1}^{k=m} \varphi_k Z_k. \quad (299)$$

Here the relation $x_{k+1} = x_k + \varphi_k$ will also be of service.

272. We may consider somewhat more in detail the special case of an optical system consisting of *Two Infinitely Thin Lenses in Contact*. The condition that the Longitudinal Aberration of a combination of this kind shall vanish is:

$$\varphi_1 Z_1 + \varphi_2 Z_2 = 0.$$

If the focal lengths of the two Lenses, or their reciprocals φ_1, φ_2, are assigned, and if also we know the reciprocal x_1 of the distance u_1 of the axial object-point from the first Lens, then, since $x_2 = x_1 + \varphi_1$, the analytical condition for the abolition of the spherical aberration will be an equation of the 2nd degree in c_1 and c_2. We may, therefore, choose arbitrarily the value of one of these two magnitudes; in which case there will always be two values of the other, real or imaginary, which will fulfil the above requirement.

Since, therefore, we have here two arbitrary variables c_1 and c_2 and only one equation to determine them, we may impose one other condition. For example, a very natural idea would be to make the curvatures of the second surface of the front Lens and the first surface of the following Lens identical ($c'_1 = c_2$), so that the two Lenses could be cemented together. However, if the two Lenses are made of different kinds of glass, with unequal co-efficients of dilatation, a combination of two cemented Lenses might become distorted under the influence of changes of temperature.

It has, therefore, been suggested that the other requirement should be the so-called HERSCHEL-*Condition;* that is, that the function

$$\varphi_1 Z_1 + \varphi_2 Z_2$$

should vanish not only for the particular value of x_1 but also for object-points on the axis very near to the point M_1 to which the value x_1 belongs. This condition will be expressed analytically by the equation:

$$\frac{\partial}{\partial x_1}(\varphi_1 Z_1 + \varphi_2 Z_2) = 0;$$

and, thus, we obtain a second equation between c_1 and c_2, as follows:

$$4\left(\frac{n_1+1}{n_1}\varphi_1 c_1 + \frac{n_2+1}{n_2}\varphi_2 c_2\right) - 2x_1\left(\frac{3n_1+2}{n_1}\varphi_1 + \frac{3n_2+2}{n_2}\varphi_2\right)$$
$$- \frac{3n_1+1}{n_1-1}\varphi_1^2 - \frac{2(3n_2+2)}{n_2}\varphi_1\varphi_2 - \frac{3n_2+1}{n_2-1}\varphi_2^2 = 0.$$

These two equations, taken simultaneously, will determine completely the forms of the two Lenses.

If now we impose still a third condition, viz., that the combination of the Two Thin Lenses in Contact shall be free from spherical aberration for all positions of the object-point on the axis, so that the function $\varphi_1 Z_1 + \varphi_2 Z_2$ shall vanish for all values of x_1, then, in addition to the two equations above, we must have also:

$$\frac{3n_1+2}{n_1}\varphi_1 + \frac{3n_2+2}{n_2}\varphi_2 = 0.$$

Evidently, in order to satisfy this last requirement, φ_1 and φ_2 must have opposite signs; that is, the combination must consist of a positive Lens and a negative Lens. Moreover, with the actual kinds of glass which are at our disposal it will be found necessary to make the curvatures of the Lenses exceedingly great in order to comply with this last requirement.

ART. 84. NUMERICAL ILLUSTRATION OF METHOD OF USING FORMULÆ FOR CALCULATION OF SPHERICAL ABERRATION.

273. In Chapter X, Art. 67, we computed by the methods of exact trigonometrical calculation the Longitudinal Aberration of a large Telescope Object-Glass, the data of which will be found in that place. Merely to illustrate the use of the formulæ which we have obtained here, it is proposed now to calculate for this same system the First Term of the Spherical Aberration of the Edge-Ray (§ 265), and the Lack of Detail in the Image (§ 266). The formulæ employed are the following:

$$M'_m L'_m = -\frac{u'^2_m h_1^2}{2n'_m}\left(\frac{h_1}{h_m}\right)^2 \sum_{k=1}^{k=m} P_k,$$

where

$$P_k = \left(\frac{h_k}{h_1}\right)^4 J_k^2 \cdot \Delta\left(\frac{1}{nu}\right)_k,$$

$$J_k = n'_{k-1}\left(\frac{1}{r_k} - \frac{1}{u_k}\right),$$

$$\Delta\left(\frac{1}{nu}\right)_k = \frac{1}{n'_k u'_k} - \frac{1}{n'_{k-1} u_k}.$$

Also, for the Angular Value of the Indistinctness or Lack of Detail in the Image, according to ABBE, we have the following formula:

$$\epsilon_{1,m} = -\frac{h_1}{2n_1}\sum_{k=1}^{k=m} P_k.$$

Moreover, in order to find the value of h_k/h_1 for each surface, we have:

$$\frac{h_k}{h_1} = \frac{u_k \cdot u_{k-1} \cdots u_2}{u'_{k-1} \cdot u'_{k-2} \cdots u'_1}.$$

Thus, for the first surface ($k = 1$), we have: $h_1/h_1 = 1$; for the second surface ($k = 2$):

$$h_2/h_1 = u_2/u'_1;$$

for the third surface ($k = 3$):

$$\frac{h_3}{h_1} = \frac{h_2}{h_1}\frac{u_3}{u'_2};$$

and for the fourth (and last) surface ($k = 4 = m$):

$$\frac{h_4}{h_1} = \frac{h_3}{h_1}\frac{u_4}{u'_3}$$

Accordingly, using the values of the u's as found in Chapter X, Art. 67, we obtain:

$$\begin{aligned} \lg u_2 &= 2.2430549 + \\ \text{clg}\, u'_1 &= \underline{7.7544706 +} \\ \lg h_2/h_1 &= 9.9975255 + \end{aligned}$$

$$\begin{aligned}
\lg h_2/h_1 &= 9.9975255 + \\
\lg u_3 &= 1.8426804 + \\
\operatorname{clg} u_2' &= \underline{8.1572385 +} \\
\lg h_3/h_1 &= 9.9974444 + \\
\lg u_4 &= 2.3518454 + \\
\operatorname{clg} u_3' &= \underline{7.6462272 +} \\
\lg h_4/h_1 &= 9.9955170 +
\end{aligned}$$

The following scheme exhibits the process of the calculation:

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$\operatorname{clg} u_k$		7.7569451 +	8.1573196 +	7.6481546 +
$\operatorname{clg} n_{k-1}'$		9.8197020 +	0.0000000	9.7926080 +
$\operatorname{clg} n_{k-1}' u_k$		7.5766471 +	8.1573196 +	7.4407626 +
$\operatorname{clg} u_k'$	7.7544706 +	8.1572385 +	7.6462272 +	7.7541648 +
$\operatorname{clg} n_k'$	9.8197020 +	0.0000000	9.7926080 +	0.0000000
$\operatorname{clg} n_k' u_k'$	7.5741726 +	8.1572385 +	7.4388352 +	7.7541648 +
$1/n_k' u_k'$	+0.0037512	+0.0143628	+0.0027469	+0.0056776
$-1/n_{k-1}' u_k$	0.0000000	−0.0037727	−0.0143655	−0.0027591
$\dfrac{1}{n_k' u_k'} - \dfrac{1}{n_{k-1}' u_k}$	+0.0037512	+0.0105901	−0.0116186	+0.0029185
$\operatorname{clg} r_k$	8.2232988 +	8.0450343 −	8.0721166 −	7.3872161 +
$1/r_k$		−0.0110926	−0.0118064	+0.0024390
$-1/u_k$		+0.0057141	+0.0143655	+0.0044479
$1/r_k - 1/u_k$		−0.0168067	−0.0261719	−0.0020089
$\lg (1/r_k - 1/u_k)$	8.2232988 +	8.2254825 −	8.4178352 −	7.3029583 −
$\lg n_{k-1}'$	0.0000000	0.1802980 +	0.0000000	0.2073920 +
$\lg J_k$	8.2232988 +	8.4057805 −	8.4178352 −	7.5103503 −
$\lg J_k^2$	6.4465976 +	6.8115610 +	6.8356704 +	5.0207006 +
$\lg \Delta(1/nu)_k$	7.5741726 +	8.0249001 +	8.0651538 −	7.4651597 +
$\lg (h_k/h_1)^4$	0.0000000	9.9901020 +	9.9897776 +	9.9820680 +
$\lg P_k$	4.0207702 +	4.8265631 +	4.8906018 −	2.4679283 +

$$\begin{aligned}
P_1 &= +\ 104.899 \cdot 10^{-8} \\
P_2 &= +\ 670.754 \cdot 10^{-8} \\
P_4 &= \underline{+\quad 2.937 \cdot 10^{-8}} \\
&\ +\ 778.590 \cdot 10^{-8} \\
P_3 &= \underline{-\ 777.324 \cdot 10^{-8}} \\
\Sigma P_k &= +\quad 1.266 \cdot 10^{-8}
\end{aligned}
\qquad\qquad
\begin{aligned}
\lg \Sigma P_k &= 2.1024337 + \\
\lg (h_1/h_4)^2 &= 0.0089660 + \\
\lg h_1^2 &= 1.5563025 + \\
\lg u_4'^2 &= 4.4916704 + \\
\operatorname{clg} 2n_4' &= \underline{9.6989700 +} \\
&\ \underline{7.8583426 +}
\end{aligned}$$

Accordingly, we find:

$$M_4' L_4' = -\ \underline{0.0072 \text{ inches.}}^{1}$$

---

[1] TAYLOR, computing the Spherical Aberration by a formula equivalent to the one employed by us, obtains a different value and one which agrees very closely with the exact value. But there appears to be a numerical error in his calculation of what he calls the "first parallel plate correction".

It will be perceived that the value of the longitudinal aberration $M'_4L'_4$ thus obtained is in fact rather more than twice as great as the exact value obtained in Art. 67 by the rigourous process of trigonometric computation, and at first sight it might appear, therefore, that the approximate value was utterly unreliable. However, the two values are of the same order of magnitude, and a little reflection will convince anyone that in this particular example, at least, we have no right to expect an agreement between the two values beyond the second place of decimals. At least one of the values of θ'_k is very nearly equal to 5°, and if we bear in mind that when we use the formula of the first approximation we are neglecting all terms involving the powers of this angle above the second, we can easily see that the agreement above in the first two figures to the right of the decimal-point is all that we could look for here.

In order to find ABBE's measure of the Angular Value of the Lack of Detail in the Image on account of the Spherical Aberration, we proceed as follows:

$$\begin{aligned} \lg \Sigma P_k &= 2.1024337 + \\ 3 \lg h_1 &= 2.3344538 + \\ \operatorname{clg} 2n_1 &= \underline{9.6989700 +} \\ \lg \epsilon_{1,m} &= 4.1358575 - \end{aligned}$$

This angle is expressed here in radians. It will be found to be less than 0″.3.

ART. 85. CONCERNING THE TERMS OF THE HIGHER ORDERS IN THE SERIES-DEVELOPMENT OF THE LONGITUDINAL ABERRATION.

274. The formulæ derived in Art. 82 were based on the assumption that we could put $v - u = a\theta^2$; thereby in the series-development of the expression for the Longitudinal Aberration neglecting all the terms after the first. So long as the slope-angle θ is relatively small, this procedure is fairly justified, and even though the formulæ thus obtained cannot claim to be entirely accurate, they will often enable us to compute very approximately the magnitude of the Spherical Aberration. Applied to optical systems of relatively narrow aperture, the formulæ will be found to be extremely serviceable in so far as they exhibit clearly the effect that will be produced by a variation of any one of the factors (radii, intervals, etc.) that are involved in the problem: so that the optical designer, instead of having to grope his way by means of tedious trial-calculations, can proceed methodically to make such alterations as he sees will tend to diminish the Spherical

Aberration. Especially, in the design of the Objectives of Telescopes —a problem which ever since the time of GALILEO has engaged the attention of some of the greatest mathematicians of the world— these approximate formulæ have proved to be of the greatest value.

If the Longitudinal Aberration δu is developed in a series of ascending powers of one of the variables α, θ, φ or h, it is obvious that the greater the relative magnitude of this variable, the more terms of the series will it be necessary to take account of. Thus, provided the slope-angle θ is not too great, it may suffice to take account of only the first two terms of the development, and then we may write:

$$\delta u = a\theta^2 + b\theta^4.$$

The development of the formulæ for the co-efficients a and b, by ABBE's Method of Invariants, is given by KOENIG and VON ROHR in their treatise on *Die Theorie der sphaerischen Aberrationen*.[1] The recurrent formula obtained in this way for the aberration-co-efficient of the second term, viz. b'_m, is not too complex to be often very serviceable in the practical design of optical instruments; but the co-efficients of the succeeding terms of the series lead to exceedingly complicated algebraic expressions, and are not usually of much value on this account, especially also as we begin to encounter well-nigh insurmountable numerical difficulties in trying to evaluate by means of these expressions the radii of the spherically corrected system. In case it is necessary to take account of these higher terms, the only satisfactory procedure is to resort to the laborious method of trigonometrical calculation of the ray-paths. After a number of trials it is nearly always possible by suitable alterations of the radii, thicknesses, etc., to contrive so that some selected ray shall emerge from the system so as to cross the optical axis approximately at the same point as the paraxial image-rays; and although this by no means implies that any other ray of the same meridian section will also intersect the axis at this point, it is usually a first step in the direction of diminishing the Longitudinal Aberration. The method is very fully explained, with a great number of actual numerical illustrations, in STEINHEIL & VOIT's *Handbuch der angewandten Optik* (Leipzig, 1891).

275. The Aberration Curve. If the Longitudinal Aberration δu of a ray of incidence-height h is developed in a series of ascending powers of h, and if we take account of only the first two terms, we may write:

$$\delta u = ah^2 + bh^4;$$

---

[1] This is Chapter V of VON ROHR's *Die Theorie der optischen Instrumente* (Berlin, 1904); see pages 217–219 and pages 235–239.

where a and b are co-efficients independent of the variable h.[1] If a and b both vanish, the Longitudinal Aberration will be zero for all values of h, and in such a case (which never actually occurs) the optical system would be entirely free from aberration for the axial object-point in question.

If we suppose that the co-efficients a, b have opposite signs, we shall find that the above equation represents a curve, of the general form shown in Figs. 134 and 135, which is symmetrical with respect to the x-axis, and which is tangent to the h-axis at the origin. This curve is called the *Aberration Curve*. For the value $\delta u = 0$, we obtain:

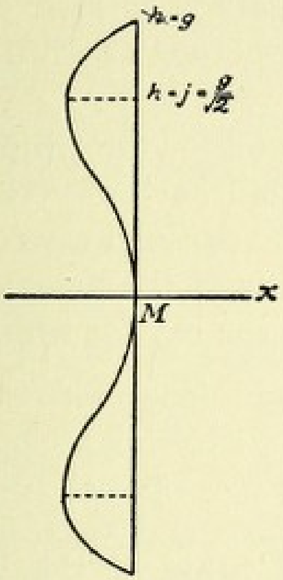

FIG. 134.
ABERRATION-CURVE: CASE OF UNDER-CORRECTION.

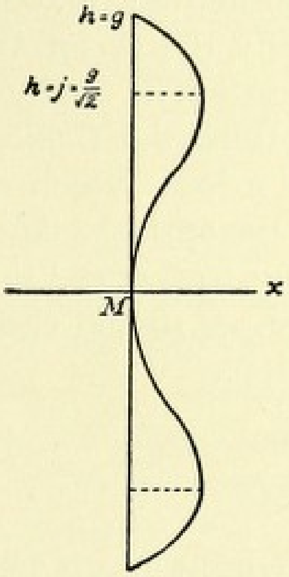

FIG. 135.
ABERRATION-CURVE: CASE OF OVER-CORRECTION.

$$h = g = \pm \sqrt{-a/b};$$

consequently, the ray whose incidence-height is equal to g will cross the optical axis at the point M where the paraxial rays converge, and the system is, therefore, said to be spherically corrected for this ray. For all values of h comprised between $h = 0$ and $h = g$, the sign of δu remains unchanged, so that all the intermediate rays will be either spherically under-corrected ($\delta u < 0$), as in Fig. 134, or spherically over-corrected ($\delta u > 0$), as in Fig. 135.

Moreover, we have a maximum (or minimum) value of the Longitudinal Aberration δu at the origin and also at the points whose ordinates are:

$$h = j = \pm \sqrt{-a/2b}.$$

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[1] These co-efficients a and b are, of course, not the same as the co-efficients denoted by these same letters in the development of u in a series of ascending powers of θ.

The absolute value of the Longitudinal Aberration will be greatest, therefore, for the ray whose incidence-height is $j = g/\sqrt{2}$, and this value is nearly equal to $-a^2/4b$. The smaller this greatest value is, the more nearly will the system be spherically corrected.

Without knowing the values of the co-efficients a and b, the Aberration Curve can be plotted by calculating by the trigonometrical formulæ the values of u corresponding to given values of the incidence-height h, and by practical opticians this method is used to exhibit graphically the performance in respect to spherical aberration of the optical system as finally completed.

Concerning the *Choice of a Suitable Aperture* for the Objective, the question arises, Which ray of the bundle shall be "corrected" so as to cross the optical axis at the point where the paraxial rays converge? According to GAUSS,[1] if H denotes the radius of the aperture of the objective, we should choose for this purpose the ray for which

$$h = g = H\sqrt{6/5}.$$

The value

$$h = g = H\sqrt{1/2}$$

has also been recommended as a suitable value of the incidence-height of the corrected ray; in this case the working part of the spherical refracting surface will be divided by the circle of radius g into two equal zones, so that half of the refracted rays will be under-corrected and half will be over-corrected.

III. THE SINE-CONDITION. (OPTICAL SYSTEMS OF WIDE APERTURE AND SMALL FIELD OF VISION.)

ART. 86. DERIVATION AND MEANING OF THE SINE-CONDITION.

276. We have seen that it is possible to design an optical system of centered spherical surfaces which for a pair of conjugate axial points is free, or practically free, from spherical aberration; so that to a homocentric bundle of object-rays proceeding from a point M on the optical axis there will correspond a homocentric bundle of image-rays with its vertex at the GAUSSian image-point M'. If the optical system consists of a single spherical refracting surface, it will be recalled that it was the pair of so-called *Aplanatic Points* Z, Z' that were thus characterized by the property that to an incident chief ray crossing the axis at Z at any angle θ corresponded a refracted ray crossing the axis at the conjugate point Z' (§ 207). But this was not the only characteristic of this remarkable pair of points, for we found, also (§ 211, Note 3), that the slope-angles θ, θ' of the incident and refracted rays were connected by the relation:

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[1] See GAUSS's Letter to BRANDES, given in GEHLERS *Physik. Woerterbuch* (Leipzig, 1831), Bd. vi., I. Abt., S. 437. This letter is quoted at length in CZAPSKI's *Theorie der optischen Instrumente* (Breslau, 1893), p. 96.

$$ny \sin \theta = n'y' \sin \theta',$$

where $y'/y = Y$ denoted the Lateral Magnification of the imagery by means of paraxial rays with respect to the pair of conjugate axial points Z, Z'. If the relation between the Object-Space and the Image-Space were a collinear relation (as it would be if all the rays concerned were paraxial rays), the slope-angles θ, θ' would be connected by the Law of ROBERT SMITH (§ 194), viz.:

$$ny \tan \theta = n'y' \tan \theta';$$

but, since for finite values of θ, θ' these two equations cannot both be true at the same time, it is manifest that the correspondence by means of wide-angle bundles of rays between the Aplanatic Points of a single spherical refracting surface is not the same kind of correspondence as we have in the ideal case of optical imagery. Here is a matter, therefore, that requires to be investigated.

The mere fact that an optical system has been so contrived that for a pair of conjugate axial points M, M' the Spherical Aberration is sensibly negligible, by no means implies also that the system will be free from aberration for any other object-point, for example, for a point Q very near to M. If the aperture of the system is so narrow that the rays which are concerned in producing the image may be regarded as altogether paraxial rays, we know that to an infinitely small object-line MQ perpendicular to the optical axis at M there will correspond, point by point, an infinitely small image-line $M'Q'$ perpendicular to the optical axis at M'; but, in general, if the incident rays which come from the object-point Q constitute a wide-angle bundle of rays, only those rays which proceed very close to the axis will emerge from the system so as to meet in the corresponding GAUSSian image-point Q'. Even in those cases where the Spherical Aberration with respect to the axial points M, M' has been most completely abolished, the points of the image which are not on the axis will appear so blurred and indistinct that the diameters of their aberration-circles are actually comparable in magnitude with their distances from the axis. According to ABBE,[1] the explanation of this indistinctness is to be found in the fact that the images of the object-line MQ (Fig. 136) produced by the different zones of the spherically corrected system have different magnifications; and, thus, although all these images will lie along the same line perpendicular to the optical axis at M', being of unequal lengths, they will overlap each other and produce therefore a confused image. If the angular aperture of the objective is pretty large, the differences in these magnification-ratios may amount to as much as 50 per cent. or more of the lateral magnification produced by the central or paraxial rays. Evidently, under such circumstances there will be no imagery at all in any practical sense. The problem consists, therefore, in finding the condition that the magnifications of all the different zones of the objective shall be equal to each other, that is, equal to the magnification

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[1] E. ABBE: Ueber die Bedingungen des Aplanatismus der Linsensysteme: *Sitzungsber. der Jenaischen Gesellschaft für Med. u. Naturw.*, 1879, 129–142; also, *Gesammelte Abhandlungen*, Bd. I, 213–226.

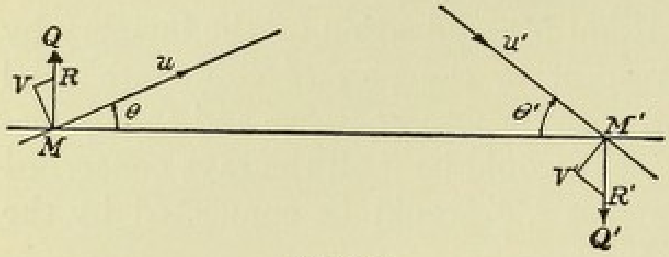

FIG. 136.

SINE-CONDITION.

$$Y = M'Q'/MQ$$

of the imagery by means of paraxial rays.

277. Consider an object-ray u proceeding from the axial object-point M to which corresponds an image-ray u' crossing the optical axis at the point M' conjugate to M; and let θ, θ' denote the slope-angles of this pair of corresponding rays. Since the optical system is supposed to be spherically corrected with respect to the points M, M', to an infinitely narrow bundle of object-rays whose chief ray is u will correspond an infinitely narrow homocentric bundle of image-rays whose chief ray is u'; so that the I. and II. image-points coincide with each other at the axial point M'. We saw (§ 245) that within the infinitely narrow region of space surrounding the "mean" incident chief ray in the object-space and the corresponding emergent chief ray in the image-space, there was a collinear correspondence between the plane-fields π, π' of the Meridian Rays and also between the plane-fields $\bar{\pi}$, $\bar{\pi}'$ of the Sagittal Rays; of such a character that to an infinitely small object-line MV lying in the plane of the meridian section and perpendicular at M to the "mean" incident chief ray u there corresponds an infinitely small image-line $M'V'$ in the same plane and perpendicular at M' to the emergent chief ray u'; and, similarly, to an infinitely small object-line $M\overline{W}$ lying in the plane $\overline{\pi}$ of the pencil of Sagittal object-rays and perpendicular at M to the "mean" incident chief ray u there corresponds an infinitely small image-line $M'\overline{W}'$ in the plane $\overline{\pi}'$ of the pencil of Sagittal image-rays and perpendicular at M' to the chief image-ray u'.

We shall use the symbols

$$Y_u = M'V'/MV, \quad \overline{Y}_u = M'\overline{W}'/M\overline{W},$$

to denote the lateral magnifications of the Meridian and Sagittal Rays, respectively. The line-elements $M\overline{W}$ and $M'\overline{W}'$ are perpendicular to the optical axis at M and M', respectively; but the same thing is not true with respect to the line-elements MV and MV'. If in the meridian plane we draw VR, $V'R'$ perpendicular at V, V' to MV, $M'V'$ and meeting in R, R' the axis-ordinates erected at M, M', respectively, so that

$$MV = MR \cdot \cos\theta, \quad M'V' = M'R' \cdot \cos\theta';$$

then

$$\frac{M'R'}{MR} = Y_u \frac{\cos\theta}{\cos\theta'};$$

and in order that the image at M' of a plane element perpendicular to the optical axis at M shall be identical with the GAUSSian image, or the image produced by means of the central (paraxial) rays, we must have:

$$\frac{M'R'}{MR} = \frac{M'\overline{W}'}{M\overline{W}} = \frac{M'Q'}{MQ},$$

that is,

$$Y_u \frac{\cos\theta}{\cos\theta'} = \overline{Y}_u = Y$$

for all values of the slope-angle θ.

If

$$Z_u = \frac{d\lambda'}{d\lambda}, \quad \overline{Z}_u = \frac{d\overline{\lambda}'}{d\overline{\lambda}}$$

denote the angular magnifications, or "convergence-ratios", of the incident and emergent pencils of Meridian and Sagittal Rays, respectively, then, since the formulæ which were deduced in the case of Collinear Imagery are applicable here, we have (see Chap. VII, § 179, and Chap XI, § 246) the following relations:

$$Y_u \cdot Z_u = \overline{Y}_u \cdot \overline{Z}_u = n/n',$$

where n and n' denote the refractive indices of the media of the incident and emergent rays, respectively.

Let us consider, first, *the Imagery in the Plane of the Meridian Section* of the infinitely narrow bundle of incident rays whose chief ray is u. Obviously,

$$Z_u = \frac{d\theta'}{d\theta},$$

and from the above relations we obtain:

$$\frac{M'R'}{MR} = \frac{n \cos\theta \, d\theta}{n' \cos\theta' \, d\theta'} = \frac{n \cdot d(\sin\theta)}{n' \cdot d(\sin\theta')}.$$

This equation shows that the lateral magnification perpendicular to the optical axis at the points M, M' produced by the Meridian Rays depends on the slope-angle θ of the chief incident ray u; and, hence, the condition that this magnification shall have the same value for all values of the slope-angle θ, between the value $\theta = 0$ and the value of θ for the edge-ray is:

$$\frac{n \cdot d(\sin\theta)}{n' \cdot d(\sin\theta')} = Y;$$

and, since this equation must be satisfied by all values of θ, θ', including very small values, it may be written:

$$\frac{\sin\theta}{\sin\theta'} = \frac{n'}{n} Y. \tag{300}$$

In the next place, we proceed to consider the *Imagery of the Sagittal Rays* of the same infinitely narrow bundle of rays. The value of the angular magnification in the Sagittal Section may easily be found by imagining the figure to be rotated about the optical axis through a very small angle, in which case the angles between the initial and final positions of the chief incident and emergent rays u, u' will be the angles $d\bar{\lambda}$, $d\bar{\lambda}'$ whose ratio $d\bar{\lambda}'/d\bar{\lambda}$ is equal to $\bar{Z}_u$. According to formula (251) of Chap. XI and formula (185) of Chap. IX, we have for the kth spherical surface:

$$\bar{Z}_{u,k} = \frac{\bar{s}_k}{\bar{s}'_k} = \frac{l_k}{l'_k} = \frac{\sin\theta'_k}{\sin\theta'_{k-1}},$$

and, since

$$\bar{Z}_u = \prod_{k=1}^{k=m} \bar{Z}_{u,k},$$

we shall find:

$$\overline{Z}_u = \frac{\sin\theta'_m}{\sin\theta_1} = \frac{\sin\theta'}{\sin\theta},$$

since here we write θ and θ' in place of θ_1 and θ'_m, respectively.

The lateral magnification $\overline{Y}_u$ of the the Imagery by means of the Sagittal Rays must be equal to the lateral magnification Y of the imagery by means of the Paraxial Rays; and, hence, since

$$\overline{Y}_u \cdot \overline{Z}_u = Y \cdot \overline{Z}_u = n/n',$$

we obtain here also:

$$\frac{\sin\theta}{\sin\theta'} = \frac{n'}{n} Y$$

as the condition that the magnification of the Imagery by means of the Sagittal Rays shall be constant and equal to that by means of the Paraxial Rays; and this condition is seen to be precisely the same as was found above for the Imagery by means of the Meridian Rays. It will be observed also that it is likewise identical with the characteristic relation which we found to be true always in regard to the pair of aplanatic points of a single spherical refracting surface (§ 211, Note 3).

The law here derived, known as the *Sine-Condition*, is one of the most important of the valuable contributions of ABBE[1] to the theory of Optical Instruments. It may be stated as follows:

The necessary and sufficient condition that all the zones of the spherically corrected optical system shall produce equal-sized images at the axis-point M', conjugate to the axial object-point M, is that, for all rays traversing the system, the ratio of the sines of the slope-angles of each pair of corresponding incident and emergent rays shall be constant; that is,

$$\sin\theta / \sin\theta' = \text{constant}.$$

The value of this constant, as we see from formula (300), is $n'Y/n$.

278. Other Proofs of the Sine-Law. The so-called Sine-Condition as enunciated by ABBE, in 1873, for the special case of a centered system of spherical refracting surfaces might have been seen to be contained in a far more general law of CLAUSIUS's[1] based on the Second Fundamental Principle of Thermodynamics; which may be stated thus:

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[1] E. ABBE: Beitraege zur Theorie des Mikroskops und der mikroskopischen Wahrnehmung: M. SCHULTZES *Archiv für mikroskopische Anatomie*, IX (1873), 413–468. Also, *Gesammelte Abhandlungen*, Bd. I, 45–100. See also paper entitled: Ueber die Bedingungen des Aplanatismus der Linsensysteme: *Sitzungsber. der Jenaischen Gesellschaft für Med. u. Naturw.*, 1879, 129–142; reprinted in CARLS *Repertorium der Exper.-Phys.*, XVI (1880), 303–316, and in *Gesammelte Abhandlungen*, Bd. I, 213–226.

If the energy radiated by an element of surface $d\sigma$, in a medium of refractive index n, by a bundle of rays of solid angle $d\omega$, is transmitted entirely to an element of surface $d\sigma'$, in a medium of refractive index n', by a bundle of rays of solid angle $d\omega'$, then we must have the following equation:

$$\frac{n^2 \cdot \cos\theta \cdot d\omega}{n'^2 \cdot \cos\theta' \cdot d\omega'} = \frac{d\sigma'}{d\sigma},$$

where θ, θ' denote the angles between the chief rays and the corresponding surface-normals.

Applied to the case of an optical system of centered spherical surfaces, to the axis of which the surface-elements $d\sigma$, $d\sigma'$ are supposed to be perpendicular, this equation is easily reducible to the form given by formula (300). For in this special case the magnitudes θ, θ' evidently denote the slope-angles of the incident and refracted rays, and

$$\frac{d\omega}{d\omega'} = \frac{\sin\theta \cdot d\theta}{\sin\theta' \cdot d\theta'},$$

so that CLAUSIUS's equation becomes:

$$\frac{n^2 \cdot d(\sin^2\theta)}{n'^2 \cdot d(\sin^2\theta')} = \frac{d\sigma'}{d\sigma};$$

and, since $d\sigma'/d\sigma = Y^2$, we obtain by integration:

$$\frac{\sin\theta}{\sin\theta'} = \frac{n'}{n} Y.$$

Applying the Law of the Conservation of Energy to the Radiation of Light, HELMHOLTZ[2] has given also another mode of deducing ABBE's Sine-Condition, which is interesting, inasmuch as this important result is thus obtained from still another point of view.

Finally, let us mention here the extremely simple and elegant proof of the Sine-Law published by Mr. Hockin,[1] which is based on the general law of the equality of the optical lengths (§ 38) of all the ray-paths between the pair of conjugate axial points M, M' for which the system is assumed to be spherically corrected.

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[1] See BROWNE's English Translation of CLAUSIUS's *Mechanical Theory of Heat* (London, 1879), p. 321. The law of CLAUSIUS's here referred to was first published in the celebrated paper, Die Concentration von Waerme und Lichtstrahlen und die Grenzen ihrer Wirkung: POGG. *Ann.*, cxxi. (1864), S. 1.

[2] H. HELMHOLTZ: Die theoretische Grenze für die Leistungsfaehigkeit der Mikroskope: POGG. *Ann.*, Jubelband, 1874, 557–584. See also *Wissenschaftliche Abhandlungen*, II, p. 185.

ART. 87. APLANATISM.

279. We must explain here the meaning that is to be attached to the term "*aplanatic*", as it is employed by Abbe and modern writers on Optics. Formerly, this word was applied to an optical system merely to mean that it was free from spherical aberration, and this is the sense in which the term is used by Coddington, Herschel, etc. But, according to Abbe, in order for an optical system to be aplanatic, it must fulfil each of two requirements, viz.: (1) It must be free from spherical aberration for a pair of conjugate axis-points M, M'; and (2) The sine-condition must also be satisfied for this pair of points M, M'. Thus, the aplanatic pair of points Z, Z' of a spherical refracting surface are rightly so-called, because not merely are these points free from aberration, but, as we have seen, they fulfil the Sine-Condition also. On the other hand, the focal points of a reflecting ellipsoidal surface are not aplanatic, because they do not satisfy the Sine-Condition, and the same observation applies also with respect to the infinitely distant axial point and the focal point of a parabolic reflector.

Accordingly, *the* **Aplanatic Points** *of an optical system are the points on the axis for which the spherical aberration is abolished, and which at the same time satisfy the Sine-Condition.*

Abbe[2] has described a very ingenious and simple mode of testing the aplanatism of a lens-system; consisting in viewing through the system a certain sheaf of concentric hyperbolæ, the plane of the object-figure being placed perpendicularly to the axis with the common centre at the proper distance from the aplanatic point; which should yield as image two sheaves of mutually perpendicular, equidistant parallel lines (see § 291). By means of this device, Abbe has investigated the older types of microscopes, and he has shown that, long before the publication, in 1873, of the Sine-Condition, microscope-designers, without knowing it, had all more or less perfectly fulfilled this essential requirement along with the abolition of the spherical aberration. As LUMMER[1] observes, this is only another of the many instances in which correct practice has preceded theory.

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[1] Charles Hockin: On the estimation of aperture in the microscope: *Journ. Royal Mic. Soc.*, (2), IV (1884), 337–346. See also J. D. Everett's note on Hockin's proof of the Sine Condition, *Phil. Mag.*, (6), IV (1902), p. 170. Hockin's Proof of the Sine-Condition will be found given also in the 9th edition of Mueller-Pouillet's *Lehrbuch der Physik*, Bd. II, *Optik*, and in Drude's *Lehrbuch der Optik*.

[2] E. Abbe: Ueber die Bedingungen des Aplanatismus der Linsensysteme: *Sitzungsber. der Jenaischen Gesellschaft für Med. u. Naturw.*, 1879, 129–142; also, *Gesammelte Abhandlungen*, I, 213–226; also, reprinted in Carls *Rep. der Exper.-Phys.*, XVI (1880), 303–316.

ART. 88. THE SINE-CONDITION IN THE FOCAL PLANES.

280. It has been pointed out (§ 276) that the imagery which we obtain when the Sine-Condition is fulfilled is not governed by the same laws as we have in the case of Collinear Imagery. This difference is made strikingly manifest, for example, if the aplanatic pair of points are the infinitely distant point of the optical axis and one of the Focal Points of the optical system; as we shall proceed to show. Let MB be an incident ray proceeding from the axial point M and meeting the first surface of the system at B, and let us put:

$$BM = l, \quad \angle AMB = \theta,$$

where A designates the vertex of the first spherical surface. If h denotes the incidence-height of this ray at this surface, then

$$\sin\theta = -\frac{h}{l}.$$

Moreover, if $x = FM$ denotes the abscissa of the object-point M with respect to the Primary Focal Point F, then (see Chap. VII, § 179) the lateral magnification of the imagery by means of paraxial rays is:

$$Y = \frac{f}{x},$$

where f denotes the Primary Focal Length of the optical system. And since (§ 193)

$$f = -\frac{n}{n'}e',$$

where e' denotes the Secondary Focal Length of the system, evidently, we may write:

$$Y = -\frac{n}{n'}\frac{e'}{x}.$$

If M, M' are the pair of aplanatic points of the system, the Sine-Condition expressed by formula (300) may be put in the following form:

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[1] See MUELLER-POUILLET'S *Lehrbuch der Physik*, Bd. II, *Optik*, neunte Auflage, Art. 191.

$$\frac{h}{\sin \theta'} = \frac{le'}{x}.$$

And if we suppose now that the object-point M is the infinitely distant point E of the optical axis, and, consequently, the image-point M' coincides with the secondary focal point E', then $l = x = \infty$, in which case we find:

$$\frac{h}{\sin \theta'} = e'.$$

Similarly, for the case of an infinitely distant image-point F' corresponding to an object-point at the Primary Focal Point F, we should obtain:

$$\frac{h}{\sin \theta} = f.$$

If, therefore, supposing that the aplanatic pair of points is the pair E, E', to which the first of these two equations applies, we describe around the Secondary Focal Point E' as centre a sphere of radius e', all the points of intersection of the parallel object-rays with their corresponding image-rays will lie on the surface of this sphere, whereas in the case of Collinear Imagery, these points of intersection of the incident and emergent rays all lie in the Secondary Principal Plane which touches the above-mentioned sphere at its vertex.

ART. 89. ONLY ONE PAIR OF APLANATIC POINTS POSSIBLE.

281. When an optical system is so contrived that for a certain pair of points M, M' on the optical axis not only is the spherical aberration abolished but at the same time the Sine-Condition is fulfilled, a flat element of luminous surface placed normally to the axis at M will be distinctly delineated as a flat surface-element at M' by bundles of rays of any angular width (not exceeding the angular aperture of the system): but it by no means follows that the system will give at M' a distinct image of a plane area at M of finite dimensions; nor, indeed, that it will produce such an image even of an element of surface if it is situated at any other place on the axis. In fact, an optical system cannot have even two pairs of adjacent aplanatic points; for if this were possible, the system would have to be spherically corrected for both pairs of points, and this requirement, as we shall show, is incompatible with the condition that either of the two pairs of points is aplanatic.

In the diagram (Fig. 137) M, M' are supposed to be a pair of aplanatic points of the optical system. A ray MB emanating from the object-point M and inclined to the axis at an angle θ will, after traversing the system, emerge so as to cross the axis at the image-point M', the slope of the image-ray being denoted by θ'. This pair of corresponding rays may be regarded as the chief rays of two infinitely narrow pencils of corresponding Meridian Rays; let I' designate the position on the emergent chief ray of the Secondary Focal Point of this pencil of Meridian Rays (see §§ 235, 246). The image $M'R'$ of an infinitely small object-line MR perpendicular to the optical axis at M will be determined by constructing the path through the system of a ray proceeding from R parallel to MB which will emerge in a direction very nearly the same as that of the emergent chief ray, and which will intersect this ray at I', and which, by its intersection with the normal to the optical axis at M' will determine the image-point R' corresponding to R. Let P, P' designate the points where this ray crosses the axis before and after refraction through the optical system. The pair of axial points P, P' are adjacent to the aplanatic pair of points M, M'; and, therefore, let us write:

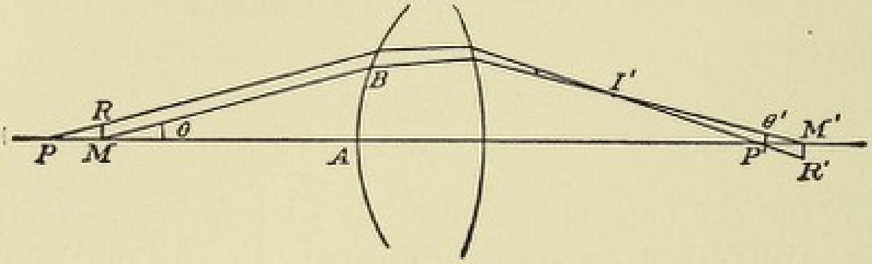

FIG. 137.

AN OPTICAL SYSTEM CAN HAVE ONLY ONE PAIR OF APLANATIC POINTS M, M'.

$$MP = dx, \quad M'P' = dx'.$$

Let us now assume also that the optical system is spherically corrected for the points P, P', so that they also are a pair of conjugate points; in which case the ratio dx'/dx will be the value of the axial magnification, at the points M, M', of the imagery by means of paraxial rays. Hence (see Chap. VII, § 179), we find:

$$\frac{dx'}{dx} = \frac{n'}{n} Y^2,$$

where Y denotes the lateral magnification, at the conjugate points M, M', of the imagery by means of paraxial rays.

Now from the figure we obtain:

$$MR = -dx \cdot \tan\theta, \quad M'R' = -dx' \cdot \tan\theta',$$

since $\angle M'P'R'$ differs from the angle θ' by only an infinitesimal magnitude; and, hence,

$$\frac{dx'}{dx} = \frac{M'R'}{MR} \frac{\tan\theta}{\tan\theta'}.$$

Here, we may recall that in § 277 we found:

$$\frac{M'R'}{MR} = \frac{n \cdot \cos\theta \cdot d\theta}{n' \cdot \cos\theta' \cdot d\theta'};$$

and therefore equating the two expressions above for dx'/dx, and at the same time introducing this last relation, we obtain the following equation:

$$\sin\theta \cdot d\theta = \frac{n'^2}{n^2} Y^2 \cdot \sin\theta' \cdot d\theta';$$

which, being integrated, gives:

$$\cos\theta = \frac{n'^2}{n^2} Y^2 \cdot \cos\theta' + C,$$

where C denotes the integration-constant. The value of C can be found by putting $\theta = \theta' = 0$; thus, we obtain:

$$C = 1 - \frac{n'^2}{n^2} Y^2.$$

Substituting this value of C in the above result, we find:

$$1 - \cos\theta = \frac{n'^2}{n^2} Y^2 (1 - \cos\theta'),$$

which can be written finally as follows:

$$\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta'}{2}} = \frac{n'}{n} Y.$$

Evidently, this equation cannot be satisfied at the same time with the Sine-Condition expressed by equation (300). Consequently, an optical system can have only one pair of aplanatic points.

This result might have been established immediately by merely remarking again (§ 276) that the Sine-Condition Imagery is essentially different from Collinear Imagery. Now we know that if as many as two elements of surface perpendicular to the axis are portrayed by similar surface-elements also perpendicular to the optical axis, the Imagery must be Collinear, and hence it follows that the Sine-Condition cannot be satisfied for two pairs of axial points. Thus, for example, the objective of a microscope must always be computed for that pair of aplanatic points for which it is to be used; and in order to obtain a distinct image of the object, the latter must be placed at the aplanatic point of the Object-Space.

It appears, therefore, that with all the means at his disposal, the utmost that the practical optician, employing *wide-angle bundles of rays*, can hope to achieve is the approximate realization of one or other of two theoretical possibilities: *To produce a perfectly sharp image* (1) *Either of an indefinitely small element of surface perpendicular to the axis*, (2) *Or, else, of an indefinitely small element of the axis itself.* It is practically impossible to obtain a sharp image of even an indefinitely small axial element of volume; for the conditions which are required to be fulfilled in order to portray distinctly its dimension parallel to the optical axis are at variance with the conditions that must be satisfied in order to produce a distinct image of its lateral dimensions.

ART. 90. DEVELOPMENT OF THE FORMULA FOR THE SINE-CONDITION ON THE ASSUMPTION THAT THE SLOPE-ANGLES ARE COMPARATIVELY SMALL.

282. Let us assume now that the effective bundles of rays are limited by a suitable stop so that the slope-angles θ are all comparatively small—so small that we may neglect powers of θ above the third. The following method of development is practically the same as that given by KOENIG and VON ROHR.[1]

Since, according to formula (185),

$$\frac{\sin\theta'_k}{\sin\theta'_{k-1}} = \frac{l_k}{l'_k},$$

we have, evidently:

$$\frac{n'_m}{n_1}\,\frac{\sin\theta'_m}{\sin\theta_1} = \frac{n'_m}{n_1}\prod_{k=1}^{k=m}\frac{\sin\theta'_k}{\sin\theta'_{k-1}} = \frac{n'_m}{n_1}\prod_{k=1}^{k=m}\frac{l_k}{l'_k}.$$

283. Let us first obtain the development of the ray-length in a series of ascending powers of the central angle φ. Since, by the second of formulæ (180),

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[1] A. KOENIG und M. VON ROHR: Die Theorie der sphaerischen Aberrationen: Chapter V of VON ROHR's *Die Theorie der optischen Instrumente* (Berlin, 1904), Bd. I, 302–304.

$$v - r = l \cdot \cos\theta - r \cdot \cos\varphi,$$

we obtain, neglecting powers of θ and φ above the third,

$$l\left(1 - \frac{\theta^2}{2}\right) = v - r\frac{\varphi^2}{2},$$

or

$$l = \left(v - r\frac{\varphi^2}{2}\right)\left(1 + \frac{\theta^2}{2}\right) = v\left(1 + \frac{\theta^2}{2} - \frac{r}{u}\cdot\frac{\varphi^2}{2}\right).$$

Now

$$\theta^2 = \frac{r^2\varphi^2}{u^2};$$

and, hence, finally, we obtain:

$$l = v\left(1 - \frac{r^2\varphi^2}{2}\cdot\frac{J}{nu}\right), \tag{301}$$

where J denotes the so-called zero-invariant (§ 126).

Since

$$v = u + \delta u, \quad r^2\varphi^2 = h^2,$$

we may also write this formula as follows:

$$l = u\left(1 + \frac{\delta u}{u} - h^2\frac{J}{2nu}\right). \tag{302}$$

And for the ray-length l' of the refracted ray we have merely to prime the letters n and u in this formula.

To the same degree of approximation, we obtain, therefore, for the ratio l/l' the following formula:

$$\frac{l}{l'} = \frac{u}{u'}\left(1 + h^2\frac{J}{2}\Delta\frac{1}{nu} - \Delta\frac{\delta u}{u}\right). \tag{303}$$

284. Thus, re-introducing the subscripts, we obtain:

$$\frac{n'_m}{n_1}\cdot\frac{\sin\theta'_m}{\sin\theta_1} = \frac{n'_m}{n_1}\left[1 + \sum_{k=1}^{k=m}\tfrac{1}{2}h_k^2 J_k\cdot\Delta\left(\frac{1}{nu}\right)_k - \sum_{k=1}^{k=m}\Delta\left(\frac{\delta u}{u}\right)_k\right]\prod_{k=1}^{k=m}\frac{u_k}{u'_k}.$$

If this expression is to be constant for all values of θ_1 (or h_1), then we must have:

$$\frac{1}{2}\sum_{k=1}^{k=m} h_k^2 J_k\cdot\Delta\left(\frac{1}{nu}\right)_k - \sum_{k=1}^{k=m}\Delta\left(\frac{\delta u}{u}\right)_k = 0.$$

Now, since (see formulæ (270))

$$J - \boldsymbol{J} = n\left(\frac{1}{\boldsymbol{u}} - \frac{1}{u}\right) = n'\left(\frac{1}{\boldsymbol{u}'} - \frac{1}{u'}\right),$$

we have:

$$\Delta\frac{\delta u}{u} = \frac{1}{J - \boldsymbol{J}}\left(\Delta\frac{n\cdot\delta u}{u\boldsymbol{u}} - \Delta\frac{n\cdot\delta u}{u^2}\right),$$

and

$$\Delta\frac{n\cdot\delta u}{u\boldsymbol{u}} = (J - \boldsymbol{J})\Delta\frac{\delta u}{u - \boldsymbol{u}};$$

and, hence,

$$\sum_{k=1}^{k=m}\Delta\left(\frac{\delta u}{u}\right)_k = \sum_{k=1}^{k=m}\Delta\left(\frac{\delta u}{u - \boldsymbol{u}}\right)_k - \sum_{k=1}^{k=m}\frac{1}{J_k - \boldsymbol{J}_k}\Delta\left(\frac{n\cdot\delta u}{u^2}\right)_k.$$

But, since

$$u_k - u'_{k-1} = -d_{k-1} = \boldsymbol{u}_k - \boldsymbol{u}'_{k-1},$$

we have evidently:

$$\sum_{k=1}^{k=m}\Delta\left(\frac{\delta u}{u - \boldsymbol{u}}\right) = \frac{\delta u_1}{u_1 - \boldsymbol{u}_1} + \frac{\delta u'_m}{u'_m - \boldsymbol{u}'_m} = 0,$$

since in the present case, in which the system is supposed to be spherically corrected for the two axial points M_1, M'_m, we must have:

$$\delta u_1 = 0 = \delta u'_m.$$

According to formula (281*a*), we have:

$$\Delta\frac{n\cdot\delta u}{u^2} = -\tfrac{1}{2}h^2J^2\cdot\Delta\frac{1}{nu};$$

and, hence, we find:

$$\sum_{k=1}^{k=m}\Delta\left(\frac{\delta u}{u}\right)_k = \frac{1}{2}\sum_{k=1}^{k=m}h_k^2\frac{J_k^2}{J_k - \boldsymbol{J}_k}\Delta\left(\frac{1}{nu}\right)_k.$$

Accordingly, the Sine-Condition may be expressed as follows:

$$\sum_{k=1}^{k=m}h_k^2J_k\left(1 - \frac{J_k}{J_k - \boldsymbol{J}_k}\right)\cdot\Delta\left(\frac{1}{nu}\right)_k = 0;$$

or

$$\sum_{k=1}^{k=m}h_k^2\frac{J_k\boldsymbol{J}_k}{J_k - \boldsymbol{J}_k}\Delta\left(\frac{1}{nu}\right)_k = 0.$$

Let Q_1 designate the end-point of the infinitely short object-line M_1Q_1 perpendicular to the optical axis at M_1, and let $\boldsymbol{h}_1$ denote the incidence-height of the paraxial object-ray which, proceeding from Q_1

is directed towards the centre $\boldsymbol{M}_1$ of the Entrance-Pupil (§ 257), and let $\boldsymbol{h}_k$ denote the incidence-height of this ray at the kth spherical surface. Introducing the relation given by formula (155) of Chapter VIII, viz.:

$$h_k \boldsymbol{h}_k (J_k - \boldsymbol{J}_k) = h_1 \boldsymbol{h}_1 (J_1 - \boldsymbol{J}_1),$$

which, in the way it is employed here, is admissible, since we neglect magnitudes above the third order, we obtain finally the formula for the Sine-Condition in the following form:

$$\sum_{k=1}^{k=m} h_k^3 \boldsymbol{h}_k J_k \boldsymbol{J}_k \Delta \left(\frac{1}{nu} \right)_k = 0. \qquad (304)$$

SEIDEL[1] notes the fact that FRAUNHOFER in his characteristic construction of the telescope-objective, appears to have satisfied this condition, and he, therefore, calls formula (304) the FRAUNHOFER *Condition*.

If this condition is fulfilled, along with the condition of the abolition of the spherical aberration for the conjugate axial points M_1, M'_m, we shall have (*cf.* Chapter VI, § 138):

$$\frac{n'_m}{n_1} \frac{\sin \theta'_m}{\sin \theta_1} = \frac{n'_m}{n_1} \prod_{k=1}^{k=m} \frac{u_k}{u'_k} = \frac{1}{Y}.$$

IV. ORTHOSCOPY. CONDITION THAT THE IMAGE SHALL BE FREE FROM DISTORTION.

ART. 91. DISTORTION OF THE IMAGE OF AN EXTENSIVE OBJECT FORMED BY NARROW BUNDLES OF RAYS.

285. In case the object to be depicted is, say, a plane surface of finite dimensions placed perpendicular to the optical axis of the Lens-System, our only chance of obtaining an approximately correct image will be by introducing a small circular stop, or diaphragm, whose duty will be to limit the angular widths of the operative bundles of rays emanating from the various points of the object. It is obvious that this mode of producing an image will be attended also by a number of difficulties of one kind and another, which may be described in a general way as aberrations due to the obliquity of the rays proceeding from the lateral parts of the object. In general, a plane object will not be reproduced by a plane image, but on account of the astigmatism of the narrow bundles of rays, the image will be resolved into a double image, symmetrically situated with respect to the optical axis on two curved surfaces, called the "astigmatic image-surfaces" (§ 295). However, passing over for the present both of these difficulties, and assuming that the aberrations which produce astigmatism and curvature of the image have been eliminated to some extent at least, so that we have a fairly sharp, flat image, even under these conditions we may still find that the image does not reproduce the object faithfully, but is *distorted*. This latter defect, which is quite distinct from the other aberrations that have to do more with the sharpness of the image, will be explained more fully in the following investigation.

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[1] L. SEIDEL: Zur Dioptrik. Ueber die Entwicklung der Glieder 3ter. Ordnung, welche den Weg eines ausserhalb der Ebene der Axe gelegenen Lichtstrahles durch ein System brechenden Medien, bestimmen: *Astr. Nach.*, No. 1029, xliii. (1856). See Section 9 of SEIDEL's paper.

286. Image-Points regarded as lying on the Chief Rays. If there were collinear correspondence between Object-Space and Image-Space, the image of an object-plane σ perpendicular at M to the optical axis of the centered system of spherical surfaces would not only be reproduced, point by point, in the image-plane σ' perpendicular to the optical axis at the point M', which, by GAUSS's Theory, is conjugate to the axial object-point M, but the image would be in every respect precisely similar to the object. The actual rays, however, being subject, as we say, to aberrations, pursue routes which, in general, are quite different from the paths that they would take if the image were ideal; so that, for example, an outgoing ray, proceeding from an object-point P in the plane σ, and traversing the optical system, will emerge finally and cross the image-plane σ' at a point P', which will be identical with the GAUSSian image-point only under exceptional circumstances. Moreover, another ray proceeding from the same object-point P will generally determine a different point P' in the image-plane σ'. This latter difficulty may be partially overcome by the use of a very small stop, whereby the effective rays emanating from the object-point P are all comprised within the limits of a very narrow bundle, all the rays of which have nearly the same inclinations and, consequently, cross the image-plane at approximately the same point, so that we do have there in a certain sense a more or less indistinct image of P.

Taking the more general case, and one that is, in fact, very common in actual optical instruments, let us suppose that this stop is interposed somewhere in the interior of the optical system, with its centre on the optical axis at a point which we shall designate by the letter O, and which coincides with the point where paraxial rays, which in the Object-Space go through the centre $\boldsymbol{M}$ of the Entrance-Pupil (see § 257), cross the optical axis in their progress through the medium in which the stop is situated. As has been already explained (§ 258), the *chief ray* emanating from the object-point P is that one of the bundle which leaves P in such a direction that it will, in traversing the medium where the stop is, go through the centre O of the stop. The path of this ray will lie in the meridian plane containing the object-point P, which is here the plane of the diagram (Fig. 138). If the image-plane σ' is supposed to be occupied by a screen, what actually appears on this screen may now be called the practical image of the plane object perpendicular to the optical axis at M. Immediately around the axial image-point M' there will be other sharp image-points, but at a little distance from the axis we shall have image-spots instead of image-points, and these images will be more and more indistinct, the farther they are from the axis. If the diameter of the stop is reduced, the effect will be to diminish the dimensions of the image-spots, and as a limiting case we may even suppose that the stop contracts into a mere point or pinhole-opening at O, so that *only the chief rays emanating from the points of the object-plane σ succeed in getting past the stop.* These chief rays, which constitute a sort of skeleton of the bundles of effective rays, will determine by their intersections with the image-plane σ' the positions in this plane of the image-points corresponding to the points of the object-plane σ. Thus, in this view of the matter, the point P', where the chief ray emanating from P finally crosses the image-plane σ', is to be considered as *the image of the object-point P.*

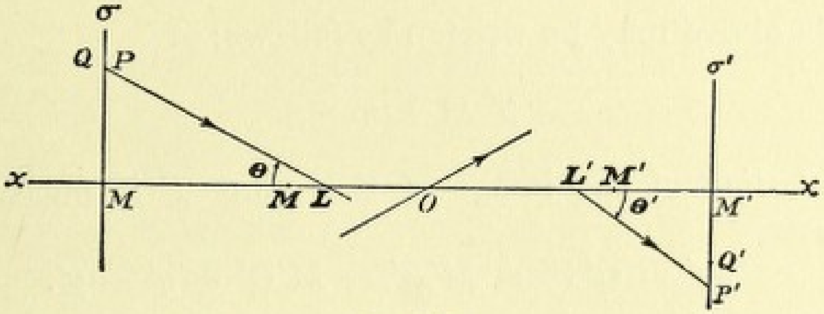

Fig. 138.

Distortion of the Image. xx represents the Optical Axis of a centered system of spherical surfaces. The position of the stop-centre is marked by O. The Object-Plane and the Image-Plane are designated by σ, σ'. The straight lines with arrow-heads show the directions of portions of the path of the chief ray which has its origin at the Object-Point P (or Q) in the Object-Plane σ.

$$MP = \eta = MQ = y, \quad M'Q' = y', \quad M'P' = \eta', \quad Q'P' = \delta y',$$
$$M\mathbf{M} = \mathbf{u} - u, \quad M'\mathbf{M}' = \mathbf{u}' - u', \quad \angle M\mathbf{L}P = \theta, \quad \angle M'\mathbf{L}'P' = \theta'.$$

287. Measure of the Distortion. The position in the image-plane σ' of the point Q', which, by Gauss's Theory, is conjugate to a point Q in the object-plane σ, is defined by the equation:

$$M'Q' = Y \cdot MQ,$$

where Y denotes the magnitude of the Lateral Magnification of the optical system for the pair of conjugate axial points M, M'. If we suppose that the point Q coincides with the object-point P, and if we put:

$$MQ = y = MP = \eta, \quad M'Q' = y',$$

the formula above may be written as follows:

$$y' = Y \cdot \eta.$$

The *Distortion* of the image of the point P is measured by the aberration

$$\delta y' = Q'P' = M'P' - M'Q' = \eta' - y'.$$

An image which is free from distortion, and which, therefore, is exactly *similar* to the object in its entire extent, is called *orthoscopic*, or "angle-true", because straight lines are reproduced as straight lines and homologous angles in the object and image are equal. A lens which casts an orthoscopic image is called a "*rectilinear*" *lens*.

ART. 92. CONDITIONS OF ORTHOSCOPY.

288. General Case: When the Centres of the Pupils are Affected with Aberrations. In the diagram (Fig. 138) the optical axis of the system is represented, but the actual refracting surfaces are not shown. The axial points $\boldsymbol{M}$, $\boldsymbol{M}'$ are the centres of the Entrance-Pupil and Exit-Pupil, respectively, and O designates the position of the stop-centre. The chief object-ray proceeding from the point P of the object-plane σ crosses the axis (really or virtually) at a point $\boldsymbol{L}$, which, in general, will not be very far from the point $\boldsymbol{M}$, since, during its progress through the system, this ray must pass (really) through the stop-centre O. Emerging after refraction at the last surface, this same ray will finally cross the axis (really or virtually) at a point $\boldsymbol{L}'$, and determine by its intersection with the image-plane σ' the point P', which we have agreed to consider as the image of the object-point P. If

$$\theta = \angle M\boldsymbol{L}P, \quad \theta' = \angle M'\boldsymbol{L}'P'$$

denote the slope-angles of the chief ray before and after passing through the optical system, we have from the diagram:

$$MP = \boldsymbol{L}M \cdot \tan\theta, \quad M'P' = \boldsymbol{L}'M' \cdot \tan\theta'.$$

If A, A' designate the vertices of the first and last spherical surfaces of the system, and if we put

$$AM = u, \quad A'M' = u', \quad A\boldsymbol{M} = \boldsymbol{u}, \quad A'\boldsymbol{M}' = \boldsymbol{u}', \quad \boldsymbol{ML} = \delta\boldsymbol{u}, \quad \boldsymbol{M}'\boldsymbol{L}' = \delta\boldsymbol{u}',$$

then, since,

$$LM = \boldsymbol{LM} + \boldsymbol{MA} + AM = u - \boldsymbol{u} - \delta\boldsymbol{u},$$
$$L'M' = \boldsymbol{L'M'} + \boldsymbol{M'A'} + A'M' = u' - \boldsymbol{u}' - \delta\boldsymbol{u}',$$

we obtain:

$$\frac{\eta'}{\eta} = \frac{u' - \boldsymbol{u}' - \delta\boldsymbol{u}'}{u - \boldsymbol{u} - \delta\boldsymbol{u}} \cdot \frac{\tan\boldsymbol{\theta}'}{\tan\boldsymbol{\theta}}.$$

Now if the image is to be free from distortion, the point P' must coincide with the ideal image-point Q'; that is, $\eta' = M'P'$ must be identical with $y' = M'Q'$, which means that we must have:

$$\frac{\eta'}{\eta} = \frac{y'}{y} = Y;$$

and, hence, the *Condition of Orthoscopy*, which requires that all pairs of conjugate chief rays shall trace similar figures on the object-plane and image-plane, may be expressed by the following formula:

$$\frac{\tan\boldsymbol{\theta}'}{\tan\boldsymbol{\theta}} = \frac{u - \boldsymbol{u} - \delta\boldsymbol{u}}{u' - \boldsymbol{u}' - \delta\boldsymbol{u}'} Y; \tag{305}$$

which involves, therefore, not merely the ratio of the tangents of the slope-angles $\boldsymbol{\theta}, \boldsymbol{\theta}'$ of the chief ray, but also the Longitudinal Aberrations $\delta\boldsymbol{u}$, $\delta\boldsymbol{u}'$ at the centres of the Pupils.

If the Lateral Magnification of the system with respect to the Pupil-Centres $\boldsymbol{M}$, $\boldsymbol{M}'$ is denoted by $\boldsymbol{Y}$, it may readily be shown, by the aid of formulæ (127) and (153), that we have always the following relation between Y and $\boldsymbol{Y}$:

$$Y = \frac{n}{n'} \cdot \frac{M'\boldsymbol{M}'}{M\boldsymbol{M}} \cdot \frac{1}{\boldsymbol{Y}}, \tag{306}$$

where n, n' denote the indices of refraction of the first and last media. Hence, we may also write formula (305) above in the following form:

$$\frac{\tan\boldsymbol{\theta}'}{\tan\boldsymbol{\theta}} = \frac{n}{n'} \cdot \frac{u' - \boldsymbol{u}'}{u - \boldsymbol{u}} \cdot \frac{u - \boldsymbol{u} - \delta\boldsymbol{u}}{u' - \boldsymbol{u}' - \delta\boldsymbol{u}'} \cdot \frac{1}{\boldsymbol{Y}}.$$

In case the object-point P is infinitely distant, the image-plane σ' will coincide with the secondary focal plane ϵ' and the point M' will coincide with the secondary focal point E'. Under these circumstances, we find:

$$\frac{\tan\boldsymbol{\theta}'}{\tan\boldsymbol{\theta}} = \frac{n}{n'} \cdot \frac{\boldsymbol{M}'E'}{\boldsymbol{M}'E' - \boldsymbol{M}'\boldsymbol{L}'} \cdot \frac{1}{\boldsymbol{Y}};$$

so that in this instance the Longitudinal Aberration of the ray at the

Entrance-Pupil does not matter. And if, as in the case of the Telescope, both object and image are infinitely distant, so that E' is also the infinitely distant point of the optical axis, then:

$$\frac{\tan\theta'}{\tan\theta}=\frac{n}{n'}\cdot\frac{1}{\boldsymbol{Y}}=\text{constant},$$

and the Condition of Orthoscopy in this special case is independent of the aberrations at both Pupil-Centres.

It is sometimes stated that the constancy of the tangent-ratio $\tan\theta'/\tan\theta$, known as AIRY's *Tangent-Condition*,[1] is the necessary and sufficient condition of freedom from distortion; but, as M. VON ROHR[2] has pointed out, this is evidently by no means the case except under special circumstances. For example, if, as is the case with a certain class of Photographic Objectives, the stop coincides with the Exit-Pupil, so that the three points designated by O, $\boldsymbol{M}'$, $\boldsymbol{L}'$ are all coincident, then for an infinitely distant object-point, just as also in the case of the Astronomical Telescope, the constancy of the Tangent-Ratio is the condition of orthoscopy.

289. Case when the Pupil-Centres are without Aberration. If the stop is placed, say, between the kth and the $(k+1)$th spherical surfaces, the optical system will be divided into two parts, an anterior part (I) composed of the first k spherical surfaces and a posterior part (II) composed of all the spherical surfaces after the kth. If the part (I) is spherically corrected for the centre of the Entrance-Pupil and the centre of the stop, and if, similarly, the posterior part is spherically corrected for the centre of the stop and the centre of the Exit-Pupil, the chief rays which are obliged to go through O will also go through the points $\boldsymbol{M}, \boldsymbol{M}'$. In this case the Longitudinal Aberrations at $\boldsymbol{M}, \boldsymbol{M}'$ will vanish, that is, $\delta\boldsymbol{u}=\delta\boldsymbol{u}'=0$; and now the condition of orthoscopy is:

$$\frac{\tan\theta'}{\tan\theta}=\frac{n}{n'}\cdot\frac{1}{\boldsymbol{Y}}=\text{constant}.$$

This is AIRY's Tangent-Condition above-mentioned, viz., that the ratio of the tangents of the slope-angles of every pair of conjugate chief rays must be the same. The requirement of spherical correction of the stop-centre for the two parts (I) and (II) of the optical system is called by von Rohr the Bow-Sutton *Condition*.[1] When both conditions are satisfied, the points $\boldsymbol{M}$, $\boldsymbol{M'}$ are called the *Orthoscopic Points* of the system.

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[1] G. B. AIRY: On the spherical aberration of the eye-pieces of telescopes: *Camb. Phil. Trans.*, III (1830), 1–64. This paper was published separately in Cambridge three years before it appeared in the *Phil. Trans.*

[2] M. VON ROHR: Beitrag zur Kenntniss der geschichtlichen Entwicklung, der Ansichten ueber die Verzeichnungsfreiheit photographischer Objektive: *Zft. f. Instr.*, xviii (1898), 4–12. See also A. KOENIG und M. VON ROHR: Die Theorie der sphaerischen Aberrationen: Chapter V of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR; see page 241.

290. The Two Typical Kinds of Distortion. If when the two partial systems are spherically corrected with respect to the stop-centre the ratio $\tan\theta' : \tan\theta$ is not constant, the magnification of the image close to the optical axis will be constant, but out towards the edges there will be distortion. For example, if with increasing values of θ, the ratio $\tan\theta' : \tan\theta$ also increases, the magnification η'/y will increase towards the margin of the field, so that spaces of equal area in the object-plane will appear distorted in the image into spaces of gradually increasing size as we go out from the axis. If the object consists of a network of two mutually perpendicular systems of equidistant parallel lines, as in Fig. 139 (*a*), the image will appear as in Fig. 139 (*b*). This case is known as "*Cushion-Shaped* Distortion", sometimes called also Positive Distortion. On the other hand, if $\tan\theta' : \tan\theta$ decreases as the slope-angle increases, the magnification η'/y will diminish out from the centre of the image; and then we have the case known as "*Barrel-Shaped* Distortion", or Negative Distortion (Fig. 139 (*c*)).

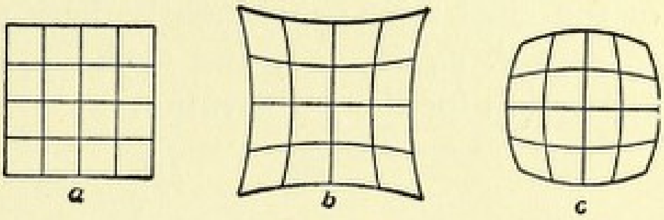

Fig. 139.
Showing the Typical Kinds of Distortion.

291. Distortion when the Pupil-Centres are the Pair of Aplanatic Points of the System. If the points $\boldsymbol{M}$, $\boldsymbol{M'}$ are the pair of Aplanatic Points of the system, they must satisfy the Sine-Condition, viz., $\sin\theta'/\sin\theta =$ constant; and since this condition is necessarily opposed to the Tangent-Condition, the image in this case will be distorted in such fashion that η' will be less than the ideal value y'. Moreover, since the tangent of an angle increases faster than its sine, the difference $y' - \eta'$ will increase as y increases, and therefore the distortion will be "barrel-shaped" (Fig. 139 (*c*)). If the object consists of two sheaves of hyperbolæ resembling Fig. 139 (*b*), and if $\boldsymbol{M}$, $\boldsymbol{M'}$ are the pair of Aplanatic Points, the image in this case will be the two systems of parallel straight lines (Fig. 139 (*a*)). This is the test which Abbe invented to determine whether a microscope fulfils the Sine-Condition, and which was alluded to in § 279.

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[1] R. H. Bow: On Photographic Distortion: *Brit. Journ. of Photography*, VIII (1861), pages 417–419 and 440–442.

T. Sutton: Distortion Produced by Lenses: *Phot. Notes*, VII (1862), No. 138, 3–5.

ART. 93. DEVELOPMENT OF THE APPROXIMATE FORMULA FOR THE DISTORTION-ABERRATION IN CASE THE SLOPE-ANGLES OF THE CHIEF RAYS ARE SMALL.

292. Here we have to do with the image of an extensive object formed by infinitely narrow bundles of rays; that is, with a large visual field and very small aperture. Accordingly, in the series-development of the aberrations of the 3rd order (see Art. 80, especially § 259), the terms involving the co-ordinates $\mathbf{y}_1$, $\mathbf{z}_1$ will be negligible, whereas the most important term will be the one involving y_1^3. Since the image is determined entirely by the chief rays, and since the path of every such ray lies in a meridian plane of the optical system, the image-point P'_m determined by the intersection of the chief ray with the image-plane σ'_m (§ 254) will lie in the meridian plane which contains the path of the chief ray; consequently, here the z-aberration (§ 256) is equal to zero, *i. e.*, $\delta z'_m = 0$, and we have, therefore, merely to develop the expression for the y-aberration:

$$\delta y'_m = \eta'_m - y'_m.$$

The method of obtaining this development, which is given below, is the same as that given by KOENIG and VON ROHR.[1]

According to GAUSS's Theory, the Lateral Magnification Y of a centered system of m spherical surfaces, with respect to the pair of axial conjugate points M_1, M'_m, is given by formula (93) of Chap. VI, as follows:

$$\frac{y'_m}{y_1} = \frac{n_1}{n'_m}\prod_{k=1}^{k=m}\frac{u'_k}{u_k},$$

where $y_1 = \eta_1 = M_1P_1 = M_1Q_1$, $y'_m = M'_mQ'_m$, $u_k = A_kM'_{k-1}$, $u'_k = A_kM'_k$. Accordingly, if

$$M'_mP'_m = \eta'_m, \quad Q'_mP'_m = \delta y'_m,$$

we have:

$$\frac{\eta'_m}{y'_m} = \frac{n'_m\eta'_m}{n_1y_1}\prod_{k=1}^{k=m}\frac{u_k}{u'_k};$$

or, since

$$y_1 = \eta_1 = \eta'_0,$$

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[1] See *Die Theorie der optischen Instrumente* (Berlin, 1904), edited by M. VON ROHR; Chapter V, Die Theorie der sphaerischen Aberrationen, by A. KOENIG and M. VON ROHR, pages 241–246.

and

$$\frac{\eta'_m}{y'_m} = 1 + \frac{\delta y'_m}{y'_m},$$

we obtain:

$$\delta y'_m = -y'_m\left\{1 - \prod_{k=1}^{k=m} \frac{n'_k \eta'_k u_k}{n'_{k-1}\eta'_{k-1}u'_k}\right\}.$$

We proceed, therefore, to develop an expression for the quotient

$$\frac{n'\eta' u}{n\eta u'}.$$

This expression relates to the *k*th spherical surface, but for the present it will be convenient to drop the subscripts. The subscripts are likewise omitted from the letters in the diagram (Fig. 140), which represents the path of the chief ray before refraction at the *k*th surface, the vertex and centre of which are designated accordingly by the letters *A* and *C*, respectively. This ray crosses the axis at the point designated in the figure by ***L*** and is incident on the *k*th surface at the point designated by ***B***. The place where the transversal plane of the σ-system belonging to the medium immediately in front of the *k*th surface is cut by the optical axis is marked by the letter *M*, and the point where the ray crosses this plane is designated by *P*. Finally, the foot of the perpendicular let fall on the optical axis from the incidence-point ***B*** is designated by ***D***. Let us also use the following symbols:

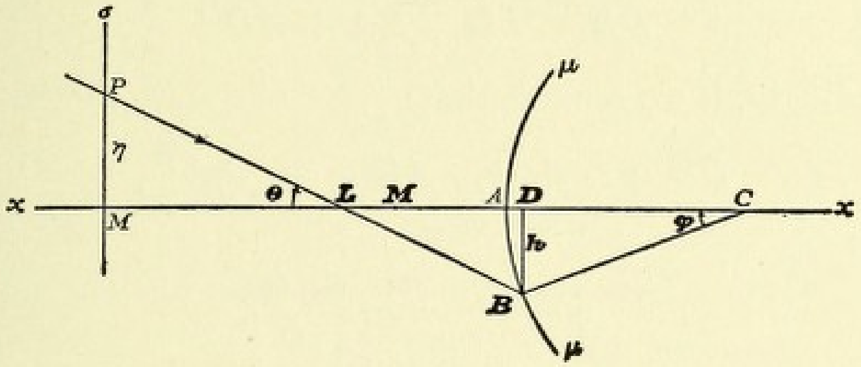

FIG. 140.

FIGURE USED IN THE DERIVATION OF THE DISTORTION-ABERRATION FORMULA. The figure represents the path of a chief ray incident on the *k*th surface of a centered system of spherical refracting surfaces.

$AC = r$, $AM = u$, $A\mathbf{M} = \mathbf{u}$. $A\mathbf{L} = \mathbf{v}$, $MP = \eta$, $\angle \mathbf{B}CA = \phi$, $\mathbf{DB} = \mathbf{h}$.

$$AC = r,\quad AM = u,\quad A\mathbf{L} = \mathbf{v},\quad MP = \eta,\quad \angle M\mathbf{L}P = \theta,$$
$$\angle \mathbf{B}CA = \phi.$$

The function $\eta = MP$ changes its sign when the slope-angle θ (or the central angle ϕ) changes sign, and hence this function may be developed in a series of odd powers of θ (or ϕ). Since the central angle ϕ remains constant during the refraction, it will be better to take this angle as the independent variable. Accordingly, assuming that the central angles ϕ of the chief rays are all so small that we can neglect the powers of these angles above the 3rd, we may write:

$$\eta = l\phi + m\phi^3,$$

where the co-efficients denoted by l, m are constants, the magnitudes of which will depend on the positions of the transversal planes σ, $\boldsymbol{\sigma}$ belonging to the medium which the ray is here traversing.

From the figure, we see that:

$$\frac{MP}{\boldsymbol{DB}} = \frac{\boldsymbol{LM}}{\boldsymbol{LD}} = \frac{\boldsymbol{LA} + AM}{\boldsymbol{LA} + \boldsymbol{AD}};$$

and, accordingly, if we remark that:

$$\boldsymbol{DB} = r \cdot \sin\phi, \quad \boldsymbol{AD} = 2r \cdot \sin^2\frac{\phi}{2},$$

we obtain:

$$\eta = \frac{(\boldsymbol{v} - u)r \cdot \sin\phi}{\boldsymbol{v} - 2r \cdot \sin^2\frac{\phi}{2}}. \tag{307}$$

In this formula we may substitute for $\boldsymbol{v}$ the development:

$$\boldsymbol{v} = \boldsymbol{u} + \mathbf{c}\phi^2,$$

where $\boldsymbol{u} = \boldsymbol{AM}$ denotes the abscissa of the point $\boldsymbol{M}$ where paraxial rays, which in the first medium go through the centre of the Entrance-Pupil, cross the axis before refraction at the surface here under consideration; and where $\boldsymbol{c}$ denotes an undetermined coefficient. Obviously,

$$\boldsymbol{ML} = \delta\boldsymbol{u} = \mathbf{c}\phi^2$$

is the expression for the Longitudinal Aberration of the chief ray before refraction at this spherical surface (see § 263). Making the substitution above mentioned, and equating the two expressions of the function η, we obtain after clearing fractions, expanding the trigonometric functions in series, and arranging according to powers of ϕ, the following equation:

$$\{(\boldsymbol{u} - u)r - l\boldsymbol{u}\}\,\phi + \left\{\left(\frac{r}{2} - \mathbf{c}\right)l - m\boldsymbol{u} + r\mathbf{c} - \frac{r(\boldsymbol{u} - u)}{6}\right\}\phi^3 = 0;$$

and since this equation must be true for all the chief rays, that is, for all values of the central angle ϕ (as far as the extreme value permitted by this approximation), we may equate to zero the co-efficients of ϕ and ϕ^3; whereby the magnitudes of the co-efficients l, m are determined as follows:

$$l = \frac{\boldsymbol{u} - u}{\boldsymbol{u}} r,$$

$$m = \frac{\boldsymbol{u} - u}{\boldsymbol{u}} r^3 \left\{ \frac{1}{2r\boldsymbol{u}} + \frac{u}{\boldsymbol{u}(\boldsymbol{u} - u)} \cdot \frac{\boldsymbol{c}}{r^2} - \frac{1}{6r^2} \right\}.$$

Substituting these expressions for the co-efficients l, m in the series-development of the function η, and at the same time using the invariant-relation obtained by combining formulæ (270):

$$\boldsymbol{J} - J = \frac{n(\boldsymbol{u} - u)}{u\boldsymbol{u}} = \frac{n'(\boldsymbol{u}' - u')}{u'\boldsymbol{u}'},$$

we derive the following formula:

$$\frac{n\eta}{u} = -(J - \boldsymbol{J}) r\phi \left[1 + r^2\phi^2 \left\{ \frac{1}{2r\boldsymbol{u}} - \frac{n\boldsymbol{c}}{r^2\boldsymbol{u}^2(J - \boldsymbol{J})} - \frac{1}{6r^2} \right\} \right]. \quad (308)$$

Similarly, for the ray refracted at this surface, we obtain the corresponding formula for $n'\eta'/u'$ by merely priming the letters $\boldsymbol{u}$ and $\boldsymbol{c}$ on the right-hand side of equation (308). Doing this, and dividing the latter equation by the former, we obtain:

$$\frac{n'\eta' u}{n\eta u'} = 1 + r^2\phi^2 \left\{ \frac{1}{2r} \Delta \frac{1}{\boldsymbol{u}} - \frac{1}{r^2(J - \boldsymbol{J})} \Delta \frac{n\boldsymbol{c}}{\boldsymbol{u}^2} \right\}.$$

Now

$$\phi^2 \Delta \frac{n\boldsymbol{c}}{\boldsymbol{u}^2} = \Delta \frac{n \cdot \delta\boldsymbol{u}}{\boldsymbol{u}^2} = -\frac{r^2\phi^2}{2} \boldsymbol{J}^2 \cdot \Delta \frac{1}{n\boldsymbol{u}},$$

as we found in § 263. Moreover, according to formula (77),

$$\Delta \frac{1}{\boldsymbol{u}} = -\boldsymbol{J} \cdot \Delta \frac{1}{n}.$$

Thus, we obtain:

$$\frac{n'\eta' u}{n\eta u'} = 1 - \frac{r^2\phi^2}{2} \left(\frac{\boldsymbol{J}}{r} \Delta \frac{1}{n} - \frac{\boldsymbol{J}^2}{J - \boldsymbol{J}} \Delta \frac{1}{n\boldsymbol{u}} \right).$$

If $\boldsymbol{h} = \boldsymbol{DB}$ denotes the incidence-height of the chief ray corresponding to the central angle ϕ, we may, neglecting magnitudes above the 3rd order, put

$$\boldsymbol{h}^2 = r^2\phi^2.$$

Re-introducing the subscripts, we shall write, therefore, the above equation as follows:

$$\frac{n'_k \eta'_k u_k}{n'_{k-1}\eta'_{k-1}u'_k} = 1 - \frac{\boldsymbol{h}_k^2}{2}\left\{\frac{\mathbf{J}_k}{r_k}\Delta\left(\frac{1}{n}\right)_k - \frac{\mathbf{J}_k^2}{J_k - \mathbf{J}_k}\Delta\left(\frac{1}{n\boldsymbol{u}}\right)_k\right\}.$$

Before substituting this expression in the equation for $\delta y'_m$, we shall put it in a form rather more convenient for actual use. Thus, if h_1 denotes the incidence-height at the first spherical surface of a paraxial object-ray proceeding from the axial object-point M_1, and if h_k denotes the incidence-height at the kth surface of this same paraxial ray, we may, without neglecting magnitudes of the 3rd order, use the relation given by formula (155) of Chapter VIII, viz.:

$$h_k\boldsymbol{h}_k(J_k - \mathbf{J}_k) = h_1\boldsymbol{h}_1(J_1 - \mathbf{J}_1);$$

in which case the expression on the right-hand side of the above equation may be transformed as follows:

$$\frac{n'_k \eta'_k u_k}{n'_{k-1}\eta'_{k-1}u'_k} = 1$$

$$-\frac{1}{2}\cdot\frac{1}{h_1\boldsymbol{h}_1(J_1 - \mathbf{J}_1)}h_k\boldsymbol{h}_k^3\left\{\frac{\mathbf{J}_k(J_k - \mathbf{J}_k)}{r_k}\Delta\left(\frac{1}{n}\right)_k - \mathbf{J}_k^2\cdot\Delta\left(\frac{1}{n\boldsymbol{u}}\right)_k\right\}.$$

Substituting this expression in the formula for the aberration $\delta y'_m$, we obtain:

$$\delta y'_m = \frac{1}{2}\cdot\frac{y'_m}{h_1\boldsymbol{h}_1(J_1 - \mathbf{J}_1)}\sum_{k=1}^{k=m}h_k\boldsymbol{h}_k^3\left\{\frac{\mathbf{J}_k(J_k - \mathbf{J}_k)}{r_k}\Delta\left(\frac{1}{n}\right)_k - \mathbf{J}_k^2\cdot\Delta\left(\frac{1}{n\boldsymbol{u}}\right)_k\right\}. \quad (309)$$

Thus, the condition that $\delta y'_m = 0$, or that there shall be no distortion of the image-point P'_m is as follows:

$$\sum_{k=1}^{k=m}h_k\boldsymbol{h}_k^3\left\{\frac{\mathbf{J}_k(J_k - \mathbf{J}_k)}{r_k}\Delta\left(\frac{1}{n}\right)_k - \mathbf{J}_k^2\cdot\Delta\left(\frac{1}{n\boldsymbol{u}}\right)_k\right\} = 0. \quad (310)$$

Formula (309) may be put also in a different form. Thus, since (as may be easily shown, see § 126) we have the following relation:

$$\Delta\frac{1}{n\boldsymbol{u}} = \frac{J - \mathbf{J}}{J}\cdot\frac{1}{r}\Delta\frac{1}{n} + \frac{\mathbf{J}}{J}\Delta\frac{1}{nu},$$

we may substitute this expression within the large brackets, whereby, after simple reductions, we obtain:

$$\delta y'_m = \frac{\boldsymbol{h}_1^2}{2}\cdot\frac{y'_m}{(J_1 - \mathbf{J}_1)}\sum_{k=1}^{k=m}\frac{h_k}{h_1}\frac{\boldsymbol{h}_k^3}{\boldsymbol{h}_1^3}\left\{\frac{\mathbf{J}_k(J_k - \mathbf{J}_k)^2}{J_k}\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k - \frac{\mathbf{J}_k^3}{J_k}\Delta\left(\frac{1}{nu}\right)_k\right\}. \quad (311)$$

Finally, according to the Law of ROBERT SMITH (§ 194), the relation between the conjugate ordinates y_1, y'_m may be expressed evidently as follows:

$$\frac{n_1 y_1 h_1}{u_1} = \frac{n'_m y'_m h_m}{u'_m};$$

and, moreover, we have also:

$$\boldsymbol{h}_1 = -\frac{\boldsymbol{u}_1 y_1}{u_1 - \boldsymbol{u}_1},$$

and

$$\frac{1}{J_1 - \boldsymbol{J}_1} = \frac{u_1 \boldsymbol{u}_1}{n_1(u_1 - \boldsymbol{u}_1)}.$$

And, hence the formula above may be written:

$$\delta y'_m = \frac{1}{2} \cdot \frac{h_1}{h_m} \cdot \frac{u'_m}{n'_m} \cdot \frac{\boldsymbol{u}_1^3}{(u_1 - \boldsymbol{u}_1)^3} y_1^3$$
$$\times \sum_{k=1}^{k=m} \frac{h_k}{h_1} \frac{\boldsymbol{h}_k^3}{\boldsymbol{h}_1^3} \left\{ \frac{\boldsymbol{J}_k (J_k - \boldsymbol{J}_k)^2}{J_k} \frac{1}{r_k} \Delta\left(\frac{1}{n}\right)_k - \frac{\boldsymbol{J}_k^3}{J_k} \Delta\left(\frac{1}{nu}\right)_k \right\}; \quad (312)$$

whence it is seen how the Distortion-Aberration $\delta y'_m$ is proportional to the cube of the ordinate y_1.

ART. 94. THE DISTORTION-ABERRATION IN SPECIAL CASES.

293. Case of Single Spherical Refracting Surface.

When the optical system is composed of a single spherical surface, formula (309) gives for the Distortion-Aberration

$$\delta y' = \frac{\boldsymbol{h}^2 \boldsymbol{J}}{2(J - \boldsymbol{J})} y' T,$$

where, for the sake of brevity, we write:

$$T = \frac{J - \boldsymbol{J}}{r} \Delta \frac{1}{n} - \boldsymbol{J} \Delta \frac{1}{n\boldsymbol{u}},$$

or

$$\frac{n'^2}{n - n'} T = \frac{n' + n}{\boldsymbol{u}^2} - \frac{2n}{r\boldsymbol{u}} - \frac{n'}{ru} + \frac{n}{r^2}.$$

If the image is to be free from distortion, we must have $\delta y' = 0$; which implies here one of two things: Either $\boldsymbol{J} = 0$, or else, $T = 0$. If $\boldsymbol{J} = 0$, then $\boldsymbol{u} = \boldsymbol{u}' = r$; which means in this case that the stop-centre coincides with its image at the centre C of the spherical surface, and under these circumstances the image will be free from distortion for all object-distances.

If the stop-centre is not at C, the condition that the image shall be without distortion is $T = 0$, or

$$\frac{1}{u} = r\left\{\frac{1}{\mathbf{u}^2} + \frac{n}{n'}\left(\frac{1}{r} - \frac{1}{\mathbf{u}}\right)^2\right\};$$

whence it is seen that for a given position of the centre $\boldsymbol{L}$ of the stop, that is, for a given (real) value of the abscissa $\mathbf{u}$, there will always be one certain object-distance u for which the spherical surface will give an image free from distortion.

If, on the other hand, the object-distance u is given, we shall find that there are always two positions of the stop-centre that will give an image without distortion; that is, corresponding to a given value of u, we obtain two values of $\mathbf{u}$, viz.:

$$\frac{1}{\mathbf{u}} = \frac{1}{n + n'}\left\{\frac{n}{r} \pm \sqrt{\frac{n'(n + n')}{ru} - \frac{nn'}{r^2}}\right\}.$$

If these two values of $\mathbf{u}$ are to be real, u and $\mathbf{r}$ must have the same signs and

$$\frac{\mathbf{r}}{u} > \frac{n}{n + n'}.$$

If the object-point M coincides with the aplanatic point Z (§ 207), then $u = \mathbf{u} = (n + n')r/n$, which means that the stop-centre also coincides with the aplanatic object-point Z; this, however, would have no practical meaning, and, hence, the distortion cannot be abolished when the object-point M coincides with the aplanatic point Z. This is in agreement with the results which we found when we were investigating the Sine-Condition (§ 276; *cf.* § 289).

294. Case of Infinitely Thin Lens.

The distortion produced by an optical system consisting of an Infinitely Thin Lens may be investigated by a method precisely similar to that used in the case of the Longitudinal Aberration (see § 268). If here also we use the same special Lens-Notation as was employed there, the expression within the large brackets in formula (309) may be put equal to $-\varphi X$, where φ denotes here the reciprocal of the Primary Focal Length (f) of the Lens, and where the function denoted by X will have the following expression:

$$X = \frac{n + 2}{n}c^2 - \left(\frac{2n + 1}{n - 1}\varphi + \frac{3(n + 1)}{n}\mathbf{x} + \frac{n + 1}{n}x\right)c$$
$$+ \frac{n^2}{(n - 1)^2}\varphi^2 + \frac{1}{n}x\mathbf{x} + \frac{1}{n - 1}\varphi x + \frac{3n}{n - 1}\varphi\mathbf{x} + \frac{3n + 1}{n}\mathbf{x}^2.$$

The condition that X shall be a minimum for given values of φ, x and $\boldsymbol{x}$ will be found to be:

$$c = \frac{3(n+1)}{2(n+2)}\boldsymbol{x} + \frac{n+1}{2(n+2)}x + \frac{n(2n+1)}{2(n-1)(n+2)}\varphi.$$

V. Astigmatism and Curvature of The Image.

ART. 95. THE PRIMARY AND SECONDARY IMAGE-SURFACES.

295. In the imagery of extended objects by means of narrow bundles of rays whose chief rays all meet at a prescribed point on the optical axis of the centered system of spherical surfaces, there will, in general, be astigmatic deformation of the bundles of image-rays; in consequence whereof to an object-point P lying outside the axis there will correspond, not a sharp image-point, but two short image-lines perpendicular to the chief ray of the bundle at the so-called I. and II. Image-Points S' and $\bar{S}'$ (see Chapter XI). Thus, in case the image-rays are received on a focussing-screen, the image of the object-point as seen on the screen will generally be a small patch of light corresponding to the cross-section of the bundle of image-rays at that place, the dimensions of which, in one direction at least, will always be comparable with the diameter of the narrow stop; so that such an image formed by an astigmatic bundle will always be more or less blurred and indistinct, and not to be compared in this respect with the sharp image which is obtained when the object-point is on the axis. The farther the object-point is from the axis, the more pronounced this defect will be. In two special positions of the focussing-screen the image will be deformed into a short line, which is vertical, say, for one of the positions, and horizontal for the other position —corresponding to the places of the two image-lines of the astigmatic bundle (§ 230). Somewhere between these two positions the bundle of rays will have its narrowest cross-section, which, in the case of a centered system of spherical surfaces, will be approximately circular in form. This is the place of the so-called "Circle of Least Confusion" (§ 244)—a somewhat misleading phrase, inasmuch as the convergence of the rays in either of the two image-lines is of a higher order. However, we do obtain here perhaps the nearest approach to a *true* image of the object-point.

If on every chief image-ray corresponding to such points of the object as are contained in a meridian plane of the optical system, we mark the I. and II. image-points S' and $\bar{S}'$, the loci of these two sets of image-points will be two curved lines which touch each other at their common vertex M' where they both cross the optical axis; so that this point M' is an accurate image of the axial object-point M. Assuming that the points of the object lie on a surface of revolution described around the optical axis, for example, in a plane perpendicular to the optical axis, we readily perceive that these curves are the traces in the meridian plane of the *I.* and *II. Image-Surfaces*, which latter will, therefore, be generated by revolving these curves around the optical axis as axis of rotation. One of these surfaces will contain all the I. Image-Lines, and the other will contain all the II. Image-Lines. A third surface of revolution, lying between these two, will contain the circles of least confusion.

Even if the bundles of image-rays were made stigmatic, so that the I. and II. image-surfaces coincided into a single image-surface corresponding with the object-surface point by point, the image would, in general, still be curved, so that if the image-rays were received on a flat screen perpendicular to the optical axis, the definition of the image-points as seen on the screen would still be more or less faulty depending on the degree of curvature of the image-surface. In the case of most optical instruments, and especially in the case of the photographic objective and of the lantern-projection system, a flat image is an essential requirement; so that closely connected with the abolition of the astigmatism of the oblique bundles of image-rays is the removal of the so-called "aberration of curvature" of the image-surface.

The methods employed in the following investigation will be found to be similar to the treatment of this subject in KOENIG and VON ROHR's *Die Theorie der sphaerischen Aberrationen.*[1]

ART. 96. THE ABERRATION-LINES, IN A PLANE PERPENDICULAR TO THE AXIS, OF THE MERIDIAN AND SAGITTAL RAYS.

296. At the image-point M' (Figs. 141 and 142), corresponding to the axial object-point M, let us suppose that the GAUSSian image-plane σ' is erected. The chief ray of the bundle of rays proceeding from an object-point P, not on the optical axis, will, in traversing the medium where the infinitely narrow stop is placed, go through the centre O of this stop, and, finally, after refraction at the last surface of the system, will cross the axis (really or virtually) at a point L' and meet the image-plane σ' in the point P'. Let H' (Fig. 141) and G' (Fig. 142) designate the points where the outermost meridian ray and the outermost sagittal ray, respectively, of the infinitely narrow astigmatic bundle of image-rays cross the transversal plane at L'

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[1] *Die Theorie der optischen Instrumente:* Bearbeitet von wissenschaftlichen Mitarbeitern an der optischen Werkstaette von CARL ZEISS; Bd. I (Berlin, 1904), herausgegeben von M. VON ROHR. V. Kapitel. 250–265.

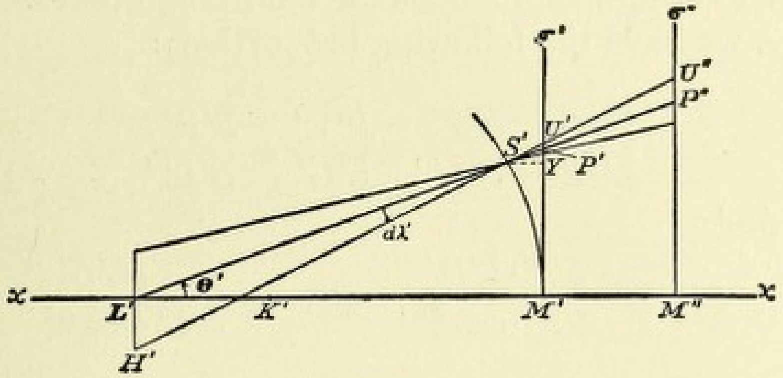

FIG. 141.

NARROW PENCIL OF MERIDIAN IMAGE-RAYS. The chief ray $L'P'$ of the pencil crosses the optical axis at the point L', and meets the GAUSSIAN Image-Plane σ' at P'. $H'U'$ is the extreme ray of the pencil crossing at H' and U' the planes perpendicular to the optical axis at L' and M', respectively. These rays intersect each other in the I. Image-Point S', and the locus of the I. Image-Points S' is the Primary Image-Curve whose centre of curvature with respect to the point M' is at the point K'. The chief ray meets a plane σ'' parallel to σ' at the point P''. $P'U'$ is the aberration line of the pencil of meridian rays in the GAUSSIAN Image-Plane σ'.

$$L'M' = u' - v', \quad M'K' = R', \quad M'M'' = e, \quad M'P' = \eta'. \quad \angle M'L'P' = \theta', \quad \angle L'S'H' = d\lambda'.$$

which is parallel to the image-plane σ'. The extreme ray of the pencil of meridian rays will intersect the chief ray at the I. image-point S', and will meet the image-plane σ' in a point U' (Fig. 141) lying in the

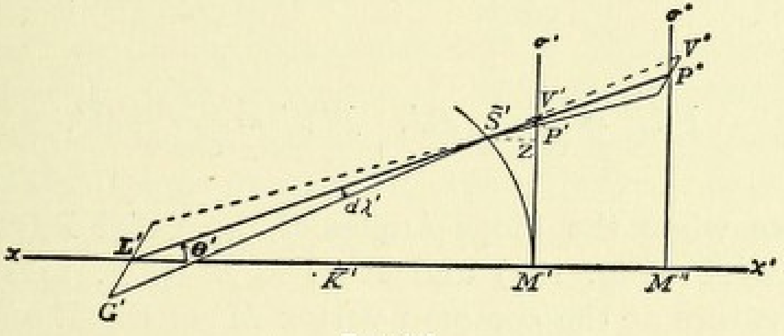

FIG. 142.

NARROW PENCIL OF SAGITTAL IMAGE-RAYS. The chief ray $L'P'$ of the pencil crosses the optical axis at the point L' and meets the GAUSSIAN Image-Plane σ' at the point P'. $G'V'$ is the extreme ray of the pencil crossing at G' and V' the planes perpendicular to the optical axis at L' and M', respectively. These rays intersect each other in the II. Image-Point $\overline{S}'$, and the locus of the II. Image-Points $\overline{S}'$ is the Secondary Image-Curve, whose centre of curvature with respect to the axial Image-Point M' is at the point marked $\overline{K}'$. The chief ray meets a plane σ'' parallel to σ' at the point P'', and the ray $G'V'$ meets this plane in the point V''. $P'V'$ is the aberration-line of the pencil of sagittal rays in the GAUSSIAN Image-Plane σ'.

$$L'M' = u' - v', \quad M'\overline{K}' = \overline{R}', \quad M'M'' = e, \quad M'P' = \eta', \quad \angle M'L'P' = \theta', \quad \angle L'\overline{S}'G' = d\overline{\lambda}'.$$

meridian plane; and, similarly, the extreme ray of the sagittal pencil will intersect the chief ray at the II. image-point $\overline{S}'$ and will meet the plane σ' in a point V' (Fig. 142) in the plane of the sagittal rays.

The line-segments $P'U'$ and $P'V'$ are the *aberration-lines, in the image-plane* σ', *of the meridian and sagittal rays*, respectively; and we proceed now to obtain expressions for their magnitudes.

Evidently, we have the following proportions:

$$\frac{P'U'}{\boldsymbol{L}'H'} = \frac{S'P'}{S'\boldsymbol{L}'}, \quad \frac{P'V'}{\boldsymbol{L}'G'} = \frac{\bar{S}'P'}{\bar{S}'\boldsymbol{L}'},$$

whence we find:

$$P'U' = \frac{S'Y \cdot \boldsymbol{L}'H'}{S'\boldsymbol{L}' \cdot \cos\theta'}, \quad P'V' = \frac{\bar{S}'Z \cdot \boldsymbol{L}'G'}{\bar{S}'\boldsymbol{L}' \cdot \cos\theta'},$$

where $\theta' = \angle M'\boldsymbol{L}'P'$ denotes the slope-angle of the chief image-ray of this bundle, and Y (Fig. 141) and Z (Fig. 142) designate the feet of the perpendiculars let fall from S' and $\bar{S}'$, respectively, on the plane σ'. If here we introduce the aperture-angles of the meridian and sagittal pencils, viz.,

$$d\lambda' = \angle\, \boldsymbol{L}'S'H', \quad d\bar{\lambda}' = \angle\, \boldsymbol{L}'\bar{S}'G',$$

these angles being supposed here so small that we may neglect any terms involving their squares, then:

$$\boldsymbol{L}'H' = \frac{S'\boldsymbol{L}'}{\cos\theta'}\,d\lambda', \quad \boldsymbol{L}'G' = \bar{S}'\boldsymbol{L}' \cdot d\bar{\lambda}';$$

and, hence:

$$P'U' = \frac{S'Y}{\cos^2\theta'}\,d\lambda', \quad P'V' = \frac{\bar{S}'Z}{\cos\theta'}\,d\bar{\lambda}'. \qquad (313)$$

297. Case when the Slope-Angles of the Chief Rays are Small.

If $R' = M'K'$ (Fig. 141) and $\bar{R}' = M'\bar{K}'$ (Fig. 142) denote the radii of curvature at the common vertex M' of the I. and II. image-surfaces, and if we neglect powers of the slope-angles θ' above the second, we may write:

$$S'Y = -\frac{M'Y^2}{2R'}, \quad \bar{S}'Z = -\frac{M'Z^2}{2\bar{R}'}.$$

Let Q' designate the position in the plane σ' of the point, which, by GAUSS's Theory, is conjugate to the object-point P; then the ordinate $M'Q' = y'$ is of the same order of magnitude as $M'P'$ and θ', and thus it is obvious that if we neglect powers of θ' above the second, we may write:

$$M'Y^2 = M'Z^2 = y'^2;$$

and, hence, to the required degree of exactness, we obtain:

$$S'Y = -\frac{y'^2}{2R'}, \quad \bar{S}'Z = -\frac{y'^2}{2\bar{R}'}.$$

Accordingly, we derive the following *approximate* expressions for the magnitudes of the aberration-lines, in the Gaussian image-plane σ', of the meridian and sagittal rays:

$$P'U' = -\frac{y'^2}{2R'}d\lambda', \quad P'V' = -\frac{y'^2}{2\bar{R}'}d\bar{\lambda}'. \qquad (314)$$

298. Moreover, let σ'' be any plane parallel to the Gaussian image-plane σ', and at a distance from it $M'M'' = e$ (say), and let P'', U'' and V'' designate the points where the rays $L'S'\bar{S}'$, $H'S'$ and $G'\bar{S}'$, respectively, cross the plane σ''; so that $P''U''$ and $P''V''$ will be the linear aberrations in this transversal plane of the meridian and sagittal rays of the astigmatic bundle of image-rays. Evidently, if we neglect the second powers of the aperture-angles $d\lambda'$, $d\bar{\lambda}'$ and the powers of the slope-angle θ' above the second, we shall have:

$$P''U'' = \left(e - \frac{y'^2}{2R'}\right)d\lambda', \quad P''V'' = \left(e - \frac{y'^2}{2\bar{R}'}\right)d\bar{\lambda}'. \qquad (315)$$

If, therefore, supposing that we have $d\lambda' = d\bar{\lambda}'$, we wish to determine the position of the focussing-plane σ'' somewhere between the I. and II. image-points S' and $\bar{S}'$ for which *the linear aberrations $P''U''$ and $P''V''$ are of equal magnitudes but of opposite signs*, the two equations (315) give the following formula for this particular value of the abscissa e:

$$e = \frac{y'^2}{4}\left(\frac{1}{R'} + \frac{1}{\bar{R}'}\right);$$

and, under these circumstances, we obtain:

$$P''U'' = V''P'' = \frac{y'^2 \cdot d\lambda'}{4}\left(\frac{1}{\bar{R}'} - \frac{1}{R'}\right).$$

If the bundle of rays is received on a plane screen coinciding with this position of the plane σ'', we shall obtain on the screen, as was stated above (§ 295), perhaps the nearest approach to a true image of the object-point.

In case the astigmatism was entirely abolished, so that

$$R' = \overline{R}',$$

by placing the plane screen in the position for which $e = y'^2/2R'$, we should obtain on it an actual point-image of the object-point P. But it will be remarked that the value of e depends on that of y', and in order to obtain point-images of the different points of the object, we should have to "focus" the screen so that its intersection with the curved stigmatic image-surface would contain the point to be observed.

ART. 97. DEVELOPMENT OF THE FORMULÆ FOR THE CURVATURES $1/R'$, $1/\overline{R}'$.

299. The Invariants of Astigmatic Refraction. The curvatures at M' of the two image-surfaces have now to be expressed in terms of the curvature of the object-surface at M and of the given constants of the centered system of spherical surfaces. In the development of these expressions we shall use ABBE's Invariant-Method, as given by KOENIG and VON ROHR in their treatise on *Die Theorie der sphaerischen Aberrationen*.

In Chapter XI, §§ 236 and 240, we derived two formulæ (246) and (250), which may be written as follows:

$$\left.\begin{aligned} Q &= n\left(\frac{\cos\alpha}{r} - \frac{\cos^2\alpha}{s}\right) = n'\left(\frac{\cos\alpha'}{r} - \frac{\cos^2\alpha'}{s'}\right), \\ \overline{Q} &= n\left(\frac{\cos\alpha}{r} - \frac{1}{\overline{s}}\right) = n'\left(\frac{\cos\alpha'}{r} - \frac{1}{\overline{s}'}\right); \end{aligned}\right\} \qquad (316)$$

where the functions denoted here by Q and $\overline{Q}$, which have the same values before and after refraction at a given spherical surface, are called the *Invariant-Functions of the Chief Ray of the Infinitely Narrow Bundle of Rays*. Each of these functions may evidently be developed in a series of ascending powers of the central angle ϕ of the following forms:

$$\left.\begin{aligned} Q &= J + B\frac{\phi^2}{2} = J + B'\frac{\phi^2}{2}, \\ \overline{Q} &= J + \overline{B}\frac{\phi^2}{2} = J + \overline{B}'\frac{\phi^2}{2}, \end{aligned}\right\} \qquad (317)$$

wherein the coefficients B, $\overline{B}$, etc. are as yet undetermined, and where, as usual, the terms involving powers of ϕ higher than the second are neglected. The relations, which we wish to find, will then be given by writing:

$$B' - B = 0, \quad \overline{B}' - \overline{B} = 0.$$

The easiest method of obtaining the expansions of Q and $\overline{Q}$ will be to develop the functions $1/s$, $1/\overline{s}$ and $\cos\alpha$ each in a series of ascending powers of ϕ, and to introduce these expressions in the formulæ (316) above.

300. Developments of $1/s$, $1/\overline{s}$ and $\cos\alpha$ in a series of powers of ϕ.

In the diagram (Fig. 143) the straight line $\boldsymbol{SB}$ represents the path

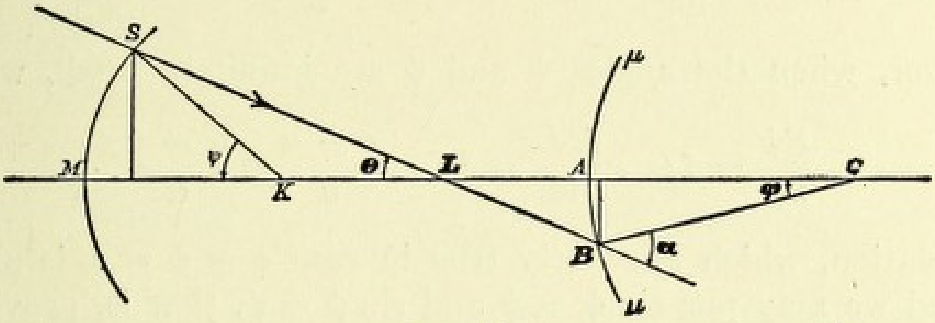

FIG. 143.

PATH OF CHIEF RAY OF PENCIL OF MERIDIAN RAYS INCIDENT ON kTH SURFACE OF CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES.

$AC = r,\ MK = R,\ \boldsymbol{B}S = s,\ AM = u,\ A\boldsymbol{L} = v,\ \boldsymbol{BL} = l,\ \angle\boldsymbol{B}CA = \phi,\ \angle SKM = \psi,\ \angle A\boldsymbol{LB} = \theta.$

of the chief ray before its refraction at (say) the kth spherical surface. In its progress through this medium the ray crosses the axis at the point designated by $\boldsymbol{L}$ and is incident on the spherical surface at the point $\boldsymbol{B}$. The point designated by S is the I. Image-Point of the astigmatic bundle of rays in the medium between the $(k - 1)$th and kth spherical surfaces, and the curved line MS represents the section in the meridian plane (or plane of the figure) of the I. image-surface which is the locus of the I. image-points S. The primes and subscripts which naturally belong to these letters are suppressed for the present; they will re-appear, as usual, at the end of the investigation. For the purpose of these developments, we shall employ, therefore, the following symbols:

$$AC = r, \quad MK = R, \quad AM = u, \quad A\boldsymbol{L} = v, \quad \boldsymbol{B}S = s,$$
$$\angle\boldsymbol{B}CA = \phi, \quad \angle SKM = \psi, \quad \angle A\boldsymbol{LB} = \theta.$$

The letter K is used here to designate the centre of curvature at M of the meridian section of the I. image-surface.

From the figure we obtain easily the following relation:

$$\frac{1}{s} = \frac{\cos\theta}{u + 2R\cdot\sin^2\frac{\psi}{2} - 2r\cdot\sin^2\frac{\phi}{2}},$$

which, provided we neglect the powers of the angles θ, ϕ and ψ above the second, may be written:

$$\frac{1}{s} = \frac{1 - \frac{\theta^2}{2}}{u + R\frac{\psi^2}{2} - r\frac{\phi^2}{2}}.$$

Moreover, when the angles ϕ and ψ are infinitely small, we have:

$$\frac{R\psi}{r\phi} = Lt_{\phi=0}\frac{ML}{AL} = Lt_{\phi=0}\frac{\mathbf{v} - u}{\mathbf{v}} = \frac{\mathbf{u} - u}{\mathbf{u}}.$$

This relation, which is strictly true in case $\phi = \psi = 0$, is also true provided we may put $\sin\phi = \phi$ and $\sin\psi = \psi$, that is, provided we neglect the powers of these angles above the first; and even when we retain, as here, the second powers of these angles, we may write:

$$\frac{R^2\psi^2}{r^2\phi^2} = \frac{(\mathbf{u} - u)^2}{\mathbf{u}^2}.$$

In the same way, also:

$$\frac{r^2\phi^2}{\mathbf{u}^2} = \theta^2.$$

Hence, eliminating θ and ψ from these equations, we obtain:

$$\frac{1}{s} = \frac{1 - \frac{\phi^2}{2}\cdot\frac{r^2}{\mathbf{u}^2}}{u\left[1 + \frac{\phi^2}{2}\left\{\frac{r^2}{uR}\left(\frac{\mathbf{u} - u}{\mathbf{u}}\right)^2 - \frac{r}{u}\right\}\right]},$$

or, finally:

$$\frac{1}{s} = \frac{1}{u} + \frac{r^2}{u^2}\left\{\frac{1}{r} - \frac{u}{\mathbf{u}^2} - \frac{u^2}{R}\left(\frac{1}{u} - \frac{1}{\mathbf{u}}\right)^2\right\}\frac{\phi^2}{2}. \qquad (318)$$

The development of the reciprocal of $\overline{s} = \boldsymbol{B}\overline{S}$ will obviously have precisely the same form as that obtained here for $1/s$; the only difference being that we shall have $\overline{R}$ in place of R in formula (318).

Again, since

$$\cos\alpha = 1 - \frac{\alpha^2}{2},$$

and since

$$\boldsymbol{\alpha}^2 = (\theta + \phi)^2 = \left(-\frac{r}{\mathbf{u}}\phi + \phi\right)^2,$$

we obtain:

$$\cos\boldsymbol{\alpha} = 1 - \left(\frac{\mathbf{u} - r}{\mathbf{u}}\right)^2 \frac{\phi^2}{2} = 1 - \frac{r^2\boldsymbol{J}^2}{n^2} \cdot \frac{\phi^2}{2}. \qquad (319)$$

301. The expressions for the co-efficients B, $\bar{B}$ and B', $\bar{B}'$.

If now we substitute in the formulæ (316) the series-developments for $1/s$, $1/\bar{s}$ and $\cos\boldsymbol{\alpha}$, as found above in formulæ (318) and (319), we obtain the following expressions for the co-efficients B and $\bar{B}$ in formulæ (317):

$$B = r^2(J - \boldsymbol{J})^2 \frac{1}{nR} - \frac{r\boldsymbol{J}^2}{n} - \frac{nr}{u^2} + \frac{nr^2}{u\mathbf{u}^2} + \frac{2r^2\boldsymbol{J}^2}{nu},$$

$$\bar{B} = r^2(J - \boldsymbol{J})^2 \frac{1}{n\bar{R}} - \frac{r\boldsymbol{J}^2}{n} - \frac{nr}{u^2} + \frac{nr^2}{u\mathbf{u}^2}.$$

These expressions can be obtained in a more convenient form. Thus, by simple transformations:

$$\begin{aligned} -\frac{r\boldsymbol{J}^2}{n} - \frac{nr}{u^2} + \frac{nr^2}{u} \cdot \frac{1}{\mathbf{u}^2} &= -\frac{r}{n}(J^2 + \boldsymbol{J}^2) + 2J - \frac{n}{r} + \frac{nr^2}{u}\left(\frac{1}{r} - \frac{\boldsymbol{J}}{n}\right)^2 \\ &= -\frac{r}{n}(J^2 + \boldsymbol{J}^2) + \frac{r^2}{nu}\boldsymbol{J}^2 + J - 2\boldsymbol{J}\left(1 - \frac{r}{n}J\right) \\ &= -\frac{r}{n}(J - \boldsymbol{J})^2 + \frac{r^2}{nu}\boldsymbol{J}^2 - 2\boldsymbol{J} + J; \end{aligned}$$

and hence:

$$\left.\begin{aligned} B &= r^2(J - \boldsymbol{J})^2\left(\frac{1}{nR} - \frac{1}{nr}\right) + 3\frac{r^2}{nu}\boldsymbol{J}^2 - 2\boldsymbol{J} + J, \\ \bar{B} &= r^2(J - \boldsymbol{J})^2\left(\frac{1}{n\bar{R}} - \frac{1}{nr}\right) + \frac{r^2}{nu}\boldsymbol{J}^2 - 2\boldsymbol{J} + J. \end{aligned}\right\} \qquad (320)$$

The expressions for the co-efficients B', $\bar{B}'$ will evidently have the same forms as the expressions found above for B, $\bar{B}$, and can be obtained directly from formulæ (320) by merely priming the symbols n, R, $\bar{R}$ and u.

302. Imposing now the conditions $B' - B = 0$ and $\bar{B}' - \bar{B} = 0$, and at the same time introducing the subscripts and employing Abbe's difference-notation, we derive the following formulæ for the relations between the curvatures of the image-surfaces before and after refraction at the *k*th surface of the centered system of spherical surfaces:

$$\left.\begin{aligned} \Delta\left(\frac{1}{nR}\right)_k &= \frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k - 3\,\frac{\boldsymbol{J}_k^2}{(J_k-\boldsymbol{J}_k)^2}\Delta\left(\frac{1}{nu}\right)_k, \\ \Delta\left(\frac{1}{n\overline{R}}\right)_k &= \frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k - \frac{\boldsymbol{J}_k^2}{(J_k-\boldsymbol{J}_k)^2}\Delta\left(\frac{1}{nu}\right)_k. \end{aligned}\right\} \qquad (321)$$

In the case of a centered system of *m* spherical refracting surfaces, we obtain, therefore, by the usual method of addition the following convenient forms of the relations between the curvatures of the object- and image-surfaces:

$$\left.\begin{aligned} \frac{1}{n'_m R'_m} - \frac{1}{n_1 R_1} &= \sum_{k=1}^{k=m}\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k - 3\sum_{k=1}^{k=m}\frac{\boldsymbol{J}_k^2}{(J_k-\boldsymbol{J}_k)^2}\Delta\left(\frac{1}{nu}\right)_k, \\ \frac{1}{n'_m \overline{R}'_m} - \frac{1}{n_1 \overline{R}_1} &= \sum_{k=1}^{k=m}\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k - \sum_{k=1}^{k=m}\frac{\boldsymbol{J}_k^2}{(J_k-\boldsymbol{J}_k)^2}\Delta\left(\frac{1}{nu}\right)_k. \end{aligned}\right\} (322)$$

If the bundles of object-rays are homocentric, as will usually be the case, the radii R_1 and $\overline{R}_1$ will be identical. For a stigmatic plane object perpendicular to the optical axis, we shall have $R_1 = \overline{R}_1 = \infty$; in which case the second term on the left-hand side of each of the above equations (322) will vanish; and, if, moreover, the image is formed in air ($n'_m = 1$), the expressions on the right-hand side of the two formulæ (322) will give at once the curvature of the two image-surfaces.[1]

If, assuming the usual case of a stigmatic object, we subtract the two equations (322), we obtain:

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[1] These formulæ, practically in the same form as they are here given, were published by H. ZINKEN gen. SOMMER in a treatise entitled *Untersuchungen ueber die Dioptrik der Linsen-Systeme* (Braunschweig, 1870); see also an article by the same writer on the same subject in POGG. *Ann.*, cxxii. (1864), 563–574. Also, L. SEIDEL: Zur Dioptrik. Ueber die Entwicklung der Glieder 3ter Ordnung, welche den Weg eines ausserhalb der Ebene der Axe gelegenen Lichtstrahles durch ein System brechender Medien, bestimmen: *Astr. Nachr.*, xliii. (1856), Nos. 1027, 1028 and 1029, paragraph 8.

H. CODDINGTON in his celebrated treatise on the *Reflexion and Refraction of Light* (Cambridge, 1829) had derived equivalent formulæ for the curvatures of both the I. and II. Image-Surfaces under the same restrictions as we have here imposed. CODDINGTON'S methods, which are always highly ingenious, are employed in H. DENNIS TAYLOR'S *A System of Applied Optics* (London, 1906). Prior to CODDINGTON, G. B. AIRY had published a small volume, *On the Spherical Aberration of the Eye-Pieces of Telescopes* (Cambridge, 1827), afterwards reprinted in the *Cambridge Phil. Trans.*, iii. (1830), in which he investigated the curvature of the image-surface. We must not omit to refer also to the investigations of P. BRETON DE CHAMP, published in the *Comptes Rendus* in 1855, '6 (Tome xl., No. 4, 189–192; tome xlii., No. 12, 542–545 and No. 16, 741–744 and No. 20, 960–963).

As to the celebrated formula published, in 1843, by J. PETZVAL, reference will be made to that in the text.

$$\frac{1}{R'_m} - \frac{1}{\overline{R}'_m} = -2n'_m \sum_{k=1}^{k=m} \frac{\boldsymbol{J}_k^2}{(J_k - \boldsymbol{J}_k)^2}\Delta\left(\frac{1}{nu}\right)_k; \tag{323}$$

and, hence, *the condition of the abolition of the astigmatism of the bundles of image-rays*, viz., $R'_m = \overline{R}'_m$, becomes:

$$\sum_{k=1}^{k=m} \frac{\boldsymbol{J}_k^2}{(J_k - \boldsymbol{J}_k)^2}\Delta\left(\frac{1}{nu}\right)_k = 0; \tag{324}$$

and, exactly, as in § 292, we may employ here also formula (155) of Chap. VIII, viz.:

$$h_1\boldsymbol{h}_1(J_1 - \boldsymbol{J}_1) = h_k\boldsymbol{h}_k(J_k - \boldsymbol{J}_k),$$

whereby formula (324) may evidently be put in the following form:

$$\sum_{k=1}^{k=m} \left(\frac{h_k}{h_1}\right)^2 \left(\frac{\boldsymbol{h}_k}{\boldsymbol{h}_1}\right)^2 \boldsymbol{J}_k^2 \cdot \Delta\left(\frac{1}{nu}\right)_k = 0; \tag{325}$$

a formula of great simplicity and convenience, since, exactly as in the case of the formula for the Longitudinal Aberration along the axis, it enables us to see distinctly the effect of each single refraction, and thereby to ascertain the factors which have the most influence on the astigmatism.

303. Curvature of the Stigmatic Image. If the astigmatism is abolished, we obtain for the curvature of the image:

$$\frac{1}{R'_m} = n'_m \sum_{k=1}^{k=m} \frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k; \tag{326}$$

whence it is seen that *the curvature of the stigmatic image is independent of the position of the stop.*

This is the so-called "PETZVAL *Formula*", which was published, unfortunately without proof, by JOSEPH PETZVAL, in his celebrated paper, *Bericht ueber die Ergebnisse einiger dioptrischer Untersuchungen* (Pesth, 1843. Verlag von C. A. HARTLEBEN).[1] The formula is applicable only in case the image is stigmatic, and although PETZVAL does not expressly even allude to this pre-requisite condition, it is hardly to be supposed that he was ignorant of it.[2]

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[1] See also J. PETZVAL: Bericht ueber optische Untersuchungen. *Sitzungsber. der math.-naturwiss. Cl. der kaiserl. Akad. der Wissenschaften*, Wien, xxvi (1857), 50–75, 92–105, 129–145. The PETZVAL-formula is given here also without proof, on p. 95, but the remainder of this contribution is chiefly devoted to a discussion of this equation, which is shown to hold for a number of simple special cases.

[2] In regard to this question, see especially M. VON ROHR's *Theorie und Geschichte des photographischen Objektivs* (Berlin, 1899), p. 270. L. SEIDEL, in his paper, "Zur Dioptrik.

The conditions that an optical apparatus consisting of a centered system of spherical refracting surfaces, provided with a narrow stop to limit the widths of the bundles of effective rays, shall, as a first approximation, produce a stigmatic plane image of a plane object, are the following:

$$\sum_{k=1}^{k=m} \frac{1}{r_k} \Delta\left(\frac{1}{n}\right)_k = 0, \quad \sum_{k=1}^{k=m} \left(\frac{h_k}{h_1}\right)^2 \left(\frac{\boldsymbol{h}_k}{\boldsymbol{h}_1}\right)^2 \boldsymbol{J}_k^2 \cdot \Delta\left(\frac{1}{nu}\right)_k = 0. \tag{327}$$

304. Formulæ for the Magnitudes of the Aberration-Lines.

Assuming that we have a plane object, we obtain, by means of formulæ (314) and (322), the following expressions for the magnitudes of the aberration-lines, in the GAUSSian image-plane σ'_m, of the meridian and sagittal rays of a narrow astigmatic bundle of image-rays:

$$\left.\begin{aligned} P'_m U'_m &= -\frac{n'_m {y'_m}^2 \cdot d\lambda'_m}{2} \left\{ \sum_{k=1}^{k=m} \frac{1}{r_k} \Delta\left(\frac{1}{n}\right)_k - 3 \sum_{k=1}^{k=m} \frac{\boldsymbol{J}_k^2}{(J_k - \boldsymbol{J}_k)^2} \Delta\left(\frac{1}{nu}\right)_k \right\} \\ P'_m V'_m &= -\frac{n'_m {y'_m}^2 \cdot d\bar{\lambda}'_m}{2} \left\{ \sum_{k=1}^{k=m} \frac{1}{r_k} \Delta\left(\frac{1}{n}\right)_k - \sum_{k=1}^{k=m} \frac{\boldsymbol{J}_k^2}{(J_k - \boldsymbol{J}_k)^2} \Delta\left(\frac{1}{nu}\right)_k \right\}. \end{aligned}\right\} \tag{328}$$

In order to be able to compare these formulæ with SEIDEL's general formulæ, to be derived hereafter, we shall transform them by the aid of several approximate relations, which may be introduced here without neglecting the magnitudes of the 3rd order of smallness.

Since the I. and II. image-points of the infinitely narrow bundle of image rays, which are designated by S'_m and $\bar{S}'_m$, respectively, are here supposed to be not very far from the axial image-point M'_m, we may put

$$d\lambda'_m = \angle \boldsymbol{L}'_m S'_m H'_m = \angle \boldsymbol{L}'_m M'_m H'_m, \quad d\bar{\lambda}'_m = \angle \boldsymbol{L}'_m \bar{S}'_m G'_m = \angle \boldsymbol{L}'_m M'_m G'_m;$$

and, thus, without neglecting magnitudes of the 3rd order, we may write here the following formulæ:

$$d\lambda'_m = \frac{\boldsymbol{y}'_m}{\boldsymbol{u}'_m - u'_m}, \quad d\bar{\lambda}'_m = \frac{\boldsymbol{z}'_m}{\boldsymbol{u}'_m - u'_m}, \tag{329}$$

where the symbols $\boldsymbol{u}$, $\boldsymbol{y}$, $\boldsymbol{z}$ have the same meanings as in § 255. Moreover, in connection with these equations, we may employ here the Law of ROBERT SMITH (§ 194), and write, therefore, according to

Ueber die Entwicklung der Glieder 3ter Ordnung, welche" u. s. w., *Astr. Nachr.*, xliii (1856), Nos. 1027, 1028 and 1029, pointed out (see No. 1029) that the PETZVAL-Equation implied the abolition of astigmatism; as was remarked, likewise, by H. ZINKEN gen. SOMMER in a paper entitled, Ueber die Berechnung der Bildkruemmung bei optischen Apparaten, POGG. *Ann.*, cxxii. (1864), 563-574.

formulæ (92) of Chap. V:

$$\frac{n'_m \boldsymbol{h}_m \boldsymbol{y}'_m}{\boldsymbol{u}'_m} = \frac{n_1 \boldsymbol{h}_1 \boldsymbol{y}_1}{\boldsymbol{u}_1}, \quad \frac{n'_m \boldsymbol{h}_m \boldsymbol{z}'_m}{\boldsymbol{u}'_m} = \frac{n_1 \boldsymbol{h}_1 \boldsymbol{z}_1}{\boldsymbol{u}_1}.$$

Finally, also, by formula (155) of Chapter VIII, we have:

$$n'_m h_m \boldsymbol{h}_m \left(\frac{1}{\boldsymbol{u}'_m} - \frac{1}{u'_m} \right) = n_1 h_1 \boldsymbol{h}_1 \left(\frac{1}{\boldsymbol{u}_1} - \frac{1}{u_1} \right);$$

and, thus, we obtain:

$$d\lambda'_m = \frac{1}{u'_m} \cdot \frac{h_m}{h_1} \cdot \frac{u_1}{\boldsymbol{u}_1 - u_1} \boldsymbol{y}_1, \quad d\overline{\lambda}'_m = \frac{1}{u'_m} \cdot \frac{h_m}{h_1} \cdot \frac{u_1}{\boldsymbol{u}_1 - u_1} \boldsymbol{z}_1. \qquad (330)$$

If, therefore, employing the relation:

$$h_1 \boldsymbol{h}_1 (J_1 - \boldsymbol{J}_1) = h_k \boldsymbol{h}_k (J_k - \boldsymbol{J}_k),$$

we take from under the two summation-signs in each of the formulæ (328) the term

$$\frac{1}{(J_1 - \boldsymbol{J}_1)^2} = \frac{u_1^2 \boldsymbol{u}_1^2}{n_1^2 (u_1 - \boldsymbol{u}_1)^2},$$

and if we multiply both sides of these equations by n'_m / u'_m, at the same time eliminating $d\lambda'_m$ and $d\overline{\lambda}'_m$ on the right-hand sides of the two equations by means of the formulæ (330), and also expressing y'_m in terms of y_1 by means of SMITH's Formula:

$$\frac{n'_m h_m y'_m}{u'_m} = \frac{n_1 h_1 y_1}{u_1},$$

we obtain, finally, the formulæ (328) in the following forms:

$$\left.\begin{aligned} \frac{n'_m}{u'_m} P'_m U'_m &= \frac{1}{2} \cdot \frac{h_1}{h_m} \cdot \frac{u_1 \boldsymbol{u}_1^2}{(u_1 - \boldsymbol{u}_1)^3} y_1^2 \boldsymbol{y}_1 \cdot S, \\ \frac{n'_m}{u'_m} P'_m V'_m &= \frac{1}{2} \cdot \frac{h_1}{h_m} \cdot \frac{u_1 \boldsymbol{u}_1^2}{(u_1 - \boldsymbol{u}_1)^3} y_1^2 \boldsymbol{z}_1 \cdot \overline{S}, \end{aligned}\right\} \qquad (331)$$

where, for brevity, we put:

$$\left.\begin{aligned} S &= \sum_{k=1}^{k=m} \left(\frac{h_k}{h_1} \right)^2 \left(\frac{\boldsymbol{h}_k}{\boldsymbol{h}_1} \right)^2 \left\{ (J_k - \boldsymbol{J}_k)^2 \frac{1}{r_k} \Delta \left(\frac{1}{n} \right)_k - 3 \boldsymbol{J}_k^2 \cdot \Delta \left(\frac{1}{nu} \right)_k \right\}, \\ \overline{S} &= \sum_{k=1}^{k=m} \left(\frac{h_k}{h_1} \right)^2 \left(\frac{\boldsymbol{h}_k}{\boldsymbol{h}_1} \right)^2 \left\{ (J_k - \boldsymbol{J}_k)^2 \frac{1}{r_k} \Delta \left(\frac{1}{n} \right)_k - \boldsymbol{J}_k^2 \cdot \Delta \left(\frac{1}{nu} \right)_k \right\}. \end{aligned}\right\} \qquad (332)$$

ART. 98. SPECIAL CASES.

305. Case of a Single Spherical Refracting Surface.

The relations between the curvatures of the image-surfaces and the curvature of the object-surface are given, in the case of a single spherical refracting surface, by formulæ (321). For a plane object ($R = \infty$), these formulæ may be written as follows:

$$\frac{1}{R'} = -\frac{n'-n}{nr} - 3n'\frac{u^2(\mathbf{u}-r)^2}{r^2(\mathbf{u}-u)^2}\left(\frac{1}{n'u'}-\frac{1}{nu}\right),$$

$$\frac{1}{\bar{R}'} = -\frac{n'-n}{nr} - n'\frac{u^2(\mathbf{u}-r)^2}{r^2(\mathbf{u}-u)^2}\left(\frac{1}{n'u'}-\frac{1}{nu}\right).$$

In each of the three following cases we shall have a *stigmatic image* whose curvature will be:

$$\frac{1}{R'} = \frac{1}{\bar{R}'} = -\frac{n'-n}{nr}:$$

(1) When $u = 0$, which is a case that possesses no practical interest;

(2) When $nu = n'u'$, in which case the conjugate axial points M, M' coincide with the aplanatic points Z, Z' of the spherical refracting surface (§ 207). Under these circumstances, it does not matter where the stop is placed. And, finally:

(3) When $\mathbf{u} = r$; that is, when the centre O (or $\boldsymbol{L}$) of the stop coincides with the centre C of the spherical surface. In this case the chief rays proceed in straight lines from the points in the plane object to the conjugate points in the image.

If the stop-centre is situated at the vertex A of the spherical surface ($\mathbf{u} = 0$), the curvatures of the two image-surfaces are:

$$\frac{1}{R'} = -\frac{n'-n}{nr} - 3n'\left(\frac{1}{n'u'}-\frac{1}{nu}\right),$$

$$\frac{1}{\bar{R}'} = -\frac{n'-n}{nr} - n'\left(\frac{1}{n'u'}-\frac{1}{nu}\right).$$

Lastly, in case the object is at infinity ($u = \infty$), the curvatures of the image-surfaces are:

$$\frac{1}{R'} = -\frac{n'-n}{r}\left\{\frac{1}{n}+\frac{3}{n'}\left(\frac{\mathbf{u}-r}{r}\right)^2\right\},$$

$$\frac{1}{\bar{R}'} = -\frac{n'-n}{r}\left\{\frac{1}{n}+\frac{1}{n'}\left(\frac{\mathbf{u}-r}{r}\right)^2\right\}.$$

306. Case of an Infinitely Thin Lens.

For the case of an Infinitely Thin Lens, we can write:

$$\sum_{k=1}^{k=m} \frac{\boldsymbol{J}_k^2}{(J_k-\boldsymbol{J}_k)^2}\Delta\left(\frac{1}{nu}\right)_k = \frac{1}{(J_1-\boldsymbol{J}_1)^2}\sum_{k=1}^{k=2}\boldsymbol{J}_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k;$$

wherein, employing the same special Lens-Notation as in § 268, we may put:

$$J_1 = c - x, \quad \boldsymbol{J}_1 = c - \boldsymbol{x}, \quad \boldsymbol{J}_2 = \frac{(n-1)(c-\boldsymbol{x}) - n\varphi}{n-1}.$$

Introducing these symbols, we shall find:

$$\frac{1}{(J_1-\boldsymbol{J}_1)^2}\sum_{k=1}^{k=2}\boldsymbol{J}_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k = \frac{\varphi}{(x-\boldsymbol{x})^2}\cdot U,$$

where the symbol U is used as an abbreviation for the following function:

$$U = \frac{n+2}{n}c^2 - \left\{\frac{2(n+1)}{n}(x+\boldsymbol{x}) + \frac{2n+1}{n-1}\varphi\right\}c$$
$$+ \frac{n^2}{(n-1)^2}\varphi^2 + \boldsymbol{x}^2 + \frac{2(n+1)}{n}x\boldsymbol{x} + \frac{2n}{n-1}\boldsymbol{x}\varphi + \frac{n+1}{n-1}x\varphi.$$

Thus, the curvatures of the images produced by a Thin Lens will be for the case of a plane object:

$$\frac{1}{R'} = -\varphi\left[\frac{1}{n} + \frac{3U}{(x-\boldsymbol{x})^2}\right], \quad \frac{1}{\overline{R}'} = -\varphi\left[\frac{1}{n} + \frac{U}{(x-\boldsymbol{x})^2}\right].$$

(1) When the centre of the stop coincides with the centre of the Infinitely Thin Lens ($\boldsymbol{x} = \infty$), we find $U/(x-\boldsymbol{x})^2 = 1$, and hence:

$$\frac{1}{R'} = -\frac{3n+1}{n}\varphi, \quad \frac{1}{\overline{R}'} = -\frac{n+1}{n}\varphi,$$

whence it appears, that under such circumstances, the curvatures of the image-surfaces are independent of the distance of the object from the Lens, and the chief rays proceed in straight lines from the points of the object to the conjugate points of the image. The curvatures, in fact, depend only on the focal length of the Lens and the value of the relative index of refraction (n), but not on the form of the Lens. If $n = 3/2$, we find $R' = -3f/11$ and $\overline{R}' = -3f/5$.

(2) The condition of the stigmatic image is

$$U = 0,$$

in which case the curvature of the image is:

$$\frac{1}{R'} = \frac{1}{\overline{R}'} = -\frac{\varphi}{n}.$$

(3) In the special case of a *System of Infinitely Thin Lenses in Contact, with the centre of the stop situated at the common vertex of the Lenses* ($\mathbf{x} = \infty$ for each Lens), the function $U/(x - \mathbf{x})^2$ is equal to unity for each Lens, and, hence, the curvatures of the image-surfaces will be:

$$\frac{1}{R'} = -\Sigma\frac{n+3}{n}\varphi, \quad \frac{1}{\overline{R}'} = -\Sigma\frac{n+1}{n}\varphi.$$

Accordingly, the condition of a flat stigmatic image in the neighbourhood of the axis ($R' = \overline{R}' = \infty$) requires that we shall have in this case:

$$\Sigma\varphi = 0,$$

which means that the combination of Lenses must act like a slab with plane parallel faces.

VI. Aberrations in the Case of Imagery by Bundles of Rays of Finite Slopes and of Small Finite Apertures.

ART. 99. COMA.

307. The Coma-Aberrations in General. Heretofore, in the investigations of the aberrations in the case of object-points not on the optical axis, it has been assumed always that the rays were limited by a stop of infinitely narrow dimensions. In actual optical construction this condition can never, of course, be absolutely realized; nor, indeed, in the case of certain optical instruments is it necessary that it should be, so long as the diameter of the stop is relatively very small. On the other hand, when it is required to produce the image of a fairly extensive object by means of somewhat wide-angled bundles of rays, as, for example, is often the case with photographic objectives, the diameter of the stop will enter as a chief factor in the study of the aberrations of the rays. Thus, whereas we saw (§ 304) that the aberration-lines in the case of infinitely narrow bundles of astigmatic rays were proportional to the first powers of the aperture-co-ordinates $\mathbf{y}_1$, $\mathbf{z}_1$ (§ 259), we must now advance a step farther, and assume here that the aperture is so wide that we will not be justified in leaving out of account the *second powers* and products of these co-ordinates.

A bundle of rays of finite aperture, emanating from a point outside the optical axis, may show aberrations of a character similar to the spherical aberration along the axis of a direct bundle of rays (see § 208 and § 260). These aberrations will be manifest in both the meridian and sagittal sections of the bundles of rays, but here a very important difference is to be remarked, as will now be explained.

The rays of the sagittal section are symmetrically situated on opposite sides of the meridian plane, so that the point of intersection of every pair of symmetrical rays in this section will lie in the plane of the meridian section, for example, as shown in Fig. 144. But in the meridian section (Fig. 145) it is obvious that, in general, there will be no symmetry at all. The chief ray of the bundle will depend on the position on the optical axis of the centre of the stop. If the rays are received on a screen placed perpendicularly to the optical axis, and if a straight radial line is drawn in the plane of the screen through the point where the screen meets the optical axis and intersecting the light-pattern on the screen, there will be no symmetry in the pencil of rays which meet the screen at points lying along this line: whereas in the case of a pencil of rays which meet the screen at points lying along a line at right angles to this radial line there will be symmetry. The light-pattern on the screen sometimes presents the appearance of a comet, with its tail turned either towards or away from the optical axis; which accounts for the origin of the name "*coma*".[1]

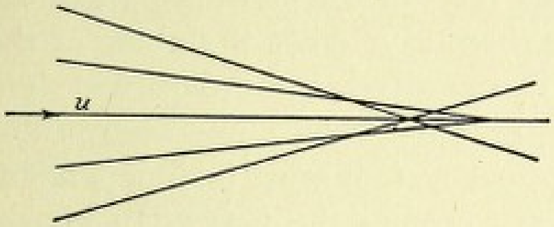

FIG. 144.

SYMMETRICAL CHARACTER OF THE ABERRATIONS OF THE RAYS OF THE SAGITTAL SECTION OF AN INCLINED BUNDLE OF RAYS OF FINITE APERTURE. The chief ray of the bundle is the ray marked *u*. The plane of the meridian section is the plane containing *u* which is perpendicular to the plane of the paper.

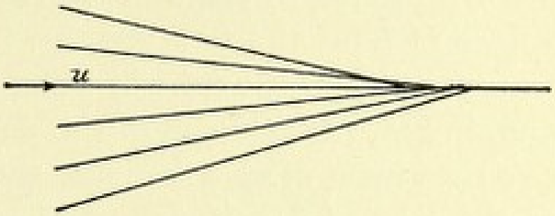

FIG. 145.

UNSYMMETRICAL CHARACTER OF THE ABERRATIONS OF THE RAYS OF THE MERIDIAN SECTION OF AN INCLINED BUNDLE OF FINITE APERTURE. The chief ray of the bundle is the ray marked *u*. This is the ray which at some stage of its progress goes through the centre of the stop. The rays of the meridian section are in general not symmetrical with respect to the chief ray.

So far as the meridian rays are concerned, we have to ascertain only the y-aberrations (§ 256), because, by the Laws of Refraction, the paths of these rays throughout their progress through the centered system of spherical surfaces will lie wholly in the meridian or xy-plane, so that their z-aberrations will all be equal to zero. But if the path of the ray lies outside of this plane, we shall have to determine its z-aberration as well as its y-aberration. In general, the y-aberration of any ray of a bundle of rays may be considered as compounded by summation of the y-aberrations of the meridian rays and of the sagittal rays.

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[1] Some excellent drawings exhibiting these appearances are to be found in H. DENNIS TAYLOR'S *A System of Applied Optics* (London, 1906). This work contains several chapters in regard to Coma. Especially interesting in the diagrams are the drawings by Prof. S. P. THOMPSON, Plate XVI.

Evidently, in the case of a pair of rays of the sagittal section which are symmetrically situated on opposite sides of the meridian plane the values of $\mathfrak{z}$ for the two points where these rays cross the plane of the Entrance-Pupil (§ 257) will be equal in magnitude but opposite in sign; and, hence, the position of the point in the meridian plane where these two rays meet after traversing the optical system must be independent of the sign of $\mathfrak{z}$. If one of these rays crosses the GAUSSian image-plane σ' in a point whose co-ordinates are given by η', ζ', the other ray will cross this plane at the point η', $-\zeta'$; and, hence, η', and, therefore, also, the y-aberration $\delta y'$, will be independent of the sign of $\mathfrak{z}$ (or of $\boldsymbol{z}$). Accordingly, in the series-development of the y-aberration of a ray belonging to the sagittal section, there can be no term involving the odd powers of the co-ordinate $\boldsymbol{z}$; and, as we propose to consider here no terms involving powers of the aperture-co-ordinates $\boldsymbol{y}$, $\boldsymbol{z}$ above the second, obviously, the only terms that can occur in the series-developments of the y-aberrations will be terms involving $\boldsymbol{y}^2$ and $\boldsymbol{z}^2$ (*cf.* § 259).

When we come to consider the z-aberration, we find that the case is exactly opposite to that of the y-aberration; for, since the aberration $\delta z'$ changes sign along with change of the sign of $\boldsymbol{z}$, the series-development of $\delta z'$ can contain terms which involve only the odd powers of the aperture-co-ordinate $\boldsymbol{z}$; so that, within the limits prescribed for the present investigation, the only term in the series-development of $\delta z'$ will be the term involving the product $\boldsymbol{yz}$.

The complete investigation of these so-called "Comatic" Aberrations is quite laborious. We shall consider here only the y-aberration of a ray lying in the meridian plane, the series-development of which will contain only the term involving $\boldsymbol{y}^2$. The reader who is interested in the investigation of the y- and z-aberrations of a ray belonging to the sagittal section of the bundle of rays will find the whole subject exhaustively treated by KOENIG and VON ROHR.[1]

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[1] A. KOENIG und M. VON ROHR: Die Theorie der sphaerischen Aberrationen: Chapter V of Volume I of *Die Theorie der optischen Instrumente* (Berlin, 1904), edited by M. VON ROHR; see pages 265–292.

308. The Lack of Symmetry of a Pencil of Meridian Rays of Finite Aperture. In the special case when the chief ray of the bundle coincides with the optical axis, there will be symmetry in the pencil of meridian rays, as is exhibited in the diagram (Fig. 146), which represents the meridian section of an optical system consisting of a single spherical surface. The centre of the stop is supposed here to be situated at the vertex A of the spherical surface, and the object is infinitely distant, so that the object-rays emanating from any point of the object are parallel.

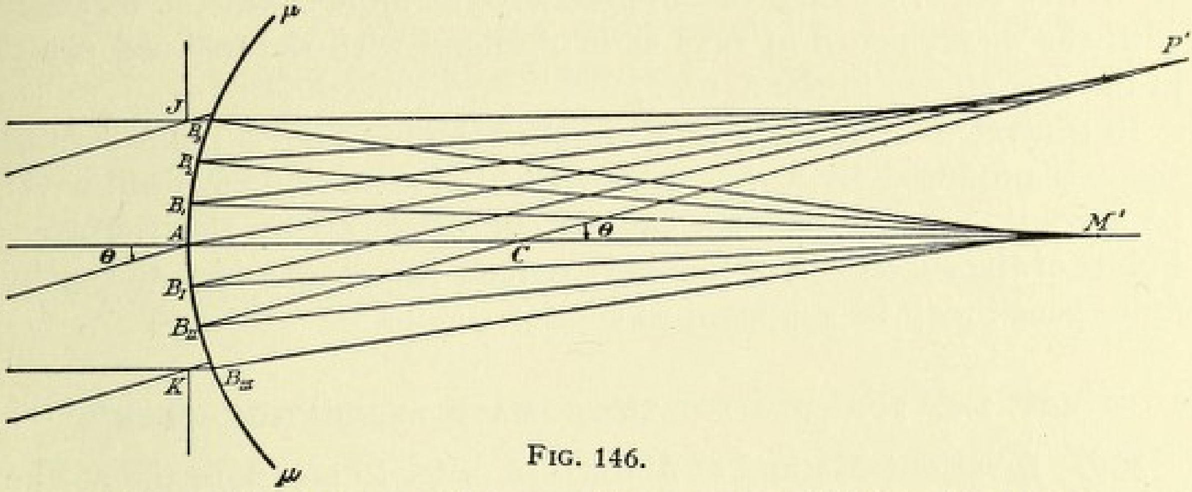

FIG. 146.

LACK OF SYMMETRY OF A PENCIL OF MERIDIAN RAYS OF FINITE APERTURE.

If the object-point is not on the optical axis, the chief ray of the bundle of object-rays will be inclined to the optical axis at some angle, say θ; and it is evident by an inspection of the figure that the meridian rays of this bundle produce an effect quite different from that which we perceived in the case of a bundle of rays emanating from an axial object-point. In the first place, the chief ray is no longer the ray which meets the spherical refracting surface normally; and, generally, this will always be a distinguishing peculiarity of such a pencil of meridian rays, so that the chief ray will not (except for certain special positions of the stop) go through the centre C of the spherical surface; and even in case it did happen to pass through the centre of one surface, it would not pass through the centre of the next following surface of a centered system of spherical surfaces. The straight line drawn through C parallel to the incident rays (which may, or may not, be the path of an actual ray of the pencil), is in a certain sense, an axis of symmetry for the refracted rays in the same way as the optical axis is an axis of symmetry for the direct pencil of refracted meridian rays: but, since the stop cuts off more rays on one side of this line than it does on the other, the actual pencil of refracted rays is not symmetrical with respect to this straight line of slope-angle θ drawn through the centre C of the spherical surface. Almost exactly the same unsymmetrical effect would be obtained with the direct pencil of meridian rays if the centre of the stop, instead of lying on the optical axis, were situated above or below the axis. In fact, if the diameter of the stop is increased in the ratio $1 : \cos\theta$, and if at the same time the centre of the stop is displaced at right angles to the axis by an amount equal to $r \cdot \sin\theta$, we shall obtain precisely the same character of effect with the direct pencil of rays as is obtained with the inclined pencil in the case shown in the figure.

In general, therefore, we can say that the image of a point outside the axis produced by a wide-angle pencil of meridian rays will never be a point, but a piece of a caustic curve formed by the I. Image-Points of the succession of infinitely narrow pencils of which the entire finite pencil may be supposed to consist.

ART. 100. FORMULÆ FOR THE COMATIC ABERRATION-LINES.

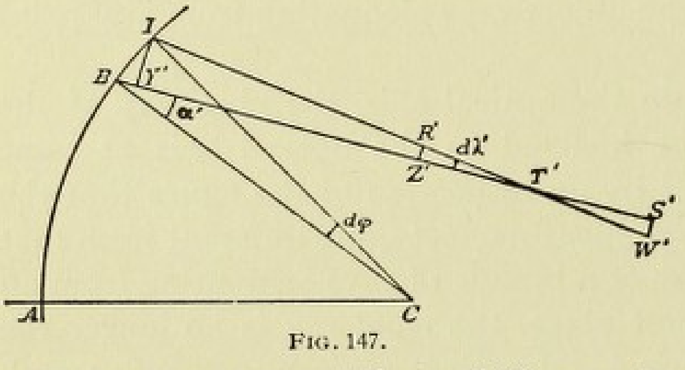

FIG. 147.

COMATIC ABERRATIONS $S'T'$ AND $S'W'$ OF AN INFINITELY NARROW PENCIL OF MERIDIAN RAYS.

$B'S' = s'$, $IR' = s' + ds'$, $S'Z' = dq'$, $S'W' = dw'$,

$\angle BCA = \phi$, $\angle ICB = d\phi$, $\angle CBS' = \alpha'$,

$\angle CIR' = \alpha' + d\alpha'$, $\angle BT'I = d\lambda'$, arc $BI = j$.

309. Invariant-Method of Abbe. In order to ascertain the nature of an element of this caustic curve, we shall employ the method of ABBE, as given both by CZAPSKI[1] and by KOENIG and VON ROHR.[2] In Fig. 147 the plane of the paper represents the plane of the meridian section of the bundle of rays; and the letters C and A designate the centre and vertex, respectively, of one of the surfaces of the centered system of spherical surfaces ($AC = r$). The points B and I lying in the meridian section of the surface are two incidence-points very near together. BS' and IR' represent the paths of two refracted meridian rays corresponding to two incident meridian rays SB and RI, respectively (which latter, however, are not shown in the diagram). The points S' and R' designate the positions on BS' and IR' of the I. Image-

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[1] S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), pages 115–118.

[2] A. KOENIG und M. VON ROHR: Die Theorie der sphaerischen Aberrationen: Chapter V of Vol. I of *Die Theorie der optischen Instrumente* (Berlin, 1904), pages 270–273.

Points corresponding to the points S and R on the incident rays SB and RI, respectively; the actual positions of S' and R' being, of course, dependent on the positions of S and R, respectively. The angles of incidence at B and I are supposed to differ from each other by an infinitely small magnitude of the 1st order; and, consequently, the points designated by S' and R' are two infinitely near points on the caustic curve of the meridian rays. The point of intersection of the refracted rays BS' and IR' is designated in the figure by T'; and we may consider $S'T'$ as the longitudinal aberration along BS' of the infinitely narrow pencil of meridian rays which are refracted at the points lying in the arc BI.

The following symbols may be conveniently employed:

$$\angle CBS' = \alpha', \quad \angle CIT' = \alpha' + d\alpha', \quad \angle ICB = d\varphi, \quad \angle BT'I = d\lambda',$$
$$BS' = s', \quad IR' = s' + ds'.$$

With T' as centre and with radii equal to $T'I$ and $T'R'$, describe two circular arcs meeting BS' in the points designated in the figure by Y' and Z', respectively. The variation $ds' = IR' - BS'$ may be considered as consisting of a displacement $S'Z'$ together with a displacement $Z'R'$. The latter may be said, in a certain sense, to be due to the variation of the point of incidence from B to I; whereas the former is the displacement depending on the angle $d\lambda'$ between the refracted rays leaving B and I. We shall try now to obtain an expression for the magnitude of the component

$$S'Z' = dq'$$

of the total variation; because, since BT' and IT' are tangents to the caustic curve at the two infinitely near points S' and R', and since, therefore, the lengths $S'T'$ and $T'R'$ can differ from each other only by an infinitesimal magnitude of an order higher than either of them, so that we can put

$$S'T' = T'R' = T'Z',$$

the magnitude denoted by dq' is equal to twice the aberration $S'T'$.

Incidentally, also, we may observe that since (neglecting infinitesimals of the 2nd order) $S'T' + T'R' = dq' =$ the length of the element of the caustic, the radius of curvature of the caustic at S' is equal to $dq'/d\lambda'$.

Throughout this present investigation we shall retain magnitudes of the order $d\varphi$. Hence, provided we neglect only small magnitudes of an order higher than the 1st, we shall obtain from the figure the following useful relations:

$$IY' = -r \cdot \cos \alpha' \cdot d\varphi;$$

also,

$$IY' = Y'T' \cdot d\lambda' = s' \cdot d\lambda';$$

and, hence,

$$\frac{d\lambda'}{d\varphi} = -\frac{r \cdot \cos \alpha'}{s'}.$$

Moreover,

$$d\alpha' = d\lambda' + d\varphi, \quad BY' = r \cdot \sin \alpha' \cdot d\varphi.$$

Now

$$ds' = IR' - BS' = Y'Z' - BS' = S'Z' - BY';$$

that is,

$$ds' = dq' - BY';$$

and, since

$$\frac{dq'}{d\varphi} = \frac{dq'}{d\lambda'} \frac{d\lambda'}{d\varphi} = -\frac{r \cdot \cos \alpha'}{s'} \frac{dq'}{d\lambda'},$$

we obtain:

$$\frac{ds'}{d\varphi} = -\frac{r \cdot \cos \alpha'}{s'} \frac{dq'}{d\lambda'} - r \cdot \sin \alpha'.$$

In order to obtain now an expression for $ds'/d\varphi$, we must employ the Law of Refraction, which Abbe does by introducing here the invariant-function of astigmatic refraction (§ 299), viz.:

$$Q = n \cdot \cos \alpha \left(\frac{1}{r} - \frac{\cos \alpha}{s} \right) = n' \cdot \cos \alpha' \left(\frac{1}{r} - \frac{\cos \alpha'}{s'} \right).$$

Accordingly, differentiating Q with respect to φ, we obtain:

$$\frac{dQ}{d\varphi} = -n' \cdot \sin \alpha' \left(\frac{1}{r} - \frac{2 \cos \alpha'}{s'} \right) \frac{d\alpha'}{d\varphi} + \frac{n' \cdot \cos^2 \alpha'}{s'^2} \frac{ds'}{d\varphi};$$

wherein let us put:

$$\frac{d\alpha'}{d\varphi} = 1 + \frac{d\lambda'}{d\varphi} = 1 - \frac{r \cdot \cos \alpha'}{s'},$$

and let us, also, substitute for $ds'/d\varphi$ the expression which we obtained above in terms of $dq'/d\lambda'$. Thus, after several simple transformations, we derive the following equation:

$$\frac{1}{r} \frac{dQ}{d\varphi} = -\frac{K}{r^2} + \frac{3K \cdot Q}{n's'} - \frac{n' \cdot \cos^3 \alpha'}{s'^3} \frac{dq'}{d\lambda'},$$

where

$$K = n \cdot \sin \alpha = n' \cdot \sin \alpha'$$

denotes the so-called "optical invariant".

The above formula has been derived for the rays after refraction at the spherical surface here considered; but it is obvious that we shall obtain in the same way a precisely similar relation connecting the corresponding magnitudes before refraction, viz.:

$$\frac{1}{r}\frac{dQ}{d\varphi} = -\frac{K}{r^2} + \frac{3K \cdot Q}{ns} - \frac{n \cdot \cos^3 \alpha}{s^3}\frac{dq}{d\lambda}.$$

Combining, therefore, these two formulæ, and using ABBE's difference-notation, we obtain:

$$\Delta\left(\frac{n \cdot \cos^3 \alpha}{s^3}\frac{dq}{d\lambda}\right) = 3K \cdot Q \cdot \Delta\left(\frac{1}{ns}\right). \qquad (333)$$

Thus, knowing the values of the magnitudes denoted by α, s, dq and $d\lambda$, which relate to the narrow pencil of meridian rays before refraction at the spherical surface, we can calculate the magnitudes denoted by α' and s', and determine, by means of the formula just obtained, the magnitude of the ratio $dq'/d\lambda'$, which relates to the pencil of rays after refraction.

310. Instead of a single spherical surface, let us suppose now that the optical system consists of m spherical surfaces with their centres ranged all along one straight line. Introducing in our notation the surface-subscripts, we must write:

$$dq_{k+1} = dq'_k, \quad d\lambda_{k+1} = d\lambda'_k;$$

and, hence, for a centered system of m spherical surfaces, we obtain by formula (333) the following recurrent formula:

$$\frac{n'_m \cdot \cos^3 \alpha'_m}{s'^3_m}\frac{dq'_m}{d\lambda'_m} = n_1\frac{dq_1}{d\lambda_1}\left(\frac{s'_1 \cdot s'_2 \cdots s'_{m-1}}{s_1 \cdot s_2 \cdots s_m}\right)^3\left(\frac{\cos \alpha_1 \cdot \cos \alpha_2 \cdots \cos \alpha_m}{\cos \alpha'_1 \cdot \cos \alpha'_2 \cdots \cos \alpha'_{m-1}}\right)^3$$
$$+3\sum_{k=1}^{k=m}\left(\frac{s'_k \cdot s'_{k+1} \cdots s'_{m-1}}{s_{k+1} \cdot s_{k+2} \cdots s_m}\right)^3\left(\frac{\cos \alpha_{k+1} \cdot \cos \alpha_{k+2} \cdots \cos \alpha_m}{\cos \alpha'_k \cdot \cos \alpha'_{k+1} \cdots \cos \alpha'_{m-1}}\right)^3 K_k \cdot Q_k \cdot \Delta\left(\frac{1}{ns}\right)_k.$$

If we write

$$\text{arc } B_k I_k = j_k,$$

then

$$j_k = -\frac{s_k \cdot d\lambda_k}{\cos \alpha_k} = -\frac{s'_k \cdot d\lambda'_k}{\cos \alpha'_k};$$

and, hence:

$$\frac{j_k}{j_{k+1}} = \frac{s'_k}{s_{k+1}} \cdot \frac{\cos \alpha_{k+1}}{\cos \alpha'_k}.$$

Therefore,

$$\frac{j_1}{j_m} = \frac{j_1 \cdot j_2 \cdots j_{m-1}}{j_2 \cdot j_3 \cdots j_m} = \frac{s_1' \cdot s_2' \cdots s_{m-1}'}{s_2 \cdot s_3 \cdots s_m} \cdot \frac{\cos \alpha_2 \cdot \cos \alpha_3 \cdots \cos \alpha_m}{\cos \alpha_1' \cdot \cos \alpha_2' \cdots \cos \alpha_{m-1}'}.$$

Thus, the recurrent formula obtained above may be put in the following form:

$$\frac{n_m' \cdot \cos^3 \alpha_m'}{s_m'^3} \cdot \frac{dq_m'}{d\lambda_m'} = n_1 \left(\frac{j_1}{j_m}\right)^3 \frac{\cos^3 \alpha_1}{s_1^3} \cdot \frac{dq_1}{d\lambda_1} + 3 \left(\frac{j_1}{j_m}\right)^3 \sum_{k=1}^{k=m} \left(\frac{j_k}{j_1}\right)^3 K_k \cdot Q_k \cdot \Delta\left(\frac{1}{ns}\right)_k. \quad (334)$$

If, as is usual, the bundle of object-rays is homocentric ($dq_1 = 0$), the formula above may be written as follows:

$$\frac{n_m' \cdot \cos^3 \alpha_m'}{s_m'^3} \frac{dq_m'}{d\lambda_m'} = 3 \left(\frac{j_1}{j_m}\right)^3 \sum_{k=1}^{k=m} \left(\frac{j_k}{j_1}\right)^3 K_k \cdot Q_k \cdot \Delta\left(\frac{1}{ns}\right)_k. \quad (335)$$

311. If a screen is placed perpendicularly to BS' at S', the pencil of meridian rays will intersect this screen in the aberration-line

$$S'W' = dw';$$

where

$$dw' = -\frac{dq' \cdot d\lambda'}{2}$$

denotes a magnitude of the second order of smallness as compared with $d\lambda'$. Hence,

$$\frac{n' \cdot \cos^3 \alpha'}{s'^3} \frac{dq'}{d\lambda'} = \frac{2}{j^3} n' \cdot dw' \cdot d\lambda'.$$

Accordingly, by formula (333), for the kth spherical surface we have:

$$\Delta(n \cdot dw \cdot d\lambda)_k = \tfrac{3}{2} j_k^3 \cdot Q_k \cdot K_k \cdot \Delta\left(\frac{1}{ns}\right)_k.$$

The product $n_k' \cdot d\lambda_k'$ is the so-called "numerical aperture" (*cf.* § 364) of the pencil of rays after refraction at the kth spherical surface, and dw_k' here is analogous to the Lateral Aberration in the case of a direct bundle of rays (see § 262 and § 266).

If we give k in succession all integral values from $k = 1$ to $k = m$, and put $dw_1 = 0$, we obtain, by addition:

$$n_m' \cdot dw_m' \cdot d\lambda_m' = \frac{3}{2} \sum_{k=1}^{k=m} j_k^3 \cdot Q_k \cdot K_k \cdot \Delta\left(\frac{1}{ns}\right)_k;$$

and if here we substitute:

$$d\lambda'_m = -\frac{j_m \cdot \cos \boldsymbol{\alpha}'_m}{s'_m} \quad \text{and} \quad j_1^2 = \left(\frac{s_1 \cdot d\lambda_1}{\cos \boldsymbol{\alpha}_1}\right)^2,$$

we can write finally:

$$n'_m \frac{dw'_m}{s'_m} = -\frac{3}{2} \cdot \frac{j_1}{j_m} (s_1 \cdot d\lambda_1)^2 \frac{1}{\cos^2 \boldsymbol{\alpha}_1 \cdot \cos \boldsymbol{\alpha}'_m} \sum_{k=1}^{k=m} \left(\frac{j_k}{j_1}\right)^3 Q_k \cdot K_k \cdot \Delta\left(\frac{1}{ns}\right)_k. \quad (336)$$

312. Let us now impose *the condition that the slope-angles* θ, θ' *of the chief rays are small magnitudes of the first order*—of the same order as the aperture-angles λ, λ', as we shall now denote these latter angles, instead of denoting them, as above, by the symbols $d\lambda$, $d\lambda'$. Without neglecting ultimately the magnitudes of the 3rd order of smallness, we may obviously introduce in the above formula (336) the *approximate* values of the magnitudes denoted by the symbols s, j, Q and K. Thus, we may employ here the *approximate relations:*

$$\cos \boldsymbol{\alpha} = 1, \quad \sin \boldsymbol{\alpha} = \boldsymbol{\alpha}, \quad \theta = -\boldsymbol{h}/\boldsymbol{u} \quad \text{and} \quad \phi = \boldsymbol{h}/\boldsymbol{r};$$

where $\boldsymbol{h}$ denotes the incidence-height of the chief ray and $\boldsymbol{u} = \boldsymbol{AM}$. And, hence, since

$$\boldsymbol{\alpha} = \theta + \phi,$$

we can put:

$$\boldsymbol{\alpha} = \frac{\boldsymbol{hJ}}{n},$$

and, therefore:

$$K = n \cdot \sin \boldsymbol{\alpha} = n\boldsymbol{\alpha} = \boldsymbol{hJ}.$$

Moreover, approximately, also:

$$s = u,$$

and, hence, if h denotes the incidence-height of a paraxial object-ray emanating from the axial object-point M_1, we may use here also the following relation:

$$\frac{j_k}{j_1} = \frac{u_2 \cdot u_3 \cdots u_k}{u'_1 \cdot u'_2 \cdots u'_{k-1}} = \frac{h_k}{h_1}.$$

Finally, we may put here $Q = J$. Accordingly, introducing these values in formula (336), and at the same time writing now $\delta w'$ in place of dw', we obtain:

$$n'_m \cdot \frac{\delta w'_m}{u'_m} = -\frac{3}{2} \cdot \frac{h_1}{h_m} \boldsymbol{h}_1 (u_1 \lambda_1)^2 \sum_{k=1}^{k=m} \left(\frac{h_k}{h_1}\right)^3 \frac{\boldsymbol{h}_k}{\boldsymbol{h}_1} J_k \cdot \boldsymbol{J}_k \cdot \Delta\left(\frac{1}{nu}\right)_k. \quad (337)$$

313. If the focussing-screen is placed perpendicularly to the chief image-ray $\boldsymbol{L}'S'$ (Fig. 148), not at the I. Image-Point S', but at some other point S'', the Lateral Aberration of the meridian rays will now be $S''W''$ and from the diagram we obtain:

$$\frac{S''W''}{S'W'} = 1 + \frac{S''S'}{S'T'}.$$

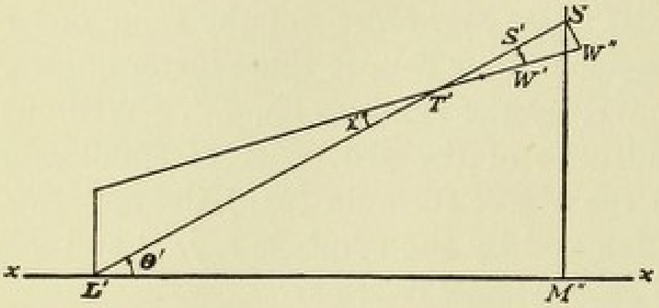

Fig. 148.

Comatic Aberration of Meridian Rays measured in a plane perpendicular to the optical axis.

Now if the screen in its new position has been displaced so little with respect to its first position that $S''S'$ is of the same order of smallness as $S'W'$, that is, if $S''S'$ is of a higher order of smallness than $S'T'$, we may put

$$S''W'' = S'W'.$$

And, moreover, if now the focussing-screen is rotated about an axis perpendicular to the plane of the diagram at S'' until it is perpendicular to the optical axis at the point M'', then, since

$$\angle M''S''W'' = \theta',$$

we have, neglecting magnitudes of orders higher than θ'^3:

$$S''W'' \cdot \cos \angle M''S''W'' = S'W';$$

and, hence, the formula (337) derived above is valid also in case the aberration-line is measured in a transversal plane $M''S''$ at right angles to the optical axis, provided this plane is not too far removed from the I. Image-Point S'. In particular, the formula (337) is valid if the aberration-line $\delta w'$ is measured in the Gaussian Image-Plane σ' perpendicular to the optical axis at M', since the distance from this plane of the I. Image-Point S' is of a higher order of smallness than

$$M'P' = \eta',$$

which is of the same order as θ'.

Finally, if we introduce the approximate relations:

$$\lambda_1 = \frac{y_1}{\boldsymbol{u}_1 - u_1} \quad \text{and} \quad \boldsymbol{h}_1 = \frac{\boldsymbol{u}_1 y_1}{\boldsymbol{u}_1 - u_1},$$

we may write formula (337) in the following form:

$$\frac{n'_m \cdot \delta w'_m}{u'_m} = -\frac{3}{2} \cdot \frac{h_1}{h_m} \cdot \frac{y_1 \boldsymbol{y}_1^2}{(\boldsymbol{u}_1 - u_1)^3} u_1^2 \boldsymbol{u}_1 \sum_{k=1}^{k=m} \left(\frac{h_k}{h_1}\right)^3 \frac{\boldsymbol{h}_k}{\boldsymbol{h}_1} J_k \cdot \boldsymbol{J}_k \cdot \Delta\left(\frac{1}{nu}\right)_k . \quad (338)$$

Thus, on the assumption that the slope-angles of the chief rays are small magnitudes, the condition of the abolition of the so-called "Comatic" Aberration of the meridian rays is:

$$\sum_{k=1}^{k=m} h_k^3 \boldsymbol{h}_k J_k \boldsymbol{J}_k \Delta\left(\frac{1}{nu}\right)_k = 0. \quad (339)$$

Moreover, if the reader will investigate also the y-aberration and the z-aberration of a ray of the sagittal section, as is done, for example, by Messrs. KOENIG and VON ROHR,[1] he will discover that equation (339) is likewise the condition of the abolition of both aberrations of the sagittal rays.

It will be recalled that precisely this same equation was obtained also as the expression of the *Sine-Condition* (formula 304).

ART. 101. SPECIAL CASES.

314. Case of Single Spherical Surface. The condition that the comatic aberration, in the case of a single spherical refracting surface, shall vanish is evidently:

$$J\boldsymbol{J}(1/n'u' - 1/nu) = 0;$$

which will be satisfied in each of the three following cases:

(1) $J = 0$, or $u = u' = r$: that is, when the object and image coincide at the centre of the spherical surface—a case possessing no practical interest;

(2) $\boldsymbol{J} = 0$, or $\boldsymbol{u} = r$: that is, when the stop-centre is situated at the centre of the spherical surface; and

(3) $nu = n'u'$: that is, when the pair of conjugate axial points M, M' are the *aplanatic* pair of points of the spherical surface.

315. Case of Infinitely Thin Lens. Employing the usual special Lens-Notation (see § 268), we may write the expression on the left-hand side of formula (339) as follows:

$$J_1\boldsymbol{J}_1(1/nu'_1 - x) + J_2\boldsymbol{J}_2(x' - 1/nu'_1) = \varphi V;$$

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[1] A. KOENIG und M. VON ROHR: Die Theorie der sphaerischen Aberrationen: Chapter V of Vol. I of *Die Theorie der optischen Instrumente* (Berlin, 1904); edited by M. VON ROHR. See pages 275–289.

where

$$J_1 = c - x, \quad \boldsymbol{J}_1 = c - \mathbf{x},$$

$$\boldsymbol{J}_2 = \frac{(n-1)(c-x) - n\varphi}{n-1}, \quad \boldsymbol{J}_2 = \frac{(n-1)(c-\mathbf{x}) - n\varphi}{n-1},$$

$$\frac{1}{nu_1'} = \frac{x + (n-1)c}{n^2}, \quad x' = x + \varphi.$$

Thus, we find:

$$V = \frac{n+2}{n}c^2 - \left\{\frac{2n+1}{n-1}\varphi + \frac{3(n+1)}{n}x + \frac{n+1}{n}\mathbf{x}\right\}c$$
$$+ \frac{n^2}{(n-1)^2}\varphi^2 + \frac{2n+1}{n-1}\varphi x + \frac{n}{n-1}\varphi\mathbf{x} + \frac{2n+1}{n}x\mathbf{x} + \frac{n+1}{n}\mathbf{x}^2.$$

The value of V will be a minimum when:

$$c = \frac{n(2n+1)}{2(n-1)(n+2)}\varphi + \frac{3(n+1)}{2(n+2)}x + \frac{n+1}{2(n+2)}\mathbf{x}.$$

For real values of c, we must have:

$$\frac{(n+1)(5n+1)}{n^2}x^2 - \frac{4n-1}{(n-1)^2}\varphi^2 + \frac{(n+1)^2}{n^2}\mathbf{x}^2$$
$$+ \frac{2(2n+1)}{n}\varphi x - \frac{2}{n}\varphi\mathbf{x} - \frac{2(n^2+4n+1)}{n^2}x\mathbf{x} \geqq 0.$$

VII. SEIDEL'S THEORY OF THE SPHERICAL ABERRATIONS OF THE THIRD ORDER.

ART. 102. DEVELOPMENT OF SEIDEL'S FORMULÆ FOR THE *y*- AND *z*-ABERRATIONS.

316. Gaussian Parameters of Incident and Refracted Rays. If we take the vertex A of the spherical refracting surface as the origin of a system of rectangular axes, and choose the positive direction of the optical axis as the positive direction of the x-axis, then, adopting the method of GAUSS,[1] we can write the equations of the incident ray as follows:

$$y = \frac{Bx}{n} + P, \quad z = \frac{Cx}{n} + Q,$$

where the two pairs of constants B, P and C, Q are the four parameters which are used here to determine the position of the incident ray. And, similarly, the equations of the corresponding refracted ray may be written as follows:

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[1] C. F. GAUSS: *Dioptrische Untersuchungen* (Goettingen, 1841), page 3.

$$y = \frac{B'x}{n'} + P', \quad z = \frac{C'x}{n'} + Q',$$

where B', P' and C', Q' denote the corresponding parameters of the refracted ray. In these equations n, n' denote the absolute indices of refraction of the first and second medium, respectively. The relations between the parameters of the incident ray and those of the refracted ray, whereby, knowing the former, we can determine the latter, are obtained by GAUSS very simply as follows:

The abscissa of the incidence-point B is:

$$AD = r(1 - \cos\varphi) = 2r\cdot\sin^2\frac{\varphi}{2},$$

where D designates the foot of the perpendicular let fall from B on the optical axis, and where $r = AC$ denotes the abscissa of the centre C of the spherical surface, and $\varphi = \angle BCA$ denotes the central angle. Since the point B is common to both the incident and refracted rays, the value $x = r(1 - \cos\varphi)$ must satisfy both sets of equations; and, consequently, we obtain:

$$\left.\begin{aligned} 2\frac{B}{n}r\cdot\sin^2\frac{\varphi}{2} + P &= 2\frac{B'}{n'}r\cdot\sin^2\frac{\varphi}{2} + P', \\ 2\frac{C}{n}r\cdot\sin^2\frac{\varphi}{2} + Q &= 2\frac{C'}{n'}r\cdot\sin^2\frac{\varphi}{2} + Q'. \end{aligned}\right\} \qquad (340)$$

Moreover, let H, H' designate the points where the incident and refracted rays, produced if necessary, cross the transversal plane perpendicular to the optical axis at the centre C of the spherical surface. Since, according to the Laws of Refraction, BH' lies in the plane containing BH and BC, the three points C, H and H' must lie all in a straight line: and if in the triangles BHC, $BH'C$ the angles at H, H' are denoted by μ, μ', the following relation can easily be deduced (see Chap. IX, formula (209)) from the law connecting the angles of incidence and refraction:

$$n\cdot CH\cdot\sin\mu = n'\cdot CH'\cdot\sin\mu'.$$

Accordingly, if the co-ordinates of H, H' are (r, y_h, z_h), (r, y'_h, z'_h), respectively, we shall have:

$$y_h = \frac{Br}{n} + P, \quad z_h = \frac{Cr}{n} + Q$$

and

$$y'_h = \frac{B'r}{n'} + P', \quad z'_h = \frac{C'r}{n'} + Q';$$

and since

$$\frac{y'_h}{y_h} = \frac{z'_h}{z_h} = \frac{CH'}{CH} = \frac{n \cdot \sin \mu}{n' \cdot \sin \mu'},$$

we obtain:

$$\left.\begin{aligned} (Br + nP) \sin \mu &= (B'r + n'P') \sin \mu', \\ (Cr + nQ) \sin \mu &= (C'r + n'Q') \sin \mu'. \end{aligned}\right\} \qquad (341)$$

By means of these formulæ (340) and (341), we can obtain the values of the parameters B', P' and C', Q' of the refracted ray in terms of those of the incident ray.[1]

317. Approximate Values of the Gaussian Parameters, and the Correction-Terms of the 3rd Order. In the following investigation it is assumed that *the aperture of the optical system is relatively small*, so that none of the effective rays are very far from the optical axis. This being the case, we may regard the parameters denoted here by B, P, C, Q and B', P', C', Q' as being all small magnitudes of the first order. For the same reason, the magnitudes $\sin \varphi$, $\cos \mu$, $\cos \mu'$ are likewise to be considered as small magnitudes of the 1st order. We propose, according to L. SEIDEL,[2] to neglect here all terms of orders higher than the 3rd; and, hence, if A denotes a small magnitude of the first order, we may write this as follows:

$$A = a + \delta a;$$

where the small letter a denotes the part of A which is of the 1st order, and δa denotes the correction-term of the 3rd order; for, as was explained in § 254, if the parameters of the ray are regarded as magnitudes of the 1st order, the series-developments will contain only terms of the *odd* orders.

If, therefore, in the exact formulæ (340) and (341) we substitute for B, P, etc., $b + \delta b$, $p + \delta p$, etc., respectively, we shall obtain a set of approximate formulæ which are accurate except for residual errors of the 5th and higher orders. Moreover, each of the new equations thus obtained will break up at once into two others, since, evidently, the terms of the 1st order on one side of the equation must be equal to the terms of the same order on the other side; and since the same is true also in respect to the terms of the 3rd order. Thus between the approximate values b, p, etc., and b', p', etc., of the parameters of the ray before and after refraction we obtain the following set of relations:

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[1] See also OSCAR ROETHIG: *Die Probleme der Brechung und Reflexion* (Leipzig, 1876), pages 15–26.

[2] L. SEIDEL: Zur Dioptrik. Ueber die Entwicklung der Glieder 3ter Ordnung, welche den Weg eines ausserhalb der Ebene der Axe gelegenen Lichtstrahles durch ein System brechenden Medien, bestimmen: *Astronomische Nachrichten*, xliii. (1856), Nos. 1027, 1028, 1029.

$$\left.\begin{aligned} p' = p; \quad q' = q; \\ b' + \frac{n'p'}{r} = b + \frac{np}{r}; \quad c' + \frac{n'q'}{r} = c + \frac{nq}{r}; \end{aligned}\right\} \tag{342}$$

and between the correction-terms of the 3rd order the following relations:

$$\left.\begin{aligned} \delta p' - \delta p = \frac{r\varphi^2}{2}\left(\frac{b'}{n'} - \frac{b}{n}\right); \quad \delta q' - \delta q = \frac{r\varphi^2}{2}\left(\frac{c'}{n'} - \frac{c}{n}\right); \\ \left(\delta b' + \frac{n'\cdot\delta p'}{r}\right) - \left(\delta b + \frac{n\cdot\delta p}{r}\right) = \frac{1}{2}\left(b + \frac{np}{r}\right)(\cos^2\mu - \cos^2\mu'); \\ \left(\delta c' + \frac{n'\cdot\delta q'}{r}\right) - \left(\delta c + \frac{n\cdot\delta q}{r}\right) = \frac{1}{2}\left(c + \frac{nq}{r}\right)(\cos^2\mu - \cos^2\mu'). \end{aligned}\right\} \tag{343}$$

Obviously, in the further development, it will be sufficient to obtain the formulæ for the magnitudes b, p, b', p' which relate to the xy-plane; then all we shall have to do to find the corresponding formulæ for the magnitudes c, q, c', q' which relate to the xz-plane will be to substitute in the first formulæ the latter magnitudes in place of the former.

318. Relations between the Ray-Parameters of Gauss and Seidel. Instead of the GAUSSian parameters

$$B = b + \delta b, \quad P = p + \delta p \quad \text{and} \quad C = c + \delta c, \quad Q = q + \delta q,$$

we have now to introduce the parameters

$$\eta = y + \delta y, \quad \zeta = z + \delta z \quad \text{and} \quad \boldsymbol{\eta} = \boldsymbol{y} + \delta \boldsymbol{y}, \quad \boldsymbol{\zeta} = \boldsymbol{z} + \delta \boldsymbol{z},$$

which are employed by SEIDEL (§ 255), and which are the co-ordinates of the points P, $\boldsymbol{P}$ where the ray crosses the two fixed transversal planes σ, $\boldsymbol{\sigma}$, respectively. The abscissæ of the points M, $\boldsymbol{M}$ where the optical axis meets the transversal planes σ, $\boldsymbol{\sigma}$ will be denoted by u, $\boldsymbol{u}$, respectively; thus,

$$AM = u, \quad A\boldsymbol{M} = \boldsymbol{u};$$

and, similarly, for the pair of axial points M', $\boldsymbol{M}'$ conjugate to M, $\boldsymbol{M}$, respectively, let us put:

$$AM' = u', \quad A\boldsymbol{M}' = \boldsymbol{u}'.$$

Moreover, as in § 255,

$$J = n\left(\frac{1}{r} - \frac{1}{u}\right) = n'\left(\frac{1}{r} - \frac{1}{u'}\right),$$

$$\mathbf{J} = n\left(\frac{1}{r} - \frac{1}{\mathbf{u}}\right) = n'\left(\frac{1}{r} - \frac{1}{\mathbf{u}'}\right).$$

Then, since the incident ray must go through the points $P(u, \eta, \zeta)$ and $\mathbf{P}(\mathbf{u}, \boldsymbol{\eta}, \boldsymbol{\zeta})$, we must have:

$$B = \frac{n^2}{(J - \mathbf{J})} \cdot \frac{\eta - \boldsymbol{\eta}}{u\mathbf{u}}, \quad P = \frac{n}{J - \mathbf{J}}\left(\frac{\boldsymbol{\eta}}{\mathbf{u}} - \frac{\eta}{u}\right);$$

$$C = \frac{n^2}{(J - \mathbf{J})} \cdot \frac{\zeta - \boldsymbol{\zeta}}{u\mathbf{u}}, \quad Q = \frac{n}{J - \mathbf{J}}\left(\frac{\boldsymbol{\zeta}}{\mathbf{u}} - \frac{\zeta}{u}\right);$$

and, hence, for the approximate values we have the following relations:

$$\left.\begin{aligned} b &= \frac{n^2}{J - \mathbf{J}} \cdot \frac{y - \mathbf{y}}{u\mathbf{u}}, \quad p = \frac{n}{J - \mathbf{J}}\left(\frac{\mathbf{y}}{\mathbf{u}} - \frac{y}{u}\right); \\ c &= \frac{n^2}{J - \mathbf{J}} \cdot \frac{z - \mathbf{z}}{u\mathbf{u}}, \quad q = \frac{n}{J - \mathbf{J}}\left(\frac{\mathbf{z}}{\mathbf{u}} - \frac{z}{u}\right); \end{aligned}\right\} \qquad (344)$$

and for the correction-terms of the 3rd order:

$$\left.\begin{aligned} \delta b &= \frac{n^2}{J - \mathbf{J}} \cdot \frac{\delta y - \delta \mathbf{y}}{u\mathbf{u}}, \quad \delta p = \frac{n}{J - \mathbf{J}}\left(\frac{\delta \mathbf{y}}{\mathbf{u}} - \frac{\delta y}{u}\right); \\ \delta c &= \frac{n^2}{J - \mathbf{J}} \cdot \frac{\delta z - \delta \mathbf{z}}{u\mathbf{u}}, \quad \delta q = \frac{n}{J - \mathbf{J}}\left(\frac{\delta \mathbf{z}}{\mathbf{u}} - \frac{\delta z}{u}\right); \end{aligned}\right\} \qquad (345)$$

and by priming all the letters in formulæ (344) and (345), except the letters J, $\mathbf{J}$, we shall obtain also the corresponding relations for the refracted ray.

Formulæ (342) and (343) lead to the following invariant relations between the approximate values of the parameters of the incident and refracted rays:

$$\left.\begin{aligned} \frac{n'y'}{u'} &= \frac{ny}{u}, \quad \frac{n'z'}{u'} = \frac{nz}{u}; \\ \frac{n'\mathbf{y}'}{\mathbf{u}'} &= \frac{n\mathbf{y}}{\mathbf{u}}, \quad \frac{n'\mathbf{z}'}{\mathbf{u}'} = \frac{n\mathbf{z}}{\mathbf{u}}; \end{aligned}\right\} \qquad (346)$$

which will be recognized as equivalent to the well-known law of

ROBERT SMITH for a single spherical refracting surface (Chap. VIII, § 194).

Moreover, we find:

$$b + \frac{np}{r} = \frac{n}{J - \mathbf{J}}\left(J\frac{\mathbf{y}}{\mathbf{u}} - \mathbf{J}\frac{y}{u}\right),$$

$$\delta b + \frac{n \cdot \delta p}{r} = \frac{n}{J - \mathbf{J}}\left(J\frac{\delta \mathbf{y}}{\mathbf{u}} - \mathbf{J}\frac{\delta y}{u}\right);$$

and, hence, substituting these values in the first and third of formulæ (343), we obtain, after some obvious reductions:

$$\left(n'\frac{\delta \mathbf{y}'}{\mathbf{u}'} - n\frac{\delta \mathbf{y}}{\mathbf{u}}\right) - \left(n'\frac{\delta y'}{u'} - n\frac{\delta y}{u}\right) = \frac{r\varphi^2}{2} n \cdot \Delta\frac{1}{n} \cdot \left(J\frac{\mathbf{y}}{\mathbf{u}} - \mathbf{J}\frac{y}{u}\right),$$

$$J\left(n'\frac{\delta \mathbf{y}'}{\mathbf{u}'} - n\frac{\delta \mathbf{y}}{\mathbf{u}}\right) - \mathbf{J}\left(n'\frac{\delta y'}{u'} - n\frac{\delta y}{u}\right)$$
$$= \frac{n}{2}\left(J\frac{\mathbf{y}}{\mathbf{u}} - \mathbf{J}\frac{y}{u}\right)(\cos^2\mu - \cos^2\mu').$$

Combining these two equations so as to eliminate the difference $\Delta(n \cdot \delta \mathbf{y}/\mathbf{u})$, we find:

$$\Delta\frac{n \cdot \delta y}{u} = -\frac{n}{2(J - \mathbf{J})}\left(J\frac{\mathbf{y}}{\mathbf{u}} - \mathbf{J}\frac{y}{u}\right)\left(Jr\varphi^2 \cdot \Delta\frac{1}{n} + \cos^2\mu' - \cos^2\mu\right). \quad (347)$$

319. It only remains now to obtain expressions for the small magnitudes φ, $\cos\mu$, $\cos\mu'$; wherein, however, we need consider only the terms of the 1st order, since these alone will have any influence of the 3rd order on the value of the expression for $\Delta(n \cdot \delta y/u)$.

In order to obtain the approximate expression for the central angle φ, we shall proceed as follows: The distance from the vertex A of the spherical surface of the point where the incident ray meets the yz-plane of co-ordinates is approximately equal to $\sqrt{p^2 + q^2}$, and since the length of the arc AB is equal to $r\varphi$, we may, if we neglect the magnitudes of the 3rd order, put:

$$r^2\varphi^2 = p^2 + q^2;$$

and, hence, we obtain:

$$r\varphi^2 = \frac{1}{r}\frac{n^2}{(J - \mathbf{J})^2}\left\{\frac{y^2 + z^2}{u^2} + \frac{\mathbf{y}^2 + \mathbf{z}^2}{\mathbf{u}^2} - 2\frac{y\mathbf{y} + z\mathbf{z}}{u\mathbf{u}}\right\}. \quad (348)$$

We must now derive an expression for $\cos^2\mu' - \cos^2\mu$.

If we neglect terms of the 3d order, the direction-cosines of the incident ray may be regarded as 1, b/n, c/n; so that the approximate equations of the incident ray BH are:

$$\frac{x}{n} = \frac{y-p}{b} = \frac{z-q}{c};$$

and the equations of the straight line CH are:

$$x = r, \quad \frac{y}{y_h} = \frac{z}{z_h},$$

and hence for the angle μ between these two straight lines, we have:

$$\cos^2\mu = \frac{(by_h + cz_h)^2}{n^2(y_h^2 + z_h^2)}.$$

Now since the point $H(r, y_h, z_h)$ is a point on the incident ray, we have:

$$y_h = \frac{br}{n} + p = \frac{r}{J - \boldsymbol{J}}\left(J\frac{\boldsymbol{y}}{\boldsymbol{u}} - \boldsymbol{J}\frac{y}{u}\right) = \frac{r}{J - \boldsymbol{J}}Y,$$

$$z_h = \frac{cr}{n} + q = \frac{r}{J - \boldsymbol{J}}\left(J\frac{\boldsymbol{z}}{\boldsymbol{u}} - \boldsymbol{J}\frac{z}{u}\right) = \frac{r}{J - \boldsymbol{J}}Z,$$

if, for the sake of brevity, we write temporarily:

$$Y = J\frac{\boldsymbol{y}}{\boldsymbol{u}} - \boldsymbol{J}\frac{y}{u}, \quad Z = J\frac{\boldsymbol{z}}{\boldsymbol{u}} - \boldsymbol{J}\frac{z}{u}.$$

Hence, since by formula (346):

$$Y' = \frac{n}{n'}Y, \quad Z' = \frac{n}{n'}Z,$$

we obtain:

$$\cos^2\mu = \frac{n^2}{(J - \boldsymbol{J})^2 u^2 \boldsymbol{u}^2} \cdot \frac{\{(y - \boldsymbol{y})Y + (z - \boldsymbol{z})Z\}^2}{Y^2 + Z^2},$$

$$\cos^2\mu' = \frac{n'^2}{(J - \boldsymbol{J})^2 u'^2 \boldsymbol{u}'^2} \cdot \frac{\{(y' - \boldsymbol{y}')Y + (z' - \boldsymbol{z}')Z\}^2}{Y^2 + Z^2}.$$

Now evidently:

$$\frac{n'}{u'\boldsymbol{u}'}\left\{(y' - \boldsymbol{y}')Y + (z' - \boldsymbol{z}')Z\right\} = n\left\{\left(\frac{y}{u\boldsymbol{u}'} - \frac{\boldsymbol{y}}{u'\boldsymbol{u}}\right)Y + \left(\frac{z}{u\boldsymbol{u}'} - \frac{\boldsymbol{z}}{u'\boldsymbol{u}}\right)Z\right\};$$

and hence we find:

$$\cos^2\mu' - \cos^2\mu = \frac{n^2}{(J - \boldsymbol{J})^2} \cdot \frac{A \cdot B}{Y^2 + Z^2},$$

where for brevity we write:

$$A = \left\{\frac{y}{u}\left(\frac{1}{\boldsymbol{u}} + \frac{1}{\boldsymbol{u}'}\right) - \frac{\boldsymbol{y}}{\boldsymbol{u}}\left(\frac{1}{u} + \frac{1}{u'}\right)\right\}Y + \left\{\frac{z}{u}\left(\frac{1}{\boldsymbol{u}} + \frac{1}{\boldsymbol{u}'}\right) - \frac{\boldsymbol{z}}{\boldsymbol{u}}\left(\frac{1}{u} + \frac{1}{u'}\right)\right\}Z,$$

$$B = \left\{\frac{y}{u}\left(\frac{1}{\boldsymbol{u}'} - \frac{1}{\boldsymbol{u}}\right) - \frac{\boldsymbol{y}}{\boldsymbol{u}}\left(\frac{1}{u'} - \frac{1}{u}\right)\right\}Y + \left\{\frac{z}{u}\left(\frac{1}{\boldsymbol{u}'} - \frac{1}{\boldsymbol{u}}\right) - \frac{\boldsymbol{z}}{\boldsymbol{u}}\left(\frac{1}{u'} - \frac{1}{u}\right)\right\}Z$$

$$= (Y^2 + Z^2)\Delta\frac{1}{n}.$$

Now

$$\frac{1}{u} + \frac{1}{u'}, = \frac{2}{r} - J\cdot\Delta\frac{1}{n} - \frac{2J}{n},$$

$$\frac{1}{\boldsymbol{u}} + \frac{1}{\boldsymbol{u}'} = \frac{2}{r} - \boldsymbol{J}\cdot\Delta\frac{1}{n} - \frac{2\boldsymbol{J}}{n};$$

and thus we can write:

$$A = -\left\{\frac{y^2 + z^2}{u^2}\boldsymbol{J}\left(\frac{2}{r} - \boldsymbol{J}\cdot\Delta\frac{1}{n} - \frac{2\boldsymbol{J}}{n}\right) + \frac{\boldsymbol{y}^2 + \boldsymbol{z}^2}{\boldsymbol{u}^2}J\left(\frac{2}{r} - J\cdot\Delta\frac{1}{n} - \frac{2J}{n}\right)\right.$$
$$\left. - 2\frac{y\boldsymbol{y} + z\boldsymbol{z}}{u\boldsymbol{u}}\left(\frac{J + \boldsymbol{J}}{r} - J\boldsymbol{J}\cdot\Delta\frac{1}{n} - \frac{2J\boldsymbol{J}}{n}\right)\right\}. \qquad (349)$$

Accordingly, we obtain finally:

$$\cos^2\mu' - \cos^2\mu = \frac{n^2}{(J - \boldsymbol{J})^2}\cdot\Delta\frac{1}{n}\cdot A, \qquad (350)$$

where A is defined by (349).

320. If now we substitute in formula (347) the expressions (348) and (350), we shall obtain on the right-hand side of the equation:

$$\frac{n^3}{2(J - \boldsymbol{J})^3}\left(J\frac{\boldsymbol{y}}{\boldsymbol{u}} - \boldsymbol{J}\frac{y}{u}\right)\cdot R,$$

where

$$R = \frac{y^2 + z^2}{u^2}\left(\frac{2\boldsymbol{J} - J}{r} - \boldsymbol{J}^2\cdot\Delta\frac{1}{n} - 2\frac{\boldsymbol{J}^2}{n}\right)\Delta\frac{1}{n}$$
$$+ \left(\frac{\boldsymbol{y}^2 + \boldsymbol{z}^2}{\boldsymbol{u}^2}J - 2\frac{y\boldsymbol{y} + z\boldsymbol{z}}{u\boldsymbol{u}}\boldsymbol{J}\right)\left(\frac{1}{r} - J\cdot\Delta\frac{1}{n} - 2\frac{J}{n}\right)\Delta\frac{1}{n};$$

which latter expression may also be written as follows:

$$R = \frac{y^2 + z^2}{u^2}\left(\frac{\boldsymbol{J}^2}{J}\Delta\frac{1}{nu} - \frac{(J - \boldsymbol{J})^2}{J}\frac{1}{r}\Delta\frac{1}{n}\right)$$
$$+ \frac{\boldsymbol{y}^2 + \boldsymbol{z}^2}{\boldsymbol{u}^2}J\cdot\Delta\frac{1}{nu} - 2\frac{y\boldsymbol{y} + z\boldsymbol{z}}{u\boldsymbol{u}}\boldsymbol{J}\cdot\Delta\frac{1}{nu}. \qquad (351)$$

Thus, we obtain finally:

$$\Delta\left(\frac{n\cdot\delta y}{u}\right) = \frac{n^3}{2(J - \boldsymbol{J})^3}\left(J\frac{\boldsymbol{y}}{\boldsymbol{u}} - \boldsymbol{J}\frac{y}{u}\right)\cdot R; \qquad (352)$$

and, similarly:

$$\Delta\left(\frac{n\cdot\delta z}{u}\right)=\frac{n^3}{2(J-\mathbf{J})^3}\left(J\frac{\mathbf{z}}{\mathbf{u}}-\mathbf{J}\frac{z}{u}\right)\cdot R; \tag{353}$$

where R is defined by (351).

321. Thus far the directions of the axes of y and z are entirely arbitrary, except that it has been assumed they are both perpendicular to the optical axis. We may *select as the xy-plane the meridian plane which contains the point Q*, and which, according to GAUSS's Theory, will contain also the conjugate point Q'. This evidently will not affect at all the generality of the treatment, and it will lead to some simplification, inasmuch as we shall have then $z = z' = 0$. Thus if we put $z = 0$ in the formulæ (351), (352) and (353), we obtain the following set of formulæ:

$$\left.\begin{aligned}
&R=\frac{y^2}{u^2}\left(\frac{\mathbf{J}^2}{J}\Delta\frac{1}{nu}-\frac{(J-\mathbf{J})^2}{J}\frac{1}{r}\Delta\frac{1}{n}\right)\\
&\qquad\qquad+\frac{\mathbf{y}^2+\mathbf{z}^2}{\mathbf{u}^2}J\cdot\Delta\frac{1}{nu}-2\frac{y\mathbf{y}}{u\mathbf{u}}\mathbf{J}\cdot\Delta\frac{1}{nu};\\
&\textit{The y-aberration:}\\
&\Delta\left(\frac{n\cdot\delta y}{u}\right)=\frac{n^3}{2(J-\mathbf{J})^3}\left(J\frac{\mathbf{y}}{\mathbf{u}}-\mathbf{J}\frac{y}{u}\right)R;\\
&\textit{The z-aberration:}\\
&\Delta\left(\frac{n\cdot\delta z}{u}\right)=\frac{n^3}{2(J-\mathbf{J})^3}J\frac{\mathbf{z}}{\mathbf{u}}R.
\end{aligned}\right\} \tag{354}$$

These formulæ give the variations of $n\cdot\delta y/u$, $n\cdot\delta z/u$ which result in consequence of the refraction of the ray at a single spherical surface. In case we have a centered system of m spherical surfaces, we must introduce the subscript k to indicate that the formulæ apply to the kth surface, and then the formulæ will be written:

$$\Delta\left(\frac{n\cdot\delta y}{u}\right)_k=\frac{n_{k-1}'^{3}}{2(J_k-\mathbf{J}_k)^3}\left(J_k\frac{\mathbf{y}_{k-1}'}{\mathbf{u}_k}-\mathbf{J}_k\frac{y_{k-1}'}{u_k}\right)R_k,$$

$$\Delta\left(\frac{n\cdot\delta z}{u}\right)_k=\frac{n_{k-1}'^{3}}{2(J_k-\mathbf{J}_k)^3}J_k\frac{\mathbf{z}_{k-1}'}{\mathbf{u}_k}R_k;$$

where

$$R_k=\frac{y_{k-1}'^{2}}{u_k^2}\left\{\frac{\mathbf{J}_k^2}{J_k}\Delta\left(\frac{1}{nu}\right)_k-\frac{(J_k-\mathbf{J}_k)^2}{J_k}\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k\right\}$$
$$\qquad+\frac{\mathbf{y}_{k-1}'^{2}+\mathbf{z}_{k-1}'^{2}}{\mathbf{u}_k^2}J_k\cdot\Delta\left(\frac{1}{nu}\right)_k-2\frac{y_{k-1}'\mathbf{y}_{k-1}'}{u_k\mathbf{u}_k}\mathbf{J}_k\cdot\Delta\left(\frac{1}{nu}\right)_k.$$

Now if h, $\boldsymbol{h}$ denote the incidence-heights of a pair of paraxial rays emanating from the axial object-points M_1, $\boldsymbol{M}_1$, respectively, we have, by ROBERT SMITH's Law (§ 194):

$$\frac{n'_{k-1}h_k y'_{k-1}}{u_k}=\frac{n_1 h_1 y_1}{u_1},\quad \frac{n'_{k-1}\boldsymbol{h}_k \boldsymbol{y}'_{k-1}}{\boldsymbol{u}_k}=\frac{n_1 \boldsymbol{h}_1 \boldsymbol{y}_1}{\boldsymbol{u}_1},\quad \frac{n'_{k-1}\boldsymbol{h}_k \boldsymbol{z}'_{k-1}}{\boldsymbol{u}_k}=\frac{n_1 \boldsymbol{h}_1 \boldsymbol{z}_1}{\boldsymbol{u}_1}.$$

Moreover, we have also SEIDEL's Formula (Chapter VIII, § 195):

$$J_k-\boldsymbol{J}_k=\frac{h_1\boldsymbol{h}_1}{h_k\boldsymbol{h}_k}(J_1-\boldsymbol{J}_1)=\frac{h_1\boldsymbol{h}_1}{h_k\boldsymbol{h}_k}\cdot\frac{n_1(u_1-\boldsymbol{u}_1)}{u_1\boldsymbol{u}_1}.$$

If we introduce these relations in the above equations, we shall obtain the following formulæ:

$$\left.\begin{aligned}
\Delta\left(n\frac{\delta y}{u}\right)_k=&\frac{1}{2}\frac{(\boldsymbol{y}_1^2+\boldsymbol{z}_1^2)\boldsymbol{y}_1}{(\boldsymbol{u}_1-u_1)^3}u_1^3\left\{\frac{h_k^3}{h_1^3}J_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k\right\}\\
&-\frac{1}{2}\frac{(3\boldsymbol{y}_1^2+\boldsymbol{z}_1^2)y_1}{(\boldsymbol{u}_1-u_1)^3}u_1^2\boldsymbol{u}_1\left\{\frac{h_k^2}{h_1^2}\frac{\boldsymbol{h}_k}{\boldsymbol{h}_1}J_k\boldsymbol{J}_k\cdot\Delta\left(\frac{1}{nu}\right)_k\right\}\\
&+\frac{1}{2}\frac{\boldsymbol{y}_1y_1^2}{(\boldsymbol{u}_1-u_1)^3}u_1\boldsymbol{u}_1^2\left\{3\frac{h_k}{h_1}\frac{\boldsymbol{h}_k^2}{\boldsymbol{h}_1^2}\boldsymbol{J}_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k\right.\\
&\qquad\qquad\left.-\frac{h_k}{h_1}\frac{\boldsymbol{h}_k^2}{\boldsymbol{h}_1^2}(J_k-\boldsymbol{J}_k)^2\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k\right\}\\
&+\frac{1}{2}\frac{y_1^3}{(\boldsymbol{u}_1-u_1)^3}\boldsymbol{u}_1^3\left\{\frac{\boldsymbol{h}_k^3}{\boldsymbol{h}_1^3}\frac{\boldsymbol{J}_k}{J_k}(J_k-\boldsymbol{J}_k)^2\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k\right.\\
&\qquad\qquad\left.-\frac{\boldsymbol{h}_k^3}{\boldsymbol{h}_1^3}\frac{\boldsymbol{J}_k^3}{J_k}\Delta\left(\frac{1}{nu}\right)_k\right\},\\
\Delta\left(n\frac{\delta z}{u}\right)_k=&\frac{1}{2}\frac{(\boldsymbol{y}_1^2+\boldsymbol{z}_1^2)\boldsymbol{z}_1}{(\boldsymbol{u}_1-u_1)^3}u_1^3\left\{\frac{h_k^3}{h_1^3}J_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k\right\}\\
&-\frac{y_1\boldsymbol{y}_1\boldsymbol{z}_1}{(\boldsymbol{u}_1-u_1)^3}u_1^2\boldsymbol{u}_1\left\{\frac{h_k^2}{h_1^2}\frac{\boldsymbol{h}_k}{\boldsymbol{h}_1}J_k\boldsymbol{J}_k\cdot\Delta\left(\frac{1}{nu}\right)_k\right\}\\
&+\frac{1}{2}\frac{y_1^2\boldsymbol{z}_1}{(\boldsymbol{u}_1-u_1)^3}u_1\boldsymbol{u}_1^2\left\{\frac{h_k}{h_1}\frac{\boldsymbol{h}_k^2}{\boldsymbol{h}_1^2}\boldsymbol{J}_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k\right.\\
&\qquad\qquad\left.-\frac{h_k}{h_1}\frac{\boldsymbol{h}_k^2}{\boldsymbol{h}_1^2}(J_k-\boldsymbol{J}_k)^2\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k\right\}.
\end{aligned}\right\}\quad(355)$$

Now

$$\Delta\left(\frac{n\cdot\delta y}{u}\right)_k=\frac{n'_k\cdot\delta y'_k}{u'_k}-\frac{n'_{k-1}\cdot\delta y'_{k-1}}{u_k}=\frac{n'_k\cdot\delta y'_k}{u'_k}-\frac{h_{k-1}}{h_k}\,\frac{n'_{k-1}\cdot\delta y'_{k-1}}{u'_{k-1}};$$

or

$$h_k\cdot\Delta\left(\frac{n\cdot\delta y}{u}\right)_k=h_k\frac{n'_k\cdot\delta y'_k}{u'_k}-h_{k-1}\frac{n'_{k-1}\cdot\delta y'_{k-1}}{u'_{k-1}};$$

and, hence, if we suppose that the object, situated in the first medium, is free from aberration, so that the object-point P_1 coincides with Q_1, and therefore

$$\delta y_1=\delta z_1=0,$$

we find:

$$\sum_{k=1}^{k=m}h_k\cdot\Delta\left(\frac{n\cdot\delta y}{u}\right)_k=h_m\frac{n'_m\cdot\delta y'_m}{u'_m};$$

that is,

$$\frac{n'_m\cdot\delta y'_m}{u'_m}=\frac{h_1}{h_m}\sum_{k=1}^{k=m}\frac{h_k}{h_1}\Delta\left(\frac{n\cdot\delta y}{u}\right)_k;$$

and, similarly:

$$\frac{n'_m\cdot\delta z'_m}{u'_m}=\frac{h_1}{h_m}\sum_{k=1}^{k=m}\frac{h_k}{h_1}\Delta\left(\frac{n\cdot\delta z}{u}\right)_k.$$

Let us now employ the following abbreviations:

$$\left.\begin{aligned}
S^{\mathrm{I}}&=\sum_{k=1}^{k=m}\frac{h_k^4}{h_1^4}\,J_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k;\\
S^{\mathrm{II}}&=\sum_{k=1}^{k=m}\frac{h_k^3}{h_1^3}\,\frac{\boldsymbol{h}_k}{\boldsymbol{h}_1}\,J_k\boldsymbol{J}_k\cdot\Delta\left(\frac{1}{nu}\right)_k;\\
S^{\mathrm{III}}&=\sum_{k=1}^{k=m}\frac{h_k^2}{h_1^2}\,\frac{\boldsymbol{h}_k^2}{\boldsymbol{h}_1^2}\,\boldsymbol{J}_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k;\\
S^{\mathrm{IV}}&=\sum_{k=1}^{k=m}\frac{h_k^2}{h_1^2}\,\frac{\boldsymbol{h}_k^2}{\boldsymbol{h}_1^2}\,(J_k-\boldsymbol{J}_k)^2\,\frac{1}{r_k}\,\Delta\left(\frac{1}{n}\right)_k;\\
S^{\mathrm{V}}&=\sum_{k=1}^{k=m}\frac{h_k}{h_1}\,\frac{\boldsymbol{h}_k^3}{\boldsymbol{h}_1^3}\left\{\frac{\boldsymbol{J}_k}{J_k}(J_k-\boldsymbol{J}_k)^2\,\frac{1}{r_k}\,\Delta\left(\frac{1}{n}\right)_k-\frac{\boldsymbol{J}_k^3}{J_k}\,\Delta\left(\frac{1}{nu}\right)_k\right\};
\end{aligned}\right\}\quad(356)$$

so that we may write finally:

$$\left.\begin{aligned}
\frac{n'_m \cdot \delta y'_m}{u'_m} &= \frac{1}{2}\frac{(\boldsymbol{y}_1^2+\boldsymbol{z}_1^2)\boldsymbol{y}_1}{(\boldsymbol{u}_1-u_1)^3}\,u_1^3\,\frac{h_1}{h_m}\,S^{\mathrm{I}} - \frac{1}{2}\frac{(3\boldsymbol{y}_1^2+\boldsymbol{z}_1^2)y_1}{(\boldsymbol{u}_1-u_1)^3}\,u_1^2\boldsymbol{u}_1\,\frac{h_1}{h_m}\,S^{\mathrm{II}}\\
&\quad+\frac{1}{2}\frac{\boldsymbol{y}_1 y_1^2}{(\boldsymbol{u}_1-u_1)^3}\,u_1\boldsymbol{u}_1^2\,\frac{h_1}{h_m}\,(3S^{\mathrm{III}}-S^{\mathrm{IV}})+\frac{1}{2}\frac{y_1^3}{(\boldsymbol{u}_1-u_1)^3}\,\boldsymbol{u}_1^3\,\frac{h_1}{h_m}\,S^{\mathrm{V}};\\
\frac{n'_m \cdot \delta z'_m}{u'_m} &= \frac{1}{2}\frac{(\boldsymbol{y}_1^2+\boldsymbol{z}_1^2)\boldsymbol{z}_1}{(\boldsymbol{u}_1-u_1)^3}\,u_1^3\,\frac{h_1}{h_m}\,S^{\mathrm{I}} - \frac{y_1\boldsymbol{y}_1\boldsymbol{z}_1}{(\boldsymbol{u}_1-u_1)^3}\,u_1^2\boldsymbol{u}_1\,\frac{h_1}{h_m}\,S^{\mathrm{II}}\\
&\qquad\qquad+\frac{1}{2}\frac{y_1^2\boldsymbol{z}_1}{(\boldsymbol{u}_1-u_1)^3}\,u_1\boldsymbol{u}_1^2\,\frac{h_1}{h_m}\,(S^{\mathrm{III}}-S^{\mathrm{IV}}).
\end{aligned}\right\}\quad(357)$$

322. Conditions of the Abolition of the Spherical Aberations of the 3rd Order. The expressions denoted here by S^{I}, S^{II}, S^{III}, S^{IV}, S^{V} are practically equivalent to the famous *five sums* of SEIDEL, although SEIDEL's expressions in their final form are different from these.

The equation $S^{\mathrm{I}} = 0$ will be recognized as the condition of the abolition of the spherical aberration at the centre of the visual field; that is, the condition that the axial points M_1, M'_m shall be a pair of "aberrationless" points (§ 265).

The equation $S^{\mathrm{II}} = 0$ is at the same time the condition of the fulfilment of ABBE's Sine-Condition (§ 284) and of the abolition of Coma (§ 313).

The condition of the abolition of the astigmatism of narrow oblique bundles of rays is $S^{\mathrm{III}} = 0$ (§ 302), and the conditions necessary for a plane, stigmatic image are $S^{\mathrm{III}} = 0$ and $S^{\mathrm{IV}} = 0$; see formulæ (332), § 304.

Finally, the condition that the image shall be without Distortion is $S^{\mathrm{V}} = 0$; see formula (311) or formula (312), § 292.

The image will be perfectly faultless (except for residual errors of the 5th order) provided all five sums S^{I}, S^{II}, S^{III}, S^{IV}, and S^{V} vanish together, and these five conditions are necessary if the image is to have this degree of perfection in every respect.

SEIDEL's Formulæ (357), which give the magnitudes of the y- and z-aberrations of the 3rd order in the image-plane σ'_m, are derived by A. KERBER[1] by the employment of KERBER's Formulæ given in Chapter IX, §§ 214, 216 for the refraction of a ray at a spherical surface; wherein the trigonometrical functions are replaced by their series-developments. KERBER's process is also given by KOENIG and VON ROHR[1] in their treatise on the Theory of Spherical Aberrations.

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[1] A. KERBER: *Beitraege zur Dioptrik.* Zweites Heft (Leipzig, 1896); pages 9–15.

ART. 103. ELIMINATION OF THE MAGNITUDES DENOTED BY h, u.

323. The natural determination-data of an optical system are the radii (r) of the spherical surfaces, the thicknesses (d) of the intervening media and the refractive indices (n). If in addition to these magnitudes we know also the positions of the object and of the stop, which is equivalent to knowing the values of u_1 and $\boldsymbol{u}_1$, we can compute the values of the two systems of magnitudes h, u and $\boldsymbol{h}$, $\boldsymbol{u}$ which occur in SEIDEL's Aberration-Formulæ (357). So long as these formulæ are to be employed to investigate the defects of an image produced by a given optical system, they answer their purpose excellently. But in case the problem is to design an optical instrument which is to fulfil certain prescribed conditions, the fact that the equations contain two sets of magnitudes which are not independent of each other is a disadvantage which must be got rid of by eliminating one of these sets of magnitudes by means of the other set. In SEIDEL's final forms of the aberration-formulæ the magnitudes denoted here by $\boldsymbol{h}$, $\boldsymbol{u}$ do not appear.

This elimination is performed with the aid of the two formulæ (155) and (156) of Chapter VIII, which are also due to SEIDEL, and which, by the introduction of the convenient abbreviating symbol T, may be written here as follows:

$$\left.\begin{aligned} T = h_k\boldsymbol{h}_k(J_k - \boldsymbol{J}_k) &= h_1\boldsymbol{h}_1(J_1 - \boldsymbol{J}_1), \\ \frac{\boldsymbol{h}_1}{h_1} - \frac{\boldsymbol{h}_k}{h_k} &= T\sum_{k=2}^{k=k}\frac{d_{k-1}}{n'_{k-1}\cdot h_k\cdot h_{k-1}}. \end{aligned}\right\} \tag{358}$$

The magnitude denoted here by T depends only on the initial values of the magnitudes h, u and $\boldsymbol{h}$, $\boldsymbol{u}$. If we introduce, also by way of abbreviation, another symbol and write:

$$X_k = -\frac{h_1}{\boldsymbol{h}_1}T\left(\frac{1}{h_k^2 J_k} + \sum_{k=2}^{k=k}\frac{d_{k-1}}{n'_{k-1}\cdot h_k\cdot h_{k-1}}\right), \tag{359}$$

formulæ (358) may be put in the following forms convenient for direct application to the expressions contained in the aberration-formulæ (357):

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[1] A. KOENIG und M. VON ROHR: Die Theorie der sphaerischen Aberrationen: Chapter V of M. VON ROHR's *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), pages 317–323.

$$\left.\begin{aligned}\boldsymbol{h}_k &= h_k\left\{\frac{\boldsymbol{h}_1}{h_1}(1+X_k)+\frac{T}{h_k^2J_k}\right\},\\ \boldsymbol{h}_k\boldsymbol{J}_k &= \frac{\boldsymbol{h}_1}{h_1}h_kJ_k(1+X_k).\end{aligned}\right\}\qquad(360)$$

Proceeding now to eliminate the magnitudes $\boldsymbol{h}_k$, $\boldsymbol{u}_k$ from the expressions under the summation-signs in the formulæ (356), we remark, in the first place, that the sum S^{I}, which is the expression of the Co-efficient of the Spherical Aberration along the axis, does not contain these magnitudes at all. Passing, therefore, to the Coma-Co-efficient, we obtain from the second of equations (360):

$$h_k^3\boldsymbol{h}_kJ_k\boldsymbol{J}_k = \frac{\boldsymbol{h}_1}{h_1}h_k^4J_k^2(1+X_k);$$

and hence:

$$S^{\mathrm{II}} = \sum_{k=1}^{k=m}\frac{h_k^4}{h_1^4}J_k^2\cdot\Delta\left(\frac{1}{nu}\right)_k\cdot(1+X_k).\qquad(361)$$

The first of the two terms on the right-hand side of this equation is the co-efficient S^{I} which is concerned with the spherical aberration along the axis. If the optical system satisfies Abbe's Sine-Condition, it must be spherically corrected for the object-point M_1 (§ 277 and § 279); that is, $S^{\mathrm{I}} = 0$; consequently, the formula for Abbe's Sine-Condition, which is identical with what Seidel has called the Fraunhofer-*Condition* (§ 284), is:

$$\sum_{k=1}^{k=m}\frac{h_k^4}{h_1^4}J_k^2\cdot X_k\cdot\Delta\left(\frac{1}{nu}\right)_k = 0.\qquad(362)$$

Again, we find:

$$h_k^2\boldsymbol{h}_k^2\boldsymbol{J}_k^2 = \frac{\boldsymbol{h}_1^2}{h_1^2}h_k^4J_k^2(1+X_k)^2;$$

hence, for the Astigmatic-Co-efficient:

$$S^{\mathrm{III}} = \sum_{k=1}^{k=m}\frac{h_k^4}{h_1^4}J_k^2(1+X_k)^2\cdot\Delta\left(\frac{1}{nu}\right)_k;\qquad(363)$$

and, since

$$h_k^2\boldsymbol{h}_k^2(J_k-\boldsymbol{J}_k)^2 = T^2,$$

we find also:

$$S^{\mathrm{IV}} = \frac{T^2}{h_1^2\boldsymbol{h}_1^2}\sum_{k=1}^{k=m}\frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k.\qquad(364)$$

The co-efficients of the expressions for the curvatures of the two image-surfaces formed by the infinitely narrow pencils of meridian and sagittal rays can be obtained by combining the two equations (363) and (364).

Finally, since

$$h_k \boldsymbol{h}_k^3 \frac{\boldsymbol{J}_k}{J_k} (J_k - \boldsymbol{J}_k)^2 = T^2 \frac{\boldsymbol{h}_1}{h_1} (1 + X_k),$$

and

$$h_k \boldsymbol{h}_k^3 \frac{\boldsymbol{J}_k^3}{J_k} = \frac{\boldsymbol{h}_1^3}{h_1} h_k^4 J_k^2 (1 + X_k)^3,$$

we have the following expression for the Distortion-Co-efficient:

$$S^{\mathrm{V}} = \frac{T^2}{h_1^2 \boldsymbol{h}_1^2} \sum_{k=1}^{k=m} (1 + X_k) \frac{1}{r_k} \Delta \left(\frac{1}{n} \right)_k - \sum_{k=1}^{k=m} \frac{h_k^4}{h_1^4} J_k^2 (1 + X_k)^3 \cdot \Delta \left(\frac{1}{nu} \right)_k. \quad (365)$$

ART. 104. REMARKS ON SEIDEL'S FORMULÆ: AND REFERENCES TO OTHER GENERAL METHODS.

324. In a masterly discussion of his formulæ, SEIDEL draws also a number of important conclusions of a general kind, which, however, can only be referred to here very briefly. Thus, for example, he points out that it is impossible (except in certain special cases that have comparatively little practical interest) to construct an optical apparatus which will produce a correct image of the 3rd order for all distances of the object. If it is required to form such images of objects at all distances, in addition to SEIDEL's five equations we shall have other conditions also, one of which, known as HERSCHEL's *Equation*, is, in general, in curious contradiction to the so-called FRAUNHOFER- or Sine-Condition expressed by formula (362): so that the two conditions can be satisfied at the same time only in particular cases, one of which is that the image shall be of the same size as the object.

An image of this degree of perfection even in the case of one special object-distance can only be attained by combining in the system of lenses a sufficient number of *separated* surfaces. If the distances between the spherical surfaces are all so small as to be negligible (so that in the formulæ we may put $d_k = 0$), it is easy to show that the conditions of the abolitions of all the errors of the 3rd order are as follows:

$$\Sigma J^2 \cdot \Delta \frac{1}{nu} = 0, \text{ (abolition of aberration along axis);}$$

$$\Sigma J \cdot \Delta \frac{1}{nu} = 0, \text{ (abolition of comatic aberration);}$$

$$n'_m u'_m = n_1 u_1, \quad \Sigma \frac{1}{r} \Delta \frac{1}{n} = 0, \text{ (condition of plane, stigmatic image);}$$

$$n'^2_m - n^2_1 = 0, \text{ (abolition of distortion).}$$

This last condition is compatible with the condition $n'_m u'_m = n_1 u_1$ only in case the optical system is a plane mirror or an infinitely thin plate of glass: and, hence, for an optical system which shall produce images of the 3rd order it is necessary that some of the *d*'s at least shall be different from zero.

325. In connection with the excellent exposition of SEIDEL's theories which is given by Professor SILVANUS P. THOMPSON in an appendix to his English Translation of Dr. O. LUMMER's *Beitraege zur photographischen Optik*,[1] he directs attention to a remarkable memoir published by FINSTERWALDER[2] in 1892, wherein the author, employing SEIDEL's Formulæ, derives the equation of the *Focal Surface*, which is the envelope of the bundle of emergent rays which have their origin at a point outside the optical axis of a centered system of spherical surfaces, and proceeds then to show in a very simple and elegant manner how the definition of the image and the distribution of the light in it depends on the extent of the visual field and on the aperture of the system and also, in the case when the image is real, on the position of the focussing screen.[3] FINSTERWALDER not only obtains by his method results which are in complete accord with those of SEIDEL, but, as Professor THOMPSON states, he has "also investigated the distribution of the light in the coma, and its changes of shape when the position and size of the stop are changed".

326. With regard to other general methods of investigation in Optics, the following paragraphs, also quoted from Professor THOMPSON's chapter on "SEIDEL's Theory of the Five Aberrations", may be appropriately inserted at this place:

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[1] O. LUMMER: *Beitraege zur photographischen Optik: Zft. f. Instr.*, xvii (1897), 208–219; 225–239; 264–271.

SILVANUS P. THOMPSON: Translation of OTTO LUMMER's *Contributions to Photographic Optics* (London, 1900).

[2] S. FINSTERWALDER: Die von optischen Systemen groesserer Oeffnung und groesseren Gesichtsfeldes erzeugten Bilder: *Muench. Abhand. der k. bayer. Akademie der Wiss.* II Cl., XVII Bd., III Abth., 519–587. Published also separately in Muenchen in 1891 by G. FRANZ.

[3] SEIDEL himself had already determined the equation of the Focal Surface, without, however, showing how the equation was obtained. See SEIDEL's paper entitled: Ueber die Theorie der caustischen Flaechen, welche in Folge der Spiegelung oder Brechung von Strahlenbueschelн an den Flaechen eines optischen Apparates erzeugt werden: *Gelehrte Anzeigen k. bayr. Akad. d. Wiss.*, xliv (1857), 241–251. See also a letter written by SEIDEL to KUMMER, and published, so FINSTERWALDER states, in *Sitzungber. der k. Akad. d. Wiss. zu Berlin*, 1867.

"Remarkable as these researches of von Seidel are, it is of interest to note that an even more general method of investigation into lens aberrations had been previously propounded. This is the fragmentary paper of Sir W. Rowan Hamilton,[1] introducing into optics the idea of a 'characteristic function' [see § 39], namely the *time* taken by the light to pass from one point to another of its path. True, he did not work out the relations between the constants of his formulæ and the data of the optical system. Yet the method, as a mathematical method of investigation, is unquestionably more powerful. It has recently, and independently, been revived by Thiesen,[2] whose equations include those of von Seidel.

"The latest development of advanced geometrical optics is due to Professor H. Bruns,[3] who has shown that in general the formulæ that govern the formation of images can be deduced from an originating function of the co-ordinates of the rays—a function termed by him the *eikonal*—by differentiating the same, just as in theoretical mechanics the components of the forces can be deduced by differentiation from the potential function. Bruns's work is based upon the theory of contact-transformations of Sophus Lie. But as yet neither the formulæ of Bruns nor those of Thiesen have been reduced to such shape as to be available for service in the numerical computation of optical systems."

In this connection it may be stated that the applications of Seidel's aberration-formulæ to the calculation and design of optical systems are attended with much difficulty, and on this account practical opticians seem still to prefer to resort to the methods of trigonometrical calculations of the paths of the rays, whereby with relatively less trouble they arrive at safer results and are also able to keep track more easily of the effects of each single surface. The complete solution of the Seidel formulæ is indeed only possible in the case of systems of comparatively simple structure. The greatest practical value of these general formulæ is to guide the optician to a correct basis for the design of his instrument and to supply him, so to speak, with a starting-point for a trigonometrical calculation of the particular system which he aims to achieve. Concerning the use of these formulæ the reader is referred to a valuable and interesting article by A. KOENIG, entitled *Die Berechnung optischer Systeme auf Grund der Theorie der Aberrationen.*[1]

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[1] On some Results of the View of a Characteristic Function in Optics, *B. A. Report* for 1833, p. 360.

[2] M. Thiesen: Beitraege zur Dioptrik: *Berl. Ber.*, 1890; 799–813. See also: Ueber vollkommene Diopter: Wied. *Ann.* (2) xlv (1892), 821–823; Ueber die Construction von Dioptern mit gegebenen Eigenschaften: Wied. *Ann.* (2) xlv (1892), 823–824. Also, J. Classen: *Mathematische Optik* (Schubertsche Sammlung 40), Leipzig, 1901, Chapter XI entitled "Thiesens Theorie der Abbildungsfehler."

[3] H. Bruns: Das Eikonal: *Abhandlungen der math.-phys. Cl. der k. saechsischen Akad. d. Wiss.*, xxi (1895), 321–436. Also published by S. Hirzel, Leipzig, 1895.

In a series of learned papers C. V. L. CHARLIER[2] has given also a method of investigating the spherical aberrations of a centered system of spherical surfaces, which is said to be especially adapted to the practical design of optical instruments. But it is impossible here to do more than merely refer to this work.

326a. In connection with the reference to HAMILTON's work on page 472, attention is directed also to a valuable and comparatively recent paper by Lord RAYLEIGH on "HAMILTON's Principle and the Five Aberrations of VON SEIDEL" (*Phil. Mag.*, June 1908, pages 677–687), in which he shows "how the number and nature of the five constants of aberration can be deduced almost instantaneously from HAMILTON's principle, at any rate if employed in a somewhat modified form." In this paper Lord RAYLEIGH also states that "the rule relating to the curvature of images, generally named after PETZVAL [see § 303], so far, at any rate, as it refers to combinations of *thin* lenses" is due originally to AIRY and CODDINGTON; and, moreover, that "four out of the five aberrations were pretty fully discussed" by these writers before the time of SEIDEL.

Here also may be mentioned a volume of A. PELLETAN entitled *Optique appliquée* (Paris, 1910), in which the author develops the theory of aberrations by means of the function called the "eikonal" introduced by BRUNS (see page 472).

A noteworthy contribution to the theory of the spherical aberrations is contained in a still more recent paper by R. A. SAMPSON,[3] who, developing and extending the method of GAUSS in his *Dioptrische*

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[1] See Chapter VII (pages 373–408) of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR. See also A. KERBER's *Beitraege zur Dioptrik*, published in Leipzig from 1895 to 1899.

[2] C. V. L. CHARLIER: Ueber den Gang des Lichtes durch ein System von sphaerischen Linsen: Upsala, *Nova Acta*, xvi (1893), 1–20; Zur Theorie der optischen Aberrationscurven: *Astr. Nachr.*, cxxxvii (1895), No. 3265, 1–6; Entwurf einer analytischen Theorie zur Construction von astronomischen u. photographischen Objectiven: *Vierteljahrsschrift der astronomischen Gesellschaft*, 31. Jahrgang (1896), Leipzig, pages 266–278. See also a paper by R. STEINHEIL: Ueber die Berechnung zweilinsiger Objektive: *Zft. f. Instr.*, xvii (1897), 338–344, in which the writer says that "Die Arbeit des Hrn. CHARLIER bedeute einen Schritt vorwaerts."

[3] R. A. SAMPSON: A new treatment of optical aberrations: *Phil. Trans.*, Royal Society, London, Series A, Vol. 212 (1912), pp. 149–185.

Untersuchungen, has derived formulæ which, as he says, are particularly appropriate for the numerical calculation of lenses "if the calculations are made with any ordinary type of multiplying machine and not with logarithms." By way of illustration, he gives a complete calculation of the celebrated object-glass of the Fraunhofer heliometer at Koenigsberg which affords indeed a remarkable proof not merely of the accuracy and reliability of the method, but of the ease with which the results are obtained as compared with the usual long and laborious logarithmic processes. This latter circumstance in itself should be sufficient to commend the method to optical engineers.

But the man who, above all others in recent years, has enriched and extended the domain of theoretical optics is the illustrious Allvar Gullstrand, who was awarded the Nobel prize in Medicine in 1911. Discarding as more or less fictitious and misleading all notions of collinear correspondence between object and image, and thus in a sense representing a revolt from the school of Abbe (§ 156), he insists that the fundamental realities of optical imagery are to be found only by investigating the geometric relations of the wave-surface and the orthotomic system of rays (§ 43) which belongs to it (see Art. 15). Other writers before Gullstrand, to be sure, have started with these premises, but it is doubtful whether any one (with the possible exception of Helmholtz) has possessed in such high degree the admirable combination of mathematical insight together with a thorough knowledge of practical optics in all of its ramifications which he has brought to bear in this work. Employing with consummate skill the powerful methods of the differential geometry, Gullstrand has developed a complete general theory of optical imagery, from which it is possible not only to deduce all the Seidel formulæ in the special case of a centered system of spherical surfaces, but also to derive results which are applicable to the more complex optical systems of modern times. These remarkable investigations, begun apparently as long ago as 1890, are contained in a series of notable papers which no student of optics can afford to neglect.[1]

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[1] The following is a partial list of the more recent and important of the optical writings of Allvar Gullstrand:

Ueber Astigmatismus, Koma und Aberration: Drude's *Ann. der Phys.*, xviii (1905), 941–973.—Die reelle optische Abbildung: *Kungl. Sv. Vet. Akad. Handl.*, xli (1906), 1–119. —Tatsachen und Fiktionen in der Lehre von der optischen Abbildung: *Arch. f. Optik*, i (1907–8), 2–41; 81–97.—See also the third edition of Helmholtz's *Handbuch der physiologischen Optik*, Bd. I (Hamburg u. Leipzig, 1909); this volume is largely written by Gullstrand, and contains a summary of his optical theories.

APPENDIX TO CHAPTER XII.

NOTE ON THE CALCULATION OF THE SPHERICAL ERRORS OF AN OPTICAL SYSTEM OF CENTERED LENSES, BY MEANS OF THE SEIDEL FORMULÆ.

I. The General Formulæ.

The five spherical aberrations of the 3d order as originally distinguished by SEIDEL may be again enumerated here in the following order:

1. *The Spherical Aberration* proper, or the aberration of the image of an axial object-point; this error will be denoted by S_1.

2. The defect called *Coma*, in consequence of which the definition of parts of the image not on the axis is impaired; which will be denoted by S_2. The equation $S_2 = 0$ is equivalent also to the *Sine-Condition* which is so important in the optical theories of HELMHOLTZ and ABBE. The so-called FRAUNHOFER-Condition discovered by SEIDEL is included also in the Sine-Condition.

3 and 4. *The Curvatures* of the primary and secondary image-surfaces in consequence of *Astigmatism.* This pair of spherical errors will be denoted by S_3 and S_4. If the two faults S_3, S_4 are both abolished, we obtain a flat, stigmatic image. It may be mentioned, however, that the curvatures of the image depend essentially on the refractivities of the lenses, so that with unsuitable kinds of glass it is impossible by any choice of the geometrical dimensions of the lens-system (radii, thicknesses, distances) to obtain a plane, stigmatic image, as was first recognized by PETZVAL and more clearly still by SEIDEL.

In order to remedy as far as possible this defect of curvature, designers of optical instruments sometimes compromise on a kind of artificial flattening of the image-field; which consists in contriving so that the so-called "circles of least confusion" of the astigmatic bundles of rays shall all be made to fall in a definite transversal image-plane. The curvature of the surface which is the locus of the circles of least confusion is approximately equal to the arithmetical mean of the curvatures of the primary and secondary image-surfaces; accordingly, if the curvature of this surface is denoted by S_3', we may write:

$$S_3' = \frac{S_3 + S_4}{2}.$$

If $S_3' = 0$, the image will be more or less plane, but in general not stigmatic.

On the other hand, again supposing that we cannot obtain an image which is both perfectly stigmatic and perfectly plane ($S_3 = S_4 = 0$), we may be content to disregard the curvature-error and to effect a compromise as to the *Astigmatism;* in which case we can endeavour to make $S_4' = 0$, this latter symbol being defined as follows:

$$S_4' = \frac{S_3 - S_4}{2}.$$

5. *The Distortion* of the marginal parts of the image, which will be denoted by S_5.

The symbols S_1, S_2, etc., which are here introduced are slightly different from the symbols S^{I}, S^{II}, etc., defined by formulæ (356); the connections between the two sets of symbols are as follows:

$$S_1 = S^{\mathrm{I}}; \quad S_2 = S^{\mathrm{II}}; \quad S_3 = 3S^{\mathrm{III}} - S^{\mathrm{IV}}; \quad S_3' = 2S^{\mathrm{III}} - S^{\mathrm{IV}};$$
$$S_4 = S^{\mathrm{III}} - S^{\mathrm{IV}}; \quad S_4' = S^{\mathrm{III}}; \quad S_5 = -S^{\mathrm{V}}.$$

It will be convenient to write

$$\boldsymbol{J}_1 - J_1 = a,$$

so that according to the first of formulæ (358):

$$a = -\frac{T}{h_1 \boldsymbol{h}_1};$$

and, hence, from the first of formulæ (360), we obtain:

$$1 + X_k = \frac{h_1}{\boldsymbol{h}_1}\left(\frac{a h_1 \boldsymbol{h}_1}{h_k^2 J_k} + \frac{\boldsymbol{h}_k}{h_k}\right).$$

In addition to X_k, let us also introduce the following abbreviation-symbols:

$$Y_k = h_k^4 J_k^2 \Delta\left(\frac{1}{nu}\right)_k, \quad Z_k = \frac{1}{r_k}\Delta\left(\frac{1}{n}\right)_k;$$

and, moreover, let the symbols N, P, Q and R be used to denote the following sums:

$$N = -\frac{1}{a h_1^4}\sum_{k=1}^{k=m} Y_k X_k;$$

$$P = \sum_{k=1}^{k=m} Z_k - \frac{1}{a^2 h_1^4}\sum_{k=1}^{k=m} Y_k X_k^2;$$

$$Q = \sum_{k=1}^{k=m} Z_k - \frac{3}{a^2 h_1^4}\sum_{k=1}^{k=m} Y_k X_k^2;$$

$$R = \frac{1}{a}\sum_{k=1}^{k=m} Z_k X_k - \frac{1}{a^3 h_1^4}\sum_{k=1}^{k=m} Y_k X_k^3.$$

Then from the first of formulæ (356) and from formulæ (361), (363), (364) and (365), we can write the expressions of the SEIDEL spherical errors in the following form:

$$\begin{aligned}
S_1 &= \frac{1}{h_1^4}\sum_{k=1}^{k=m} Y_k;\\
S_2 &= S_1 - aN;\\
S_3 &= 3S_1 - 6aN - a^2Q;\\
S_4 &= S_1 - 2aN - a^2P;\\
S_3' &= 2S_1 - 4aN - a^2\frac{P+Q}{2};\\
S_4' &= S_1 - 2aN + a^2\frac{P-Q}{2};\\
S_5 &= S_1 - 3aN - a^2Q - a^3R.
\end{aligned}$$

The practical problem consists in determining the radii, thicknesses, distances, etc., and the kinds of glass in such way that one or more of SEIDEL's five expressions for the spherical errors will be made to vanish. If we could calculate an optical system of centered spherical refracting surfaces for which

$$S_1 = N = P = Q = R = 0,$$

then the monochromatic image of a plane object at right angles to the axis will be at once sharply defined, flat and true throughout, so that it will indeed coincide completely with the so-called GAUSSian or theoretical (collinear) image. Although it is impossible to attain this degree of perfection, we may at least endeavour to design the optical system so as to abolish those faults, which, for the particular type of instrument in view (telescope, microscope, photographic objective, etc.), are the most detrimental to the image, and perhaps also to minimize the residual errors. Expressions for the magnitudes of these residual errors can be derived from the above equations.

When the so-called PETZVAL Condition (§ 303), viz.,

$$\sum_{k=1}^{k=m} Z_k = 0,$$

is satisfied, we have $Q = 3P$. If in addition one of the sums denoted by S_3, S_4, S_3', S_4' vanishes also, all of them will vanish.

On account of the algebraic difficulties involved in the solution of these equations, the method, as has been stated (§ 326), is applicable only to optical systems of comparatively simple structure. The analytical hindrances are very greatly reduced provided we can neglect the thicknesses of the lenses, and a still further simplification will be introduced if we may also disregard the intervals between each pair of successive lenses.

II. Formulæ for a System of Infinitely Thin Lenses.

The expressions for the spherical errors of a centered system of infinitely thin lenses, separated by finite intervals, may be easily derived from formulæ (356). Employing the above symbols for the spherical errors, and supposing that the ith lens is included between the kth and $(k+1)$th spherical surfaces, we may conveniently introduce also the following abbreviation-symbols:

$$\sum_{k=k}^{k=k+1} J_k^2 \Delta\left(\frac{1}{nu}\right)_k = A_i;$$

$$\sum_{k=k}^{k=k+1} J_k \boldsymbol{J}_k \Delta\left(\frac{1}{nu}\right)_k = B_i;$$

$$\sum_{k=k}^{k=k+1} \left\{ 3\boldsymbol{J}_k^2 \Delta\left(\frac{1}{nu}\right)_k - (\boldsymbol{J}_k - J_k)^2 \frac{1}{r_k} \Delta\left(\frac{1}{n}\right)_k \right\} = C_i;$$

$$\sum_{k=k}^{k=k+1} \left\{ \boldsymbol{J}_k^2 \Delta\left(\frac{1}{nu}\right)_k - (\boldsymbol{J}_k - J_k)^2 \frac{1}{r_k} \Delta\left(\frac{1}{n}\right)_k \right\} = D_i;$$

$$\sum_{k=k}^{k=k+1} \frac{1}{J_k} \left\{ \boldsymbol{J}_k^3 \Delta\left(\frac{1}{nu}\right)_k - (\boldsymbol{J}_k - J_k)^2 \frac{1}{r_k} \Delta\left(\frac{1}{n}\right)_k \right\} = E_i.$$

Considering each lens as surrounded by air, so that

$$n'_{k-1} = n'_{k+1} = 1, \quad n_k' = n_i,$$

and using the ordinary lens-notation as given by formulæ (291), let us also introduce the symbols c and x in place of r and u according to the following scheme:

$$\frac{1}{r_k} = c_i, \quad \frac{1}{r_{k+1}} = c_i';$$

$$\frac{1}{u_k} = x_i, \quad \frac{1}{u_k'} = \frac{1}{u_{k+1}} = \frac{x_i + (n_i - 1)c_i}{n_i}, \quad \frac{1}{u'_{k+1}} = x_i';$$

$$\frac{1}{\boldsymbol{u}_k} = \boldsymbol{x}_i, \quad \frac{1}{\boldsymbol{u}_k'} = \frac{1}{\boldsymbol{u}_{k+1}} = \frac{\boldsymbol{x}_i + (n_i - 1)c_i}{n_i}, \quad \frac{1}{\boldsymbol{u}'_{k+1}} = \boldsymbol{x}_i'.$$

Since in an infinitely thin lens

$$x_i' - x_i = \boldsymbol{x}_i' - \boldsymbol{x}_i = (n_i - 1)(c_i - c_i') = \phi_i,$$

we obtain:

$$J_k = c_i - x_i; \quad J_{k+1} = c_i' - x_i' = c_i - x_i - \frac{n_i}{n_i - 1}\phi_i;$$

$$\mathbf{J}_k = c_i - \mathbf{x}_i; \quad \mathbf{J}_{k+1} = c_i' - \mathbf{x}_i' = c_i - \mathbf{x}_i - \frac{n_i}{n_i - 1}\phi_i.$$

Employing these relations, we can find the following expressions for A_i, B_i, etc., in terms of ϕ_i, c_i, x_i, $\mathbf{x}_i$ and n_i:

$$\begin{aligned} A_i = {} & \frac{n_i + 2}{n_i}\varphi_i c_i^2 - \left\{ \frac{4(n_i + 1)}{n_i} x_i + \frac{2n_i + 1}{n_i - 1}\varphi_i \right\} \varphi_i c_i \\ & + \frac{3n_i + 2}{n_i}\varphi_i x_i^2 + \frac{3n_i + 1}{n_i - 1}\varphi_i x_i + \left(\frac{n_i}{n_i - 1} \right)^2 \varphi_i^3; \end{aligned}$$

$$\begin{aligned} B_i = {} & \frac{n_i + 2}{n_i}\varphi_i c_i^2 - \left\{ \frac{n_i + 1}{n_i}(3x_i + \mathbf{x}_i) + \frac{2n_i + 1}{n_i - 1}\varphi_i \right\} \varphi_i c_i \\ & + \frac{n_i + 1}{n_i}\varphi_i x_i^2 + \frac{2n_i + 1}{n_i}\varphi_i x_i \mathbf{x}_i + \frac{2n_i + 1}{n_i - 1}\varphi_i^2 x_i \\ & + \frac{n_i}{n_i - 1}\varphi_i^2 \mathbf{x}_i + \left(\frac{n_i}{n_i - 1} \right)^2 \varphi_i^3; \end{aligned}$$

$$\begin{aligned} C_i = {} & \frac{3(n_i + 2)}{n_i}\varphi_i c_i^2 - \left\{ \frac{6(n_i + 1)}{n_i}(x_i + \mathbf{x}_i) + \frac{3(2n_i + 1)}{n_i - 1}\varphi_i \right\} \varphi_i c_i \\ & + \frac{1}{n_i}\varphi_i x_i^2 + \frac{2(3n_i + 2)}{n_i}\varphi_i x_i \mathbf{x}_i + \frac{3n_i + 1}{n_i}\varphi_i \mathbf{x}_i^2 \\ & + \frac{3(n_i + 1)}{n_i - 1}\varphi_i^2 x_i + \frac{6n_i}{n_i - 1}\varphi_i^2 \mathbf{x}_i + 3\left(\frac{n_i}{n_i - 1} \right)^2 \varphi_i^3; \end{aligned}$$

$$\begin{aligned} D_i = {} & \frac{n_i + 2}{n_i}\varphi_i c_i^2 - \left\{ \frac{2(n_i + 1)}{n_i}(x_i + \mathbf{x}_i) + \frac{2n_i + 1}{n_i - 1}\varphi_i \right\} \varphi_i c_i + \frac{1}{n_i}\varphi_i x_i^2 \\ & + 2\varphi_i x_i \mathbf{x}_i + \frac{n_i + 1}{n_i}\varphi_i \mathbf{x}_i^2 + \frac{n_i + 1}{n_i - 1}\varphi_i^2 x_i \\ & + \frac{2n_i}{n_i - 1}\varphi_i^2 \mathbf{x}_i + \left(\frac{n_i}{n_i - 1} \right)^2 \varphi_i^3; \end{aligned}$$

$$\begin{aligned} E_i = {} & \frac{n_i + 2}{n_i}\varphi_i c_i^2 - \left\{ \frac{n_i + 1}{n_i}(x_i + 3\mathbf{x}_i) + \frac{2n_i + 1}{n_i - 1}\varphi_i \right\} \varphi_i c_i + \frac{3n_i + 1}{n_i}\varphi_i \mathbf{x}_i^2 \\ & + \frac{1}{n_i}\varphi_i x_i \mathbf{x}_i + \frac{1}{n_i - 1}\varphi_i^2 x_i + \frac{3n_i}{n_i - 1}\varphi_i^2 \mathbf{x}_i + \left(\frac{n_i}{n_i - 1} \right)^2 \varphi_i^3. \end{aligned}$$

Concerning these expressions, it may be observed that $A_i = \phi_i Z_i$, where Z is defined in § 269 as the expression within the square brackets on the right-hand side of formula (292); $B_i = \phi_i V_i$, where V is defined in § 315; $\frac{1}{2}(C_i - D_i) = \phi_i U_i$, where U is defined in § 306; and, finally, $E_i = \varphi_i X_i$, where X is defined in § 294.

Thus for a system of m infinitely thin lenses, we have the following expressions for the spherical errors:

$$S_1 = \sum_{i=1}^{i=m} \frac{h_i^4}{h_1^4} A_i; \qquad S_2 = \sum_{i=1}^{i=m} \frac{h_i^3}{h_1^3} \frac{\boldsymbol{h}_i}{\boldsymbol{h}_1} B_i;$$

$$S_3 = \sum_{i=1}^{i=m} \frac{h_i^2}{h_1^2} \frac{\boldsymbol{h}_i^2}{\boldsymbol{h}_1^2} C_i; \quad S_3' = \sum_{i=1}^{i=m} \frac{h_i^2}{h_1^2} \frac{\boldsymbol{h}_i^2}{\boldsymbol{h}_1^2} \frac{C_i + D_i}{2};$$

$$S_4 = \sum_{i=1}^{i=m} \frac{h_i^2}{h_1^2} \frac{\boldsymbol{h}_i^2}{\boldsymbol{h}_1^2} D_i; \quad S_4' = \sum_{i=1}^{i=m} \frac{h_i^2}{h_1^2} \frac{\boldsymbol{h}_i^2}{\boldsymbol{h}_1^2} \frac{C_i - D_i}{2};$$

$$S_5 = \sum_{i=1}^{i=m} \frac{h_i}{h_1} \frac{\boldsymbol{h}_i^3}{\boldsymbol{h}_1^3} E_i.$$

III. Abolition of Two of the Spherical Errors of a Thin Lens-System.[1]

In the special case when the lenses are all in contact, and the entire thickness of the system may be regarded as negligible, that is, in the case of a *thin lens-system*, we have:

$$h_k = h_1, \quad \boldsymbol{h}_k = \boldsymbol{h}_1,$$

and the expressions denoted by N, P, Q and R will be found to have the following forms:

$$N = \sum_{i=1}^{i=m} \left\{ \frac{n_i}{n_i - 1} \varphi_i^2 + \frac{2n_i + 1}{n_i} \varphi_i x_i - \frac{n_i + 1}{n_i} \varphi_i c_i \right\},$$

$$P = -\sum_{i=1}^{i=m} \frac{n_i + 1}{n_i} \varphi_i, \quad Q = -\sum_{i=1}^{i=m} \frac{3n_i + 1}{n_i} \varphi_i, \quad R = 0.$$

Moreover, for a system of infinitely thin lenses in contact we have:

$$x_i - x_1 = \sum_{r=1}^{r=i-1} \phi_r = \boldsymbol{x}_i - \boldsymbol{x}_1,$$

and, hence:

$$x_i - \boldsymbol{x}_i = x_1 - \boldsymbol{x}_1 = a.$$

The general expression of a spherical error of a thin lens-system may now be written in the following form:

$$S = pS_1 - a \left\{ qN - a \sum_{i=1}^{i=m} L_i \phi_i \right\},$$

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[1] J. P. C. SOUTHALL: Abolition of Two of the Spherical Errors of a Thin Lens-System: *Astrophys. Journ.*, xxxiii (1911), 330–337.

where p, q have certain integral positive values (between 0 and $+6$), and L_i denotes a function of n_i. The values of p, q and L_i corresponding to each of the spherical errors S_1, S_2, etc., are exhibited in the subjoined table:

S	S_1	S_2	S_3	S_4	S_3'	S_4'	S_5
p	1	1	3	1	2	1	1
q	0	1	6	2	4	2	3
L_i	0	0	$\frac{3n_i+1}{n_i}$	$\frac{n_i+1}{n_i}$	$\frac{2n_i+1}{n_i}$	1	$\frac{3n_i+1}{n_i}$

Let S, S' and S'' denote three of the spherical errors of a thin lens-system, and according to the above let us write:

$$S = pS_1 - a\left\{qN - a\sum_{i=1}^{i=m} L_i\phi_i\right\},$$

$$S' = p'S_1 - a\left\{q'N - a\sum_{i=1}^{i=m} L_i'\phi_i\right\},$$

$$S'' = p''S_1 - a\left\{q''N - a\sum_{i=1}^{i=m} L_i''\phi_i\right\}.$$

If the first error (S) has been corrected ($S = 0$), the condition of the simultaneous abolition of the second error (S'), that is, the condition of the fulfilment of the additional requirement $S' = 0$, will be given by the following convenient equation:

$$\sum_{i=1}^{i=m}\left\{l\frac{n_i}{n_i-1}\varphi_i^2 + \left(l\frac{2n_i+1}{n_i} + M_i\right)\varphi_i x_i - M_i\varphi_i\mathbf{x}_i - l\frac{n_i+1}{n_i}\varphi_i c_i\right\} = 0,$$

where

$$l = p'q - pq',$$

$$M_i = pL_i' - p'L_i.$$

If two spherical errors have been abolished ($S = S' = 0$), the magnitude of a third residual spherical error (S'') will depend only on the strengths of the lenses and their refractive indices, and will, therefore, not be affected by any further bending of the lenses or by any alteration in the order of sequence of the lenses. Under these circumstances we find in fact:

$$S'' = a^2\sum_{i=1}^{i=m} K_i\phi_i, \quad (S = S' = 0),$$

where

$$K_i = \frac{(p'q'' - p''q')L_i + (p''q - pq'')L_i' + (pq' - p'q)L_i''}{pq' - p'q}.$$

The subjoined table exhibits the values of l and M_i for the simultaneous abolition of any pair of the spherical errors denoted by S, S' and also the expressions for K_i for each one of the spherical errors denoted by S''. For convenience of printing, the subscript i has been

l	M_i	K_i						
		1	2	3	4	$3'$	$4'$	5
-1	0	0	0	$\frac{3n+1}{n}$	$\frac{n+1}{n}$	$\frac{2n+1}{n}$	$+1$	$\frac{3n+1}{n}$
-6	$\frac{3n+1}{n}$	0	$-\frac{3n+1}{6n}$	0	$\frac{2}{3n}$	$\frac{1}{3n}$	$-\frac{1}{3n}$	$\frac{3n+1}{2n}$
-2	$\frac{n+1}{n}$	0	$-\frac{n+1}{2n}$	$-\frac{2}{n}$	0	$-\frac{n+1}{n}$	$-\frac{1}{n}$	$\frac{3n-1}{n}$
-4	$\frac{2n+1}{n}$	0	$-\frac{2n+1}{4n}$	$-\frac{1}{2n}$	$\frac{1}{2n}$	0	$-\frac{1}{2n}$	$\frac{6n+1}{4n}$
-2	$+1$	0	$-\frac{1}{2}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	0	$\frac{3n+2}{2n}$
-3	$\frac{3n+1}{n}$	0	$-\frac{3n+1}{3n}$	$-\frac{3n+1}{n}$	$-\frac{3n-1}{3n}$	$-\frac{6n+1}{3n}$	$-\frac{3n+2}{3n}$	0
-3	$\frac{3n+1}{n}$	$\frac{3n+1}{3n}$	0	0	$\frac{2}{3n}$	$\frac{1}{3n}$	$-\frac{1}{3n}$	$\frac{3n+1}{3n}$
-1	$\frac{n+1}{n}$	$\frac{n+1}{n}$	0	$-\frac{2}{n}$	0	$-\frac{1}{n}$	$-\frac{1}{n}$	$\frac{n-1}{n}$
-2	$\frac{2n+1}{n}$	$\frac{2n+1}{n}$	0	$-\frac{1}{2n}$	$\frac{1}{2n}$	0	$-\frac{1}{2n}$	$+1$
-1	$+1$	$+1$	0	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	0	$\frac{n+1}{n}$
-2	$\frac{3n+1}{n}$	$\frac{3n+1}{2n}$	0	$-\frac{3n+1}{2n}$	$-\frac{n-1}{2n}$	-1	$-\frac{n+1}{2n}$	0
-3	$\frac{6n+2}{n}$	$\frac{3n+1}{n}$	$\frac{3n+1}{3n}$	0	$\frac{2}{3n}$	$\frac{1}{3n}$	$-\frac{1}{3n}$	0
-1	$+2$	$\frac{3n-1}{n}$	$\frac{n-1}{n}$	$-\frac{2}{n}$	0	$-\frac{1}{n}$	$-\frac{1}{n}$	0
-2	$\frac{4n+1}{n}$	$\frac{6n+1}{2n}$	$+1$	$-\frac{1}{2n}$	$\frac{1}{2n}$	0	$-\frac{1}{2n}$	0
-1	$\frac{2n+1}{n}$	$\frac{3n+2}{n}$	$\frac{n+1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	0	0

omitted from the letter n in this table. The numerals 1, 2, 3, 4, 3′, 4′, 5 at the heads of the columns under K_i refer to the spherical errors denoted by S_1, S_2, S_3, S_4, S_3', S_4', S_5, respectively. It will be observed that in each horizontal row of the table there are always two values $K_i = 0$; the first or left-hand one of these corresponds to $S = 0$ and the right-hand one to $S' = 0$.

As an illustration of the method of procedure in a comparatively simple case, suppose that we wish to design an ordinary telescope-objective composed of two thin lenses cemented together, so that $c_1' = c_2$. The kinds of glass (n_1, n_2) will be selected with a view to the correction of the colour-faults or chromatic aberrations, but we need not concern ourselves with this phase of the problem at present. In this connection, however, it may be remarked in passing that with regard to the correction of the colour-faults (which are treated in Chapter XIII) the modern optician possesses an immense advantage over his predecessors on account of the variety of the new kinds of optical glass that are now at his disposal. The favourable circumstance that varieties of glass of equal refractivity but of different dispersion are now available, enables the designer of a system of lenses to postpone the colour-correction until after he has effected the correction of the spherical errors.

Since in the optical system above proposed there are three radii to be determined, we can impose three other conditions, which would probably be:

1. A prescribed focal length; so that in the equation

$$\phi_1 + \phi_2 = \phi$$

the strength of the combination, denoted by $\phi = 1/f$, has an assigned value, which is usually put equal to unity; whereby all linear magnitudes are thus expressed in terms of the focal length of the system as unity.

2. Abolition of the spherical aberration in the centre of the field of view ($S_1 = 0$); which will be expressed by the equation:

$$\sum_{i=1}^{i=2} \left\{ \frac{n_i + 2}{n_i} \phi_i c_i^2 - \left(\frac{4(n_i + 1)}{n_i} x_i + \frac{2n_i + 1}{n_i - 1} \varphi_i \right) \varphi_i c_i + \frac{3n_i + 2}{n_i} \varphi_i x_i^2 \right.$$
$$\left. + \frac{3n_i + 1}{n_i - 1} \varphi_i x_i + \left(\frac{n_i}{n_i - 1} \right)^2 \varphi_i^3 \right\} = 0.$$

3. Abolition of Coma ($S_2 = 0$), whereby not only the centre but the adjacent surrounding parts of the field of view will be portrayed

distinctly. This condition in conjunction with the preceding involves the additional equation:

$$\sum_{i=1}^{i=2}\left\{\frac{n_i}{n_i-1}\varphi_i^2+\frac{2n_i+1}{n_i}\varphi_i x_i-\frac{n_i+1}{n_i}\varphi_i c_i\right\}=0.$$

According to the very practical method of calculation given by E. VON HÖEGH,[1] it is best to regard ϕ_1 as the unknown to be first determined; whence the required curvatures of the lens-surfaces can be determined without difficulty. The first step is to eliminate x_2 and c_2 by means of the relations

$$x_2 = x_1 + \phi_1, \quad c_2 = c_1 - \frac{\phi_1}{n_1 - 1};$$

so that finally, after eliminating ϕ_2 by means of the first of the equations above, we shall obtain an equation of the fifth degree in ϕ_1 of the following form:

$$A\phi_1^5 + B\phi\phi_1^4 + C\phi^2\phi_1^3 + D\phi^3\phi_1^2 + E\phi^4\phi_1 + F\phi^5$$
$$+ (G\phi_1^4 + H\phi\phi_1^3 + I\phi^2\phi_1^2 + J\phi^3\phi_1 - \phi^4 - \phi^3 x_1)x_1 = 0;$$

where the coefficients denoted by A, B, C, etc., are functions of n_1 and n_2. If the object is infinitely distant, we must put $x_1 = 0$, and then (after putting also $\phi = 1$, as above stated) the equation will be simplified as follows:

$$A\phi_1^5 + B\phi_1^4 + C\phi_1^3 + D\phi_1^2 + E\phi_1 + F = 0.$$

If, for example, adopting the illustration given by GLEICHEN, we choose for the front (crown-glass) lens the value $n_1 = 1.58$ and for the second or flint-glass lens the value $n_2 = 1.62$, we find the following equation:

$$\phi_1^5 + 228.0248\,\phi_1^4 + 6058.5514\,\phi_1^3 - 8977.2146\,\phi_1^2$$
$$- 3696.4855\,\phi_1 - 55787.2820 = 0.$$

This equation will be found to have only one real root, viz.:

$$\phi_1 = 2.696\,954;$$

and hence also: $\phi_2 = -1.696\,954$. Thus, for the curvatures of the lens-surfaces we find:

$$c_1 = +1.565\,99;\ c_1' = c_2 = -3.083\,93;\ c_2' = -0.346\,91.$$

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[1] E. VON HÖEGH: Zur Theorie der zweitheiligen verkitteten Fernrohrobjective: *Zft. f. Instr.*, xix (1899), 37–39. See also: A. GLEICHEN: *Lehrbuch der geometrischen Optik* (Leipzig und Berlin, 1902), 331–334.

An incidental advantage connected with v. HÖEGH's method of calculation deserves to be pointed out. Having found the equation of the fifth degree in ϕ_1 for given values of n_1, n_2 and x_1, we can derive from it a second equation similar to the first which will enable us to calculate a second set of values of ϕ_1 for a combination of the same two kinds of glass in the reverse order, merely by putting $-(x_1 + \phi)$ in place of x_1 in the first equation. The reason of this is because $x_2' = x_1 + \phi$. For example, suppose that we are calculating a telescope-objective for which $x_1 = 0$, and that we have obtained v. HÖEGH's formula for a given combination of crown glass for the first lens and flint glass for the second lens; we can obtain thence a second equation by substituting $-\phi$ in place of x_1 in the first equation; whereby we can determine the dimensions of a second optical system composed of the same two kinds of glass, but with the flint glass for the first lens and the crown glass for the second lens, which will satisfy the given conditions for the new object-distance.

In the case of the optical system here under consideration, we have abolished the spherical aberration and the coma-error ($S_1 = S_2 = 0$); the magnitudes of the residual aberrations can be found from the following expressions:

$$S_3 = S_5 = a^2 \sum_{i=1}^{i=2} \frac{3n_i + 1}{n_i} \phi_i; \quad S_4 = a^2 \sum_{i=1}^{i=2} \frac{n_i + 1}{n_i} \phi_i;$$

$$S_3' = a^2 \sum_{i=1}^{i=2} \frac{2n_i + 1}{n_i} \phi_i; \quad S_4' = a^2 \sum_{i=1}^{i=2} \phi_i; \quad S_1 = S_2 = 0.$$

The above illustration will suffice to give some idea of the methods of optical calculation on the basis of the SEIDEL formulæ for the spherical aberrations of the third order. Lack of space forbids us to treat this subject more at length; but for a complete and admirable discussion of the whole matter the reader is again referred to A. KOENIG's article entitled "Die Berechnung optischer Systeme auf Grund der Theorie der Aberrationen," Chapter VII of v. ROHR's *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904).

CHAPTER XIII.

COLOUR-PHENOMENA.

I. DISPERSION AND PRISM-SPECTRA.

ART. 105. INTRODUCTORY AND HISTORICAL.

327. Relation between the Refractive Index and the Wave-Length. In the preceding chapters it has been tacitly assumed that the index of refraction (n) of an isotropic optical medium was a constant magnitude; which assumption was permissible so long as we were concerned only with light of some definite kind or colour. The length (λ) of a light-wave depends on two factors, the speed of propagation (v) and the vibration-number or frequency (N), according to the familiar formula:

$$\lambda = v/N.$$

Light of a definite colour is characterized by a definite value of the frequency N, which is not altered when the light is refracted from one medium into another. On the other hand, the speed (v) with which the light is propagated is different in different media, and, consequently, the wave-length (λ) must vary also. However, if we select some standard medium (§ 24), as, for example, the free ether of empty space (wherein also light of all colours is propagated with the same speed), the wave-length of the light in this medium may be employed also to characterize the colour of the light. In this chapter, therefore, the symbol λ will be used to denote always *the wave-length of the light in vacuo*.

The refractive index of a given medium is a function of the wave-length λ; so that we may write:

$$n = f(\lambda).$$

The exact character of this relation has never been definitely ascertained, although a number of formulæ have been proposed. The earliest and best known of such formulæ is the one suggested by CAUCHY,[1] as follows:

$$n = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots,$$

where A, B, C, etc., denote constants depending on the nature of the medium and diminishing rapidly in magnitude as we proceed to the higher terms of the series. The formula shows that the waves of the shorter wave-lengths are the more highly refracted. In media which exhibit the so-called phenomenon of "*anomalous dispersion*" it is, however, not true that the shorter waves have the higher indices of refraction, so that the formula is by no means general; but within certain limits it is found to represent fairly well the results of experiments. An investigation of the experimental data in regard to this matter shows that, in general, as many as three coefficients A, B, C will be required in order to express completely the relation between n and λ for all optical media; although, as SCHMIDT[1] has shown, in the case of a number of substances, the relation may be right well expressed by a series with only two constants.

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[1] A. L. CAUCHY: *Mémoire sur la dispersion de la lumière;* published in Prague in 1836.

We see, therefore, that until we specify the kind of light that is being used, the refractive index of a medium is a phrase without meaning; for a medium has just as many indices of refraction as there are different kinds of light. If, for example, a given straight line is the common path of rays of two or more kinds of light, these rays will, in general, be separated by refraction and made to take different routes when they enter a new medium. This phenomenon is called *Dispersion* of the Light, sometimes called also the "chromatic dispersion".

328. Newton's Prism-Experiments and the Fraunhofer Lines of the Solar Spectrum. The discovery and explanation of the fact that the light of the sun is composite and consists of light of a great variety of colours is unquestionably the greatest of NEWTON's contributions to optical science. Admitting the rays of the sun through a small circular opening in the window-shutter, NEWTON caused these rays to pass through a glass prism, and was surprised to find that the image on the opposite wall, instead of being a circular spot of white light (as was produced before the interposition of the prism in the path of the beam) was an elongated *spectrum*, with vivid colours, and about five times as long as it was broad. NEWTON's remarkable series of prism-experiments was begun in the year 1666: a complete description of them was afterwards published in his treatise on Optics.[2] He was led to conclude that sun-light is not homogeneous, but is composed of rays of different colours, some of which are more refrangible than others, the red rays being the least refracted and the violet rays the most refracted; so that the coloured spectrum varied by imperceptible gradations of colour from red at one end to violet at the other;

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[1] W. SCHMIDT: *Die Brechung des Lichts in Glaesern* (Leipzig, 1874).

[2] ISAAC NEWTON: *Opticks: or a treatise of the reflexions, refractions, inflexions and colours of light* (London, 1704). The discovery of Dispersion and the explanation of the colours of the Spectrum was communicated to the Royal Society in 1672.

the order of the colours (as they were distinguished by NEWTON) being red, orange, yellow, green, blue, indigo and violet.

The important practical problem of abolishing, if possible, the chromatic aberrations of optical instruments, especially in the case of the telescope, raised the question as to whether the dispersions of different substances were such as to allow of combinations which neutralized the dispersion without at the same time neutralizing the refraction. NEWTON himself conceived that he had proved by experiment (*Opticks*, Book i, Part ii, Prop. 3) that achromatism involved necessarily the abolition of ray-deviation also; so that in an achromatic combination the emergent rays must needs be parallel to the corresponding incident rays. NEWTON concluded, therefore, that it was impossible to produce an achromatic image by refraction, and it was this error that "made him despair of improving refracting telescopes and led him to turn his attention to the application of mirrors to these instruments".[1] NEWTON's authority on such questions was so great that for a long time his view was accepted as settling the matter.

EULER,[2] approaching the subject from a theoretical stand-point, and basing his argument on the erroneous assumption that the human eye is an achromatic combination of lenses, deduced the correct conclusion that such combinations were possible, and calculated the conditions that were necessary therefor, although he lacked sufficient experimental data. In 1754 KLINGENSTIERNA,[3] in Sweden, succeeded in showing by a combination of two prisms not only the deviation of the rays without dispersion, but also the dispersion of the rays without deviation.

HEATH[4] states that the mistake in NEWTON's experiment (above referred to) "was first discovered by a gentleman of Worcestershire named HALL, who made the first achromatic telescope"; but that "this discovery was allowed to fall into oblivion, until the experiment was again tried by DOLLOND, an optician in London, who found that the dispersion could be corrected without destroying the refraction and therefore that NEWTON's conclusion was not correct". In 1757, DOLLOND was able to construct an achromatic telescope by the use of two kinds of glass called "crown glass" and "flint glass", of which the former is the weaker in respect to both refraction and dispersion.

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[1] See HEATH's *Geometrical Optics* (Cambridge, 1887), Art. 179.

[2] L. EULER: Sur la perfection des verres objectifs des lunettes: *Mém. de Berlin*, iii (1747), 274–296.

[3] S. KLINGENSTIERNA: Anmerkung ueber das Gesetz der Brechung bei Lichtstrahlen von verschiedener Art, wenn sie durch ein durchsichtiges Mittel in verschiedene andere gehen: *Svensk. Vet. Acad. Handl.*, xv (1754), 300–306.

[4] HEATH's *Geometrical Optics* (Cambridge, 1887), Art. 179.

In this combination the convergent lens was made of crown glass and the divergent lens of flint glass.

DOLLOND's success revived interest in the question, and a number of mathematicians, for example, EULER, CLAIRAUT and D'ALEMBERT, proceeded to investigate formulæ for calculating optical systems; but so long as the numerical constants of the different kinds of glass were not available, these labours were necessarily unproductive; and no farther progress worth recording was achieved until the era of FRAUNHOFER (1814), whose brilliant researches marked the dawn of a new day in optical science. By looking through a prism at a very narrow slit, formed by the window-shutters of a darkened room, WOLLASTON[1] had detected in 1802 that the solar spectrum was crossed by dark bands; but it was not until these so-called FRAUNHOFER Lines were independently re-discovered by FRAUNHOFER[2] in a far more thorough and scientific manner that their real significance and value were recognized.

In the Prism-Spectroscope, such as was afterwards used by KIRCHHOFF and BUNSEN, the source of the light is an illuminated slit placed parallel to the edge of the prism in the focal plane of a collimating lens; whereby the rays incident on the first face of the prism are rendered parallel. If, after emerging from the prism, the rays are made to pass through a second convergent lens, there will be formed in the focal plane of this lens a series of images of the slit, each image corresponding to light of a definite colour or wave-length (§ 327). If the slit is illuminated by monochromatic light, there will be only one image, but if the incident rays are composed of light, say, of two kinds, of wave-lengths λ_1 and λ_2, we shall have two slit-images side by side and more or less separated from each other depending, among other things, on the magnitude of the interval $\lambda_1 - \lambda_2$. If

$$\lambda_2 = \lambda_1 + d\lambda_1,$$

the two slit-images will be immediately adjacent to each other, and they may partly overlap and blur each other. If the slit is illuminated by white light emitted originally by an incandescent solid, for example, the light of an electric arc, there will be formed in the focal plane of the receiving lens a *continuous spectrum*, consisting of an innumerable series of coloured images of the slit of every gradation of shade from red to violet, one image for each of the infinite varieties of the light that is emitted by the source. A definite wave-length (λ) is associated with each colour, and to each wave-length there corresponds also a definite value of the refractive index (n), which increases continuously from its greatest value for the extreme red light to its least value for the extreme violet light.

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[1] W. H. WOLLASTON: A method of examining refractive and dispersive powers, by prismatic reflection: *Phil. Trans.*, ii (1802), 365–380.

[2] A preliminary report of FRAUNHOFER's work was communicated to the academy of sciences in Munich in the years 1814 and 1815. See also: JOSEPH FRAUNHOFER: Bestimmung des Brechungs- und Farbenzerstreuungsvermoegens verschiedener Glassorten, in Bezug auf die Vervollkommnung achromatischer Fernroehre: GILBERTS *Ann.*, lvi (1817), 264–313.

However, the *solar spectrum* obtained when the slit is illuminated by sun-light is *not continuous*, as NEWTON supposed, but is crossed by a vast number of dark bands parallel to the slit, corresponding, as we know now, to those radiations which are absent from the light that comes to us from the sun. It would be more correct to say that these dark places indicate a relative deficiency of intensity of certain definite kinds of light in what we call sun-light. These FRAUNHOFER Lines are irregularly distributed over the entire extent of the solar spectrum, and although their actual positions will be altered if we replace the prism of the spectroscope by another one of different material, the order of the lines and of the coloured intervals between them is always the same, so that any line can be readily recognized. The great importance of these lines for optical science consists, as FRAUNHOFER was quick to perceive, in the fact that each line corresponds to a definite wave-length of light, and hence we can employ them in the determinations of the refractive indices of a substance. The more conspicuous of the lines in the different parts of the spectrum were designated by FRAUNHOFER by the capital letters of the Latin alphabet from A to H; the violet end of the spectrum, as nearly as he could locate it, being designated by the letter J. The indices of refraction of a given substance for rays of light of wave-lengths corresponding to the FRAUNHOFER Lines A, B, C, $\cdots$ are usually denoted by the symbols n_A, n_B, n_C, $\cdots$.

329. The Jena Glass. Now that it was possible to determine accurately the optical properties of different media, the great obstacle in the way of perfecting optical instruments so as to fulfil as far as possible the theoretical requirements was found to be the lack of suitable kinds of glass. This deficiency, which FRAUNHOFER and others had tried to supply by the manufacture of new kinds of optical glass, began to be realized more and more with the development of the microscope and in the construction of the photographic objective. Finally, in 1881, Professor E. ABBE, who has been rightly called the "GALILEO of the Microscope", undertook, in conjuction with Dr. O. SCHOTT, a systematic investigation of the "optical properties of all known substances which undergo vitreous fusion and solidify in non-crystalline transparent masses".[1] The success of these ingenious and exhaustive experiments, in which entirely new and remarkable compositions of glass were obtained by using a far greater number of chemical elements than had ever been essayed before and, especially, by employing in the manufacture both boric and phosphoric acids as well as the usual silicic acid, was almost immediate and beyond all expectations, and a few years later (1886) the "Glastechnisches Laboratorium" of Messrs. Schott und Gen., in Jena, was established, where the now world-famous "Jena Glass" is manufactured.

The important practical problem, suggested first by Fraunhofer, of producing pairs of crown glass and flint glass such that the dispersions of the different parts of the spectrum should be as nearly as possible equal for both kinds of glass, with the object of abolishing or diminishing the so-called *secondary spectrum* (Art. 112), was successfully solved by the labours of Abbe and Schott. Another problem of not less importance consisted in producing a large variety of kinds of optical glass of graduated properties, so that in the design of an optical system the optician might be able to find a combination more or less exactly adapted to his particular requirements. This result was likewise achieved.

The optical properties of the different varieties of glass are described in the Jena-Glass Catalogue with reference to five bright lines of the spectrum which are all easily obtained by artificial sources of light, viz.: The red potassium line, which is very close to the Fraunhofer Line A, and which may be designated, therefore, by A'; the yellow sodium line which coincides with the Fraunhofer Line D; and, finally, the bright lines of the spectrum of hydrogen, the first two of which are identical with the Fraunhofer Lines C and F, while the third, designated by G', is very near the Fraunhofer Line G. The wave-lengths of the light corresponding to these lines are as follows:

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[1] See E. Abbe und O. Schott: Productionsverzeichniss des glastechnischen Laboratoriums von Schott und Genossen in Jena: published as a "prospectus" in July, 1886, and re-printed in *Gesammelte Abhandlungen von* Ernst Abbe, Bd. II (Jena, 1906), 194–201. See also: E. Abbe: Ueber neue Mikroskope: *Sitz.-Ber. Jen. Ges. Med. u. Natw.*, 1886, 107–128; reprinted in *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 450–472.

Especially, see S. Czapski: Mittheilungen ueber das glastechnische Laboratorium in Jena und die von ihm hergestellten neuen optischen Glaeser: *Zft. f. Inst.*, vi (1886), 293–299 and 335–348. See also the very complete history of optical glass-manufacture given in M. Von Rohr's valuable and learned work, *Theorie und Geschichte des photographischen Objektivs* (Berlin, 1899), 325–341.

$$\begin{aligned} \lambda_{A'} &= 767.7\ \mu\mu, \\ \lambda_C &= 656.3\ \mu\mu, \\ \lambda_D &= 589.3\ \mu\mu, \\ \lambda_F &= 486.1\ \mu\mu, \\ \lambda_{G'} &= 434.1\ \mu\mu. \end{aligned}$$

By the aid of these data, the dispersion of the glass for any interval of the spectrum comprised between lines of known wave-lengths may be obtained, closely enough at any rate for practical purposes, by the method of graphical interpolation, wherein the abscissæ denote the reciprocals of the wave-lengths.

The following list, selected somewhat arbitrarily from the "Table of Optical Glasses made in Jena", given in CZAPSKI's paper in the *Zeitschrift für Instrumentenkunde* (vi, 338–9), will serve to give an idea not only of the remarkable range and variety of the properties of the new kinds of optical glass, but also of the fundamental constants that were employed by ABBE for describing these properties:

SOME VARIETIES OF THE JENA OPTICAL GLASS

Factory Number	Name	n_D	Mean Dispersion $n_F - n_C$	$\nu = \dfrac{n_D - 1}{n_F - n_C}$	Partial Dispersions			Specific Gravity
					$n_D - n_{A'}$	$n_F - n_D$	$n_{G'} - n_F$	
O.225	Light Phosphate-Crown	1.5159	0.00737	70.0	0.00485 0.658	0.00515 0.698	0.00407 0.552	2.58
S.30	Heavy Barium-Phosphate Crown	1.5760	0.00884	65.2	0.00570 0.644	0.00622 0.703	0.00500 0.565	3.35
O.60	Calcium-Silicate-Crown	1.5179	0.00860	60.2	0.00553 0.643	0.00605 0.703	0.00487 0.566	2.49
O.138	Sil. Crown of high ref. ind.	1.5258	0.00872	60.2	0.00560 0.642	0.00614 0.704	0.00494 0.566	2.53
S.52	Light Borate-Crown	1.5047	0.00840	60.0	0.00560 0.667	0.00587 0.700	0.00466 0.555	2.24
S.35	Borate-Flint	1.5503	0.00996	55.2	0.00654 0.656	0.00699 0.702	0.00561 0.563	2.56
O.152	Silicate Glass	1.5159	0.01049	51.2	0.00659 0.628	0.00743 0.708	0.00610 0.582	2.76
S.8	Borate-Flint	1.5736	0.01129	50.8	0.00728 0.645	0.00795 0.704	0.00644 0.571	2.82
O.164	Boro-Silicate-Flint	1.5503	0.01114	49.4	0.00710 0.637	0.00786 0.706	0.00644 0.578	2.81
S.7	Borate-Flint	1.6086	0.01375	44.3	0.00864 0.628	0.00974 0.708	0.00802 0.583	3.17
O.154	Light Silicate-Flint	1.5710	0.01327	43.0	0.00819 0.617	0.00943 0.710	0.00791 0.596	3.16
S.57	Heaviest Sil.-Flint	1.9626	0.04882	19.7	0.02767 0.567	0.03547 0.726	0.03252 0.666	6.33

The index of refraction of each kind of glass for the D-Line is given in the first column of the table. Since this line is about at the brightest part of the spectrum, and since also this radiation is especially convenient to obtain, the value of n_D is usually employed to characterize the refrangibility of an optical medium.

The next column of the table gives the value of the so-called *mean dispersion*, that is, the difference $(n_F - n_C)$ of the indices of refraction for the light corresponding to the lines C and F. This difference is about proportional to the length of the spectrum, since the greater part of the visible spectrum is included between the lines C and F.

The third column gives the value of the magnitude

$$\nu = \frac{n_D - 1}{n_F - n_C}. \tag{366}$$

The numerator of this fraction is the difference between the mean index of refraction (n_D) of the material and the index of refraction of air ($n = 1$); which difference occurs so frequently, for example, in the formulæ of Thin Lenses. The reciprocal of this fraction, viz., $1/\nu$, is called the *relative dispersion;* and, hence, the greater the value of ν, the smaller will be the relative dispersion. It will be remarked that the series of glasses are arranged in the table with respect to the magnitude of this constant ν from the greatest value of ν to its least value in descending order. This is due to the fact that the optical character of a given specimen of glass is seen most clearly by a consideration of its ν-value.

The values of the *partial dispersions* for the three intervals A'–D, D–F and F–G', which appear in the next three columns of the table, enable us to perceive also the behaviour of the glass as regards dispersion; so that we can compare the dispersions of two different kinds of glass for the various parts of the spectrum with a view to ascertaining the degree of achromatism that is possible by a combination of the pair. For this same purpose also the value obtained by dividing the partial dispersion of one of these intervals by the value of the mean dispersion $n_F - n_C$ is entered in the same column immediately under the value of the partial dispersion to which it belongs. It will be seen from the table that the partial dispersions of different kinds of glass are, in general, quite different. Moreover, comparing the spectra produced by two different optical media, we may find that the dispersion of the red region is relatively greater, and at the same time the dispersion of the blue region is relatively less, for the first substance than the corresponding partial dispersions for the second substance. This phenomenon is known as the *irrationality of dispersion*, in consequence whereof we are unable to compare the spectra produced by prisms of different materials, since there is no law of proportionality between them. This fundamental fact in regard to prism-spectra NEWTON failed to perceive; and when LUCAS of Liege, attempting to repeat NEWTON's first prism-experiment (§ 328), declared that he could never obtain a spectrum whose length was more than three and one-half times its breadth, NEWTON persisted in asserting that, if the experiment were properly performed, the spectrum would be found to be five times as long as it was broad; whereas, no doubt, the real explanation of the discrepancy in the two observations was to be found in the fact that the English prism and the Dutch prism were made of different kinds of glass.

By comparing the corresponding values of the relative partial dispersions of two different specimens of glass, say, crown and flint, we can tell immediately what will be the character and extent of the residual or secondary spectrum obtained by a combination of the two materials. Thus, for example, a large value of the relative partial dispersion for the interval $A'–D$ will mean that the red part of the spectrum produced with this kind of glass will be relatively extensive. The difference of the values of corresponding ratios for two specimens of glass will be a measure of the dissimilarity of the two spectra in the region or interval to which the ratio applies. On the other hand, the equality of these corresponding pairs of ratios for two materials will indicate the possibility of employing these kinds of glass for achromatic combinations that are free from secondary colour-effects, provided also the ν-values are sufficiently different to warrant this selection. Referring to the table, we see that there are several pairs of varieties of the Jena-glass, which have approximately equal relative partial dispersions and at the same time quite different ν-values, and which, therefore, enable us to make achromatic combinations that are practically free from secondary spectrum; for example, the pairs O.225 and S.35; S.40 and S.35; S.30 and S.8; and O.60 and O.164. On the other hand, we can find also in the table pairs of glasses with approximately equal ν-values, which show, however, considerable differences in their relative partial dispersions; for example, compare O.138 and S.52; O.152 and S.8; and S.7 and O.154.

Prior to the time of ABBE, there was no kind of glass available for the design of an optical instrument which, with a high refractive index, possessed at the same time a low dispersive power, or *vice versa*. Thus, for example, FRAUNHOFER's flint glass had both a greater refractive index and a greater dispersive power than his crown glass. But a high refractive index does not necessarily imply also a great dispersive power, as was formerly supposed, as will be seen by comparing the following pair of products of the Jena-Glass Laboratory:

		n_D	$n_F - n_C$
O.1209.	Densest Baryta Crown	1.6112	0.01068
O.7260.	Extra Light Flint	1.5398	0.01142

Here it will be remarked that the more highly refracting of these two specimens is at the same time the less strongly dispersive one of the pair. It is easy to understand how the production of different kinds of glass with such properties as we have noted marked an epoch in optical engineering and made possible the extraordinary perfections of modern optical instruments.

330. Combinations of Thin Prisms.

In connection with this subject it will be of service to consider here briefly two combinations which have been mentioned above and which have great practical importance, viz., the case of *deviation without dispersion* and the case of *dispersion without deviation*. Suppose that we have two prisms made of substances whose indices of refraction for light of a given wave-length λ may be denoted by n and n'; and, for the sake of simplicity, let us assume, for the present, that the refracting angles β and β' are exceedingly small, and also that the rays which we employ meet the surfaces of the prisms at very nearly normal incidence. Of course, these assumptions are widely different from the conditions that we have in an actual case; but that need not affect the object which we have here in view.

If ϵ denotes the total deviation of the ray of wave-length λ that is produced by the pair of prisms in combination, then, by formula (28) of § 72, we can write:

$$\epsilon = (n - 1)\beta + (n' - 1)\beta'. \tag{367}$$

The variation $\partial\epsilon$ of the deviation in consequence of a variation of the wave-length of light from the value λ to the value $\lambda + d\lambda$ will be a measure of the dispersion. Thus, by differentiation, we obtain:

$$\partial\epsilon = \beta \cdot dn + \beta' \cdot dn'. \tag{368}$$

(1) If the combination of the two thin prisms is to be *achromatic* with respect to light of wave-lengths λ and $\lambda + d\lambda$, then we must put $\partial\epsilon = 0$, and, hence, the condition of achromatism requires that

the angles β, β' shall be related as follows:

$$\frac{\beta'}{\beta} = -\frac{dn}{dn'}.$$

In order to obtain with this combination a given deviation

$$\epsilon_D = (n_D - 1)\beta + (n'_D - 1)\beta'$$

of the *D*-ray, we must have therefore:

$$\beta = \frac{\epsilon_D}{(\nu - \nu')\cdot dn}, \quad \beta' = -\frac{\epsilon_D}{(\nu - \nu')\cdot dn'}, \tag{369}$$

where

$$\frac{1}{\nu} = \frac{dn}{n_D - 1}, \quad \frac{1}{\nu'} = \frac{dn'}{n'_D - 1} \tag{370}$$

denote the so-called relative dispersions of the two optical media. In the achromatic prism-combination it is usual to superpose the lines *C* and *F* (red and blue); in which case:

$$dn = n_F - n_C, \quad dn' = n'_F - n'_C.$$

(2) On the other hand, if we are to have *dispersion without deviation* (as in the so-called "direct-vision" combination of prisms), then, assuming that the *D*-ray is the ray which is to emerge without deviation ($\epsilon_D = 0$), we shall have:

$$\frac{\beta'}{\beta} = -\frac{n_D - 1}{n'_D - 1};$$

and, hence, for a given value $\partial\epsilon$ of the dispersion of the rays of wave-lengths λ and $\lambda + d\lambda$, we find:

$$\beta = -\frac{\partial\epsilon}{dn}\cdot\frac{\nu'}{\nu - \nu'}, \quad \beta' = \frac{\partial\epsilon}{dn'}\cdot\frac{\nu}{\nu - \nu'}, \tag{371}$$

where ν, ν' are the magnitudes defined according to equations (370).

ART. 106. THE DISPERSION OF A SYSTEM OF PRISMS.

331. When a ray of light is refracted in succession through a series of optical media, the angular deviation ϵ is a function of the indices of refraction n_1, n'_1, n'_2, etc., and each of these latter magnitudes is itself a function of the wave-length λ. The change of the angular deviation corresponding to a given change of the wave-length λ is a measure of the dispersion of the system for this interval. Accordingly, the dispersion for the interval comprised between the values λ and λ + *d*λ will be expressed analytically by the following formula:

$$\frac{\partial\epsilon}{\partial\lambda}=\frac{\partial\epsilon}{\partial n_1'}\frac{dn_1'}{d\lambda}+\frac{\partial\epsilon}{\partial n_2'}\frac{dn_2'}{d\lambda}+\cdots+\frac{\partial\epsilon}{\partial n_m'}\frac{dn_m'}{d\lambda}, \tag{372}$$

wherein it is assumed that there is no dispersion of the light in the first medium ($\partial\epsilon/\partial n_1 = 0$). In this formula m denotes the number of refracting surfaces. The partial differential co-efficients $\partial\epsilon/\partial n$ are not only functions of the refractive indices n_1, n_1', n_2', etc., but these magnitudes depend also on the forms and position-relations of the refracting surfaces; whereas the magnitudes $dn/d\lambda$ depend only on the form of the function connecting the variables n and λ (§ 327) and on the values of the numerical constants of the medium in question; and, hence, it has been suggested that the differential co-efficient $dn/d\lambda$ might properly be called the "characteristic dispersion" of the medium. Accordingly, the problem of finding the dispersion in the case of a given optical system consists in determining the values of the magnitudes $\partial\epsilon/\partial n$ for each medium. We propose now to investigate this problem in the case of a system of prisms with their refracting edges all parallel.[1]

According to formulæ (43) of § 93, we have, for the refraction at the kth plane refracting surface of a ray lying in a principal section of the prism-system, the following equations:

$$\left.\begin{aligned} n_k'\cdot\sin\alpha_k' &= n_{k-1}'\cdot\sin\alpha_k, \\ \epsilon_k &= \alpha_k-\alpha_k'; \end{aligned}\right\} \tag{373}$$

where n_k' denotes the index of refraction of the $(k+1)$th medium for light of the given wave-length λ; α_k, α_k' denote the angles of incidence and refraction at the kth surface; and ϵ_k denotes the angular deviation of the ray produced by this refraction. Moreover, if β_k denotes the refracting angle of the kth prism (that is, the dihedral angle between the kth and the $(k+1)$th refracting planes, as in § 93), we have also:

$$\alpha_{k+1}=\alpha_k'-\beta_k; \tag{374}$$

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[1] See S. Czapski: *Theorie der optischen Instrumente nach* Abbe (Breslau, 1893), pages 145, foll.; H. Kayser: *Handbuch der Spectroscopie*, Bd. I (Leipzig, 1900), Arts. 297, foll.; and F. Loewe's "Die Prismen und die Prismensysteme" which is Chapter VIII of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. Von Rohr; pages 455-457.

whence we obtain for the total deviation (ϵ) of the ray of wave-length λ:

$$\epsilon = \sum_{k=1}^{k=m} \epsilon_k = \alpha_1 - \alpha'_k - \sum_{k=1}^{k=m-1} \beta_k,$$

where, as above stated, m denotes the total number of plane refracting surfaces.

The total deviation of a ray of wave-length $\lambda + d\lambda$ will be $\epsilon + \partial\epsilon$, where according to the formula above:

$$\partial\epsilon = d\alpha_1 - d\alpha'_m.$$

If, as is usually the case, there is no dispersion of the light in the first medium, that is, if α_1 has the same value for the rays λ and $\lambda + d\lambda$, then we must put $d\alpha_1 = 0$, in which case we have therefore:

$$\partial\epsilon = - d\alpha'_m.$$

The magnitude $\partial\epsilon$ is a measure of the dispersion of the light of wave-lengths λ and $\lambda + d\lambda$.

Differentiating the first of equations (373), we obtain:

$$n'_k \cdot \cos \alpha'_k \cdot d\alpha'_k + \sin \alpha'_k \cdot dn'_k = n'_{k-1} \cdot \cos \alpha_k \cdot d\alpha_k + \sin \alpha_k \cdot dn'_{k-1},$$

wherein, according to formula (374), we have:

$$d\alpha_k = d\alpha'_{k-1}.$$

This equation may evidently be written in the following recurrent form:

$$d\alpha'_k = \frac{n'_{k-1}}{n'_k} \cdot \frac{\cos \alpha_k}{\cos \alpha'_k} d\alpha'_{k-1} - \frac{1}{n'_k \cdot \cos \alpha'_k} X_k,$$

where, for brevity, we have written:

$$X_k = n'_{k-1} \cdot \sin \alpha_k \left(\frac{dn'_k}{n'_k} - \frac{dn'_{k-1}}{n'_{k-1}} \right). \tag{375}$$

Thus, we obtain the following formula:

$$d\alpha'_m = \frac{n_1}{n'_m} d\alpha_1 \prod_{k=1}^{k=m} \frac{\cos \alpha_k}{\cos \alpha'_k} - \frac{1}{n'_m} \sum_{k=1}^{k=m} X_k \prod_{r=k}^{r=m} \frac{\cos \alpha_{r+1}}{\cos \alpha'_r}; \tag{376}$$

wherein it is to be understood that *we must put always* $\cos \alpha_{m+1} = 1$. In this equation n'_k and $n'_k + dn'_k$ denote the indices of refraction of the ($k + 1$)th medium for light of wave-lengths λ and $\lambda + d\lambda$, respectively.

If the first and last media are both air, we can put:

$$n'_m = n_1 = 1;$$

and, if, moreover, there is *no initial dispersion*, we can put also:

$$d\alpha_1 = 0, \quad \partial\epsilon = -\,d\alpha'_m.$$

Accordingly, under these circumstances, we have:

$$\left.\begin{aligned} \partial\epsilon &= \sum_{k=1}^{k=m} X_k \prod_{r=k}^{r=m} \frac{\cos\alpha_{r+1}}{\cos\alpha'_r}, \\ \cos\alpha_{m+1} &= 1. \end{aligned}\right\} \qquad (377)$$

332. Dispersion of a Single Prism in Air.

Assuming that there is no initial dispersion ($d\alpha_1 = 0$) and that the prism is surrounded by air, so that we may write:

$$n_1 = n'_2 = 1, \quad n'_1 = n,$$

and putting $m = 2$ in formulæ (377), we obtain for the dispersion of a single prism:

$$\frac{\partial\epsilon}{\partial\lambda} = \frac{\sin\beta}{\cos\alpha'_1 \cdot \cos\alpha'_2}\frac{dn}{d\lambda}, \qquad (378)$$

where

$$\beta = \alpha'_1 - \alpha_2$$

denotes the refracting angle of the prism. According to this formula, the dispersion of a single prism for light of wave-lengths λ and $\lambda + d\lambda$ depends not only on the value of the refractive index n but also on the refracting angle β and on the angle of incidence α_1. When the angle of emergence $\alpha'_2 = 90°$, the dispersion $\partial\epsilon/\partial\lambda = \infty$ has its maximum value. As the angle α'_2 decreases (in consequence of a corresponding variation of the incidence-angle α_1), the dispersion $\partial\epsilon/\partial\lambda$ diminishes until it reaches a *minimum* value, after which farther decrease of the angle α'_2 is accompanied by increase of the dispersion. The fact that for a certain value of the incidence-angle α_1 *the dispersion* $\partial\epsilon/\partial\lambda$ *is a minimum* was first remarked by J. F. W. HERSCHEL,[1] who found also that this position was different from that of minimum deviation. The dispersion will be a minimum for that value of the incidence-angle α_1 for which $\cos\alpha'_1 \cdot \cos\alpha'_2$ is a maximum; but the solution of this problem leads to a cubic equation for the determination of α_1.[1] According to THOLLON,[2] the condition of minimum dispersion is given *approximately* by the following equation:

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[1] J. F. W. HERSCHEL: Article "On Light" in the *Encyc. Metropolitana* (London, 1828).

$$\alpha_1' = -n^2\alpha_2. \tag{379}$$

If the prism is in the position of *minimum deviation* ($\epsilon = \epsilon_0$), we have (see § 71):

$$\beta = 2\alpha_1', \quad \alpha_2' = -\alpha_1;$$

and if these values are introduced in formula (378), we find for the dispersion of the prism in this special position:

$$\frac{\partial\epsilon_0}{\partial\lambda} = \frac{2}{n}\tan\alpha_1 \cdot \frac{dn}{d\lambda}. \tag{380}$$

333. Another special case, which is of interest from a practical standpoint is **the dispersion of a train of prisms composed alternately of glass and air,** so that we are concerned with only two media. Here also we shall assume that there is no initial dispersion ($d\alpha_1 = 0$) and also that the dispersion of the air-prisms is negligible. If we put

$$n_1 = n_2' = \cdots = n_{2i}' = \cdots = 1, \quad n_1' = n_3' = \cdots = n_{2i-1}' = \cdots = n,$$

we shall have for the refraction from air to glass at the $(2i-1)$th surface:

$$X_{2i-1} = \sin\alpha_{2i-1}' \cdot dn,$$

and for the next following refraction, that is, from glass to air:

$$X_{2i} = -\sin\alpha_{2i} \cdot dn,$$

where i denotes here any positive integer from $i = 1$ to $i = m/2$. Substituting these values in formula (377), we obtain:

$$\frac{\partial\epsilon}{\partial\lambda} = \frac{dn}{d\lambda}\left\{\sin\alpha_1' \frac{\cos\alpha_2 \cdot \cos\alpha_3 \cdots \cos\alpha_m}{\cos\alpha_1' \cdot \cos\alpha_2' \cdots \cos\alpha_m'} - \sin\alpha_2 \frac{\cos\alpha_3 \cdot \cos\alpha_4 \cdots \cos\alpha_m}{\cos\alpha_2' \cdot \cos\alpha_3' \cdots \cos\alpha_m'} \right.$$
$$\left. + \sin\alpha_3' \frac{\cos\alpha_4 \cdot \cos\alpha_5 \cdots \cos\alpha_m}{\cos\alpha_3' \cdot \cos\alpha_4' \cdots \cos\alpha_m'} - \cdots \right\}.$$

In the special case when the path of the ray through the system of prisms which are composed alternately of glass and air is *symmetrical,* we shall have:

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[1] See H. KAYSER: *Handbuch der Spectroscopie*, Bd. I (Leipzig, 1900), Art. 300.

[2] L. THOLLON: Minimum de dispersion des prismes; achromatisme de deux lentilles de même substance: *Comptes Rendus*, lxxxix (1879), 93–97.

$$\alpha_1 = -\alpha'_2 = \alpha_3 = \cdots = \alpha_{2i-1} = -\alpha'_{2i} = \cdots = -\alpha'_m,$$
$$\alpha'_1 = -\alpha_2 = \alpha'_3 = \cdots = \alpha'_{2i-1} = -\alpha_{2i} = \cdots = -\alpha_m;$$

and evidently now each of the m terms within the brackets on the right-hand side of the above equation will be equal to

$$+\frac{\sin\alpha'_1}{\cos\alpha_1},$$

the signs of the terms being all positive. Since, moreover, $n\cdot\sin\alpha'_1 = \sin\alpha_1$, we obtain for the magnitude of the dispersion under these circumstances:

$$\frac{\partial\epsilon_0}{\partial\lambda} = \frac{m}{n}\tan\alpha_1\frac{dn}{d\lambda}, \tag{381}$$

where $m/2$ denotes the number of glass prisms.

Comparing this result with formula (380), we see that the dispersion of a train of glass prisms adjusted as above described is equal to the sum of the dispersions of the prisms taken separately. This formula (381) is a useful one, because in actual practice the prisms of a prism-spectroscope are usually adjusted in this way.

334. Achromatic Prism-Systems.

The condition that the rays of wave-lengths λ and $\lambda + d\lambda$ shall emerge from the optical system along the same identical path is $\partial\epsilon/\partial\lambda = 0$; in which case the deviations of the two rays will have the same value ϵ. This is the case of *Deviation Without Dispersion.* Assuming, as is usually the case, that the incident rays are themselves without dispersion, we find by formula (377) the following *condition of achromatism* of a system of prisms for light of wave-lengths λ and $\lambda + d\lambda$:

$$\left.\begin{aligned}\sum_{k=1}^{k=m} X_k \prod_{r=k}^{r=m}\frac{\cos\alpha_{r+1}}{\cos\alpha'_r} = 0,\\ \cos\alpha_{m+1} = 1.\end{aligned}\right\} \tag{382}$$

If, for example, the system is composed of three prisms ($m = 4$), and if the first, third and last media are air ($n_1 = n'_2 = n'_4 = 1$), so that the system consists, let us say, of *two glass prisms separated by air,* the combination will be achromatic for light of wave-lengths λ and $\lambda + d\lambda$, provided we have:

$$\sin\beta_1\cdot\cos\alpha_3\cdot\cos\alpha_4\cdot dn'_1 + \sin\beta_3\cdot\cos\alpha'_1\cdot\cos\alpha'_2\cdot dn'_3 = 0, \tag{383}$$

where

$$\beta_1 = \alpha_1' - \alpha_2, \quad \beta_3 = \alpha_2' - \alpha_3$$

denote the refracting angles of the two glass prisms. In this equation the magnitudes α_1', α_2', α_3 and α_4 are connected by the relations:

$$\sin \alpha_2' = n_1' \cdot \sin (\alpha_1' - \beta_1), \quad \sin \alpha_3 = n_3' \cdot \sin (\alpha_4 + \beta_3);$$

and, hence, if the first prism is supposed to be known, that is, if the magnitudes denoted by n_1' and β_1 are given, and if also the angle of incidence α_1 and the index of refraction n_3' of the second prism are given, there will still remain two arbitrary magnitudes, viz., β_3 and α_3. Under these circumstances, therefore, the condition expressed by equation (383) may be satisfied in either of two ways, as follows: (1) Any arbitrary value may be assigned to the refracting angle (β_3) of the second glass prism, and we shall have then to determine the corresponding value of the angle α_3, that is, we shall have to find the orientation of the second glass prism with respect to the first in order that the combination may be achromatic; or (2) Assuming an arbitrary value of the angle α_3, we may then employ equation (383) to determine what value the refracting angle of the second glass prism must have. The two glass prisms may even be made of the same kind of glass ($n_1' = n_3'$).

As a concrete illustration, let us assume that the ray of wave-length λ traverses each of the glass prisms symmetrically, that is, with *minimum deviation;* in which case we have the following relations (§ 71):

$$\alpha_1 = -\alpha_2', \quad \alpha_1' = -\alpha_2 = \frac{\beta_1}{2}, \quad \alpha_3' = -\alpha_4 = \frac{\beta_3}{2}.$$

Introducing these values in equation (383), we obtain the condition of achromatism for this special case in the following form:

$$\tan \alpha_1 \frac{dn_1'}{n_1'} + \tan \alpha_3 \frac{dn_3'}{n_3'} = 0,$$

where

$$\sin \alpha_1 = n_1' \cdot \sin \frac{\beta_1}{2}, \quad \sin \alpha_3' = n_3' \cdot \sin \frac{\beta_3}{2}.$$

The simplest case is that in which the system is composed of *two glass prisms* (usually *cemented together* along their common face), the first and last media being air, so that $n_1 = n_3' = 1$. For this case $m = 3$, and by formula (375) we find:

$$X_1 = \sin \alpha_1' \cdot dn_1', \quad X_2 = \sin \alpha_2' \cdot dn_2' - \sin \alpha_2 \cdot dn_1', \quad X_3 = -\sin \alpha_3 \cdot dn_2';$$

whence, employing formulæ (377), we obtain after several obvious reductions:

$$\partial\epsilon = \frac{\sin\beta_1\cdot\cos\alpha_3\cdot dn_1' + \sin\beta_2\cdot dn_2'}{\cos\alpha_2'\cdot\cos\alpha_3'}, \tag{384}$$

where

$$\beta_1 = \alpha_1' - \alpha_2, \quad \beta_2 = \alpha_2' - \alpha_3$$

denote the refracting angles of the prisms.

If, therefore, this combination is to be *achromatic* for light of wave-lengths λ and $\lambda + d\lambda$, we must have:

$$\sin\beta_1\cdot\cos\alpha_3\cdot dn_1' + \sin\beta_2\cdot dn_2' = 0. \tag{385}$$

By means of this formula, the angle β_2 of the second prism can be calculated, so soon as we assign the value of the angle of incidence (α_1) and the value of the deviation-angle (ϵ).

335. Direct-Vision Prism-System. We may consider briefly also the important practical case of a system which is constructed so that, although the rays of wave-lengths λ and $\lambda + d\lambda$ are dispersed, the standard ray of wave-length λ traverses the system without being deviated ($\epsilon = 0$)—prism-system *à vision directe*. If, as in the special case considered in § 334, the system is composed of *two cemented glass prisms* ($m = 3$) surrounded by air, the dispersion is given by formula (384) above. If we specialize the problem still farther by supposing that the ray of wave-length λ emerges from the system in a direction perpendicular to the third plane refracting surface (Fig. 149), we have evidently the following system of equations for this *Direct-Vision Combination:*

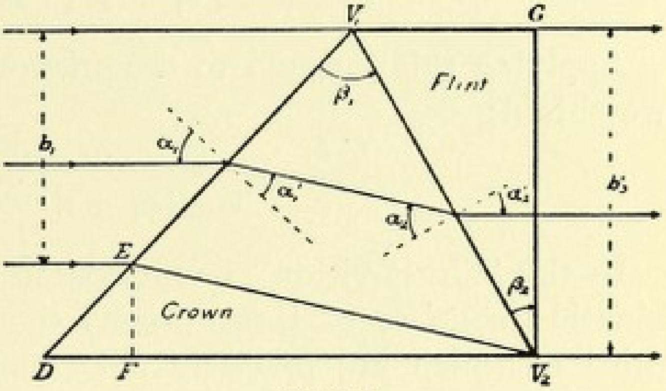

FIG. 149.

DIRECT-VISION COMBINATION OF TWO CEMENTED GLASS PRISMS. The portion *DEF* of the Crown-Glass Prism can be cut away, as no rays traverse this part. See diagram of AMICI-Prism, Fig. 153.

$$\left.\begin{aligned}
&n_1 = n_3' = 1, \quad \alpha_3' = \alpha_3 = 0, \quad \alpha_2' = \beta_2,\\
&\sin\alpha_2 = \frac{n_2'}{n_1'}\sin\beta_2,\\
&\epsilon_1 = -\epsilon_2 = \alpha_1 - \alpha_1', \quad \epsilon_3 = \epsilon = 0,\\
&\tan\alpha_1 = \frac{n_1'\cdot\sin\epsilon_1}{n_1'\cdot\cos\epsilon_1 - 1},\\
&\beta_2 = \alpha_1 - \beta_1;
\end{aligned}\right\} \tag{386}$$

whence the magnitude of the refracting angle β_2 of the second prism can be calculated.

The dispersion of the system is given by the following formula:

$$\partial\epsilon = \frac{\sin\beta_1}{\cos(\alpha_1 - \beta_1)}dn_1' + \tan(\alpha_1 - \beta_1)dn_2'.$$

When a beam of parallel rays is refracted at a plane surface, the widths b, b' of the pencils of incident and refracted rays are in the same ratio to each other as the cosines of the angles of incidence and refraction; thus,

$$\frac{b_k'}{b_{k-1}'} = \frac{\cos\alpha_k'}{\cos\alpha_k};$$

and, hence, also, in the case of a system of plane refracting surfaces:

$$\frac{b_m'}{b_1} = \prod_{k=1}^{k=m}\frac{\cos\alpha_k'}{\cos\alpha_k}. \tag{387}$$

Applying this formula to the prism-system represented in Fig. 149, we obtain:

$$\frac{b_1}{b_3'} = \frac{\cos\alpha_1\cdot\cos\alpha_2}{\cos(\alpha_1 - \epsilon_1)\cdot\cos(\alpha_2 + \epsilon_1)}. \tag{388}$$

In the "direct-vision" prism-system designed by AMICI,[1] the combination consists of three prisms cemented together, the first and third of which are precisely alike and both made of crown glass; whereas the second, or middle, prism is made of flint glass. If the AMICI-system is divided into two equal halves by a plane containing the refracting edge of the flint glass prism, the first half will be exactly similar to the combination represented in Fig. 149. In the AMICI-combination the widths of the incident and emergent beams of rays are equal.

ART. 107. PURITY OF THE SPECTRUM. RESOLVING POWER OF PRISM-SYSTEM.

336. Measure of the Purity of the Spectrum. In the investigation of the spectra produced by a prism-system, the problem

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[1] J. B. AMICI: Museo florentino, i (1860), p. 1. (This reference has not been verified by the author).—Concerning AMICI's prism-system, see especially S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), pages 149, 150 and 153; and F. LOEWE's "Die Prismen und die Prismensysteme", pages 461, 462 of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR.

Note on § 335.

In a foot-note on page 492 of the first edition of this book it was stated that the author had been unable to obtain formula (388) in the form which was given by both CZAPSKI and LOEWE. As the trigonometric transformation is exceedingly obvious and simple, the author must confess that he is completely at a loss to explain how this statement occurred. The matter was brought to his attention by the review of the book in the *Zeitschrift für Instrumentenkunde*, xxxi (1911), 168–170. The formula in question may be derived from formula (388) as follows:

$$\begin{aligned}\cos(\alpha_1 - \epsilon_1)\cos(\alpha_2 + \epsilon_1) &= \tfrac{1}{2}\{\cos(\alpha_1 + \alpha_2) + \cos(\alpha_1 - \alpha_2 - 2\epsilon_1)\}\\ &= \tfrac{1}{2}\{\cos(\alpha_1 + \alpha_2) + \cos(\beta_1 - \epsilon_1)\}\\ &= \tfrac{1}{2}(\cos\alpha_1\cos\alpha_2 + \sin\epsilon_1\sin\beta_1 + \cos\epsilon_1\cos\beta_1\\ &\qquad - \sin\alpha_1\sin\alpha_2).\end{aligned}$$

Now since

$$\beta_1 + \epsilon_1 = \alpha_1 - \alpha_2,$$

and hence

$$\cos(\beta_1 + \epsilon_1) = \cos(\alpha_1 - \alpha_2),$$

we find:

$$\cos\alpha_1\cos\alpha_2 + \sin\epsilon_1\sin\beta_1 = \cos\epsilon_1\cos\beta_1 - \sin\alpha_1\sin\alpha_2;$$

and therefore:

$$\cos(\alpha_1 - \epsilon_1)\cos(\alpha_2 + \epsilon_1) = \cos\alpha_1\cos\alpha_2 + \sin\epsilon_1\sin\beta_1.$$

Accordingly, formula (388) may be written:

$$\frac{b_1}{b_3'} = \frac{\cos\alpha_1\cos\alpha_2}{\cos\alpha_1\cos\alpha_2 + \sin\epsilon_1\sin\beta_1},$$

which is equivalent to the formula given by CZAPSKI.

is not merely to increase the dispersion $\partial\epsilon/\partial\lambda$, but rather to obtain as nearly as possible a *pure spectrum*, wherein the light to be analyzed is resolved into its simplest components, so that at any given part of the spectrum the difference $d\lambda$ of the wave-lengths that are superposed shall be as small as possible. The spectrum is composed of a series of images of the slit, each of which corresponds to light of a definite kind or colour; and if the apparent width of the slit-image for light of wave-length λ is greater than the angular dispersion $\partial\epsilon$ of the rays of wave-lengths λ and $\lambda + d\lambda$, the slit-images corresponding to these two radiations will partly overlap each other, and, accordingly, the spectrum in this region will be more or less impure. If the slit itself were a mathematical line of light, and if there were perfect collinear correspondence between object and image, the spectrum would be absolutely pure, and the image of the line-source for a given wave-length of light would be itself a line occupying a perfectly definite and distinct position in this ideal spectrum.

Evidently, the purity of the spectrum will depend on the width of the slit-image and on the length of the spectrum. Let $\delta\alpha$ denote the apparent size of the slit as viewed from the first face of the prism, and, similarly, let $\delta\alpha'$ denote the apparent size of the slit-image for light of wave-length λ as viewed from the last refracting plane. The greater the dispersion $\partial\epsilon/\partial\lambda$ of the light of wave-lengths λ and $\lambda + d\lambda$, and the smaller the magnitude of the angular width $\delta\alpha'$ of the slit-image, the greater will be the purity of the spectrum at this place in it; and, hence, as a *measure of the purity of the spectrum*, HELMHOLTZ[1] proposed that we employ the following expression:

$$P = \frac{\partial\epsilon}{\partial\lambda} : \delta\alpha'. \tag{389}$$

What is here meant by the image of the slit is not the actual or "diffraction" image, but the image as determined on the assumption of the rectilinear propagation of light according to the laws of Geometrical Optics.

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[1] H. VON HELMHOLTZ: *Handbuch der physiologischen Optik* (zweite umgearbeitete Auflage, Hamburg u. Leipzig, 1886), p. 294. In regard to this subject see also:

H. KAYSER: *Handbuch der Spectroscopie*, Bd. I (Leipzig, 1900), pages 305 & foll. and pages 548 & foll.;

S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), pages 148 & foll.; and

F. LOEWE'S "Die Prismen und die Prismensysteme" in *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR, pages 448 & foll.

If, therefore, we leave out of account the diffraction-effects, then, according to formula (49) of § 97, the angular width of the slit-image formed by a system of prisms, which is composed of m plane refracting surfaces and in which the first and last media are both air ($n_1 = n'_m = 1$), is given by the following formula:

$$\delta\alpha'_m = \delta\alpha_1 \prod_{k=1}^{k=m} \frac{\cos\alpha_k}{\cos\alpha'_k}, \tag{390}$$

where $\delta\alpha_1$ denotes the angular width of the slit itself. Accordingly, assuming that there is no initial dispersion, and employing therefore formulæ (377), we find for the purity of the spectrum produced by a system of prisms, as it is defined by equation (389), the following expression:

$$P = \frac{1}{d\lambda} \cdot \frac{\partial\epsilon}{\delta\alpha'_m} = \frac{1}{d\lambda} \cdot \frac{1}{\delta\alpha_1} = \sum_{k=1}^{k=m} X_k \prod_{r=1}^{r=k} \frac{\cos\alpha'_{r-1}}{\cos\alpha_r}, \tag{391}$$

wherein *the term* $\cos\alpha'_0$ *must be put equal to unity always.* Thus, we see that the magnitude P depends not merely on the properties of the prism-system but on the width of the slit itself; and, hence, the purity of the spectrum, as defined by HELMHOLTZ, is not by itself a sufficient criterion for the comparison of the spectra produced by different prism-systems.

337. Purity of Spectrum in Case of a Single Prism. Consider the spectrum of a *single prism surrounded by air.* In this special case let us write according to our custom:

$$n_1 = n'_2 = 1, \quad n'_1 = n.$$

According to formula (375), we have here:

$$X_1 = \sin\alpha'_1 \cdot dn, \quad X_2 = -\sin\alpha_2 \cdot dn;$$

and, hence, putting $m = 2$ in formula (391), we obtain for the purity of the spectrum of a single prism in the region corresponding to the light of wave-length λ:

$$P = \frac{\sin\beta}{\cos\alpha_1 \cdot \cos\alpha_2} \frac{dn}{d\lambda} \frac{1}{\delta\alpha_1}, \tag{392}$$

where $\beta = \alpha'_1 - \alpha_2$ denotes the refracting angle of the prism. We see, therefore, that the purity of the spectrum of a single prism surrounded by air is proportional to the so-called "characteristic dispersion" (§ 331) of the prism-medium and is inversely proportional to the width of the slit. The purity of the spectrum varies also with the angle of incidence (α_1), and when the prism is adjusted so that the incident ray "grazes" the first face ($\alpha_1 = 90°$), we find $P = \infty$. The advantage of using the prism in this position is enormously discounted, however, on account of the great loss of light by reflexion. As the angle α_1 decreases from the value $\alpha_1 = 90°$, the purity P diminishes also until it attains a minimum value determined by that value of the angle α_1 for which the function $\cos\alpha_1 \cdot \cos\alpha_2$ is a maximum. This value of α_1 may be found by a process entirely analogous to that employed by THOLLON in ascertaining the position of minimum dispersion, which was alluded to in § 332; in fact, by merely interchanging the symbols α_1' and α_2 in formula (379), we obtain immediately the following relation:

$$\alpha_2 = -n^2\alpha_1'; \tag{393}$$

which gives approximately the position of the prism for minimum purity of the spectrum in the region corresponding to the light of wave-length λ.

If the prism is adjusted in the *position of minimum deviation* for the rays of wave-length λ (which, in addition to other advantages, is also the position in which the loss of light by reflexion is least), we must introduce in formula (392) the following relations:

$$\beta = 2\alpha_1' = -2\alpha_2;$$

whereby we obtain:

$$P = \frac{2}{n}\tan\alpha_1 \frac{dn}{d\lambda}\frac{1}{\delta\alpha_1};$$

a result which may be derived also directly from formula (389) by merely remarking that for the position of minimum deviation we have (see § 86) $\delta\alpha' = \delta\alpha_2' = \delta\alpha_1$, whereas the value of $\partial\epsilon/\partial\lambda$ is given here by formula (380). The purity of this part of the spectrum depends only on the refractive index n, the width of the slit and the form of the prism. It is called the "*normal purity*".

338. Diffraction-Image of the Slit. The methods of Geometrical Optics alone are not sufficient to enable us to ascertain the character of the slit-image; this problem involves not merely the theory of refraction but the theory of diffraction also. According to this latter theory, the image of a luminous line (or very narrow rectangular aperture) parallel to the edge of the prism is never itself a line, but a far more complicated effect which we have not space to investigate here, especially too as a complete exposition of the matter can be found in almost any standard work on Physical Optics. In the accompanying diagram (Fig. 150) the plane of the paper represents a principal section of the prism-system, of which the traces in this plane of the first and last surfaces, $\mu_1\mu_1$ and $\mu_m\mu_m$, are shown in the figure. The source of the light is supposed to be a small luminous line perpendicular at S to the plane of the paper. The rays emanating from this line-source are made parallel by a "collimating" lens; so that the straight line $PQ = b_1$ is the trace in the plane of the paper of the portion of the plane-wave which is due to arrive later at the first surface $\mu_1\mu_1$ of the prism-system. The straight line $P'Q' = b'_m$ shows the trace in this plane also of the corresponding emergent plane-wave for light of wave-length λ. A convex lens interposed in the path of the beam of emergent rays will produce on a screen situated in the focal plane of the lens an image of the slit. If this image is investigated by the methods of Physical Optics, we find that the image of a vertical line at S consists mainly of a so-called "central band" of light of a certain finite horizontal width (which depends on the focal length of the lens, for one thing) and of maximum brightness along a vertical line perpendicular to the plane of the paper at the point designated in the diagram by S'. On either side of this vertical median line the brightness of the central band diminishes very rapidly to absolute darkness. There is also a series of much fainter bands situated symmetrically on both sides of the central band, but for all practical purposes the central band alone may be considered as the actual and effective image of the very narrow rectangular aperture at S.

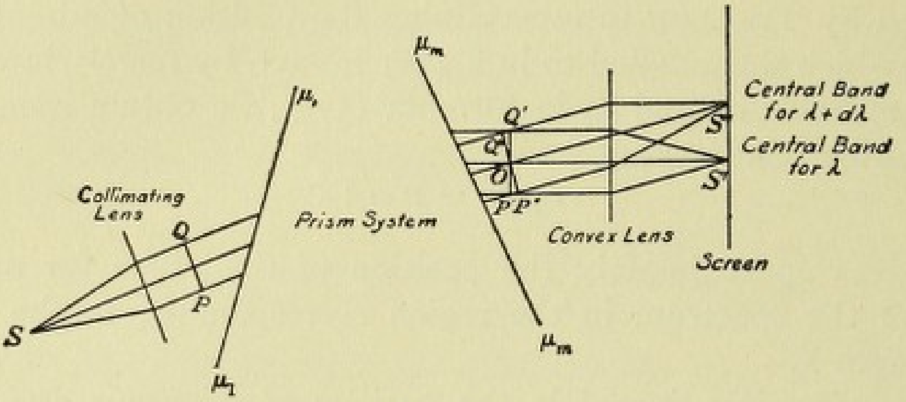

FIG. 150.

RESOLUTION OF LINES IN PRISMATIC SPECTRUM. The plane of the paper represents the plane of a principal section of the prism-system. The source of light is a small luminous line perpendicular to plane of paper at the point marked S. The straight lines $\mu_1\mu_1$ and $\mu_m\mu_m$ are the traces in the plane of the paper of the first and last refracting planes, respectively. $PQ = b_1$ = width of beam of parallel incident rays; $P'Q' = b_m'$ = width of beam of parallel emergent rays of wave-length λ. $\angle S'OS'' = \delta\epsilon$ = angular distance of slit-images corresponding to light of wave-lengths λ and $\lambda + d\lambda$.

If the slit is illuminated by light of wave-lengths λ and $\lambda + d\lambda$, the rectangular beam of parallel incident rays will be resolved by the prism-system into two such beams, one for each of the two colours. Thus, we shall have at the points S' and S'' on the screen the maxima of brightness of the two slit-images corresponding to the light of wave-lengths λ and $\lambda + d\lambda$. Now, in order that these images whose central portions are at S' and S'' may be far enough apart to be distinguished by the eye as separate and distinct images, Lord RAYLEIGH[1] has shown that the $\angle S'OS'' = \delta\epsilon$ must be at least equal to λ/b'_m, where $b'_m = P'Q'$ is the width of the beam of emergent rays of wave-length λ. The width of the beam of emergent rays will depend on the orientation of the prism-system, as is evident from formula (387), and the angular interval $\delta\epsilon$ of the centres of the two images will depend on this also.

In order, therefore, to *resolve a "double line"* of wave-lengths λ and $\lambda + d\lambda$, it is necessary that the angular interval $\delta\epsilon$ shall have the following value at least:

$$\delta\epsilon = \frac{\lambda}{b_1} \prod_{k=1}^{k=m} \frac{\cos\alpha_k}{\cos\alpha'_k}. \qquad (394)$$

In the special case when the rays corresponding to the light of wave-length λ traverse the prism-system with *minimum deviation*, we have, according to the formula at the end of § 94:

$$\prod_{k=1}^{k=m} \frac{\cos\alpha_k}{\cos\alpha'_k} = 1;$$

and hence the condition (394) becomes in this special case:

$$\delta\epsilon = \frac{\lambda}{b_1}.$$

In the case of a *single prism* ($m = 2$), formula (394) is as follows:

$$\delta\epsilon = \frac{\lambda}{b_1} \cdot \frac{\cos\alpha_1 \cdot \cos\alpha_2}{\cos\alpha'_1 \cdot \cos\alpha'_2}. \qquad (395)$$

339. Ideal Purity of Spectrum. According to RAYLEIGH's investigations, the least value of the angular interval $\delta\epsilon$ necessary in order to resolve a "double line" is equal to half the width of the central band of the diffraction-image of the slit; and, hence, on the supposition that the object is a luminous line, the methods of Physical Optics show that the angular width of the image $= 2\delta\epsilon = 2\lambda/b'_m$. HELMHOLTZ defines the purity of the spectrum as equal to the ratio of the dispersion to the angular width of the image (§ 336); and, hence, the so-called Ideal Purity[1] of the spectrum (which may be distinguished by the symbol P_0, where the zero-subscript is written to indicate that the slit in this case is *infinitely narrow*) may be defined by the following equation:

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[1] Lord RAYLEIGH: Investigations in Optics: *Phil. Mag.* (5) viii (1879), pages 261-274, 403-411, 477-486; and (5) ix (1880), pages 40-55. See also article "Wave Theory", 9th ed. of *Encyclopædia Britannica*, xxiv, 430-434.

$$P_0 = \frac{b'_m}{2\lambda}\frac{\partial\epsilon}{\partial\lambda}. \tag{396}$$

Accordingly, *the Ideal Purity of Spectrum produced by a prism, in the case when the slit is infinitely narrow, is proportional to the product of the dispersion by the width of the emergent beam;* both of which factors, as above remarked, depend on the orientation of the prism (or prism-system).

Thus, in the case of a single prism, we obtain by formulæ (378), (387) and (396):

$$P_0 = \frac{b_1}{2\lambda}\cdot\frac{\sin\beta}{\cos\alpha_1\cdot\cos\alpha_2}\frac{dn}{d\lambda}, \tag{397}$$

which should be compared with formula (392).

If s and t denote the lengths of the ray-paths within the prism of the rays of wave-length λ which are nearest to the refracting edge of the prism and farthest from it, respectively (see Fig. 151), it is easy to show that

$$t - s = b_1\frac{\sin\beta}{\cos\alpha_1\cdot\cos\alpha_2};$$

and, hence, we may write:

$$P_0 = \frac{t-s}{2\lambda}\frac{dn}{d\lambda}. \tag{398}$$

340. Resolving Power of Prism-System. The magnitude denoted above by the symbol P_0 is closely related to what Lord RAYLEIGH calls the *Resolving Power* of the Prism-System. Thus, if $d\lambda$ is the difference of wave-length of two "lines" of the spectrum whose mean wave-length is λ and which are just barely separated in the spectrum, the resolving power p is defined by RAYLEIGH as follows:

$$p = \frac{\lambda}{d\lambda}. \tag{399}$$

RAYLEIGH's investigation of the joint effect of dispersion and width of beam on the resolving power of a prism is as follows:

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[1] See H. KAYSER: *Handbuch der Spectroscopie*, Bd. I (Leipzig, 1900), p. 307. See also F. L. O. WADSWORTH: The Modern Spectroscope: *Astrophys. Journ.*, i (1895), 52–79.

The straight line PQ (Fig. 151) represents the trace in the plane of a principal section (plane of the paper) of the plane wave-front of the light at some instant prior to its arrival at the first face of the prism; and the straight line $P'Q'$ represents in the same way the position of the wave-front of the light of wave-length λ at some subsequent instant after the waves have traversed the prism. Similarly, also, the straight line $P''Q''$ represents at this same later instant the position of the plane wave-front for the light of wave-length $\lambda + d\lambda$. As a matter of fact, the two rays of wave-lengths λ and $\lambda + d\lambda$, which meet the first face of the prism at the same point, will thereafter pursue slightly different geometrical paths; but by virtue of FERMAT's Minimum Principle (Art. 11), this difference will be entirely negligible in comparison with the actual distances traversed by the rays; and, hence, we may consider that the rays pursue the same routes both within and without the prism, as represented in the diagram. Thus, for example, if s and t denote the lengths of the ray-paths within the prism of the rays of wave-length λ which are nearest to the refracting edge and farthest from it, respectively, these same magnitudes will denote also the lengths of the ray-paths within the prism of the corresponding pair of rays for light of wave-length $\lambda + d\lambda$. The refractive indices of the prism for light of wave-lengths λ and $\lambda + d\lambda$ will be denoted by n and $n + dn$, respectively. Finally, the prism is supposed to be surrounded on both sides by air whose dispersion is so slight as to be negligible.

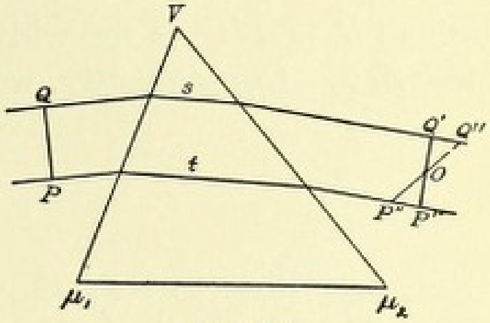

FIG. 151.

RESOLVING POWER OF A PRISM.

According to Art. 11, the optical lengths of the paths from P to P' and from Q to Q' for light of wave-length λ are equal; as is true, likewise, with respect to the optical lengths of the paths from Q to Q'' and from P to P'' for light of wave-length $\lambda + d\lambda$. Evidently, the optical length of the path from Q to Q'' for the light of wave-length $\lambda + d\lambda$ is longer than that from Q to Q' for the light of wave-length λ by the amount $s \cdot dn + Q'Q''$; and, in the same way, the optical length of the path from P to P'' for the light of wave-length $\lambda + d\lambda$ exceeds that from P to P' for the light of wave-length λ by the amount $t \cdot dn - P''P'$; and since these excesses must be equal, we find:

$$(t - s) \cdot dn = P''P' + Q'Q'' = 2P''P'.$$

Now $\angle P''OP' = \delta\epsilon$ is the angular interval between the maxima of intensity of the diffraction-images corresponding to the light of wave-lengths λ and $\lambda + d\lambda$; and

$$\delta\epsilon = \frac{P''P'}{P'O'} = 2\frac{P''P'}{P'Q'} = 2\frac{P''P'}{b'},$$

where $b' = P'Q'$ is the width of the beam of emergent rays of wave-length λ. Consequently, we have here:

$$\delta\epsilon = \frac{t - s}{b'} dn.$$

But, according to RAYLEIGH's conclusion, the least angle subtended by a "double line" of mean wave-length λ must be equal to λ/b' in order that it may be fairly resolved in the spectrum (§ 338); that is, the very smallest value of the angle $\delta\epsilon$ under these circumstances is:

$$\delta\epsilon = \frac{\lambda}{b'};$$

and, consequently, RAYLEIGH finds:

$$\lambda = (t - s)\cdot dn,$$

or

$$p = \frac{\lambda}{d\lambda} = (t - s)\frac{dn}{d\lambda}, \tag{400}$$

and, hence, referring to formula (398), we see that we have the following relation between the Ideal Purity P_0 of the spectrum and the Resolving Power p of the prism:

$$p = 2\lambda P_0. \tag{401}$$

According to formula (400), the Resolving Power of a prism is proportional to the product of the difference of the lengths $(t - s)$ of the ray-paths within the prism of the two extreme rays of the pencil and the characteristic dispersion $dn/d\lambda$ of the dispersive material of the prism. If, for example, the value of the difference $(t - s)$ is the same for two prisms of the same material, the Resolving Power of both prisms will be the same, although the prisms may have different refracting angles and may be oriented differently. It is assumed only that the slit is infinitely narrow.

If one of the extreme rays passes through the prism-edge, then

$s = 0$, and formula (400) reduces to the following:

$$p = t\frac{dn}{d\lambda}. \tag{402}$$

If the prism is adjusted in the position of *minimum deviation*, and if the entire extent of the prism is traversed by the rays, *the Resolving Power of the prism depends only on the thickness of the prism at the base.*

The formulæ obtained above may be extended at once to a system of glass prisms separated from each other by air, provided the glass prisms are all made of the same kind of glass. For such a system, $(t - s)$ in formula (400) will denote the difference in aggregate thickness of the dispersive material through which the extreme rays of the pencil have passed. If the prisms are all adjusted so that the rays traverse them symmetrically, and if the upper extreme ray passes through the edges of all the prisms, then t in formula (402) denotes the sum of the "bases" of the prisms.

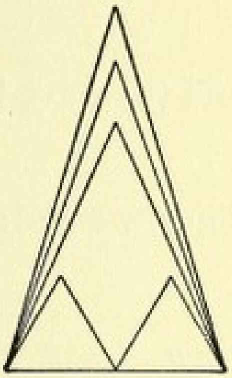

FIG. 152.

SCHUSTER[1] remarks that "the resolving power of prisms depends on the total thickness of glass, and not on the number of prisms, one large prism being as good as several small ones". Thus, all the prisms shown in Fig. 152 "would have the same resolving power, though they would show very considerable differences in dispersion".

341. According to CAUCHY's Dispersion-Formula (see § 327), we may write approximately:

$$n = A + B\lambda^{-2};$$

and, hence, by formula (402) the resolving power of a prism of base t is:

$$p = -2B\frac{t}{\lambda^3}. \tag{403}$$

We may say, therefore, roughly speaking, that the Resolving Power of a prism is inversely proportional to the cube of the wave-length; and hence the Resolving Power is much greater for light of short wave-lengths. The Resolving Power of a grating is the same for all wave-lengths, and hence a grating-spectroscope is not so good as a prism-spectroscope for resolving the ultra-violet lines of the spectrum.

The value of the co-efficient B in formula (403) depends on the material of the prism. Lord RAYLEIGH gives the following calculation of the thickness t of a prism made of the "extra dense flint" glass of Messrs. CHANCE Bros. that is necessary in order to resolve the FRAUNHOFER double D-line. The indices of refraction of this glass for light corresponding to the FRAUNHOFER lines C and D are:

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[1] A. SCHUSTER: *An Introduction to the Theory of Optics* (London, 1904), p. 144.

$$n_C = 1.644866, \quad n_D = 1.650388;$$

and the wave-lengths in centimetres are:

$$\lambda_C = 6.562 \cdot 10^{-5}, \quad \lambda_D = 5.889 \cdot 10^{-5}.$$

Thus, we find:

$$B = \frac{n_D - n_C}{\lambda_D^{-2} - \lambda_C^{-2}} = 0.984 \cdot 10^{-10}.$$

Now

$$t = p\frac{\lambda^3}{2B} = \frac{\lambda}{d\lambda} \cdot \frac{\lambda^3}{2B} = \frac{\lambda^4}{2B \cdot d\lambda}.$$

For the D-line: $\lambda = 5.889 \cdot 10^{-5}$, $d\lambda = 0.006 \cdot 10^{-5}$ (difference between D_1 and D_2).

Accordingly, we find $t = 1.02$ cm., which is, therefore, the necessary thickness of a prism of this material in order to resolve the double D-line. Moreover, Lord RAYLEIGH, testing this result by experiment, found that, as a matter of fact, a prism-thickness of between 1.2 and 1.4 cm. was needed for this purpose.

342. The Resolving Power of a system of prisms of different materials is given by the following formula:

$$p = \frac{\lambda}{d\lambda} = \sum_{k=1}^{k=m-1} (t_k - s_k) \frac{dn'_k}{d\lambda}; \tag{404}$$

where s_k and t_k denote the lengths of the ray-paths of the extreme rays between the kth and the $(k + 1)$th plane refracting surfaces.

For example, in an AMICI Direct-Vision Prism (§ 335), consisting of two prisms of crown glass cemented to a prism of flint glass, as represented in Fig. 153, we have:

$$s_1 = s_3 = 0, \quad s_2 = s, \quad t_1 = t_3 = t, \quad t_2 = 0, \quad n'_1 = n'_3;$$

and hence by formula (404):

$$p = 2t \cdot \frac{dn'_1}{d\lambda} - s \cdot \frac{dn'_2}{d\lambda}.$$

In this combination the dispersion of the crown glass is opposed to

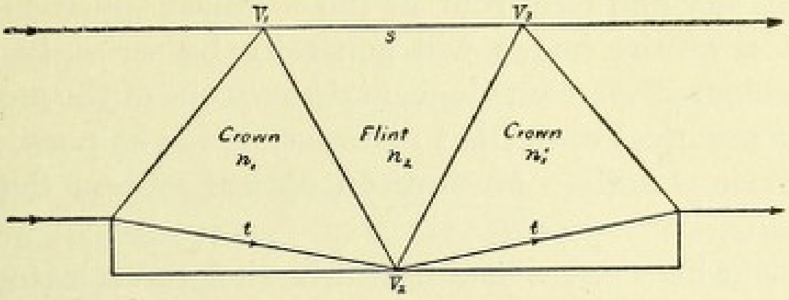

FIG. 153.

RESOLVING POWER OF AMICI "DIRECT-VISION" PRISM.

that of the flint glass, and the Resolving Power of the system is not great.

II. THE CHROMATIC ABERRATIONS.

ART. 108. THE DIFFERENT KINDS OF ACHROMATISM.

343. When a ray of white light is incident on a refracting surface, it will, in general, be resolved at the point of incidence into a pencil of coloured rays, since, as we have seen, the index of refraction depends on the colour of the light. Thus, for example, if P designates the position of a radiant point emitting, say, red and blue light, and if B_1 designates the position of a point on the first surface of a centered system of spherical refracting surfaces, and, finally if π designates a transversal plane perpendicular to the optical axis, then corresponding to an incident ray proceeding along the straight line PB_1, there will be a red image-ray which will cross the plane π (really or virtually) at a point P' and likewisealso a blue image-ray which will cross the plane π at a point $\overline{P}'$, which, in general, will be different from the point P'.[1] Since the positions of the focal points and the magnitudes of the focal lengths of an optical system depend also on the indices of refraction of the media traversed by the rays, and since the values of these indices depend on the colour of the light, it is evident that the same optical system will produce as many coloured images of a given object as there are colours in the light emitted by the object; and, in general, also, these images will be formed at different places and will be of different sizes. The entire series of images may be described as an image affected with *chromatic aberrations*. Even if the image were otherwise perfect and free from all the so-called spherical aberrations, the definition of the image will generally be seriously impaired on account of colour-dispersion alone, and hence one of the most important problems of practical optics is to correct, as far as possible, the chromatic aberrations and to produce an optical system that is more or less *achromatic*.

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[1] In the following pages of this chapter, whenever we have to deal with two colours, the letters and symbols which relate to the second colour will be distinguished from the corresponding letters and symbols which relate to the first colour by means of a dash written immediately above the character. It is true this same method of notation was used in the theory of astigmatism to distinguish between the meridian and sagittal rays; but no confusion is likely to occur on this account, and for various reasons it is convenient to use this same device here in a new sense.

The problem here mentioned is still further complicated by the fact that not only are the fundamental characteristics of the optical system (viz., the positions of the focal points and the magnitudes of the focal lengths) dependent on the indices of refraction of each medium, but the various spherical aberrations, which are encountered when the rays are not infinitely near to the optical axis, are likewise functions of the indices of refraction; so that we may have also *chromatic variations of the spherical aberrations*, even though the optical system has been corrected so that the focal points and the focal lengths are the same for all wave-lengths of light. As a matter of fact, all the properties of a centered system of spherical refracting surfaces are dependent in some way or other on the indices of refraction, and hence they are all variable with the colour of the light. The term "achromatism" by itself is, therefore, entirely indefinite, for the system may be achromatic in one sense and not at all so in other senses. For example, the images corresponding to the different colours may all be formed at the same place, and yet be of different sizes; or the system may be achromatic with respect to Distortion or with respect to the Sine-Condition, etc., and at the same time affected with colour-dispersion in a variety of other ways. Obviously, it will not be possible to correct all these different kinds of chromatic aberrations at the same time; and, in fact, in order to have a distinct image (which is the primary aim of an optical instrument), this will not be necessary, as some of the chromatic aberrations are comparatively unimportant, depending on the purpose which the apparatus is intended to fulfil.

An optical system which produces two coloured images of a given object at the same place and of the same size is said to be *completely achromatic* for these two colours. The images of other colours will, however, generally be different as to both size and position, and the effect on the resultant image usually appears in a coloured margin or "*secondary spectrum*" (§ 329).

But usually the best we can do is to contrive to obtain a *partial achromatism* of some sort, and especially one of the two following kinds: *achromatism with respect to place* (so that the two coloured images, although of unequal sizes, are both formed in the same image-plane), or *achromatism as to magnification* (so that the two coloured images, although differently situated, are of equal size). In many cases, indeed, it will be found quite sufficient to effect a partial achromatism of one or other of these two kinds. Thus, for example, it is essential that the coloured images formed by the objectives of telescopes and microscopes shall be situated as nearly as possible at the same place; whereas, since the images do not extend far from the optical axis, the unequal colour-magnifications are comparatively negligible. On the other hand, in the case of the eye-pieces of these instruments, whose particular office is to produce extended images of the small images formed by the objectives, the main point is to obtain achromatism as to magnification, whereas the slight differences in the distances of the coloured images are of relatively small importance. As a rule, it may be stated that for an optical system which produces a real image, it is more desirable to have achromatism with respect to the place of the image; whereas if the image is virtual, achromatism with respect to the magnification is likely to be the more important requirement.

In the following pages it is proposed to develop the formulæ for the numerical calculation of the more important of the chromatic aberrations, and to determine the conditions that are necessary in order to abolish or diminish them. In this investigation it will be assumed (except in the brief treatment, at the end of the chapter, of the chromatic variations of the spherical aberrations) that we are concerned only with *paraxial rays*, so that between the object and its image in any one definite colour there is complete collinear correspondence.

As to notation, let us state here that the change of a magnitude x in consequence of a *finite variation* of the wave-length of the light from the value λ to the value $\bar{\lambda}$ will be indicated by the capital letter D written immediately in front of the symbol of the magnitude, thus:

$$Dx = \bar{x} - x.$$

ART. 109. THE CHROMATIC VARIATIONS OF THE POSITION AND SIZE OF THE IMAGE, IN TERMS OF THE FOCAL LENGTHS AND FOCAL DISTANCES OF THE OPTICAL SYSTEM.

344. Let A_1 and A_m designate the positions of the vertices of the first and last surfaces, respectively, of a centered system of m spherical refracting surfaces, and let F, E' and $\bar{F}$, $\bar{E}'$ designate the positions on the optical axis of the focal points of the system for the two colours corresponding to light of wave-lengths λ and $\bar{\lambda}$, respectively. At a point M on the optical axis, erect the perpendicular MQ; and let $M'Q'$ and $\bar{M}'\bar{Q}'$ be the GAUSSian images in the two colours corresponding to the object MQ. We shall employ here the following symbols:

$$\begin{array}{llll} z = A_1F, & \bar{z} = A_1\bar{F}, & Dz = \bar{z} - z = F\bar{F}, & \\ z' = A_mE', & \bar{z}' = A_m\bar{E}', & Dz' = \bar{z}' - z' = E'\bar{E}', & \\ u = A_1M, & u' = A_mM', & \bar{u}' = A_m\bar{M}', & Du' = \bar{u}' - u' = M'\bar{M}', \\ x = FM, & x' = E'M', & \bar{x} = \bar{F}M, & \bar{x}' = \bar{E}'M', \\ y = MQ, & y' = M'Q', & \bar{y}' = \bar{M}'\bar{Q}'. & \end{array}$$

Denoting the focal lengths of the system for the two colours by f, e' and $\bar{f}$, $\bar{e}'$, we have the following set of equations (§ 178):

$$\left.\begin{array}{ll} xx' = fe', & \bar{x}\bar{x}' = \bar{f}\bar{e}', \\ Y = \dfrac{y'}{y} = \dfrac{f}{x}, & \bar{Y} = \dfrac{\bar{y}'}{y} = \dfrac{\bar{f}}{\bar{x}}, \end{array}\right\} \tag{405}$$

where Y, $\bar{Y}$ denote the lateral magnifications for the two colours. Now

$$\bar{x} = x - Dz, \quad \bar{x}' = x' + Du' - Dz';$$

and, hence, eliminating $\bar{x}$ and $\bar{x}'$ and solving for Du', we obtain from the upper pair of equations (405):

$$Du' = Dz' + \frac{x' \cdot Dz + D(fe')}{x - Dz}. \tag{406}$$

Similarly, eliminating $\bar{x}$ from the lower pair of equations (405), we obtain:

$$DY = \frac{f}{x - Dz}\left(\frac{Df}{f} + \frac{Dz}{x}\right). \tag{407}$$

These Difference-Formulæ, which are given by KOENIG,[1] give the variations of the position and magnification of the image of an object corresponding to any arbitrary variations of the fundamental characteristics of the optical system.

345. We may consider several *special cases* as follows:

(1) If the optical system is achromatic with respect to the position of the image, then we shall have $Du' = 0$, and from equation (406) we obtain in this case the following quadratic equation with respect to x:

$$x^2 \cdot Dz' + \{D(fe') - Dz \cdot Dz'\}\, x + fe' \cdot Dz = 0;$$

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[1] See A. KOENIG: "Die Theorie der chromatischen Aberrationen", Chapter VI of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR. See p. 345.

so that, in general, there will be two positions of the object for which its images in the two given colours will be formed in the same transversal plane; but if the roots of the quadratic are imaginary, there will be no position of the object for which the system can have this kind of achromatism. In the special case when

$$Dz = Dz' = D(fe') = 0,$$

the quadratic equation will be satisfied for all values of x; and we have then what is sometimes called *stable achromatism* with respect to the place of the image. If, also, the first and last media are identical, we have $DY = 0$, and the system will be completely achromatic, in the sense in which this term was defined in § 343.

(2) If the system is achromatic with respect to the lateral magnification, then $DY = 0$. This condition is satisfied by $x = \bar{x} = \infty$, $Y = \bar{Y} = 0$; and, also, according to equation (407), by the following value of x:

$$x = -\frac{f}{Df} \cdot Dz.$$

If $Df = Dz = 0$, then $DY = 0$ for all values of x.

If g, $\bar{g}$ denote the ordinates of the points where an incident ray emanating from the axial object-point M crosses the primary focal planes corresponding to the two colours, then

$$x : \bar{x} = g : \bar{g};$$

and if the two magnifications Y, $\bar{Y}$ are equal, then

$$x : \bar{x} = f : \bar{f};$$

and hence:

$$g : \bar{g} = f : \bar{f};$$

and, since (§ 178)

$$g = e' \cdot \tan \theta', \quad \bar{g} = \bar{e}' \cdot \tan \bar{\theta}',$$

where θ', $\bar{\theta}'$ denote the slopes of the pair of coloured image-rays corresponding to the given incident ray, we have:

$$e' \cdot \tan \theta' : \bar{e}' \cdot \tan \bar{\theta}' = f : \bar{f}.$$

If, therefore, the first and last media are identical, so that $e' = -f$, $\bar{e}' = -\bar{f}$, we obtain $\theta' = \bar{\theta}'$; and hence when $Y = \bar{Y}$, and $n = n'$, $\bar{n} = \bar{n}'$, *the pair of coloured emergent rays corresponding to a given incident ray will be parallel.*

(3) If the optical system is achromatic with respect to the positions of the Focal Points, then $Dz = Dz' = 0$, and we have:

$$Du' = \frac{Df^2}{f^2}x' = 2\frac{Df}{f}x', \text{ approximately;}$$

which shows that the variation Du' of the position of the image will depend on the variation of the Focal Length. If $Df = 0$, or if $x' = 0$, the two coloured images of the given object will lie in the same plane ($Du' = 0$).

Moreover, we have here:

$$DY = \frac{Df}{x}, \quad \text{or} \quad Dy' = \frac{Df}{f}y';$$

and hence the variation Dx' in the size of the image is independent of the value of x. Accordingly, there is no value of x that will make $Dy' = 0$; except the value $x = \infty$, for which we have $y' = \bar{y}' = 0$. Since, however, Dx' is proportional to y', the variation Dy' will be more and more nearly negligible, the smaller y' is; and hence this variation will be very slight in the central part of the field of view.

(4) Finally, if the system is achromatic with respect to the Focal Lengths ($f = \bar{f}$, $e' = \bar{e}'$), the positions of the Focal Points F, $\bar{F}$ and E', $\bar{E}'$ will, in general, be different. Putting $Df = De' = 0$ in formulæ (406) and (407), we obtain:

$$\frac{Dz}{x} + \frac{Du' - Dz'}{f^2}(x - Dz) = 0, \quad DY = -\frac{Du' - Dz'}{f}.$$

ART. 110. FORMULÆ ADAPTED TO THE NUMERICAL CALCULATION OF THE CHROMATIC VARIATIONS OF THE POSITION AND MAGNIFICATIONS OF THE IMAGE OF A GIVEN OBJECT IN A CENTERED SYSTEM OF SPHERICAL REFRACTING SURFACES.

346. Chromatic Longitudinal Aberration.

The relation between the intercepts u, u' of a paraxial ray before and after refraction, respectively, at a spherical surface, of radius r, which separates media whose indices of refraction for light of wave-length λ are denoted by n and n' is given by the following equation:

$$n\left(\frac{1}{r} - \frac{1}{u}\right) = n'\left(\frac{1}{r} - \frac{1}{u'}\right) = J;$$

and, hence, after elimination of r, we obtain:

$$DJ = J\frac{Dn}{n} + \frac{\bar{n} \cdot Du}{u\bar{u}} = J\frac{Dn'}{n'} + \frac{\bar{n}' \cdot Du'}{u'\bar{u}'},$$

where the symbols with dashes above them relate to the same incident ray for light of wave-length $\bar{\lambda}$. Accordingly, we find:

$$J \cdot \Delta\left(\frac{Dn}{n}\right) = -\Delta\left(\frac{\bar{n} \cdot Du}{u\bar{u}}\right),$$

where, according to Abbe's system of notation, the operator Δ, written before an expression, indicates, as always heretofore, the difference of the values of the expression before and after refraction. Thus, for the kth surface of a centered system of spherical refracting surfaces, we have:

$$\Delta\left(\frac{\bar{n} \cdot Du}{u\bar{u}}\right)_k = -J_k \cdot \Delta\left(\frac{Dn}{n}\right)_k. \tag{408}$$

Since the distance measured along the optical axis between the kth and the $(k+1)$th spherical surfaces is

$$d_k = u'_k - u_{k+1} = \bar{u}'_k - \bar{u}_{k+1},$$

we have:

$$Du_{k+1} = Du'_k.$$

Moreover, if h_k, $\bar{h}_k$ denote the incidence-heights, for the rays of the two colours, at the kth surface, then:

$$\frac{h_k}{u'_k} = \frac{h_{k+1}}{u_{k+1}}, \quad \frac{\bar{h}_k}{\bar{u}'_k} = \frac{h_{k+1}}{\bar{u}_{k+1}}.$$

Taking note of these relations, and multiplying both sides of equation (408) by $h_k \cdot \bar{h}_k$, and then giving k in succession all integral values from $k = 1$ to $k = m$, and adding together all the equations thus formed, we obtain:

$$\bar{n}'_m h_m \bar{h}_m \frac{Du'_m}{u'_m \bar{u}'_m} - \bar{n}_1 h_1 \bar{h}_1 \frac{Du_1}{u_1 \bar{u}_1} = -\sum_{k=1}^{k=m} h_k \bar{h}_k J_k \cdot \Delta\left(\frac{Dn}{n}\right)_k;$$

and if the object is without dispersion, that is, if $Du_1 = 0$, we derive finally the following formula for the so-called *chromatic longitudinal aberration*[1] of a centered system of m spherical surfaces for a given position of the axial object-point:

$$Du'_m = -\frac{u'_m \cdot \bar{u}'_m}{\bar{n}'_m} \sum_{k=1}^{k=m} \frac{h_k}{h_m} \cdot \frac{\bar{h}_k}{\bar{h}_m} \cdot J_k \cdot \Delta\left(\frac{Dn}{n}\right)_k. \tag{409}$$

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[1] See A. Koenig: Die Theorie der chromatischen Aberrationen: Chapter VI of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. von Rohr. See page 341.

The chromatic lateral aberration, with respect to a given axial object-point, is defined by ABBE as the radius of the cross-section of the bundle of image-rays of the second colour made by the image-plane of the first colour. Its magnitude is:

$$\frac{\bar{h}_m}{\bar{u}'_m} Du'_m;$$

and we can obtain here also, exactly as in § 266, a more or less artificial measure of the "indistinctness of the image" due to the chromatic aberration.

The condition that the optical system shall be achromatic with respect to the place of the image of a given object is:

$$\sum_{k=1}^{k=m} h_k \bar{h}_k J_k \cdot \Delta\left(\frac{Dn}{n}\right)_k = 0. \qquad (410)$$

347. Differential Formulæ for the Chromatic Variations.

If we assume that the variation of n is infinitely small, we shall obtain for the chromatic variations of the place and magnifications of the image a series of differential formulæ, which are much simpler than the corresponding difference-formulæ, and which are also sufficiently accurate in most cases even for a finite colour-interval. Thus, for example, by differentiation, we easily derive the following formula[1] for the *Chromatic Longitudinal Aberration :*

$$du'_m = -\frac{u'_m}{n'_m} \sum_{k=1}^{k=m} \left(\frac{h_k}{h_1}\right)^2 J_k \cdot \Delta\left(\frac{dn}{n}\right)_k. \qquad (411)$$

In a centered system of m spherical surfaces, the Lateral Magnification is (§ 138):

$$Y = \frac{n_1}{n'_m} \prod_{k=1}^{k=m} \frac{u'_k}{u_k}; \qquad (412)$$

the Angular Magnification is (*cf.* § 179 and § 193):

$$Z = \prod_{k=1}^{k=m} \frac{u_k}{u'_k}; \qquad (413)$$

and the Axial Magnification is:

$$X = \frac{Y}{Z} = \frac{n_1}{n'_m} \prod_{k=1}^{k=m} \left(\frac{u'_k}{u_k}\right)^2. \qquad (414)$$

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[1] See L. SEIDEL: Zur Theorie der Fernrohr-Objective: *Astr. Nachr.*, xxxv (1853), No. 835, pages 301–316.

By differentiation, we obtain:

$$\left.\begin{aligned} \frac{dY}{Y} &= \frac{dn_1}{n_1} - \frac{dn'_m}{n'_m} + \sum_{k=1}^{k=m} \Delta\left(\frac{du}{u}\right)_k, \\ \frac{dZ}{Z} &= -\sum_{k=1}^{k=m} \Delta\left(\frac{du}{u}\right)_k, \\ \frac{dX}{X} &= \frac{dn_1}{n_1} - \frac{dn'_m}{n'_m} + 2\sum_{k=1}^{k=m} \Delta\left(\frac{du}{u}\right)_k. \end{aligned}\right\} \tag{415}$$

Since

$$du'_{k-1} = du_k \quad \text{and} \quad d_{k-1} = u'_{k-1} - u_k,$$

we have evidently:

$$\frac{du'_m}{u'_m} - \frac{du_1}{u_1} - \sum_{k=2}^{k=m} \frac{d_{k-1}}{u_k \cdot u'_{k-1}} du'_{k-1} = \sum_{k=1}^{k=m} \Delta\left(\frac{du}{u}\right)_k;$$

and if we assume that the object is without dispersion ($du_1 = 0$), we may, therefore, write formulæ (415) as follows:

$$\left.\begin{aligned} \frac{dY}{Y} &= \frac{dn_1}{n_1} - \frac{dn'_m}{n'_m} + \frac{du'_m}{u'_m} - \sum_{k=2}^{k=m} \frac{d_{k-1}}{u_k \cdot u'_{k-1}} du'_{k-1}, \\ \frac{dZ}{Z} &= -\frac{du'_m}{u'_m} + \sum_{k=2}^{k=m} \frac{d_{k-1}}{u_k \cdot u'_{k-1}} du'_{k-1}, \\ \frac{dX}{X} &= \frac{dn_1}{n_1} - \frac{dn'_m}{n'_m} + 2\frac{du'_m}{u'_m} - 2\sum_{k=2}^{k=m} \frac{d_{k-1}}{u_k \cdot u'_{k-1}} du'_{k-1}; \end{aligned}\right\} \tag{416}$$

which forms are more convenient than equations (415) in case we have to determine the chromatic variations of the magnification for the special case when $du'_m = 0$.

If the first and last media are identical ($n_1 = n'_m$), we have, according to formulæ (416):

$$\frac{dY}{Y} = -\frac{dZ}{Z} = \frac{1}{2}\frac{dX}{X} = \frac{du'_m}{u'_m} - \sum_{k=2}^{k=m} \frac{d_{k-1}}{u_k \cdot u'_{k-1}} du'_{k-1},$$

and the condition that $dZ = 0$ is also in this case identical with the conditions that dY and dX shall vanish. Under these circumstances (see § 345), *the pair of coloured emergent rays corresponding to a given incident ray will be parallel.* If, moreover, the distances (d) between each pair of successive surfaces are all vanishingly small (system of infinitely thin lenses in contact), the condition

$$dX = dY = dZ = 0$$

is identical with the condition $du'_m = 0$.

KOENIG,[1] in his excellent treatise on the Chromatic Aberrations, derives also the formulæ for DX/X, DY/Y, and DZ/Z; but these difference-formulæ, although easily obtained, are rather complicated in form.

348. KOENIG remarks also that, in the case of optical systems affected with chromatic longitudinal aberration, instead of trying to abolish the variation of the chromatic lateral magnification, it is sometimes preferable to adjust the transversal focussing plane in the position $M''Q''$ (Fig. 154), where Q'' designates the point of intersection of the chief image-rays of the two bundles of coloured rays corresponding to the bundle of incident rays emanating from the extra-axial object-point Q; so that the projections on the focussing plane of the two coloured images $M'Q'$ and $\bar{M}'\bar{Q}'$ of the object MQ are of the same size, viz., $M''Q''$. If $\mathbf{M}'$, $\bar{\mathbf{M}}'$ designate the positions of the points where the chief rays of the bundles of image-rays of the two colours cross the optical axis, then evidently, as the diagram shows:

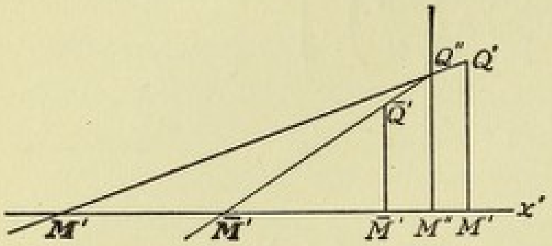

FIG. 154.

CHROMATIC VARIATION OF THE LATERAL MAGNIFICATION. $M'Q'$, $\bar{M}'\bar{Q}'$ are the two coloured images of an object MQ (not shown in the diagram). $M''Q''$ is the position of the focussing plane adjusted so that the two coloured images are projected on it from the points $\mathbf{M}'$ and $\bar{\mathbf{M}}'$ in a piece $M''Q''$ which is the same for both images.

$$\frac{\mathbf{M}'M''}{\mathbf{M}'M'} = \frac{M''Q''}{M'Q'}, \quad \frac{\bar{\mathbf{M}}'M''}{\bar{\mathbf{M}}'\bar{M}'} = \frac{M''Q''}{\bar{M}'\bar{Q}'};$$

and hence, if

$$Y = \frac{M'Q'}{MQ}, \quad \bar{Y} = \frac{\bar{M}'\bar{Q}'}{MQ}$$

denote the lateral magnifications for the two colours, we have:

$$Y \cdot \frac{\mathbf{M}'M''}{\mathbf{M}'M'} = \bar{Y} \cdot \frac{\bar{\mathbf{M}}'M''}{\bar{\mathbf{M}}'\bar{M}'},$$

or

$$\frac{DY}{Y} = \frac{\mathbf{M}'M''}{\mathbf{M}'M'} \cdot \frac{\bar{\mathbf{M}}'\bar{M}'}{\bar{\mathbf{M}}'M''} - 1.$$

If now the segments $M'M''$, $\bar{M}'M''$, $M'\bar{M}'$ and $\mathbf{M}'\bar{\mathbf{M}}'$ may be regarded as small magnitudes of the first order, and if we neglect magnitudes of the second order of smallness, then

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[1] A. KOENIG: Die Theorie der chromatischen Aberrationen: Chapter VI of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR. See pages 342 and 343.

$$\begin{aligned}\frac{DY}{Y} &= \left(1+\frac{M'M''}{\boldsymbol{M}'M'}\right)\left(1+\frac{M''\bar{M}'}{\bar{\boldsymbol{M}}'M''}\right)-1\\ &= \frac{M'M''}{\boldsymbol{M}'M'}+\frac{M''\bar{M}'}{\bar{\boldsymbol{M}}'M''}\\ &= \frac{M'\bar{M}'}{\boldsymbol{M}'M'}+\bar{M}'M''\left(\frac{\bar{\boldsymbol{M}}'M''-\boldsymbol{M}'M'}{\boldsymbol{M}'M'\cdot\bar{\boldsymbol{M}}'M''}\right)\\ &= \frac{M'\bar{M}'}{\boldsymbol{M}'M'};\end{aligned}$$

so that if the abscissæ, with respect to the vertex of the last surface of the centered system of spherical surfaces, of the points M', $\bar{M}'$ and $\boldsymbol{M}'$ be denoted by u', $\bar{u}'$ and $\boldsymbol{u}'$, respectively, we have approximately:

$$\frac{DY}{Y}=\frac{Du'}{u'-\boldsymbol{u}'}. \tag{417}$$

This formula is given by KOENIG.[1]

ART. 111. CHROMATIC VARIATIONS IN SPECIAL CASES.

349. Optical System consisting of a Single Lens, surrounded on both sides by air.

If the optical system consists of a single lens ($m = 2$), surrounded on both sides by air ($n_1 = n'_2 = 1$, $n'_1 = n$), the formulæ for the chromatic longitudinal aberration, as derived from the difference-equation (409), is as follows:

$$Du'_2 = -u'_2\cdot\bar{u}'_2\left(\frac{h_1}{h_2}\cdot\frac{\overline{h}_1}{\overline{h}_2}J_1-J_2\right)\frac{Dn}{n}; \tag{418}$$

which vanishes when $u'_2 = 0$ or $\bar{u}'_2 = 0$ (neither of which cases need be considered), and also when:

$$h_1\overline{h}_1J_1 = h_2\overline{h}_2J_2. \tag{419}$$

Since (*cf.* § 126) $h_1J_1 = \alpha_1$, $h_2J_2 = n\alpha_2 = \alpha'_2$, this condition may also be written as follows:

$$\frac{\alpha_1}{\alpha'_2}=\frac{\overline{h}_2}{\overline{h}_1};$$

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[1] A. KOENIG: Die Theorie der chromatischen Aberrationen: Chapter VI of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR. See page 345.

and, hence, we may say that the condition of the abolition of the chromatic longitudinal aberration with respect to an object-point on the axis of the lens is that the angles of incidence at the first surface and of emergence at the second surface of a ray of the first colour shall be inversely proportional to the incidence-heights of the corresponding ray of the second colour.

Equation (419) may also be written as follows:

$$u'_1\bar{u}'_1 J_1 = u_2\bar{u}_2 J_2. \tag{420}$$

Now

$$J_1 = \frac{1}{r_1} - \frac{1}{u_1} = n\left(\frac{1}{r_1} - \frac{1}{u'_1}\right) = \bar{n}\left(\frac{1}{r_1} - \frac{1}{\bar{u}'_1}\right);$$

and hence:

$$r_1 = \frac{Dn}{D\dfrac{n}{u'_1}};$$

moreover,

$$J_2 = n\left(\frac{1}{r_2} - \frac{1}{u_2}\right), \quad d = u'_1 - u_2 = \bar{u}'_1 - \bar{u}_2,$$

where d denotes the thickness of the lens. Eliminating J_1, J_2 from equation (420) by means of the above relations, and, finally, solving for r_2, we obtain:

$$r_2 = \frac{(u'_1 - d)(\bar{u}'_1 - d)(\bar{n} - n)}{\bar{n}(u'_1 - d) - n(\bar{u}'_1 - d)}; \tag{421}$$

which is a formula due to KESSLER.[1] Thus, if we know the radius r_1 of the first surface of the lens and the lens-thickness d, this formula enables us to calculate the radius r_2 of the second surface, so that the images of a given axial radiant point M_1, of colours corresponding to the given values of n and $\bar{n}$, will be formed at one and the same point.

In case the axial object-point is infinitely distant ($u_1 = \infty$), so that the focal points of the lens (E', $\bar{E}'$) for the two colours are coincident, we find:

$$r_2 = \frac{\{nr_1 - (n-1)d\}\{\bar{n}r_1 - (\bar{n}-1)d\}}{n\bar{n}r_1 - (n-1)(\bar{n}-1)d};$$

and it is easy to show that for such a lens the secondary principal point of the lens (A') for the colour corresponding to the value n coincides with the secondary focal point of the first surface of the lens ($\bar{E}'_1$) for the colour corresponding to the value $\bar{n}$; and, similarly, that the point $\bar{A}'$ (secondary principal point of the lens for the colour corresponding to the value $\bar{n}$) coincides with the point E'_1 (secondary focal point of the first surface of the lens for the colour corresponding to the value n).

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[1] F. KESSLER: Ueber Achromasie: *Zft. f. Math. u. Phys.*, xxix (1884), 1–24. This admirable paper contains a complete treatment of the conditions of the achromatism of a thick lens, with a number of elegant geometrical constructions by the methods of projective geometry.

In the case of a lens surrounded by the same medium on both sides, the lateral magnifications for the two colours corresponding to the values n and $\bar{n}$ are as follows:

$$Y = \frac{u'_1 \cdot u'_2}{u_1 \cdot u_2}, \quad \bar{Y} = \frac{\bar{u}'_1 \cdot \bar{u}'_2}{u_1 \cdot \bar{u}_2};$$

whence we derive the following difference-formula:

$$\frac{DY}{Y} = \frac{\bar{u}'_1}{u'_1} \cdot \frac{nr_2 - (n - 1)(u'_1 - d)}{\bar{n}r_2 - (\bar{n} - 1)(\bar{u}'_1 - d)} - 1. \qquad (422)$$

In order, therefore, that the chromatic variation of the lateral magnification shall vanish ($DY = 0$), we have the following condition:

$$r_2 = \frac{(\bar{n} - n)u'_1 \cdot \bar{u}'_1 - \{(\bar{n} - 1)u'_1 - (n - 1)\bar{u}'_1\}d}{\bar{n}u'_1 - n\bar{u}'_1}; \qquad (423)$$

so that, provided the position of the axial object-point is assigned, and the radius of the first surface of the lens is given, together with the values of n and $\bar{n}$, this formula (423) gives r_2 and d as linear functions of each other.

Eliminating u'_1 and $\bar{u}'_1$ by means of the relations given above just after formula (420), we derive from formula (423) the following equation:

$$u_1 = \frac{r_1 d}{n\bar{n}(r_1 - r_2) - (n\bar{n} - 1)d}; \qquad (424)$$

whereby for a given lens we can determine the position of the object-plane which for two given colours is portrayed by the lens in images of equal dimensions.

If the thickness of the lens is

$$d = \frac{n\bar{n}(r_1 - r_2)}{n\bar{n} - 1},$$

then, according to formula (424), we find $u_1 = \infty$; so that for such a lens the object must be situated at infinity. We find also that the focal lengths are equal; thus:

$$f = \bar{f} = \frac{n\bar{n} - 1}{(n - 1)(\bar{n} - 1)} \cdot \frac{r_1 r_2}{r_2 - r_1};$$

and it may also be shown that the primary principal point of this lens (A) for the first colour coincides with the secondary focal point of the first surface of the lens ($\bar{E}'_1$) for the second colour; and, similarly, that $\bar{A}$ coincides with E'_1.

350. Infinitely Thin Lens.

If the lens is infinitely thin, we may put:

$$h_1 = \bar{h}_1 = h_2 = \bar{h}_2$$

in formula (418); and if also we introduce here our special notation for the case of an infinitely thin lens (§ 268), so that $x = 1/u$, $x' = 1/u'$, and $\bar{x}' = 1/\bar{u}'$ denote the reciprocals of the intercepts on the axis of the incident and emergent paraxial rays for the two colours; and if $\varphi = 1/f$ denotes the "power" of the lens for the colour corresponding to the value n; and, finally, if $c = 1/r_1$, $c' = 1/r_2$ denote the curvatures of the bounding surfaces of the lens, then the lens-formulæ may be written as follows:

$$\varphi = (n - 1)(c - c'), \quad x' = x + \varphi.$$

Accordingly, we derive the following formula for *the chromatic longitudinal aberration of an infinitely thin lens:*

$$Dx' = \frac{Dn}{n - 1}\varphi = \frac{\varphi}{\nu}, \tag{425}$$

where

$$\nu = \frac{n - 1}{Dn} \tag{426}$$

is the magnitude defined in § 329. The reciprocal value $1/\nu$ is sometimes called the *dispersor* of the lens, and the quotient φ/ν is called the *dispersive strength* of the lens.

If M', $\bar{M}'$ designate the positions of the image-points in the two colours corresponding to the axial object-point M, then

$$Du' = M'\bar{M}' = -u'\bar{u}'\,\frac{\varphi}{\nu};$$

and hence (except in the merely theoretical case when $u' = \bar{u}' = 0$) it is not possible to abolish the chromatic longitudinal aberration of an infinitely thin lens.

When the incident ray is parallel to the axis, we have $u' = f$, $\bar{u}' = \bar{f}$, and in this case:

$$Du' = E'\bar{E}' = -\frac{\bar{f}}{\nu}.$$

If the lens is convergent ($\bar{f} > 0$), and if, as is assumed throughout, $\bar{n} > n$, then $Du' < 0$; so that the more refrangible (blue) rays are brought to a focus $\bar{E}'$ nearer to the lens than the focus E' of the less refrangible (red) rays; which is the case known as *Chromatic Under-Correction*. The opposite effect, viz., *Chromatic Over-Correction*, is exhibited by an infinitely thin divergent lens.

The chromatic aberration of the lateral magnification of an infinitely thin lens is:

$$DY = \frac{Du'}{u'} = -\frac{u' \cdot \bar{u}'}{u} \cdot \frac{\varphi}{\nu}. \tag{427}$$

351. Chromatic Aberration of a System of Infinitely Thin Lenses. The formula for the chromatic longitudinal aberration of a system of infinitely thin lenses with the centres of their surfaces ranged along one and the same straight line may be derived very easily from the general formula (409). However, as we use here a special notation corresponding to that employed above in the case of a single infinitely thin lens, and as *the subscripts attached to the symbols relate now to the number of the lens, and not to the number of the refracting surface*, it is more convenient to deduce the formula independently. Consider, therefore, the kth lens of the system, and let A_k designate the position of the optical centre of this lens. Also, let M'_{k-1}, M'_k designate the points where a paraxial ray (emanating originally from the axial object-point M_1), of colour corresponding to the value n_k, crosses the optical axis before and after refraction, respectively, through this lens; and let

$$\frac{1}{x_k} = u_k = A_k M'_{k-1}, \quad \frac{1}{x'_k} = u'_k = A_k M'_k;$$

and, similarly, for a paraxial ray of colour corresponding to the value $\bar{n}_k$, which emanates from the same axial object-point M_1, we shall write:

$$\frac{1}{\bar{x}_k} = \bar{u}_k = A_k \bar{M}'_{k-1}, \quad \frac{1}{\bar{x}'_k} = \bar{u}'_k = A_k \bar{M}'_k.$$

Denoting the strength or "power" of the kth lens for the two colours by φ_k and $\bar{\varphi}_k$, we derive easily the following difference-relation:

$$Dx'_k = Dx_k + D\varphi_k = Dx_k + \frac{\varphi_k}{\nu_k},$$

where

$$\nu_k = \frac{n_k - 1}{Dn_k}$$

denotes the value of ν for the substance of the kth lens. Moreover,

$$Dx_k = \frac{x_k}{x'_{k-1}} \cdot \frac{\bar{x}_k}{\bar{x}'_{k-1}} \cdot Dx'_{k-1} = \frac{h_{k-1}}{h_k} \cdot \frac{\bar{h}_{k-1}}{\bar{h}_k} \cdot Dx'_{k-1},$$

where h_k, $\bar{h}_k$ denote the incidence-heights of the rays of the two colours at the kth lens. Hence, we obtain the following recurrent formula:

$$Dx'_k = \frac{h_{k-1}}{h_k} \cdot \frac{\bar{h}_{k-1}}{\bar{h}_k} \cdot Dx'_{k-1} + \frac{\varphi_k}{\nu_k};$$

whence, putting $Dx_1 = 0$, $Dx'_1 = \varphi_1/\nu_1$, we deduce easily the following equation for *the chromatic longitudinal aberration of a system of m infinitely thin lenses:*

$$Dx'_m = \sum_{k=1}^{k=m} \frac{h_k}{h_m} \cdot \frac{h_k}{\bar{h}_m} \cdot \frac{\varphi_k}{\nu_k}. \tag{428}$$

It may be remarked that the actual forms of the lenses which compose the system are entirely immaterial; so that any lens in the system may be replaced by an equivalent one of different form but of the same dispersive strength (§ 350), without affecting the magnitude of the chromatic longitudinal aberration.

If the distance between each pair of lenses of the system is negligible, that is, if we have a *system of m infinitely thin lenses in contact*, the magnitudes denoted by the letters h will all be equal, and in this case we shall have:

$$Dx'_m = \sum_{k=1}^{k=m} \frac{\varphi_k}{\nu_k}; \tag{429}$$

so that for such a system the chromatic longitudinal aberration is independent of the position of the axial object-point; and, moreover, the order in which the lenses are placed does not matter.

A single infinitely thin lens of strength

$$\varphi = \sum_{k=1}^{k=m} \varphi_k, \tag{430}$$

and made of material whose ν-value is:

$$\nu = \frac{\sum_{k=1}^{k=m} \varphi_k}{\sum_{k=1}^{k=m} \frac{\varphi_k}{\nu_k}} \tag{431}$$

will be equivalent to the system of m thin lenses in contact, in respect both to the refraction and the dispersion of paraxial rays. Thus, for a given value of φ and for a given value of ν_k, we may vary the strength φ_k of the kth lens, so that ν has any arbitrary value whatever. This fact is of immense importance to the optician; for although he has at his disposal only a limited series of optical glasses with values of ν ranging from, say, $\nu = 20$ to $\nu = 70$, yet in case he needs for a certain lens a certain ν-value that does not belong to any actual kind of glass, he has merely to substitute for this lens a suitable combination of two or more lenses.[1]

If the system of thin lenses in contact is achromatic, we have:

$$\sum_{k=1}^{k=m} \frac{\varphi_k}{\nu_k} = 0; \tag{432}$$

in which case the ν-value of the combination is equal to infinity.

352. In particular, let us consider, first, a system consisting of **Two Infinitely Thin Lenses in Contact** ($m = 2$). The condition of the abolition of the chromatic aberration with respect to the place of the image, as derived from formula (429), is as follows:

$$\frac{\varphi_1}{\nu_1} + \frac{\varphi_2}{\nu_2} = 0.$$

If the differences of the curvatures of the two surfaces of the lenses be denoted by C_1 and C_2, that is, if

$$C_1 = c_1 - c'_1, \quad C_2 = c_2 - c'_2, \tag{433}$$

then, since

$$\varphi_1 = (n_1 - 1)\,C_1, \quad \varphi_2 = (n_2 - 1)\,C_2,$$

the condition above may be expressed in the following form also:

$$C_1 \cdot Dn_1 + C_2 \cdot Dn_2 = 0. \tag{434}$$

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[1] See A. Koenig: Die Theorie der chromatischen Aberrationen, Chapter VI of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. von Rohr. See page 349.

Moreover, if

$$\varphi = \varphi_1 + \varphi_2$$

denotes the strength of the combination of lenses, we find that, in order to fulfil the condition of achromatism, the strengths of the two lenses must be as follows:

$$\varphi_1 = \frac{\nu_1}{\nu_1 - \nu_2}\varphi, \quad \varphi_2 = -\frac{\nu_2}{\nu_1 - \nu_2}\varphi;$$

also we obtain:

$$C_1 = \frac{\varphi}{(\nu_1 - \nu_2)\cdot Dn_1}, \quad C_2 = -\frac{\varphi}{(\nu_1 - \nu_2)\cdot Dn_2}.$$

According to these results, a system of two infinitely thin lenses in contact can be achromatic only in case the ν-values of the two lenses are different; that is, the two lenses must be made of different kinds of glass. Moreover, the focal lengths of the lenses must have opposite signs, and must be inversely proportional to the ν-values. The focal length of the combination has the same sign as that of the lens with the greater ν-value; thus, the strength (φ) of the achromatic combination will be positive when the strength of the positive lens exceeds that of the negative lens. For a prescribed value of φ, the strengths of the individual lenses are smaller in proportion as the ν-values are smaller, and also in proportion as the difference of the ν-values is greater. By selecting two kinds of glass with as great difference of ν-values as possible, we can reduce the differences of the curvatures of the two surfaces of the lenses.

353. System of Two Infinitely Thin Lenses Separated by a Finite Interval (d).

According to formula (428), the chromatic longitudinal aberration of a system of two infinitely thin lenses is as follows:

$$Dx_2' = \frac{h_1}{h_2}\cdot\frac{\bar{h}_1}{\bar{h}_2}\cdot\frac{\varphi_1}{\nu_1} + \frac{\varphi_2}{\nu_2},$$

which, since

$$\frac{h_2}{h_1} = 1 - (x_1 + \varphi_1)d, \quad \frac{\bar{h}_2}{\bar{h}_1} = 1 - (x_1 + \varphi_1)d - \frac{\varphi_1}{\nu_1}d$$

(where $d = A_1A_2$ denotes the distance between the lenses), may be written in the following form:

$$Dx_2' = \frac{1}{\{1 - (x_1 + \varphi_1)d\}\left\{1 - (x_1 + \varphi_1)d - \frac{\varphi_1}{\nu_1}d\right\}}\cdot\frac{\varphi_1}{\nu_1} + \frac{\varphi_2}{\nu_2}. \quad (435)$$

It can be easily shown that, except in the case when the second lens is so placed as to separate the two coloured images M'_1, $\bar{M}'_1$ formed by the first lens, the strengths of the two lenses must have opposite signs in order that Dx'_2 shall vanish. If we may assume that the variations $D\varphi_1 = \varphi_1/\nu_1$ and $D\varphi_2 = \varphi_2/\nu_2$ are so small that their product $D\varphi_1 \cdot D\varphi_2$ may be neglected, the condition that the system of two thin lenses shall have the same focal point for the two colours is found by putting $x_1 = 0$ in the equation $Dx'_2 = 0$; which condition is, therefore, as follows:

$$\frac{\varphi_1}{\nu_1} + \frac{\varphi_2}{\nu_2}(1 - \varphi_1 d)^2 = 0.$$

If the two lenses are made of the same kind of glass ($\nu_1 = \nu_2$), the condition that $Dx'_2 = 0$ becomes:

$$h_1 \bar{h}_1 \varphi_1 + h_2 \bar{h}_2 \varphi_2 = 0;$$

which is analogous to the condition, expressed in formula (419), for the abolition of the chromatic longitudinal aberration of a thick lens.

The angular magnification (Z) of a system of two infinitely thin separated lenses is:

$$Z = \frac{x'_1}{x_1} \cdot \frac{x'_2}{x_2},$$

whence, since

$$x'_1 = x_1 + \varphi_1, \quad x'_2 = x_2 + \varphi_2, \quad x_2 = \frac{x'_1}{1 - x'_1 \cdot d},$$

we obtain:

$$x_1 Z = (x_1 + \varphi_1)(1 - \varphi_2 d) + \varphi_2.$$

Accordingly, the formula for the chromatic magnification-difference (DZ) is as follows:

$$x_1 \cdot DZ = \frac{\varphi_1}{\nu_1} + \frac{\varphi_2}{\nu_2} - \varphi_1\varphi_2\left(\frac{1}{\nu_1} + \frac{1}{\nu_1\nu_2} + \frac{1}{\nu_2}\right)d - \frac{\varphi_2}{\nu_2}x_1 d. \quad (436)$$

Assuming here also that the variations $D\varphi_1$ and $D\varphi_2$ are so small that we may neglect their product, we can write the condition of the abolition of the chromatic magnification-difference as follows:

$$\varphi_2\left\{\varphi_1\left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right) + \frac{x_1}{\nu_2}\right\}d = \frac{\varphi_1}{\nu_1} + \frac{\varphi_2}{\nu_2}; \quad (437)$$

which, in general, will not be independent of the position of the object. If this condition is fulfilled, not only will the two coloured rays emerge from the system in parallel directions, but the two coloured images will be of equal size; since DY vanishes along with DZ (see § 347).

If, therefore, the two lenses are made of the same material ($\nu_1 = \nu_2$), they must be separated by the distance

$$d = \frac{\varphi_1 + \varphi_2}{(2\varphi_1 + x_1)\varphi_2},$$

if the two coloured images of the object are to be equal in size. It will be observed that here d is independent of the value of ν; and herein consists the great advantage of employing a combination of lenses of the same kind of glass; for in such a case if two coloured images are made of equal size, all the coloured images will be of equal size.

When the object is infinitely distant ($x_1 = 0$), the formula above will be simplified as follows:

$$d = \frac{\varphi_1 + \varphi_2}{2\varphi_1\varphi_2} = \frac{f_1 + f_2}{2};$$

that is, the distance between the two lenses must be equal to half the sum of their focal lengths.

The most important application of this last result is in the construction of the *oculars of telescopes*.[1] If the actual paths of the rays through a telescope are supposed to be reversed in direction, then the ocular will produce in the focal plane of the object-glass a real image of an infinitely distant object, and the lens farthest from the object-glass will be the first lens of the system of two lenses which form the ocular. The ocular called after HUYGENS is composed of two lenses as follows:

$$f_1 = \tfrac{3}{4}f, \quad f_2 = 2f_1 = \tfrac{3}{2}f, \quad d = \tfrac{9}{8}f;$$

and the construction-data of the RAMSDEN-ocular are as follows:

$$f_1 = f_2 = d = f;$$

so that both of these types satisfy the condition expressed in the last formula.

It may be added, in conclusion, that it is not possible to have complete achromatism of a system of two lenses separated by a finite interval.

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[1] Concerning this matter, see E. VON HOEGH: Die achromatische Wirkung der HUYGHENS'schen Okulare. *Central-Zeitung f. Opt. u. Mech.*, vii (1886), 37, 38.

ART. 112. THE SECONDARY SPECTRUM.

354. In consequence of the so-called "irrationality of the dispersion" (§ 329), it is evident that even when an optical system has been designed so as to be achromatic with respect to one pair of colours, it will, in general, not be achromatic for all colours. If, for example, it is contrived so that the red and blue rays are again united in the image, there will still be, perhaps, a slight dispersion of the yellow and green rays; that is, an uncorrected residual colour-error, or, as BLAIR termed it, a "*secondary spectrum*".[1] The residual chromatic longitudinal aberration of a system of thin lenses which is achromatic with respect to two principal colours has been thoroughly investigated by KOENIG;[2] whose methods will be used in the brief and elementary treatment of this matter that is given here, wherein we consider only *the secondary spectrum of a system of thin lenses in contact.*

If n, $\bar{n}$ denote the indices of refraction of an optical medium for the two principal colours with respect to which the optical system is assumed to be achromatic, the difference $\bar{n} - n = Dn$ is called the *fundamental dispersion* of the medium; and if $\mathfrak{n}$ denotes the index of refraction of the same medium for a third colour, the difference $\mathfrak{D}n = \mathfrak{n} - n$ is called the *partial dispersion;* and the ratio

$$\frac{\mathfrak{D}n}{Dn} = \frac{\mathfrak{n} - n}{\bar{n} - n} = \beta \tag{438}$$

is called the *relative partial dispersion* of the medium (see § 329). In general, the relative partial dispersion β_1 of one medium will be different from the relative partial dispersion β_2 of another medium. The ratio

$$\frac{n - 1}{\mathfrak{n} - n} = \frac{\nu}{\beta}$$

will be the ν-value of the medium for the interval from the first to the third colour.

If the chromatic longitudinal aberration with respect to the two principal colours has been abolished, the paraxial image-rays corresponding to these two colours which emanate originally from the axial

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[1] See S. CZAPSKI: Mittheilungen ueber das glastechnische Laboratorium in Jena und die von ihm hergestellten neuen optischen Glaeser: *Zft. f. Instrumentenkunde*, vi (1886), 293–299, 335–348. Also, see S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), pages 128–132.

[2] A. KOENIG: Die Theorie der chromatischen Aberrationen: Chapter VI of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR. See pages 357–366.

object-point M, will cross the optical axis at one and the same point M'; and if $\mathfrak{M}'$ designates the point where the paraxial image-rays corresponding to the third colour cross the axis, then, following KOENIG, let us agree to call the line-segment

$$M'\mathfrak{M}' = \mathfrak{u}' - u'$$

(where u', $\mathfrak{u}'$ denote the distances from the system of thin lenses in contact of the points M', $\mathfrak{M}'$, respectively) the *secondary spectrum* with respect to this third colour.

For example, consider an achromatic combination of m thin lenses in contact, for which the condition

$$\sum_{k=1}^{k=m} \frac{\varphi_k}{\nu_k} = 0$$

is satisfied. According to formula (429), the magnitude of the secondary spectrum will be:

$$\mathfrak{D}u'_m = \mathfrak{u}'_m - u'_m = -u'^2_m \sum_{k=1}^{k=m} \frac{\varphi_k}{\nu_k} \cdot \beta_k, \tag{439}$$

wherein we have put $u'^2_m = u'_m \cdot \mathfrak{u}'_m$; and, accordingly, the condition of the abolition of the secondary spectrum of an achromatic combination of m thin lenses in contact is as follows:

$$\sum_{k=1}^{k=m} \frac{\varphi_k \beta_k}{\nu_k} = 0. \tag{440}$$

If the ν-values are all equal, this equation is equivalent to the condition of achromatism with respect to the pair of principal colours. Evidently, a system composed of two thin lenses in contact, made of different kinds of glass, cannot be achromatic with respect to three different colours. In the case of a system of three thin lenses in contact, the conditions to be fulfilled for the abolition of the secondary spectrum are as follows:

$$\sum_{k=1}^{k=3} \varphi_k = \varphi, \quad \sum_{k=1}^{k=3} \frac{\varphi_k}{\nu_k} = 0, \quad \sum_{k=1}^{k=3} \frac{\varphi_k \cdot \beta_k}{\nu_k} = 0.$$

The magnitude of the secondary spectrum of an achromatic combination of two thin lenses in contact is as follows:

$$\mathfrak{D}u'_2 = -\frac{u'^2_2}{f} \cdot \frac{\beta_1 - \beta_2}{\nu_1 - \nu_2};$$

whence we see that the smaller the difference of the relative partial dispersions of the two kinds of glass and the greater the difference of their ν-values, the less will be the magnitude of the secondary spectrum. A number of pairs of glasses fulfilling these requirements will be found listed in the catalogue of the "glastechnische Laboratorium" in Jena.

355. The character and extent of the secondary spectrum of an achromatic combination of lenses will evidently depend on *the choice of the two principal colours* with respect to which the conditions of achromatism are satisfied. A chief consideration in the determination of the two colours that are to be united will be the mode of using the instrument. Thus, if it is designed to be an *optical* instrument in the literal sense of that term, we shall be concerned primarily with the physiological actions of the rays on the retina of the eye; whereas in the case, for example, of a photographic objective, in which the rays are to be focussed on a sensitive plate, achromatism with respect to the so-called actinic rays will be extremely desirable.

The rays that are most effective in their actions on the retina of the eye are comprised between the FRAUNHOFER lines C and F, with a distinct maximum of brightness in the region between the lines D and E. If, therefore, the instrument is intended to be used by the eye, it is usual to design it so as to be achromatic with respect to the colours corresponding to C and F. Assuming that the system is a convergent combination of two thin lenses in contact, we shall find then that the focal points corresponding to the colours between C and F will lie nearer to the lens-system, and the focal points corresponding to the other colours will lie farther from it, than the common focal point of the two principal colours. Moreover, the secondary spectrum will be approximately least for some colour very nearly corresponding to the D-line, which is a very favourable circumstance, since this is the brightest region of the spectrum for visual purposes.

For the purposes of astrophotography, it is found best to obtain as great a concentration as possible of the actinic rays, especially as here the object will usually be of relatively feeble light-intensity. Moreover, since the celestial objects are infinitely distant, the focussing of the instrument may be done once for all, so that the eye does not have to judge of the perfection of this adjustment, and consequently we may disregard the visual rays here entirely. Such an instrument will be designed, therefore, to unite the rays corresponding (say) to the FRAUNHOFER line F and the violet line in the spectrum of mercury. The secondary spectrum with respect to the longer wave-lengths will be very extensive, but in this case this will not matter.

In the case of the ordinary photographic objective, however, the conditions are different, and here it is found necessary to effect a compromise between the visual and the actinic rays; because the image has to be focussed first on the ground-glass plate by the eye, and afterwards the rays have to be received on the sensitive plate. Under these circumstances the usual procedure is to make the system achromatic with respect to the colours corresponding to the lines D and G'.[1]

ART. 113. CHROMATIC VARIATIONS OF THE SPHERICAL ABERRATIONS.

356. If the aperture of the optical system is not infinitely narrow (as has been assumed throughout in the preceding investigation of the Chromatic Aberrations), not only the paraxial rays but also the rays that meet the objective at points lying outside this small central zone will be concerned in the formation of the image; so that even if the chromatic aberrations of the central rays and the spherical aberrations of the outer rays could all be abolished, the image would still not be perfect on account of faults due to the colour-dispersion of the latter rays. These faults, it is true, will not be very objectionable so long as the aperture of the system is relatively small; but with systems of large aperture the definition of the image may be seriously impaired, and if we wish to obtain an image that is entirely free from colour-faults, the chromatic aberration must be abolished for each zone of the objective.

It would be a waste of time to develop here difference-formulæ for the chromatic variations of SEIDEL's approximate expressions of the five spherical aberrations; because it is only when the rays make considerable angles with the optical axis that the colour-faults under consideration assume serious importance. Indeed, it will only be necessary to consider briefly the two of these faults that are the most troublesome in the ordinary case of optical systems of large aperture, viz., the chromatic variations (1) of the Longitudinal or so-called "Spherical" Aberration and (2) of the Aplanatism or Sine-Ratio.

357. Let us begin with an investigation of the colour-variation from zone to zone of the place where the rays cross the optical axis, called by ABBE[2] "the chromatic difference of the spherical aberration"; which, as KRUESS[3] remarks, may also be called with equal right "the spherical difference of the chromatic aberration", whereby this colour-fault is regarded as due to the variation of the chromatic longitudinal aberration (§ 346) from zone to zone.

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[1] This whole matter is thoroughly explained and discussed in M. VON ROHR's *Theorie und Geschichte des photographischen Objektivs* (Berlin, 1899), pages 60–64. See also A. KOENIG: Die Theorie der chromatischen Aberrationen, Chapter VI of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. VON ROHR; pages 366–369.

[2] E. ABBE: On New Methods for Improving Spherical Correction, applied to the Construction of Wide-Angled Object-glasses: *Roy. Mic. Soc. Journal*, (2) 2, (1879), 812–824. See also: *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 196–212.

[3] H. KRUESS: Die Farben-Correction der Fernrohr-Objective von GAUSS und von FRAUNHOFER: *Zft. f. Instr.*, viii (1888), 7–13, 53–63 and 83–95.

The adjoining diagrams (Figs. 155 and 156), similar to those given by Lummer[1] in his treatment of this subject, will help to make the matter clear. In both figures the red rays and the blue rays representing the light of the longer wave-lengths and the shorter wave-lengths, respectively, are shown on opposite sides of the optical axis; thus, above the axis the two rays selected are a red paraxial ray and a red edge-ray; whereas below the axis the two corresponding rays are blue. For some colour, say, yellow (as being optically the most intensive), intermediate between red and blue, the optical system in both cases is supposed to be spherically corrected, so that the edge-rays corresponding to this mean colour cross the axis at the same point as the central rays of this colour. In both cases also there is spherical under-correction of the red rays and spherical over-correction of the blue rays. In Fig. 155, however, the chromatic longitudinal aberration of the central red and blue rays is abolished, whereas in Fig. 156 the chromatic longitudinal aberration of the red and blue edge-rays is abolished. In both illustrations there is a residual chromatic aberration, which in the case of systems of relatively large aperture may be more injurious than the so-called "secondary spectrum" (Art. 112) due to the disproportionality of the dispersion-ratios for the different parts of the spectrum.

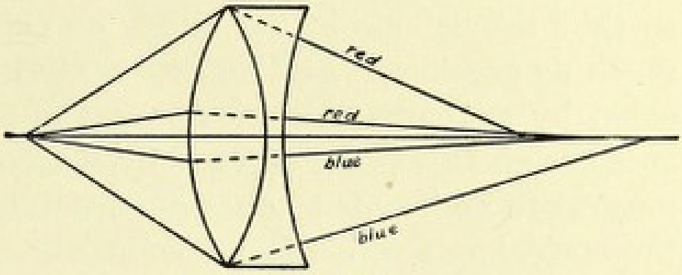

Fig. 155.

Optical System in which the Chromatic Longitudinal Aberration of the Central Red and Blue Rays is abolished.

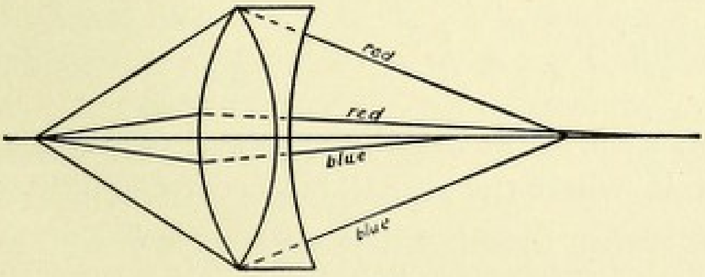

Fig. 156.

Optical System in which the Chromatic Longitudinal Aberration of the Red and Blue Edge-Rays is abolished.

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[1] O. Lummer: See Mueller-Pouillet's *Lehrbuch der Physik und Meteorologie*, Bd. II, zehnte Auflage (Braunschweig, 1909), Art. 169.

As far back as 1817, GAUSS[1] "suggested a plan for getting rid of the residual aberration in binary lenses made of ordinary crown and flint, for telescopic use".[2] Thus, if the optical system is so contrived that there is spherical correction of the extreme red rays and chromatic correction of the red and blue paraxial rays, GAUSS's *Condition*, as it is called,[3] requires that we shall have also the extreme blue rays crossing the optical axis at the same point as the extreme red and the paraxial red and blue rays.[4] As a rule, the fulfilment of GAUSS's Condition will necessitate lenses of very deep curvatures.

Since it is not possible to have perfect chromatic correction for the entire aperture of the objective, the question arises for what zone is it best to abolish the chromatic difference of the spherical aberration, so that the defects in the image on this score may be diminished as much as possible. As KOENIG[5] remarks, this question cannot be decided by geometrical optics alone; but the matter has been investigated from this point of view by KERBER[6] who suggested that it was natural to endeavor to obtain chromatic correction for that place on the optical axis which is determined by the position of the transversal plane of least cross-section of the bundle of image-rays corresponding to the principal colour. Thus, if H denotes the incidence-height at the last spherical surface of the extreme outside rays of a bundle of image-rays of a given colour, emanating originally from an axial object-point, and if the point where the optical axis meets the transversal plane of least cross-section of this bundle of image-rays is designated by N', it may be shown, by simple deductions from the results of § 351, that

$$AN' = u' + \frac{3}{4}\frac{a'}{u'^2} \cdot H^2;$$

where u' denotes the abscissa, with respect to the vertex A of the last spherical surface, of the point M' where the paraxial rays corresponding to this colour cross the optical axis ($u' = AM'$); and where a' is the characteristic aberration-co-efficient employed in the series-development in formula (273). If here we use the symbol h to denote the incidence-height of the cone of rays of this bundle of image-rays that meet the axis at the point N', we have also:

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[1] C. F. GAUSS: Ueber die achromatischen Doppelobjective besonders in Ruecksicht der vollkommenern Aufhebung der Farbenzerstreuung: *Zft. f. Astron. u. verwandte Wissenschaften*, IV (1817), 343–351. See also: GAUSS's *Werke*, Bd. V (zweiter Abdruck, Goettingen, 1877), 504–508.

[2] See E. ABBE: On New Methods for Improving Spherical Correction, etc.: *Roy. Mic. Soc. Journ.*, (2) 2 (1879), p. 814.

[3] H. KRUESS: Die Farben-Correction der Fernrohr-Objective von GAUSS und von FRAUNHOFER: *Zft. f. Instr.*, viii (1888), 7–13, 53–63 and 83–95.

[4] See also S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), p. 134.

[5] A. KOENIG: Die Theorie der chromatischen Aberrationen, Chapter VI of M. VON ROHR's *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), p. 370.

[6] A. KERBER: Ueber die chromatische Korrektur von Doppelobjektiven: *Central-Zeitung f. Opt. u. Mech.*, vii (1886), 157–158.

$$AN' = u' + \frac{a'}{u'^2} \cdot h^2;$$

and, hence, by equating these two expressions for the abscissa AN', we find:

$$h = \frac{H}{2} \sqrt{3};$$

as given by KERBER; who concludes, according to this process of reasoning, that the chromatic correction should be made for the zone whose height above the axis is $h = 0.866 \cdot H$.

A comprehensive view of the performance of a given optical system with respect to the chromatic variations of the spherical aberrations for rays of different colours and of different incidence-heights can be obtained by means of the so-called *isoplethic curves* employed by M. VON ROHR.[1] The wave-lengths of the light (expressed in $\mu\mu$) are laid off along the axis of abscissæ, whereas the incidence-heights (in mm.) are represented along the other of the two rectangular axes of the diagram; so that to each point in the plane of the figure there corresponds a certain ray of a definite colour and of a definite incidence-height. In the object-space of the optical system the rays are all assumed to be parallel to the optical axis. If M' designates the point where a paraxial image-ray of mean refrangibility, corresponding, say, to the FRAUNHOFER D-line, crosses the optical axis, and if $\overline{L}'$ designates the point where a ray of some other colour, say λ, and of finite incidence-height h, crosses the optical axis, we calculate (in thousandths of a millimetre) the length $M'\overline{L}'$; and if the point (λ, h) in the diagram is designated by P, we ascribe to this point P the number corresponding to the numerical value of $M'\overline{L}'$. The curve drawn through all points P which have the same numerical value will be one of the system of isoplethic curves of the optical system. M. VON ROHR gives diagrams showing the system of isoplethic curves from the value $M'\overline{L}' = +0.050$ mm. to the value $M'\overline{L}' = -0.050$ mm. for a PETZVAL portrait-objective and for the so-called "Planar" type of photographic objective of P. RUDOLPH.

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[1] M. VON ROHR: *Theorie und Geschichte des photographischen Objektivs* (Berlin, 1899), 65-68.

The errors due to the chromatic variations of spherical aberration are especially objectionable in the case of microscope-objectives on account of their large apertures; but we have not space to describe here the ingenious and successful methods that have been devised by ABBE[1] to overcome this defect in such optical systems.

358. If the wide-angle optical system is to produce a good image not merely of an axial object-point, but also of adjacent points not on the axis, for example, of a surface-element at right angles to the axis, in addition to the abolition of the spherical aberration, the *Sine-Condition*, as expressed by formula (300), must also be fulfilled, viz.:

$$\frac{\sin\theta}{\sin\theta'} = \frac{n'}{n} Y.$$

Obviously, it will be meaningless to investigate the chromatic variation of the sine-ratio unless both the spherical aberration along the axis and the chromatic differences thereof have been abolished. Assuming, therefore, that these prerequisite conditions have been satisfied, the condition of Aplanatism for two adjacent colours of the spectrum, corresponding to the wave-lengths λ and $\lambda + d\lambda$, requires not only that the above equation for λ shall be true, but that the equation obtained therefrom by variation with respect to λ shall also be satisfied, viz.:

$$\frac{d\theta'}{\tan\theta'} - \frac{d\theta}{\tan\theta} + \frac{dn'}{n'} - \frac{dn}{n} = -\frac{dY}{Y},$$

which may be written in the form given by CZAPSKI[2] as follows:

$$\Delta\left(\frac{d\theta}{\tan\theta}\right) + \Delta\left(\frac{dn}{n}\right) = -\frac{dY}{Y}.$$

An optical system which is without secondary spectrum and aplanatic for two colours has been called by ABBE[3] an "apochromatic" system.

Even when the last equation is satisfied, there will still be an error due to the difference in the magnification of the two coloured images for each zone of the objective; and if this is to be abolished also, we must put $dY = 0$; and, hence (supposing also that the object itself is free from aberrations, so that $d\theta = 0$) we obtain:

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[1] E. ABBE: On New Methods for Improving Spherical Correction, applied to the Construction of Wide-angled Object-glasses: *Roy. Mic. Soc. Jour.*, (2) 2 (1879), 812–824. See also: *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 196–212.

[2] S. CZAPSKI: *Die Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), p. 134.

[3] E. ABBE: Ueber neue Mikroskope: *Sitz.-Ber. Jen. Ges. Med. u. Natw.*, 1886, 107–128; see p. 111. Also: *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), p. 454.

$$\frac{d\theta'}{\tan\theta'} = \frac{dn}{n} - \frac{dn'}{n'}.$$

359. If v, θ denote the ray-co-ordinates of a ray of wave-length λ before refraction at a given spherical surface of radius r, and if α, α' denote the angles of incidence and refraction, respectively, we have the following system of equations for determining the corresponding ray-co-ordinates (v', θ') of the refracted ray (§ 211):

$$\sin\alpha = -\frac{v-r}{r}\sin\theta, \quad \sin\alpha' = \frac{n}{n'}\sin\alpha,$$

$$\theta' = \theta + \alpha' - \alpha, \qquad v' - r = -\frac{r\cdot\sin\alpha'}{\sin\theta'};$$

whence for an adjacent ray of wave-length $\lambda + d\lambda$ we derive immediately a series of differential formulæ as follows:

$$\left.\begin{aligned}
\frac{d\alpha}{\tan\alpha} &= \frac{d\theta}{\tan\theta} + \frac{dv}{v-r},\\
\frac{d\alpha'}{\tan\alpha'} &= \frac{d\alpha}{\tan\alpha} - \left(\frac{dn'}{n'} - \frac{dn}{n}\right),\\
d\theta' &= d\theta + d\alpha' - d\alpha,\\
\frac{dv'}{v'-r} &= \frac{d\alpha'}{\tan\alpha'} - \frac{d\theta'}{\tan\theta'};
\end{aligned}\right\} \qquad (441)$$

so that if we know the values of dv, $d\theta$ before refraction at a given spherical surface, we can find the corresponding variations dv', $d\theta'$ after refraction.

CHAPTER XIV.

THE APERTURE AND THE FIELD OF VIEW. BRIGHTNESS OF OPTICAL IMAGES.

ART. 114. THE PUPILS.

360. Effect of Stops.

In the ideal case of perfect collinear correspondence between Object-Space and Image-Space we found that it was sufficient to know the constants of the optical system, for example, the magnitudes of the focal lengths and the positions of the focal points, in order to construct the image of a given object; but the rays that are employed in such graphical constructions are seldom, if ever, the actual rays that are utilized by the optical instrument in the formation of an image. From a purely geometrical point of view, one pair of rays was as good as another in locating the position of an image-point, and we were not at all concerned to inquire whether the rays so employed were really operative or not. But when it was attempted to realize an optical image by the aid of a so-called optical instrument consisting of a centered system of spherical refracting or reflecting surfaces, we encountered, first of all, the spherical aberrations; and at once the question as to the *actual rays* that are concerned in the phenomena assumed a place of fundamental importance, so that not only the degree of perfection of the image, but the region of validity of the imagery, and the possibility of extending it, are found to depend essentially on the slopes of the rays, and on the heights above or below the optical axis of the points where they meet the spherical surfaces, and on the aperture-angles of the ray-bundles, etc.

As a matter of fact, even under ideal conditions of collinear correspondence between two infinitely extended space-systems, these same factors would be concerned also in certain other properties of optical images (brightness, resolving-power, etc.) besides those above-mentioned. Throughout this chapter it will be tacitly assumed (unless expressly stated otherwise) that the optical system is free from aberrations both spherical and chromatic, so that we have to do with the case of so-called GAUSSian *Imagery*, without colour-faults. Of course, it must be borne in mind that the formulæ and results derived on these assumptions will have to be applied with due caution to the more or less imperfect imagery that is realized in the case of actual optical instruments; but, on the other hand, it would lead us too far and tend only to confuse the matter in hand if we attempted here to go into all the intricate and special questions that are involved when the different aberrations are taken into account.

The bundles of rays that traverse an optical instrument are limited either by the physical dimensions of the lenses themselves or by perforated diaphragms or "*stops*" interposed specially for this purpose. In all cases that possess interest for us such stops are circular in form and concentric with the optical axis. The direct and obvious effect of a stop or lens-rim is two-fold, viz., first, to *restrict the apertures of the bundles of effective rays*, and, second, to *limit the extent of the object that is reproduced in the image*. The mode and measure of these restrictions will depend on the sizes and positions of the stops and also on the type of the optical apparatus itself.

361. The Aperture-Stop.

In the general and at the same time the most usual case, the diaphragm or stop is placed with its centre on the optical axis at some point lying between two consecutive lenses of the optical system L; which is thereby divided into two parts, a *front component* (L_1) consisting of the part of the lens-system in front of the *interior stop*, and a *hinder component* (L_2) consisting of the remainder of the lens-system lying on the other or far side of the stop. There may be also not merely one but several such interior stops, either actual perforated diaphragms or the rims of the lenses themselves; each of which, according to its position, will divide the lens-system into two parts, as above-mentioned. Frequently a stop is placed in front of the entire system, in which case it is called a *front stop*. And, similarly, a stop which is placed behind, or towards the image-side of, the optical system (as is also not uncommon) is called a *rear stop*. With respect to a front stop, $L_2 = L$, and with respect to a rear stop, $L_1 = L$.

The apertures of the bundles of effective rays are conditioned by these stops. In the simplest case of all when the optical system consists of a single lens whose two surfaces intersect in the circular rim of the lens, this circle is the common base of the cones of incident and refracted rays that take part in the image-phenomena; and here the bundles of effective rays are limited by the surface of the lens itself.

If now we interpose between the axial object-point M (Fig. 157) and the lens a front stop with its centre on the axis at the point designated in the figure by $\boldsymbol{M}$ whose diameter subtends at M an angle smaller than that subtended at the same point by the diameter of the lens, this stop will evidently limit the aperture of the bundle of object-rays emanating from the axial object-point M; and if the position on the axis of the point which is conjugate to $\mathbf{M}$ is designated by $\mathbf{M}'$, the GAUSSian image of the circular stop in the transversal plane $\boldsymbol{\sigma}$ of the Object-Space with its centre at $\mathbf{M}$ will be a circle with its centre at $\mathbf{M}'$ lying in the transversal plane $\boldsymbol{\sigma}'$ conjugate to $\boldsymbol{\sigma}$. Since all the

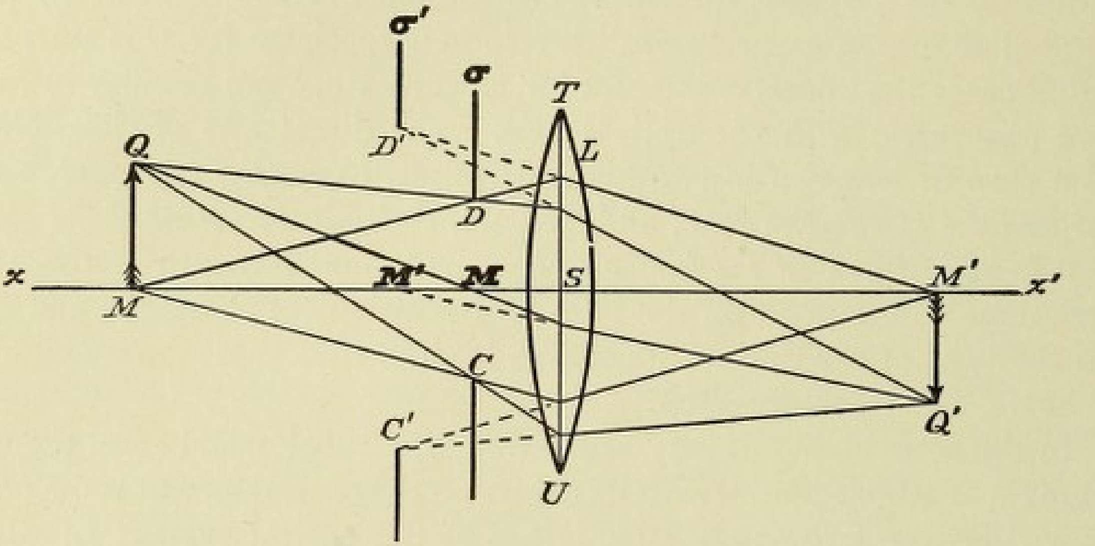

FIG. 157.

INFINITELY THIN CONVEX LENS WITH FRONT STOP. CD is the Aperture-Stop with its centre on the optical axis at $\mathbf{M}$. CD is here also the Entrance-Pupil; $C'D'$ the Exit-Pupil. $M'Q'$ is the image of the object MQ.

$$\mathbf{M}M = \xi,\quad \mathbf{M}'M' = \xi',\quad \mathbf{M}D = p,\quad \mathbf{M}'D' = p',\quad \angle \mathbf{M}MD = \Theta,\quad \angle \mathbf{M}'M'D' = \Theta'.$$

rays that before refraction go through the stop at $\mathbf{M}$ must after refraction pass through the stop-image at $\mathbf{M}'$, we see that, whereas the material stop placed in front of the lens at $\mathbf{M}$ limits the apertures of the bundles of effective rays in the Object-Space, the stop-image at $\mathbf{M}'$ performs the same office for the bundles of rays in the Image-Space; or, in other words, the front stop lying in the transversal plane $\boldsymbol{\sigma}$ is the common base of all the cones of object-rays, and, similarly, the stop-image in the transversal plane $\boldsymbol{\sigma}'$ is the common base of all the cones of image-rays.

Proceeding now to the most general case, let us suppose that the optical system L is composed of several lenses and provided with one or more interior stops, either perforated diaphragms or lens-rims. We begin by constructing the GAUSSian image of each stop O (Fig. 158) formed by that part L_1 of the system that lies in front of (or to the left of) O. The stop that corresponds to that one of these images that subtends the smallest angle at the selected axial object-point M is called the *aperture-stop;* because this is evidently the stop that, with respect to M, conditions the apertures of the bundles of effective rays. In the figure the aperture-stop is represented as the one with its centre located at the point O, whose image formed by the front part L_1 of the optical system L in the transversal plane σ that is crossed by the axis at the point $\mathbf{M}$ subtends a smaller angle at M than the corresponding image of any of the other stops. Which one of the perforated diaphragms or lens-rims plays the rôle of aperture-stop will depend essentially on the position of the axial object-point M.

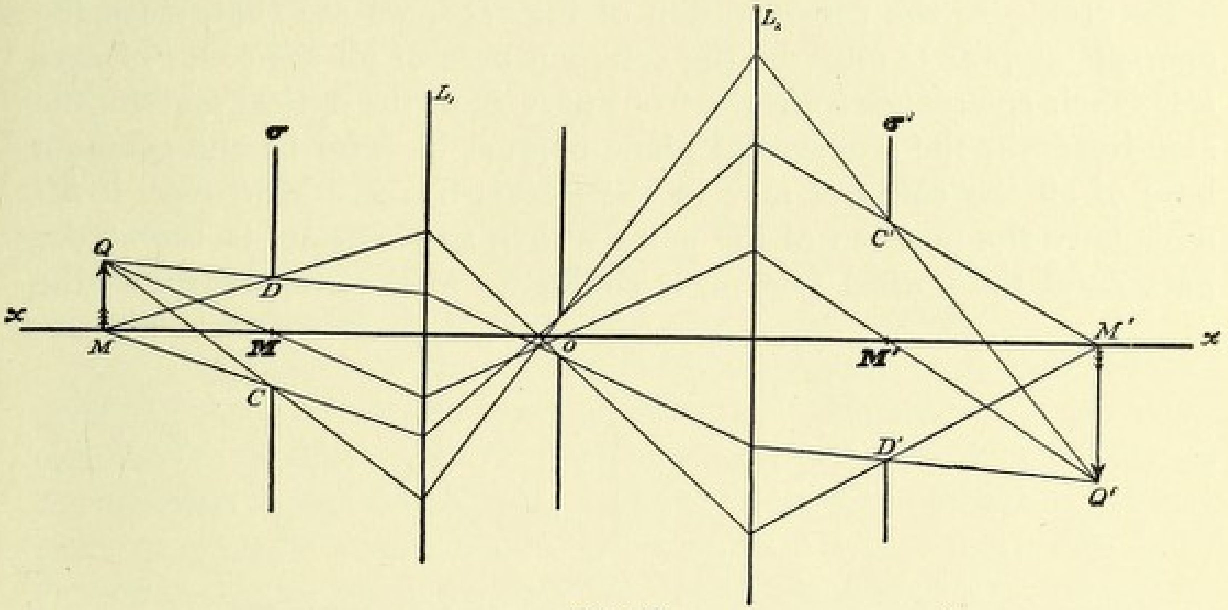

FIG. 158.

COMPOUND OPTICAL SYSTEM CONSISTING OF TWO THIN LENSES L_1, L_2, SEPARATED BY INTERIOR APERTURE-STOP WITH CENTRE AT O. The axial point $\mathbf{M}$ conjugate to O with respect to L_1 and the axial point $\mathbf{M}'$ conjugate to O with respect to L_2 (and therefore conjugate also to $\mathbf{M}$ with respect to $L_1 + L_2$) are the centres of the Entrance-Pupil and Exit-Pupil, respectively. $M'Q'$ is the image of the object MQ.

$$\mathbf{M}M = \xi, \quad \mathbf{M}'M' = \xi', \quad \mathbf{M}D = p, \quad \mathbf{M}'D' = p', \quad \angle \mathbf{M}MD = \Theta, \quad \angle \mathbf{M}'M'D' = \Theta'.$$

In passing, it may be observed that a case may occur, such as that shown in Fig. 159, in which the images of two (or more) of the material stops formed by the parts of the optical system lying in front of them subtend at the axial object-point M angles of equal magnitude; so that (if this angle is also the smallest of all such angles) either of these two stops may be regarded as the aperture-stop. The point of intersection of the pair of straight lines joining the upper extremity of one stop-image with the lower extremity of the other determines a second point K on the optical axis at which the two stop-images also subtend angles of equal magnitude. With respect to an axial object-point situated anywhere between the two extreme positions M and K, the stop-image marked II in the diagram will subtend the smaller angle of the two; whereas for an axial object-point lying anywhere outside the segment MK the stop-image marked I will subtend the smaller angle.[1] It is apparent that the stop that acts as the aperture-stop for an object in one position on the axis may not be the aperture-stop for another position of the object. We must assume, therefore, that the object has a fixed position or at any rate that it is movable within certain prescribed limits if the stops are to retain their functions, as is necessary, for example, in the case of such optical instruments as the telescope and the microscope.

Returning to the consideration of Fig. 158, we see that, since the aperture-stop at O must be the common base of all the cones of rays after their emergence from the front part L_1 of the optical system, the stop-image in the transversal plane σ must likewise be the common base of all the cones of rays in the Object-Space. Moreover, if $\boldsymbol{M}'$ designates the position of the point which, with respect to the hinder part L_2 of the optical system, is conjugate to the stop-centre O, the image of the stop formed by L_2 will lie in the transversal plane σ' determined by the axial point $\boldsymbol{M}'$; and, similarly, this stop-image will evidently be the common base of all the cones of image-rays after having traversed the entire compound system $L = L_1 + L_2$. Evidently, also, the transversal planes σ, σ' are a pair of conjugate planes, so that the stop-images at $\boldsymbol{M}$ and $\boldsymbol{M}'$ are images of each other with respect to the whole system L. Together they constitute a pair of *virtual stops* (as distinguished from actual or material stops) that are the measures of the apertures of the ray-bundles in the Object-Space and Image-Space. A material stop of the same size and position as

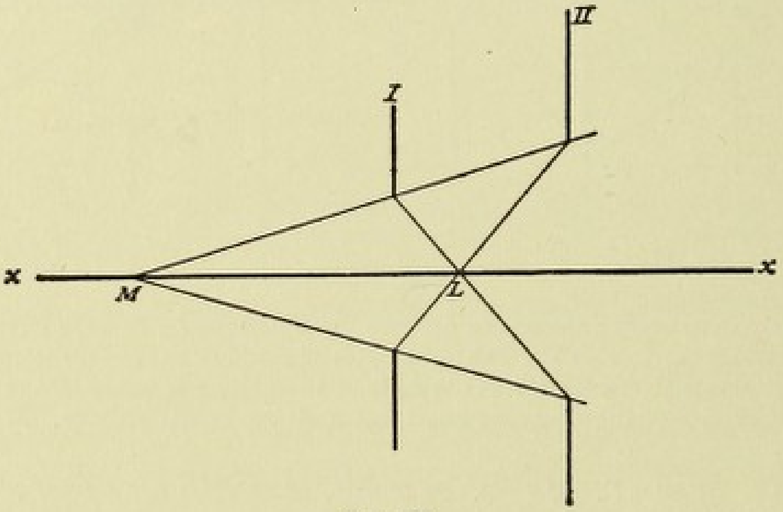

Fig. 159.

Case of Two Entrance-Pupils.

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[1] See M. von Rohr: "Die Strahlenbegrenzung in optischen Systemen", Chapter IX of *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. von Rohr. See p. 469.

the stop-image at **M** will act exactly in the same way with respect to the limiting of the bundles of rays as a material stop identical in size and position with the stop-image at **M'**; and either of them or both together, so far as this effect is concerned, would be precisely equivalent to the actual stop that we suppose to be situated at O. ABBE,[1] who has done most to develop the theory of stops, calls the stop-images at **M** and **M'**, from an analogy with the optical system of the human eye, the *pupils* of the system. The pupil of the eye is the contractile aperture of the iris, the image of which produced by the cornea and the aqueous humour lies in front of the eye (as can be seen by looking directly into the eye); so that only such rays as are directed towards this image can enter the eye through the iris-opening. From this same analogy, ABBE calls also the aperture-stop at O the *iris* of the optical system. The two pupils at **M** and **M'** are distinguished by the names *Entrance-Pupil* and *Exit-Pupil*, respectively.[2]

362. An imagery is completely determined so soon as we know the positions of the two pairs of conjugate transversal planes σ, σ' and $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$, together with the values of the magnification-ratios Y and $\boldsymbol{Y}$ that characterize these two pairs of planes. Thus, if the pupils of the system are given in both size and position, and if also the image $M'Q'$ corresponding to a given object-line MQ at right angles to the optical axis has been constructed, the procedure of every ray that traverses the system can be ascertained immediately without taking farther account of the special construction of the apparatus. For example, to an object-ray QD which, originating at the object-point Q crosses the $\boldsymbol{\sigma}$-plane at a point D in the circumference of the entrance-pupil there must correspond an image-ray directed toward the image-point Q' and going through the point D' of the circumference of the exit-pupil that is conjugate to the point D. It is evident also that the totality of the effective rays in the Object-Space may be regarded in either of two ways, viz.: (1) As cones of rays emanating from points in the object MQ and having the entrance-pupil as a common cross-section; or (2) As cones of rays with their vertices at points of the entrance-pupil and a common base in the object MQ; so that the rôles of object and entrance-pupil are interchangeable. This same reciprocity exists likewise between the image and the exit-pupil.

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[1] See E. ABBE: Beitraege zur Theorie des Mikroskops und der mikroskopischen Wahrnehmung: *Archiv f. mikr. Anat.*, ix (1873), 413–468. Also, *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 45–100.

[2] See also E. ABBE: Ueber die Bestimmung der Lichtstaerke optischer Instrumente: *Jen. Zft. f. Med. u. Naturw.*, vi (1871), 263–291. Also, *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 14–44.

363. The Aperture-Angle.

The angle $\boldsymbol{M}MD = \Theta$, defined more precisely by the relation

$$\tan\Theta = \frac{\boldsymbol{M}D}{M\boldsymbol{M}},$$

where D designates the position of a point in the meridian plane lying in the circumference of the entrance-pupil, is called the *aperture-angle* of the optical system. If $p = \boldsymbol{M}D$ denotes the radius of the entrance-pupil (reckoned positive or negative according as D lies above or below the optical axis), and if $\boldsymbol{M}M = \xi$ denotes the abscissa of the axial object-point M with respect to the centre $\boldsymbol{M}$ of the entrance-pupil as origin, we may write:

$$\tan\Theta = -\frac{p}{\xi}. \tag{442}$$

Similarly, if $\Theta' = \angle \boldsymbol{M}'M'D'$, $p' = \boldsymbol{M}'D'$, $\xi' = \boldsymbol{M}'M'$, where the points designated by M', $\boldsymbol{M}'$, D' are conjugate to the points in the Object-Space designated by the same letters without the primes, we have also:

$$\tan\Theta' = -\frac{p'}{\xi'}. \tag{443}$$

364. The Numerical Aperture.

Although the size of the aperture-angle Θ is in a certain more or less geometrical sense a measure of the number of effective rays emanating from the axial object-point M, this angle by itself, from an optical standpoint, is not a true criterion of the aperture of the optical system. All the rays of a bundle are not of equal optical value, and on this account the quantity of light-energy that is transmitted through the optical system from an object-point to its conjugate image-point depends on something more than just the size of the aperture angle. A luminous surface-element emits more energy along some directions than along others, the intensity of radiation (§ 388), according to LAMBERT's law, being proportional to the cosine of the angle of emission; so that the most energetic ray is the one that is directed along the normal to the surface-element at the origin-point of the ray. Consequently different rays emanating from the same object-point will be the routes through the entrance-pupil of the optical system of different cargoes of light-energy.

According to ABBE,[1] the proper and rational measure of the aperture of an optical system—the only one indeed that affords a just idea of its efficiency—is given by the product of the refractive index of the first medium (n) and the sine of the aperture-angle; this product, to which ABBE gives the name *numerical aperture*, and which is denoted here by the symbol A, has therefore the following expression:

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[1] E. ABBE: Die optischen Huelfsmittel der Mikroskopie: *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 119–164; especially, p. 142. (This paper was published originally

$$A = n \cdot \sin \Theta. \tag{444}$$

It would derange too much the plan of this treatise if we paused here to explain fully the basis of this definition, especially also as such an exposition belongs rather to the special theory of optical instruments and to the theory of the microscope in particular where the numerical aperture has an exceedingly important rôle. In the case of the instrument just mentioned, the conjugate axial points M, M' are the aplanatic pair of points of the optical system (§ 279), and under these circumstances it would be easy to show that the quantity of radiant energy transmitted from M to M' is proportional to the numerical aperture.

It may be remarked that the magnitude denoted by A is proportional, not to the aperture-angle Θ, but to the sine of this angle; so that, for example, if Θ were increased from, say, 30° to 90°, the numerical aperture would be only doubled, since $\sin 90^\circ : \sin 30^\circ = 2 : 1$. The numerical aperture is also proportional to the refractive index, so that its value can be altered merely by immersing the object in a different medium for which n has a different value; and, hence, as ABBE has observed, this measure A enables us to compare the apertures of the so-called "dry" and "immersion" optical systems.

The relation between the numerical aperture and the radius (p) of the entrance-pupil and the abscissa $\xi = \boldsymbol{M}M$ is exhibited by the formula:

$$A = \frac{n \cdot p}{\sqrt{\xi^2 + p^2}}; \tag{445}$$

whence also we can see the effect on the aperture of a displacement $\delta\xi$ of the object-point M. Whether the aperture will be increased or diminished by such a variation of the position of the axial object-point M, will depend on the signs of both ξ and $\delta\xi$.

If $\boldsymbol{Z}$ denotes the angular magnification of the system with respect

in Braunschweig in 1878.) Also:

E. ABBE: Ueber die Bedingungen des Aplanatismus der Linsensysteme: *Sitzungsber. d. Jen. Gesellschaft f. Med. u. Naturw.*, 1879, 129–142. See *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 213–226; especially, pages 225 and 226. Also:

E. ABBE: On the Estimation of Aperture in the Microscope: *Journ. Roy. Micr. Soc.*, (2), i (1881), 388–423: especially, pages 395 & 396. (A German translation of this paper is in *Gesammelte Abhandlungen*, Bd. I, 325–374.)

to the pupil-centres $\boldsymbol{M}$, $\boldsymbol{M'}$, and if Y denotes the lateral magnification with respect to the pair of conjugate points M, M', then, according to the last of the image-equations (127), we shall have:

$$\frac{\xi'}{\xi} = \frac{Y}{\boldsymbol{Z}}; \tag{446}$$

and, hence, in the special case when the points M, M' are the aplanatic pair of points, so that

$$\frac{n \cdot \sin \Theta}{n' \cdot \sin \Theta'} = \frac{A}{A'} = Y,$$

we obtain the relation:

$$\frac{A}{A'} = \frac{\xi'}{\xi} \cdot \boldsymbol{Z}; \tag{447}$$

which will be found to be a very useful formula in the special theory of optical instruments.

ART. 115. THE CHIEF RAYS AND THE RAY-PROCEDURE.

365. Chief Ray as Representative of Bundle of Rays. The rays which, emanating from all the points of the object, are directed towards the centre $\boldsymbol{M}$ of the entrance-pupil constitute the bundle of so-called *chief rays* in the Object-Space; to which in the Image-Space there corresponds also a conjugate bundle of chief rays which all meet at the centre $\boldsymbol{M'}$ of the exit-pupil. Accordingly, the pupil-centres $\boldsymbol{M}$, $\boldsymbol{M'}$ are to be considered as the centres of perspective of the Object-Space and Image-Space, since to any object-point P lying on the chief object-ray $P\boldsymbol{M}$ there corresponds an image-point P' lying on the conjugate chief image-ray $P'\boldsymbol{M'}$. The chief ray is the axis of symmetry of the cone of rays, and, therefore, especially when the circular aperture-stop is very small, it may be regarded as the representative ray of the bundle (*cf.* § 286); and, hence, a knowledge of the procedures of the chief rays will often afford an accurate idea of the entire image-process.

Since the pupil-centres are the centres of perspective of the Object-Space and Image-Space, object-points which lie along a chief ray in the Object-Space will be reproduced by image-points which lie along the conjugate chief ray in the Image-Space, and which, therefore, if viewed by an eye placed at the exit-pupil (which is the usual place for the eye in order that the entire image may be all commanded at the same time), will appear to lie all at the same place. If the image is received on a plane screen, placed at right angles to the optical axis, and if this screen does not coincide exactly with the transversal image-plane σ' which is conjugate to the transversal plane σ in the Object-Space that contains the object-point Q (Fig. 160), the image of Q on

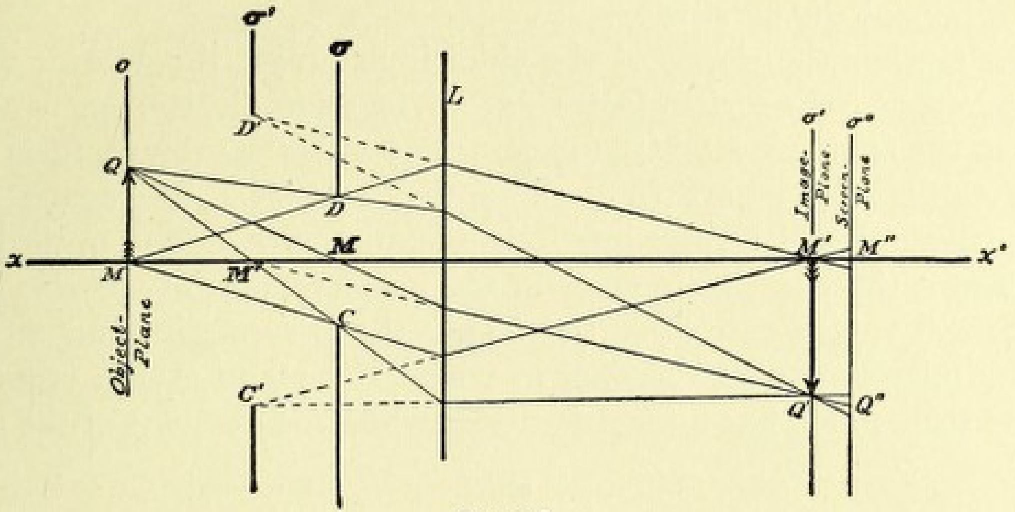

FIG. 160.

BLUR-CIRCLES IN THE SCREEN-PLANE DUE TO IMPERFECT FOCUSSING. xx' is the optical axis of the system L. CD, $C'D'$ diameters of Entrance-Pupil and Exit-Pupil. $M'Q'$ is the image of MQ, and M'' and Q'' are the centres of the blur-circles in the Screen-Plane corresponding to the object-points M and Q, respectively.

$$MQ = y,\quad M'Q' = y',\quad M''Q'' = y''.\quad \mathbf{M}D = p,\quad \mathbf{M}'D' = p',$$
$$\mathbf{M}M = \xi,\quad \mathbf{M}'M' = \xi',\quad \angle\,\mathbf{M}MD = \Theta,\quad \angle\,\mathbf{M}'M'D' = \Theta'.$$

the screen will not be a point but an aberration-figure coinciding with the section of the bundle of image-rays made by the screen-plane. If the aperture-stop is circular in form, this aberration-figure will be a circle (so-called "*blur-circle*"), and the centre of the circle where the chief ray crosses the screen-plane will be regarded as the place on the screen of the image corresponding to the object-point Q. The smaller the diameter of the exit-pupil, the smaller will be the diameter of the blur-circle; and if the diameter of the aperture-stop is infinitely small, the blur-circles will all contract into points at their centres.

366. Optical Measuring Instruments.

The importance of taking into consideration the procedures of the chief rays may be illustrated by investigating the class of optical instruments that are especially contrived for determining the size of an object by measuring the size of the image. The image may be cast on a screen which is provided with a scale or the image may be formed in the air in a plane containing a material scale or a scale-image. But here, owing partly perhaps to the unavoidable dioptric imperfections of the image itself but above all to the difficulty of focussing the instrument exactly so that the true image-plane coincides with the scale-plane, there is a source of error in the method, since, instead of measuring the size of the true image, we may be measuring that of the apparent and more or less blurred image as viewed by the eye in the scale-plane. The size of this image in the scale-plane is determined by the procedures of the chief rays.

The steeper the slopes of the chief image-rays, the greater will be the error due to imperfect focussing. If, for example, the centre O (Fig. 161) of the aperture-stop (§ 361) coincides with the primary focal point F_2 of the posterior part L_2 of the optical system L, the centre $\mathbf{M}'$ of the exit-pupil will be the infinitely distant point of the optical axis, and hence the parallactic error in the measurement of the size of the image will vanish entirely. In an optical measuring instrument, such as the micrometer-microscope, in which the plane of the cross-hairs or the scale-plane has a fixed position, while the distance of the object is adjustable, we must contrive therefore so that the exit-pupil will be infinitely distant, in which case the image will appear of the same size whether it lies in the scale-plane or not. On the other hand, if the object is fixed and the position of the scale-plane variable (as, for example, in the case of a telescope in which the object is usually immovable and the adjustment is accomplished by the eye-piece), the endeavour is to contrive so that the distance of the object from the entrance-pupil will have no effect on the measurement of the lateral magnification $Y = y'/y$; which can be achieved in a similar way by designing the instrument so that the centre O of the aperture-stop coincides with the secondary focal point E_1' of the part L_1 of the optical system that lies in front of it, whereby the centre $\mathbf{M}$ of the

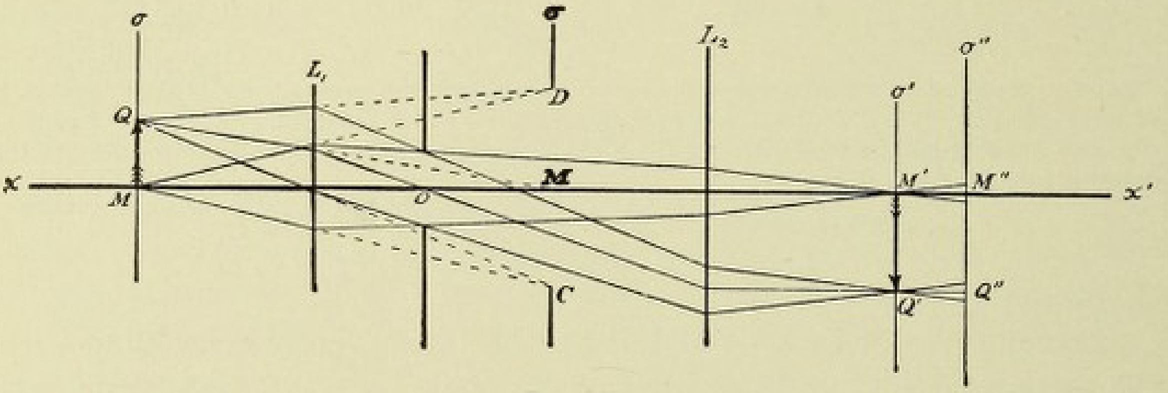

Fig. 161.

Optical System Telecentric on the Side of the Image. The centre O of the Aperture-Stop is at the primary focal point F_2 of the Lens-System L_2; so that the centre $\mathbf{M}'$ of the Exit-Pupil is the infinitely distant point of the optical axis xx'. The blurred image $M''Q''$ of the object MQ is of the same size as the true image $M'Q'$, no matter how we adjust the screen-plane σ''.

$$MQ = y,\quad M'Q' = M''Q'' = y',\quad \mathbf{M}M = \xi,\quad \mathbf{M}'M' = \xi',$$
$$\mathbf{M}D = p,\quad \mathbf{M}'D' = p',\quad \angle \mathbf{M}MD = \Theta,\quad \angle \mathbf{M}'M'D' = \Theta'.$$

entrance-pupil will be the infinitely distant point of the optical axis, and the chief rays in the Object-Space will, therefore, be parallel to the axis.

An optical system in which the centre O of the aperture-stop coincides with one or other of the two focal points that are here designated by E_1' and F_2 is called by ABBE[1] a *telecentric system*. According as it is the entrance-pupil or the exit-pupil which is the infinitely distant one of the two pupils, the system is said to be "telecentric on the side of the object" or "telecentric on the side of the image", respectively. In the special case when the focal points E_1' and F_2 coincide with each other the system will be telescopic (§ 186, Case 1); and if, moreover, the centre O of the aperture-stop coincides with both of these focal points, the system will be "telecentric on both sides".

367. If the positions of the two focal points of the optical system are designated by F and E', and if the magnitudes of the focal lengths are denoted by f and e', and if, finally, $x = FM$, $x' = E'M'$ denote the abscissæ, with respect to the focal points, of the pair of conjugate axial points M, M'; then, on the assumption of perfect collinear correspondence, we have, according to the second of formulæ (115), for the lateral magnification of the system with respect to the points M, M':

$$Y = \frac{f}{x} = \frac{x'}{e'}.$$

In the special case, therefore, when the centre $\boldsymbol{M}$ of the entrance-pupil coincides with the position F of the primary focal point, so that $x = FM = \boldsymbol{M}M = \xi$, we obtain:

$$Y = \frac{f}{\xi};$$

and, hence, when the system is telecentric on the side of the image, the magnification Y will not depend on the position of the scale-plane, but only on the position of the object-plane σ. Similarly, when the centre $\boldsymbol{M}'$ of the exit-pupil coincides with the position E' of the secondary focal point ($x' = E'M' = \boldsymbol{M}'M' = \xi'$), we find:

$$Y = \frac{\xi'}{e'};$$

which shows that when the system is telecentric on the side of the object, the magnification Y is independent of the position of the object-plane σ.

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[1] E. ABBE: Ueber mikrometrische Messung mittelst optischer Bilder: *Sitzungsber. d. Jen. Gesellschaft f. Med. u. Naturw.*, 1878, 11–17. See also: *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 165–172.

ART. 116. MAGNIFYING POWER.

368. The Objective Magnifying Power.

If the optical image is received upon a screen, as in the case of the image produced by the photographic objective or the projection-microscope, or if the dimensions of the image suspended in air are to be determined by comparison with a scale (optical measuring instrument, § 366), the lateral magnification

$$Y = \frac{y'}{y},$$

defined by the ratio of the corresponding linear dimensions y, y' of object and image, which, theoretically at least, as we have seen (§ 179), may have any value depending on the position of the axial object-point M, is the measure of the actual or so-called *objective magnification*.

In case the object $MQ = y$ is very distant or inaccessible, instead of comparing the actual dimensions of the image on the screen with those of the object, we may have to measure the magnifying power of the optical projection-system in some other way, for example, by means of the ratio of the linear size of the image ($M'Q' = y'$) to the apparent size of the object; which latter expression needs however to be explained more precisely. By the *apparent size of the object* is meant here the angle (or the trigonometric tangent of the angle) $M\boldsymbol{M}Q = \theta$ subtended by the object at the centre $\boldsymbol{M}$ of the entrance-pupil, which is also the same thing as the slope-angle of the chief ray proceeding from the extra-axial object-point Q. This ratio

$$\frac{y'}{\tan\theta}$$

is always a more or less important factor in the determination of the objective magnifying power of an optical projection-system. In the limiting case when *the object is infinitely distant*, and the image is formed therefore in the secondary focal plane of the optical system, the size of the image $y' = M'Q' = E'Q'$ (since the axial image-point M' coincides with the secondary focal point E') will depend only on the slope θ of the cylindrical bundle of object-rays coming from the infinitely distant object-point Q; so that under these circumstances the ratio above defined is the real measure of the objective magnification produced by the optical projection-system.

Since

$$\tan\theta = \frac{y}{\xi},$$

where $\xi = \boldsymbol{M}M$ denotes the abscissa of the axial object-point M with respect to the centre $\boldsymbol{M}$ of the entrance-pupil, and since, moreover,

$$\frac{y'}{y} = \frac{f}{x} = \frac{f}{\boldsymbol{x} + \xi},$$

where $x = FM$ and $\boldsymbol{x} = F\boldsymbol{M}$ denote the abscissæ, with respect to the primary focal point F, of the points M and $\boldsymbol{M}$, respectively, and where f denotes the primary focal length of the optical system; we obtain finally:

$$\frac{y'}{\tan\theta} = \frac{\xi}{\boldsymbol{x} + \xi} f. \tag{448}$$

If the object is at a great distance, $\boldsymbol{x}$ will be very small compared with ξ, so that the fraction $\xi/(\boldsymbol{x} + \xi)$ will be very nearly equal to unity; and hence we may write:

$$\frac{y'}{\tan\theta} = f, \text{ approximately};$$

which will be not only approximately but strictly true in case either the object is infinitely distant ($\xi = \infty$) or the plane $\boldsymbol{\sigma}$ of the entrance-pupil coincides with the primary focal plane ($\boldsymbol{x} = 0$).

In making geodetic measurements it often happens that one wishes to determine the distance of the object (a surveyor's rod, for example) by measuring the size of its image. If the entrance-pupil of the optical instrument is situated in the primary focal plane, the angle θ can be determined by the relation found above:

$$\tan\theta = \frac{y'}{f},$$

and hence the distance of the object may be computed by the formula:

$$\xi = \frac{y}{\tan\theta} = f \cdot \frac{y}{y'},$$

provided we know the values of the magnitudes denoted by y, y' and f.

369. The Subjective Magnifying Power.

If, however, the optical instrument is designed to be used subjectively in conjunction with the eye for the purpose of reinforcing vision (as, for example, in the case of the ordinary magnifying glass or the microscope, etc.), the magnitude of the image produced by the instrument will not, by itself, serve as a measure of the magnification, but the question here is rather with respect to *the size of the image that is formed on the retina of the eye;* and since this latter cannot be subjected to direct measurement, the formula $Y = y'/y$ (where y' denotes the size of the retina-image) is not applicable in this case.

In the ordinary acceptation of the term, the subjective magnifying power of an optical instrument which, as in the case of the microscope, is to be used in conjunction with the eye, is not the ratio of the actual dimensions of the image and object, but the ratio of their apparent sizes as seen by the eye of the observer. Here also it is necessary to explain distinctly what is meant by the "apparent size" of both the image and the object.

The apparent size of the image $M'Q' = y'$ as viewed by an eye placed at a point J on the axis of the optical system is measured by the angle (or, rather, by the tangent of the angle) subtended at J by $M'Q'$; that is, by

$$\tan \angle M'JQ' = \frac{M'Q'}{JM'}.$$

The place of the eye here designated by J is actually the centre of the entrance-pupil of the eye, which is usually placed so as to coincide with the centre $\boldsymbol{M}'$ of the exit-pupil of the optical system. If this is the adjustment, then $JM' = \boldsymbol{M}'M' = \xi'$, and

$$\angle M'JQ' = \angle M'\boldsymbol{M}'Q' = \boldsymbol{\theta}',$$

where $\boldsymbol{\theta}'$ denotes the slope of the chief image-ray of the bundle of rays that go through the image-point Q'. Introducing these symbols, we obtain the following expression for the apparent size of the image:

$$\tan \boldsymbol{\theta}' = \frac{y'}{\xi'}.$$

If now the optical instrument is removed, and the eye is focussed so as to view directly and distinctly the object $MQ = y$, the apparent size of the object as viewed by the eye at the distance of distinct vision $JM = a$ is measured by the angle $MJQ = \eta$, that is, by

$$\tan \eta = \frac{y}{a}.$$

Thus, according to the usual definition, what is meant by the subjective magnifying power of an optical instrument belonging to the same general class as the microscope is *the ratio of the visual angles* (or trigonometric tangents of the angles) *subtended at the eye, on the one hand, by the image as viewed in the instrument, and, on the other hand, by the object as seen by the naked eye at the distance of distinct vision.* Denoting this ratio by the symbol W, we have therefore:

$$W = \frac{\tan\theta'}{\tan\eta} = \frac{a}{\xi'} \cdot \frac{y'}{y}. \tag{449}$$

Although this definition of the subjective magnifying power combines the two merits of simplicity and clearness, it is open to objection on account of the fact that it involves essentially the magnitude denoted here by a, the so-called "distance of distinct vision", which has no connection with the instrument itself and which is different for different individuals. It is a well-known fact of experience that by virtue of its power of accommodation the normal eye is capable of seeing distinctly at almost any distance; but what is here meant by the distance of distinct vision is the distance from the eye at which an observer would naturally place an object in order to view it intently; which in the case of a normal eye is usually reckoned as about 25 cm. or 10 in. Accordingly, whereas the magnification as defined by the ratio W will be different for a near-sighted observer for whom $a = 10$ cm. and for a far-sighted observer for whom $a = 50$ cm., yet, as ABBE[1] has pointed out, both observers looking through the instrument will, as a matter of fact, view the image of the same object under the same visual angle; so that whatever difference there may be in the magnification is to be found, not in the instrument itself, but in the different organs of sight that are employed in conjunction with the apparatus.

Eliminating the angle η which has nothing to do with the optical instrument, we may write the formula for W in the following form:

$$W = a \cdot \frac{\tan\theta'}{y} = a \cdot V, \tag{450}$$

whereby the magnifying power W is expressed now as the product of two factors, viz., the factor a, which depends entirely on the eye of the observer, and the factor

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[1] E. ABBE: Note on the Proper Definition of the Amplifying Power of a Lens or a Lens-system: *Journ. Roy. Micr. Soc.*, (2), iv (1884), 348–351. See German translation in *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 445–449.

See also S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), 160–164.

$$V = \frac{\tan \theta'}{y}, \tag{451}$$

which, notwithstanding the fact that the distance of the image from the eye is involved in the definition of the apparent size $\tan \theta'$ of the image, depends essentially, as we shall show, on the structure of the optical system alone.

Since

$$\tan \theta' = \frac{y'}{\xi'} \quad \text{and} \quad \frac{y'}{y} = \frac{x'}{e'} = \frac{\mathbf{x}' + \xi'}{e'},$$

where $x' = E'M'$, $\mathbf{x}' = E'\mathbf{M}'$ denote the abscissæ, with respect to the secondary focal point E', of the points M' and $\mathbf{M}'$ respectively, and where e' denotes the secondary focal length of the optical system, we obtain:

$$V = \frac{1}{e'}\left(1 + \frac{\mathbf{x}'}{\xi'}\right). \tag{452}$$

Now almost without exception in the case of all optical instruments that are employed subjectively in conjunction with the eye, no matter how the image may be focussed by the eye, the distance $\mathbf{x}'$ is so small in comparison with the distance ξ that the fraction $\mathbf{x}'/\xi'$ is practically negligible. Under these circumstances we may write therefore:

$$V = \frac{\tan \theta'}{y} = \frac{1}{e'}, \text{ approximately;} \tag{453}$$

and in the special case when the plane σ' of the exit-pupil coincides with the secondary focal plane ($\mathbf{x}' = 0$) and the eye is situated at the secondary focal point E', the formula $V = 1/e'$ will be strictly true. Accordingly, as above stated, the magnitude denoted by V depends solely on the structure of the optical instrument provided it is to be used subjectively.

According to ABBE, this magnitude V defined as *the ratio of the visual angle subtended at the eye by the image viewed through the instrument to the corresponding linear dimension of the object* is therefore a proper measure of the characteristic or intrinsic magnifying power of an optical system on the order of the microscope. For every such system it has a perfectly definite value, viz., $1/e'$, and thus is entirely independent of all the more or less accidental circumstances that may affect the magnification, such as the distance from the image of the observer's eye, the distance from the focal plane of the exit-pupil, etc.

ABBE's definition V of the Subjective Magnifying Power is obtained from the ordinary definition W by merely dividing W by the distance a of distinct vision of the observer; thus,

$$V = \frac{W}{a}. \tag{454}$$

Since W is proportional to a, the popular use of the term "magnifying power", which corresponds to the magnitude W, expresses the fact that the advantage gained by the use of an optical instrument is proportional to the observer's distance of distinct vision and is therefore greater for a far-sighted than for a near-sighted observer. From the scientific point of view, ABBE's definition V is far superior, inasmuch as V is a constant of the instrument itself. The subjective magnifying power V in the case of an instrument on the order of the microscope is seen to be completely analogous to the objective magnifying power $y'/\tan\theta$ in the case of the image of an infinitely distant object formed by an optical instrument on the order of the photographic objective or the objective of the telescope.

ART. 117. THE FIELD OF VIEW.

370. Entrance-Port and Exit-Port.

The limiting of the bundles of rays that are permitted to traverse the optical system is not the only duty performed by the stops and lens-fastenings; but these serve also to define the extent of the object that is reproduced in the image. For the sake of simplicity, let us assume for the present that *the aperture-stop at O is infinitely small*, so that the pupil-openings at $\boldsymbol{M}$ and $\boldsymbol{M}'$ (Fig. 162) are reduced to mere points ($\theta = \theta' = 0$, $p = p' = 0$). In this case the chief ray of a bundle will be the only effective ray, and the bundle of chief rays will constitute therefore the entire system of effective rays.

In order now to ascertain which one of the stops present is the one that determines the expanse of object that will be depicted, we construct, as before (§ 361), the image of each stop formed by that part of the optical system which is in front of it. That one whose image thus constructed subtends at the centre $\boldsymbol{M}$ of the entrance-pupil the smallest angle is the stop that limits *the field of view of the object.* In the diagram this stop-image is represented as situated with its centre on the optical axis at the point designated by S. The cone of chief object-rays whose transversal cross-section at S coincides with this stop-image divides the transversal object-plane σ into two regions, an inner circular space comprising the so-called field of view of the object and an outer region containing points so situated that no rays emanating from them can go through the instrument. The actual stop that is thus responsible for the limiting of the field of view of the object may be called the *field-stop;* as seen from the centre O of the aperture-stop it is the material stop or lens-fastening that subtends the smallest visual angle. The image of the field-stop formed by the part of the optical system that lies in front of it may be appropriately called the *Entrance-Port.* Its radius ST (reckoned plus or minus according as the circumference-point T is above or below the axis) will be denoted by q. Finally, the angle $SMT = \Theta$ subtended at the centre $\boldsymbol{M}$ of the entrance-pupil by the radius of the entrance-port, defined more precisely by the relation

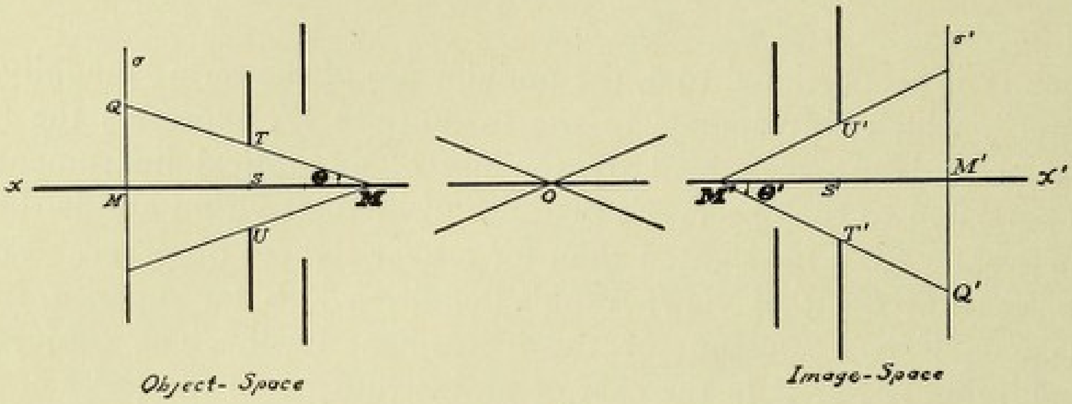

FIG. 162.

ENTRANCE-PORT AND EXIT-PORT OF OPTICAL SYSTEM. xx' represents the optical axis of the system, which latter is not shown in the diagram. The Aperture-Stop is here assumed to be a pin-hole opening at O; so that the Entrance-Pupil and Exit-Pupil are likewise contracted into mere points at $\boldsymbol{M}$, $\boldsymbol{M}'$, respectively. In the Object-Space are shown two stop-images, of which the one whose centre is at the point marked S subtends at $\boldsymbol{M}$ the smaller angle. This stop-image is the Entrance-Port. The Exit-Port with its centre at S' subtends at $\boldsymbol{M}'$ an angle smaller than that subtended there by any other stop-image on the image-side.

$\boldsymbol{M}S = c$, $\boldsymbol{M}'S' = c'$, $ST = q$, $S'T' = q'$, $\boldsymbol{M}M = \xi$, $\boldsymbol{M}'M' = \xi'$, $\angle S\boldsymbol{M}T = \Theta$, $\angle S'\boldsymbol{M}'T' = \Theta'$.

$$\tan\Theta = \frac{ST}{\boldsymbol{M}S} = \frac{q}{c}, \qquad (455)$$

where $c = \boldsymbol{M}S$, is called *the angular measure* (or the semi-angular diameter) *of the field of view of the object.*

Analogously, the image of the field-stop produced by the part of the optical system L that lies on the other side of it (which will obviously be identical also with the image of the entrance-port produced by the action of the entire compound system L) will define likewise the extent of the image and may be called the *Exit-Port.*[1] Thus, also, the angle $S'\mathbf{M}'T' = \Theta'$, where S', T' designate the positions of the points conjugate, with respect to the entire system, to the points designated above by S, T, respectively, is *the angular measure* (or the semi-angular diameter) *of the field of view of the image.*

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[1] The names "Entrance-Port" and "Exit-Port" introduced here were suggested by the corresponding terms *Eintrittsluke* and *Austrittsluke* used by VON ROHR in his treatise

It is possible, of course, that an optical system may have two or more entrance-ports. An obvious illustration is suggested by the familiar type of photographic double-objective in which the two parts of the system are symmetrical with respect to the aperture-stop in the middle (as in the case of the "Aplanats"), so that the rims of the two lens-systems subtend equal angles at the centre O of the aperture-stop; and hence, since the rim of the front component and the image of the rim of the hinder component produced by the front component subtend equal angles at the centre $\mathbf{M}$ of the entrance-pupil, either of these two may be regarded as the entrance-port. This fact will be found to possess a certain importance in the case of an optical system of finite aperture, as we shall have occasion to see (§ 383).

371. In the special case when the extent of the object MQ is so small that the angle subtended at the centre $\mathbf{M}$ of the entrance-pupil is smaller than the angle subtended at the same point by the entrance-port (that is, $\angle M\mathbf{M}Q < \angle S\mathbf{M}T$), the field of view is limited by the object itself. In any case if we designate by Q the object-point in the transversal plane σ that is farthest from the axis, the angular measure of the field of view of the object is $\angle M\mathbf{M}Q = \Theta$, where Θ denotes always the slope-angle of the outermost ray of the bundle of chief rays in the Object-Space. If we put $MQ = y$, $\mathbf{M}M = \xi$, we can write:

$$\tan\Theta = \frac{y}{\xi}. \tag{456}$$

ART. 118. PROJECTION-SYSTEMS WITH INFINITELY NARROW APERTURE ($\Theta = 0$).

372. Focus-Plane and Screen-Plane. According to the geometrical theory of collinear correspondence, the image of a 3-dimensional object is itself 3-dimensional; but by the image produced by an optical instrument is usually meant not this geometrical image-relief in space, but almost without exception the projection thereof on some specified surface, such as the retina of the eye itself in the case of the class of optical instruments that are used subjectively in conjunction with the eye, or such as a screen or sensitive photographic plate in

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on *Die Strahlenbegrenzung in optischen Systemen.* (See *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), edited by M. von Rohr: Chapter IX, 466-507.)

that other large class of optical instruments, the so-called projection-systems. Only at such points of this selected surface as are conjugate to actual points of the object will the definition of the projected image be sharp; these points being the vertices of bundles of rays which emanated originally from the corresponding points of the object. But since the image-points conjugate to all the other points of the object will lie on one side or the other of this projection-surface, these object-points will be represented on this surface not by points at all, but by the sections that are cut out of the corresponding bundles of image-rays by the projection-surface. Thus, there will be impressed on the surface (retina of the eye or screen) a certain approximate effect or vicarious image, so to speak, which represents the relief-image and enables us to form a more or less correct conception of it.

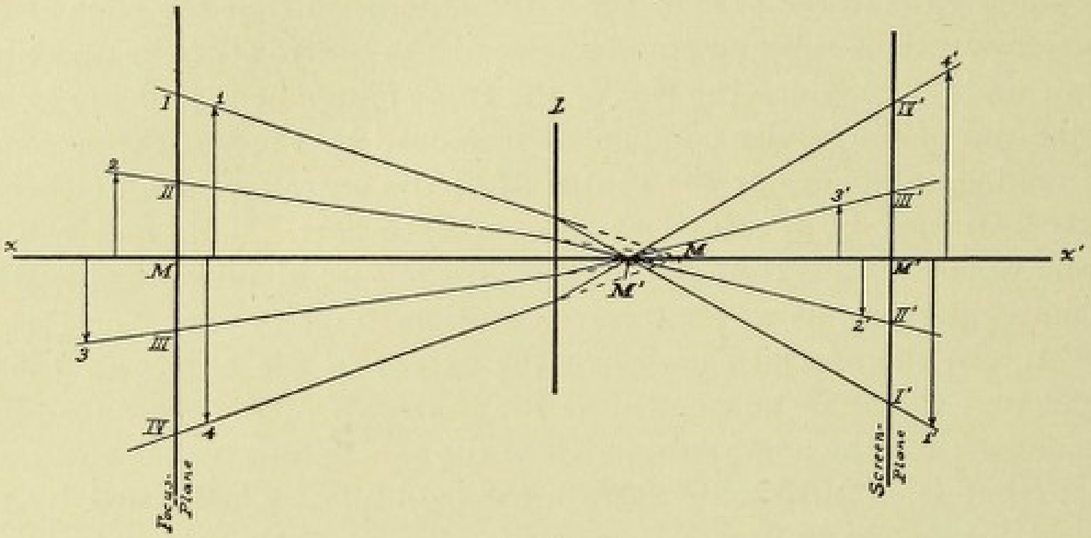

FIG. 163.

PROJECTION-SYSTEM WITH INFINITELY NARROW APERTURE. The transversal planes σ, σ' conjugate to each other with respect to the convex lens L represent here the Focus-Plane and the Screen-Plane, respectively. The centres of the infinitely narrow pupils are at M, M', and the only effective ray of each bundle is the chief ray. If the numerals 1, 2, 3, 4 designate the positions of certain points of the object, and if 1', 2', 3', 4' designate the positions of the corresponding points of the image-relief, the Roman numerals without primes show the positions in the Focus-Plane of the corresponding vicarious or projected object-points, and the Roman numerals with primes show the positions in the Screen-Plane of the corresponding projected points of the image.

In the case of the photographic objective, which may be considered as a typical example of the projection-system, the image is cast upon a plane surface placed at right angles to the optical axis, and the instrument is focussed on a definite axial object-point M by adjusting the ground-glass screen so that its front surface coincides with the transversal plane σ' conjugate to the transversal plane σ of the axial object-point M. In the present discussion it will be convenient to distinguish the pair of conjugate transversal planes σ and σ' as the *Focus-Plane* and the *Screen-Plane*, respectively. In case the aperture is infinitely narrow, as is here assumed, the chief ray is the only ray of the bundle that is effective; and the figure in the focus-plane corresponding to that which is actually visible on the screen-plane may be constructed point by point by tracing backwards the path of each chief ray from the point P' where it crosses the screen-plane to the point P where the corresponding ray in the Object-Space crosses the focus-plane. Practically, this process amounts simply to projecting all the points of the object from the centre $\boldsymbol{M}$ of the entrance-pupil on to the chosen focus-plane; and this projection-figure, which may be called the "*projected object*", is the object that is in reality reproduced in the "*projected image*" in the screen-plane; which latter may also be constructed in the same way by projecting all the points of the relief-image from the centre $\boldsymbol{M}'$ of the exit-pupil on to the screen-plane, as shown in Fig. 163.

373. Perspective-Elongation.

If MQ (Fig. 164) is the projection from $\boldsymbol{M}$ on to the focus-plane σ of an object-line NR perpendicular to the optical axis at N, we have:

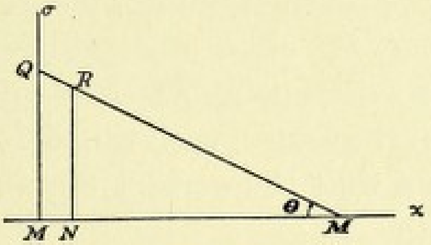

FIG. 164.

PERSPECTIVE ELONGATION OF THE OBJECT. Q is the projection from centre $\boldsymbol{M}$ of the Entrance-Pupil of the object-point R on to the Focus-Plane σ.

$\angle M\boldsymbol{M}Q = \theta$.

$$MQ = NR \cdot \frac{\boldsymbol{M}M}{\boldsymbol{M}N} = NR \cdot \frac{\boldsymbol{M}M}{\boldsymbol{M}M + MN};$$

or, since MN is usually small in comparison with $M\boldsymbol{M}$,

$$\frac{MQ}{NR} = 1 + \frac{MN}{M\boldsymbol{M}}, \text{ approximately.}$$

The difference $MQ - NR$ is the measure of the perspective elongation of the object NR, and the ratio

$$\frac{MQ - NR}{NR} = \frac{MN}{N\boldsymbol{M}}$$

is called *the relative perspective elongation* of the object NR.

374. Correct Distance of Viewing a Photograph.

Viewed from the centre $\boldsymbol{M}$ of the entrance-pupil, the object NR and the projected object MQ have the same apparent size; and this is the case no matter where the focus-plane σ is situated. In order to get the correct impression, the projected image $M'Q' = Y \cdot MQ$, where Y denotes the value of the magnification-ratio for the pair of conjugate transversal planes σ, σ', should be inspected under the same visual angle θ that is subtended by NR or MQ at the centre $\boldsymbol{M}$ of the entrance-pupil.

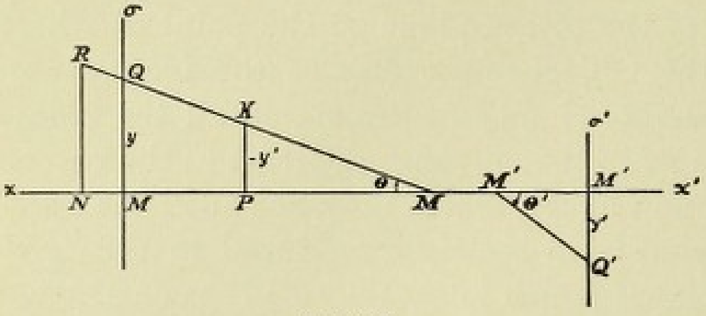

Fig. 165.

Correct Distance for Viewing a Photograph. Here $y' = M'Q'$ is the image on the ground-glass screen σ' of the object NR. If the eye is placed at $\boldsymbol{M}$, the photograph or copy placed in the position $PK = -y'$ will have the same perspective as the object.

$\angle M\boldsymbol{M}Q = \theta$, $\angle M'\boldsymbol{M}'Q' = \theta'$, $P\boldsymbol{M} = d$, $\boldsymbol{M}M = \xi$, $\boldsymbol{M}'M' = \xi'$.

If $y' = M'Q'$ (Fig. 165) is the image projected by a photographic objective on the ground-glass screen of the object NR, the photograph $PK = Q'M' = -y'$ should be viewed at such a distance $d = P\boldsymbol{M}$ that it subtends at $\boldsymbol{M}$ the same angle θ as the object subtends there. From the figure, with the aid of the image-equations, it can readily be shown that the correct distance of viewing a photograph is:

$$d = e'(Y\boldsymbol{Z} - 1),$$

where $e'(= -f)$ denotes the secondary focal length of the photographic objective, $\boldsymbol{Z}$ denotes the angular magnification at the pupil-centres $\boldsymbol{M}$, $\boldsymbol{M}'$, and Y denotes the linear magnification with respect to the given focus-plane σ. Of course, in using this formula, attention must be paid to the signs of the magnitudes represented by the symbols therein.[1] In the case of a landscape-lens, $Y = 0$, and then $d = -e'$; which means that the photograph of a landscape should be viewed at a distance equal to the focal length of the lens. Ordinarily, we have $\boldsymbol{Z}$ equal very nearly to unity, and in such cases ($\theta = \theta'$) the photograph should be viewed at a distance $d = \boldsymbol{M}'M' = \xi'$.

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[1] See M. von Rohr: *Theorie und Geschichte des photographischen Objektivs* (Berlin, 1899), 16.

ART. 119. OPTICAL SYSTEMS WITH FINITE APERTURE.

375. Projected Object and Projected Image in the case of Projection-Systems of Finite Aperture.

So long as the aperture of the system was infinitely narrow, we had to consider merely the procedures of the chief rays; but advancing now to the study of optical projection-systems of finite aperture, we must take account of other rays besides just those that in the Object-Space are directed towards the centre of the entrance-pupil. Every point of the object is the vertex of a cone of rays whose paths lie along straight lines which, produced if necessary, must first of all go through points in the transversal plane $\boldsymbol{\sigma}$ contained within the circular opening of the entrance-pupil. Some of the rays of such a bundle, possibly all of them, may be intercepted at the entrance-port, and in this event only a portion of the bundle at most will be effective. To each cone of rays in the Object-Space corresponds also a cone of rays in the Image-Space, whose paths likewise lie along straight lines which, produced if necessary, must pass through points in the transversal plane $\boldsymbol{\sigma}'$ comprised within the circular opening of the exit-pupil; and to an incomplete cone of object-rays corresponds, of course, an incomplete cone of image-rays. The relief-image of a 3-dimensional object is the configuration of image-points which are at the vertices of all these cones or partial cones of image-rays. Some of these vertices may fall in the transversal screen-plane σ'; and these will be the image-points corresponding to such of the points of the object as lie in the transversal focus-plane σ. But all the other points of the object, which lie to one side or other of the focus-plane, will be represented in the projected image on the screen-plane, not by points at all, but by the circular discs or patches—so-called "diffusion-circles" or "blur-circles" (see § 365)—which are the sections of the cones of image-rays made by the screen-plane. In the case of an incomplete cone of image-rays, the image of the corresponding object-point will be represented on the screen-plane by only a piece of a blur-circle. These ideas will be made clear by the consideration of the diagram (Fig. 166) which represents a meridian section of an optical system consisting of an infinitely thin convex lens UT with a front stop CD with its centre on the optical axis at $\boldsymbol{M}$. In this illustration the rim of the lens is the circumference of both the entrance-port and the exit-port.

The real object corresponding to the above-described projected image in the screen-plane σ' is the figure in the focus-plane σ obtained by projecting the entrance-pupil on to this plane from each point of the actual object. In the case of those object-points so situated that, on account of the limited opening of the entrance-port, they can utilize only a part of the area of the entrance-pupil, we must project on to the focus-plane only the part of the entrance-pupil that is utilized. The centres of these circular discs and disc-portions which are the sections of the bundles of effective rays made by the focus-plane σ and the screen-plane σ' in the Object-Space and Image-Space, respectively, are at the points where the chief rays cross these planes.

This last statement suggests also, that in regard to this vicarious object-figure in the focus-plane σ, there is an important difference to be remarked between the case of a point inside of one of these object-side blur-circles and the case of an ordinary object-point lying in the focus-plane; for whereas the latter emits rays in all directions, the former is to be regarded as sending out only one single ray coinciding with the actual object-ray which crosses the focus-plane at this point.

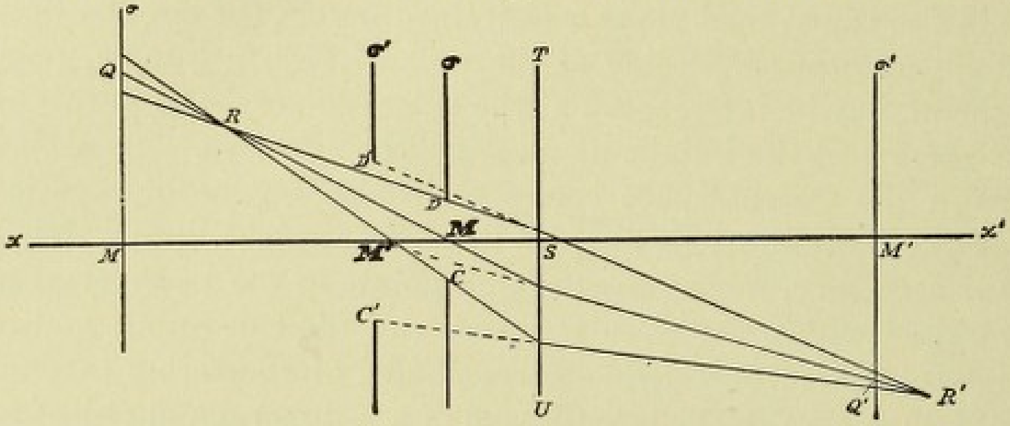

Fig. 165.

Projected Object and Image in Projection-System of Finite Aperture. The Entrance-Pupil *CD* is projected from the object-point *R* on to the Focus-Plane σ in the blur-circle with centre at *Q*; and, similarly, the Exit-Pupil *C'D'* is projected from the image-point *R'* on to the Screen-Plane σ' in the blur-circle with centre at the point *Q'* conjugate to *Q*.

If the aperture of the optical system is not only finite but relatively large, the transversal planes σ, σ' must be a pair of aplanatic planes in order that there may be a point-to-point correspondence between the focus-plane and the screen-plane; and when this is the case, the image of an object-point which lies outside the focus-plane will not be a point, since the so-called Herschel-Condition (*cf.* § 324) is incompatible with the Sine-Condition. Under such circumstances, where, in general, the bundles of image-rays are no longer homocentric, it is particularly advantageous to represent the image of a 3-dimensional object by means of its projected image on the screen-plane.

376. The centres of the blur-circles on the screen-plane are to be regarded as the positions of the image-points; and since, even in the extreme case just mentioned of a system of very large aperture, these are the places where the chief image-rays cross this plane, the *perspective* is exactly the same here as for the case of a system of infinitely narrow aperture (§ 373), so that nothing needs to be added to what has been said already in the treatment of the perspective in the preceding case.

377. Focus-Depth of Projection-System of Finite Aperture.

With regard to the distinctness of the image on the screen-plane, that is a matter that will depend very largely on the acuteness of vision of the observer. If the resolving power of the eye were absolutely perfect, this screen-image composed partly of image-points and partly of blur-circles and pieces of such circles would appear faulty on the mere ground that it was not a faithful reproduction of the original. But the resolving power of the eye is limited (*cf.* § 252), depending on a variety of conditions, both physical and physiological. Under average conditions the human eye is able to distinguish as separate and distinct two points whose angular distance apart varies for different individuals between the limits of one and five minutes of arc;[1] and hence the blur-circles in the projection-image will not be distinguishable from points provided their angular diameters do not exceed this limiting angular measure (ϵ) of the resolving power of the eye. Similarly, also, in regard to the projection-figure of the object on the focus-plane, in order that this may appear sharp and distinct as viewed by an eye at the centre $\boldsymbol{M}$ of the entrance-pupil, the diameters of the blur-circles must subtend at $\boldsymbol{M}$ angles that are smaller than the limiting angle ϵ. Since the diameters of these blur-circles will depend on the distances of the actual object-points from the focus-plane, the question arises how far from this plane can such an object-point be in order that its image in the screen-plane shall still appear to be a point and not a fleck of light. This distance, as we shall see, will be different according as the object-point lies on one side or the other of the focus-plane, so that all object-points which are comprised within the space between two determinate transversal planes at unequal distances from the focus-plane and on opposite sides of it will be reproduced distinctly in the projection-image in the screen-plane. The distance between this pair of transversal planes, called the *Focus-Depth*, we propose now to investigate.

378. Let Q_1 (Fig. 167) designate the position of the point where the chief ray $R_1\boldsymbol{M}$ of the object-point R_1 crosses the focus-plane σ, so that Q_1 is therefore the centre of the blur-circle that represents R_1

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[1] See, for example, E. ABBE: Beschreibung eines neuen stereoskopischen Oculars: CARLS *Rep. f. Exp.-Phys.*, xvii (1881), 197–224. See p. 219. This paper will be found also in *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 244–272.

in the projection-figure on the focus-plane. The object-ray R_1D which, lying in the meridian plane of the figure, is directed towards the point D of the circumference of the entrance-pupil determines by its intersection with the focus-plane σ the point Q_2 in the circumference of the above-mentioned blur-circle. If N_1 designates the point where the optical axis crosses the transversal plane of the object-point R_1, we obtain from the figure:

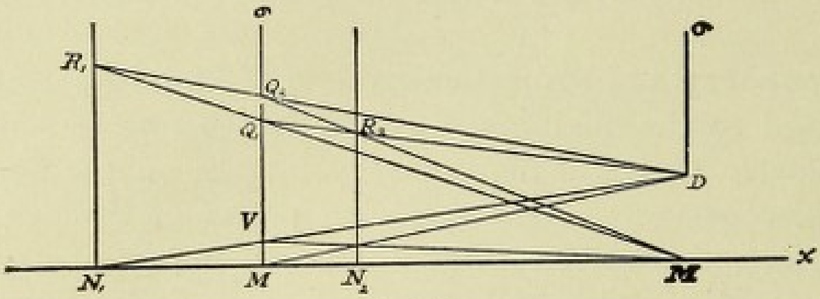

FIG. 167.

FOCUS-DEPTH OF PROJECTION-SYSTEM OF FINITE APERTURE. Diagram represents upper half of meridian section of Object-Space.

$Q_1Q_2 = dy$, $\boldsymbol{M}M = \xi$, $MN_1 = \delta\xi_1$, $MN_2 = \delta\xi_2$, $\boldsymbol{M}D = p$, $\angle \boldsymbol{M}MD = \Theta$.

$$\frac{Q_1Q_2}{\boldsymbol{M}D} = \frac{MN_1}{\boldsymbol{M}N_1} = \frac{MN_1}{\boldsymbol{M}M + MN_1};$$

which, if we put

$$Q_1Q_2 = dy, \quad \boldsymbol{M}M = \xi, \quad MN_1 = \delta\xi_1, \quad \boldsymbol{M}D = p,$$

may be written as follows:

$$\frac{dy}{p} = \frac{\delta\xi_1}{\xi + \delta\xi_1}. \tag{457}$$

Hence, since

$$p = -\,\xi \cdot \tan\Theta;$$

where $\Theta = \angle \boldsymbol{M}MD$ denotes the angular measure of the aperture of the system (§ 363), we derive the following expression for the radius dy of the blur-circle on the focus-plane:

$$dy = -\frac{\xi \cdot \delta\xi_1}{\xi + \delta\xi_1} \tan\Theta. \tag{458}$$

This formula, which is given by CZAPSKI,[1] shows that the size of the blur-circle depends not only on the aperture-angle Θ and the distance $\delta\xi_1$ of the object-point from the focus-plane, but also on the distance ξ of the focus-plane from the entrance-pupil. Moreover, the size of the blur-circle does not depend on the distance of the object-point from the optical axis, so that all object-points in the same transversal plane will be represented in the projection-figure on the focus-plane by blur-circles of equal diameters. Thus, for example, the blur-circle of the axial object-point N_1 is equal to that of R_1; in the figure

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[1] S. CZAPSKI: *Die Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), p. 170.

$$MV = Q_1Q_2 = dy.$$

The straight lines Q_1D and $Q_2\boldsymbol{M}$ determine by their intersection a point R_2 on the opposite side of the focus-plane from R_1, which, regarded as an object-point, will be represented in the projection-figure on the focus-plane by a blur-circle whose centre is at the point Q_2 and whose radius $Q_2Q_1 = -dy$ has the same absolute magnitude as that of the object-point R_1. Thus, on either side of the focus-plane there is a certain transversal plane characterized by the fact that all object-points in this plane will be projected on to the focus-plane in blur-circles all of the same prescribed size. If we put $MN_2 = \delta\xi_2$, where N_2 is used to designate the point where the optical axis crosses the transversal plane of the object-point R_2, we obtain from the figure, exactly as in the case of the similar formula above:

$$\frac{dy}{p} = -\frac{\delta\xi_2}{\xi + \delta\xi_2}.$$

Hence, also, we find for the distances from the focus-plane of this pair of transversal planes:

$$\delta\xi_1 = \frac{\xi \cdot dy}{p - dy}, \quad \delta\xi_2 = -\frac{\xi \cdot dy}{p + dy}; \qquad (459)$$

and, accordingly, we see also that the two transversal planes determined by these formulæ are at unequal distances from the focus-plane, and, in fact, that the front one of the two planes (in the figure the one containing the object-point R_1) is always nearer to the focus-plane than the other plane.

Now if the magnitude dy is such that

$$\frac{dy}{\xi} = \tan\frac{\epsilon}{2},$$

where ϵ denotes the angular measure of the resolving power of the eye (in the figure $\epsilon/2 = \angle V\boldsymbol{M}M$), the blur-circle on the focus-plane corresponding to an object-point lying anywhere in the space comprised between the pair of transversal planes belonging to R_1 and R_2

will be so small that the eye placed at the centre **M** of the entrance-pupil could not distinguish them from points; so that, practically speaking, all object-points lying within this region on either side of the focus-plane, will be sharply defined in the projection-figure.

The distance

$$N_1N_2 = N_1M + MN_2, = \delta\xi_2 - \delta\xi_1$$

between the pair of transversal planes determined by this critical value $dy = (\epsilon\cdot\xi)/2$ is, as was stated above, the Focus-Depth of the projection-system for a given position of the focus-plane σ. Thus, we find the following expression for the Focus-Depth:

$$N_1N_2 = -\frac{2p\cdot\xi\cdot dy}{p^2 - dy^2}. \tag{460}$$

The reciprocal of the focus-depth may be regarded as a measure of the exactness of the focus.

379. Lack of Detail in the Image due to the Focus-Depth.

As to the detail or distinctness of the image projected on the screen-plane, this is a question that involves not merely the absolute sizes of the blur-circles but the magnification-ratio also. Thus, for example, the blur-circles of the image of a piece of hand-writing which is magnified to double the size of the original may have blur-circles of twice as great diameters as would be permissible if it were not magnified at all; the former would be just as legible or distinct as the latter. Accordingly, as a measure of the indistinctness due to being out of focus, it has been proposed[1] to take the ratio of the radius dy' of the blur-circle in the screen-plane σ' corresponding to an object-point R (see Fig. 165) to the magnification $Y_R = M'Q'/NR$. Since $Y = M'Q'/MQ$, $dy' = Y\cdot dy$ and (§ 373)

$$\frac{MQ}{NR} = \frac{\xi}{\xi + \delta\xi},$$

the measure of the indistinctness, according to the above definition, is given as follows:

$$\frac{dy'}{Y_R} = \frac{\xi + \delta\xi}{\xi}dy = -\delta\xi\cdot\tan\Theta. \tag{461}$$

380. Focus-Depth of Optical Systems of Finite Aperture used in Conjunction with the Eye.

If the optical system is to be employed, not for the purpose of casting objective images on a screen, but in conjunction with the eye for the reinforcement of vision, and if the pupil of the passive eye is supposed to be placed at the exit-pupil of the instrument, the image is presented to the eye at the distance $\boldsymbol{M}'M' = \xi'$.

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[1] See S. Czapski: *Theorie der optischen Instrumente nach* Abbe (Breslau, 1893), p. 171.

The absolute linear diameter of the blur-circle in the image-plane corresponding to a non-focussed point is:

$$2dy' = 2Y \cdot dy = -2Y \cdot \frac{\xi \cdot \delta\xi}{\xi + \delta\xi} \cdot \tan\Theta = -2Y \cdot \delta\xi \cdot \tan\Theta, \text{ approx.,}$$

where Y denotes the lateral magnification of the aplanatic pair of axial points M, M', and where, in obtaining the final approximate expression, the distance $\delta\xi$ is supposed to be small as compared with ξ, as is the fact with an optical instrument of high magnifying power. If

$$\epsilon = \frac{2dy'}{\xi'} = -2Y \cdot \frac{\delta\xi}{\xi'} \cdot \tan\Theta$$

denotes the visual angle subtended at the eye by the blur-circle, and if we recall from § 369 that

$$V = \frac{\tan\theta'}{y} = \frac{y'}{y} \cdot \frac{1}{\xi'} = \frac{Y}{\xi'},$$

we find

$$\epsilon = -2V \cdot \delta\xi \cdot \tan\Theta,$$

or

$$2\delta\xi = -\frac{\epsilon}{V \cdot \tan\Theta}. \tag{462}$$

Thus, if $y'/y = 100$, and if the absolute value of ξ' is equal to the conventional distance of distinct vision, viz., 250 mm., so that

$$V = \frac{y'}{y} \cdot \frac{1}{\xi} = 0.4,$$

and if we take $\Theta = -30°$, $\epsilon = 3' = 0.00087$ radian, we obtain for the Focus-Depth: $2\delta\xi = 0.0037$ mm.

381. Accommodation-Depth.

By virtue of its power of accommodation, the eye can be focussed at will on different points of the image-relief, and provided these image-points are within the range of distinct vision, and also provided the imagery is ideal, the different parts of the image can be viewed with perfect exactness; so that, owing to this property inherent in the eye to a greater or less degree in different individuals, a certain depth of the object called the *accommodation-depth* will be seen distinctly in its image, which measured along the axis may be denoted by M_1M_2. The depth of vision is extended beyond these points by the focus-depth $\delta\xi_1$ in one direction from M_1 and the focus-depth $\delta\xi_2$ in the other direction from M_2, since within these extended parts the blur-circles are too small to be resolved by the eye; and hence the *Entire Depth of Vision* is equal to the sum of the Accommodation-Depth and Focus-Depth, viz. $= M_1M_2 + \delta\xi_1 + \delta\xi_2$.

If the eye is placed at the exit-pupil of the instrument, whose centre is at the point designated by $\boldsymbol{M'}$, and if the positions on the optical axis in the Image-Space of the "near-point" and "far-point" of the eye of the observer are designated by M'_1 and M'_2, respectively, the range of distinct vision is equal to the piece $M'_1M'_2$ of the optical axis. The points designated above by M_1 and M_2 are the axial object-points conjugate to M'_1 and M'_2, respectively. If the focal points of the optical system are designated by F and E', and if we put

$$x_1 = FM_1, \quad x_2 = FM_2, \quad x'_1 = E'M'_1, \quad x'_2 = E'M'_2,$$

and, finally, if the focal lengths are denoted by f and e', then, on the assumption of collinear correspondence, we have:

$$x_1x'_1 = x_2x'_2 = fe';$$

and hence:

$$M_1M_2 = \delta x = x_2 - x_1 = -fe'\frac{x'_2 - x'_1}{x'_1x'_2} = -fe'\frac{\delta x'}{x'_1x'_2},$$

where $\delta x' = M'_1M'_2$. If (§ 179)

$$Y_1 = \frac{x'_1}{e'}, \quad Y_2 = \frac{x'_2}{e'}$$

denote the magnification-ratios of the two pairs of conjugate axial points, and if we introduce also the relation (§ 193):

$$n'f + ne' = 0,$$

where n, n' denote the indices of refraction of the first and last media of the optical system, we obtain:

$$\delta x = \frac{n}{n'} \cdot \frac{\delta x'}{Y_1 \cdot Y_2}.$$

If $\boldsymbol{M'}M'_1 = \xi'_1$ and $\boldsymbol{M'}M'_2 = \xi'_2$ denote the least and greatest distances of distinct vision of the eye, then, according to DONDERS, the magnitude

$$A = \frac{1}{\xi'_1} - \frac{1}{\xi'_2} = \frac{M'_1M'_2}{\xi'_1 \cdot \xi'_2} = \frac{\delta x'}{\xi'_1 \cdot \xi'_2} \qquad (463)$$

is the rational measure of the power of accommodation of the eye;[1] and hence we obtain the following expression for the accommodation-depth:

$$\delta x = \frac{n}{n'} \cdot A \cdot \frac{\xi_1' \cdot \xi_2'}{Y_1 \cdot Y_2}; \tag{464}$$

and if Y_1 and Y_2 are not much different from each other, we can replace each of them by a certain mean value Y, which, to be perfectly accurate, should be the geometric mean between Y_1 and Y_2; and, similarly, we can introduce in place of ξ_1' and ξ_2' a mean value ξ'; so that the final form of the expression becomes:

$$x = \frac{n}{n'} \cdot A \cdot \left(\frac{\xi'}{Y}\right)^2, \tag{465}$$

where usually ξ' is put $= 250$ mm., the conventional distance of distinct vision. Thus, for example, in the case of a myopic eye, for which $\xi_1' = 150$ mm., $\xi_2' = 300$ mm., so that $A = 1/300$, we obtain for a magnification of $Y = 100$ (assuming $n = n' = 1$):

$$x = 1/48 = 0.021 \text{ mm.}$$

ABBE,[2] who has investigated this subject very exhaustively, especially in connection with the microscope, gives several tables (which are given also by CZAPSKI[3]) exhibiting the relations between the Focus-Depth and the Accommodation-Depth for different values of the magnification-ratio Y; whereby it appears that, although for low magnifications the accommodation-depth is far more important than the focus-depth, the reverse is true in the case of high magnifications.

ART. 120. THE FIELD OF VIEW IN THE CASE OF PROJECTION-SYSTEMS OF FINITE APERTURE.

382. Case of a Single Entrance-Port.

The characteristic effect of a finite aperture in dividing the field of view into separate regions distinguished by the different magnitudes of the apertures of the bundles of rays that have their vertices at points comprised within these regions was remarked by J. PETVZAL[1] in the case of a photographic double-objective in which there was no material diaphragm other than the lens-fastenings themselves. The investigation of this effect in the general case of an optical projection-system of finite aperture will be different according as the field of view is limited by one or by two ports; and hence we shall treat, first, the simpler case of an optical system with a *single entrance-port*.

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[1] The measure of the power of accommodation of the eye is the strength of an infinitely thin lens, placed where the eye is, for which the far-point and near-point are conjugate points.

[2] E. ABBE: Beschreibung eines neuen stereoskopischen Oculars: CARLS *Rep. f. Exp.-Phys.*, xvii (1881), 197–224: also *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 244–272. Section III of this paper treats of the special matters here referred to. See also: E. ABBE: *Journ. Roy. Micr. Soc.* (2), I (1881), 687–689.

[3] S. CZAPSKI: *Theorie der optischen Instrumente nach* ABBE (Breslau, 1893), p. 173.

In the diagram (Fig. 168) the plane of the paper represents a meridian section in the Object-Space; so that, in order to have a complete representation, the entire figure should be imagined as revolved around the optical axis xx. The positions on the axis of the centres of the entrance-pupil and entrance-port are designated by $\mathbf{M}$ and S, respectively. The end-points, on the same side of the axis, of the diameters, in the meridian plane of the figure, of the entrance-pupil and entrance-port are designated by D and T, respectively. Finally, the point M designates the point where the optical axis crosses the focus-plane σ.

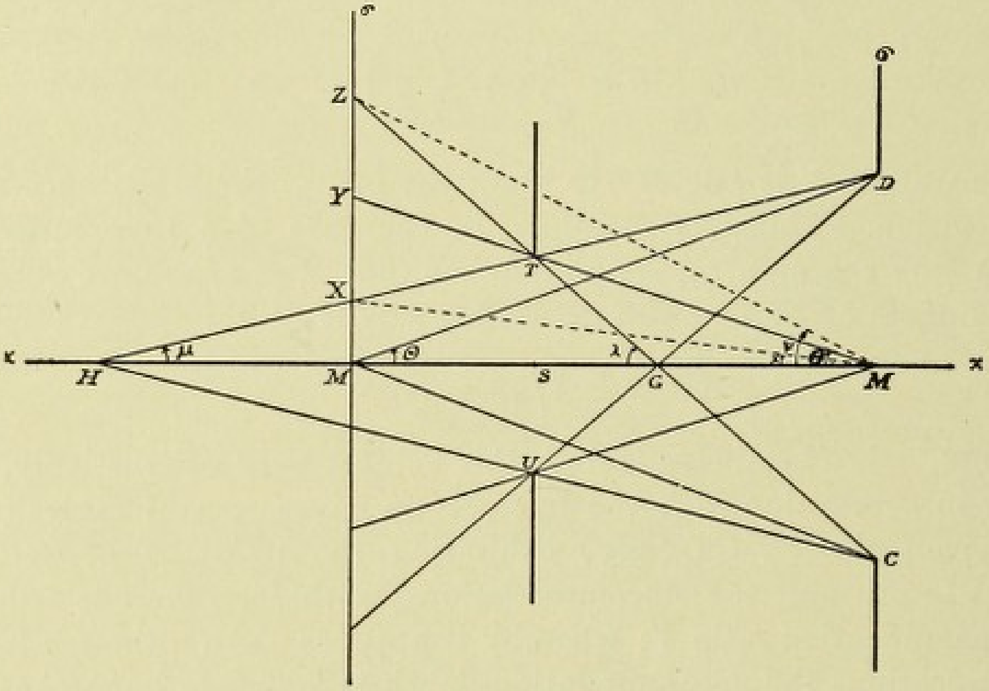

FIG. 168.

FIELD OF VIEW OF OBJECT IN CASE OF PROJECTION-SYSTEM OF FINITE APERTURE WITH A SINGLE ENTRANCE-PORT.

$\mathbf{M}M = \xi$, $\mathbf{M}S = c$, $\mathbf{M}D = p$, $ST = c$, $\angle S\mathbf{M}T = \angle M\mathbf{M}Y = \Theta$, $\angle SHT = \angle MHX = \mu$, $\angle SGT = \angle MGZ = \lambda$, $\angle M\mathbf{M}X = \chi$, $\angle M\mathbf{M}Z = \psi$, $\angle \mathbf{M}MD = \theta$.

The straight line DT joining the end-points, on the same side of the axis, of the diameters of the entrance-pupil and entrance-port meets the optical axis at the point designated by H and crosses the focus-plane at the point designated by X. The region of the field of view of the object defined by the circle described in the focus-plane around M as centre with radius equal to MX is distinguished by the fact that within this circular space are contained all the points of the focus-plane σ that are the vertices of cones that have the entire opening of the entrance-pupil as common base; so that no object-ray emanating from a point of this central region of the focus-plane and directed towards a point of the circular opening of the entrance-pupil will be intercepted.

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[1] J. PETZVAL: Bericht ueber dioptrische Untersuchungen: *Sitzungsberichte der math.-naturw. Cl. der kaiserl. Akad. der Wissenschaften* (Wien), xxvi (1857), 33-90. See p. 57.

The straight line $\boldsymbol{MT}$ crosses the focus-plane at a point designated by Y, which, since the entrance-pupil, in consequence of its definition (§ 361), must subtend at M a smaller angle than is subtended there by the entrance-port, will lie always on the same side of the optical axis as the point X and at a distance MY greater than MX. The annular region of the field of view comprised between the circumferences of the two concentric circles described around M as centre with radii equal to MX and MY contains all points which, regarded as object-points, are in a position to utilize one half or more of the total aperture of the entrance-pupil. Not more than half of the rays of a bundle of rays emitted from an object-point in this annular region of the focus-plane and directed towards all the points of the entrance-pupil will be intercepted, and in general less than half.

Finally, the straight line joining the extremity T of the diameter of the entrance-port with the opposite extremity of the diameter of the entrance-pupil will determine by its intersection with the focus-plane σ a third point Z, also on the same side of the axis as the points X and Y, but the most distant one of the three, which marks the extreme limit on that side of the axis of the field of view. More than half of the rays emitted by an object-point lying within this outside annular space of the focus-plane that are directed towards all the points of the entrance-pupil will be cut off; and a point lying in the focus-plane at a distance from the axis greater than MZ can send through the system no ray at all.

The $\angle MHX$ is called by VON ROHR[1] the *vignette-angle*. Employing symbols as follows:

$$\angle MHX = \angle SHT = \mu, \quad \boldsymbol{MM} = \xi, \quad \boldsymbol{MS} = c, \quad \boldsymbol{MD} = p, \quad ST = q,$$

we obtain from the figure:

$$\tan\mu = -\frac{p-q}{c} = -\frac{p-MX}{\xi};$$

---

[1] M. VON ROHR: *Die Theorie der optischen Instrumente*, Bd. I (Berlin, 1904), p .485.

whence also we find the following expression for the radius of the central region of the field of view:

$$MX = p - \frac{p-q}{c}\cdot\xi. \tag{466}$$

The abscissa of the point H with respect to the centre $\boldsymbol{M}$ of the entrance-pupil is:

$$\boldsymbol{M}H = -\frac{p}{\tan\mu} = \frac{pc}{p-q}. \tag{467}$$

From the figure also we obtain the following relations:

$$\begin{aligned}\tan\angle M\boldsymbol{M}X = \frac{MX}{\boldsymbol{M}M} = \frac{\boldsymbol{M}D}{\boldsymbol{M}M}\cdot\frac{MH}{\boldsymbol{M}H} = \frac{\boldsymbol{M}D}{\boldsymbol{M}M} - \frac{\boldsymbol{M}D}{\boldsymbol{M}H} \\ = -\tan\angle\boldsymbol{M}MD + \tan\angle MHX.\end{aligned}$$

The $\angle\boldsymbol{M}MD = \Theta$ is the aperture-angle, and if we put $\angle M\boldsymbol{M}X = \chi$, the result just obtained may be written as follows:

$$\tan\chi = \tan\mu - \tan\Theta; \tag{468}$$

and hence the tangent of the angle χ subtended at the centre $\boldsymbol{M}$ of the entrance-pupil by the radius MX of the central region of the field of view is equal to the algebraic difference of the tangents of the vignette-angle μ and the aperture-angle Θ. In terms of the given linear magnitudes, we can write also:

$$\tan\chi = -\frac{p-q}{c} + \frac{p}{\xi}. \tag{469}$$

If G designates the position of the point where the straight line TZ crosses the optical axis, and if we put $\angle SGT = \angle MGZ = \lambda$, we obtain from the figure exactly as above:

$$\tan\lambda = \frac{p+q}{c} = \frac{p+MZ}{\xi},$$

and hence for the radius of the entire field of view we find:

$$MZ = \frac{p+q}{c}\xi - p. \tag{470}$$

The abscissa of the point G with respect to the centre $\boldsymbol{M}$ of the entrance-pupil is:

$$\boldsymbol{M}G = \frac{p}{\tan\lambda} = \frac{pc}{p+q}. \tag{471}$$

Moreover, from the figure:

$$\tan\angle M\mathbf{M}Z = \frac{MZ}{M\mathbf{M}} = \frac{\mathbf{M}D}{M\mathbf{M}}\cdot\frac{MG}{\mathbf{M}G} = \frac{\mathbf{M}D}{\mathbf{M}G} - \frac{\mathbf{M}D}{M\mathbf{M}}$$
$$= \tan\angle SGT + \tan\angle M\mathbf{M}D.$$

If we put $\angle M\mathbf{M}Z = \psi$, this result may be written as follows:

$$\tan\psi = \tan\lambda + \tan\Theta. \qquad (472)$$

Hence, the tangent of the angle ψ subtended at the centre $\mathbf{M}$ by the radius MZ of the entire extent of the field of view is equal to the algebraic sum of the tangent of the angle λ subtended at G by the same radius and the tangent of the aperture-angle Θ. In terms of the given linear magnitudes, we can write also:

$$\tan\psi = -\frac{p}{\xi} + \frac{p+q}{c}. \qquad (473)$$

In the case of an object-point in the focus-plane σ whose chief ray has a slope-angle θ greater than the angle denoted by χ, a part of the bundle of object-rays will be intercepted at the entrance-port. The chief rays will be absent from the bundles of effective rays that come from the object-points in the focus-plane that are farther from the axis than the point designated by Y.

Of all possible object-rays that pass unimpeded through the centre $\mathbf{M}$ of the entrance-pupil, those (such as YT) that graze the "rim" of the entrance-port have the greatest slopes, viz.:

$$\angle M\mathbf{M}Y = \angle S\mathbf{M}T = \Theta,$$

where Θ, in the case of an infinitely narrow aperture, is the angular measure of the field of view (§ 370). The three angles at $\mathbf{M}$ denoted by χ, Θ and ψ define the limits of the three parts of the field of view of the object.

In connection with the case of an optical system of finite aperture with a single entrance-port, one point remains to be particularly mentioned, viz., with respect to the projection-figures on the focus-plane of object-points that lie outside this plane. If the slope-angle θ of the chief ray from an object-point R not in the focus-plane is greater than the angle χ, the projection-figure on the focus-plane will not be a circle but a lune, and the chief ray will not be the representative ray of the bundle, and indeed, if $\angle M\mathbf{M}R > \Theta$, the so-called chief ray will be absent from the bundle of effective rays emitted by the object-point R. Hence, also, in the case of object-points thus situated, it is obviously not correct to consider the points where their chief rays cross the focus-plane as the representative points of their projection-figures, especially since the former may not even lie within the boundaries of the corresponding projection-figures at all.

383. Case of Two Entrance-Ports.

Proceeding now to consider the case of an optical projection-system of finite aperture with *two entrance-ports*, we may regard as typical thereof the case shown in the diagram (Fig. 169), which represents a meridian section of the Object-Space. Let S_1, S_2 designate the centres and T_1, T_2 the extremities above the optical axis xx of the diameters, in the meridian plane of the figure, of the two entrance-ports; and let U_1, U_2 designate the other two ends of these diameters. According to our previous definitions, the centre of the entrance-pupil must lie at the point $\mathbf{M}$ where the straight line T_1U_2 which joins a pair of opposite ends of the diameters U_1T_1, U_2T_2 crosses the optical axis. The point M where the optical axis crosses the focus-plane σ is determined by the intersection of the straight line T_2T_1 with the straight line xx. It is obvious from the figure that the points M, $\mathbf{M}$ are harmonically separated by the points S_1, S_2, so that we have the relation:

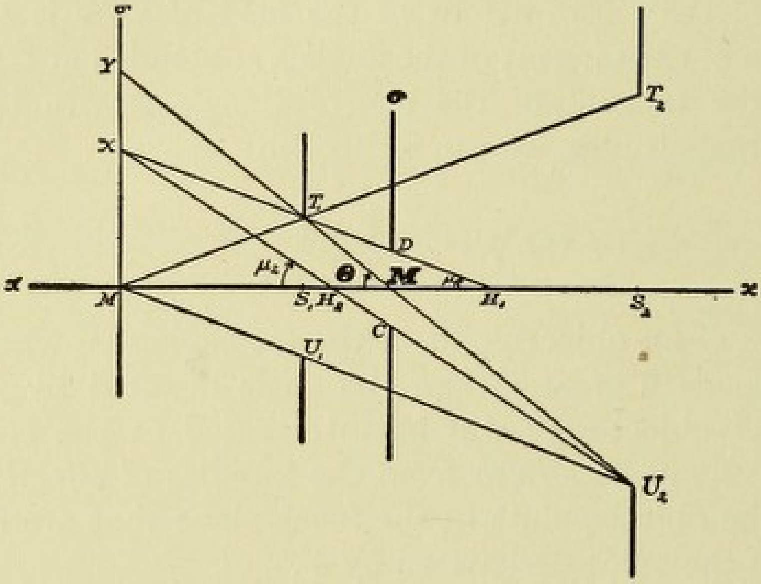

Fig. 169.

Field of View of Object in case of Projection-System of Finite Aperture with Two Entrance-Ports.

$$\mathbf{M}M = \xi,\quad \mathbf{M}S_1 = c_1,\quad \mathbf{M}S_2 = c_2,\quad \mathbf{M}D = p,\quad S_1T_1 = q_1,\quad S_2T_2 = q_2,$$
$$\angle S_1H_1T_1 = \angle MH_1\lambda = \mu_1,\quad \angle S_2H_2T_2 = \angle MH_2X = \mu_2,\quad \angle M\mathbf{M}D = \Theta,\quad \angle M\mathbf{M}X = \chi,\quad \angle M\mathbf{M}Y = \Theta.$$

$$(M\mathbf{M}S_1S_2) = -1. \tag{474}$$

If, therefore, the positions of the two entrance-ports with reference to the entrance-pupil are given, and if we put:

$$\boldsymbol{M}S_1 = c_1, \quad \boldsymbol{M}S_2 = c_2,$$

the position of the focus-plane is determined by the relation:

$$\xi = \frac{2c_1c_2}{c_1 + c_2},$$

where $\xi = \boldsymbol{M}M$ denotes the abscissa of the point M with respect to the centre $\boldsymbol{M}$ of the entrance-pupil.

Let C and D designate the lower and upper ends, respectively, of the diameter of the entrance-pupil which lies in the meridian plane of the figure, and through the upper ends D and T_1 of the diameters CD and U_1T_1 draw the straight line DT_1 crossing the optical axis at the point designated by H_1; and, similarly, through the lower ends of the diameters CD and U_2T_2 draw a straight line CU_2 crossing the optical axis at the point designated by H_2. Let X designate the point of intersection of the straight lines DT_1 and CU_2; we wish to show that this point X will fall in the focus-plane σ. Suppose it does not, and that we draw through X a straight line parallel to CD meeting the optical axis in a point not marked in the diagram which we shall call N, and meeting the straight line $U_2\boldsymbol{M}T_1$ in a point Y. According to this construction, it is plain that the pair of points N, $\boldsymbol{M}$ will be harmonically separated by the pair of points H_2, H_1, and that from the point X the harmonic point-range N, $\boldsymbol{M}$, H_2, H_1 will be projected on to the straight line U_2T_1 in the harmonic point-range Y, $\boldsymbol{M}$, U_2, T_1; which latter projected on to the optical axis from the infinitely distant point of the straight line CD will give:

$$(N\boldsymbol{M}S_2S_1) = -1.$$

But since, as a matter of fact, we know that

$$(M\boldsymbol{M}S_2S_1) = -1,$$

it follows that the point designated by N must be coincident with the point M, and hence the point X must lie in the focus-plane σ, as shown in the figure. Moreover,

$$(M\boldsymbol{M}H_2H_1) = -1. \qquad (475)$$

384. Any object-point lying in the focus-plane within the central region defined by the circle described around M as centre with radius equal to MX will be in a position to send out rays that will go through every point of the opening of the entrance-pupil; whereas any point in the focus-plane at a greater distance from M than X will be in a position to send out rays that will go through some, but not all, of the points of the entrance-pupil, provided its distance from M does not exceed the distance MY; in which latter case it cannot send any rays through the optical system.

If we put

$$\angle S_1H_1T_1 = \angle MH_1X = \mu_1, \quad \angle S_2H_2T_2 = \angle MH_2X = \mu_2,$$
$$\boldsymbol{M}D = p, \quad S_1T_1 = q_1, \quad S_2T_2 = q_2,$$

we obtain from the figure:

$$\tan\mu_1 = \frac{q_1 - p}{c_1} = -\frac{p}{\boldsymbol{M}H_1} = \frac{MX - p}{\xi},$$
$$\tan\mu_2 = -\frac{q_2 - p}{c_2} = \frac{p}{\boldsymbol{M}H_2} = \frac{MX + p}{\xi};$$

whence also we find for the radius of the central region of the field of view:

$$MX = p + \frac{q_1 - p}{c_1}\xi = -p - \frac{q_2 - p}{c_2}\xi; \tag{476}$$

and for the abscissæ, with respect to $\boldsymbol{M}$, of the points H_1, H_2:

$$\boldsymbol{M}_1H = -\frac{pc_1}{q_1 - p}, \quad \boldsymbol{M}H_2 = -\frac{pc_2}{q_2 - p}. \tag{477}$$

Likewise from the figure we obtain also the following relations:

$$\tan\angle M\boldsymbol{M}X = \frac{MX}{\boldsymbol{M}M} = -\frac{\boldsymbol{M}D}{\boldsymbol{M}H_1} + \frac{\boldsymbol{M}D}{\boldsymbol{M}M} = \frac{\boldsymbol{M}D}{\boldsymbol{M}H_2} - \frac{\boldsymbol{M}D}{\boldsymbol{M}M};$$

and hence if $\angle \boldsymbol{M}MD = \theta$, $\angle M\boldsymbol{M}X = \chi$, we have here:

$$\tan\chi = \tan\mu_1 - \tan\theta = \tan\mu_2 + \tan\theta. \tag{478}$$

If we put $\angle M\boldsymbol{M}Y = \Theta$, we obtain evidently also:

$$\tan\Theta = \frac{q_1}{c_1} = \frac{q_2}{c_2};$$

and for the radius of the entire field of view:

$$MY = \frac{q_1}{c_1}\xi = \frac{q_2}{c_2}\xi. \tag{479}$$

Thus, we see that, whereas the position of the point Y is entirely independent of the diameter of the entrance-pupil, this is not true with regard to the position of the point X; for the greater this diameter is, the nearer X will be to the axial point M; and in the limiting case when the end-point D of the diameter of the entrance-pupil lies in the straight line MT_1T_2, the point X will coincide with M.

385. By placing in the focus-plane a circular diaphragm with its centre at M and with an opening of radius equal to MX, all of the field of view outside the central part will be screened off; and then, provided the object lies wholly in the focus-plane, all the points of the object will send through the system cones of rays that fill completely the opening of the entrance-pupil. The same result will be obtained by placing in the screen-plane or image-plane σ' a diaphragm with its centre at the point M' conjugate to M and with an opening of radius $M'X' = Y \cdot MX$, where Y denotes the magnification-ratio of the pair of conjugate transversal planes σ and σ'. Thus, for example, in the case of the astronomical telescope, a diaphragm of this kind is placed in the focal plane of the objective. This simple method is applicable to all cases in which the depth of the object is negligible, especially when the object-distance is prescribed and the points M, M' are the pair of aplanatic points of the optical system (§ 279). But if the points of the object are situated at finite distances from the focus-plane, a stop such as above described will not avail for this purpose.

386. Consider an object-point in the plane of the figure above the optical axis; if it lies to the right of the focus-plane within the angle $MH_2X = \mu_2$ or on the other side of this plane within the angle $MH_1X = \mu_1$, it will be in a position to send through the optical system a cone of rays completely filling the opening of the entrance-pupil. If the object-point lies within the angle subtended at X by the diameter CD of the entrance-pupil, some of the rays of the cone which has the opening of the entrance-pupil for its base will be intercepted at the entrance-port S_2 if the vertex of the cone lies to the right of the focus-plane, and at the entrance-port S_1 if the vertex of the cone lies on the other side of the focus-plane. And, finally, if the object-point lies to the right of the focus-plane and outside the angle μ_1, or on the other side of the focus-plane outside the angle μ_2, all the rays will be intercepted.

INTENSITY OF ILLUMINATION AND BRIGHTNESS.

ART. 121. FUNDAMENTAL LAWS OF RADIATION.

387. Radiation of Point-Source.

Regarding the light-rays as the routes of propagation of light-energy, we may call a bundle of rays a "tube of light";[1] and it is assumed in the theory of radiation that with a steady source of light equal quantities of light-energy traverse every cross-section of such a tube in unit-time. If the source is a radiant point P or a luminous body of such relatively minute dimensions that it may be considered as physiologically a mere point (or centre) of light, the light-tubes will be cones with their vertices at the point-source. The quantity of light radiated in a given time from a steady source may be expressed generally as the product of two factors, one of which has to do with the purely geometrical relations, whereas the other depends on the physical nature and condition of the radiating body. Thus, in the simplest case, when we have a point-source at the point P, the quantity of light which in unit-time "flows" through any cross-section of an elementary tube of light may be represented as follows:

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[1] See P. Drude: *Lehrbuch der Optik* (Leipzig, 1900), p. 72. See also P. G. Tait: *Light* (Edinburgh, 1889), Chapter V.

$$dL = C \cdot d\omega, \tag{480}$$

where $d\omega$ denotes the magnitude of the solid angle of the narrow cone of rays emanating from P, and where C denotes a certain magnitude called the "*candle-power*" of the point-source in the direction of the axis of the cone. If around P as centre a sphere of unit-radius is described, the quantity of light that falls on a unit area of this sphere will be numerically equal to the factor here denoted by C. In general, the value of C will vary with the directions of the light-rays; but if we may assume that the point-source radiates light-energy at approximately the same rate in all directions, the total quantity of light-energy per second that traverses any closed surface surrounding the point P will be equal to $4\pi C$.

If P' designates the position of a point within the elementary conical light-tube of solid angle $d\omega$ which lies on a surface σ' at a distance from the radiant point P denoted by $r = PP'$, and if $d\sigma'$ denotes the area of the surface-element that is cut out of the surface σ' by the cone, and, finally, if φ' denotes the acute angle between the normal to the surface σ' at the point P' and the straight line PP', then

$$r^2 \cdot d\omega = d\sigma' \cdot \cos\varphi';$$

and accordingly we can write:

$$dL = C \cdot d\omega = C \cdot \frac{d\sigma' \cdot \cos\varphi'}{r^2}, \tag{481}$$

where dL denotes the quantity of light emanating from P that falls every second on the surface-element $d\sigma'$. The quantity of light-energy which is received by unit-area of the illuminated surface in unit-time is called the *intensity of illumination* of the surface σ' at the point P'; and, since this magnitude is defined by the equation

$$\frac{dL}{d\sigma'} = C \cdot \frac{\cos \varphi'}{r^2}, \tag{482}$$

we see that the intensity of illumination is inversely proportional to the square of the distance from the point-source and directly proportional to the cosine of the angle of incidence and to the candle-power of the source in the given direction.

388. Radiation of a Luminous Surface-Element.

If the light-source at P must be regarded as a luminous element of surface ($d\sigma$) rather than as a mathematical point, the quantity of light-energy dL that is emitted in a given direction in the unit of time will depend not only on the magnitude of $d\sigma$ but also on the angle of emission (φ) between the normal to $d\sigma$ at P and the given direction PP'. Thus, according to LAMBERT's Law, the specific energy of the radiation of the luminous surface-element $d\sigma$ in the direction PP' will be expressed by the formula:

$$C = i \cdot d\sigma \cdot \cos \varphi, \tag{483}$$

where the co-efficient i denotes a magnitude depending on the physical nature of the light-source (for example, its temperature, radiating power, etc.) which is called the *specific intensity* or the *intensity of radiation* of the luminous surface σ at the point P. The apparent uniformity of the brightness of the sun's disc is in agreement with this "cosine-law". Thus, near the margin of the sun's disc, areas which appear to be of the same size as areas nearer the centre, but which in reality are larger than their oblique projections, do not radiate any more energy than the smaller but more central areas of the same apparent size.

Hence, according to the so-called "cosine-law of emission", the quantity of light-energy radiated per unit of time from the luminous surface-element $d\sigma$ to the illuminated element $d\sigma'$ in the direction PP' is:

$$dL = i \cdot \frac{d\sigma \cdot d\sigma' \cdot \cos \varphi \cdot \cos \varphi'}{r^2}. \tag{484}$$

By means of this fundamental formula of photometry, due originally to LAMBERT,[1] the factor denoted by i may also be defined as the quantity of light which in the unit of time is radiated from a unit-area of the radiating surface to another unit-area at unit-distance from it, when the line PP' is a common normal to the radiating surface at P and to the illuminated surface at P'. As a matter of fact, it is found by experiment that the specific intensity i varies with the angle of emission φ and according to a peculiar law for each different substance; but in the following discussion it will be simpler to disregard this variation and to assume therefore that the value of i is independent of the angle φ.

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[1] J. H. LAMBERT: *Photometria sive de mensura et gradibus luminis colorum et umbrae* (Augsburg, 1760). See also German translation by E. ANDING in Nos. 31–33 of OSTWALD's "Klassiker der exakten Wissenschaften" (Leipzig, 1892). Also, see A. BEER: *Grundriss des photometrischen Calcueles* (Braunschweig, 1854).

The symmetry of the expression on the right-hand side of the above equation cannot fail to be remarked. Thus, for example, the quantity of light conveyed from $d\sigma$ to $d\sigma'$ in a given time is the same as would be transmitted in this same time from $d\sigma'$ to $d\sigma$ in case the rôles of the two surfaces were interchanged, so that $d\sigma'$ was the radiating element of specific intensity equal to i and $d\sigma$ was the illuminated element.

Since

$$d\omega = \frac{d\sigma' \cdot \cos\varphi'}{r^2},$$

and since, also, if $d\omega'$ denotes the solid angle subtended at P' by the radiating surface-element $d\sigma$,

$$d\omega' = \frac{d\sigma \cdot \cos\varphi}{r^2},$$

formula (484) may be written likewise in either of the two following forms:

$$dL = i \cdot d\sigma \cdot \cos\varphi \cdot d\omega = i \cdot d\sigma' \cdot \cos\varphi' \cdot d\omega'. \tag{485}$$

389. Equivalent Light-Source. The *intensity of illumination* at P' due to the radiating element $d\sigma$ at P, viz.,

$$\frac{dL}{d\sigma'} = i \cdot \cos\varphi' \cdot d\omega', \tag{486}$$

is proportional to the specific intensity (i) of the source and also to the solid angle ($d\omega'$) subtended at P' by the radiating surface-element ($d\sigma$) and the cosine of the angle of incidence (φ') of the rays. With respect to the illumination at P', the most important deduction to be made here is that, so far as the resultant effect at P' is concerned, the surface-element $d\sigma$ may be supposed to be replaced by its central projection from P' on to any other surface in the same optical medium, provided we ascribe the same specific intensity i to the corresponding points of the projection-surface.[1] Accordingly, a fictitious source of light, or rather an imaginary distribution of the specific intensity, can be thus substituted in place of the actual distribution so as to have precisely the same effect at a prescribed point P'. However, this so-called equivalent surface-distribution of the specific intensity—or "*equivalent light-source*"—will, in general, produce a different effect from that produced by the actual light-source at any point other than the given point P'.

ART. 122. INTENSITY OF RADIATION OF OPTICAL IMAGES.

390. Optical System of Infinitely Narrow Aperture (Paraxial Rays). Let M, M' designate the positions of a pair of conjugate axial points of a centered system of spherical surfaces, and let us suppose, at first, that the aperture of the system is infinitely narrow, so that only the so-called paraxial rays emanating from the luminous point-source M can traverse the system. To the bundle of paraxial rays in the Object-Space of solid angle $d\omega$ corresponds also a bundle of paraxial rays in the Image-Space of solid angle $d\omega'$; and if C denotes the candle-power of the point-source, the quantity of light radiated from M in unit-time will be $dL = C \cdot d\omega$; and, similarly, the quantity of light radiated in the same time from the conjugate image-point M' will be $dL' = C' \cdot d\omega'$, where C' denotes the candle-power of the image-point M' regarded as a source of light in the Image-Space. Moreover, for the sake of simplicity, let us assume here that no light-energy is "lost" either by absorption in traversing the various media or by undesirable reflexions at the spherical surfaces; and although this assumption is notoriously contrary to the fact, it will not materially affect the conclusions which we have here in view. Accordingly, putting $dL' = dL$, we obtain therefore:

$$C \cdot d\omega = C' \cdot d\omega'.$$

The following relation may easily be deduced:

$$\frac{d\omega'}{d\omega} = \prod_{k=1}^{k=m} \left(\frac{u_k}{u'_k} \right)^2,$$

where u_k, u'_k denote the abscissæ of the points where the rays cross the optical axis before and after refraction, respectively, at the kth surface of the centered system of m spherical surfaces. If, therefore,

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[1] See E. ABBE: Ueber die Bestimmung der Lichtstaerke optischer Instrumente: *Jen. Zft. f. Med. u. Natw.*, vi (1871), 263–291. Also, *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 14–44.

Y denotes the lateral magnification of the system with respect to the pair of conjugate axial points M, M', and if also n, n' denote the indices of refraction of the media of the Object-Space and Image-Space, respectively, we obtain by the employment of formula (93):

$$\frac{d\omega'}{d\omega} = \frac{1}{Y^2} \cdot \frac{n^2}{n'^2};$$

and, hence:

$$\frac{C'}{C} = Y^2 \cdot \frac{n'^2}{n^2}; \tag{487}$$

whereby, knowing the candle-power (C) of the point-source on the axis of the optical system, and knowing also the constants of the system, we are enabled to determine the corresponding candle-power (C') of the image-point M'.

If, instead of a point-source at the axial point M, we have a luminous surface-element $d\sigma$ at right angles to the optical axis at M, the image thereof will be a surface-element $d\sigma'$ at right angles to the optical axis at the point M', of such dimensions that

$$d\sigma' = Y^2 \cdot d\sigma.$$

Hence, since here we have

$$dL = i \cdot d\sigma \cdot d\omega = dL' = i' \cdot d\sigma' \cdot d\omega',$$

where i, i' denote the specific intensities in the direction of the axis of the radiating elements $d\sigma$, $d\sigma'$, respectively, we obtain in this case the following striking relation:

$$\frac{i'}{i} = \frac{n'^2}{n^2}.$$

391. Optical System of Finite Aperture.

Finally, let us now proceed to the more general case and assume that the aperture of the optical system is finite; and let us denote by i the specific intensity of radiation, in a direction defined by the slope-angle θ, of a luminous surface-element $d\sigma$ placed at right angles to the optical axis at the point M. The quantity of light radiated in unit-time from the element $d\sigma$ to an elementary annular ring of the entrance-pupil whose inner and outer radii subtend at the axial object-point M angles denoted by θ and $\theta + d\theta$, respectively, may be easily calculated from the fundamental formula (484) and will be found to be:

$$dL = 2\pi i \cdot d\sigma \cdot \sin\theta \cdot d\,(\sin\theta).$$

Employing the same symbols with primes to denote the corresponding magnitudes in the Image-Space, we shall find also a precisely analogous expression for the quantity of light that is radiated per unit of time from the image-element $d\sigma'$ to the corresponding elementary annular ring of the exit-pupil, viz.:

$$dL' = 2\pi i' \cdot d\sigma' \cdot \sin\theta' \cdot d\,(\sin\theta').$$

Now if $d\sigma'$ is to be a correct image of the object-element $d\sigma$, it is necessary to suppose that M, M' are an aplanatic pair of points, so that the Sine-Condition is satisfied, whereby we must have (§ 277):

$$n \cdot \sin\theta = n' \cdot Y \cdot \sin\theta'.$$

Introducing this condition, and employing here also the relation

$$d\sigma' = Y^2 \cdot d\sigma,$$

and, finally, assuming, as before, that $dL' = dL$, we derive again the same relation as above, viz.:

$$\frac{i'}{i} = \frac{n'^2}{n^2}. \tag{488}$$

Accordingly, *no matter how the specific intensities of radiation of object and image may vary for different angles of emission, their ratio is the same for every pair of values of θ and θ'.* This constant ratio depends only on the indices of refraction of the media in which the object and image are situated; and *the specific intensity of radiation (i') of any element of the image in a given direction (θ') is equal always to $(n'/n)^2$ times that of the corresponding object-element in the conjugate direction (θ).*[1]

392. In deriving the above results, it was assumed that there were no losses of light by absorption, reflexion, etc., so that we could put $dL' = dL$. It would have been more correct to have written:

$$dL' = (1 - \eta)dL,$$

where η denotes the fraction of the original quantity of light that is dissipated in its passage through the system, and is a function of the angle of emission (θ), which may be determined in any given special case. Under these circumstances, formula (488) would be modified as follows:

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[1] This result is identical with KIRCHHOFF's well-known law of radiation. See G. KIRCHHOFF: Ueber das Verhaeltniss zwischen dem Emissionsvermoegen und dem Absorptionsvermoegen der Koerper für Waerme und Licht. POGG. *Ann.*, cix (1860), 275–301. Also, R. CLAUSIUS: Ueber die Concentration von Waerme- und Lichtstrahlen und die Grenzen ihrer Wirkung. POGG. *Ann.*, cxxi (1864); also, BROWNE's English translation of CLAUSIUS's *Mechanical Theory of Heat* (London, 1879), Chapter XII.

Starting from KIRCHHOFF's law of radiation, HELMHOLTZ deduced the Sine-Law; see H. HELMHOLTZ: Die theoretische Grenze für die Leistungsfaehigkeit der Mikroskope: POGG. *Ann. Jubelband*, 1874, 557–584.

$$\frac{i'}{i} = (1 - \eta)\frac{n'^2}{n^2}; \tag{489}$$

which shows also that the ratio i'/i is in reality a function of the angle θ.[1]

In nearly all actual optical instruments the first and last media are both air ($n = n' = 1$); even in the so-called "immersion-systems" the source is not the object immersed in the fluid, inasmuch as the object is illuminated from without. The case when $n' > n$ is hardly realizable. Thus, under the most favourable conditions, the specific intensity of radiation from a definite part of the image in a given direction will always be less than the specific intensity of radiation from the corresponding part of the object in the conjugate direction. For example, the intensity of radiation of the sun's image at the focus of a convex lens can never be greater than that of the sun itself, although the intensity of illumination of a screen placed at the focus of the glass may be much greater with the lens than without it.

393. The Illumination in the Image-Space.

The image $M'Q'$ of a luminous object MQ may be regarded as the source of all the illumination in the Image-Space; and in case we wish to ascertain the intensity of the illumination produced at any point R' of the Image-Space by an element of the image at P', we have merely to trace backwards through the optical system the path of the image-ray $P'R'$ and thereby determine the point P of the object that is conjugate to the image-point P'. The specific intensity of the radiation from P' in the direction $P'R'$ is $(n'/n)^2$ times that from the object-point P in the conjugate direction in the Object-Space; provided we assume that none of the light is dissipated in its passage through the system.

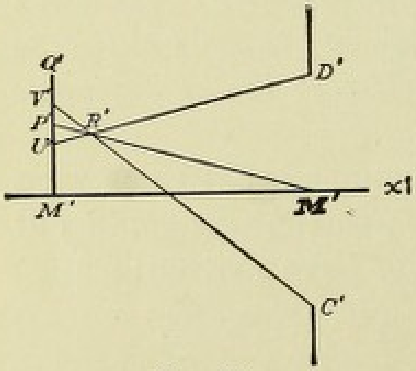

Fig. 170.

Exit-Pupil as Equivalent Luminous Surface.

The part of the image $M'Q'$ (Fig. 170) that is effective in producing illumination at a point R' of the Image-Space is easily found by projecting the exit-pupil $C'D'$ on to the image-plane σ'; thus, in the diagram, $U'V'$ represents the effective part of the image with respect to the illumination at the point R'. In place of the portion of the image $U'V'$, we may substitute an equivalent distribution of light (§ 389) by considering the specific intensity of the parts of the image comprised between U' and V' as localized at the corresponding parts of the exit-pupil; and this distribution of light supposed to be spread over the exit-pupil would produce exactly the same effect at R' as is produced there by the image of the luminous object. This ingenious method, due to ABBE,[1] enables us to determine the intensity of illumination at any point of the image itself. For example, the nearer the point R' is to the point P' of the image, the smaller will be the circular space around P' that is obtained by projecting the exit-pupil on to the plane of the image; and, finally, when the point R' coincides with P', so that the exit-pupil is projected on to the image-plane in the point P' itself, the intensity of the illumination at P' can be found by regarding the illumination there as due to a distribution of light over the exit-pupil of the same specific intensity of radiation as that of the point P', viz., $(n'/n)^2 i$, where i denotes the specific intensity of radiation in any given direction (θ) of the object-point P conjugate to P'.

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[1] See S. Czapski: *Theorie der optischen Instrumente nach* Abbe (Breslau, 1893), p. 179.

ART. 123. BRIGHTNESS OF OPTICAL IMAGES.

394. Brightness of a Luminous Object.

In connection with the definition of the objective intensity of illumination (§ 389) at a given place of an illuminated surface, we can derive also an idea of what is meant by the *Brightness* of the source as seen by an eye situated at the place in question. The brightness of an element $d\sigma$ of a radiating surface is defined as the quantity of light-energy which in the unit of time falls on unit-area of the image $d\sigma'$ that is formed on the retina of the eye; in other words, it is the intensity of illumination of the element of the retina-surface that is affected by the given element of the luminous body. Thus, if dL denotes the quantity of light which is radiated per unit of time from the element $d\sigma$ into the eye, the brightness of this element is defined by the equation:

$$B = \frac{dL}{d\sigma'}. \qquad (490)$$

If we assume that there is no loss of light in traversing the optical media of the eye, then

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[1] E. ABBE: Ueber die Bestimmung der Lichtstaerke optischer Instrumente. *Jen. Zft. f. Med. u. Natw.*, vi (1871), 263–291. Also *Gesammelte Abhandlungen*, Bd. I (Jena, 1904), 14–44.

$$2\pi i' \cdot d\sigma' \cdot \sin\theta' \cdot d\,(\sin\theta')$$

will be the quantity of light that is radiated per unit of time across an elementary annular ring of the exit-pupil of the eye, where θ' denotes the angle subtended at the retina by the inner radius of this ring and i' denotes the specific intensity of the image $d\sigma'$ on the retina; and hence the total quantity of light that enters the pupil will be

$$dL = \pi i' \cdot d\sigma' \cdot \sin^2\Theta_0',$$

where Θ_0' denotes the angle subtended at the image on the retina by the radius of the exit-pupil of the eye, which usually does not exceed about 5°. If, therefore, the object is viewed by the unaided eye, we find for the so-called *natural brightness* (B_0) of the luminous surface-element $d\sigma$ (supposed to be situated in air, so that $n = 1$):

$$B_0 = \frac{dL}{d\sigma'} = \pi \cdot n'^2 \cdot i \cdot \sin^2\Theta_0', \tag{491}$$

where i denotes the specific intensity of radiation of the source, and n' denotes the refractive index of the vitreous humour of the eye. It follows immediately from this expression that the natural brightness of a uniformly radiating surface depends only on the intensity of radiation of the light-source and is entirely independent of the distance of the luminous object from the eye; as is found to be practically the case.

395. In the next place, let us suppose that this same object is viewed through an optical instrument by an eye placed at the exit-pupil of the instrument. Everything is the same as before, except that now, instead of the mere optical system of the eye, we have a compound optical system formed by the combination of the eye with the optical instrument. If we disregard all losses of light by reflexion and absorption, and assume, as before, that the luminous object is in air, the brightness B of the optical image as seen through the instrument will be equal to the natural brightness B_0 of the object as viewed by the naked eye, provided the exit-pupil of the eye is smaller than that of the instrument. But if, on the other hand, the diameter of the exit-pupil of the instrument is smaller than that of the eye, the aperture-angle will be an angle $\Theta' < \Theta_0'$, so that in this case we shall have:

$$B : B_0 = \sin^2\Theta' : \sin^2\Theta_0'.$$

Since the angles Θ_0', Θ' are so small that we may substitute the tangents of these angles in place of their sines, and since, moreover, the exit-pupil of the eye coincides very nearly with the eye-pupil (or iris), we obtain:

$$B : B_0 = p'^2 : p_0^2, \tag{492}$$

where p_0, p' denote the radii of the iris-opening and exit-pupil of the instrument, respectively; so that the brightness of the image compared with the natural brightness of the object is diminished in the ratio of the size of the exit-pupil of the instrument to the size of the eye-pupil. It is, therefore, impossible by means of any optical instrument to increase the natural brightness of an object as seen by the unaided eye. Thus, the only function of an optical instrument is by means of a light-source either of small dimensions or very far away to produce an effect equal to that which could be produced without the instrument only by a larger or nearer source of light radiating with equal specific intensity.[1]

396. Brightness of a Point-Source.

If the luminous object is so small or so far away that it has no sensible apparent size, the definition of brightness given above (§ 394) ceases to have any meaning; for the image on the retina of the eye will in this case be itself a mere point without appreciable area. If, therefore, the source of illumination is a point, for example, a fixed star, the brightness is defined as equal or proportional to the quantity of light which comes to us from it. Thus, when we speak of a star of the "first magnitude", this expression refers merely to the amount of light we receive from it and has nothing to do with the size of the star.

If in formula (481) we put $d\sigma' = \pi p_0^2$ (where p_0 denotes the radius of the eye-pupil) and $\cos\varphi' = 1$ (since the rays are supposed to fall normally on the retina when the eye is directed towards the point-source), we obtain:

$$B = C\pi\frac{p_0^2}{r^2}. \tag{493}$$

Hence, *the brightness of an object which appears like a point is inversely proportional to the square of its distance from the eye, and directly proportional to the size of the eye-pupil.*

Thus, stars which are invisible to the naked eye may be brought to view by the aid of a telescope, whereby the eye receives a greater quantity of light from the star than before, so that the brightness (in this latter sense of the term) is increased; whereas, on the other hand, the brightness of the background of the sky (using the word "brightness" in its original sense, as defined in § 394) will be diminished. This is the reason why a powerful telescope, of large aperture and great magnifying power, may enable an observer to view the stars even in the noon-day glare.

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[1] Lord RAYLEIGH, in his brilliant article on *Optics* in the ninth edition of the *Encyclopaedia Britannica*, has pointed out that "the general law that the apparent brightness depends only on the area of the pupil filled with light" was stated and demonstrated by ROBERT SMITH. See SMITH's *Optics* (Cambridge, 1738), Vol. I, Sections 255 and 261.

APPENDIX.

EXPLANATIONS OF LETTERS, SYMBOLS, ETC.

The meanings of the principal letters and symbols both in the text and in the diagrams are here set forth as briefly as possible; but such uses as are occasional or merely incidental are generally not noted at all. In consulting these tables, it is important to bear in mind this last statement.

I. DESIGNATIONS OF POINTS IN THE DIAGRAMS.

As a rule (but not without exception), the positions of the points in the diagrams are designated by Latin capital letters. The most important uses of these letters are explained below.

A

1. A, A' are used to designate the primary and secondary *principal points*, respectively, of two collinear space-systems; see Fig. 92. Similarly, as in Fig. 99, A_k, A'_k designate the principal points of the kth component of a compound optical system.

In Chap. XIII, A, A' and $\bar{A}$, $\bar{A}'$ designate the positions on the optical axis of the two pairs of principal points of the system for rays of light of wave-lengths λ and $\bar{\lambda}$, respectively.

In the case of two *centrally collinear plane-fields*, A designates the position of the point of intersection with the axis of collineation (y) of the self-corresponding ray (x, x') that meets this axis at right angles.

2. Especially, the letter A is used to designate the *vertex* of a spherical refracting (or reflecting) surface. Similarly, A may be used to designate the position of the foot of the perpendicular let fall on to a *plane* refracting (or reflecting) surface from a point on the incident ray regarded as object-point, as in Fig. 8.

The letter A designates the *optical centre* of an Infinitely Thin Lens.

The vertex of the kth surface of a centered system of spherical surfaces is designated by A_k; also, the optical centre of the kth lens of a centered system of Infinitely Thin Lenses.

3. In Chap. IX, in the determination of the path of a ray *refracted obliquely* at a spherical surface, A_g, A_i are used to designate the points of intersection with the surface of the radii drawn through the points designated by G (14) and I (18), respectively (Fig. 122). In

Chap. X, in the case of a ray refracted obliquely through a centered system of spherical surfaces, $A_{g,k}$, $A_{i,k}$ have the same meanings as above with respect to the kth surface.

B, $\boldsymbol{B}$

4. B, B' are used to designate the points of intersection of a pair of conjugate rays with the *principal planes* of two collinear space-systems. In Fig. 99, for example, B_k, B'_k designate the points where a meridian ray crosses the principal planes of the kth component of a compound optical system.

In particular, B designates the point of intersection of a ray with the *axis of collineation* (y) of two centrally collinear plane-fields.

5. Especially, B designates the position on the refracting (or reflecting) surface of the *incidence-point* of a ray. In the case of a centered system of spherical surfaces or a prism-system, B_k designates the incidence-point of the ray at the kth surface.

In Chap. XIII, B_k, $\overline{B}_k$ designate the incidence-points at the kth spherical surface of rays of light of wave-lengths λ, $\overline{\lambda}$, respectively, whose paths in the Object-Space are identical.

If we are concerned with a pair of rays from two different sources, whose paths lie in the plane of a principal section, their incidence-points may be designated by B and $\boldsymbol{B}$ (or by B_k and $\boldsymbol{B}_k$), as, for example, in Chap. VIII.

Usually, however, $\boldsymbol{B}$ or $\boldsymbol{B}_k$ designates the position of the incidence-point of the *chief ray* of a bundle.

C

6. C is used primarily to designate the *centre* of a spherical refracting or reflecting surface. The centre of the kth surface of a centered system of spherical surfaces is designated by C_k.

This letter is used also to designate the *centre of collineation* of two centrally collinear plane-fields, as in Fig. 66.

7. In Chap. XIV, C, C' are used to designate corresponding extremities of conjugate diameters of the *entrance-pupil* and *exit-pupil*, respectively, of an optical system. C designates the lower extremity of the diameter, in the meridian plane, of the entrance-pupil.

D, $\boldsymbol{D}$

8. D designates the *foot of the perpendicular* BD let fall from the incidence-point B on to the optical axis; in a centered system of spherical surfaces, D_k designates the foot of the perpendicular let fall on to the optical axis from the point B_k where the ray meets the kth surface.

The foot of the perpendicular let fall on to the optical axis from the point **B** (see 5) is designated by **D**, as in Fig. 140.

The feet of the perpendiculars let fall on to the optical axis from the points B_k, $\overline{B}_k$ (see 5) are designated by D_k, $\overline{D}_k$, respectively.

9. D, D' are used (in Chap. XIV) to designate corresponding extremities of conjugate diameters of the *entrance-pupil* and *exit-pupil*, respectively, of an optical system. Generally, D designates the upper extremity of the diameter, in the meridian plane, of the entrance-pupil (see 7).

10. In certain of the *prism-diagrams* of Chap. IV, D designates the point of intersection of the incident and emergent rays.

E

11. E, E' are used to designate the infinitely distant point of the optical axis (x) in the Object-Space and its conjugate point in the Image-Space, respectively. E' designates, therefore, the *secondary focal point* of the optical system.

E', $\overline{E}'$ designate the secondary focal points of the optical system for rays of light of wave-lengths λ, $\overline{\lambda}$, respectively.

E'_k is used to designate the secondary focal point of the kth component of a compound optical system, as in Fig. 99.

F

12. F, F' designate the *primary focal point* in the Object-Space and the infinitely distant point of the optical axis (x') in the Image-Space, respectively.

F, $\overline{F}$ designate the primary focal points of the optical system for rays of light of wave-lengths λ, $\overline{\lambda}$, respectively.

F_k designates the primary focal point of the kth component of a compound optical system, as in Fig. 99.

G

13. G designates the *incidence-point* of a ray in the *meridian* section of an infinitely narrow bundle of rays (Fig. 127).

14. In Chap. IX, G, G' are used to designate the points where an *oblique ray* crosses the plane of the principal section (xy-plane) of the spherical refracting surface, before and after refraction, respectively (Figs. 122 and 123).

H

15. H, H' designate the points where an oblique ray crosses the *central transversal plane* (yz-plane), before and after refraction, respectively, at a spherical surface (Fig. 123). Similarly, H_k, H'_k are used as above stated, with respect to the kth surface of a centered system of spherical surfaces.

In particular, H, H' designate the points where a ray, lying in the principal section of a spherical refracting surface, crosses the *central perpendicular*, before and after refraction, respectively (Fig. 120).

I

16. In certain diagrams in Chapters V, VI and VII, I, I' are used to designate the infinitely distant point of an object-ray s and the "*Flucht*" Point of the conjugate image-ray s', respectively, of two collinear plane-fields. See, for example, Fig. 67.

17. Especially, in Chap. XI, in connection with the theory of the refraction of an *infinitely narrow bundle* of rays at a spherical surface, I, I' designate the infinitely distant point of the range of primary object-points lying on the chief incident ray and the "Flucht" Point of the conjugate range of primary image-points lying on the chief refracted ray, respectively. Thus, I' designates the secondary *focal point* of the collinear plane-fields of the *meridian sections* of the bundles of incident and refracted rays. See Fig. 128.

Similarly, $\bar{I}$, $\bar{I}'$ designate the infinitely distant point of the range of secondary object-points lying on the chief incident ray and the "Flucht" Point of the conjugate range of secondary image-points lying on the chief refracted ray, respectively. Thus, also $\bar{I}'$ designates the secondary *focal point* of the collinear plane-fields of the *sagittal sections* of the bundle of incident and refracted rays. See Fig. 128.

According, therefore, as the chief incident ray is regarded as the base of a range of primary or secondary object-points, the infinitely distant point of this ray is designated by I or $\bar{I}$.

Moreover, I'_k, $\bar{I}'_k$ designate the focal points of the systems of meridian and sagittal rays, respectively, with respect to a given chief ray refracted at the kth surface of a centered system of spherical surfaces.

18. In Chap. IX, I, I' designate the points where an *oblique ray* crosses the (horizontal) meridian xz-plane, before and after refraction, respectively, at a spherical surface (Fig. 122).

Similarly, in Chap. X, I_k, I'_k (or I_{k+1}) are used in the same way as above, with respect to the kth surface of a centered system of spherical refracting surfaces.

J

19. In certain diagrams in Chapters V, VI and VII, J, J' designate the "*Flucht*" Point of an object-ray s and the infinitely distant point of the conjugate image-ray s', respectively, of two collinear plane-fields. See, for example, Fig. 67.

20. In Chap. XI, in connection with the theory of the refraction of an *infinitely narrow bundle of rays* at a spherical surface, J, J' designate the "*Flucht*" Point of the range of primary object-points lying on the chief incident ray and the infinitely distant point of the corresponding range of primary image-points lying on the chief refracted ray, respectively. Thus, J designates the primary *focal point* of the collinear plane-fields of the *meridian sections* of the bundles of incident and refracted rays (Fig. 128).

Similarly, $\bar{J}$, $\bar{J}'$ designate the "*Flucht*" Point of the range of secondary object-points lying on the chief incident ray and the infinitely distant point of the corresponding range of secondary image-points lying on the chief refracted ray, respectively. Thus, also, $\bar{J}$ designates the primary focal point of the collinear plane-fields of the *sagittal sections* of the bundles of incident and refracted rays (Fig. 128).

According, therefore, as the chief refracted ray is regarded as the base of a range of primary or secondary image-points, the infinitely distant point of this ray is designated by J' or $\bar{J}'$.

Moreover, J_k, $\bar{J}_k$ designate the primary focal points of the systems of meridian and sagittal rays, respectively, with respect to a given chief ray refracted at the kth surface of a centered system of spherical surfaces.

K

21. In Fig. 128, K designates the *centre of perspective* of the range of object-points lying on the chief incident ray and the range of primary image-points lying on the corresponding refracted ray.

22. In Chap. XII, K, $\bar{K}$ and K', $\bar{K}'$ designate the *centres of curvature* of the two *astigmatic image-surfaces*, before and after refraction, respectively, at one of a centered system of spherical surfaces (Figs. 141, 142).

L, $\mathbf{L}$

23. L, L' designate the points where a ray, lying in the principal section of a spherical refracting surface, *crosses the axis*, before and after refraction, respectively (Fig. 120).

$\mathbf{L}$, $\mathbf{L}'$ designate the points where the *chief ray* of a bundle crosses the axis, before and after refraction, respectively, at a spherical surface.

In certain cases, also, L, L' (or $\mathbf{L}$, $\mathbf{L}'$) are used to designate the points where an object-ray and the corresponding image-ray, respectively, cross the optical axis of a centered system of spherical refracting surfaces.

Sometimes L'' is used to designate the point where a second ray emanating from the axial object-point L crosses the axis after emerging from the optical system (Fig. 117).

L'_k (or L_{k+1}) designates the point where a ray lying in the principal section crosses the optical axis after refraction at the kth surface of a centered system of spherical surfaces. If the ray is the *chief ray* of the bundle, the point in question is designated by $\boldsymbol{L}'_k$ (or $\boldsymbol{L}_{k+1}$). The point where the ray crosses the axis in the Object-Space is designated by L_1 (or $\boldsymbol{L}_1$).

M, $\boldsymbol{M}$, $\mathfrak{M}$

24. M, M' designate a pair of *conjugate axial points* of an optical system; especially, a pair of points where a *paraxial ray* crosses the optical axis in the Object-Space and Image-Space, respectively.

In Chap. XIII, M', $\bar{M}'$ and $\mathfrak{M}'$ designate the points where paraxial rays of light of wave-lengths λ, $\bar{\lambda}$ and $\mathfrak{l}$, respectively, all emanating originally from the same axial object-point M, cross the optical axis in the Image-Space.

M'_k (or M_{k+1}) designates the point where a paraxial ray, emanating from the axial object-point M_1, crosses the optical axis of a centered system of spherical surfaces after refraction at the kth surface (see Fig. 71). Here also $\bar{M}'_k$ has a meaning corresponding to that of $\bar{M}'$ above.

25. Especially, M, M' designate the points where the optical axis crosses the transversal *object-plane* (σ) and the conjugate (GAUSSian) image-plane (σ'); or the points where the optical axis crosses the *focus-plane* and the *screen-plane*, respectively (Chap. XIV). See 69. M may be defined as the foot of the perpendicular let fall on to the optical axis from the extra-axial object-point Q; and if Q' designates the GAUSSian image-point corresponding to Q, M' will designate also the foot of the perpendicular let fall on to the optical axis from Q'.

26. $\boldsymbol{M}$, $\boldsymbol{M}'$ designate a second pair of *conjugate axial points*, with respect either to a single spherical surface or a centered system of spherical surfaces. The meanings of $\boldsymbol{M}'$, $\bar{\boldsymbol{M}}'$; $\boldsymbol{M}'_k$ (or $\boldsymbol{M}_{k+1}$); and $\bar{\boldsymbol{M}}'_k$ correspond exactly with the meanings given above (24) of M', $\bar{M}'$; M'_k (or M_{k+1}); and $\bar{M}'_k$, respectively.

27. Especially, $\boldsymbol{M}$, $\boldsymbol{M}'$ designate the positions on the axis of the centres of *entrance-pupil* and *exit-pupil*, respectively, of the optical system. In an optical system of m centered spherical surfaces, the centres of the pupils may be designated by $\boldsymbol{M}_1$ and $\boldsymbol{M}'_m$.

If $\boldsymbol{M}$, $\boldsymbol{M}'$ designate the pupil-centres of the optical system for rays of wave-length λ, $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{M}}'$ may be used to designate the pupil-centres for rays of wave-length $\bar{\lambda}$.

28. In an *Infinitely Thin Lens*, M, M' are used (as in Fig. 75) to designate the points where a paraxial ray crosses the optical axis, before and after passing through the lens. And, especially, in the case of a centered *System of Infinitely Thin Lenses*, M_k, M'_k are used in this way with respect to the kth lens. Exactly, the same statements can be made here with reference to the use of $\mathbf{M}$, $\mathbf{M}'$ and $\mathbf{M}_k$, $\mathbf{M}'_k$.

29. M'' is used in various ways; for example, to designate the point where the focussing screen is crossed by the optical axis, or, as in Fig. 161 (and elsewhere), to designate the centre of the "*blur-circle*" corresponding to an axial object-point M.

N

30. N, N' are used to designate points on the *normal* to a refracting surface at the incidence-point B in the first and second medium, respectively (as in Fig. 5).

31. Especially, N, N' designate the pair of *nodal points* of an optical system (Fig. 92).

O

32. O designates the position on the optical axis of the centre of the *aperture-stop*.

33. In the *prism-diagrams* of Chap. IV, O designates the point of intersection of the normals to the two faces of the prism at the points of entry and exit. In the case of a train of prisms, a numerical subscript indicates the prism to which the letter refers (as in Fig. 45).

34. O is used also to designate the position of the *optical centre* of a thick lens (Fig. 74).

35. In Chapters V, VI and VII, O, O' occur frequently to designate a *pair of conjugate points*.

P, $\mathbf{P}$

36. P, P' designate the point where the object-ray crosses the *transversal object-plane* σ and the point where the corresponding refracted ray crosses the *transversal image-plane* σ', respectively (69).

P'_k (or P_{k+1}) designates the point where a ray crosses the transversal plane σ'_k after refraction at the kth surface of a centered system of spherical surfaces.

P_1 designates the point where the rectilinear path of the ray in the Object-Space crosses the transversal object-plane σ_1; especially, it designates the position of the *object-point* in this plane, and, in general, the same extra-axial object-point as is designated by Q_1 (see 39). In a certain sense (see Chap. XII) the point P'_m may be regarded as the image of the object-point P_1.

37. $\boldsymbol{P}$, $\boldsymbol{P}'$ are used in Chap. XII to designate the point where the object-ray crosses the transversal plane $\boldsymbol{\sigma}$ in the Object-Space and the point where the corresponding image-ray crosses the conjugate plane $\boldsymbol{\sigma}'$ in the Image-Space, respectively (71). The object-ray here mentioned is a ray that goes through the point P (36).

Similarly, also, $\boldsymbol{P}'_k$ (or $\boldsymbol{P}_{k+1}$) designates the point where a ray which in the Object-Space goes through P_1 crosses the transversal plane $\boldsymbol{\sigma}'_k$ after refraction at the kth surface of a centered system of spherical surfaces. The point where the object-ray crosses the first one ($\boldsymbol{\sigma}_1$) of this series of transversal planes is designated by $\boldsymbol{P}_1$ (or $\boldsymbol{Q}_1$).

38. P, P'; Q, Q'; R, R'; S, S' and $\overline{P}$, $\overline{P}'$; $\overline{Q}$, $\overline{Q}'$; $\overline{R}$, $\overline{R}'$; $\overline{S}$, $\overline{S}'$ are used frequently to designate pairs of *corresponding points of projective point-ranges*. Thus, for example, in Chap. XI, P, P' designate a pair of corresponding points of the ranges of primary object-points and image-points lying along the chief incident ray of a narrow bundle of rays and the corresponding refracted ray, respectively; and, similarly, $\overline{P}$, $\overline{P}'$ designate corresponding points of the ranges of secondary object-points and image-points lying along the same chief incident and refracted rays, respectively.

Q, $\boldsymbol{Q}$

39. Q, Q' designate a pair of conjugate points, especially a pair of *extra-axial conjugate points*, of two collinear systems.

In general, Q, Q' designate a pair of points, lying outside the axis of the optical system, which, by GAUSS's Theory, are conjugate to each other, with respect to either a single spherical surface or a centered system of spherical surfaces. Especially, Q, Q' designate a pair of conjugate points lying in the transversal planes σ, σ', respectively.

Q, Q' are used also (Chap. XIV) to designate the centres of the "*blur-circles*" in the focus-plane (σ) and the screen-plane (σ'), respectively.

In case we have to do with rays of light of two *different colours* (as in Chap. XIII), Q', $\overline{Q}'$ designate the points conjugate to Q for rays of light of wave-lengths λ, $\overline{\lambda}$, respectively.

Q'_k (or Q_{k+1}) designates the point where, according to GAUSS's Theory, a ray, emanating originally from the object-point Q_1, crosses the transversal plane σ'_k after refraction at the kth surface of a centered system of spherical surfaces.

40. The point where an object-ray, which goes through the object-point Q (or P), crosses the plane ($\boldsymbol{\sigma}$) of the entrance-pupil is designated by $\boldsymbol{Q}$ or $\boldsymbol{P}$; and the point in the plane ($\boldsymbol{\sigma}'$) of the exit-pupil, which, by GAUSS's theory, is conjugate to $\boldsymbol{Q}$ (or $\boldsymbol{P}$), is designated by $\boldsymbol{Q}'$.

In a centered system of spherical surfaces, $\boldsymbol{Q}_1$ (or $\boldsymbol{P}_1$) designates the point where an object-ray, containing the object-point $\boldsymbol{Q}_1$ (or $\boldsymbol{P}_1$), crosses the plane $\boldsymbol{\sigma}_1$ of the entrance-pupil; and $\boldsymbol{Q}'_k$ (or $\boldsymbol{Q}_{k+1}$) designates the position, in the transversal plane $\boldsymbol{\sigma}'_k$ (71), of the point which, by GAUSS's Theory, is conjugate to the point $\boldsymbol{Q}_1$ after refraction at the kth surface.

41. Q'', $\overline{Q}''$ are used to designate the points where a pair of paraxial rays, of colours λ, $\overline{\lambda}$, respectively, both emanating from the same extra-axial object-point Q, cross the *focussing plane* in the Image-Space of the optical system (Chap. XIII).

In Fig. 161, Q'' designates the centre of the "*blur-circle*" in the scale-plane σ'' of an optical measuring instrument, which corresponds to the extra-axial image-point Q'.

42. See use of this letter in conjunction with P, R and S (38).

R

43. The letter R is used, especially in Chap. XIV, to designate a point of a 3-dimensional object, and R' to designate the conjugate point of the relief-image.

44. See 38 for use of this letter in conjunction with P, Q, S.

S

45. S, $\overline{S}$ and S', $\overline{S}'$ are used, especially in the theory of the refraction of a narrow bundle of rays, to designate the *primary and secondary object-points and image-points*, respectively. Thus, S, S' designate a pair of conjugate points on the chief ray of a pencil of *meridian* rays, before and after refraction, respectively; and, similarly, $\overline{S}$, $\overline{S}'$ designate a pair of conjugate points on the chief ray of a pencil of *sagittal* rays, before and after refraction, respectively. If the bundle of incident rays is *homocentric*, the points S, $\overline{S}$ coincide at the vertex of the bundle.

S'_k (or S_{k+1}) and $\overline{S}'_k$ (or $\overline{S}_{k+1}$) designate the positions on the chief ray of the primary and secondary image-points, respectively, after refraction of the narrow bundle of rays at the kth surface of a system of refracting surfaces.

46. S, S' and $\overline{S}$, $\overline{S}'$ are used also as explained in 38 above. S, S' occur frequently to designate a pair of conjugate points of two collinear systems.

47. Especially, the letters S, S' designate the positions on the axis of the centres of the *entrance-port* and *exit-port*, respectively, of an optical system. If the system has two entrance-ports, the centres are designated by S_1 and S_2.

T

48. T, T' are used in Chap. IX to designate the points of intersection of a pair of incident rays lying in the plane of a principal section of the spherical refracting surface and the pair of corresponding refracted rays, respectively. See Fig. 121.

49. T is used also to designate the upper end of the diameter, in the plane of the principal section, of the *entrance-port* of the optical system; and T' designates the point in the circumference of the *exit-port* which is conjugate to T. If the system has two entrance-ports, the upper ends of the diameters, in the plane of the principal section, are designated by T_1 and T_2.

50. In certain diagrams of Chap. V, the letter T is used to designate the infinitely distant point of the y-axis.

U

51. This letter occurs in various uses. We mention here only one of these, viz.: U designates the lower end of the diameter, in the plane of the principal section, of the *entrance-port* of an optical system; and U' designates the point in the circumference of the *exit-port* which is conjugate to U. If the system has two entrance-ports, the lower ends of the diameters are designated by U_1, U_2.

V

52. In the prism-diagrams, V designates the *vertex* of the prism. In a system of prisms whose refracting edges are all parallel, V_k designates the point where the refracting edge of the kth prism meets the plane of the principal section.

There are also various other uses of this letter which it is not necessary to enumerate.

W

53. This letter occurs frequently in various ways.

X

54. This letter occurs in various ways.

Y

55. Y, Y' are used to designate the feet of the perpendiculars let fall from the centre of the spherical refracting surface on the incident and refracted rays, respectively (Fig. 120).

The letter Y occurs also in various other connections.

Z

56. In Chap. IX, Z, Z' designate the points where the incident and refracted rays cross the auxiliary concentric spherical surfaces τ, τ' (72), respectively, which are used in YOUNG's construction of the path of a ray refracted at a spherical surface (Figs. 114, 115).

In particular, Z, Z' designate the positions on the optical axis of the pair of *aplanatic points* of a spherical refracting surface (Fig. 116).

The letter Z is used also in various other ways.

II. DESIGNATIONS OF LINES.

Lines in the diagrams are designated generally by italic small letters. Without undertaking to enumerate all the uses of these letters, we may mention here the following as among the most important.

f, e'

57. In Chapters V, VI and VII, the letters f and e' are frequently used to designate the *Focal Lines* (or "*Flucht*" *Lines*) of two collinear plane-fields; or the lines in which the Focal Planes ϕ, ϵ' (73, 65) are intersected by conjugate meridian planes containing the principal axes x, x', respectively, of two collinear space-systems. See Figs. 64, *a* and *b*, and 65.

i, j

58. In Chap. VII, the letters i, i' (and similarly also the letters j, j') are used to designate the infinitely distant straight line and the "*Flucht*" Line, respectively, of two collinear plane-fields.

s, s'

59. In Chapters V, VI and VII, s, s' are used to designate a pair of conjugate rays of two collinear systems; as in Fig. 66.

u, u'

60. Throughout Chap. XI, u, u' are used to designate the chief incident ray and the corresponding refracted ray, respectively, of an infinitely narrow bundle of rays refracted at a spherical surface.

x, x'; y, y'; z, z'

61. x, x' designate the *Principal Axes* of two collinear systems. In an optical system of centered spherical surfaces, the optical axis is designated by x or x' according as it is regarded as belonging to the Object-Space or Image-Space, respectively.

In Chap. VII, x_k, x'_k designate the Principal Axes of the kth component of a compound optical system.

62. x, x'; y, y'; z, z' are used also to designate corresponding (but not necessarily conjugate) pairs of rectangular *axes of co-ordinates* in the Object-Space and Image-Space.

63. y designates the *axis of collineation* of two centrally collinear plane-fields. It designates especially the tangent-line in the meridian plane at the vertex of a spherical refracting surface.

64. It may also be mentioned that z is used (with suitable primes, subscripts, etc.) to designate the chief ray of a narrow bundle of rays refracted at the edge of a prism. See Fig. 42.

III. DESIGNATIONS OF SURFACES.

Surfaces, plane or curved, are designated by small letters of the Greek alphabet. Of these the following are the more important.

ϵ

65. ϵ, ϵ' are used to designate the infinitely distant plane of the Object-Space and the *Focal Plane* of the Image-Space, respectively, of two collinear space-systems. Similarly, in the case of a compound optical system, ϵ'_k designates the *secondary focal plane* of the *k*th component.

η

66. η, η' are used sometimes (see Chap. VII) to designate a pair of *conjugate plane-fields* of two collinear space-systems.

μ

67. μ is used to designate the *refracting* or *reflecting surface.* If there are a series of such surfaces, μ_k designates the *k*th surface of the series reckoned in the order in which they are encountered by the rays of light.

π

68. π, π' are used frequently, especially in Chap. VII, to designate two *collinear plane-fields.*

These symbols are also employed, especially in Chap. XI, to designate the coincident planes of incidence and refraction of the chief incident ray and the corresponding refracted ray, respectively, of an infinitely narrow bundle of rays refracted at a spherical (or plane) surface. In particular π, π' designate the collinear plane-fields of the meridian sections of a narrow bundle of incident rays and the bundle of corresponding refracted rays.

In the same way, also, $\bar{\pi}, \bar{\pi}'$ are used to designate the pair of planes, both at right angles to the plane of incidence of the chief ray of a narrow bundle of rays refracted at a spherical (or plane) surface, which contain the chief incident ray and the corresponding refracted ray, respectively. And, especially, $\bar{\pi}$, $\bar{\pi}'$ designate the two collinear plane-systems of the sagittal sections of the bundles of incident and refracted rays, respectively.

Moreover, the symbols π'_k, $\bar{\pi}'_k$ designate the plane-systems of the meridian and sagittal sections, respectively, after refraction of a narrow bundle of rays at the *k*th surface of a series of refracting surfaces.

σ, $\boldsymbol{\sigma}$

69. σ, σ' are used in Chap. VII to designate a pair of conjugate planes parallel to the Focal Planes.

Especially, the symbols σ, σ' are used to designate a pair of transversal planes which are conjugate, in the sense of GAUSS's Theory, with respect to either a single spherical refracting (or reflecting) surface or a centered system of spherical surfaces. In this case, σ designates the so-called *Object-Plane* (Chap. XII) which is defined as the transversal plane (perpendicular to the optical axis) which contains the object-point P (or Q); see 36, 39. The axial point M' conjugate, by GAUSS's Theory, to the point M (24) where the optical axis crosses the Object-Plane σ determines the position of the transversal *Image-Plane* σ'. In Chap. XIV, the planes σ, σ' are usually called the *Focus-Plane* and the *Screen-Plane*, respectively.

In the case of a centered system of spherical surfaces, σ'_k is used to designate the transversal plane which, by GAUSS's Theory, is conjugate to the Object-Plane σ_1 with respect to the optical system composed of the first k surfaces.

70. σ'' is employed to designate a transversal plane of the Image-Space of an optical system which is usually not far from the Image-Plane σ'. For example, in Fig. 161, σ'' designates the so-called *Scale-Plane* of an optical measuring instrument.

71. The symbols $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ are used to designate a second pair of transversal planes conjugate to each other in the same way as σ, σ' above. Generally, $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ designate (as always in Chapters XII and XIV) the planes of the *Entrance-Pupil* and *Exit-Pupil*, respectively, of the optical system.

In the case of a centered system of spherical surfaces, $\boldsymbol{\sigma}'_k$ designates the transversal plane which, by GAUSS's Theory, is conjugate to the initial plane $\boldsymbol{\sigma}_1$ in the Object-Space, with respect to the optical system composed of the first k surfaces.

τ

72. τ, τ' are used to designate the *auxiliary spherical surfaces*, concentric with the spherical refracting surface, used in YOUNG's Construction of the path of the refracted ray (Figs. 114 and 115).

φ

73. φ, φ' are used to designate the *Focal Plane* (or "*Flucht*" Plane) of the Object-Space and the infinitely distant plane of the Image-Space, respectively, of two collinear space-systems. Similarly, in the case of a compound optical system, φ_k designates the *primary focal plane* of the *k*th component.

IV. SYMBOLS OF LINEAR MAGNITUDES.

Introduction. A straight line is divided into two segments by a pair of actual (or "finite") points A, B on the line, viz., a segment of finite magnitude which is the shortest distance between the two points and another segment of unlimited length which is the "long way" between the two points *via* the infinitely distant point I of the straight line. Three actual points A, B, C lying along a straight line determine a certain "sense" ABC along the line or direction in which the line has to be traversed in order to go from A to B without passing through C. As we shall exclude infinitely great line-segments, the segment AB is to be understood therefore as meaning always the finite one of the two above-mentioned; and as indicating also not merely the distance from A to B but the segment AB in the sense ABI. Evidently, therefore, we have the following relation:

$$AB + BA = 0.$$

Also, if A, B, C are three points ranged along a straight line in any order whatever, we may write according to the above:

$$AB + BC + CA = 0;$$

and, generally, in the case of any number of points lying on one straight line, a similar relation will exist.

If A, B, C, D, $\cdots$ designate a series of points ranged along a straight line, the segments AB, AC, AD, $\cdots$ are called here (for lack of a better term) the "**abscissæ**" of the points B, C, D, $\cdots$, respectively, with respect to the point A as origin.

As a rule, to which, however, there are some notable exceptions (as will be seen in the following), linear magnitudes are denoted by italic small letters. Italic capital letters and Greek letters occur sometimes as symbols of linear magnitudes. The more important of these magnitudes will be found in the following list.

a

74. In Chap. X, the symbol a_k is used to denote the abscissa of the centre C_{k+1} of the $(k + 1)$th surface with respect to the centre C_k of the kth surface of a centered system of spherical surfaces; thus $a_k = C_kC_{k+1}$.

b

75. In Chap. IX, b, b' denote the intercepts of a ray, lying in the principal section of a spherical refracting surface, on the central perpendicular, before and after refraction, respectively; thus $b = CH$, $b' = CH'$ (6 and 15). Similarly, in Chap. X, $b_k = C_kH_k$, $b'_k = C_kH'_k$.

76. In Chap. XIII, b, b' are used to denote the widths of a pencil of parallel meridian rays before and after refraction, respectively, at a plane surface. Similarly, b'_k denotes the width of a pencil of parallel meridian rays after refraction at the kth surface of a system of prisms with their refracting edges all parallel.

c

77. In Chap. IX, c, c' denote the abscissæ, with respect to the centre C of the spherical refracting surface, of the points designated by L, L' (23); thus, $c = CL$, $c' = CL'$.

78. In Chap. XIV, c, c' denote the abscissæ, with respect to the centres of the pupils, of the centres of the ports (27 and 47); thus, $c = \boldsymbol{M}S$, $c' = \boldsymbol{M}'S'$. Also, $c_1 = \boldsymbol{M}S_1$, $c_2 = \boldsymbol{M}S_2$.

d, δ, Δ

79. d denotes the *axial thickness* of an optical medium comprised between two consecutive surfaces of a centered system of spherical surfaces. Particularly, $d_k = A_kA_{k+1}$ (see 2).

In an optical system composed of a single lens, the *thickness* of the lens is denoted by d; thus, $d = A_1A_2$.

80. In a centered *System of Infinitely Thin Lenses*, d_k denotes the distance of the $(k + 1)$th lens from the kth lens; thus, $d_k = A_kA_{k+1}$ (see 2).

81. In Chap. VIII, in an optical system consisting of a combination of two lenses, d is used to denote the abscissa, with respect to the secondary principal point of the first lens, of the primary principal point of the second lens; thus, $d = A'_1A_2$ (see 1).

82. The symbol δ_k is employed to denote the length of the ray-path comprised between the kth and the $(k + 1)$th refracting surfaces; thus, $\delta_k = B_kB_{k+1}$ (see 5).

83. Here also we note the use of the symbol Δ_k to denote the so-called "*optical interval*" between the kth and the $(k + 1)$th components of a compound optical system; thus, $\Delta_k = E'_k F_{k+1}$ (see 11 and 12). If the compound system has only two parts, we write: $\Delta = E'_1 F_2$.

e

84. The *secondary focal length* of an optical system is denoted by e'; that is, $e' = E'A'$ (1 and 11). Also, in a compound optical system, $e'_k = E'_k A'_k$. Also, in Chap. XIII, $\bar{e}' = \bar{E}'A'$.

85. In the theory of the refraction of a narrow bundle of rays at a spherical surface (or through a centered system of spherical surfaces), the symbols e'_u and $\bar{e}'_u$ are used in Chap. XI to denote the secondary focal lengths of the two collinear plane-systems π, π' and $\bar{\pi}$, $\bar{\pi}'$, respectively (68). The subscript u refers to the chief ray of the bundle of incident rays (60).

Similarly, the secondary focal lengths of the systems of meridian and sagittal rays of an infinitely narrow bundle of rays which are refracted at the kth surface of a centered system of spherical surfaces are denoted by $e'_{u,k}$, $\bar{e}'_{u,k}$, where u designates the chief ray of the bundle of object-rays.

f

86. The *primary focal length* of an optical system is denoted by f; thus, $f = FA$ (see 1 and 12). Also, in a compound optical system, $f_k = F_k A_k$. Also, $\bar{f} = \bar{F}A$ (see Chap. XIII).

87. The symbols f_u and $\bar{f}_u$ are used in the same connection as e'_u and $\bar{e}'_u$ (85) to denote the primary focal lengths of the systems π, π' and $\bar{\pi}$, $\bar{\pi}'$, respectively. Similarly also the symbols $f_{u,k}$, $\bar{f}_{u,k}$, corresponding to $e'_{u,k}$, $\bar{e}'_{u,k}$, respectively.

g

88. The symbols g, $\bar{g}$ are used in Chap. XIII to denote the ordinates of the points where an incident paraxial ray emanating from the axial object-point M crosses the primary focal planes of an optical system which correspond to light of wave-lengths λ, $\bar{\lambda}$, respectively. Also, in Chapters V, VI and VII, the symbol g is employed in a sense similar to the above. See Figs. 65 and 90, where $g = FR$.

h, ***h***

89. The symbol h is used to denote the *incidence-height* (or ordinate of the incidence-point B) of a ray refracted (or reflected) at a spherical surface; thus, $h = DB$ (5 and 8). With respect to a centered system of spherical surfaces, $h_k = D_k B_k$ denotes the incidence-height at the

kth surface of a ray lying in the principal section. In Chap. XIII, we have also $\overline{h}_k = \overline{D}_k\overline{B}_k$.

90. Similarly, $\boldsymbol{h}_k = \boldsymbol{D}_k\boldsymbol{B}_k$ (5 and 8) denotes the incidence-height of a second ray, usually the *chief ray*, at the kth surface of a centered system of spherical surfaces.

91. The symbol h is used to denote the incidence-height of a ray refracted through an *Infinitely Thin Lens*. In a centered system of Infinitely Thin Lenses, h_k denotes the incidence-height at the kth lens.

92. In the case of two collinear space-systems (Chap. VII), the symbols h, h' are used to denote the ordinates of the points where a pair of conjugate rays cross the primary and secondary principal planes, respectively; $h' = h$. In a compound optical system, h_k, h'_k $(= h_k)$ are used in this same sense with respect to the kth component; see Fig. 99.

k

93. The symbol k' is used, always in connection with the symbol g (88), to denote the ordinate of the point where a paraxial image-ray lying in the plane of the principal section crosses the secondary focal plane; see Fig. 65.

l

94. The symbols l, l' are used to denote the so-called "*ray-lengths*" of a ray lying in the principal section of a spherical refracting surface, before and after refraction, respectively; reckoned in each case from the incidence-point B to the point where the ray crosses the optical axis; thus, $l = BL$, $l' = BL'$ (23). In the case of a ray lying in the princiapl section of a centered system of spherical refracting surfaces, $l_k = B_kL'_{k-1}$, $l'_k = B_kL'_k$ denote the ray-lengths, before and after refraction, respectively, at the kth surface.

p

95. In Chap. IX, p, p' denote the radii vectores of the points H, H' (15); thus $p = CH$, $p' = CH'$ (Fig. 123). In Chap. X, $p_k = C_kH_k$, $p'_k = C_kH'_k$.

96. In Chap. XIV, p, p' denote a pair of conjugate radii of the *Entrance-Pupil* and *Exit-Pupil*, respectively, of the optical system; thus, $p = \boldsymbol{M}D$, $p' = \boldsymbol{M}'D'$ (9 and 27).

The symbol p_0 occurs to denote the radius of the iris-opening of the eye.

q

97. In Chap. XIV, q, q' denote a pair of conjugate radii of the

Entrance-Port and *Exit-Port*, respectively, of the optical system; thus, $q = ST$, $q' = S'T'$ (47 and 49). Also, $q_1 = S_1T_1$, $q_2 = S_2T_2$.

r, R

98. The symbol r is used to denote the *radius* of the spherical refracting surface; or, more exactly, to denote the abscissa of the centre C with respect to the vertex A; $r = AC$. Similarly, $r_k = A_kC_k$ (2 and 6) denotes the radius of the kth surface.

99. The symbols R, $\overline{R}$ and R', $\overline{R}'$ are used to denote the radii of curvature at the axial points M and M' of the I. and II. image-surfaces, before and after refraction, respectively, at a spherical surface (Chap. XII); $R = MK$, $R' = M'K'$, $\overline{R} = M\overline{K}$, $\overline{R}' = M'\overline{K}'$ (22 and 24). Also, R'_k, $\overline{R}'_k$ are used in the same way, with respect to the astigmatic image-surfaces after refraction at the kth spherical surface.

s

100. The symbols s, s' are used to denote the distances, reckoned in each case from the incidence-point B of the chief ray, of the vertex S of an infinitely narrow pencil of *meridian* rays and the vertex S' of the pencil of corresponding refracted rays, respectively; thus, $s = BS$, $s' = BS'$ (5 and 45). Similarly $\overline{s}$, $\overline{s}'$ denote the distances, from the incidence-point B of the chief ray, of the vertex $\overline{S}$ of an infinitely narrow pencil of *sagittal* rays and the vertex $\overline{S}'$ of the pencil of corresponding refracted rays, respectively; $\overline{s} = B\overline{S}$, $\overline{s}' = B\overline{S}'$ (5 and 45).

If the rays traverse a system of prisms or a centered system of spherical surfaces, we have with respect to the kth surface:

$$s_k = B_kS_k = B_kS'_{k-1}, \quad \overline{s}_k = B_k\overline{S}_k = B_k\overline{S}'_{k-1},$$
$$s'_k = B_kS'_k = B_kS_{k+1}, \quad \overline{s}'_k = B_k\overline{S}'_k = B_k\overline{S}_{k+1}.$$

t

101. In Chap. IX, t, t' denote the distances from the incidence-point $\mathbf{B}$ of a chief ray lying in the plane of the principal section of a spherical surface of the points T, T' of intersection with this ray of another meridian ray, before and after refraction, respectively; thus, $t = \mathbf{B}T$, $t' = \mathbf{B}T'$, as in Fig. 121. See 5 and 48.

u, $\mathbf{u}$, $\mathfrak{u}$

102. The symbols u, u' are used to denote the abscissæ, with respect to the principal points A, A' of two collinear systems, of a pair of conjugate axial points M, M' respectively; thus, $u = AM$, $u' = AM'$ (2 and 24).

103. Especially, u, u' denote the abscissæ, with respect to the vertex A of the spherical surface, of the points M, M' where a paraxial ray crosses the optical axis, before and after refraction (or reflexion), respectively; $u = AM$, $u' = AM'$ (2 and 24, 25).

If (as in Chap. VIII, § 195) we have a pair of paraxial rays of different origins, the abscissæ of the points $\boldsymbol{M}$, $\boldsymbol{M}'$ where the second ray crosses the axis before and after refraction are denoted by $\boldsymbol{u}$, $\boldsymbol{u}'$, respectively; $\boldsymbol{u} = A\boldsymbol{M}$, $\boldsymbol{u}' = A\boldsymbol{M}'$ (2 and 26).

In case we are concerned with paraxial rays of two different colours emanating from a common source, the symbols u, u' and $\bar{u}$, $\bar{u}'$ are used as above described with reference to rays of light of wave-lengths λ, $\bar{\lambda}$, respectively; and if also there is a ray of a third colour $\mathbf{l}$, the abscissæ of the points where this ray crosses the axis are denoted by $\mathbf{u}$, $\mathbf{u}'$ (see Chap. XIII).

In an optical system consisting of a centered system of spherical surfaces, the symbols u_k, u'_k; $\boldsymbol{u}_k$, $\boldsymbol{u}'_k$; $\bar{u}_k$, $\bar{u}'_k$; etc., are used precisely in the same way as described above, with respect to the kth surface of the system; so that

$$u_k = A_k M_k = A_k M'_{k-1}, \quad u'_k = A_k M'_k = A_k M_{k+1}, \text{etc.}$$

In particular, the symbol $u_1 = A_1 M_1$ denotes the abscissa, with respect to the vertex A_1, of the axial object-point M_1. Frequently, however, the symbol u, without any addition, is used to denote the abscissa of the point where a paraxial object-ray crosses the optical axis of a centered system of spherical surfaces; in which case u' denotes the abscissa, with respect to the vertex of the last surface, of the point where the conjugate image-ray crosses the axis.

104. In the case of an *Infinitely Thin Lens*, the symbols u, u' denote the abscissæ, with respect to the optical centre of the lens, of the points where a paraxial ray crosses the axis before entering the lens and after emerging from it, respectively. The symbols $\boldsymbol{u}$, $\boldsymbol{u}'$ and $\bar{u}$, $\bar{u}'$ are used also in this way.

Similarly, in the case of a centered system of Infinitely Thin Lenses, the symbols u_k, u'_k are used as just stated, with respect to the kth lens. So also $\boldsymbol{u}_k$, $\boldsymbol{u}'_k$ and $\bar{u}_k$, $\bar{u}'_k$.

105. In general, the symbol $\boldsymbol{u}_1 = A_1\boldsymbol{M}_1$ denotes the abscissa, with respect to the vertex A_1 of the centre $\boldsymbol{M}_1$ of the *Entrance-Pupil* of the system. If the centres of the Entrance-Pupil and Exit-Pupil are designated by $\boldsymbol{M}$, $\boldsymbol{M}'$ (27), then $\boldsymbol{u} = A_1\boldsymbol{M}$, $\boldsymbol{u}' = A_m\boldsymbol{M}'$ are used to denote the *abscissæ of the pupil-centres*.

v, $\boldsymbol{v}$

106. The symbols v, v' denote the abscissæ, with respect to the vertex A of the spherical refracting surface, of the points L, L' (23) where a ray lying in the plane of the principal section crosses the optical axis, before and after refraction, respectively; $v = AL$, $v' = AL'$. Also: $v_k = A_kL_k = A_kL'_{k-1}$, $v'_k = A_kL'_k = A_kL_{k+1}$.

The symbols $\boldsymbol{v}$, $\boldsymbol{v}'$ have meanings with respect to the *chief ray* precisely the same as above; thus, $\boldsymbol{v} = A\boldsymbol{L}$, $\boldsymbol{v}' = A\boldsymbol{L}'$; also, $\boldsymbol{v}_k = A_k\boldsymbol{L}'_{k-1}$, $\boldsymbol{v}'_k = A_k\boldsymbol{L}'_k$ (see 23).

In Chap. IX, in Kerber's formulæ for the path of an *oblique ray* refracted at a spherical surface, we have:

$$v_g = A_gG, \quad v_i = A_iI, \quad v'_g = A_gG', \quad v'_i = A_iI',$$

where the points designated by A_g, A_i, G, G', I and I' are points described in 3, 14 and 18. See Fig. 122. Also, in Chap. X, in the same connection we have:

$$v_{g,k} = A_{g,k}G_k, \quad v_{i,k} = A_{i,k}I_k, \quad v'_{g,k} = A_{g,k}G'_k, \quad v'_{i,k} = A_{i,k}I'_k.$$

x, $\boldsymbol{x}$

107. x, x' denote especially the abscissæ, with respect to the focal points F, E', of a pair of conjugate axial points M, M', respectively, of two collinear systems; thus, $x = FM$, $x' = E'M'$. Similarly, with reference to the kth component of a compound optical system, we have: $x_k = F_kM'_k$, $x'_k = E'_kM'_k$ (11, 12 and 24).

In Chap. XIII, $\bar{x}$, $\bar{x}'$ occur in connection with the Focal Points $\bar{F}$, $\bar{E}'$ (11 and 12).

In Chap. VII, the letters, x, x' occur also with special subscripts.

108. $\boldsymbol{x}$, $\boldsymbol{x}'$ denote the abscissæ, with respect to the focal points F, E', of the *pupil-centres;* thus, $\boldsymbol{x} = F\boldsymbol{M}$, $\boldsymbol{x}' = E'\boldsymbol{M}'$ (11, 12 and 27).

109. The letters x, y, z and x', y', z' are used to denote the rectangular co-ordinates of a pair of conjugate points of two collinear space-systems.

110. In Chap. IX, x_g, x'_g and x_i, x'_i denote the x-co-ordinates, with respect to the centre C of the spherical refracting surface, of the points designated by G, G' and I, I', respectively; and, similarly, in Chap. X, $x_{g,k}$, $x'_{g,k}$, and $x_{i,k}$, $x'_{i,k}$ denote the x-co-ordinates, with respect to C_k, of the points G_k, G'_k (or G_{k+1}) and I_k, I'_k (or I_{k+1}), respectively (6, 14 and 18).

y, $\boldsymbol{y}$

111. y, y' denote the y-co-ordinates of a pair of conjugate points of two collinear space-systems; especially, the ordinates of the extra-axial conjugate points Q, Q' lying in the meridian xy-plane; $y = MQ$, $y' = M'Q'$ (24 and 39).

y' denotes the ordinate of the vertex, after refraction at the kth surface of a centered system of spherical surfaces, of a bundle of paraxial rays which emanate originally from the extra-axial object-point Q_1 lying in the plane of the principal section; or the ordinate of the point Q'_k (or Q_{k+1}), where, according to GAUSS's Theory, a ray emanating from the object-point Q_1 (or P_1) would cross the transversal plane σ'_k after refraction at the kth surface of a centered system of spherical surfaces. See 36, 39 and 69. Thus, $y'_k = M'_kQ'_k$.

In Chap. XIII, where we have to do with rays of light of two or more different colours, the symbols $\bar{y}$, $\bar{y}'$ denote the ordinates of the pair of extra-axial conjugate points $\bar{Q}$, $\bar{Q}'$ for rays of wave-length $\bar{\lambda}$; $\bar{y} = \bar{M}\bar{Q}$, $\bar{y}' = \bar{M}'\bar{Q}'$ (24 and 39); usually, $\bar{y} = y$. In the same way, $\bar{y}'_k = \bar{M}'_k\bar{Q}'_k$.

112. $\boldsymbol{y}$, $\boldsymbol{y}'$ denote the y-co-ordinates of the points $\boldsymbol{Q}$, $\boldsymbol{Q}'$, respectively (40). The symbol $\boldsymbol{y}'_k$ denotes the y-co-ordinate of the point $\boldsymbol{Q}'_k$ (see Chap. XII).

113. In Chap. IX, y_g, y'_g and y_h, y'_h denote the y-co-ordinates of the points designated by G, G' and H, H', respectively (Figs. 122 and 123); see also Chap. XII. In Chap. X, $y_{g,k}$, $y'_{g,k}$ and $y_{h,k}$, $y'_{h,k}$ denote the y-co-ordinates of the points designated by G_k, G'_k (or G_{k+1}) and H_k, H'_k, respectively (14 and 15).

z, $\boldsymbol{z}$

114. The symbols z, z' denote the z-co-ordinates of a pair of conjugate points of two collinear space-systems.

115. Especially in Chap. XII, z, z' denote the z-co-ordinates of the points Q, Q'. If the object-point Q lies in the meridian xy-plane, $z = z' = 0$. z'_k denotes the z-co-ordinate of the point Q'_k (39).

116. $\boldsymbol{z}$, $\boldsymbol{z}'$, $\boldsymbol{z}'_k$ denote the z-co-ordinates of the points designated by $\boldsymbol{Q}$, $\boldsymbol{Q}'$, $\boldsymbol{Q}'_k$, respectively; see 40.

117. In Chap. IX, z_h, z'_h and z_i, z'_i denote the z-co-ordinates of the points designated by H, H' and I, I', respectively (Figs. 122 and 123). Also, in Chap. X, $z_{h,k}$, $z'_{h,k}$ and $z_{i,k}$, $z'_{i,k}$ denote the z-co-ordinates of the points H_k, H'_k and I_k, I'_k, respectively (15 and 18).

118. Finally, in Chap. XIII, the symbols z, z' are used in a special sense to denote the abscissæ, with respect to the vertices of the first and last surfaces, of the primary and secondary focal points, respectively, for rays of light of wave-length λ; so that $z = A_1F$, $z' = A_mE'$. Similarly, for rays of light of wave-length $\bar{\lambda}$, we have: $\bar{z} = A_1\bar{F}$, $\bar{z}' = A_m\bar{E}'$ (11 and 12).

ξ, η, ζ; $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$

119. In Chap. XIV, the Greek letters ξ, ξ' are used to denote the abscissæ, with respect to the centres of the entrance-pupil and exit-pupil, of the pair of conjugate axial points M, M', respectively; thus, $\xi = \boldsymbol{M}M$, $\xi' = \boldsymbol{M}'M'$ (24 and 27).

120. In Chap. XII, the rectangular co-ordinates of the points designated by P and P' are denoted by ξ, η, ζ and ξ', η', ζ', respectively. Also, the co-ordinates of P'_k are ξ'_k, η'_k, ζ'_k (36).

121. In Chap. XII, $\boldsymbol{\eta}$, $\boldsymbol{\eta}'$ and $\boldsymbol{\zeta}$, $\boldsymbol{\zeta}'$ are used to denote the y- and z-co-ordinates of the points designated by $\boldsymbol{P}$, $\boldsymbol{P}'$, respectively; similarly, $\boldsymbol{\eta}'_k$, $\boldsymbol{\zeta}'_k$ are used with reference to the point $\boldsymbol{P}'_k$ (37).

V. SYMBOLS OF ANGULAR MAGNITUDES.

If A, B, C designate the positions of three points not in a straight line, the $\angle ABC$ is the angle through which the straight line AB must be turned in order that the point A may be brought to lie in the same direction from the turning-point B as the point C is; thus, $\angle ABC + \angle CBA = 0$.

Throughout this volume, *counter-clockwise rotation is reckoned always as positive rotation.*

With rare exceptions, angular magnitudes are denoted by the letters of the Greek alphabet. The more important of these angles are enumerated in the following list.

α, $\boldsymbol{\alpha}$, A

122. The *angles of incidence and refraction*, as defined in Chap. I (see Fig. 5), are denoted by α, α', respectively. When a ray of light traverses a series of optically isotropic media, the symbols α_k, α'_k denote the angles of incidence and refraction, respectively, at the kth refracting surface.

123. The capital Greek letter A denotes the *critical angle of* incidence of a ray refracted into a less dense medium, and A' denotes the critical angle of refraction of a ray refracted into a more dense medium.

124. The angles of incidence and refraction at a spherical surface of the so-called *chief ray* are denoted by $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$, respectively. Similarly, with respect to the kth refracting surface of a series of such surfaces, the symbols $\boldsymbol{\alpha}_k$, $\boldsymbol{\alpha}'_k$ are employed.

β

125. The *refracting angle of a prism* is denoted by β. In a train of prisms, $\beta_k = \angle V_{k-1}V_kV_{k+1}$ denotes the refracting angle of the kth prism (52).

δ

126. In KERBER's Refraction-Formulæ (Chap. IX), δ, δ' are used to denote a certain pair of auxiliary angular magnitudes relating to the ray before and after refraction, respectively (Fig. 122). In Chap. X, δ'_k is employed in the same way with reference to the ray after refraction at the kth surface of a centered system of spherical surfaces.

ϵ

127. The acute angle through which the refracted ray has to be turned in order to bring it into coincidence with the corresponding incident ray, the so-called *angle of deviation*, is denoted by ϵ; in Fig. 9 $\angle P'BP = \epsilon$. Thus, also, ϵ_k denotes the angle of deviation at the kth refracting surface. The *total deviation* of a ray after traversing a *train of prisms* with their edges all parallel is denoted by $\epsilon = \sum_{k=1}^{k=m} \epsilon_k$, where m denotes the total number of refracting planes.

The angle of *minimum deviation* of a prism or prism-system is denoted by ϵ_0.

128. In KERBER's Refraction-Formulæ (Chap. IX), ϵ, ϵ' are used to denote a certain pair of auxiliary angular magnitudes relating to the ray before and after refraction, respectively (Fig. 122). Also, in Chap. X, in the same connection, ϵ'_k has reference to the ray after refraction at the kth surface of a centered system of spherical surfaces.

θ, Θ and $\boldsymbol{\theta}$, $\boldsymbol{\Theta}$

129. $\theta = \angle AMB$ or $\angle ALB$, $\theta' = \angle AM'B$ or $\angle AL'B$, where the points designated by A, B, M, M', L, L' have the meanings explained in 2, 5, 23 and 24. Also, $\theta'_k = \theta_{k+1} = \angle A_kM'_kB_k$ or $\angle A_kL'_kB_k$. The angles θ, θ' are the so-called *slope-angles* of the ray before and after refraction, respectively, at a spherical surface.

If we have a pair of rays of two *different colours* emanating originally from the same point on the optical axis of a centered system, θ'_k, $\overline{\theta}'_k$ denote the slope-angles, after refraction at the kth surface, of the rays of wave-lengths λ, $\overline{\lambda}$, respectively.

The symbol θ is used to denote the slope-angle of an object-ray proceeding from the axial object-point M (24 and 25) and the symbol θ' to denote the slope-angle of the conjugate image-ray, especially on the assumption of collinear correspondence between Object-Space and Image-Space.

130. The symbols $\boldsymbol{\theta}$, $\boldsymbol{\theta}'$ are used to denote the slope-angles of the *chief ray*, before and after refraction, respectively, at a spherical surface; thus, $\boldsymbol{\theta} = \angle A\boldsymbol{LB}$, $\boldsymbol{\theta}' = \angle A\boldsymbol{L'B}$. Similarly,

$$\boldsymbol{\theta}'_k = \boldsymbol{\theta}_{k+1} = \angle A_k \boldsymbol{L}'_k \boldsymbol{B}_k.$$

See 2, 5 and 23.

$\boldsymbol{\theta}$, $\boldsymbol{\theta}'$ denote the slope-angles of a chief object-ray and its conjugate image-ray, respectively; especially, on the assumption of collinear correspondence between Object-Space and Image-Space.

131. In Chap. XIV, Θ, Θ' are used to denote the semi-angular diameters of the *aperture* of the optical system in the Object-Space and Image-Space, respectively; $\Theta = \angle \boldsymbol{M}MD$, $\Theta' = \angle \boldsymbol{M}'M'D'$ (9, 24, 25, 26).

In this same chapter, Θ', Θ'_0 are used to denote the angles subtended at the centre of the image on the retina of the eye by the radius of the *exit-pupil* of the instrument and the radius of the *eye-pupil*, respectively.

132. In Chap. XIV, $\boldsymbol{\Theta}$, $\boldsymbol{\Theta}'$ denote the semi-angular diameters of the *field of view* of the object and image, respectively; thus,

$$\boldsymbol{\Theta} = \angle S\boldsymbol{M}T, \quad \boldsymbol{\Theta}' = \angle S'\boldsymbol{M}'T' \text{ (26, 47, 49).}$$

133. Finally, in connection with KERBER's Refraction-Formulæ (Chap. IX), we have:

$$\theta_g = \angle A_g GB, \quad \theta'_g = \angle A_g G'B, \quad \theta_i = \angle A_i IB, \quad \theta'_i = \angle A_i I'B,$$

where the points designated by A_g, A_i, B, G, G' and I, I' are the points explained in 3, 5, 14 and 18. See Fig. 122.

Similarly, in Chap. X,

$$\theta'_{g,k} = \angle A_{g,k} G'_k B_k, \quad \theta'_{i,k} = \angle A_{i,k} I'_k B_k.$$

λ

134. In Chap. IX, λ, λ' are used to denote the angles between a pair of meridian incident rays and the pair of corresponding refracted rays, respectively; see Fig. 121.

Especially, in Chaps. XI and XII, the symbols $d\lambda$, $d\lambda'$ are used to denote the *angular apertures* of an infinitely narrow pencil of *meridian* incident rays and the pencil of corresponding refracted rays, respectively; thus, $d\lambda = \angle BSG$, $d\lambda' = \angle BS'G$ (see 5, 13 and 45), for example, in Fig. 127.

Similarly, $d\overline{\lambda}$, $d\overline{\lambda}'$ denote the *angular apertures* of a narrow pencil of *sagittal* incident rays and the pencil of corresponding refracted rays, respectively.

The symbols $d\lambda_k'$, $d\overline{\lambda}_k'$ are employed in the same way as above, with respect to the *k*th surface of a centered optical system.

μ

135. In SEIDEL's Refraction-Formulæ, the symbols μ, μ' are employed to denote a pair of auxiliary angles, viz., the angles at H, H' of the triangles BHC, $BH'C$, respectively (5, 6 and 15). For the exact definitions of these angles, see Chap. IX. In Chap. X, the symbols μ_k, μ_k' are used in the same sense.

π

136. In Chap. IX, in SEIDEL's Refraction-Formulæ, π, π' denote the polar angles of the points H, H', respectively; thus, $\pi = \angle HCy$, $\pi' = \angle H'Cy$, where y designates a point on the positive half of the y-axis of co-ordinates and C, H, H' have the meanings given in 6 and 15. See Fig. 123. Similarly, also, in Chap. X:

$$\pi_k = \angle H_k C_k y, \quad \pi_k' = \angle H_k' C_k y.$$

τ

137. In Chap. IX, in SEIDEL's Refraction-Formulæ, τ, τ' are employed to denote the positive acute angles between the direction of the optical axis (x-axis) and the path of an oblique ray, before and after refraction, respectively, at a spherical surface (Fig. 123). Similarly, in Chap. X, the symbols τ_k, τ_k' are used.

ϕ, $\boldsymbol{\phi}$

138. ϕ is used to denote the *central angle* subtended at the centre C of the spherical refracting (or reflecting) surface by the arc BC; thus, $\phi = \angle BCA$. Also, $\phi_k = \angle B_k C_k A_k$ (2, 5 and 6).

139. Similarly, $\boldsymbol{\phi} = \angle \boldsymbol{B}CA$ or $\boldsymbol{\phi}_k = \angle \boldsymbol{B}_k C_k A_k$ denotes the central angle with respect to the so-called *chief ray* (Fig. 121). See 2, 5, 6.

140. In Chap. IX, in KERBER's Refraction-Formulæ, we have: $\phi_g = \angle ACA_g$, $\phi_i = \angle ACA_i$, where the letters A, A_g, A_i and C have the meanings given in 2, 3 and 6. See Fig. 122. Similarly, in Chap. X: $\phi_{g,k} = \angle A_k C_k A_{g,k}$, $\phi_{i,k} = \angle A_k C_k A_{i,k}$.

141. In Chap. XIV, ϕ, ϕ' are used in the radiation-formulæ to denote the angles of emission and radiation, respectively.

χ

142. In Chap. IX, the symbol χ is used to denote the angle $BC\boldsymbol{B}$, where B, $\boldsymbol{B}$ designate the incidence-points on a spherical refracting surface of a pair of meridian rays. See Fig. 121.

ψ

143. In Chap. IX, in SEIDEL's Refraction-Formulæ, ψ, ψ' denote a certain pair of angular magnitudes (see Fig. 123); also, in Chap. X, ψ_k, ψ'_k are used in same sense.

VI. SYMBOLS OF NON-GEOMETRICAL MAGNITUDES (CONSTANTS, CO-EFFICIENTS, FUNCTIONS, ETC.).

Among the more important magnitudes under this head may be mentioned the following:

A

144. The *numerical aperture* of the optical system, in the Object-Space and in the Image-Space, is denoted by A, A', respectively; thus, $A = n \cdot \sin \Theta$, $A' = n' \cdot \sin \Theta'$ (131, 155).

B

145. $B = n\alpha = hJ$ denotes the *optical invariant in the case of paraxial rays* (89, 122, 150, 155).

c, C

146. The symbols c, c' (sometimes also c_1, c_2) are used to denote the *curvatures* of the surfaces of an *Infinitely Thin Lens*; thus,

$$c = 1/r_1, \quad c' = 1/r_2 \text{ (see 98).}$$

In a centered system of infinitely thin lenses, the symbols c_k, c'_k denote the curvatures of the kth lens. Moreover, in Chap. XIII, $C_k = c_k - c'_k$.

147. In Chap. XIV, C, C' denote the *candle-powers* of a point-source of light in a given direction and the corresponding point of the image in the conjugate direction, respectively.

I, i

148. I denotes the *invariant of refraction* in the case of the refraction at a spherical surface of a ray of finite slope lying in the principal section: $I = n(v - r)/rl = n'(v' - r)/rl'$; see 94, 98, 106 and 155.

149. In Chap. XIV, i, i' denote the *specific intensities* of radiation of a luminous surface-element in a given direction and the corresponding element of the image in the conjugate direction, respectively.

J, $\boldsymbol{J}$

150. The symbols J, $\boldsymbol{J}$ denote the so-called "*zero-invariants*" in the case of the refraction of paraxial rays at a spherical surface, with respect to the two pairs of conjugate axial points M, M' and $\boldsymbol{M}$, $\boldsymbol{M}'$, respectively (24, 25, 26 and 27); thus,

$$J = n(1/r - 1/u) = n'(1/r - 1/u'); \quad \boldsymbol{J} = n(1/r - 1/\mathbf{u}) = n'(1/r - 1/\mathbf{u}');$$

(see 98, 103, 155).

In the case of a centered system of spherical surfaces, J_k, $\boldsymbol{J}_k$ denote the zero-invariants for the kth surface, with respect to the pairs of conjugate axial points M'_{k-1}, M'_k and $\boldsymbol{M}'_{k-1}$, $\boldsymbol{M}'_k$, respectively.

In Chap. XIII, J_k, $\bar{J}_k$ denote the zero-invariants, with respect to the kth surface, for paraxial rays of light of colours λ, $\bar{\lambda}$, respectively, emanating originally from the same axial object-point.

K, k

151. $K = n \cdot \sin\alpha = n' \cdot \sin\alpha'$ denotes the magnitude of the *optical invariant* in the refraction of a ray of light (122 and 155).

152. The symbol k, which occurs usually as a subscript, denotes the series-number of any one of a system of refracting (or reflecting) surfaces; or of any integral part or component of a compound optical system. In certain prism-formulæ, the subscripts i and r occur also in this same sense.

L

153. In Chap. XIV, L, L' are used to denote the quantities of light-energy emitted in unit-time by a certain portion of a luminous object and the corresponding portion of the image, respectively.

m

154. The total number of refracting surfaces of a system is denoted by m; also, the total number of components (prisms, lenses or lens-combinations) of a compound optical system.

n, $\mathbf{n}$

155. The absolute *indices of refraction* of the first and second medium are denoted by n, n', respectively.

When a ray traverses a series of media, the symbol $n'_k = n_{k+1}$ is used to denote the absolute index of refraction of the $(k + 1)$th medium. Note that $n'_0 = n_1$ denotes the absolute index of refraction of the first medium.

Often, also, the symbols n, n' are used to denote the absolute indices of refraction of the first and last medium, respectively.

The symbols n, $\bar{n}$ and $\mathfrak{n}$ are used to denote the absolute indices of refraction of a medium for rays of light of wave-lengths λ, $\bar{\lambda}$ and $\mathfrak{l}$, respectively. The symbols n, $\bar{n}$, $\mathfrak{n}$ and n', $\bar{n}'$, $\mathfrak{n}'$ refer to the first and second (or to the first and last) medium, respectively.

So, also, n_A, n_B, n_C, etc. are used to denote the absolute indices of refraction of a medium for rays of light corresponding to the FRAUNHOFER lines A, B, C, etc., respectively.

156. In an optical system wherein there are only two different media, as, for example, in a glass lens (or prism) surrounded by air, the *relative index of refraction* from the first medium to the second is usually denoted by n; thus, $n = n'_1/n_1 = n'_1/n'_2$. In this sense, the symbol n_k is regularly employed to denote the index of refraction of the material of the kth lens of a *System of Infinitely Thin Lenses*, each of which is surrounded by air.

P, p

157. In Chap. XIII, P, P_0 denote the "*purity*" and the "*ideal purity*", respectively, of the spectrum. In this chapter, also, the *resolving power* of a prism or prism-system is denoted by p.

Q

158. The *invariant-functions* of the chief ray of an infinitely narrow bundle of rays refracted at a spherical surface are denoted by Q, $\bar{Q}$ (or Q_k, $\bar{Q}_k$); see Chap. XII, § 299.

T

159. The function $T = h_k \boldsymbol{h}_k (J_k - \boldsymbol{J}_k)$ denotes a certain constant which has the same value for each surface of a centered system of spherical surfaces; see Chap. XII, § 323.

V

160. In Chap. XIV, the symbol V is used to denote the *characteristic magnifying power* of an optical instrument which is intended to be used subjectively in conjunction with the eye.

161. In Chap. XI, § 247, V_k denotes the so-called "*constant of astigmatism*" for the kth surface of a centered system of spherical surfaces. See also Chap. X, 229*a*, and Appendix to Chap. XI.

W

162. In Chap. XIV, W denotes the ratio of the visual angles subtended at the eye, on the one hand, by the image as viewed through the instrument, and, on the other hand, by the object as seen by the naked eye at the distance of distinct vision.

x, $\boldsymbol{x}$, X

163. In an *Infinitely Thin Lens*, the symbols x, x' are used to denote the *reciprocals* of the abscissæ, with respect to the optical centre A (2), of the points M, M' (28) where a paraxial ray crosses the axis before entering the lens and after leaving it, respectively; thus, $x = 1/u$, $x' = 1/u'$ (104). Similarly, in a centered system of Infinitely Thin Lenses, x_k, x'_k denote the same reciprocals with respect to the kth lens.

Similarly, also, the symbols $\bar{x}$, $\boldsymbol{x}$ are employed as follows (see 104):

$$\bar{x} = 1/\bar{u}, \quad \bar{x}' = 1/\bar{u}'; \quad \bar{x}_k = 1/\bar{u}_k, \quad \bar{x}'_k = 1/\bar{u}'_k;$$
$$\boldsymbol{x} = 1/\boldsymbol{u}, \quad \boldsymbol{x}' = 1/\boldsymbol{u}'; \quad \boldsymbol{x}'_k = 1/\boldsymbol{u}_k, \quad \boldsymbol{x}'_k = 1/\boldsymbol{u}'_k.$$

164. The so-called "*axial magnification*" (or "*depth magnification*") with respect to a pair of conjugate axial points M, M' of two collinear systems is denoted by X; thus, if $FM = x$, $E'M' = x'$ (11, 12, 24 and 107), we have: $X = dx'/dx$.

The symbols X, $\bar{X}$ denote the axial magnifications of an optical system with respect to a given axial object-point M for rays of light of wave-lengths λ, $\bar{\lambda}$, respectively.

In Chap. VII, X_0 is used to denote the axial magnification at the point O (see 35).

Y, $\boldsymbol{Y}$

165. Y denotes the so-called "*lateral magnification*" at a pair of conjugate axial points M, M' (24) of two collinear systems: $Y = y'/y$. In an optical system composed of a centered system of m spherical surfaces, $Y = y'_m/y_1$. See 111.

Y, $\bar{Y}$ denote the lateral magnifications of an optical system with respect to a given axial object-point M for rays of light of wave-lengths λ, $\bar{\lambda}$, respectively.

$\boldsymbol{Y}$ denotes the lateral magnification at the *pupil-centres* $\boldsymbol{M}$, $\boldsymbol{M}'$.

In Chap. VII, Y_0 denotes the lateral magnification at the axial object-point O (see 35).

166. In the theory of the refraction of an infinitely narrow bundle of rays Y_u, $\bar{\bar{Y}}_u$ denote the lateral magnifications of the collinear plane-systems π, π' and $\bar{\pi}$, $\bar{\pi}'$, respectively (68). See 149a.

Z, $\mathbf{Z}$

167. The so-called "*angular magnification*" (or "*convergence-ratio*") at a pair of conjugate axial points M, M' (24) of two collinear systems is denoted by Z; $Z = \tan\theta'/\tan\theta$ (129). In an optical system composed of a centered system of m spherical surfaces, $Z = \tan\theta'_m/\tan\theta_1$.

Z, $\overline{Z}$ denote the angular magnifications of an optical system with respect to a given axial object-point M for rays of light of wave-lengths λ, $\overline{\lambda}$, respectively.

$\mathbf{Z}$ denotes the angular magnification at the *pupil-centres* $\mathbf{M}$, $\mathbf{M}'$.

In Chap. VII, Z_0 denotes the angular magnification at the axial object-point O (see 35).

168. In the theory of the refraction of an infinitely narrow bundle of rays, Z_u, $\overline{Z}_u$ denote the *convergence-ratios* of the *meridian* and *sagittal* rays, respectively, where u designates the chief ray of the bundle; $Z_u = d\lambda'/d\lambda$, $\overline{Z}_u = d\overline{\lambda}'/d\overline{\lambda}$ (134).

β

169. In Chap. XIII, β is used to denote the so-called "*relative partial dispersion*" of an optical medium. For the kth medium, this magnitude is denoted by β_k.

ν

170. The symbol ν is employed to denote the so-called "*relative dispersion*" of an optical medium. See formulæ (366), (426).

φ

171. The symbol φ is used to denote the *reciprocal* of the primary focal length of an *Infinitely Thin Lens*—the so-called "*power*" or "*strength*" of the lens.

In a system of infinitely thin lenses, φ_k denotes the power of the kth lens, and φ is used to denote the power of the Lens-System.

NEW LETTERS AND SYMBOLS EMPLOYED IN THE SECOND EDITION OF THIS WORK.

H

15a. Especially, H, $\overline{H}$ and H', $\overline{H}'$ are used to designate the positions of the I. and II. Principal Points of the imageries in the meridian and sagittal planes of the chief incident and emergent rays, u, u', respectively. See Appendix to Chap. XI.

$\mathbf{S}$

45a. $\mathbf{S}$, $\overline{\mathbf{S}}$ and $\mathbf{S}'$, $\overline{\mathbf{S}}'$ are used (in the same way as S, $\overline{S}$ and S', $\overline{S}'$) to designate the positions on the incident and refracted chief rays u,

u' of the I. and II. object-points and image-points which are determined by the stop-centre through which the chief ray must ultimately pass.

c

78a. In a compound optical system consisting of two components, the symbol c is used to denote the so-called "reduced" (p. 366g) interval between the II. Principal Point (A_{I}') of the first partial system and the I. Principal Point (A_{II}) of the second partial system; thus, $c = A_{\text{I}}'A_{\text{II}}/n$, where n denotes the index of refraction of the medium in which the points A_{I}', A_{II} lie. If this interval is measured not along the optical axis but along a given chief ray which is incident at the angle α, we use the symbols c_a, $\bar{c}_a$ defined as follows: $c_a = H_{\text{I}}'H_{\text{II}}/n$, $\bar{c}_a = \overline{H}_{\text{I}}'H_{\text{II}}/n$, where the points denoted by H, H' and $\overline{H}$, $\overline{H}'$ are the Principal Points on the given ray as defined in 15a.

p

95a. Again, in Chap. X, § 229a, in the trigonometric formulae of M. Lange, p_k is used to denote the "optical length" of the ray-path between the $(k - 1)$th and kth spherical refracting surfaces.

95b. In the Appendix to Chap. XI, p, p' are used to denote the abscissae, with respect to $\boldsymbol{S}$, $\boldsymbol{S}'$ as origins (see 45a), of the I. object-point and image-point S, S' lying on the chief incident and refracted rays u, u'; thus, $p = \boldsymbol{S}S$, $p' = \boldsymbol{S}'S'$.

Similarly, $\bar{p} = \overline{\boldsymbol{S}}\overline{S}$, $\bar{p}' = \overline{\boldsymbol{S}}'\overline{S}'$.

In a centered system of spherical surfaces, the symbols p_k, p_k' and $\bar{p}_k$, $\bar{p}_k'$ are used in the same way as above explained, with respect to the kth surface, thus:

$$p_k = \boldsymbol{S}_k S_k = \boldsymbol{S}'_{k-1} S'_{k-1}, \quad p_k' = \boldsymbol{S}_k' S_k' = \boldsymbol{S}'_{k+1} S'_{k+1}, \quad \text{etc.}$$

D

147a. In the Appendix to Chap. XI, the symbols D, $\overline{D}$ are used to denote the *refractivities* of the optical system in the meridian and sagittal planes of the chief ray u; thus, $D = n/f_u$, $\overline{D} = n/\bar{f}_u$.

F

147b. In the Appendix to Chap. XI, the symbol F is used to denote the *refractivity* of the optical system; thus, $F = n/f$.

i, $\boldsymbol{i}$

149a. In the theory of the refraction of an infinitely narrow bundle of rays, i, $\bar{i}$ (or Y_u, $\overline{Y}_u$) denote the magnification-ratios of the imageries in the meridian and sagittal sections, with respect to the I. and II. object-points S, $\overline{S}$, respectively.

Similarly, also, $\boldsymbol{i}$, $\overline{\boldsymbol{i}}$ denote the magnification-ratios with respect to $\boldsymbol{S}$, $\overline{\boldsymbol{S}}$ (see 45a).

L

153a. L, L' are used to denote the Focal-Point Convergences; thus, $L = n/x$, $L' = n'/x'$ (see 107).

P

157a. In the Appendix to Chap. XI, the symbols P, P' (also $\overline{P}$, $\overline{P}'$) are used to denote the so-called "reduced convergences" (p. 366g), defined by the relations: $P = n/p$, $P' = n'/p'$, $\overline{P} = n/\overline{p}$, $\overline{P}' = n'/\overline{p}'$. (See 95b and 155.) In a centered system of spherical surfaces P_k, P_k' ($\overline{P}_k$, $\overline{P}_k'$) are used with respect to the kth surface, thus:

$$P_k = n_k/p_k, \quad P_k' = n_{k+1}/p_k', \quad \text{etc.}$$

U

159a. U, U' are used to denote the Principal-Point Convergences (pages 366g and 366l); thus, $U = n/u$, $U' = n'/u'$ (see 102 and 155). In a centered system of spherical surfaces:

$$U_k = n_k/u_k, \quad U_k' = n_{k+1}/u_k'.$$

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