GENERAL AND PRACTICAL OPTICS.

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OPTICS

BY

LIONEL LAURANCE

THE ORTHOS PRESS

21, JOHN STREET, BEDFORD ROW
LONDON, W.C.

ERRATA

Page

7.—Line 4 from bottom, *add* "green" after "yellow"

28.—Line 3 from top, *for* $>$ *put* $<$

34.—*For* "Romford" *read* "Rumford"

34.—In the formula, *for* "C a" *read* "C a^2"

58.—In the formula, *for* "A + F" *read* "A"

64.—Line 5 from top, *for* "Snellen's" *read* "Snell's"

65.—Line 13 from top, *for* "velocity" *read* "velocities"

69.—Line 11 from top, *insert* "normal to the" *before the word* "hypotheneuse"

73.—Line 4 from top. A general expression is $\frac{t \cos r}{\mu \cos i}$

82.—Line 8 from bottom, *for* "dispersion" *read* "refractivity"; and *for* "refractivity" *read* "dispersion"

82.—Fig. 71. Delete the parallel light entering the slit.

89.—Last line, *add* Δ *after* 1.745

91.—Line 12 from top, *for* "A E" *read* "M E"

100.—Line 8 from top, *for* "61.4" *read* "62.6"

113.—Line 17 from top, *for* "plano-convex" *read* "plano-concave"

115.—Line 4 from bottom, *for* "surved" *read* "curved"

116.—Top line, *for* "r = .2" *read* "t = .2"

116.—Line 3, 2nd formula, *for* ".2 + 10" *read* ".2 × 10"

143.—Line 4 from bottom, *for* "∞" *read* "0"

180.—Lines 11 and 12 from bottom, *for* "centre" *read* "central."

181.—Line 5 from top, *for* "three-leg" *read* "four-leg."

206.—Line 9 from bottom, *for* "c" *read* "C"

246.—Line 12 from top, *for* "52 Δ" *read* ".52 Δ"

256.—Last line, *for* "F" *read* "F_B"

258.—Fig. 192 should show the two *convex* surfaces facing each other.

259.—Line 5 from top, *for* "2 in." *read* "4 in."

259.—Line 15 from top, *for* "F" *read* "F_B"

259.—Line 16 from top, *for* "F_B" *read* "F"

260.—Line 15 from top, *for* "Unifocal" *read* "Unofocal"

261.—Line 10 from top, *delete* "Adon lens"

261.—Line 20 from top, *delete* whole line in italics.

267.—Line 6 from bottom, *after* "which is" *insert* "measured from the lens situated"

279.—Line 13 from top, *for* "$F_2 f_1$" *read* "$F_2 f_2$"

319.—Line 2 from bottom, *for* "F t + 1" *read* "F_2 t + 1"

322.—Line 3 from bottom, *for* "F" *read* "F_5"

323.—1st formula, *for* "F" *read* "F_5"

323.—3rd formula, *for* "F" *read* "v"

326.—Fig. 249. The ray in the prisms A and B should be undeviated.

356.—Line 9 from top, *for* "R_1" *read* "R"

391.—Table of Refractive indices, *for* "Gum Damman" *read* "Gum Dammar"; *for* "Styrex" *read* "Styrax"

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PREFACE.

WORKS on general optics are usually either too mathematical or of a nature unsuited for the purpose of reading up for the Spectacle Makers' Examination, and for understanding and working out the simple problems which continually arise in the course of an optician's business.

While the author has tried to supply this want, he has not neglected to treat more completely those subjects which are intimately connected with spectacle work, and are rarely, if indeed ever, met with in any of the books written for schools, colleges, and technical Institutes. Also an endeavour has been made to give the student a practical insight into more complicated subjects such as lens systems, aberrations, etc., sufficient, at least, to serve as a foundation for the study of higher works.

This book, although it embraces a yet wider field, has, in fact, been written to cover the syllabus of the General and Practical Examination of the Worshipful Company of Spectacle Makers, which must be passed by all candidates whether they take the Final in Visual Optics or in Higher Optics and Optical Instruments. No attempt has been made to deal with the subjects required for these latter examinations, except in the most elementary manner possible in order to satisfy the requirements of the Preliminary Examination.

No excuse is needed for having departed from the academic convention as to signs in the formulæ connected with lenses and refracting surfaces. Convex and concave curvatures are assumed to be positive and negative respectively as they are convex or concave to the medium of lower index, thus giving positive focal lengths to those lenses and surfaces which tend to produce real images, and negative focal lengths to those which can form only virtual images. This convention serves perfectly well for systems of not more than three or four surfaces and it coincides with the common usage of the optical trade.

The author takes this opportunity of acknowledging the invaluable help and advice so freely given by Dr. Geo. Lindsay Johnson. He has done so much for this work that he might almost be claimed as a joint author.

The author's thanks are equally due to Mr. Oscar Wood for his assistance generally and, indeed, the brief description of the Gauss Equation and some of the articles on instruments were written entirely by him.

To those students who wish to pursue their studies further, the author would suggest the following works as being among the most suitable for their purpose, viz., Percival's "Optics," Silvanus Thompson's "Light Visible and Invisible," Lindsay Johnson's "Photographic Optics and Colour Photography" and his other works on optical instruments, Maddox's "Ophthalmological Prisms," Glazebrook's "Physical Optics" and "Practical Physics," Edser's "Light for Students," and the numerous contributions of Dr. C. V. Drysdale and Mr. S. D. Chalmers, M.A. Also Prof. Thompson's "Optical Tables" is of great value as a book of reference for opticians.

LIONEL LAURANCE.

London, *August*, 1908.

GENERAL AND PRACTICAL OPTICS.

CHAPTER I.

LIGHT.

Light.—Everything we see around us is rendered visible by means of a form of radiant energy which is termed light. With the exception of fluorescence, phosphorescence and a few manifestations of energy which need not concern us here, all light has its source in bodies which are in a condition of white heat or incandescence. It is not necessary that the source of light should be seen. Thus the sky, moon, trees, houses, etc., are visible by means of reflected light, which can invariably be traced to the sun, or to some artificial source of incandescence. The way in which the light from these sources reaches the eye, and so renders objects visible, has occupied mankind from the earliest ages.

It was once supposed that light was something which radiated from the eye to the objects seen, that is, a sort of tentacle or invisible feeler which darted out from the eye. Later it was thought to be due to infinitely small corpuscles which streamed out from every visible object to the eye at great speeds, but other phenomena of light showed this theory to be untenable, and proved beyond a doubt that light is due to a series of vibrations set up in the luminiferous ether by the molecular agitations of an incandescent body.

Ether.—This medium is believed to occupy all space throughout the visible universe, penetrating inside and between the molecules and atoms of which all bodies are composed, so that no body exists which is not saturated with ether, nor can any vacuum, however perfect, remove the slightest fraction of it. Exceedingly little is known about the nature of the ether, since its properties are chiefly negative, and it can neither be perceived by any of the senses, nor can it be collected or weighed. It has been concluded, however, that it possesses density, rigidity, elasticity, as well as the property of propagating transverse undulations or waves, which are generated by vibrations in incandescent material bodies. These waves travel to an infinite distance without appreciable loss of energy, and it is due to the presence of ether that material bodies are capable of acting on one another at a distance, and by which the various forms of energy, e.g., light, heat, magnetism, electricity, &c., are made manifest.

Light Waves.—Since every part of a source of light generates an oscillation which travels in every direction, let one of these parts L (Fig. 1) be considered an incandescent point of vibrating matter. This forms the centre of a tiny sphere whose diameter equals a wave-length λ_1 λ_2. Every point on the circumference of this sphere forms a new centre of disturbance which generates a fresh sphere, and each of these spheres again forms fresh ones, and so on. Now, as these tiny spheres may be supposed to lie side by side overlapping each other, tangents to points on their combined

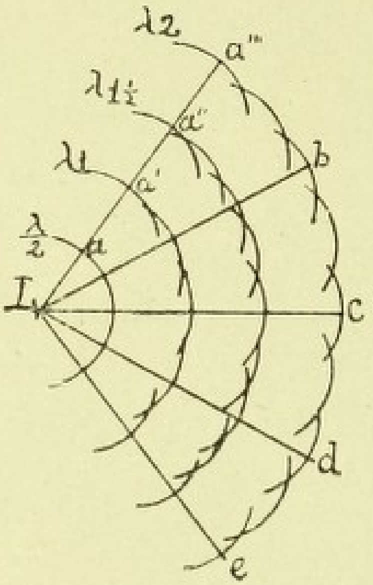


Fig. 1.

circumference (which points are ends of radii from the primary centre of disturbance) will, if taken collectively, form a wave-front (a''' b c d e). As each wave-front forms a centre for the formation of a fresh row of spheres, the diameter of each sphere is equal to a wave-length. Each successive wave-front may therefore be considered as the crest (λ_1 λ_2) and the space between it and the next wave-front as the trough of a wave ($\lambda/_2$ $\lambda_{1\frac{1}{2}}$). We may consider light as advancing in the form of a wave-front which forms part of an ever enlarging sphere.

The wave motion of the ether is always transverse, i.e., at right-angles to the direction of propagation of the light. The ether particles themselves do not travel but merely oscillate, much in the same way as a cork bobs up and down in the water as the wave passes by. Or, to employ another illustration, as the vibrations of a rope, fixed at one end, travel along it when shaken at the other extremity.

Although light is propagated from a luminous point in a series of wave-fronts, it is more simple for our purpose to consider the *direction* of the propagation, which can be shown by a straight line. From the luminous point L (Fig. 1) the light radiates in every direction, and any line of propagation, such as Lb, Lc, etc., is termed a ray of light.

Radiant Energy.—When the temperature of a body is raised, the increased molecular activity causes a generation of ether waves of diminished length and heightened frequency, which constitutes what is termed heat. If the temperature is raised still more, the activity is proportionally increased, so that the waves become shorter and the vibrations more rapid. Thus, when the temperature of a body reaches about 500° centigrade, it not only emits the relatively long waves of heat, but also the shorter waves of light; the difference between the two forms of radiant energy—heat and light—existing solely in the length of the waves. The undulations must be of a certain shortness and rapidity in order to become "light" as distinct from "heat."

In their passage to the earth, the calorific or heat rays radiated from the sun are, to a great extent, absorbed by the atmosphere.

Some bodies transmit light and not heat rays, and others the reverse. Bodies which transmit the invisible heat rays without becoming quickly warmed themselves are termed diathermanous; those which do not transmit radiant heat are termed athermanous or adiathermanous.

The longest light waves, i.e., those of least frequency, give rise to the visual sensation of red when the temperature of a body is raised to about 500° C. On further raising the temperature of a body, shorter waves are also produced which, being of different lengths and frequencies, cause the sensation of various colours, varying from red, the longest, to violet, the shortest visible waves. White is a sensation caused by the combined action of all waves ranging between red and violet. White is produced when the temperature reaches about 1000° C.

The existence of the spectrum beyond the visible red (infra red), which consists of longer waves, may be shown in various ways. Thus a blackened thermometer bulb placed just beyond where the red in the spectrum ceases will show a rise of temperature, proving the existence of heat rays. Again, by employing a lens made of rocksalt, heat waves can be demonstrated when the visible spectrum is cut off.

By painting the screen, on which the the spectrum is thrown, with a fluorescent liquid such as a solution of quinine, just beyond the violet end, or by using a quartz prism, the existence of rays beyond the visible violet end of the spectrum (ultra violet) may be shown to exist.

In addition to the effect on the eye, and the sensation of heat, it is obvious that light waves possess many other properties, such as the chemical actions which occur in photography, bleaching, the generation of carbonic acid, and the formation of chlorophyll necessary for vegetable life.

The incandescence of the sun is, of course, the principal source from which light on the earth is derived. Impact, friction, electricity, chemical combination, combustion, in fact anything which causes increased molecular motion may give rise to light. Sunlight is white, while artificial lights usually appear more or less coloured.

Velocity of Light.—Light travels in air at about 186,000 miles or 300,000 kilometres per second; the velocity is somewhat lessened in denser media, the decrease being roughly proportionate to the density; thus, in glass, the rate of progression is about one-third less and in water one-fourth less than it is in air. In space or a vacuum the speed is almost the same as in air. 186,000 miles is a distance equal to about eight times the circumference of the earth at the equator, a journey travelled by light in one second. From the sun it takes about eight minutes for light to reach the earth, some 93 million miles distant. At this rate light travels six million million miles in a year, and the distance of a fixed star is said to be so many light years, thus expressing the number of years the light from the star takes to reach the earth.

Measurement of Light-Speed.—There are at least four methods by which the velocity of light has been measured. The earliest methods, by reason of the imperfection of optical instruments, were of necessity astronomical ones.

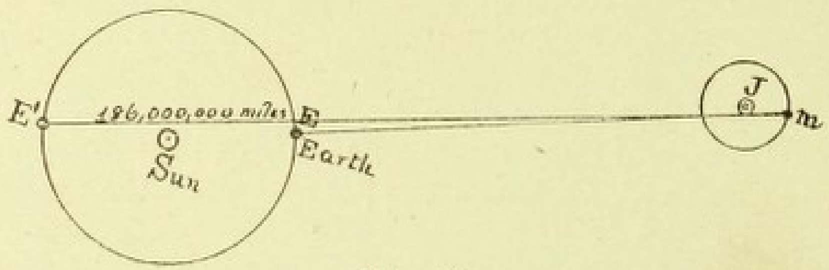


Fig. 2.

Römer's Method.—It was known that one of Jupiter's moons passed into the shadow of the planet every 48½ hours and became eclipsed. At a certain period of the earth's annual revolution the earth is in opposition to Jupiter (Fig. 2). If light were to travel instantaneously the eclipse and the observation of the same by an observer on the earth would occur simultaneously. The light, however, has to travel from Jupiter to the earth before the eclipse can be seen. Let R and r be respectively the radii of the orbits of Jupiter and the earth round the sun. Then J E (i.e., R—r) is the distance the light has to travel at a velocity V. The time therefore will be $(R - r)/V$ seconds after the eclipse has taken place. After six months the earth and Jupiter will again be in opposition, the earth now being at E′ on the other side of the sun. The eclipse will therefore be observed $(R + r)/V$ seconds after the occurrence, the difference between the two observations being equal to $2r = 186$ million miles.

Römer observed that as the earth moved from E to E′ the observed time steadily exceeded the calculated time. Thus he found that an eclipse observed when the earth was at E′ occurred 995 seconds later than when it was observed at E. Since the diameter of the earth's orbit is 186 million miles,

$$V = \frac{186,000,000}{995} = \text{about } 186,000 \text{ miles per second.}$$

Bradley's Aberration Method.—The apparent direction of light from a star owing to the earth's motion makes an angle with its true direction. The velocity of the earth being known, that of the light has been determined from observations of a star at different periods of the year.

The astronomer Bradley was led to discover this method when driving along a road in a shower of rain which happened to be falling vertically. He noticed that the latter instead of falling directly on his head beat against his face obliquely. This led him to consider that the movement of the earth round the sun will cause the calculated position of a star to vary slightly from its observed

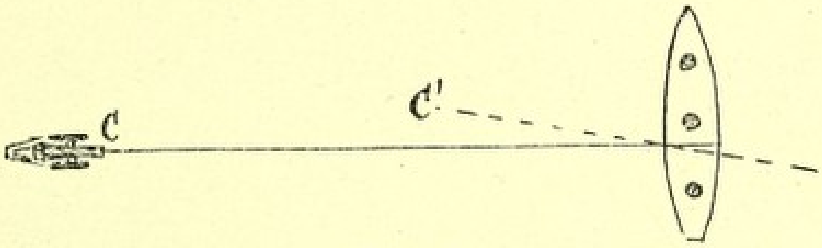


Fig. 3.

position. As the earth pursues its elliptical orbit round the sun it must move in an opposite direction to that which it took exactly six months before, so that a telescope directed to a star situated somewhere along a line at right angles to the earth's motion must be pointed slightly in front of the mean calculated position at the first period of observation, and a similar distance behind at the second observation. The angle which the telescope makes between the calculated and the observed position is called the aberration of the star.

Bradley knew the velocity of the earth's motion, and he measured the angle of aberration, and from these data he proved the velocity V of light to be about 180,000 miles per second. Thus:

$$V = \frac{\text{velocity of earth}}{\text{tan of aberration}} = \frac{18 \text{ miles}}{\tan 20''} = \frac{18}{0{,}0001}$$

= 180,000 miles per second, which is nearly correct.

This may also be illustrated by the experiment of firing a shot at a moving object. Thus, if a shot from a cannon C (Fig. 3) be directed towards a ship moving at right angles to the direction of the shot, the latter will not pass through it at right angles to the direction of the ship, but obliquely as if the shot came in the direction of the dotted line C′.

Fizeau's Method.—Fizeau's method depends on the interruption of a beam of light by the teeth of a revolving wheel. The light from a source S (Fig. 4)—rendered convergent by a lens L—falls on a plane unsilvered mirror m which is inclined at 45° and situated between the lens and its focus F, the latter being at the teeth of a wheel. Another lens L′, placed at its principal focal length on the other side of the wheel and in a line with the mirror, renders the light from F parallel. The beam of light is collected by a third lens L″, situated at a distance (say four miles), and is brought to a focus on a spherical mirror M, from which it is reflected, so as to return along the same path, finally forming a real image at F, and is viewed by the observer at E through an eyepiece.

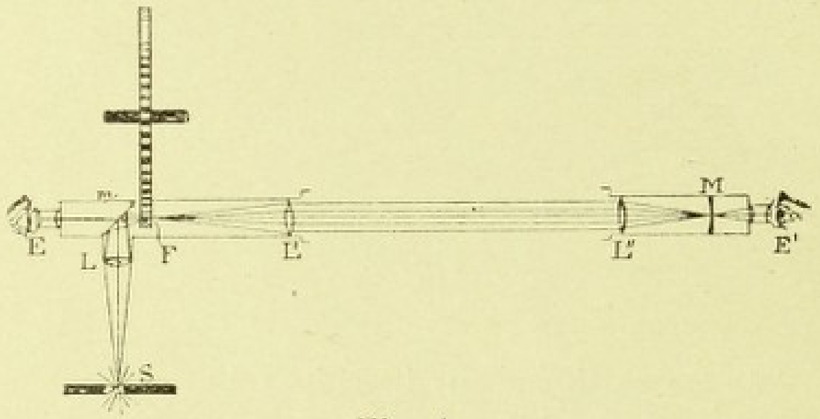


Fig. 4.

Suppose the light escapes through the first gap while the wheel is turning slowly, then it will, after travelling eight miles, pass through the same opening and a flickering image is seen. If the speed is greater the second tooth blocks out the light, but if still greater the light passes through the second gap, the wheel having revolved one tooth while the light travelled eight miles, and so reappears to an observer at E. The result is checked by another observer at E′ who sees the light through an opening in M. The speed of the wheel being further increased the light appears and disappears as an additional tooth or gap passes by before the light returns. The speed of the toothed wheel, the size of the teeth, and the distance between m and M being known, Fizeau, and later Cornu, who improved on the apparatus, found the velocity of light in air to be about 300,000 km. per second.

Foucault's Method.—A beam of light is passed through a slit S and a lens L on to a plane mirror M_1 which is made to revolve at a known speed. From M_1 the light passes to a concave mirror M_2 placed at a distance equal to its radius. From this the light is again reflected back to M_1 and retracing its path is partly reflected by the glass plate M_3 to the eye at T. If M_1 is then rapidly rotated it will have had time to turn through an appreciable angle during the time that the light has travelled from M_1 to M_2 and back again, so that it will not be reflected back to the same spot on the mirror M_3. Thus the image seen by the observer through the telescope will not be formed on the cross wires at a,

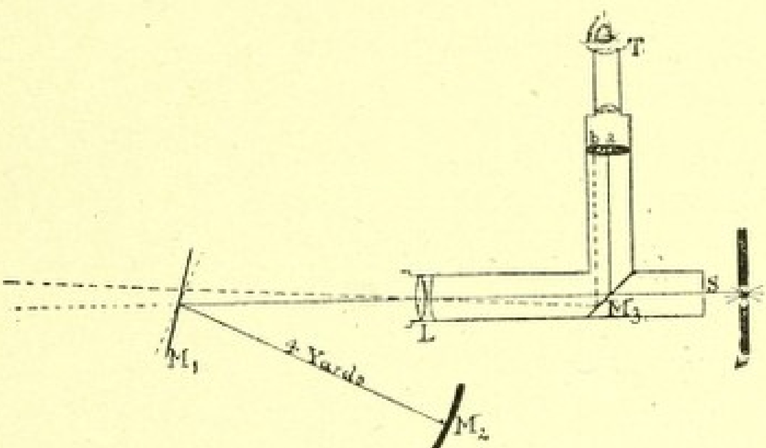


Fig. 5.

but will be found shifted to some point b. If the speed be known at which the mirror M_1 is rotated, and the distance which the light has to travel from M_1 to M_2 and back (which in this case is equal to eight yards) the velocity of light can be calculated by the displacement of the image from a to b as seen through the telescope T.

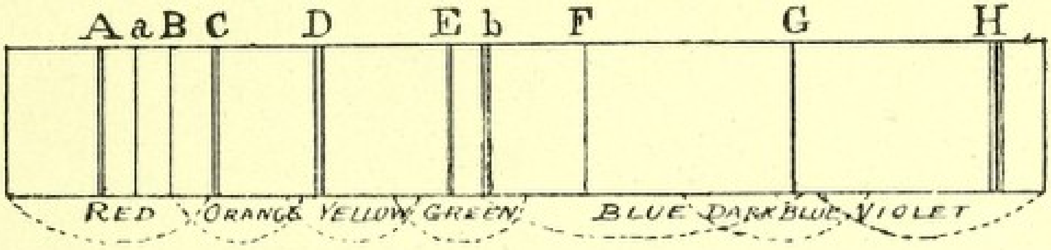
The Spectrum.—Sunlight being produced by a combination of waves of different lengths, its component parts are retarded unequally when passing through a dense medium. The shorter violet waves are more retarded and, if refracted, these are bent to a greater extent than the longer red waves so that the component colours of white light become separated. If white light be passed through a prism, the dispersed colours can be seen on a screen as a bright coloured band, called the spectrum, which consists of red, orange, yellow, green, blue, indigo and violet, but most authorities now omit the indigo and consider the spectrum to consist of only six colours. The following table shows the entire range of the spectrum as far as is known.

TABLE OF WAVE-LENGTHS AND FREQUENCIES.

Wave-lengths in $\mu\mu$	No. of vibrations in billions per second.		Character.
100,000,000 (100 mm.) 3,000,000 (3 mm.)			Electrical vibrations. Shortest are about 3 mm. Longest 1 meter to several miles.
61,000	4.8	Infra-red spectrum	Longest waves measured by Langley by his bolometer.
8,000	37		Longest waves measured by Ruebens and Snow by fluor-spar prism and bolometer.
812	370		Longest waves capable of being seen by the spectroscope, according to Helmholtz.
750	400	Ordinary visible spectrum	Red.
650	460		Orange.
590	508		Yellow.
530	566		Green.
460	652		Blue.
420	710		Indigo.
375	800		Violet.
330	909	Ultra-violet spectrum	Shortest waves visible according to Soret.
210	1430		Shortest waves visible according to Mascart.
185	1620		Shortest waves photographed through fluor-spar prism alone.
100	3000		Shortest waves photographed by means of fluor-spar prism, vacuum camera and bromide of silver plate without gelatine.

NOTE.—A billion is a million times a million. A micromillimetre $\mu\mu$ = one millionth of a millimetre. A micron μ = one thousandth of a millimetre.

Fraunhofer's Lines.—Fraunhofer, who first reduced spectroscopy to a science, discovered that incandescent gases give rise to certain bright lines in the spectrum which were characteristic of each particular element of which they are composed. Kirchoff also found that if the light were passed through gases or vapours, cooled below the temperature of the source of the same elements, these bright lines would become absorbed, and they would be replaced by dark lines, identical in position and thickness. These lines are found in great numbers in the solar spectrum, and are indicated by letters of the alphabet. As they always correspond to rays of a definite wave-length, they form a convenient means of identifying any particular part of the spectrum. The following is a diagram of the spectrum, showing the chief Fraunhofer lines in their approximate position.



Line.	Position in spectrum.	Metal or gas producing the line.	Wave-Lengths.
			$\mu\mu$
A	Red	Oxygen (O)	759
a	Red	Water Vapour	733
B	Red	Oxygen	686
C	Orange-red	Hydrogen (H)	656
D	Yellow	Sodium (Na)	589
E	Green	Iron (Fe) Calcium (Ca) ...	527
b	Blue-green	Magnesium (Mg)	518
F	Blue	Hydrogen	486
G	Dark Blue	Hydrogen Iron	430
H	Violet	Calcium (Bright Line) ...	397

The visible spectrum approximately consists of those light waves whose lengths vary between 800 and 400 $\mu\mu$, and whose vibrations respectively vary between 400 and 800 billions per second. The mean refractive index of glass or any other substance is expressed by that of the D line and written μ_D. For achromatising glasses for visual purposes the lines C and F are combined. For photographic purposes where violet light plays an important part, the lines D and G are usually combined. For astrophotographic purposes in which vision is of little consequence lines F and H (or beyond) are brought together.

Speed and Frequency of Light.—The speed of light in air is 300,000 kilometres per second, and there are a billion $\mu\mu$ in a kilometre; the frequencies of wave-lengths in a second of time varying inversely with the length of these waves. If we express the length of the waves in billionths of the kilometre, that is to say in $\mu\mu$, and the frequencies in billions per second, then by dividing 300,000 by the wave-length in $\mu\mu$ the number of billions of frequencies per second for any kind of light is obtained. In the most luminous part of the solar spectrum, the number of billionths of a kilometre of the wave-length is equal to the billions of frequencies per second, namely, about 548.

Complementary Colours.—It is usual to denominate red, green, and blue-violet as *primary* colours because their visual impressions cannot be reproduced by a mixture of other spectrum hues. The intermediate colours are termed *secondary* because they are

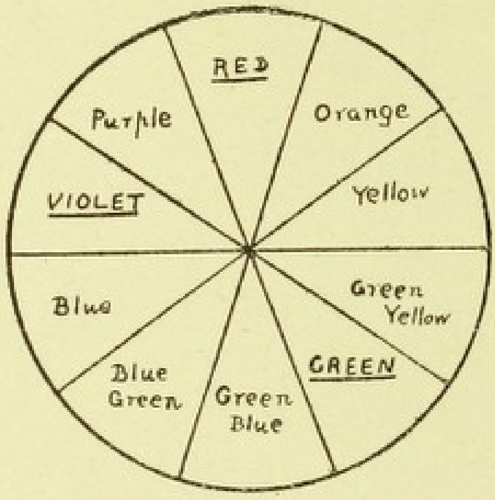


Fig. 6.

Diagram illustrating complementary colours in light.

obtained by the combination or overlapping of the primaries. A complementary colour is that which, when combined with another colour, forms white light. Thus red and green-blue form white, so also do violet and green-yellow. The complement of green is purple, which is not in the spectrum. Yellow is formed by the fusion of orange-red and green and is the narrowest of all the colours of the spectrum, and its complement is blue.

The primary green is not the brightest part, but inclines rather towards the yellow, and the primary violet inclines rather towards the blue.

Colour Sensation.—According to the theory propounded by Young and Helmholtz, there exist in the eye three primary colour sensations which are supposed to be conveyed to the brain by three sets of nerves, causing the sensation of red, green and violet respectively. Each set of nerves conveys, however, not only the sensation of its special colour but also to a slight extent that of the other two. By stimulating one or more of these nerves, in varying proportions, all the colours of the spectrum can be conveyed to the brain. These curves are shown in the diagram (Fig. 7).

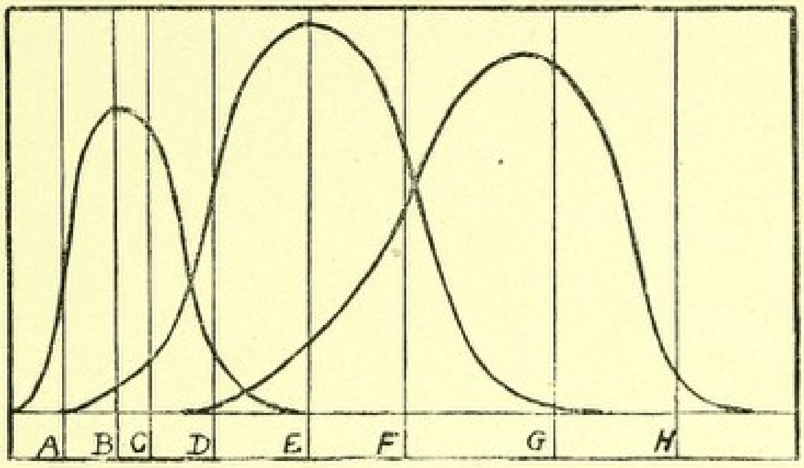


Fig. 7.

Colours of Light.—A mixture of spectrum red and spectrum green will if mixed in certain proportions produce a sensation of yellow. If spectrum red, green and blue-violet in the right proportions be mixed white light is formed. If the wave-lengths of red and green be added together the mean will give the wave-length of yellow. Thus, taking the wave-length of orange-red as equal to 656 and that of blue-green as 518, then 656 + 518 = 1174 and 1174/2 = 587, which corresponds to the wave-length of yellow light. Again, taking the wave-lengths of the three colours red, green, and blue respectively, the sum divided by three will give the wave-length for the brightest part of the yellow, which is the nearest approach to white light which the spectrum affords. Thus, 748 + 527 + 486 = 1761 and 1761/3 = 587, which is about the wave-length of the whitest part of the spectrum. The quantity of light of one colour necessary to mix with any other to produce white light, or a third colour, does not appear to follow any law, but the proportions usually remain the same for different observers; occasionally, however, the amount is found to be very different, even among persons who are not colour blind to standard tests.

Many colours, such as brown, pink, and purple, do not exist in the spectrum, the latter being a combination of extreme red and extreme violet light. A colour is said to be saturated when pure or not diluted with white. A colour such as brown, is merely orange diluted with white.

If three pieces of coloured glass corresponding to the summits of the three curves of red, green and blue-violet be superimposed, since each absorbs some of the components of white light, the three will absorb the whole of the visible spectrum and no light whatever can be seen through the combination. If, on the other hand, the three colours be thrown on a screen from three separate lanterns and made to overlap the result is an approximately white disc.

The natural colours of objects may be imitated by applying the above facts to photography. Three separate photographs are taken of an object or landscape, made up of any number of colours and shades, each through a glass selected to match as nearly as possible one of the three primary colours. A positive is taken from each on a film or paper, stained with the colour complementary to the colour of the glass used for that particular negative, and the three prints are superposed. This may be done either by laying the films exactly over each other and looking through them as a transparency, or each colour may be printed on the same piece of paper, and examined as an opaque object. In this way an approximate facsimile in colour of the original object can be obtained.

Black may be described as a sensation caused by want of colour, but it is very different from what is seen, or rather not seen, in the area occupied by the blind spot, the latter being incapable of conveying any sensation of light at all and is quite different from black, which produces a distinct sensation. That is why the area occupied by the blind spot is unnoticed when we look at the sky or other bright field.

Composition of Solar Light.—The various colours in the spectrum are not sharply separated, but merge so imperceptibly into one another that it is almost impossible to locate where one colour ends and another commences. The space in the spectrum, formed by a prism, occupied by the different colours varies with the refracting medium used for its production. If a spectrum of solar rays refracted by a given prism of flint glass be divided into 360 parts the proportional space occupied by each colour will be approximately as follows—red 50, orange 35, yellow 15, green 50, blue 60, indigo 50, violet 100; total, 360.

The red end of the spectrum is, however, crowded together by the refraction of the prism, while the blue end is spread out considerably; for actually the extent of spectrum on either side of the central green-yellow is about equal, as is seen in the spectrum obtained by a diffraction grating. For similar reasons, i.e., the crowding at the red and the spreading at the blue end of the prismatic spectrum, the red end seems brighter than the blue, but actually the luminosity of the spectrum diminishes about equally on both sides from the most luminous part, which is the yellow.

Sunlight is said to consist of about 30 parts green, 50 parts red, and 20 parts violet in 100 and has about 30 per cent. of luminous rays. Artificial light has a higher proportion of heat or red rays and the proportion of luminous rays is much smaller, varying from 20 per cent. for electricity (arc), 10 per cent. for oils and coal-gas, to one per cent. for alcohol.

Luminous Bodies.—Waves of light are termed *incident* when they fall on a body. A body is said to be luminous when it is, in itself, an original source of light. Every visible body, which is not in itself a source of light reflects some of the light received from a luminous source, but it may be convenient to consider that every visible body is luminous and therefore a source of light. A body is rendered luminous by the light emitted or radiated from every point of it. The rays diverging from these points travel without change so long as they are in the same medium.

Transparency and Translucency.—A body is said to be *transparent* when light passes freely through it with a minimum of absorption or reflection. It is called *translucent* when it transmits only a portion of the light, such as frosted glass and tortoise-shell. Much of the light incident on such a body is reflected, scattered or absorbed, and so objects cannot be seen clearly through it.

Reflection.—Reflection is the rebound of light waves from the surface, on which they are incident, into the original medium. The reflection is *regular* from a polished surface and *irregular* from a roughened surface. Irregularly reflected light causes the reflecting surface to become visible; regularly reflected light causes the image of the original source of light to be seen.

The rougher the surface, the greater is the proportion of irregularly reflected light; the smoother the surface, the greater that of regularly reflected light. The proportion of light regularly reflected from a partially roughened surface is increased as the angle of incidence of the light becomes greater, so that a reflected image may be obtained, with certain positions of light and surface, from a body which ordinarily gives no definite reflected image.

Total regular reflection never occurs, for even a silvered mirror or highly-polished surface of metal fails to reflect all the light falling on it, but the proportion reflected by metallic surfaces does not vary with the incidence of the light as is the case with glass. Polished silver reflects some 90 per cent., polished steel some 60 per cent., and mirrors reflect about 70 to 85 per cent. of the incident light. Nor is there ever total irregular reflection; even fresh snow absorbs some of the light it receives.

Opacity.—A substance is said to be opaque when all the rays of light, incident on it, are either absorbed or reflected, so that none traverse it.

Opacity, Transparency, Absorption, and Reflection.—No substance is absolutely transparent, the clearest glass or water absorbs some of the incident light. It is estimated that below 50 fathoms the sea is pitch dark, at least to the human eye, and even glass of sufficient thickness is opaque. Again any ordinary opaque object such as stone, metal, etc., may be ground or hammered into a sheet so thin as to permit the passage of some light through it.

Thus gold leaf of sufficient thinness is translucent and transmits greenish rays. It follows, therefore, that transparency and opacity depend not only on the nature of the medium, but also on its thickness.

A body which is usually opaque may be rendered translucent by making it less capable of reflection. This fact is very often made use of in practice. For instance, if a drop of Canada Balsam be dropped on to a camera focussing-screen, and a cover glass pressed over it, the screen becomes immediately transparent at that spot, so that the aerial image may be readily focussed with a magnifying-glass, and very minute details observed. The liquid occupies the spaces between the particles of the surface and, being of the same index of refraction, converts the whole into a homogeneous refracting body which transmits nearly all the light. Moistening a piece of paper with oil or water makes it much more translucent for the same reason. The fibres of which the paper is made are of a higher index of refraction than the air, so that, when the latter is replaced by oil or water, the two indices are then more nearly alike; and being homogeneous, less light is scattered.

Some of the incident light is reflected from the polished surface of a transparent body, and the proportion reflected varies with the nature of the body and with the angle of incidence, it being greater as such angle increases. The proportion reflected is very small (about eight per cent.) when the light is incident perpendicularly and it is almost totally reflected if the angle of incidence is nearly 90°. Also the proportion reflected increases as the index of refraction of the medium is greater and *vice versa*. If glass is dusty, the irregularly reflected light is increased and the glass becomes more visible. Scratches on a piece of glass roughen the surface and so tend to destroy its transparency by irregularly reflecting the light. If the scratches be multiplied indefinitely, the glass ceases to be transparent and becomes translucent.

Thus, in the case of every transparent body, some of the incident light is always transmitted, some absorbed and some reflected. Of the light falling from all sides on to a piece of well-polished transparent glass, about 75 per cent. is refracted and transmitted, 15 per cent. is regularly reflected and gives an image of the source from which the light proceeds, about five per cent. is irregularly reflected, and so makes the glass itself visible, while the remainder is lost, being absorbed and changed into heat, etc.

Polarising Angle.—If with perpendicular incidence all the light is transmitted and none reflected, and if with an extremely oblique incidence (nearly 90°) none is transmitted and all is reflected, there must be some angle of incidence at which half the light is reflected and half transmitted and refracted. This occurs when the light is incident at what is termed the polarising angle, and then the reflected and refracted rays are at right angles to each other, as shown in the following diagram—

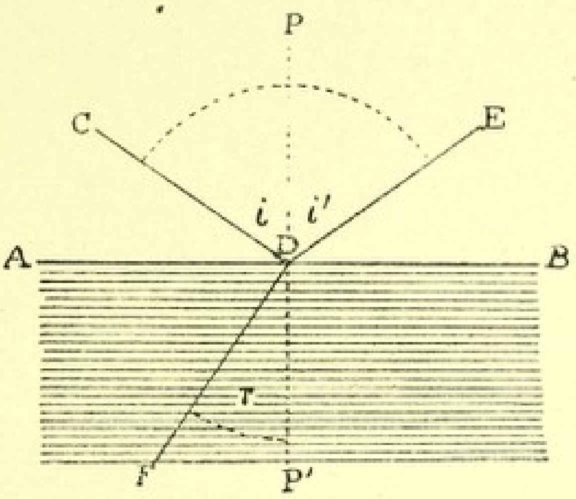


Fig. 8.

In Fig. 8 A B is a polished surface of glass, on which a beam of light E D is incident at the point D. The reflected light is represented by D C and the refracted light, at right angles to it, by D F. The perpendicular, at the point of incidence P D P′ is the normal or perpendicular to the surface, P D E is the angle of incidence, P D C the angle of reflection, and P′ D F the angle of refraction. The polarising angle of glass is about $57\frac{1}{2}°$.

Colours of Bodies.—A substance is said to be of a certain colour when it reflects or transmits rays of certain wave-lengths and absorbs the rest of the spectrum. Thus an object which absorbs the violet and green and reflects the red waves appears red. If it absorbs red waves and reflects green and violet it has a blue colour. A green body absorbs all but the green waves. One which is orange in colour reflects red and green and absorbs violet. The colour reflected by a body is usually the same as that which it transmits, but some bodies transmit the complementary colour to that which they reflect. Dark colours reflect little light, and slight differences between them are hardly appreciated in dull illumination; similarly, light colours reflect much light, and slight differences are hardly noticed in very bright illumination. All colours lose their distinctive hue in proportion as the light reflected becomes reduced. Thus with dull illumination all colours appear dark grey. The proportion of light reflected varies with the nature and colour of the body. Approximately a coloured body reflects 20 per cent to 50 per cent. of the light which falls on it.

A body which reflects light of all wave-lengths is called white; a body which has affinity for all the colours, so that all are absorbed and none reflected, is called black. No body, however, is of a nature so chemically pure as to absorb entirely or reflect all the incident light; the whitest body, such as fresh snow, reflects about

70 per cent. of light, white paper reflects from 35 to 40 per cent., while black velvet reflects about one per cent., the proportion reflected increasing with the thickness of the body.

Colour is a quality of the illuminating light itself, and not of the body which reflects it. In order to appear of a certain colour, the object must receive that colour in the light and reflect it, and at the same time absorb all the other colours.

Coloured Bodies and Lights.—The real colour of a body is that which it exhibits in white sunlight; it often appears of a different colour in ordinary artificial light. For example, some blues and greens can barely be distinguished by gaslight and still less by lamp or candle light. Moreover, a deep orange surface in bright daylight creates the same impression that a white one does when seen by lamp light. The nearer the light of the illuminant resembles that which the body absorbs, the darker the latter will appear, since it will reflect less light. Should the light be of a colour exactly corresponding to that which the body absorbs, none will be reflected, and the body will consequently appear black.

If a coloured body be viewed through a coloured glass, which absorbs the rays reflected by the body, the latter appears black. Thus a red body appears black through a green glass of the proper shade, the red rays reflected by the body not traversing the glass. A body viewed through a glass, or by light, of the same colour appears almost white, or at least barely distinguishable from a white object.

If a blue-green and a red glass be placed together, the light transmitted by the one is absorbed by the other, and the combination is rendered opaque. Cobalt glass transmits red and blue, ordinary green glass transmits blue and green; on the two being placed together original white light transmitted appears blue.

Mixing Pigment Colours.—The primary colours in pigments (paints or colouring matter) are red, yellow, and blue. Any other colour is obtained by mixing two primaries. Thus, blue and yellow pigments combined make green. A blue pigment reflects violet and green, yellow reflects red and green. If the two pigments be mixed, there are reflected a certain quantity of violet and of red and a double quantity of green. The red, the violet, and a portion of the green combine to form white light, so that there is a residue of green light which gives the nature of the colour to the mixture of the two pigments. If coloured crystals are pulverised they become lighter and more white in colour since their power of absorbtion of colour becomes less. Thus, if bichromate of potash or sulphate of copper crystals are reduced to an impalpable powder the result in either case is a nearly white dust.

In some instances the result of the mixture of pigments is surprisingly different from mixing the same colours in light. Thus if yellow and blue spectrum colours are superposed on a screen they produce the sensation of white, but if yellow and blue pigments are mixed they cause the sensation of green. If we draw yellow and blue sectors alternately on a disc, which is rapidly spun round, the result is white, provided the colours are of the right tint and of the right proportion. Otherwise, especially in artificial light, they might give a purple tint. The colour, whether white or purple, is always muddy or impure because true spectrum colours can never be produced by pigments or artificial means. In pigments, a secondary colour is obtained from the mixture of two primary colours. A tertiary colour results from a mixture of two secondary colours. Combining the three primary colours in certain proportions produces grey; also, if they are mixed in certain other proportions and diluted with white an innumerable number of intermediate colours can be formed.

Complementary Colours in Pigments.—The complement of a primary colour is that secondary colour which results from the mixture of the other two primaries. The complement of a secondary colour is that primary colour which is not contained in it. Thus the complement of red is green, which is formed by combining yellow and blue. The complement of yellow is violet, which is a combination of blue and red. The complement of blue is orange, which is produced by a mixture of red and yellow.

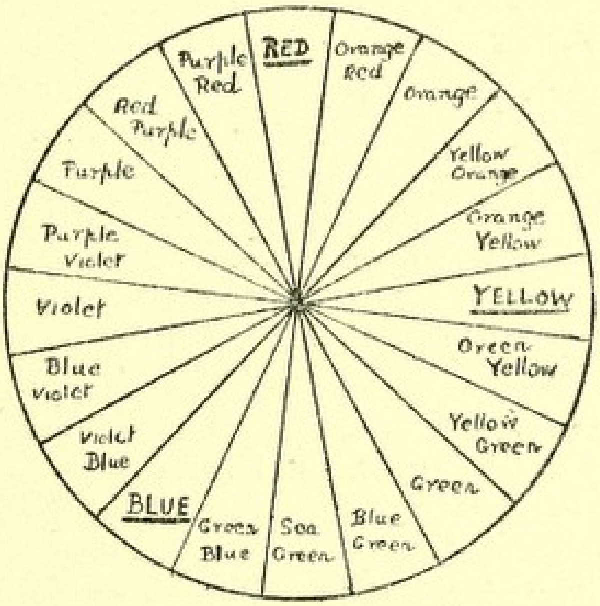


Fig. 9.

Diagram illustrating Complementary Colours in Pigments.

Qualities of Colours.—Colours in pigments possess three qualities, viz., tone, brightness, and purity. Tone or hue is that quality which differentiates between the various colours—say, red and orange; it depends on the wave-length. Brightness, intensity, or luminosity is that quality which represents the strength of a colour; it depends on the amount of light reflected; one which reflects little light is a dark colour, and one which reflects much light is a light colour. Fullness, saturation, tint, or purity is that quality which represents the depth of a colour; the less the admixture of white or black the purer is the colour. Red mixed with white forms pink, whereas red mixed with black makes a kind of maroon. Yellow or orange becomes straw or brown according as it is mixed respectively with white or black.

Density of Media.—The speed with which light travels within a certain medium depends on the nature of the medium, or, more exactly, on the elasticity of the ether within it; thus, light travels more slowly in a dense medium, i.e., one in which its component particles are crowded together like glass, than in a rare one, such as air.

Linear Propagation of Light.—The propagation of light is rectilinear, and the familiar instance of sunlight, admitted through a hole in the shutter into a darkened room, illustrates this fact by the illumination of the dust particles in the air along its path. The illuminated dust renders the course of the light visible, for, were the air to be deprived of it by filtration, the space over which the light passes would be invisible. The direction of a wave-front, as it advances, may be considered a line which is called a ray. The rays of light diverging from a luminous point form a cone, of which the point itself is the vertex, and such collection of rays is called a *pencil* of light.

Divergence of Light.—In nature, rays of light always diverge from luminous points. If the luminous point be very distant the angle of divergence becomes so small that the rays may be considered parallel to each other, and the luminous point is then said to be at infinity. A collection of parallel rays is called a *beam* of light.

As light radiates from luminous points which have no real magnitude, any body on which it falls must be larger than such points, the pencil from any given point constituting a cone of which the point of origin is the apex (or vertex) and the illuminated body the base. And from a luminant of sensible size an innumerable number of such cones of light proceed, all having their bases on the illuminated object.

The angle of divergence is that angle included between the rays, proceeding from the luminous point, which fall on the outermost edges of the object; consequently the angle of divergence varies inversely with the distance between the source of light and the illuminated body, and directly with the size of the latter. Rays of light which diverge from a very distant point are always regarded as parallel, and those from a near point as divergent. This being so, there must be some distance at which divergence merges into parallelism.

Parallel Light.—In visual optics, 20 feet or 6 metres marks the shortest distance from which light may be regarded as parallel, and this distance, or any beyond it, regarded as infinity which is written thus ∞. For some branches of optics a much greater distance is taken as the divergence limit. Thus in photographic optics it may amount to 100 yards or more, while in astronomy the nearest ∞ point may be taken at several miles. If d is the angle of divergence, a the aperture of the lens, and S the distance of the source, the divergence of light is, with sufficient exactitude, found from tan d = a/S. For example, suppose the source of light is at 6 M and the pupil of the eye 3.5 mm, then the visual angle of divergence will be 2′, for

$$\tan d = \frac{3 \cdot 5}{6{,}000} = .0005 = \tan 2'.$$

Since a divergence of 2′ is so small as to be negligible, it explains why 6 M is considered ∞. At 20 cm., with the same pupil, the divergence of the light is one degree.

Light is never naturally convergent, but can be rendered so by means of a lens or reflector. A collection of convergent rays is also called a pencil of light. The apex of the pencil, towards which they are convergent, is the focus.

Small Apertures.—If the light from a candle be allowed to pass through a small aperture on to a white screen, an inverted image of the flame is seen. The relative size of the image and of the flame itself are as their respective distances from the aperture; thus they are equal in size when the two are equi-distant from the aperture. The image is smaller if the screen be brought nearer to the aperture or if the candle be moved further away, and *vice versa*. Generally, the smaller the aperture, the sharper but less bright is the image. The shape of the small aperture does not materially affect the distinctness of the image, nor does it have any appreciable effect on its shape. This is seen when the sun shines vertically through the gaps in the foliage of a tree. Each of these gaps varies in size and shape, but the luminous images of the sun form bright discs on the ground, all identical in shape unless the gaps are large.

In order that a distinct image of a flame may be seen on a screen, it is necessary that the rays from each point of the luminous body should have a separate focus on the screen. This may be said to occur when the light passes through a minute aperture, because then only a small beam of light from each point can reach the screen, and for the same reason the image thus formed is faint. If twenty apertures be made, near one another, twenty images of the flame will be seen on the screen, and the number of images will increase with the number of holes, until the images will so overlap one another that it will be found impossible to distinguish them separately, in which case there will be a general illumination of the screen.

The Flame.—A flame consists of three cone-shaped portions, viz. :—

(A) The dark central portion surrounding the wick is called the cone of generation or obscure cone. It is of low temperature and is composed of gaseous products holding in suspension fine carbon particles which have not yet become incandescent.

(B) The luminous part surrounding A, called the cone of decomposition or luminous cone, in which the carbon is in a state of intense incandescence, and in which luminosity is greatest.

(C) The thin external envelope which is light yellow towards the summit and light blue at the base. It is the cone of complete combustion giving but little light, and is the main source of heat. The temperature is high and combustion complete on account of the free access of the oxygen of the air.

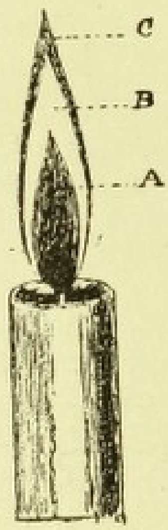


Fig. 10.

The flame in general is brighter at the top where the light predominates, and darker towards the base where heat is in excess. The outer envelope, being mixed with oxygen, is called the oxydising element, while the inner cone, consisting mainly of unconsumed gas, is called the reducing element of the flame, since at that spot metals may be reduced from their compounds.

A flame is produced by the incandescence of carbon particles which have been brought to a high temperature; the combustion, when once started, being continued owing to the heat produced by the chemical action itself. In a lamp or candle flame the material consumed is drawn up by capillarity through the wick.

Heat being produced by combustion, and luminosity being the result of the incandescence of unconsumed particles of carbon, the luminosity of a flame is low when combustion is complete, as is the case with the flame of some gases and of alcohol. It is high in a coal-gas flame, or in that produced by the combustion of oils and fats, where a considerable quantity of incandescent carbon is present. If the combustion be intensified by the introduction and intimate mixture of a sufficient supply of oxygen, as is done in the Bunsen burner (Fig. 11), in which coal-gas is consumed, luminosity is decreased and heat is increased; the flame produced is then of a faint blue instead of the usual yellowish colour. The oxyhydrogen

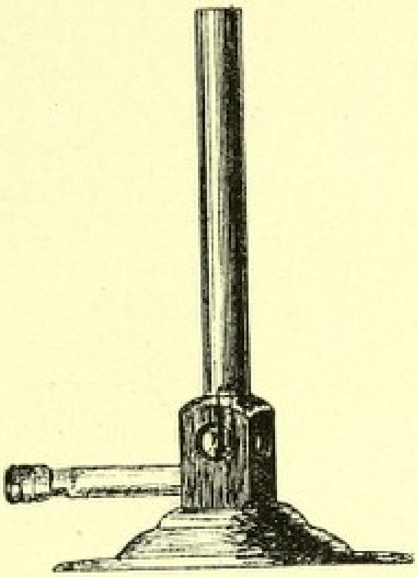


Fig. 11.

flame also gives very great heat, and yet is of a pale bluish colour and almost invisible; but when made to impinge on a lime cylinder, it renders it white hot at the point of contact, giving rise to an intensely brilliant spot of light, so that the temperature of a flame is neither indicated by the luminosity nor by the colour alone. To obtain maximum luminosity the supply of air must be neither too large nor too small. If too large the carbon is consumed too quickly, and if too small the carbon passes off unconsumed as soot.

Interference.—If from two adjacent points of light P_1 and P_2 (Fig. 12) waves of light are propagated, the crests and troughs of the waves from P_1 will coincide with those from P_2 along certain lines marked B, and they reinforce each other, thus causing

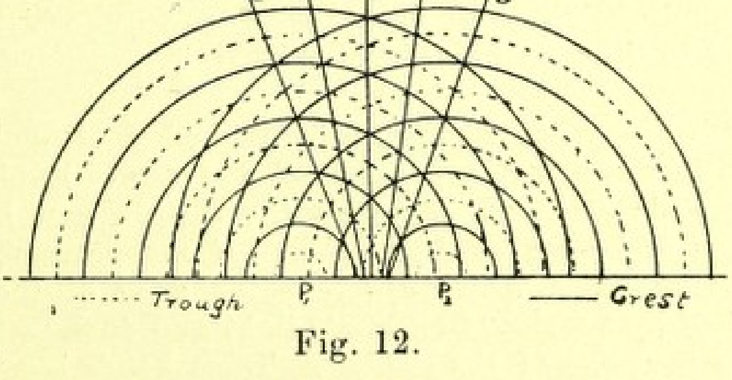


Fig. 12.

increased wave motion. Between these lines marked D the crests from the one source coincide with the troughs of the waves from the other, with the result that the wave motion is neutralised at these spots owing to the *interference* of the one set of waves with the other. Alternate lines of light and darkness, known as interference bands or fringes, are in this way produced.

The light bands are along lines so situated that any point on them is a whole number of wave-lengths from P_1 and P_2. The dark bands are along lines so situated that any point on them is one half wave-length further from the one source than the adjacent white band. The shorter the waves which interfere with each other, the less is the distance between the light and the dark bands.

If, as in white light, there are waves of different lengths, the interference bands, instead of being alternately light and dark, take the form of coloured bands which are alternately red, blue, and white, the latter occurring where all the various colour bands coincide. At certain distances from the central bright band, the fringes disappear owing to the coincidence of so many bands of different colours.

The colours of thin films, such as soap bubbles, layers of grease on water, etc., are due to interference. The light is reflected from the outer surface, and again from the inner surface of the film, and the light reflected from the two surfaces is not in the same phase—that is to say, similar points in two waves are not simultaneously moving in the same direction and to the same extent.

Newton's Rings.—When two plane or two similarly curved surfaces, the one convex and the other concave, are placed in contact, the film of air contained between them is of equal thickness, but if the one surface is not truly plane, or of exactly similar curvature to the other, the film of air is of varying thickness, and colours, due to interference, are exhibited. This constitutes a method of determining a true plane or a uniform curvature. If a convex surface is placed in contact with a plane, or another convex surface, the film of air contained between them must be of gradually increasing thickness. At the centre the film is very thin, and a central black spot results, which is surrounded by a series of alternately bright and dark rings if monochromatic light is employed, or by coloured rings if the incident light is white. These are termed *Newton's Rings.* If the rings are viewed by transmitted light the centre is bright and the surrounding rings are alternately bright and dark, or of colours which are complementary to those seen by reflected light.

Diffraction.—When light reaches the edge of a body owing to its undulatory motion, some of the waves bend round the edge of the obstacle and penetrate the shadow cast by it. This phenomenon is known as *diffraction.* If monochromatic light is admitted through a small aperture the edge of the shadow is characterised by a series of alternate light and dark bands or rings, parallel to the edge of the shadow. These bands become less and less distinct as they are further from the edge of the shadow, and they are broader in proportion to the length of the waves. If the source of illumination be white light, the diffraction fringes of the different colours overlap and a series of coloured fringes are seen.

It is essential that the aperture be narrow, or small, since otherwise the unimpeded waves so out-number the retarded ones that the diffraction effect is more or less lost. The wider the opening, the narrower do the bands become. If a very fine obstacle such as a hair or thin wire be placed between the light and a screen, a series of fringes can be seen both within and beyond the geometrical shadow. If the obstacle be circular, such as a small round patch on a piece of clear glass, the shadow is seen surrounded by alternate light and dark rings, or, if the source be sunlight, by a series of spectra. These bands encroach on the shadow, at the centre of which a bright dot can be seen.

The sun shining on twigs and leaves of trees causes a glistening appearance due to diffraction bands. A star seen through a perfectly corrected telescope, or small objects seen by the microscope, appear bordered by one or more faint rings.

Owing to diffraction, there is a limit to the possible magnifying power of a microscope, since the higher the power of the objective, the smaller the lenses, and consequently the more marked the diffraction phenomena.

The colours of many beetles, butterflies, and of mother-of-pearl are caused by diffraction and interference phenomena, and are not due to pigmentation at all.

Diffraction Grating.—If a large number of very fine equidistant lines—some thousands to the inch—be ruled parallel to each other on a plate of glass or metal it forms a *diffraction grating.* The light transmitted through it, or reflected by it, forms a series of spectra which can be thrown on a screen or be examined by a telescope, and the finer and closer the lines the purer will be the spectrum obtained. The diffraction spectrum exhibits the colours in proper proportion, the inequality and irrationality of the spectrum formed by a prism being absent, and it enables the wave-lengths of light to be measured.

Phosphorescence.—This is the name given to the property of a substance by which, without sensible rise of temperature, it becomes luminous in the dark after exposure to light. The term, however, also includes phenomena due to other causes. Phosphorescence occurs (a) in minerals, (b) in vegetable matter, (c) in animals, and (d) indirectly owing to the radio-activity.

(a) If some chloride or sulphide of calcium or barium be preserved from air in a sealed glass it will shine brilliantly for a long time. Some diamonds are said to possess this property. The rays which excite the luminosity are those of high refrangibility, but the colours of the phosphorescence are of the most varied kinds. Besides this form of phosphorescence many minerals exhibit this property under different circumstances, thus:—By heating fluorspar, quinine, etc., by applying friction to quartz or cane-sugar in the dark, or by cleaving a slab of mica. Fused boric acid or even water when rapidly crystallised or frozen may exhibit this phenomenon.

(b) A number of plants exhibit phosphorescence, as do also fungi growing in decayed wood. The phosphorescence of decaying vegetable matter is caused by oxydation, and that of phosphorus is probably due to the same cause.

(c) The brilliant phosphorescence observed on tropical seas at night is due to numberless phosphorescent organisms. The light emitted by various animals and insects, such as the glow-worm, firefly, and certain centipedes is phosphorescent, and is found of almost every colour in one or other species. Putrefying animal substances as well as vegetable often become phosphorescent.

(d) When an electric current is passed through a vacuum tube Röntgen rays are produced, and the walls of the tube emit a greenish phosphorescence. This is assumed to be due to minute electrified particles striking the wall of the tube with immense velocity and producing phosphorescence and heat by this impact, the colour of the phosphorescence depending on the nature of the glass. Radium is found to shine perpetually in the dark, and bodies exposed to the radiation of radium become themselves radioactive, i.e., phosphorescent for a time.

Fluorescence.—Fluorescence is the property possessed by certain bodies of absorbing ultra-violet waves, invisible to the eye and emitting by radiation light of longer wave-lengths by which they appear self-luminous. This property was first discovered by Stokes in fluorspar, and so named by him fluorescence. The emission of light ceases immediately the original source of light is cut off, and thus differs from phosphorescence.

The phenomenon is not confined to the ultra-violet rays, for if a solution of chlorophyl be placed in a dark room and a beam of white light allowed to fall on it, the surface of the solution emits a red fluorescent light. A solution of quinine emits a pale bluish colour in the presence of daylight. The fluorescence increases if the solution is held in the violet end of the spectrum, and is visible when held beyond the limits of the visible spectrum, the invisible ultra-violet rays exciting fluorescence and becoming changed into visible blue-violet rays. A thick plate of violet glass placed in front of a beam of light from the electric arc will cause the same phenomenon. Æsculine (the juice of the horse-chestnut bark), barium, and many other substances are fluorescent, and so are also the cornea, crystalline lens, and bacillary layer of the retina.

It has been said that the ozone of the atmosphere is fluorescent, and, by converting the ultra-violet into visible rays, makes the sky appear blue. When invisible rays are changed into luminous rays by their passage through, or reflection by, a partially transparent body, the phenomenon is termed *luminescence*. The conversion of light into heat rays is *calorescence*.

CHAPTER II.

SHADOWS AND PHOTOMETRY.

Shadows.—Since light rays travel in straight lines, and ether waves travel in a direction at right angles to their front, any opaque obstacle in their path will arrest their march and produce a negative image of the object which is called a shadow. In the same way, if sound waves meet with an obstacle some of the sound will be stopped and produce a sound shadow.

Umbra and Penumbra.—When the source of light is in a line with the centre of the obstacle, and the ground on which the shadow is cast is at right angles to the central ray of the pencil of light, the shadow has an outline exactly corresponding to that of the body, because then, as in Fig. 13, the periphery of O cuts off

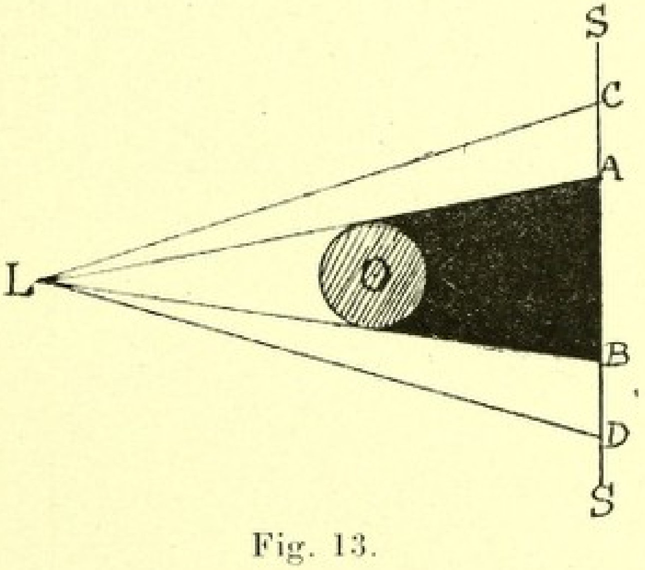


Fig. 13.

the light equally in every direction. The shape of the shadow depends upon the inclination of the screen to the opaque body and the source of light. This may be illustrated by the shadow cast by one's body along the street. If the source of light is directly overhead, no shadow is cast, while the nearer the light is to the horizon the longer the shadow becomes. If the same light approximates to a point as shown in Fig. 13 the shadow is dark and its edges clearly defined as at A B on the screen SS.

If, however, the source of light L is of definite size relative to the intercepting body (as in Fig. 14), the edges of the shadow are not so sharp and the shadow exhibits two parts, viz., a very dark centre

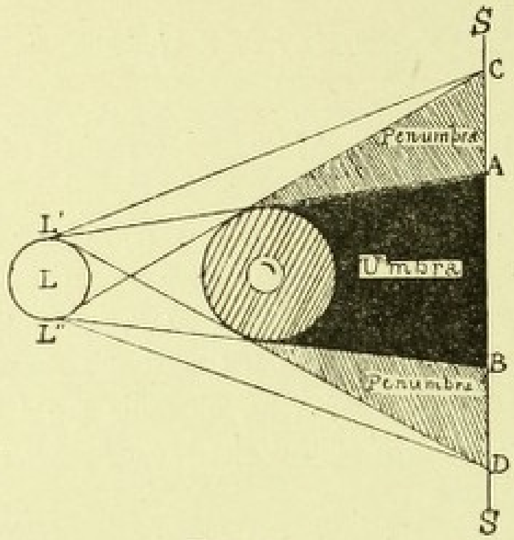


Fig. 14.

A B called the umbra, from which the light is entirely cut off, and a less black outer portion C A, B D, called the penumbra, which receives a certain amount of illumination. The space C A receives light from L′, but none from L″, while B D receives light from L″, but none from L′. A B receives light from neither L′ nor L″.

Fig. 14 shows the umbra A B and the penumbra A C, B D when the luminant L is smaller than the intercepting body. In this case the umbra A B becomes larger and the penumbra A C, B D smaller, as the shadow is further from the intercepting body. Both the umbral and the penumbral cones are divergent.

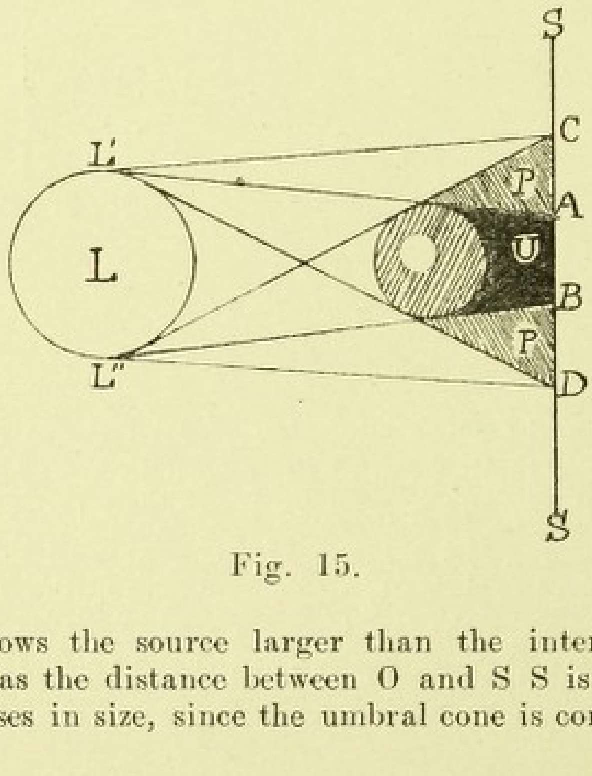


Fig. 15.

Fig. 15 shows the source larger than the intercepting body. In this case, as the distance between O and S S is increased, the umbra decreases in size, since the umbral cone is convergent, while the penumbra increases in size owing to the penumbral cone being divergent. Beyond a certain point there is no umbra. Thus, when the hand is held close to a wall, in a well-illuminated room, the projected shadow is almost entirely umbra; as the hand is moved away the umbra decreases and the penumbra increases until, at a certain distance, the whole shadow becomes penumbral. This is shown in Fig. 16.

The larger the size of the luminant as compared with that of the intercepting body, the smaller is the umbra, and the larger the penumbra, and *vice versa*. If the luminant and the obstructing body are of equal size, the umbra, which is cylindrical in section, does not vary in size with its distance from the body or screen, but the penumbra increases as the screen is further from the body.

The umbral and penumbral portions of a shadow are not separated from each other by a sharply defined line of demarcation, but imperceptibly merge into each other. Generally the brighter the light, the deeper is the shadow cast by the object, for then the contrast between the illuminated ground and the part from which the light is totally or partially obstructed is greater than in a dull light, when shadows are barely perceptible.

Shadows from Coloured Lights.—A shadow cast by a body when the light is coloured appears to be tinged with the colour complementary to that of the light. This is due to *contrast*, because the illuminated ground is coloured by the light, although this fact may be hardly appreciated. If with the Rumford Photometer the two lights employed be red and green, the shadow due to the former is green and the latter red. In this case only the one light falls on a space from which the other is excluded, and the space is therefore coloured by the light it receives. *Contrast* also helps to increase the depth of the colours.

Calculations of Umbrae and Penumbrae.—The calculations for determining the size of the umbra and penumbra are complicated, and vary so much with the conditions under which the shadow is cast that in every case they must *be worked out from general principles*.

Let U be the umbra, P the penumbra, S the source of light, B the intercepting body, and C the screen. Let sb, sc, and bc be respectively the distance between S and B, that between S and C, and that between B and C.

$$U = B \text{ when } S = B;\ U > B \text{ when } S < B;\ U < B \text{ when } S > B.$$

If $\frac{sc}{S} = \frac{bc}{B}$ as in Fig. 16 (1), there is no umbra, for then the penumbral disc has its centre axially in line with B and S. When $\frac{sc}{S} < \frac{bc}{B}$ there is also no umbra, the penumbra from each side of the body overlapping the centre as shown in Fig. 16 (2).

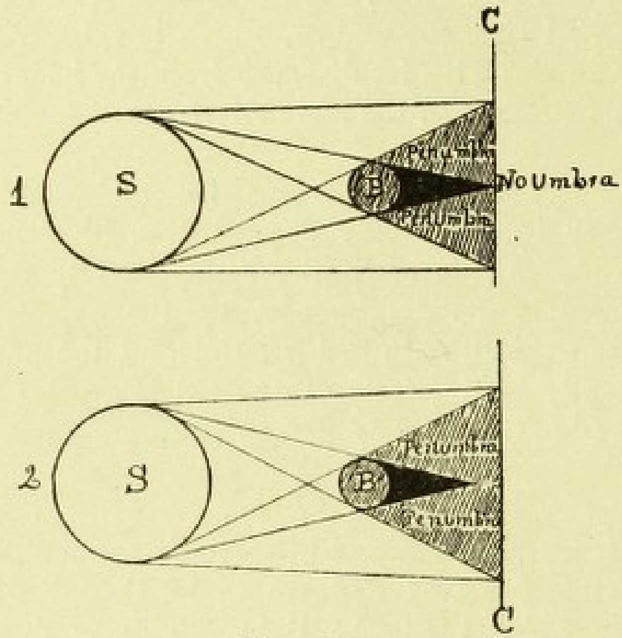


Fig. 16.

The penumbra, if S be distant, can be calculated from the angle which S subtends at B, and it may be considered subtended equally at the edge of B as at its centre. The angle of the diverging cone of light at the edge of B is equal to the angle there subtended by S. Also the central line of the penumbral cone may be considered as coinciding with the central line connecting S, B, and C. Thus, if S be a square window 2ft. in diameter, the size of U and P on a wall, 20ft. distant, cast by a coin 1 inch in diameter held 1ft. from the wall would be estimated as follows: Let A be the angle subtended by S at the edge of the coin, then

$$\frac{2}{19} = .105 = \tan a.$$

Without finding the angular dimension of a, we know that the one side of the penumbral disc is $.105 \times 12 = 1.26$ in. Since we take S as axially in line with the edge of B, as well as its centre, one half of the one side of the penumbral disc, viz.: $\frac{1.26}{2} = .63$in.

is behind the edge of B, encroaching on the geometrical umbra, and similarly on the other side. Thus there is no umbra, for 1 − .63 − .63 = − .26, which is a negative quantity, as in Fig. 16 (2).

The penumbral disc is 1.26 + 1.26 − .26 = 2.26in.

If the coin were 2in. in diameter we should have

$$U = 2 - .63 - .63 = .74\text{in.},$$

And $$P = 1.26 + 1.26 + .74 = 3.26\text{in.}$$

As another example, the length of the shadow cast by a stick 3ft. long, 20ft. from a small lamp which is 10ft. from the ground, would be found thus:—

Here the distance of the lamp to the end of the shadow is 20 + x, and 20 + x : x as 10 : 3; therefore

$$\frac{10}{3} = \frac{20 + x}{x} \qquad 10x = 60 + 3x$$

$$7x = 60, \text{ and } x = 8\tfrac{4}{7}\text{ ft.}$$

So the length of the shadow is $8\frac{4}{7}$ft.

Shadows cast by Lenses.—A concave lens, when placed between a light and a screen, casts a shadow like an opaque body. The transmitted rays being divergent, only very few impinge on the screen immediately behind the central portion of the lens. The diverged rays fall on the screen away from the axial line, on a space which is then doubly illuminated, so that the shadow is surrounded by a luminous zone. The luminous zone becomes larger and fainter as the distance between the screen and the lens is increased. A convex lens throws a very bright image on a screen if placed near the focus, because it condenses to a small area all the light passing through it. The bright area is surrounded by a shadow, this being the area from which light is excluded. If bright light be passed through a prism the space on a screen, immediately behind it, exhibits a shadow, the light deviated by the prism falling on another part of the screen, which, being also illuminated directly, exhibits there a bright area.

Intensity of Illumination.—In order to illustrate how the intensity of illumination varies with the distance between a source of light and an illuminated area, let the source of light, say a candle flame, be supposed to be at the centre of a sphere of one foot radius, and let the intensity of the light at the surface be considered as unity. The area of a sphere is equal to $4\,\pi r^2$, r being the radius.

If the radius of the spherical envelope be increased from one foot to two feet, its area will then be quadrupled, since the superficial area of a sphere varies as the square of its radius. If the sphere be three feet in radius its area will be increased nine times. In this latter case, the available light is distributed over nine times the area of the one foot sphere, so that the intensity of illumination over a given area is but one-ninth that of the first sphere, and in this way the intensity may be calculated for a sphere of any size.

The Law of Inverse Squares.—Since any flat surface virtually forms a portion of a sphere having the source of light for its centre, without much error, it may be stated that the illumination of a flat surface also varies inversely with the square of its distance from the source of light. This distribution of the illuminating agent is illustrated in Fig. 17.

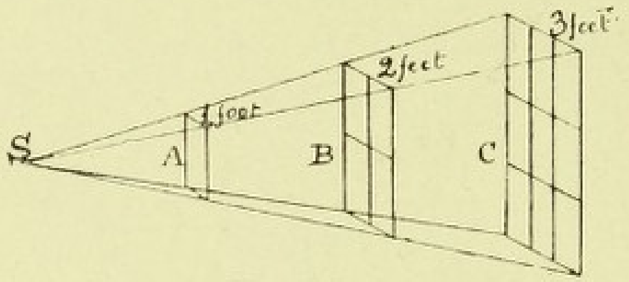


Fig. 17.

Let S be the source of light and A, B, and C screens subtending equal angles, placed vertically at a distance 1, 2, and 3 feet respectively. The same amount of light from L is received by all, but C, being at a distance from L, which is three times greater than that of A, is nine times as large; and it follows, therefore, that each unit of area of C receives only 1/9th of the quantity of light received by each similar unit of A, while B at 2 feet receives 1/4th.

If at a given distance, say 1 foot, a certain intensity of illumination I is obtained from a lamp, and the lamp be moved to a greater distance, say 9 feet, then the intensity becomes $\frac{1}{9^2} = \frac{1}{81}$ or the 81st part of the illumination received at a distance of one foot. If it be increased to 10 feet it will require $10^2 = 100$ luminants to obtain an equal intensity as at 1 foot.

Obliquity of Illuminated Surface.—The intensity of illumination depends not only on the distance of the surface from the light, with which it varies inversely, but also on the inclination of the surface to the light, with which it varies as the cosine of the angle which the surface makes with the normal.

Suppose, for example, a series of parallel rays of light impinge on a vertical screen A B. If A B be inclined to the position B D, so that the angle of inclination a or A B D = 60°, then only those rays corresponding to C B will fall on B D, as in Fig. 18.

Now $\cos a = \frac{BC}{BD} = \frac{1}{2}$. Also from inspection it is clear that B C is half of B A, which equals B D. Therefore, if the screen be inclined 60° from the vertical it will receive half the light that it does when it is vertical.

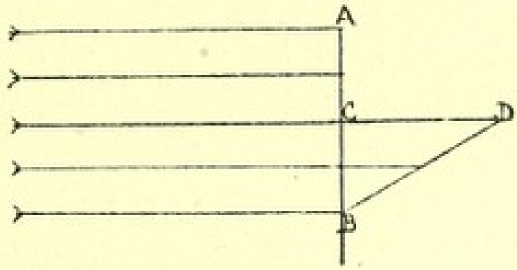


Fig. 18.

Suppose L be the amount of light falling on a unit of A, the area of the screen, 1 metre from the light. What amount of light will fall on the screen when inclined at 45° to the normal and removed to five times the distance? The total amount of light received each instant is = L A, and the amount of light received on the screen inclined at 45° is therefore equal to

$$\frac{L\,A \cos 45^\circ}{d^2}$$

The intensity of illumination per unit area is

$$\frac{\cos 45^\circ}{d^2} = \frac{.7071}{25} = 0.028\,;$$

or about $\frac{1}{36}$ of the light received on the screen at 1 M distance.

This holds good for the light reflected from a surface, as can be seen from Fig. 18.

Apparent Paradoxes.—In connection with this subject there are two apparent paradoxes which need explanation.

The first is that an object or source of light appears equally bright at all distances from the eye. It is known that the brightness of an object varies inversely as the square of the distance, so that an object at one yard is four times as bright as one at two yards, but at the same time the image on the retina occupies four times the area, which means it is only a fourth as bright as it would be if compressed into the size it would be were the object placed twice as far away. Of course if a screen were substituted for the eye the whole screen would be illuminated, as no image would be formed on it, no matter where the light is placed, so that there would be nothing to counteract the law of inverse squares. But in the former case the light gained by bringing the object nearer is exactly neutralised by spreading it over a proportionately larger area.

It may therefore be said that the law of squares holds good only for light received directly on a screen, and that if it passes through a lens system so as to form an image, as in a camera, the brightness of the image is the same whatever the distance of the object may be. This rule does not hold good for a point of light such as a star, since it can only form a point image on the retina, nor does it obtain for any object whose image is so small that it will only cover a single cone on the retina, for then its reduction in size could produce no effect on the eye. It is true that distant objects, as a rule, look more hazy than near ones, but that is due to the partial opacity of the atmosphere.

The other is that a luminous or illuminated surface appears equally bright at whatever angle it is seen. This apparently contradicts the law of cosines, but it does not really do so, for although an inclined surface receives less light, the area perceived is correspondingly diminished. Therefore its brightness as perceived by the eye is the same in both cases.

The foreshortening which the tilted reflecting surface undergoes is, like the amount of light it receives, proportional to the cosine of the angle of inclination.

The sun and moon appear as flat discs and not as hemispheres, since their surfaces are apparently equally illuminated, and in the same way a cannon ball or cylinder of metal, heated white hot, appears quite flat.

Photometry.—The measurement of the *luminosity* of a light source, or of the illumination of a surface, is termed *photometry*, and the instrument or apparatus employed is called a *photometer*.

A luminous source, unless it be a star, has a definite surface which is seldom of equal luminosity throughout. The quantity of light emitted varies at different points of the surface, but the sum of the light emitted from every point is the total luminosity, and it is this which is measured by photometers. The *intrinsic intensity* of luminosity I is the mean quantity of light emitted normally from a unit of surface.

This is found by the equation $I = \frac{\phi}{S}$ where ϕ is the total amount of light and S is the surface of the luminous source.

The intensity of *illumination* is the total amount of light which falls on a unit of the illuminated surface.

Luminosity and Illumination.—It is necessary, when considering photometry, to clearly differentiate between *luminosity*, or the illuminating power of the source light, and *illumination*, or the amount of light received from the source by a body.

Photometric Standards.—The usual standard of illumination in Great Britain is that given by a sperm candle $\frac{7}{8}$ inch in diameter, $\frac{1}{6}$ of a pound in weight, and burning 120 grains per hour. It has a variation of about 20%. The luminosity of gas, with an ordinary burner, is equal to that of from 12 to 16 candles.

There are various other photometric units, among them the following:—

In Germany the standard is the Hefner-Alteneck lamp, called a "Hefnerlamp" (H), having a cylindrical wick 8mm. in diameter burning amylacetate, the flame being 40mm. high. It is correct to about 2%.

The "Pentane" standard consists of a mixture of pentane gas C_5H_{12} and air, which is burnt at the rate of $\frac{1}{2}$ cubic foot per hour; the flame is a circular one $2\frac{1}{2}$ inches high and $\frac{1}{4}$ inch in diameter. There is no wick or chimney round the flame. Pentane is a volatile liquid, like naptha, prepared from petroleum. The form designed by Vernon Harcourt is a 10 candle-power standard, and is largely used in this country. It is said to vary less than 1%.

The French "Carcel" is a lamp of special construction burning 42 grammes of colza oil per hour.

The "Violle" or absolute unit was the standard invented by M. Violle, and adopted at the International Congress at Paris in 1884. It consists of the light emitted from a square cm. of platinum heated to its melting point. Of all the standards it is the most exact and reliable, but it has the objection of being expensive and difficult to apply.

The International Congress of 1890 adopted as the standard the "Bougie-decimale" or decimal candle, the unit illumination of a surface being that produced by one bougie-decimal at one metre.

The British candle and the bougie-decimal have about the same intensities. The "Carcel" equals about $9\frac{1}{2}$ candles, and the "Violle" unit about 20 candles. Thus 20 bougie-decimals = $19\frac{3}{4}$ B.C. = 22.8 Hefner = 2.08 Carcel = 1 Violle.

Measurement of Light Sources.—Photometry consists in making a comparison of the unknown illuminating power of any source of light with that of a standard unit. Direct comparison would be difficult, but the stronger light can be placed at a greater distance, where it produces an intensity of illumination equal to that of the standard light at some shorter distance. The illuminating powers of the two sources of light are respectively as the squares of the distance at which, on a given surface, they produce equal intensities of illumination.

Let the standard candle at one foot and another luminant at four feet give equal intensity of illumination; then the greater luminant is of 16 candle power.

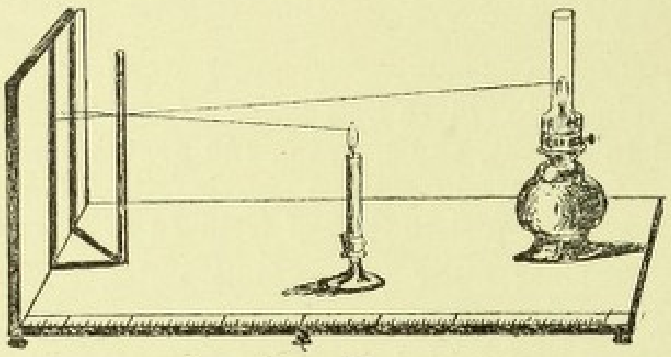


Fig. 19.

The Rumford Photometer.—The shadow or Romford Photometer consists of a vertical white screen before which is placed a rod. The standard candle is placed (preferably at one foot) in front of the screen and the rod casts a shadow. The lamp or other luminant which is to be measured is placed so far away that the shadow cast by the rod, from its light, is of equal intensity with that of the other. The space on the screen, occupied by the candle's shadow, is illuminated only by the light from the lamp, while that occupied by the lamp's shadow is illuminated only by the candle. It is these intensities of illuminations that are actually compared, although apparently it is the shadows themselves. The lights should be placed so that the two shadows lie near to each other without overlapping. The luminant measured is of so many candle power according to the distance at which the shadow pertaining to it equals in depth that pertaining to the standard candle. If we represent the respective luminosities of the lamp and candle by L and C, and the distances of the two when the shadows are equal in intensity by a and b, then

$$\frac{L}{C} = \frac{a^2}{b^2}; \qquad \text{or } L = \frac{C\,a^2}{b^2} \qquad [1]$$

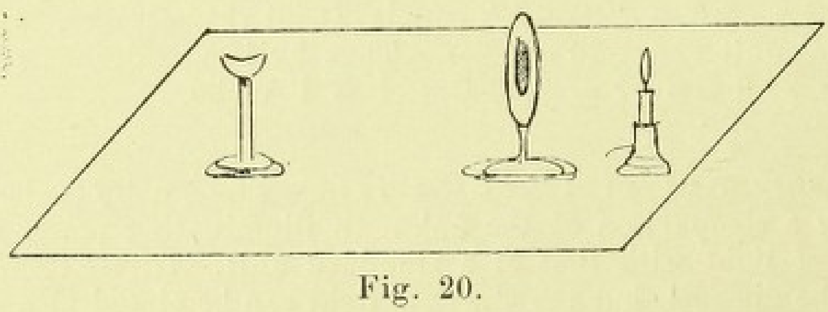


Fig. 20.

The Bunsen Photometer.—The *grease spot* or *Bunsen photometer* consists of a sheet of white paper, suitably mounted in a frame, on which there is a spot rendered semi-transparent by grease or oil. If the paper be viewed on the side remote from the candle the grease spot looks lighter than the balance of the paper, because more light penetrates. Viewed from the other side, the greased spot looks darker, because less light is reflected from it than from the rest of the paper. Used as a photometer, the paper is placed one foot from the standard candle, the light from which is totally reflected by the ungreased part of the paper and transmitted to a great extent by the grease spot. The luminant to be tested is placed on the other side of the screen at such a distance that the amount of light from it, transmitted by the grease spot, equals that passing the other way; then the paper appears of uniform brightness all over. In the above case, if we take one foot as unity, then the candle power of the light to be tested will be equal to the square of its distance from the grease spot expressed in feet.

The Slab Photometer.—The *Paraffin Slab* photometer consists of two thick slabs of solid paraffin separated by an opaque layer of tin foil. The two lights to be compared are placed one on either side, and their intensities are compared by viewing the sides of the two slabs simultaneously.

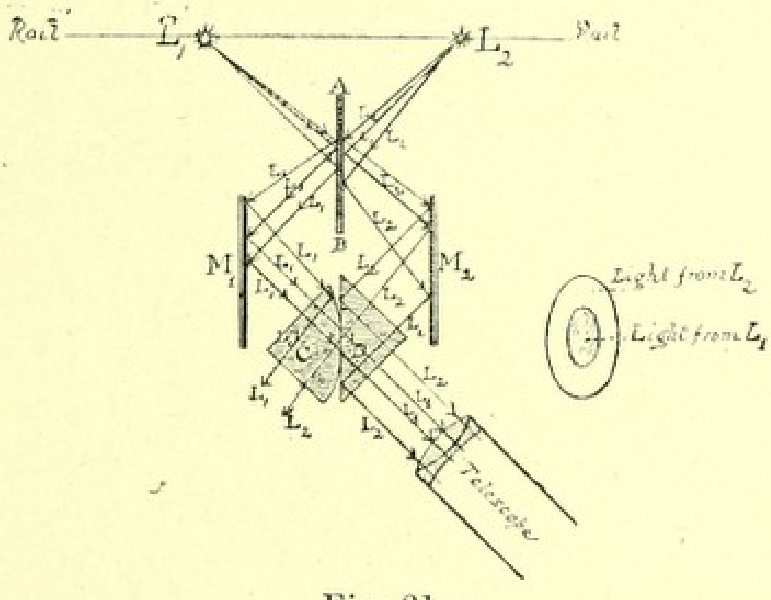


Fig. 21.

The Lummer-Brodhun Photometer.—This photometer is largely used in scientific laboratories, being accurate to about 1%. Its superior accuracy over the Bunsen and other previously described photometers is due to the fact that, with all these, the two images to be compared cannot be seen simultaneously. With the Lummer-Brodhun instrument only one combined image is seen by one eye. The instrument consists of a rail on which the two luminants L_1 and L_2 can be made to travel at right angles to the opaque screen A B, which is whitened on both sides. The observer looks through a short telescope placed in front of D and sees the light from the two sides of A B reflected from the two mirrors M_1 and M_2. The light then passes through the cube of glass C D made of two right angle prisms, the hypothenuse side of one of which is partly cut away. These prisms are cemented together in the centre.

The light which reaches the telescope from L_1 passes through the central cemented portion of C and D, that from L_2 is reflected from the peripheral portion of D. The two lights therefore enter the eye simultaneously in two concentric rings, as shown in the figure.

The lights are moved to and fro along the rail until the two circles appear equally illuminated.

The Simmance-Abady "Flicker" Photometer.—This consists essentially of a white circular disc or wheel, the edge of which is peculiarly bevelled by being "chucked" eccentrically at two positions with the turning tool set obliquely at 45°. Thus the periphery of the wheel, when revolved, presents a bevel of 45° on the one side, say the right, and no bevel on the left, then graduates to a knife edge, and finally to a bevel of 45° on the left and no bevel on the right.

This wheel is so fixed in a box that only part of it projects, and immediately in front of it, but leaving its projecting portions unobscured, there is a sighting tube carrying a Cx. lens for magnifying purposes. The box contains a clockwork arrangement by means of which the wheel is made to revolve at a rapid speed. The box itself is fixed on a bar 60 inches long, scaled in terms of a standard candle, and along which the apparatus can be freely moved.

The two illuminants which are to be compared are placed one at each end of the bar, and the light from them falls on that part of the revolving disc which projects from the box. When the light falls on the bevelled edge at 45° it is reflected, and passing through the sighting tube, is seen by the observer. When incident on the unbevelled part of the disc, the light does not pass through the sighting tube, so that each luminant is alternately *light* and *dark* to the observer's eye, and both are light at the same time when the knife edge is immediately in front of the sighting tube.

Then when the intensities are equal the light is absolutely steady, while it flickers when they are not. If there is flickering the apparatus is moved until this disappears, and the position is found where

$$\frac{L}{a^2} = \frac{C}{b^2}.$$

Then the smallest alteration of the position of the apparatus towards either light causes flicker. The test is made more sensitive, and the point of *balanced intensities* more exactly located, when the speed of revolution of the wheel is lessened. The apparatus can be set obliquely for measuring lights at any angle.

Photometry of Coloured Lights.—One of the great difficulties of photometry is the difference in the nature and colour of various lights; and the comparison or measurement of actually coloured or monochromatic lights is still more difficult, or rather impossible, by ordinary photometry. These difficulties seem, however, to be obviated by the Simmance-Abady photometer.

By the flicker photometer coloured lights, and therefore also the transmissive qualities of coloured and smoked glasses, can be compared and measured, since the flicker depends on equality of intensity of illumination on the two sides of the bevelled disc, and is independent of the colour of these illuminations. By means of it also the illuminating power or the effect of daylight can be measured as well as that of different sources of artificial light.

Coloured lights may, however, be compared by occlusion, using for the purpose a series of properly graduated smoked glasses.

Calculations in Photometry.—Having by means of the photometer made the intensities of illumination equal, the candle power of the luminant is calculated from the square of the distance of the luminant divided by the square of the distance of the standard candle. When this latter distance is 1 unit (say 1 foot), of course no division is necessary, as the square of 1 is 1. Thus if the luminant at 5 feet is equal to the standard candle at 1 foot, the former is of $5^2 = 25$ c.p. If the candle is at 2 feet and the luminant at 8 feet the latter is

$$\frac{8^2}{2^2} = \frac{64}{4} = 16 \text{ c.p.}$$

To compare the intensity of illumination l and c of two sources of light L and C of different powers.

$$l : c \text{ as } \frac{L}{a^2} : \frac{C}{b^2}, \qquad [2]$$

where a is the distance of L and b is the distance of C.

Thus four candles 4 feet from a screen have the same effect as one candle at 2 feet. For $4/16 = 1/4$.

If L be of 30 c.p. and placed at 20 feet, while C is 200 c.p. at 70 feet,

$$l = \frac{30}{400}, \qquad \text{and} \qquad c = \frac{200}{4900};$$

therefore $$c \text{ is } \frac{2}{49} \times \frac{40}{3} = \frac{80}{137} = \frac{3}{5} l \text{ (approx.)}.$$

How many candles at 100 feet would give the same illumination as 1,000 candles at 30 feet?

Now since $\frac{L}{a^2} = \frac{C}{b^2} \therefore \frac{L}{100^2} = \frac{1000}{30^2},$

or $900\ L = 10,000,000,$ so that $L = 11,111.$

At what distance should an arc lamp of 1200 c.p. be placed so as to give an illumination three times as great as that of an incandescent light of 70 c.p. at 15 feet?

$$\frac{70}{15^2} \times 3 = \frac{1200}{b^2}$$

therefore $210b^2 = 1200 \times 15^2$; that is $b = 36$ feet (approx.).

CHAPTER III.

REFLECTION AND MIRRORS.

Irregular Reflection.—When a beam of light falls on an unpolished surface, such as a ground glass globe, it is (owing to the irregular nature of the surface) incident at all conceivable angles, and each point of the surface becomes a source of light, so that the light is reflected irregularly or dispersively. No image is therefore formed either of the source of light or of any external object.

The diffused reflected light diverges in every direction, so that the surface becomes visible, no matter from what direction it is viewed, and it is either white or coloured according as some wave-lengths are, or are not, absorbed. The incident light is broken up so that each point of the surface gives rise to a fresh series of waves.

Regular Reflection.—When light falls on a smooth polished surface it is regularly reflected in definite directions according to the following laws:—

1.—The angle of reflection is equal to the angle of incidence.

2.—The incident and reflected rays are both in the same plane as the perpendicular, or normal, to the point of incidence, and lie on opposite sides of it.

Oblique Incidence.—In Fig. 22, A B is a reflecting surface at which the ray I C is incident at the point C, and reflected from C in the direction C R. P C is the perpendicular to A B at C, and

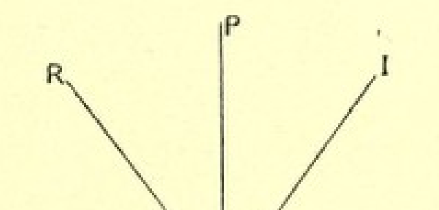


Fig. 22.

the angle of reflection I C P is equal to the angle of incidence R C P. The perpendicular divides equally the angle included between the incident and reflected rays, and all three lines are in the plane of the paper.

Perpendicular Incidence.—If the ray be incident in a direction P C, normal to the surface, the angle of incidence being zero, the angle of reflection is also zero; the ray is then reflected back along its original path.

Images.—An image is real or positive if it can be received on a screen. It is formed by rays of light which are convergent, and which therefore actually meet. An image is virtual (imaginary) or negative if it cannot be received on a screen. It is formed of rays of light which do not meet, but which, being divergent, are referred back to the point from whence they appear to diverge.

In order that an image be distinct the rays of light which diverge from various points of the object must meet, or appear to meet, as so many distinct points; each pencil of light must have its own focus, which may be defined as the image of a luminous point. If the rays diverging from a luminous point do actually meet again the focus is real or positive, and it is virtual or negative if the rays, being referred back, appear to diverge from a point from which they do not really originate.

Mirror.—A mirror is an opaque body with a highly polished surface. It is usually made of glass backed by a film of mercurial amalgam, or coated with an extremely thin layer of silver.

Reflection by Plane Mirror.—If a beam of parallel rays falls on a plane mirror all the rays having similar angles of incidence are reflected under equal angles, and are therefore reflected as parallel rays. If a pencil of divergent rays be thus incident, after reflection they are equally divergent, and appear to come from a point as far behind the mirror as the original luminous point is

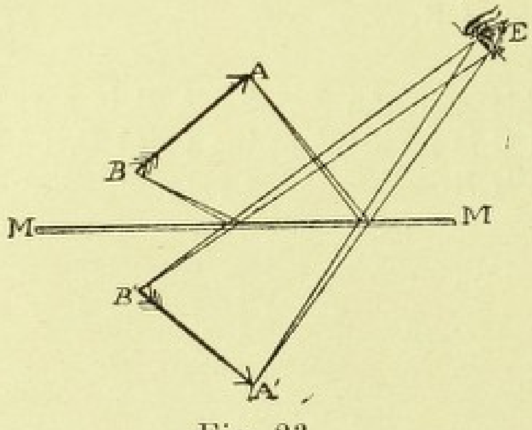


Fig. 23.

situated in front of it. Accordingly, if an object stands in front of a plane mirror the rays diverging from each point on it are reflected from the surface of the mirror and enter the eye of an observer as so many cones of light diverging from so many points behind the mirror, and these points, from which the light appears to diverge, constitute the virtual image of the original object.

If the object is parallel to the surface of the mirror the image is also parallel; if the object is oblique to the surface the image forms a similar angle with it.

In Fig. 23 A B is an object placed in front of the mirror M M. Rays diverging from A, after reflection, enter the eye E, and are projected to a virtual focus at A′, from which point they appear to diverge. Those from B are projected to B′ and A′ B′ is the image of A B. A′ is apparently as far behind M M as A is in front of it; so also B and B′ are equally distant from M M. The complete image is erect and corresponds exactly as regards shape, distance, and size to the object itself, the relative directions of the rays from each point on the object being unchanged by the reflection.

Lateral Displacement by Reflection.—The image is, however, laterally displaced, the right hand of a person becomes the left of his image in the mirror and *vice versa*. If the eye regards A B (Fig. 23) directly, A is to the right of A B, but looking into the mirror A′ is seen to the left of A′ B′.

If the top of a page of printed matter be held obliquely downwards against the mirror the letters will be in the same order from the left to right, only they will be upside down, and at the same angle to the mirror as the page, thus resembling a case of type. Engravers sometimes use a mirror in front of the letters or objects they wish to draw on a wood-block and copy the image they see in the mirror. On taking an impression of the block the letters or objects are in their right position.

Distance of Image.—If a person stands at, say 2 feet, in front of a looking-glass and looks into it he sees an image of himself at a distance of 4 feet. The light has travelled 2 feet to the mirror and then 2 feet to his eyes, and is mentally projected backwards through a distance of 4 feet. If an object is placed in contact with a glass mirror its image appears behind the silvered surface, and only twice the thickness of the glass itself separates object and image. The image appears rather nearer owing to vertical displacement by refraction. If the mirror is of polished metal the two are in contact.

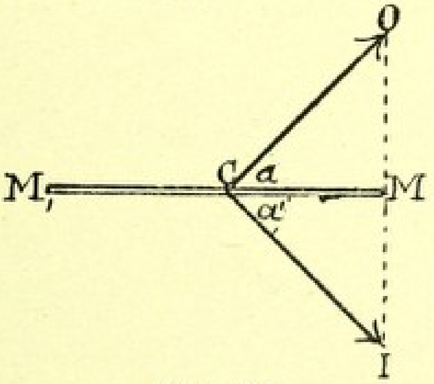


Fig. 24.

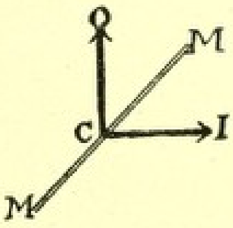


Fig. 25.

Position of Image.—Since the angle O C M, between the mirror M M′ and the object O C (Fig. 24), and the angle I C M, between the mirror and the image C I, are equal, it follows that the angle O C I between the object and the image is twice as large as either; so if the mirror be placed at an angle of 45° with the object, the object and image are at right angles to each other, as is shown in Fig. 25.

Size of Mirror.—The smallest *plane* mirror which will enable a person to see the whole of himself reflected, is one which is half his height, the top of the mirror being on a level with his head. This can be understood without explanation from Fig. 26.

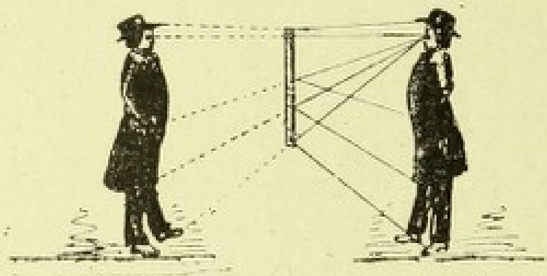


Fig. 26.

Construction of Image.—The image can be graphically constructed by drawing straight lines from the extremities of the object, perpendicular to the mirror or plane of the mirror, and continuing such lines as far behind the mirror as the object points are in front of it. Thus, in Fig. 23, if a line be drawn from B to B′, and another from A to A′, and B′ A′ be connected, the image B′ A′ is obtained.

Angular Displacement of Image.—If a mirror be turned through any angle the image will move through twice that angle. This can be shown as follows:—Let M N (Fig. 27) be a plane mirror capable of turning round a vertical axis. Suppose a pointed indicator fixed to the mirror projects forward at right angles to it,

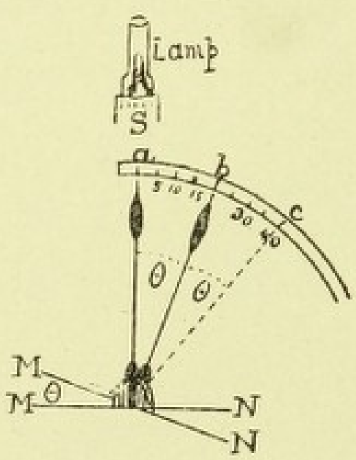


Fig. 27.

and a screen in the form of an arc be placed in front of the mirror. If a lamp be placed in a line with the pointer at S its image will be returned along the same path and be received on the scale at a. Now let the mirror be turned through the angle $\theta = 20°$, so that the pointer will be turned to b, but as the light remains in the same place its reflected image will be removed to c, $\theta°$ to the other side of the pointer. Since the angle of reflection is equal to the angle of incidence, the image of S will be at c, which is 20° from b, or 40° from a, showing that the light has turned through twice the angle of the mirror.

This fact is made use of in the construction of the sextant. Each degree of rotation of the mirror actually corresponds to two degrees of deviation of the light, and therefore to assist calculation every single degree on the arc is marked as two. The same principle is applied to the reflecting galvanometer.

Multiple Images.—When there is but one reflecting surface, as in a metal mirror, there is but one image, but in a glass mirror having two reflecting surfaces, namely, the front surface of the glass A B (Fig. 28) and its silvered back surface C D, there are multiple images of an object. Let a candle flame O be held near to a glass mirror and a series of images will be seen; the first image p, that nearest to the candle, is formed by direct reflection from the front surface of the glass along a I′; the second image p′, which is very bright, is directly reflected from the silvered surface along a′ b I″.

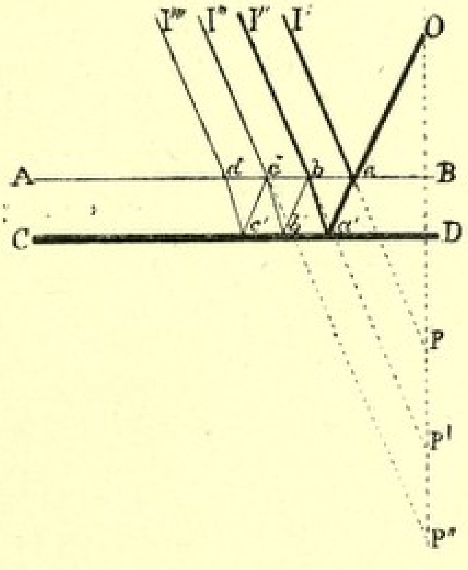


Fig. 28.

The other images formed along I‴ c b′, I⁗ d c′, etc., are progressively fainter, they being formed by reflection from the silvered surface to the front of the glass, then back to the silvered surface as a a′ b, whence the light is reflected into the observer's eyes; they are progressively faint, because by each reflection some light is lost. The last of the multiple images are barely visible, and the total number, that can be seen, depends on the luminosity of the flame.

On looking into a mirror usually only two images are noticeable, the faint one reflected from the front and the bright one from the back surface (Fig. 29).

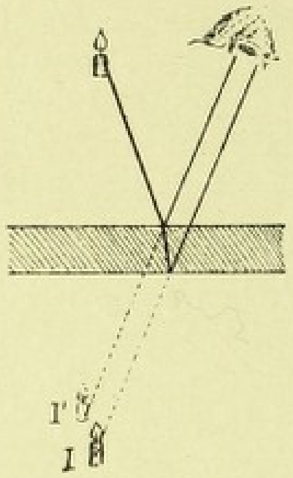


Fig. 29.

Parallel Mirrors.—If two plane mirrors M and M′ (Fig. 30) are parallel to each other, and an object O is placed between them, a series of images (the first of which are I and I′), infinite in number, is produced by reflection of the light backwards and forwards between the two mirrors. As with the single mirror, the light finally becomes so feeble that the images are too faint to be visible.

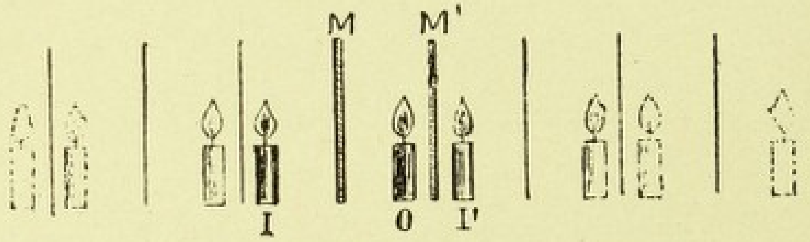


Fig. 30.

Inclined Mirrors.—When two mirrors are inclined to each other, the number of images produced, including the object itself, is found by dividing 360° by the angle between the mirrors, or the angle may be calculated by dividing 360° by the total number of images seen, including the object. Thus, if the angle is 90° there are four, if 60° there are six, and if 45° there are eight images.

Kaleidoscope.—The principle of the kaleidoscope depends on the multiple reflection caused by two inclined mirrors. The mirrors are placed lengthways in a tube, which is closed at one end by a disc of transparent glass, beyond which is one of frosted glass. Between these two glass discs there are a number of small coloured objects, or fragments of coloured glass.

Looking through the open end of the tube an image is seen which consists of a certain number of images, the whole forming a more or less symmetrical figure.

When the number of degrees between the mirrors is an exact *even* divisor of 360 as 45° or 60°, the complete figure is symmetrical; if the number is an exact *odd* divisor of 360, such as 120° and 72°, the figure is not quite symmetrical; if the number is not an exact divisor of 360, the figure is asymmetrical, as some of the images are either incomplete or overlapping.

The usual form of kaleidoscope has three mirrors inclined towards each other at 60°, and the figure is symmetrically hexagonal, or rather it looks triangular, as shown in Fig. 31.

The whole central figure, as seen in a kaleidoscope, is surrounded by others formed by repeated reflections of the light.

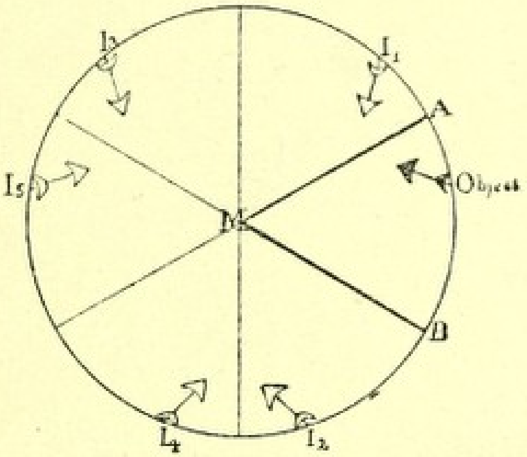


Fig. 31.

Construction of Multiple Images.—To find by construction the images formed by inclined mirrors, let M A and M B (Fig. 31) be the mirrors at any angle, and O the object between them. With M as centre and M O as radius, describe a circle; measure off O I_1 equal to twice O A, and O I_2 equal to twice O B; measure off I_2 I_3 equal to twice I_2 A, and similarly I_1 I_4 equal to twice I_1 B. Then take I_4 I_5 equal to twice I_4 A, and so on until two images coincide or overlap.

CURVED MIRRORS.

Spherical Mirrors.—A spherical mirror is a portion of a sphere, the cross section of which is an arc of a circle; its centre of curvature is the centre of the sphere of which it forms a part. It may be either concave or convex, and considered as made up of an infinite number of minute plane mirrors, each at right angles to one of the radii of the sphere.

Concave Mirror.—Let A B be a concave mirror (Fig. 32) and C its centre of curvature. Then all straight lines drawn from C to any part of A B are radii. They are therefore all of equal length and perpendicular to the surface of the mirror, in other words, normal to it.

All rays therefore starting from C, on reaching the surface of the mirror, will be reflected back along the same paths and form an united image at C.

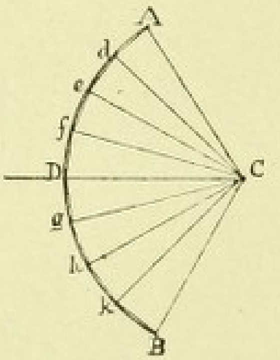


Fig. 32.

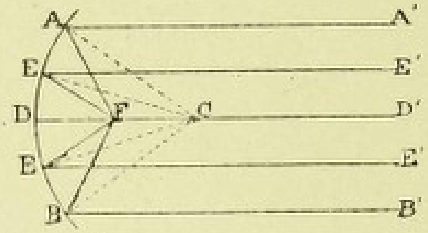


Fig. 33.

The line C D is termed the *principal axis*. It always passes through the centre of curvature C and bisects the mirror at right angles to its surface, as at D; all other lines passing through C to the surface are termed *secondary axes*. The point D is termed the *vertex* or *pole*, and the surface A B between the extremities of the reflecting surface is called the *aperture*.

Action of Concave Mirror.—The course of rays incident on the mirror can be traced after reflection. Thus, in the case of a luminous point situated infinitely far away, the angle of divergence being very small, the rays are considered parallel to each other and to the principal axis. Let A′ A, B′ B, D′ D, etc., be such rays, and let C A, C B, and C D be joined; then, since these latter are radii, they each form a right angle at A, B, and D respectively with the surface of the mirror. Therefore A C is a normal to the surface at A, and the ray A′ A will be reflected to F, making the angle of reflection F A C equal to the angle of incidence A′ A C. All the other rays, in the same way, are reflected to F, which is the common image of a luminous point situated at ∞. F is the *principal focus* of the mirror, and the distance D F is the *principal focal distance* or *focal length*. D F is equal to half the radius D C.

Since the image can be received on a screen or seen in the air in front of the mirror, the focus of a concave mirror is *real* or *positive*.

The course of a ray can be traced backwards along the same path as that by which it arrived; so that if F be the object-point, the rays F E, F A, etc., will be reflected back parallel to the axis along the lines E E′, A A′, etc. Thus, image and object are interchangeable.

Conjugate Focal Distances.—If the object-point (Fig. 34) be on the principal axis between C and ∞, say at f, the image must be at f' somewhere between F and C. An object at ∞ will have its focus at F, and it is obvious that the angle of incidence f K C is less than the angle I K C, therefore the angle formed by the reflected ray f' K C, which equals the angle of incidence f K C, must be less than the angle F K C, and therefore f', the image of f, will lie nearer to C than F.

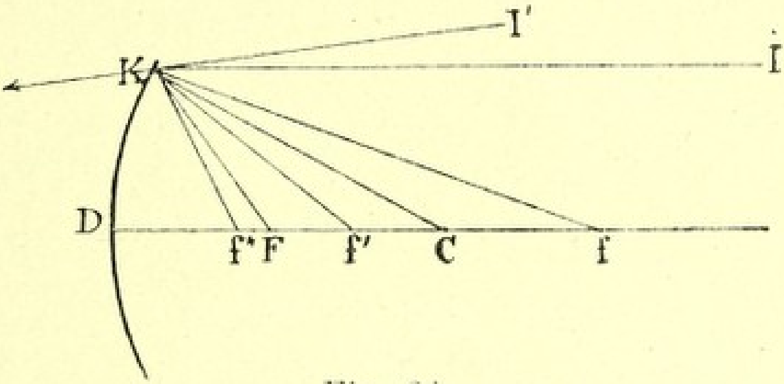


Fig. 34.

As the object-point gets nearer to C, its image also approaches C, and when the object-point arrives at C the image will also be at C, the ray C K being reflected back along its own path. When the object-point arrives at f' the image is obviously at f, and when it reaches F its image is at ∞.

Immediately the object-point passes F towards D as at f'' the reflected ray K I' will lie outside K I. Then the focus will no longer be on the same side of the mirror as the object, but will be found by prolonging the ray K I' backwards to f''' the other side of the mirror, as shown in Fig. 35. In this case the image is not formed in reality, but is virtual or negative.

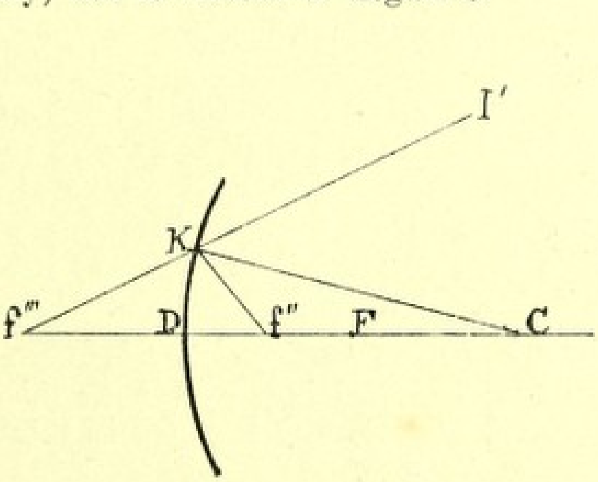


Fig. 35.

As the object-point f'' travels on towards D the image f''' also approaches until the two meet at D, when both touch the mirror together.

If the object could be supposed to travel past the mirror to the left side, it would be virtual and negative, and its image (if it could possibly exist) would be positive; so that, as the object-point would travel to the left from D to $-\infty$, the image would travel to the right from D to F.

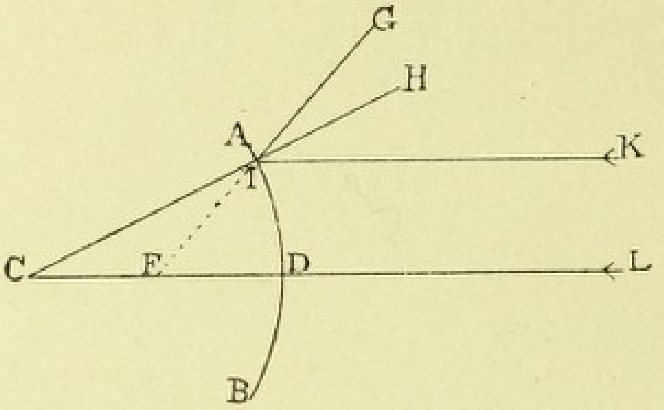


Fig. 36.

Convex Mirror.— Let A B be a convex mirror, C the centre of curvature, D the pole, and C D L the principal axis. Then if the object be at ∞ in a line with the principal axis the rays proceeding from it to the mirror are parallel. Let K I be one of these rays meeting the mirror at I, and let C H be a normal to the surface. The ray K I will be reflected at I to I G, so that the angle of reflection H I G is equal to the angle of incidence H I K, and the reflected ray I G, produced backwards, cuts the axis at F, which is the principal focus of the mirror. As the object-point approaches the mirror the image also approaches the mirror from F to D, until at D both object and image coincide.

Furthermore, in the case of a convex mirror, no matter where the object is, the image is always formed behind the mirror either at F, or between it and D, by prolongation backwards of the divergent rays, and is imaginary or virtual.

Images on Secondary Axes.—In the preceding cases the object is supposed to be on the principal axis, so that the image is also on the principal axis. If the object be situated on some secondary axis the image is on that same secondary axis. Also the object hitherto has been considered as a point; it can now be supposed to have a definite size.

The virtual image of a Cx or Cc mirror is laterally inverted as in a plane mirror. The real image of a Cc mirror is entirely reversed and therefore not laterally inverted in this sense.

Construction of Images—Concave Mirror.—It is known that (1) a ray parallel to the principal axis passes, after reflection, through the principal focus; (2) a ray passing through F, after reflection, is parallel to the principal axis; (3) a ray proceeding through C the centre of curvature is reflected along its original path.

It is possible to make a graphical construction of the image of an object placed in front of a spherical mirror by tracing any two of such rays from the extremities of the object, and their course after reflection. The point where these rays meet is the point where all the rays, which diverge from the object-point, also meet, and is therefore the image of that point.

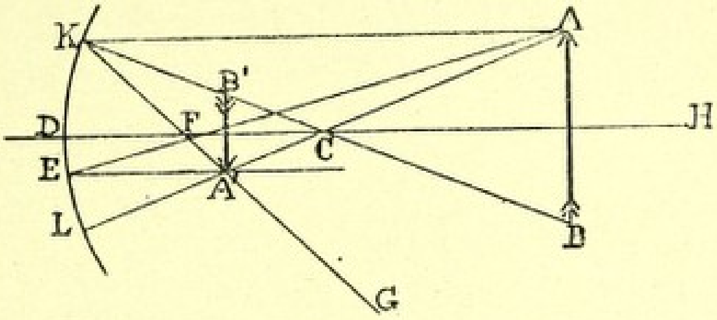


Fig. 37.

Graphical construction when the object is beyond C.

Let A B (Fig. 37) be the object, C the centre of curvature, and let F be the principal focus. Draw A K parallel to the axis, connect K F, and produce towards G; draw A L through C; draw A E through F, and draw E A′ parallel to the axis. These three lines cut each other in A′, which is therefore the image of A, situated on the secondary axis A C L.

In the same way, rays drawn from B meet at B′, and both B and B′ are on the secondary axis B C K. By connecting B′ and A′ the image of A B is obtained, and it is real, inverted, and smaller than the object. If the object were at B′ A′ within the centre of curvature and beyond F, the image would be A B, real and inverted, but larger.

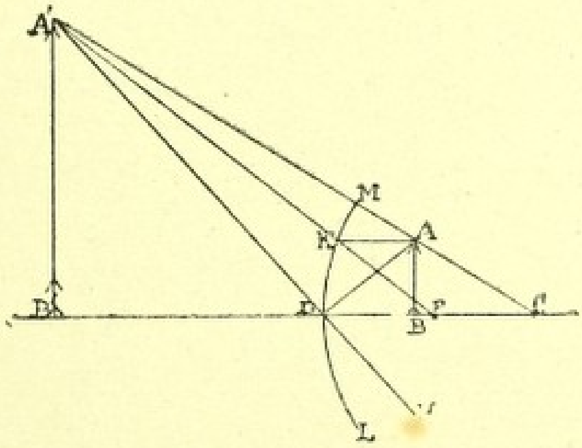


Fig. 38.

Graphical construction when the object is within F.

Let A B (Fig. 38) be the object.

Draw A K, connect F and K, and produce towards A′; draw C A, producing it similarly.

These lines meet on the secondary axis C A A′ in the point A′, which is therefore the image point of A. Any ray A D can be shown to be reflected as if proceeding from A′. In the same way B′ can be shown to be the image of B. By connecting B′ and A′ the image B′ A′ is obtained. It is virtual, erect, and enlarged.

The graphical construction of an image formed by an object at F, resolves itself into lines parallel to the axes, so that the image is at infinity (Fig. 39, 1).

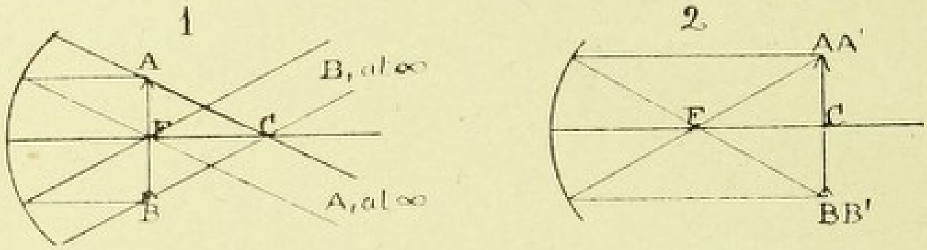


Fig. 39.

If the object is at C, then image and object coincide, but the image is inverted (Fig. 39, 2). Lastly, if the object is at D (Fig. 38) no rays can be drawn, since both image and object are in contact with the mirror and coincide.

Construction of Images—Convex Mirror.—Draw A K and connect K with F; join A C. Where these cut each other at A′ is the image of A. It is on the secondary axis AA′ C. Any ray, as A D or A F, can be shown to be reflected in the directions respectively of A′ M and A′ O as if proceeding from A′.

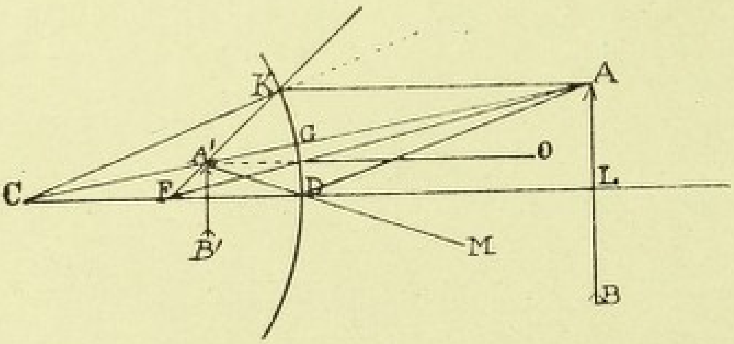


Fig. 40.

By similar construction the position of B′ the image-point of B is determined, and, connecting A′ B′ the complete image of the object A B is obtained, B being on the secondary axis B C.

In the case of a convex mirror, wherever the object may be placed, the image A′ B′ is always virtual (imaginary) erect and smaller than the object, but if A B is in contact with the mirror, the image A′ B′ coincides with it.

Size of the Image.—The relative magnitudes of object and image are proportional to their respective distances from the mirror, or from its centre of curvature, and this rule holds good for all images both virtual and real, and for convex and concave mirrors.

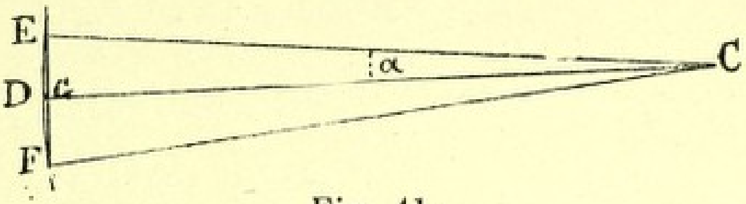


Fig. 41.

Aperture of a Mirror.—In order that a true image of a point may be obtained by a spherical mirror, it is essential that the aperture should be small compared with its radius, subtending, say, not more than 20° at C, so that the arc of the aperture may be approximately a straight line. Suppose E F (Fig. 41) be the aperture of the mirror, C D the principal axis, and C the centre of curvature. Join E F. Then if the angle E C D be small (under 10°) the distance D G will also be small, so that C G may, without much error, be taken as equal to C E; also

$$E D C = G E C = E G C = \text{a right angle.}$$

Now $\tan a = \dfrac{E G}{G C}$, $\sin a = \dfrac{E G}{E C}$, and $\cos a = \dfrac{G C}{E C}$,

Now since E C is taken as equal G C $\sin a = \tan a =$ the arc E D,

and $\cos a = \dfrac{1}{1} = 1$.

Thus all calculations involving both lenses and mirrors are greatly simplified, since the sine and tangent can be replaced by the arc, and the cosine by unity, whenever the aperture is small.

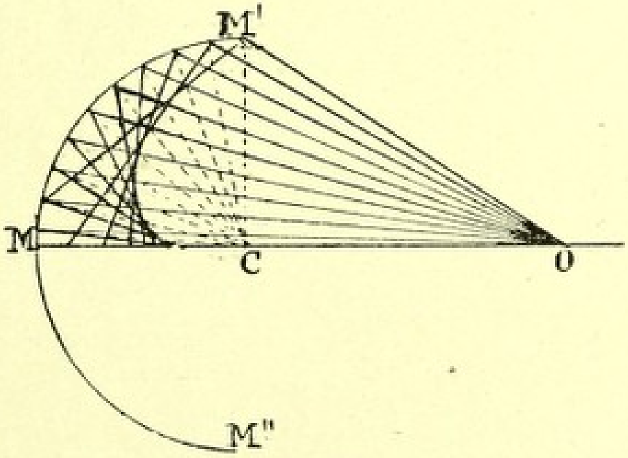


Fig. 42.

Caustics.—If the aperture be large, rays which diverge from a point O on the principal axis beyond C, and form a small angle with the axis, intersect each other and the axis, so as to form a cusp between the mirror and its centre. Here is the greatest condensation of the light and consequently the brightest spot. Those rays which form larger angles with the axis intersect each other in such a way as to form a curve. This is known as a caustic curve, and all the reflected rays are tangential to it. It can be readily seen by letting the light fall very obliquely on the inside of a cup half filled with tea.

It will be noticed that as the angles which the rays make with the axis O M become smaller, their foci approach the image-point more and more, until when the angle only amounts to a few degrees they form a common focus.

Ellipsoidal and Parabolic Mirrors.—There are two forms of curved mirrors which form sharp images of a point, i.e., images free from spherical aberration.

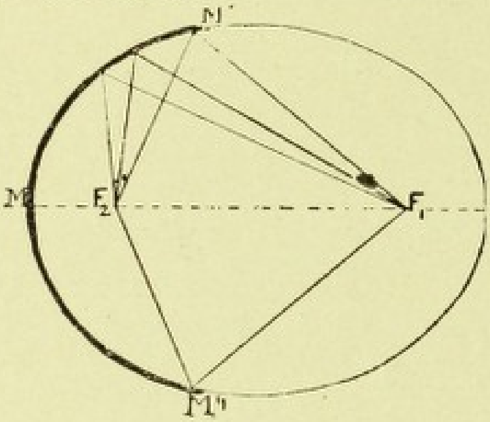


Fig. 43.

An ellipsoidal mirror (Fig. 43) is one formed by an arc of an ellipse in a section in the plane of the paper. In every ellipse there are two foci F_1 F_2 situated on the long diameter, or major axis. These have the property that any ray, which diverges from the one focus, after reflection passes through the other focus, so that any object at F_2 forms an image at F_1, and *vice versa*. If the one focus is at ∞ the curve becomes a parabola.

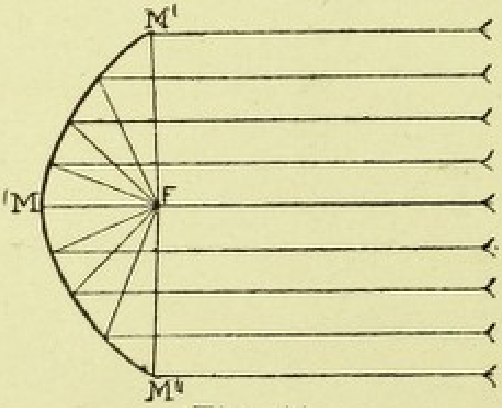


Fig. 44.

A parabolic mirror (Fig. 44) has the property that all rays falling on it parallel to the axis, after reflection, meet at a common focus F, situated on the principal axis. And if the object point be placed at this focus the rays, after reflection, form a parallel beam.

Such mirrors are employed in reflecting telescopes for bringing rays from an infinitely distant object, such as a star, to a sharp focus. Also for projecting a parallel beam of light as in lighthouse and optical lanterns, microscopic reflectors, etc. Such mirrors possess the advantage over refractors in that all the different waves of light are equally projected, and there is therefore no chromatic aberration. For this reason they are preferred to refractors for the photography of celestial bodies.

Cylindrical Mirror.—With a convex cylindrical mirror the image is distorted, being diminished across the axis while it is unchanged in dimension in the direction of the axis. The image is always virtual, and, as with the plane mirror, there is lateral inversion.

With a concave cylindrical mirror, when the object is within the focal distance, the image is distorted, being magnified across the axis, and unchanged in the direction of the axis, and there is lateral inversion.

When the object is beyond the focal distance of a concave mirror the image is real, unchanged in size in the direction of the axis, but diminished or magnified across the axis, according as the object is further from, or nearer to the mirror, than the centre of curvature.

If the concave mirror is with its axis horizontal the image is reversed vertically, and there is lateral inversion in the horizontal meridian.

If the mirror is placed with axis vertical the image is upright vertically and reversed horizontally, consequently there is no lateral inversion. Thus the right of the real image formed is the right of the object.

Determination of the Focal Length of a Mirror.—The curvature of a mirror can be obtained by means of the spherometer, and from the curvature the focal length is directly obtainable. This is the best practical method for both concave and convex mirrors.

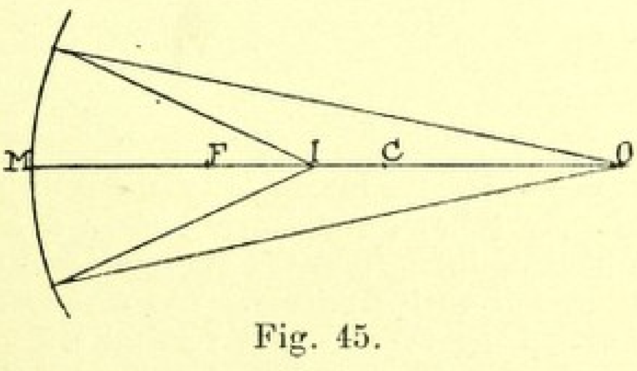


Fig. 45.

Concave Mirror.—Let O be an object placed on the axis of the mirror M (Fig. 45) beyond the centre of curvature C. The mirror forms a real image I between F and C, and this can be received on a screen which is moved until the image is as sharp as possible. Measure the distances M O and M I, which may be written f_1 and f_2 respectively. Then r, the radius of curvature, is found from the formula

$$\frac{2}{r} = \frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

Thus let the distance of the object be 45in. and the image 20in. from the pole, then

$$\frac{1}{F} = \frac{2}{r} = \frac{1}{45} + \frac{1}{20} = \frac{65}{900} \quad \therefore r = 28\text{in., or } F = 14\text{in. (approx.)}$$

Another method. Since at the centre of curvature image and object correspond in size and position, it is evident that if the object be elsewhere, the image at that same position will be blurred or invisible. Make a hole, say, ¼in. square in a card and fix a piece of wire gauze across the aperture. Place a candle so that the flame is behind the gauze, and hold the card upright, slightly to one side of the axis. Now move the candle and card to and fro in front of the mirror. A position is found at which the image of the gauze is sharply defined on the card, and by slightly shifting the card the image can be adjusted close to the hole. The image and object are then identical in size and shape, but the image is inverted. The slightest movement of the card to or from the mirror causes the image to become hazy. Measure the distance from the mirror to the card at the position of best definition, and this distance is equal to the radius of curvature, or twice the principal focal distance of the mirror.

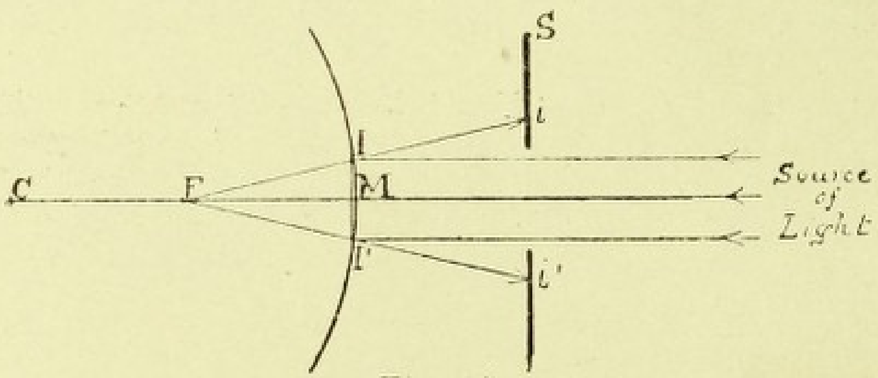


Fig. 46.

Convex Mirror.—Cover the mirror M (Fig. 46) with lamp black by means of a smoky candle, but leaving clear a small central space I I′. Let parallel light from the sun, or from a collimator in front of a flame, fall on the mirror. The rays after reflection diverge as if coming from F. Place a vertical screen S, having a central hole of slightly greater diameter than I I′, at such a distance from the mirror that the ring of light i i′ formed on S is exactly twice as large as I I′. Then the focal length of M is equal to the distance of S from the mirror; the radius of curvature being twice that distance.

CONJUGATE FOCI OF SPHERICAL MIRRORS.

The method of making simple algebraical calculations is to be found in the appendix.

Conjugate Focal Distances.—If F be the principal focal distance then $1/F$ is the reflecting power of the mirror, because these two quantities are reciprocals of each other; thus, if F be 10, then $1/F = 1/10$. Let f_1 be the distance of the object, then its reciprocal $1/f_1$ is that power which brings parallel rays to a focus at the distance of f_1. If we call f_2 the distance of the image from the mirror, then $1/f_2$ is that power which causes parallel rays to meet at the distance of f_2.

Conjugate foci are a pair of positions, occupied by the object and its image, which are so related that when the object is at the one, the image is at the other. Conjugate focal distances are the distances of the conjugate foci from the mirror (or lens).

In a concave mirror F is positive, $1/F$ is a positive quantity, and the total power of a concave mirror $1/F$ is equal to the sum of the powers which represent the distances of the object and image. In other words the reciprocal of the principal focal distance is equal to the sum of the reciprocals of any pair of conjugate foci.

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}; \text{ or } \frac{1}{F} - \frac{1}{f_1} = \frac{1}{f_2}$$

Thus if the mirror be of 20in. radius or 10in. F, the sum of $1/f_1 + 1/f_2$ is always $1/10$.

This formula is one of the most important in optics. It enables us to find the radius or focal length of a mirror if f_1 and f_2 are known; or if r and f_1 are known we can find f_2 (the image). It is universal and holds true for both concave and convex mirrors and, as will be seen, for lenses as well.

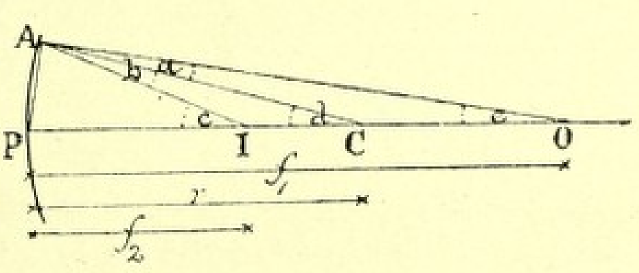


Fig. 47.

Proof of the Formula.—Let O be a point on the axis of a mirror A P. Draw the radius A C to the point of incidence A. Thus a is the angle of incidence, and if b is made equal to a, the image of O will be I. Then if the aperture A P is small compared with F, the angle A P C may be considered a right angle. For the same reason A O may be considered equal to P O, A C to P C, and A I to P I.

Now $a = d - e$

and $b = c - d$

Then, as the angles c, d and e are small

$$\sin a = \sin d - \sin e = \frac{AP}{CP} - \frac{AP}{OP}$$

and $$\sin b = \sin c - \sin d = \frac{AP}{IP} - \frac{AP}{CP}$$

But $a = b$

$$\therefore \frac{AP}{CP} - \frac{AP}{OP} = \frac{AP}{IP} - \frac{AP}{CP}$$

or $$\frac{1}{CP} - \frac{1}{OP} = \frac{1}{IP} - \frac{1}{CP}$$

That is $$\frac{1}{r} - \frac{1}{f_1} = \frac{1}{f_2} - \frac{1}{r}$$

Therefore $$\frac{1}{f_1} + \frac{1}{f_2} = \frac{2}{r} = \frac{1}{F.}$$ [3]

Since the two fractions $\frac{1}{f_1} + \frac{1}{f_2}$ added together always produce the same sum, it follows that however much the one is increased the other is decreased in the same proportion.

Another Expression for Conjugate Focal Distances.—We may also express the formula in another way. If the distance of the object from F = A and that of the image from F = B then $A\,B = F^2$.

That is, the product of the distances of image and object from the principal focus is equal to the square of the principal focal distance. It is a good practice to work out the following problems by this formula. The results should always be the same as when worked out by the other method.

Examples—Concave Mirror.—(1) If an object be situated beyond F, say at 30in. in front of a concave mirror of 10in. focal length, then

$$\frac{1}{10} - \frac{1}{30} = \frac{2}{30} = \frac{1}{15} \quad \textit{i.e.} \text{ the image is at 15in.}$$

Now 15in. and 30in. are conjugate foci in respect to a 10in. concave mirror, so that if the object be at 15in. its image is at 30in.

(2) If the object be at infinity, the formula becomes

$$\frac{1}{F} - \frac{1}{\infty} = \frac{1}{F} - 0 = \frac{1}{F}.$$

The image then is at the principal focal distance, f_2 being equal to F.

(3) If the object be at F the calculation is

$$\frac{1}{F} - \frac{1}{F} = 0 = \frac{1}{\infty}$$

so then the image is at ∞, and F and ∞ are conjugate distances.

(4) If the object be at twice F, that is at the centre of curvature, say, 20in. in front of a 10in. concave mirror, the image is at the same distance, since

$$\frac{1}{10} - \frac{1}{20} = \frac{1}{20}. \text{ or I is at 20in.}$$

(5) Suppose, lastly, the object is placed within the principal focal distance. The conjugate focus is then negative, a higher number than 1/F being deducted from it, the result is a minus quantity.

Thus if the object be placed six inches in front of a 10in. concave mirror, then,

$$\frac{1}{10} - \frac{1}{6} = -\frac{4}{60} = -\frac{1}{15}$$

so that the image is 15in. behind the mirror. The proof of this is that

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{F}; \quad \text{or } \frac{1}{6} + \left(-\frac{1}{15}\right) = \frac{1}{10}.$$

Here —15in. is the conjugate of 6in. in respect to a 10in. concave mirror, and 6in. is the conjugate of —15in. but not of 15in. That is to say, if the rays of light incident on the mirror are convergent to a point 15in. behind it, they will be reflected so as to come to a focus 6in. in front of it.

Using the formula A B = F^2 we have

$$B = \frac{F^2}{A} = \frac{100}{6-10} = -25.$$

The negative image is $-25 + 10 = -15$, or 15in. behind the mirror.

Nature of Image.—From these calculations it will be seen that a real or positive image is obtained with a concave mirror so long as the object is beyond F, and that image becomes virtual or negative when the object is nearer than F.

Relative Distances of O and I.—The nearer the object is to F, the more distant is the real image; as the object recedes from F, the image approaches it, but no positive image of an object can be nearer than F since no object can be more distant than ∞. If, however, the rays are convergent before reflection, then f_2 passes to the mirror side of F.

Thus, if light is converging to 20ins. behind a 10in. Cc mirror $1/f_2 = 1/10 + 1/20$, or the image is formed $6\frac{2}{3}$ins. in front of the mirror.

The planes of unit magnification for real images lie at the point where the object coincides with the centre of curvature of the Cc. mirror, for then the image is equal in size to the object and at the same distance.

The nearer the object is to F, the more distant also is the negative image. As the object recedes from F and approaches the mirror, the image also approaches the mirror, and when the object touches the reflecting surface, the image does so likewise, this being the plane of unit magnification for virtual images formed by a Cc. mirror.

Recapitulation of Conjugate Foci.—To recapitulate the position of f_1 and f_2 with respect to a Cc mirror,

If the object is at ∞..................the image is real, inverted, diminished and at F.

,, ,, ,, between ∞ and 2F...the image is real, inverted, diminished and between 2F and F.

,, ,, ,, at 2F ....the image is real, inverted, equal to the object and at 2F.

,, ,, ,, between 2F and F ...the image is real, inverted, enlarged and between 2F and ∞.

,, ,, ,, at F......................the image is infinitely great and at ∞.

,, ,, ,, within F.................the image is virtual, erect, and enlarged and on the other side of the mirror.

,, ,, ,, at the mirror............the image is virtual, erect, and at the mirror.

Conjugate Foci—Convex Mirror.—Remembering that in a convex mirror F and 1/F are negative, the rules governing the calculations of conjugate foci are the same as with a concave mirror.

$$-\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}; \quad \text{or} \quad -\frac{1}{F} - \frac{1}{f_1} = \frac{1}{f_2}$$

Thus, if an object is situated 30in. in front of a convex mirror of 10in. focal length

$$-\frac{1}{10} - \frac{1}{30} = -\frac{1}{7\frac{1}{2}} \text{ in.}$$

The image is virtual and $7\frac{1}{2}$in. behind the mirror.

—$7\frac{1}{2}$in. is conjugate to 30in. with respect to a 10in. convex mirror and 30in. is the conjugate of —$7\frac{1}{2}$in., but not of $7\frac{1}{2}$in. If rays were convergent to a point $7\frac{1}{2}$in. behind the surface, the convergence would be, by reflection, so much reduced that they would meet 30in. in front of a 10in. convex mirror.

Using the formula A B = F^2 we have

$$\frac{100}{30 - (-10)} = 2\tfrac{1}{2}, \text{ or } 2\tfrac{1}{2} - 10 = -7\tfrac{1}{2}\text{in.}$$

If the object be at infinity, the calculation is

$$-\frac{1}{F} - \frac{1}{\infty} = -\frac{1}{F} - 0 = -\frac{1}{F}$$

the image being at the principal focal distance.

If the object be in front of the mirror at a distance equal to F of a 10in. Cx. mirror, the image is at half F or 5in. for

$$-\frac{1}{10} - \frac{1}{10} = -\frac{1}{5}$$

Relative Distances of O and I.—The image of a real object formed by a convex mirror is then always virtual and cannot be at a greater distance from it than F, the object being then at ∞. When the object is nearer than ∞ the image recedes from F towards the mirror, and when the object touches the surface the image does likewise. This is the plane of unit magnification of a Cx. mirror. If, however, rays be convergent before reflection, f_2 will be beyond F at a distance dependent on the degree of convergency, and if they be directed towards F they would be reflected as parallel rays, while if convergent to a shorter distance than F they would be reflected as less convergent rays to a real image in front of the mirror.

Thus, if light is convergent to 4 ins. behind a 10 in. cx. mirror $1/f_2 = -1/10 + 1/4$, or the image is formed $6\frac{2}{3}$ ins. in front of the mirror.

Size of the Image formed by a Spherical Mirror.—In the case of both Cc. and Cx. mirrors the size of the image bears the same relation to the size of the object as the distance of the image does to the distance of the object from the mirror or from the centre of curvature. This rule holds good whether the image be real or virtual, and in all cases, since both object and image subtend the same angle at the centre of curvature; so that their respective sizes depend solely on their respective distances from C. The working formula is:—

$$\frac{O\ f_2}{f_1} = I \qquad [3a]$$

Where O is the size of the object, I is the size of the image, f_2 is the distance of the image, and f_1 is the distance of the object.

The application of the given formula is illustrated in the following examples:—

Let an object 2in. in diameter be placed 16in. in front of a 6in. concave mirror, then

$$\frac{1}{f_2} = \frac{1}{6} - \frac{1}{16} = \frac{10}{96}.$$

The image is at $9\frac{3}{5}$ inches in front of the mirror and

$$I = \frac{2 \times 96}{10} \times \frac{1}{16} = 1\frac{1}{5} \text{ ins.}$$

The relative sizes 2 and $1\frac{1}{5}$ are as the relative distances from the mirror 16 and $9\frac{3}{5}$ or as the distances from C, the centre of curvature, 4 and $2\frac{2}{5}$.

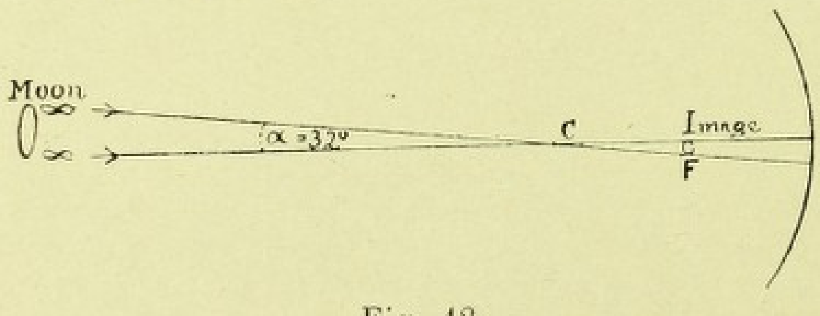


Fig. 48.

What will be the size of the image of the moon formed by a concave mirror of 16in. focus?

The object being at ∞ (Fig. 48) the rays from each point of it are parallel, and the image I will be formed at the principal focus and ∴ $f_2 = 16$ins. But although the object is at ∞ it has a definite size subtending an angle a of $32'$, and we reckon the size of the angle subtended by the object instead of the object itself. Now object and image subtend the same angle at C, and since $\tan 32' = .0093$ and $f_2 = 16$ ins, we have

$$16 \times .0093 = 0.1488\text{ins., or about } \tfrac{1}{7}\text{in.}$$

Let an object $\frac{1}{2}$in. in diameter be placed 8in. in front of a 12in. mirror. What is the size of the image?

$$\text{We find } \frac{1}{f_2} = \frac{1}{12} - \frac{1}{8} = -\frac{4}{96} = -\frac{1}{24}$$

The image is at 24in. virtual,

$$\text{therefore its size } = \frac{\frac{1}{2} \times 24}{8} = 1\tfrac{1}{2}\text{in.}$$

$\frac{1}{2} : 1\frac{1}{2}$ as $8 : 24$, or as the distance from C, $16 : 48$.

Suppose an object 2in. long placed 20in. in front of an 8in. Cx. mirror then,

$$\frac{1}{f_2} = -\frac{1}{8} - \frac{1}{20} = -\frac{28}{160}$$

The image is $5\frac{5}{7}$in. behind the mirror and

$$I = \frac{40}{7} \times \frac{2}{20} = \frac{4}{7}\text{ in.}$$

$2 : \frac{4}{7}$ as $20 : 5\frac{5}{7}$, or as the distances from C, $36 : 10\frac{2}{7}$.

CHAPTER IV.

REFRACTION AND THE REFRACTIVE INDEX.

Normal Incidence of Light.—The fact that the velocity of light is lessened in a dense medium, is the cause of refraction. When a beam of parallel rays, traversing the air, is incident normally on a refracting medium such as a sheet of glass, the whole of the wave-front is retarded simultaneously and equally. The plane of the wave after entering the glass is unchanged in direction and continues so during its progression through the denser medium. On reaching the second surface, the whole of the wave-front is again incident at the same time, and each part of it is equally increased in speed as it passes again into the rarer medium, so that its line of progression remains unchanged.

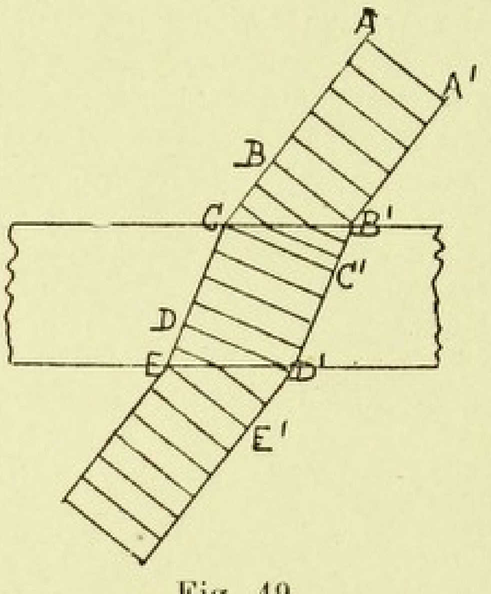


Fig. 49.

Oblique Incidence of Light.—But if the plane wave-front A A′ (Fig. 49) be incident on the first surface obliquely, one part B′ meets the denser medium sooner than the rest and this is retarded, while the others are still in the rarer medium advancing at an undiminished rate of speed. The rest of the wavelets on reaching the glass become retarded, one by one, until the whole of the wave-front has passed into the denser medium and in consequence the wave-front is changed in direction. The extent of the alteration of direction depends on the distance that the more rapidly advancing parts of the wave-front travel before their speed is also checked, that is, on the obliquity of incidence of the rays, and on the retardation itself, that is, on the refracting power of the medium. When the whole of the wave-front C C′ has arrived within the denser medium, it travels without deviation but at a diminished rate of speed. On reaching the second surface of the glass the wave-front D D′ is again incident sooner at one point D′ than at others. The wavelet at that point increases its speed, while the remainder is still moving less rapidly in the denser medium; then the other wavelets emerge and increase their speed until, having passed into a rarer medium, the entire wave-front E E′ travels with its original velocity and in a direction parallel to its original direction.

The Laws of Refraction are as follows :—

1. A ray passing obliquely from a rare into a denser medium is refracted towards the perpendicular at the boundary plane between the two media.
2. A ray passing obliquely from a dense into a rarer medium is refracted away from the perpendicular at the boundary plane between the two media.
3. A ray suffers no deviation if, at the point of incidence, it is perpendicular to the surface of the medium which it enters.
4. The extent of the deviation suffered by a ray of light as it passes from one medium into another is governed by the difference in the refracting powers of the two media. Generally, the refracting power of a medium is proportional to its density, and is measured by the retardation suffered by a ray entering it from a vacuum.
5. The incident and refracted rays are in the same plane as the perpendicular to the refracting surface and on opposite sides of it.
6. A constant ratio exists between the sine of the angle of incidence and the sine of the angle of refraction, termed *the index of refraction.* This is denoted by the Greek letter μ (mu).

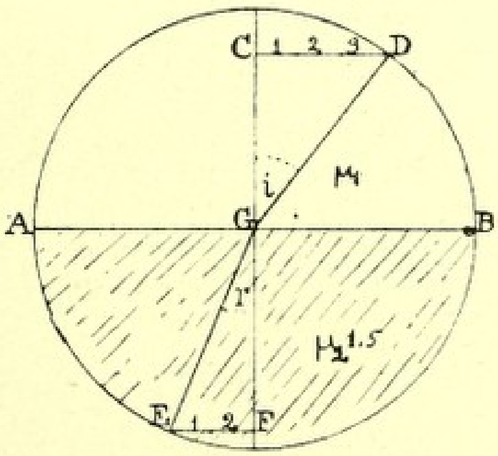


Fig. 50.

The Law of Sines.—In order to illustrate the course of a ray as it passes from one medium to another (Fig. 50), let A B be a refracting surface separating air of index $\mu_1 = 1$ from a denser medium glass of $\mu_2 = 1.5$. Let D G be a ray incident on the surface at the point G to which the line CGF is the normal; then D G C is the angle of incidence and E G F the angle of refraction, and if we call D G C the angle i and E G F the angle r, then

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}, \text{ or } \sin i \; \mu_1 = \sin r \; \mu_2 \qquad [4]$$

This is called Snell's Law or the Law of Sines, which holds true whatever may be the inclination of the incident ray.

If E G were the incident ray it would be refracted as G D as it passes from the dense to the rare medium.

If the first medium is air $\mu = 1$ then

$$\frac{\sin i}{\sin r} = \mu_2 \text{ or the } \mu \text{ of the dense medium.} \qquad [5]$$

If the angles i and r are very small, they may be taken in place of their sines and the law may be written

$$\mu = \frac{i}{r}, \text{ or } i = \mu r. \qquad [6]$$

Velocity of Light and μ.—The velocity of the light in the first medium is to the velocity in the second medium, as μ_2, the refractive index of the second medium, is to μ_1, that of the first, or the rate of progression of light in a medium is inversely proportional to its optical density.

Let V_1 be the velocity of the light in the first medium, which is air, and V_2 the velocity in the second and denser medium.

Then in Fig. 49 B B′ C is equal to the angle of incidence of the wave B B′, and is the same as i in Fig. 50. Also B′ C C′ in Fig. 49 is the angle of refraction of the wave, and is the same as r in Fig. 50.

Now B C and B′ C′ in Fig. 49 represent the relative distances that the wave has travelled in a given time in the two media, so that B C = V_1 and B′ C′ = V_2.

Then B′ C being common to the two triangles B B′ C and B′ C C′, it follows that

$$\frac{B\,C}{B'C'} = \frac{\sin i}{\sin r} = \frac{V_1}{V_2} = \mu \qquad [7]$$

Absolute μ.—The index of refraction of a vacuum is taken as unity or 1, so that any other medium (with the exception of certain metals), being denser, has an index greater than 1. This numerical expression indicates the refractivity of a medium with respect to a vacuum and is *the absolute index of refraction.*

Since the index of refraction of air is 1.000294, a figure which differs but very slightly from 1, for all practical purposes the μ of air may be taken as the standard index of refraction represented numerically by 1.

Relative μ.—*The relative index of refraction* is the expression of the refractivity when light passes from one dense medium into another, say, from water into glass or *vice versa*. It is found by dividing the index of the medium, into which the ray passes, by the index of the medium from which it proceeds; thus when light passes from water $\mu = 1.333$ into glass $\mu = 1.545$ the relative index is

$$\frac{1.545}{1.333} = 1.16$$

The sines of the angles of incidence and refraction as light passes through two such media are to each other as the velocities of the light in those two media.

Reciprocal *μ's.*—In the case of any two media A and B the index of refraction for light passing from A into B is the reciprocal of the index for light passing from B into A. Thus, when light passes from air into glass, the sines of the angles of incidence and refraction are as 3 is to 2 and the index is 3/2. If it passes from glass into air, the sines of the two angles are as 2 is to 3 and the index is 2/3.

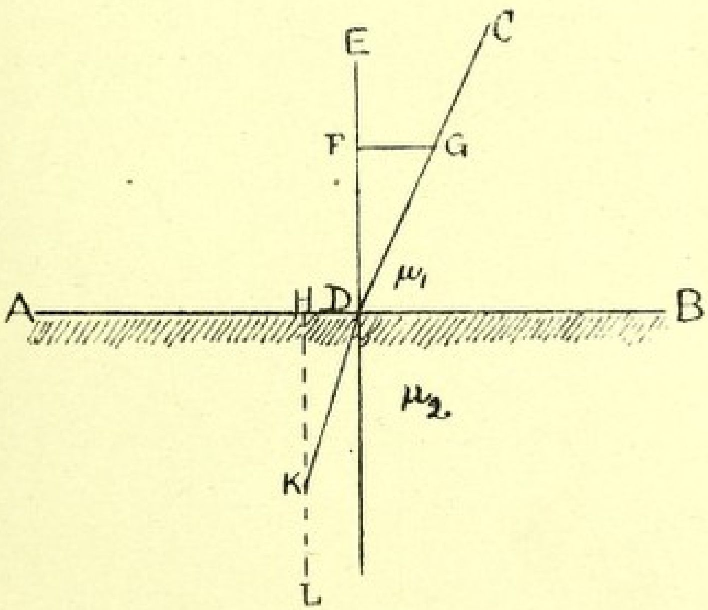


Fig. 51.

Course of a Light Ray.—Fig. 51 illustrates the method of tracing the course of a refracted ray. Let the media be respectively air, $\mu_1 = 1$, and glass, $\mu_2 = 1.5$, and let C D be a ray of light incident at D. From any point G on C D drop a perpendicular G F on to the normal E D. Measure F G and mark off D H equal to ⅔rds of F G. From H, let fall a perpendicular H L and connect

D with H L by a line D K, whose length equals G D. Then D K is the direction of the refracted ray. The method of construction for a ray K D passing from the denser medium into the rare medium is similar, but the construction is reversed.

The line F G is the numerator and H D is the denominator of the fraction representing μ; in water F G would be four parts and D H three parts, and in order to make the measurements easy F G can be taken anywhere on E D. Thus suppose μ_2 is 1.55, F G could be 15.5 mm. in length and D H would then be 10 mm.

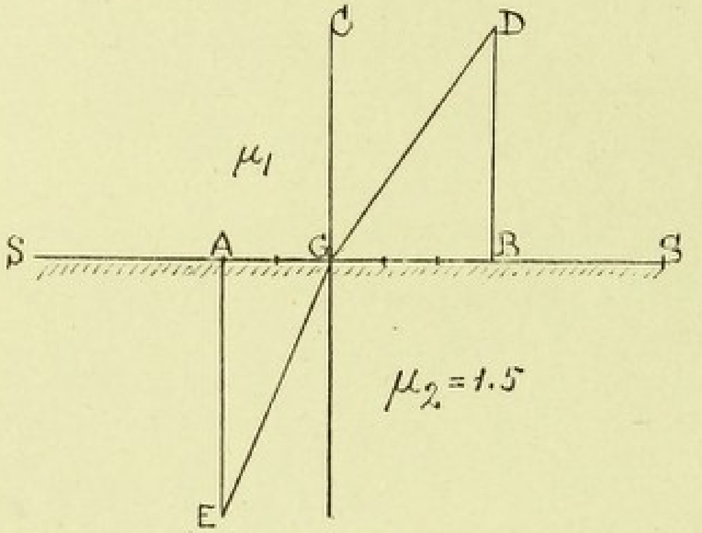


Fig. 52.

Another Method which is based on the construction shown in Fig. 50. Let D G (Fig. 52) be any ray incident at G on the surface S S of a medium whose $\mu = 1.5$ or $3/2$. From D drop the perpendicular D B and divide B G into three equal spaces. Then from G mark off G A equal to two such spaces. From A drop a perpendicular and from G draw a line G E, equal in length to G D, cutting the perpendicular from A in E. Then G E is the direction of the refracted ray.

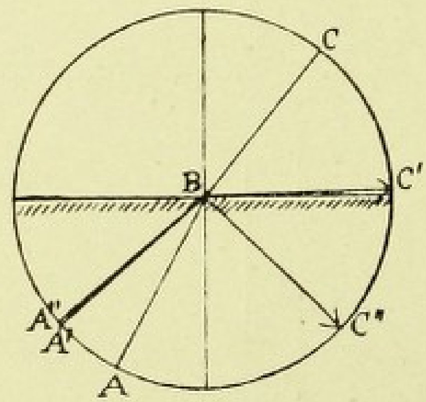


Fig. 53.

Critical Angle and Total Reflection.—When a ray of light enters a rare from a denser medium, it is bent away from the normal with which it makes a larger angle than before; the sine of the angle of refraction is greater than the sine of the angle of incidence. In Fig. 53 let μ_2 be the dense and μ_1 the rare medium.

Let A B be the incident and B C the refracted ray. As A B makes a larger angle with the normal the corresponding angle of refraction becomes still larger. Hence if the ray A′ B be incident at an angle sufficiently large, the refracted angle becomes a right angle, and the refracted ray B C′ will skim along the boundary surface. The incident angle which produces this result is termed the *Critical Angle*, because the slightest further increase of it prevents the ray from passing out of the denser medium. If the incident ray be A″ B it is reflected as B C″ and *internal reflection* takes place.

If i is the angle of incidence in the dense medium and r the angle of refraction in the rare medium and μ' the *relative index* from the dense to the rare medium, then sin i = μ' sin r. But r = 90° and the sine of 90° = 1, which is the greatest possible value that a sine can have. Therefore sin i = μ', or the sine of the critical angle equals the relative index of refraction.

Let μ_1 be the index of refraction of the rare and μ_2 that of the dense medium; let c be the critical angle. If the rare medium is air,

then $$\sin i\ \mu_2 = \sin r\ \mu_1,$$

but $$\mu_1 = 1 \text{ and } \sin r = 1, \qquad \text{also } c = i,$$

therefore $$\sin c\ \mu_2 = 1,$$

that is $$\mu_2 = \frac{1}{\sin c}, \quad \text{or } \sin c = \frac{1}{\mu_2}. \qquad [8]$$

Thus the sine of the critical angle of any medium bounded by air equals 1 divided by the refractive index.

Suppose the ray to pass from glass to air, then $1/\mu = 1/1.5 = 0.666$, so that the sine of the critical angle for glass is .666, which = sin 41° 46′. This is the greatest angle at which a ray can be incident in order to emerge from glass of $\mu = 1.5$ into air, and the emergent ray is then parallel to the surface. The critical angle of water is 48° 30′.

This principle affords a method of determining the refractive index of any medium. If the angle at which the incident ray just ceases to emerge into air (or vacuum) be measured, one divided by the sine of that angle is equal to the refractive index of the medium.

Internal reflection is termed *total* to distinguish it from ordinary reflection, which is always accompanied by a certain amount of absorption or transmission.

Critical Angle for Various Media.—Since 1.33/1.5 = .886 = sin 62°, the critical angle for light passing from glass to water is 62°, but the critical angle for light passing from water to air is about 49°, therefore light passing from glass to water and thence to air would be totally reflected at the upper surface of the water for any angle of original incidence over 41° as shown in Fig. 54.

Thus the critical angle for light passing through various media into air is the same as that from the first medium directly into air.

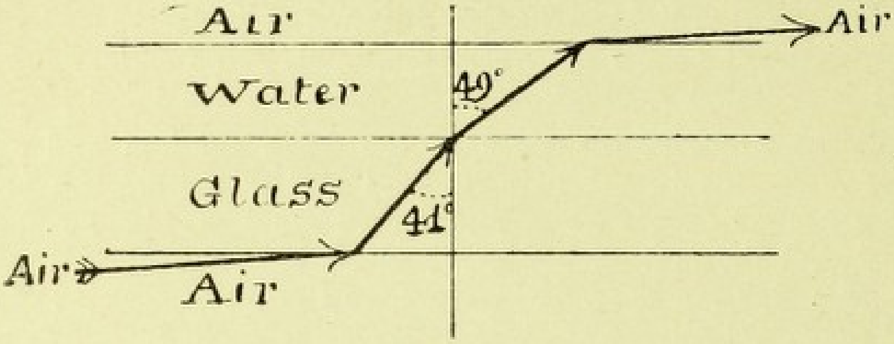


Fig. 54.

Parallel Incidence and Emergence of Light.—If a ray of light passes, at the critical angle, through a plate of glass with parallel sides, it enters and emerges parallel or nearly parallel to the surface of the glass. If a layer of water be poured on to the surface of the glass the ray passes through the water at the critical angle of water and emerges parallel to the initial ray which entered the glass at nearly 90° with the normal, as in Fig. 54. If there were any number of parallel plates superposed, all of different refractive indices, the ray finally emerges from the last plate at the same angle as that at which it entered the first plate, provided that the media outside the first and last plates be the same.

Effects of Total Reflection.—Total reflection explains why on looking into the water of a large aquarium tank the surface above one's head glistens like quicksilver, owing to all the light being reflected downwards. A metal ball, blackened by a smoky flame, immersed in water appears brilliantly polished, owing to the thin film of air surrounding it which totally reflects the light.

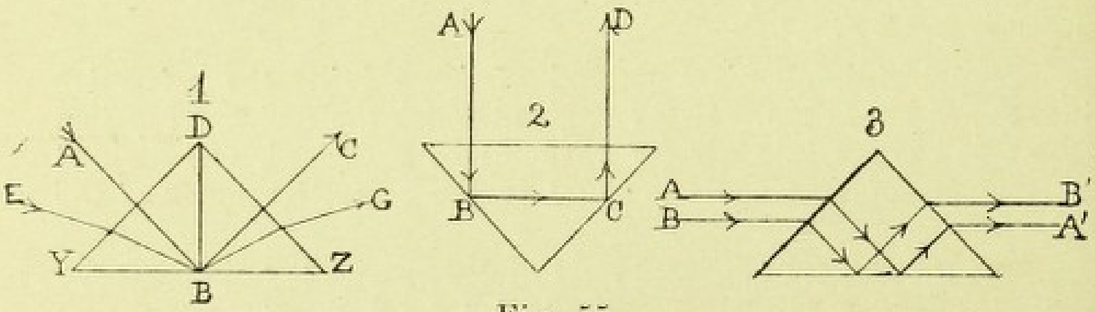


Fig. 55.

Total Reflecting Prisms.—The critical angle or angle of total reflection for crown glass is about 42°. If, therefore, a ray (A B) enters a prism and makes an angle greater than 42° with the normal to the surface Y Z, the ray will be totally reflected in the direction B C, Fig. 55 (1).

If the principal angle of a prism exceeds twice the critical angle of the medium, of which it is made, total reflection ensues for all incident light.

Right-Angled Reflection.—The ray is not refracted at the surfaces D Y and D Z of the prism because it is incident normally to each. Thus a right-angled prism serves as a total reflector when the light is incident perpendicularly to the one face, the direction of the emergent light being at right angles to the original course.

But the ray need not enter at right angles to D Y, any direction will do provided it makes, after refraction, an angle greater than 42° with the hypotheneuse Y Z. Thus the ray E B G will be totally reflected. The dispersion which takes place as the ray enters the glass is reversed as it leaves the prism, so that the emergent ray consists of white light similar to that which entered.

Parallel Reflection.—If the light falls normally on the hypotheneuse of a right-angled prism it causes total reflection twice at B and C, as in Fig. 55 (2), so that the final direction C D of the light is parallel to its original course A B. By means of a right-angled prism, as indicated in Fig. 55 (3), vertical without lateral inversion may be obtained. This prism is largely used in process photography.

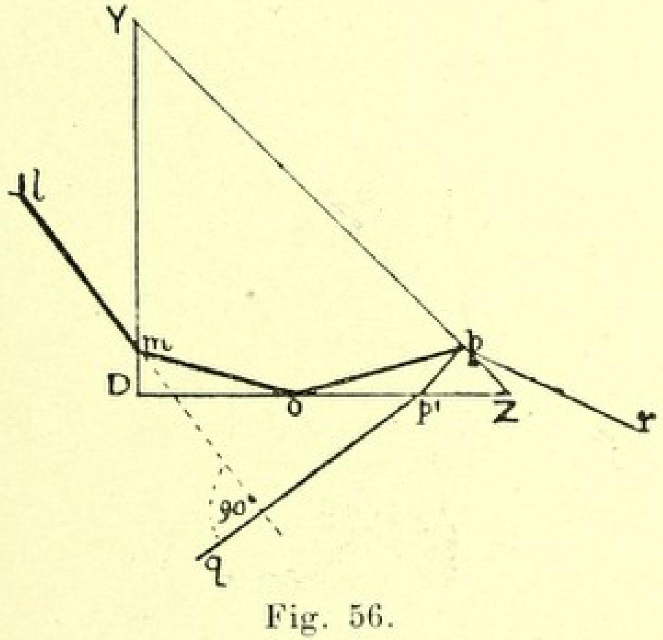


Fig. 56.

Testing Prisms.—Use is made of this property of total reflection in order to learn whether a prism is ground to a right angle. If the ray l m (Fig. 56) strikes the prism so near to D that it is reflected in the direction p, it will then be partly reflected and partly refracted. The reflected ray emerges at p′ in the direction p′ q, which makes an angle of 90° with l m produced, if the prism be truly worked, no matter the direction of l m; but if there is any error in the angles of the prism p′ q will not meet l m at right angles.

Effects of Total Reflection.—If a tank half full of water has some benzine on the top, the two liquids, owing to their different specific gravities, do not mix. As the benzine has a higher index than the water, a beam of light which is made to enter the benzine from above may be totally reflected at the surface of the water and emerge upwards, as is shown in Fig. 57 (1). The surface common to the two liquids, seen obliquely from above, will glisten like polished silver.

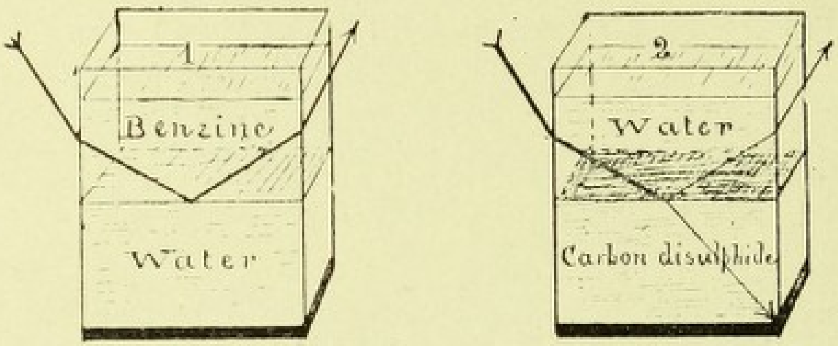


Fig. 57.

If another tank, Fig. 57 (2), containing carbon-disulphide be filled up with water, the lower liquid has the higher refractive index, so that only a few of the rays are reflected, the bulk of them being refracted downwards through the carbon-disulphide. Instead of the boundary surface glistening like silver it will appear as a dull matt surface, since most of the rays pass back into the heavier liquid.

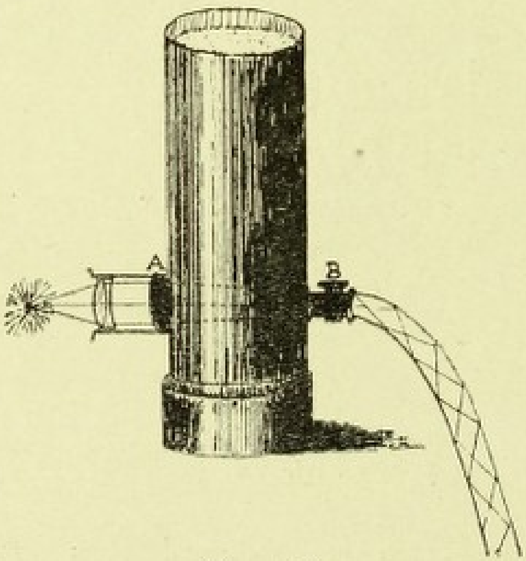


Fig. 58.

Another experiment is the cascade of silver. This consists of a tank fitted with a glass window at A and an aperture at B which can be opened by a tap. Facing A is a collimating lens. On filling the tank with water and opening the tap the light which passes in a parallel beam through A emerges at B and enters the stream of water, which it follows on account of internal reflection.

The appearance of the jet is that of glistening silver, owing to the escape of a portion of the light by scattering; but were the jet of water perfectly smooth the scattering would not occur, and the jet would appear dark.

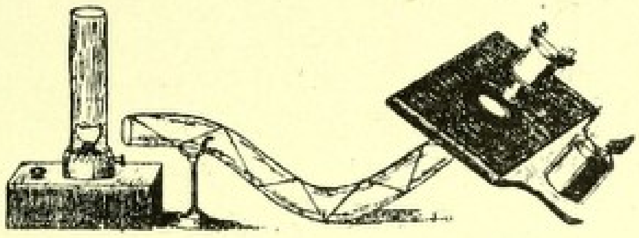


Fig. 59.

Another experiment on similar lines, but of practical value in microscopy, is as follows (Fig. 59). A solid tube of glass half an inch in diameter is bent into any desired shape. A strong light is brought close to one end, and the other is bent up against the opening of the stage of a microscope. The light by total reflection will form a powerful evenly-illuminated disc under the stage, and any slide will be uniformly illuminated from below. Here no light is scattered since the tube is smooth.

TABLE OF LIMITING OR CRITICAL ANGLES.

Medium.	Index of Refraction.	Critical Angle.
Chromate of lead	2.92	20°
Diamond	2.47	24°
Various precious stones		25° to 30°
Flint glass of about	1.60	38° to 40°
Crown glass of about	1.54	40° to 43°
Pebble	1.54	40°
Water	1.33	48° 30′

It will be seen, from the above, that the critical angle varies inversely with μ.

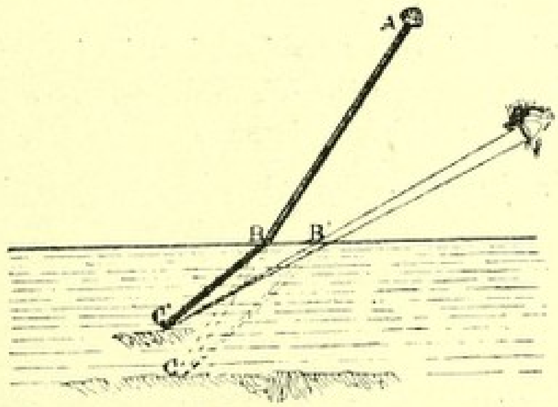


Fig. 60.

Displacement due to Refraction.—In Fig. 60 let C be the luminous point in a dense medium from which rays diverge and, after refraction, enter an observer's eye. These rays being projected backwards, intersect at C′, the virtual image of C which is situated nearer to the refracting surface, at a point dependent on the obliquity of the emergent rays and the index of refraction of the dense medium. This explains why an oar or teaspoon A B C partly immersed in the water, in an oblique direction, appears bent towards the surface, the bend commencing at the level of the water.

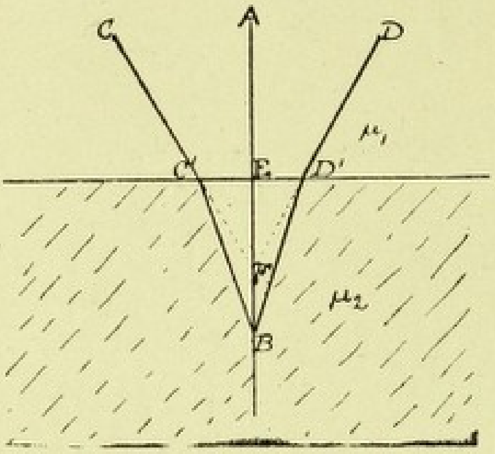


Fig. 61.

Vertical Displacement.—In Fig. 61 B is a luminous point in a dense medium observed from a position A, situated vertically above it. The ray B E passes out without deviation. B C′ and B D′ are refracted in the directions C′ C and D′ D, so the point B has its image F where C C′ and D D′ meet when referred backwards.

Let t be the actual distance B from the surface and let t′ be its apparent distance, that is, E B = t and E F = t′. Then

$$t' = \frac{t}{\mu} \qquad [9]$$

If μ_2 is water whose index of refraction is 4/3, then t′ = 3/4 t; that is to say, B appears, when viewed from above vertically, to be 1/4 nearer to the surface than it really is.

An object in a dense medium is apparently raised a distance of

$$t\,\frac{\mu - 1}{\mu} \qquad [10]$$

where t is the thickness of the medium or the actual distance of the object viewed from the bounding surface. This explains why a fish appears nearer to the surface than it really is, and if looked at obliquely with the eye near to the surface of the water it appears distorted, being thinner if viewed lengthways (parallel) to the surface and shortened if viewed with its head directed to the surface.

The apparent depth of a transparent body viewed vertically is similarly obtained by dividing its real depth by its refractive index. The apparent depth is lessened as the view becomes more oblique, but there is no definite expression by which it can be calculated.

A general expression is $\frac{t \cos r}{\mu \cos I}$

Lateral Displacement.—As already shown if a ray of light A B (Fig. 62) be incident on a denser refracting medium which has two parallel plane surfaces, it is refracted at the first point of incidence B in the direction B C and again at C in the direction of C D. Therefore rays refracted by a body, bounded by two plane parallel surfaces, emerge in a direction parallel to their original

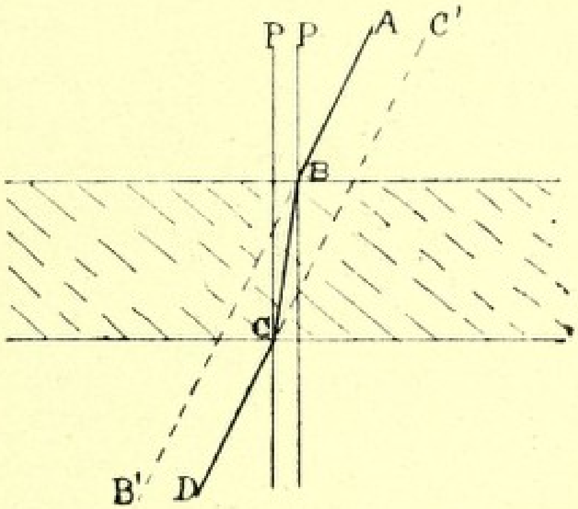


Fig. 62.

course, but they are laterally displaced, the amount of such displacement depending on the obliquity of the light, i.e., on the angle of incidence i, the index of refraction μ, and the thickness of the medium t. Let d be the lateral displacement = B′ D (Fig. 62), then

$$d = \frac{t \sin (i - r)}{\cos r} \qquad [11]$$

If the angle of incidence is small an approximate value of d is found from

$$d = \frac{t \tan i (\mu - 1)}{\mu}. \qquad [12]$$

Lateral displacement causes distortion of an object viewed, but an object seen through a plate-glass is not appreciably distorted if the thickness of the glass is small; nor is one which is at a considerable distance.

Lateral and Vertical Displacements.—In Fig. 63 the point L viewed obliquely by an eye at A A through a transparent medium N, whose two refracting surfaces are plane and parallel, is seen as L′ laterally displaced and nearer. If L is viewed from B, situated vertically above, it appears to be nearer (at I), but not latterally displaced.

Rays divergent before refraction are, after refraction, divergent as if from a nearer spot. If convergent they are, after refraction, convergent to a more distant point.

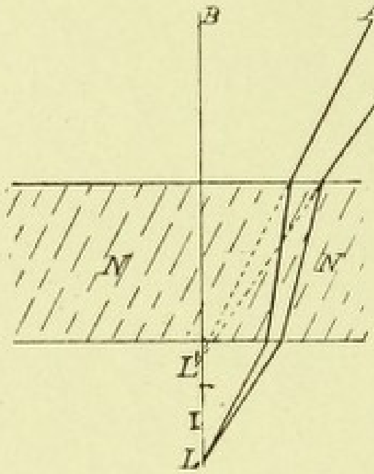


Fig. 63.

SOME OPTICAL PHENOMENA OF THE ATMOSPHERE.

Blueness of Sky.—If the air above us were absolutely transparent and of uniform density, light from the sun would reach the earth without any loss, and the sun, moon, and stars would appear set in a sky which would be black both during the day-time and at night.

But the air contains a great quantity of aqueous vapour, and the blue colour of the sky is said to be due to reflection from the minute particles of this vapour suspended in the higher layers of the atmosphere. Tyndall showed that when mastic is thrown into water the minute particles of the mastic which are insoluble emit a deep blue colour similar to that of the unclouded sky. Large quantities of so-called cosmic dust also are held in suspension in the air, and this is believed, by some scientists, to be a cause. By others the blue of the sky is said to be due to polarisation by oblique reflection from particles of vapour, salt, etc., in the air.

If a cloud of smoke be blown into the air, the smoke particles reflect the short blue waves more freely, and the cloud assumes a blue tint, and if a white screen be held, in bright sunlight, behind the smoke, the screen assumes a reddish brown hue.

Aerial Perspective.—If two objects, one light and the other dark, be seen at a considerable distance, they lose some of their contrast, the light object becoming darker by absorption of its reflected light by the intervening air and the dark object becoming lighter by the superadded light diffused through the air. This causes what is known as aerial perspective. If the air is clear and the added light is blue, distant hills throw deep shadows of a purple blue colour in bright sunshine.

Sun on Horizon.—When the sun is low down on the horizon its rays have to pass through a thicker layer of atmosphere filled with dust particles and moisture, and being deprived of some of its blue and violet rays, which become absorbed or reflected, it appears reddish. For the same reason the sun appears red in a fog.

Near the horizon the sun or moon appears larger than when higher in the heavens, because they are mentally projected beyond the horizon, being compared with terrestrial objects, whereas when seen in the zenith this cannot be done, as they stand alone. Of course they are not really larger as measurements with a telescope show. They are slightly flattened vertically when near the horizon and appear a trifle higher up than they really are, owing to the refraction of the air and the greater obliquity of the light from their lower edges. Refraction diminishes the dip of the horizon and so slightly increases its apparent distance.

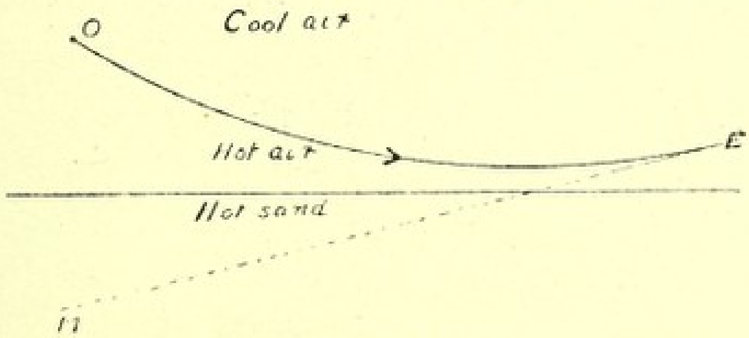


Fig. 64.

Mirage (Fata Morgana).—If the layers of the air are of markedly unequal density as is sometimes the case in hot climates, especially on a desert, where the warmest layers are the lowest, the phenomenon known as the *mirage* may be seen. Objects above the surface reflect rays which, on passing to the surface, traverse layers which become gradually less refracting. The angle of incidence accordingly increases from layer to layer so that the rays become more and more parallel to the surface, until at length the critical angle is reached, beyond which refraction becomes changed to internal reflection, and the rays are reflected in the contrary direction and ascend to reach the observer's eye. The rays which enter the eye then appear, from their direction, to proceed from a point below the ground, and the object appears inverted.

This is shown in Fig. 64, where light from an object O, on reaching the eye at E, appears to come from M below the level of the ground.

If the lowest strata of air are the densest, as in Fig. 65, they give rise to the same phenomenon, but the mirage M is in the contrary direction, so that land or ships at sea appear above the horizon. This occurs in very cold climates, in which the lowest strata are the coldest, and consequently the most dense.

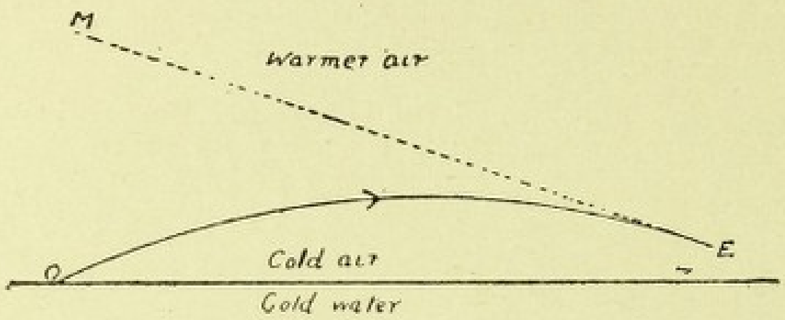


Fig. 65.

Scintillation.—The twinkling of a star is due to irregularities in the atmosphere causing variations in the path of the waves, which partially interfere. This produces variations in the apparent brightness and colour of a source of light, subtending a very small angle at the eye. It is not observed in the case of a planet, because this has a real magnitude.

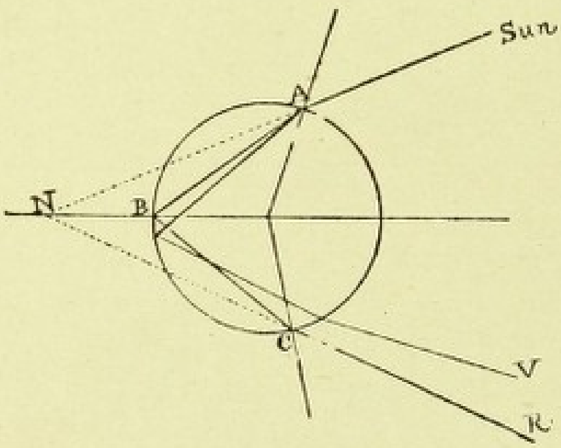


Fig. 66.

The Rainbow.—A rainbow is visible when the sun is behind the observer and a shower of rain in front of him. The bow forms a portion of the base of a cone, which has its apex situated at the observer's eye, its base in the plane drawn through the falling layer of raindrops, and its axis a prolongation of a line drawn from the sun to the observer. The same effect may be seen in the spray of a waterfall. Since the sun's rays falling on the raindrops are parallel, the course of light through all the drops must be the same, and it is therefore sufficient to trace the course of a ray through a single drop. Let a pencil of rays from the sun meet the drop at A. On entering it is refracted towards the back of the drop to B, is reflected thence to C, and is again refracted on emergence and proceeds in the direction C R. The emergent ray C R makes, with the entering ray, the angle A N C. A number of rays enter the drop parallel at A, but at different positions to its spherical surface, so that the emergent rays are divergent (see R and V, Fig. 66), the divergence varying with the different colours, so that the outside of the bow is always red, and the inside blue-violet.

The extent of the bow depends on the position of the sun; when at the horizon the bow forms a semi-circle to an observer at the sea-level. As the sun rises the arc sinks so that its centre is below the horizon, and the arc gets smaller.

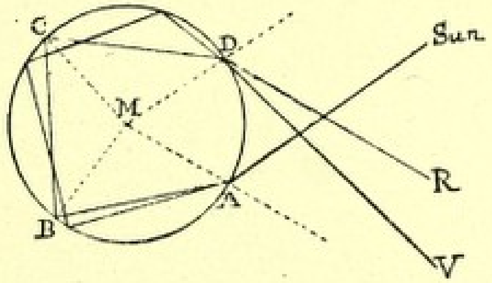


Fig. 67.

A secondary larger rainbow is often seen concentric with the first. In this the order of colours is reversed, the red being inside and the violet on the outside. This secondary bow is much fainter than the primary, because the rays undergo two reflections instead of one, as shown in Fig. 67, where the emergent rays cross the entering rays, thus giving rise to the reversal of the colours.

The rays proceed from the sun to a point A, where they undergo refraction and are reflected at B and C, and are again refracted at D. The final directions of the rays vary with the colour of the light, and therefore the bow appears blue on the outside and red within. The secondary bow is also somewhat broader than the primary.

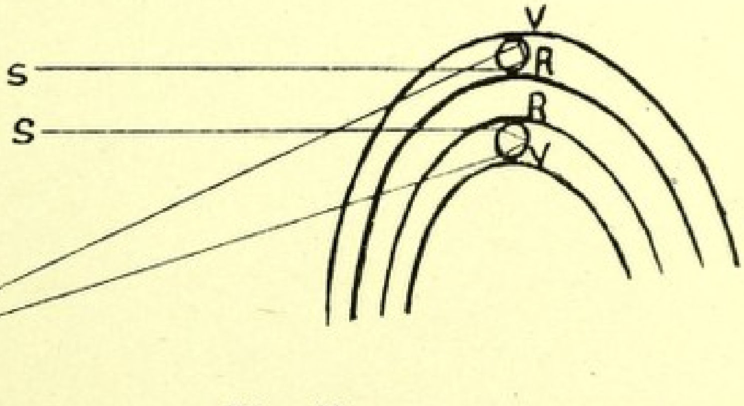


Fig. 68.

Fig. 68 shows the general appearance and relative positions of the two bows.

THE DETERMINATION OF μ.

Prism Method.—If the principal and deviating angles of a prism are known its refractive index can be found by the formula

$$\mu = \frac{\sin\left(\frac{P + d}{2}\right)}{\sin\left(\frac{P}{2}\right)}$$ [13]

where P is the principal angle of the prism, and d is the angle of minimum deviation. It should be noted that

$$\sin\left(\frac{P + d}{2}\right)$$

is the sine of half the sum and not half the sine of P + d. A proof of the above formula will be found in Chapter V.

This is the most accurate method of determining the refractive index of a transparent body. If the medium is a liquid it must be enclosed in a hollow glass prism.

The index of refraction for the different lines of the spectrum can be determined by this method, and the dispersion between any two given lines thus arrived at.

Example.—Given a certain prism whose principal angle P is found to be 59° 57′ and the angle of minimum deviation d is 48° 21′, then

$$\frac{P + d}{2} = \frac{59^\circ .57' + 48^\circ 21'}{2} = 54^\circ\ 9', \text{ and } \frac{P}{2} = 29^\circ\ 58';$$

so that

$$\mu = \frac{\sin 54^\circ\ 9'}{\sin 29^\circ\ 58'}$$

Now $\sin 54^\circ\ 9' = .81055$ and $\sin 29^\circ\ 58' = .49962$.

Therefore

$$\mu = \frac{.81055}{.49962} = 1.622.$$

If the incident light is allowed to fall perpendicularly on to one of the surfaces the formula becomes simplified to

$$\mu = \frac{\sin (P + d)}{\sin P}, \text{ or still more simply } \mu = \frac{P + d}{P}$$ [14] [15]

Lens and Critical Angle Methods.—μ can also be calculated from the focal length and radii of curvature of a lens made of the medium in question, or from the critical angle.

Microscope Method.—The microscope affords a means of finding the index of refraction of a transparent plate in the following way. A fine line is focussed and the plate is then placed above the line. Now the microscope must be raised in order that the line be clearly seen, since the rays proceeding from it are divergent, as if from a point nearer to the objective.

The distance that the microscope objective has to be raised—as read on the index attached to the millhead of the fine adjusting screw—equals the distance between the real position of the line and its apparent position when seen through the plate. Let t be the thickness of the glass and d the distance that the objective has to be raised, then

$$\mu = \frac{t}{t - d} \qquad [16]$$

Thus, if the thickness of the plate be 1mm. and the object-glass has to be raised .38mm.

$$\mu = \frac{1}{1 - .38} = \frac{1}{.62} = 1.61.$$

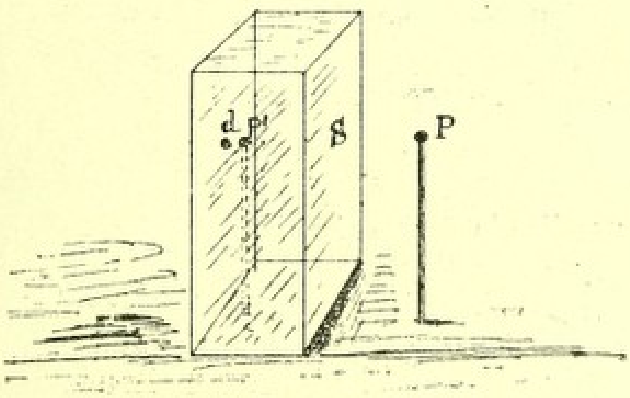


Fig. 69.

Vertical Displacement Method.—The refractive index of a transparent body, such as glass, can be roughly found as follows:—Make a dot d on the back of the block of glass; then find such a position for a pin P, placed vertically in front of the glass, that on moving one's head from side to side the virtual image of the pin, reflected from the front surface, appears to be behind that surface at such a distance P′, that it coincides with the virtual image of the dot seen through the glass. In this case the apparent thickness of the glass is P′ S which = P S.

Therefore $$\mu = \frac{dS}{PS}. \qquad [17]$$

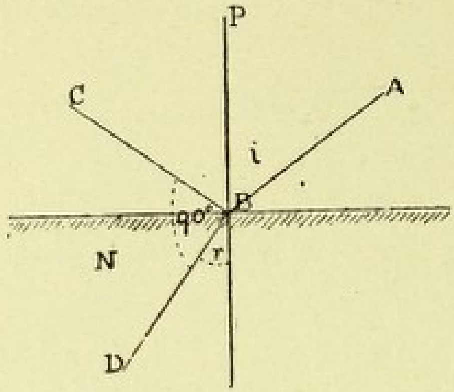


Fig. 70.

The Refractive Index of an Opaque Solid.—Let A B be a ray of light incident on a polished body N and reflected in the direction B C. If the angle of incidence A B P be the polarising angle of the medium N, then any light transmitted or absorbed is refracted in the direction B D at right angles to B C. Therefore, if the polarising angle i of an opaque body be known, its refractive index is the tangent of that angle; for in Fig. 70, if the angle of the incidence is i, then the angle of refraction r = 180 − (90 + i).

$$\text{Now } \mu = \frac{\sin i}{\sin r} = \frac{\sin i}{\sin (90 - i)} = \frac{\sin i}{\cos i} = \tan i.$$

And since i = p, the polarising angle

$$\mu = \tan p. \qquad [18]$$

By making exceedingly thin prisms of less than one minute of arc, Kundt successfully determined the refractive indices of a number of the metals. Thus, if an incident ray fall perpendicularly on to one of the surfaces of such a prism, the refractive index can be quite approximately arrived at by the formula

$$\mu = \frac{P + d}{P}.$$

The interest of these experiments lies in the fact that the results showed a refractive index for silver, gold, copper, magnesium, and sodium as being less than that of a vacuum, and this, no doubt, accounts for the absence of a polarising angle in some substances. Further, the red rays in some cases were found to be more refracted than the blue, so that metals form a good example of anomalous dispersion. Lastly, the refractive indices of the metals were found to be proportional to their electric conductivity, i.e., those metals which were the best conductors had the lowest refractive index, and *vice versa*.

The Refractive Index of a Liquid.—Take a small quantity and place it between a plate of plane glass and a convex lens of known radius and focal length F_1. The liquid then forms a plano-concave lens. If now F of the combination be found, that of the concave F_2 is learnt from

$$\frac{1}{F} - \frac{1}{F_1} = \frac{1}{F_2}.$$

Its radius is also known, it being that of the convex lens, so the refractive index μ can be calculated from

$$\mu = \frac{r + F_2}{F_2}. \quad [19]$$

TABLE OF REFRACTIVE INDICES.

For the Mean (D) Line.

(For other Media see Appendix.)

Air	1.000
Water	1.333
Alcohol	1.363
Pebble	1.544
Canada Balsam	1.535
Tourmaline	1.636
Crown glass	say 1.500 to 1.600
Flint ,,	,, 1.530 to 1.800
Diamond	,, 2.425

The index of glass varies with the materials used in its manufacture, and as a rule the higher the μ the softer is the glass.

DISPERSION.

When white light undergoes refraction its components are refracted to different extents, so that the various colours become separated, producing what is known as dispersion.

Dispersion or Chromatism.—This is due to the fact that the shorter waves, with rare exceptions, are retarded by the refracting medium, more than the longer waves.

Dispersive Index.—Each refracting medium has what may be termed an index of dispersion, which represents the differences between the indices of refraction of the lines A and H of the spectrum. Thus, water has an index of refraction for the line A of 1.3289, and for the line H of 1.3434. The difference between 1.3434 and 1.3289 is .0145, which is the index of dispersion of water. *Mean dispersion* is represented by the difference between the indices of refraction of lines C and F, and *partial dispersion* is that between the refractive indices of any two given lines of the spectrum. The dispersion of a medium is independent of its refracting power, some media of high mean refraction having low dispersion and *vice versa*, generally however high refractivity and high dispersivity accompany each other. The following table gives in the third column the mean dispersion, and in the fourth column the total dispersion of the visible spectrum:—

TABLE OF DISPERSIONS.

			Mean.	*Total.*
Water	$\mu_C = 1.3317$	$\mu_F = 1.3378$	.0061	.0145
Alcohol	$\mu_C = 1.3621$	$\mu_F = 1.3683$	.0062	.0149
Pebble				.014
Canada Balsam				.021
Tourmaline				.019
Crown Glass if	$\mu_C = 1.5376$	$\mu_F = 1.5462$	.0086	.018
Flint Glass if	$\mu_C = 1.6199$	$\mu_F = 1.6335$	.0136	.026
Diamond	$\mu_C = 2.4102$	$\mu_F = 2.4355$	.0253	.056

The dispersion of various kinds of glass differs with the materials used in their manufacture.

Dispersive Effect.—The true *dispersive power* is not, however, represented by the index of dispersion, since to compare the dispersion of one medium with that of another the amount of refraction effected by the two media should be equal. For instance, the diamond disperses the various rays more than crown glass, but it also refracts to a greater extent, and if the refractivity of these two media were equal their dispersive powers would also be nearly equal. The true dispersive effect or *efficiency, which is the relative* refractivity ~~*dispersion*~~ *for equal* dispersion ~~*refractivity*~~, is represented by the symbol ν (nu) and expressed by

$$\nu = \frac{\mu_D - 1}{\mu_F - \mu_C}.$$ [20]

Air and gases whose μ approximately $= 1$ have no appreciable dispersion, and their ν may be taken as $= 100$.

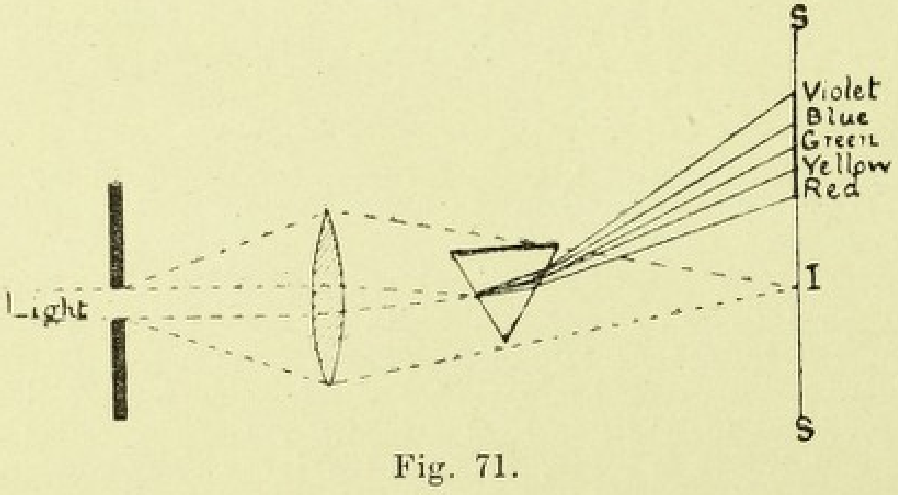


Fig. 71.

Refraction Spectrum.—In order to produce a spectrum by refraction the light should be admitted into a dark room through a small horizontal aperture, preferably about 20mm. long by 1 to

2mm. wide; if the aperture be large or round the colours overlap, and the spectrum obtained is not pure. The light, being thus admitted, is incident on a prism placed in its path in the position of minimum deviation.

If an achromatic bi-convex lens of, say, 3 feet focus is put at twice its focal length from the slit and close to the prism, a sharp pure spectrum is formed on a screen situated 6 feet beyond the lens. If the prism be placed base up the violet is above and the red below. All the different colours are seen well defined, but the red end of the spectrum is somewhat crowded, while the blue is spread out.

Refraction and Dispersion.—Refraction by a simple medium is, so far as known, always accompanied by dispersion or chromatism, and even when a prism or lens is, as is termed, fully achromatised by one or more other prisms or lenses some dispersion still obtains. Although two or even three colours may be brought to a common focus, this can never be done for every colour of the spectrum, and, as will be seen further on, with two lenses, only two colours can be accurately brought to focus, the coincidence of the others being merely approximate. This want of coincidence of all the colours of the spectrum is due to *irrationality of the spectrum*.

If we take a number of prisms of different substances, but of the same angle, it will be found that those having the higher refractive index usually, but not of necessity, possess the longer spectra. These different spectra can be made of the same length by altering the angles or the position of the prisms, or by adjusting the position of the screen.

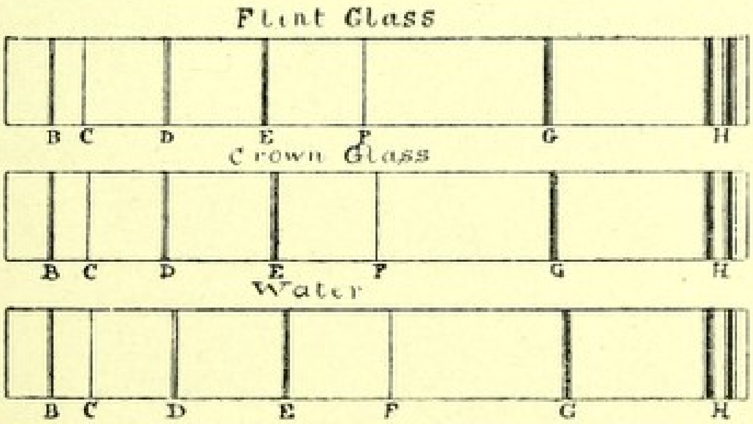


Fig. 72.

If, as in the accompanying diagram (Fig. 72), the spectra be placed one under the other so that the B lines at the red and the H lines at the blue correspond in position, it will be found that the intermediate lines do not do so. This fact renders it difficult to fix the exact position of lines in the spectrum, since a special scale has to be made for each spectroscope.

Anomalous Dispersion.—In glass, water and most substances, the order of refrangibility is from the red through the orange, yellow, green, blue, indigo to violet, which is the most refrangible, but certain substances have the property of refracting the normally more refrangible rays less and the less refrangible more. This is called *anomalous dispersion*. The substances which exhibit this peculiarity possess what is called surface colour, i.e., they have a different colour when viewed by reflected light to what they have by transmitted light. As they reflect only a certain colour, the complementary colours are transmitted, and their spectra exhibit an absorption band of more or less considerable dimensions, it being the space which would have been occupied by the reflected colour had it been transmitted. Such substances are termed *dichroic*.

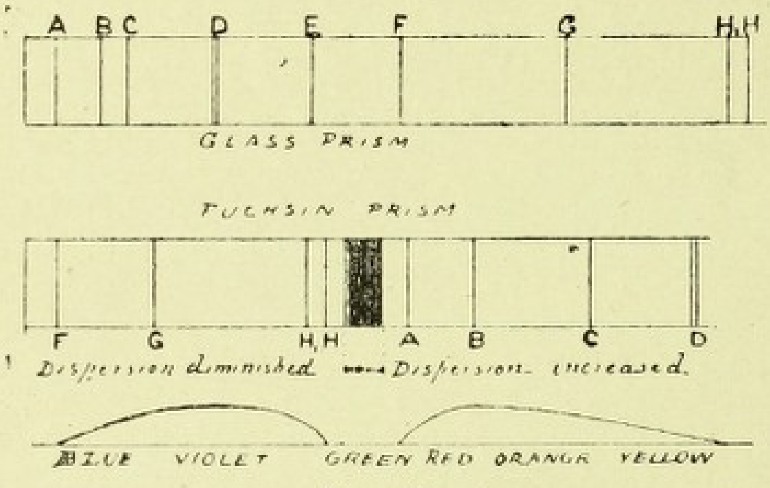


Fig. 73.

Most metals except gold and copper, as well as many of the aniline products, possess this abnormal dispersion, the order of colours being changed. Moreover, Kundt found, in these dyes, the dispersion abnormally increased on the red side of the band but diminished on the violet side; so that in the case of fuchsin, for example, the red end, usually so short, is actually more extended than the violet end.

Recomposition of Dispersed Light.—To recombine the spectrum of a prism in order to form white light we may adopt several methods.

1. By employing a prism of equal dispersive power. This is placed in the path of the dispersed light, having its base turned in the opposite direction to that of the first prism. (Newton's method).

2. A series of plane mirrors may be so arranged that each receives a different portion of the spectrum; from each the light is reflected to the same part of a screen where the colours are re-combined.

3. By receiving the dispersed light on a concave mirror, from which it is reflected on to a screen, and then by rapidly oscillating

the mirror or the screen the impression of white light is produced. Or the prism or the screen may be oscillated or rotated to produce a similar effect without the interposition of the concave mirror.

Any mechanical arrangement of rotation or oscillation by which the colours of the spectrum, whether produced by dispersion or by transmission through coloured glasses, or by reflection from pigments, are caused to successively enter the eye with sufficient rapidity, produces the impression of white. Different colour sensations result while others are still existing, and the combination of all results in a sensation of white or grey. Colour tops or discs divided into sectors of different colours are examples of this phenomenon.

Diffraction Spectrum.—Dispersion can be obtained by reflection from, or transmission through, a diffraction grating which consists of a piece of glass or metal ruled with very fine lines—some thousands to the inch—and the closer and the more exactly equidistant the lines, the purer is the spectrum produced. The lines of the grating break up the original waves into fresh series, of which some are quenched by interference. Unlike the spectrum obtained by prismatic refraction, the colours are evenly distributed in accordance with their wave-lengths, so that the red end is not condensed, nor the violet end dispersed, while the red and orange occupy more, and the blue or violet occupy less space than in a refraction spectrum; also the most luminous part is more nearly in the centre. Such diffraction gratings afford an accurate standard by which to measure the wave-lengths of light and the relative positions of the lines.

CHAPTER V.

REFRACTION BY PRISMS AND LENSES.

If the two surfaces of a refracting medium are not parallel to each other every incident ray must suffer refraction, since no ray can be perpendicular to both surfaces.

Prism.—An optical prism is a transparent body usually made of glass, but it may for special reasons consist of quartz, rocksalt, fluorspar, etc. It has two plane refracting surfaces A B, A C (Fig. 74), which meet in a line at A, termed the apex or edge of the prism. The side B C opposite this edge and joining the two refracting surfaces is called the base. The latter may slope in any direction, as it does not affect the course of the rays.

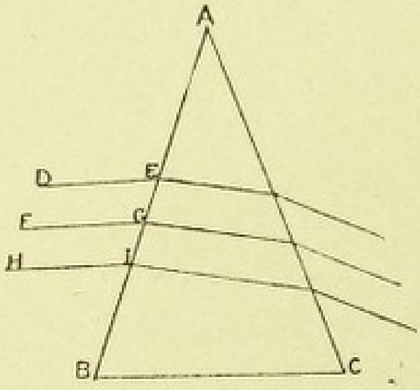


Fig. 74.

If a ray be incident in a direction perpendicular to the first surface it passes through the prism without deviation until it reaches the second surface, when it is refracted away from the perpendicular.

If a ray be incident otherwise than normally on the first surface as it passes from the rarer into the denser medium it is refracted towards the perpendicular to the first surface, and on emergence is again refracted, as it passes from the denser into the rarer medium, away from the perpendicular to the second surface.

Provided that the angle of incidence be the same, the rays are refracted to the same extent, no matter on what part of the first surface of a prism they are incident. If the rays (Fig. 74) incident on the prism are parallel before refraction they are similarly situated in relation to each other after refraction and emerge from the prism parallel. If they are divergent before refraction (Fig. 75) they emerge from the prism divergent. If they are convergent, they are convergent on passing out. Nevertheless, as will be seen later, the degree of divergence or convergence is not the same after refraction as it was before.

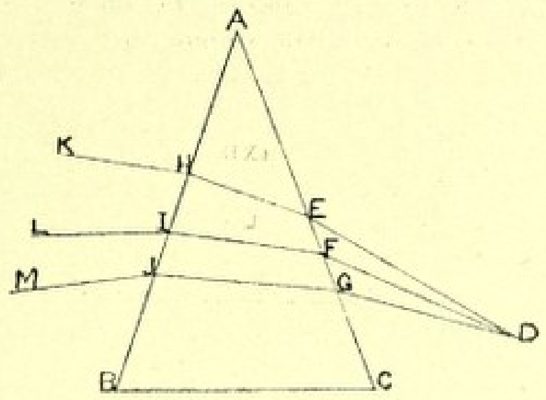


Fig. 75.

Refracting Angle.—In Fig. 76 the angle formed at A, by the two refracting surfaces, is called the principal angle, refracting angle, or angle of inclination. The refracting power of the prism is governed chiefly by the principal angle.

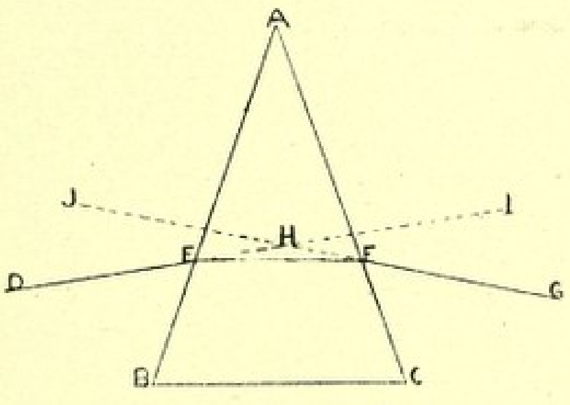


Fig. 76.

Angle of Deviation.—Let the incident ray D E (Fig. 76) be directed towards a point H in the centre of the prism. Being refracted at E it takes the direction E F and passes out in the direction F G as if proceeding from H. The angle of deviation of the prism is in this case I H G, because D E, instead of following the path H I, appears after refraction to follow the path H F G.

An object at D, when viewed through the prism from G, appears as if it were situated at J. The deviating angle constitutes the important optical property of a prism.

Shape of Prism.—A prism, as regards the outer margins of its refracting surfaces, may be of any shape—square, circular, or oval; neither the shape nor size of its surface influences the course of the light passing through it.

Defining Terms.—In the prism (Fig. 77) the line of junction A B of the two refracting surfaces is termed the edge. F C D E is the base, A B C D and A B E F are the two refracting surfaces. The plane A I K B containing the edge of the prism and situated symmetrically with respect to the two surfaces may be termed the base apex plane; generally it bisects the base. Any line, as L M, at right angles to the edge of the prism and lying in the base-apex plane is a base-apex line. The line G H, parallel to the edge and lying in the base-apex plane, may, if midway between the edge and the base, be considered the axis of the prism.

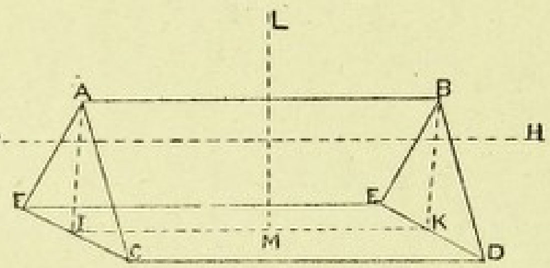


Fig. 77.

A principal section of a prism is any section, as A F C, cutting it from edge to base perpendicularly to the axis.

In a circular or oval prism the thinnest part L (Fig. 78) is considered to be the apex. M N is the base opposite to the apex. The central line L M of the plane (A B I K of Fig. 77) connecting the thinnest and thickest parts of a round or oval prism is called its base-apex line. It is usually marked on the circular prisms of the trial case by two small scratches, one at the apex and the other at the base. O P tangent to L and perpendicular to L M is the imaginary edge. P M N shows a section of such a prism along the base-apex line.

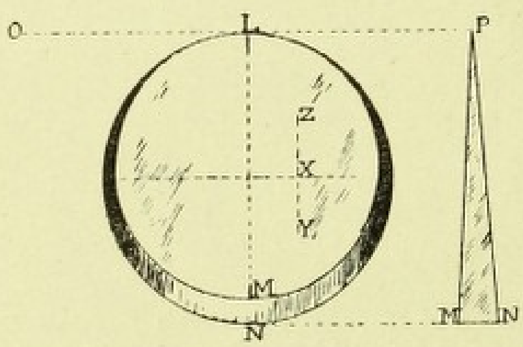


Fig. 78.

Deviation by a Prism.—The apparent deviation of an object caused by a prism is the combined result of the refraction suffered by the rays at the two surfaces, and although commonly said to be towards the apex, it is actually towards the imaginary edge of a circular prism in a direction parallel to the base-apex plane. A ray incident at X (Fig. 78), from an object beyond the prism, is refracted towards Y and is referred back towards Z, the line of deviation Z X Y being parallel to the base apex-line L M.

The Degree.—A prism is usually measured by the number of degrees included between its two inclined sides. A prism of say 10° is one in which the two sides enclose an angle of that amount.

Deviating Power of Prism.—The deviation of a ray passing through a prism depends on (1st) the angle of the prism, (2nd) the index of refraction of the medium, and (3rd) the angle of incidence of the rays.

The larger the angle formed by the two refracting surfaces the greater is the angle formed by the incident ray and the perpendicular, and therefore the greater is the deviating effect of the prism.

The deviating effect also depends on the index of refraction of the medium of which the prism is made, since the higher the index the more is a ray incident at a given angle refracted. In ophthalmic prisms the glass has usually a refractive index of 1.52. In this case a prism of 1° angle causes a deviation of .9cm. at 1 metre distance.

The deviating effect may be assumed to be equal to half the angle of the prism. This is the case if glass of μ of 1.5 is used for the weak prisms such as are employed in spectacles, so that a prism whose refracting angle is 8° may be said to have an angle of deviation of 4°; or a prism which deviates a ray 4° (when incident at minimum deviation) requires to have an angle of 8°. In the case of thick prisms, however, such calculation is inadmissible since the error would be considerable.

Degree of Deviation.—Although ophthalmic prisms have been generally numbered according to their refracting angles, they are frequently referred to in terms of their deviating angles. A prism is of 1°d (one degree angle of deviation) if, in the position of minimum deviation, an object is deviated 1.745cm. at 1 metre.

The Prism Diopter.—Still more frequently the prism diopter (symbolised thus △) is used for the unit. This signifies a prism which causes a deviation of 1cm. at 1 metre.

The value of the △ is slightly greater than that of the degree, 9△ being equal to 10°, but it is of course impossible to give any real value to the refracting degree, since the refractive index is ignored. In practice, however, the △ and ° are often considered as equal.

Relative Values.—The ratio between the three units mentioned is as follows:—

$$1^\circ = .52^\circ d = .9^\triangle$$
$$1^\triangle = .57^\circ d = 1.1^\circ$$
$$1^\circ d = 1.745\triangle = 1.9^\circ$$

Minimum Deviation.—For every prism there is one position in which an incident ray will be less deflected than in any other. From this position, if the prism be rotated round its axis so that either the edge or base is advanced towards the source of light, it will be noticed that an object viewed through the prism appears still more deviated towards the edge of the prism.

Minimum deviation obtains when the incident and emergent rays are equi-distant from the edge, and, as shown in Fig. 79, the angles of incidence and emergence (i and e) are also equal.

In this position the course of the ray, as it traverses the prism, forms with its sides the base of an isosceles triangle, and a perpendicular let fall on it from the prism apex will bisect it.

For any other incidence of the ray as i increases, e decreases less rapidly; while if i decreases, e increases more rapidly, so that, in any case, the total deviation is greater.

The angle of incidence, from which minimum deviation results, varies with the angle of the prism and its μ. It is found from

$$i = \frac{P \mu}{2} \qquad [21]$$

where P is the refracting angle and i is the angle of incidence. For approximate calculations we can substitute for the above formula

$$i = \frac{3 P}{4} \qquad [22]$$

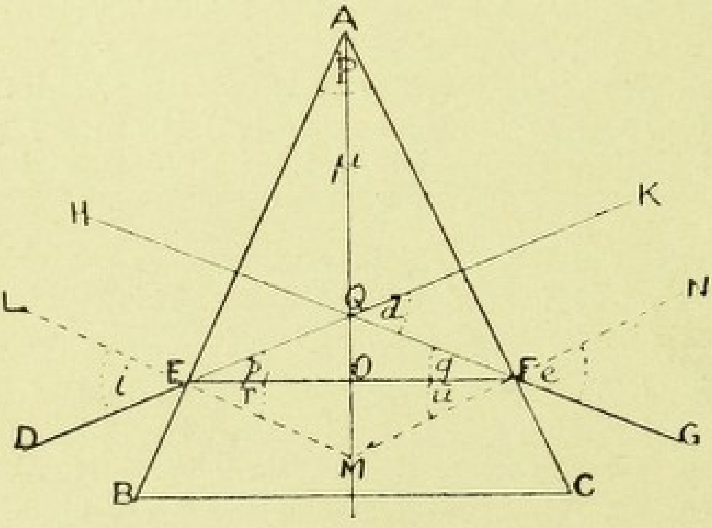


Fig. 79.

To Determine μ of a Prism.—In the prism A B C let i be the angle of incidence, r the angle of refraction, μ the index of refraction, and e the angle of emergence which the refracted ray makes with the normal F N.

Through A draw A M, making the angle E A O = F A O. Produce the incident ray D E to K, and produce the emergent ray G F backward to meet D K in Q, then the total deviation of the incident ray is equal to d. Produce the normals L E and N F to meet at M.

As the prism is in the position of minimum deviation

$$i = e, \text{ but } i = p + r, \text{ and } e = q + u.$$

We know that sin i = μ sin r, and therefore

$$\mu = \frac{\sin i}{\sin r}.$$

In the triangle Q E F the external angle d = the two opposite and equal internal angles p and q.

In the triangles E A M and O E M, M E is perpendicular to A E and A O to E O, while M E is common to both.

Therefore r (the angle of refraction) = E A M = $\frac{P}{2}$

Since the angle $p = q = \frac{d}{2}$

then i (the angle of incidence) = $r + p = r + \frac{d}{2}$;

so that $$i = \frac{P}{2} + \frac{d}{2} = \frac{P + d}{2}$$

and consequently $$\mu = \frac{\sin i}{\sin r} = \frac{\sin\left(\frac{P + d}{2}\right)}{\sin\left(\frac{P}{2}\right)}. \qquad [23]$$

This formula enables us to find the index of refraction of a prism, when the principal angle and the angle of minimum deviation are known, d and P being measured by the goniometer or spectrometer.

Example.—What is the index of refraction of a prism, the angle of minimum deviation being 5.75° and the principal angle 10.15°?

$$\frac{d + P}{2} = \frac{5.75° + 10.15°}{2} = \frac{15.90°}{2} = 7.95° = 7° 57'$$

$$\frac{P}{2} = \frac{10.15°}{2} = 5.075° = 5° 4'$$

therefore $$\mu = \frac{\sin 7° 57'}{\sin 5° 4'} = \frac{.1383}{.0837} = 1.652$$

To Determine the Angle of Deviation.—If μ and P are known d can be found thus:—

Multiply the sine of half the principal angle by the refractive index of the prism. Find from the table of natural sines the angle whose sine is this product, multiply it by two, and deduct the angle of refraction. The result is the angle of deviation. [24]

Example.—What is the angle of deviation of a glass prism whose principal angle is 4° and index of refraction 1.575?

We find that sin 2° = .0349, and .0349 × 1.575 = .0549675 = sin 3° 9′.

$$\text{Now} \qquad 3^\circ\ 9' = \frac{d + P}{2} = \frac{d + 4}{2};$$

$$\text{therefore} \qquad 6^\circ\ 18' = d + 4^\circ$$

$$\text{or} \qquad d = 6^\circ\ 18' - 4^\circ = 2^\circ\ 18'$$

To Determine the Principal Angle.—To find P the angle at which a prism of known index must be ground so that a certain angle of deviation be obtained the formula is

$$\tan\frac{P}{2} = \frac{\sin\frac{d}{2}}{\mu - \cos\frac{d}{2}} \qquad [25]$$

Example.—What angle must be given to a prism of 8° deviation when μ = 1.527? Here

$$\frac{\sin\frac{d}{2}}{\mu - \cos\frac{d}{2}} = \frac{\sin 4^\circ}{1.527 - \cos 4^\circ} = \frac{.0697}{1.527 - .9975} = \frac{.0697}{.5295} = .1316$$

Now .1316 = tan 7° 30′; P must therefore be twice this angle or 15°.

Simplified Formulæ.—When the angle of incidence or emergence is zero, i.e., when the incident ray is perpendicular to the first surface or the emergent ray is perpendicular to the second, the formula for finding μ, d, or P, when the other two values are known, becomes simplified to

$$\mu = \frac{\sin (d + P)}{\sin P} \qquad [26.]$$

whence $$\mu \sin P = \sin (d + P) \qquad [27]$$

and $$\tan P = \frac{\sin d}{\mu - \cos d}. \qquad [28]$$

By substituting angles for their sines, which can be done without serious error, when the angle of the prism is small as in ophthalmic prisms, the formulæ may be greatly simplified. Thus,

$$\mu = \frac{d + P}{P} \text{ or } \frac{d}{p} + 1 \qquad [29]$$

whence $$d = P (\mu - 1) \qquad [30]$$

and $$P = \frac{d}{\mu - 1} \qquad [31]$$

If the refractive index = 1.5 then $\mu - 1 = \frac{1}{2}$ and

$$d = \frac{P}{2}; \qquad [32]$$

so that the deviation of this prism is equal to half the principal angle. Thus for a prism of 5° the angle of deviation would be approximately 2° 30′.

If the refracting angle of a prism is 10° and the deviating angle 5.25, then

$$\mu = \frac{5.25}{10} + 1 = 1.525.$$

A prism of 10° principal angle, whose index is 1.54, has an angle of deviation of

$$d = 10 \times .54 = 5.4^\circ = 5^\circ\ 24'.$$

If a prism of 6.25° deviation is required, the index of refraction being 1.56, the prism angle is

$$P = \frac{6.25}{.56} = 11.166 \text{ or } 11^\circ\ 10'.$$

To calculate the prism P_1 made of glass of a certain index of refraction, which will neutralise the deviation of another P_2, whose index is different, we have only to put

$$P_1 = \frac{P_2 (\mu_2 - 1)}{(\mu_1 - 1)}$$ [33]

Thus if a crown glass prism of 15°, whose $\mu = 1.54$ has to be neutralised by a flint prism whose index $= 1.62$, then from the above formula

$$P_2 = \frac{15 \times .54}{.62} = 13°.$$

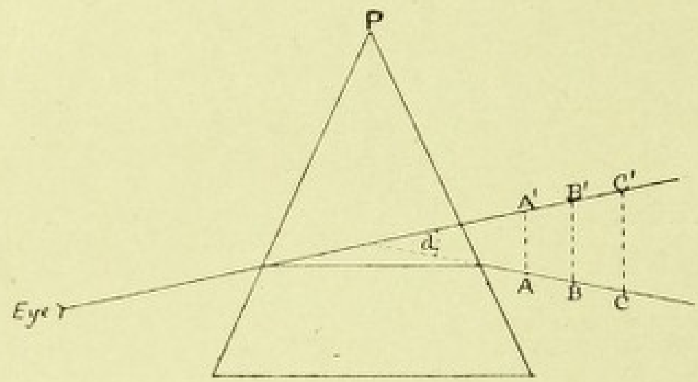


Fig. 80.

Displacement by a Prism.—In the above figure the object A is seen, through the prism, at A′. If the object is at B or at C, it is seen at B′ or C′ respectively. The apparent angular displacement of the object by a given prism depends entirely on the magnitude of d, and no matter how near or how distant the object (as may be seen from the figure) the angle is invariable. But the actual displacement AA′ BB′ CC′ is proportionate to the distance of the object, which distance is the radius of an imaginary circle within which the angle is contained. Thus the deviation AA′, etc., is represented by the tangent of the angle of deviation d.

For the practical methods of determining the deviating and refracting angles of a prism see the chapter on prism notation and measurement.

Construction.—To trace the course of a ray D E refracted by the prism A B C of μ 1.5. From D (Fig. 81) draw D F perpendicular to A C and divide E F into three equal parts. From E on A E mark off E G equal to two such parts. From G draw G I perpendicular to A E and connect E with G I by a line E H (cutting A B at the point J), whose length is equal to E D. Then E H will be the direction of the refracted ray. From E draw E K perpendicular to A B. Divide J K into two equal parts. On J B, from J, mark off J L equal to three such parts. Draw M L perpendicular to A B. Connect J with M L by a line J M, whose length is equal to that of E J. Then J M is the direction of the ray of emergence. The angle of deviation E N O is found by prolonging M J backwards and D E forwards so that they meet at N. Should the μ of the prism have any other value than 1.5 then E F and J L must have as many divisions as the numerator and E G and J K as many as the denominator of the fraction. Thus if μ = 1.6 the proportional parts would be eight and five.

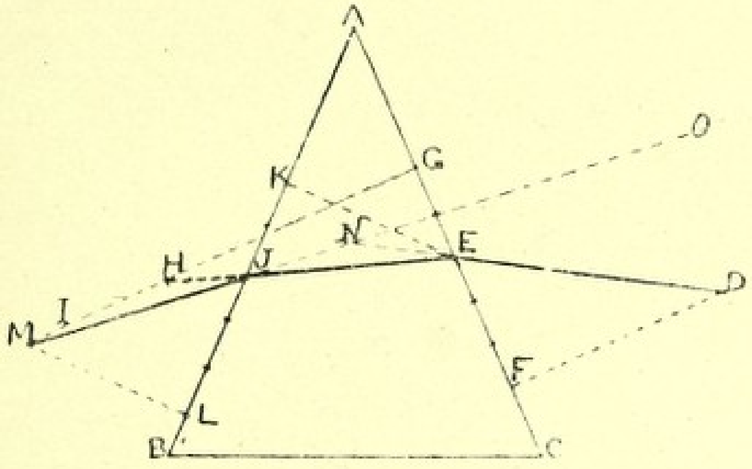


Fig. 81.

Another method is as follows. Let D E be the incident ray. Produce B A to H. From A along A H mark off any three equal parts. From A as a centre describe a circle G I N L with a radius

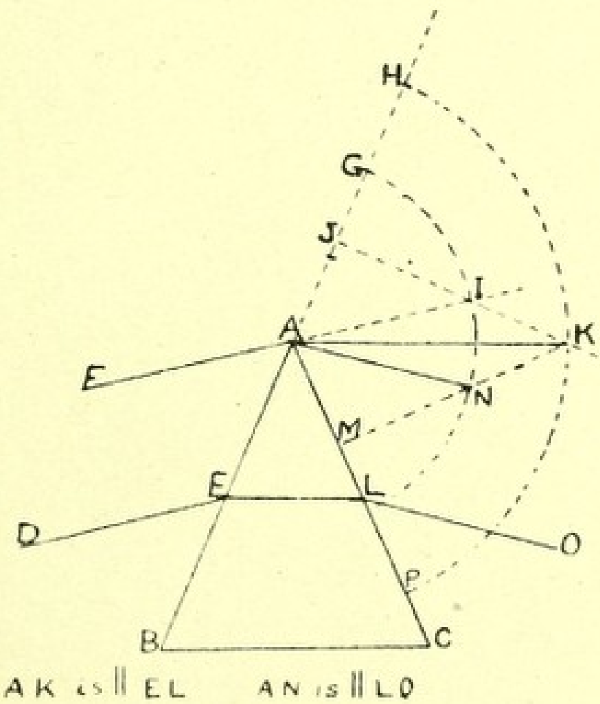


Fig. 82.

A G equal to two such parts, and another circle H K P with a radius A H equal to three such parts. Through A draw F A parallel to D E and produce F A cutting the circle G I N at I. Drop a perpendicular I J on to A H, and produce it backwards to cut the outer circle at K. Then K A will be parallel to E L, the direction of the first refracted ray. From K drop a perpendicular K M cutting the smaller circle at N, then A N will be parallel to L O, the direction of the refracted ray after leaving the prism.

In order to trace the course of a ray of light through any refracting body, with plane or curved surfaces, the procedure is the same, but in the case of the curved surface the tangent to the curve at the point on which the ray falls is considered to be the plane of incidence and of refraction.

REFRACTION AT A SPHERICAL SURFACE.

Power of Curved Surface.—The refractive power of a curved surface depends on its curvature and the refractive index of the medium, so that an increase in either is accompanied with increase of power. The focal length depends on the refractive power, the one being inversely proportional to the other, i.e., the greater the power, the shorter is the focal length.

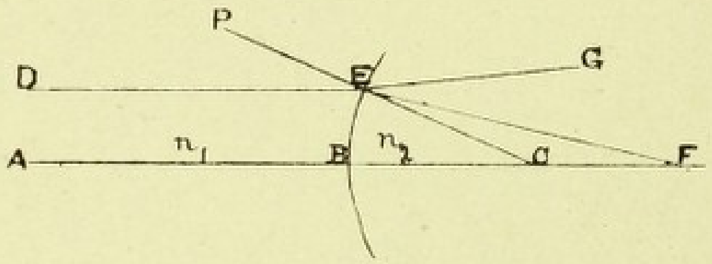


Fig. 83.

Curvature of Surface.—Since every line drawn from the centre to the circumference of a sphere is a radius of curvature, every point on the circumference may be regarded as a minute plane at right angles to the radius which reaches it. Thus C E is called a normal to the surface at E, being perpendicular thereto, and this holds true when the line is prolonged beyond the circumference to P.

Course of Light.—In Fig. 83 let n_2 represent a transparent body having a single surface with its centre of curvature at C. Any ray of light A B or P E proceeding from the medium n_1 is, when incident at the surface, directed towards C, and so is perpendicular at the point of incidence. It therefore passes into the medium n_2 without deviation. But the ray D E, incident at E, in a direction which is not perpendicular to the surface, is bent towards the perpendicular P E C in the direction E F, if n_2 is of a higher index of refraction than n_1 or it is bent away from the perpendicular, in the direction E G, if n_2 is of a lower index.

In Fig. 84 let n_2 be a mass of glass in air bounded by a convex surface. The ray f_1 A directed towards A, is normal to the refracting surface and passes onward without deviation. The rays f_1 B and f_1 D form the angles B_1 B f_1, D_1 D f_1 with the normals B_1 B C, D_1 D C to the surface, and each, on passing into the denser medium, is bent towards the perpendicular to an amount governed by the ratio between the sines of the angles of incidence and refraction. Thus the ray f_1 D is bent more than the ray f_1 B and the two meet the line f_1 A C (prolonged) at the point f_2. Similarly, all the rays diverging from f_1 are refracted to f_2; f_2 is, therefore, called the focus or the image of the object or source of light f_1. The points f_1 and f_2 are conjugate foci. If the object were at f_2, the image would be at f_1.

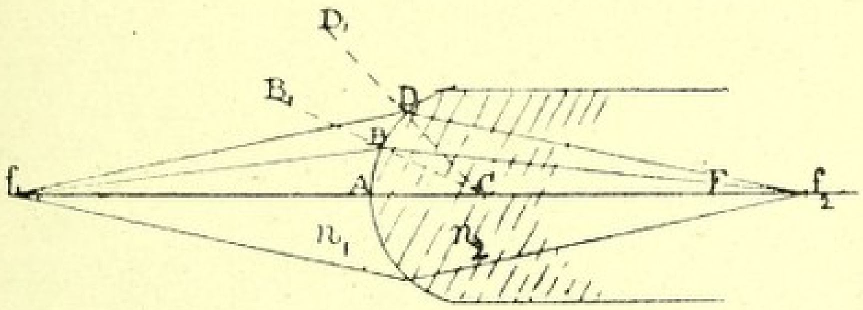


Fig. 84.

Defining Terms.—The line f_1 A f_2, which is perpendicular to the refracting surface and passes through the centre of curvature and F (the principal focus), is called the principal axis; and any point on it has its image on the same axis. The focus thus formed, by a convex surface of a medium of a higher index of refraction, is positive or real. If the medium is of lower index, rays of light entering it are rendered divergent, and their focus is negative or virtual. C is the centre of curvature, A C = r is the radius of curvature of the surface.

Formulæ Connecting f_1 and f_2.—In Fig. 84 let D_1 D f_1 = i, the angle of incidence, C D f_2 = r, the angle of refraction; let the angles D f_1 A = a, D C A = b, and D f_2 C = c; let the index of the first medium be μ_1 and of the second μ_2.

Then $$\mu_1 \sin i = \mu_2 \sin r,$$

but $$i = a + b \text{ and } r = b - c,$$

$$\therefore \quad \mu_1 \sin (a + b) = \mu_2 \sin (b - c).$$

If the pencil of rays be small, the angles i and r are small, and we can omit the sines and replace angles by their tangents.

Let A f_1 = f_1, A f_2 = f_2, and let r be the radius of curvature A C.

Then
$$\mu_1\left(\frac{1}{f_1} + \frac{1}{r}\right) = \mu_2\left(\frac{1}{r} - \frac{1}{f_2}\right)$$

or
$$\frac{\mu_1}{f_1} + \frac{\mu_1}{r} = \frac{\mu_2}{r} - \frac{\mu_2}{f_2}$$

$$\therefore \quad \frac{\mu_1}{f_1} + \frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r} \qquad [34]$$

If the object f_1 is at ∞, we have

$$\frac{\mu_1}{\infty} + \frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r}; \text{ or } f_2 = \frac{\mu_2 \, r}{\mu_2 - \mu_1} \qquad [35]$$

this being the *posterior principal focal distance* F_2 of the surface.

Again, if f_2 the image be at ∞ we have

$$\frac{\mu_1}{f_1} + \frac{\mu_2}{\infty} = \frac{\mu_2 - \mu_1}{r}; \text{ or } f_1 = \frac{\mu_1 \, r}{\mu_2 - \mu_1} \qquad [36]$$

this being the *anterior principal focal distance* F_1 of the surface.

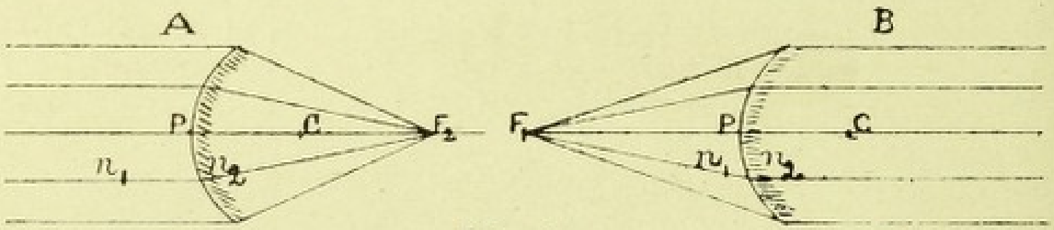


Fig. 85.

Positive Focal Length.—Fig. 85 (A) shows a beam of rays which diverge from a single point situated so far away that the angle of divergence may be neglected and the rays considered parallel to each other; these rays, on passing into the denser medium n_2, are refracted towards the principal axis and meet at the point F_2, situated beyond the centre of curvature C.

The distance P F_2 is called the posterior principal focal distance; F_2 is the posterior principal focus of the refracting surface, and it is situated on the principal axis P C F_2. The distance P F_2 is governed by the index of refraction and the curvature of the medium n_2, it being longer as the index is low and the curvature small and *vice versa*.

In Fig. 85 (B) the rays are parallel in the denser medium, and they will, on emergence into the rarer medium n_1, each be refracted away from the perpendicular at the point of emergence and meet at the point F_1 situated on the principal axis. P F_1 is called the anterior principal focal distance, and F_1 the anterior principal focus.

In the preceding figures, let n_1 be some rare medium and n_2 be a solid body of glass (or an aphakic eye, that is, an eye from which the lens has been removed), let μ_1 be the index of refraction of air and μ_2 that of medium n_2, and let r be the radius of curvature; then the distances of F_2 and F_1 from P are found by the formulæ as given in the preceding paragraph.

$$F_2 = \frac{\mu_2 r}{\mu_2 - \mu_1} \quad \text{and } F_1 = \frac{\mu_1 r}{\mu_2 - \mu_1}. \qquad [37] \; [38]$$

If n_1 is air, μ_1 = unity, and we can substitute 1 for it. Also μ_2 we can then call μ, so that the formulæ become simplified to

$$F_2 = \frac{\mu r}{\mu - 1} \quad \text{and } F_1 = \frac{r}{\mu - 1}. \qquad [39] \; [40]$$

These formulæ hold good only when the focus lies within the medium to which r the radius pertains.

Thus if the index of n_2 is 1.5 and the radius of curvature eight inches, then

$$F_2 = \frac{1.5 \times 8}{(1.5 - 1)} = \frac{12}{.5} = 24\text{in. from P},$$

or 24 − 8 = 16 inches from C.

$$F_1 = \frac{8}{1.5 - 1} = \frac{8}{.5} = 16\text{in. from P},$$

or 16 + 8 = 24 inches from C.

The anterior and posterior focal distances of the surface of a glass convex body are approximately twice and three times the radius respectively.

Negative Focal Length.—If the surface is concave the radius is negative and would be prefixed in the formulæ by a − sign, so that F_1 and F_2 become negative quantites, and they are situated on the same side of the surface as the source of light. That is F_2 is in the air and F_1 is in the dense medium. The calculations are the same as for a Cx surface.

Relative μ. — Let us suppose the rarer medium to be some other medium than air, such as water, and let μ_1 and μ_2 be the indices of refraction of water and glass respectively; we can find F_1 and F_2 by the original formulæ.

Example.—Suppose parallel rays pass from water $\mu = 1.33$ into glass $\mu = 1.5$ and let the radius be eight inches, then

$$F_2 = \frac{1.5 \times 8}{1.5 - 1.33} = \frac{12}{.17} = 70.6\text{in. from P,}$$

or $70.6 - 8 = 62.6$ inches from C.

If the rays pass from glass into water,

$$F_1 = \frac{1.33 \times 8}{1.5 - 1.33} = \frac{10.64}{.17} = 62.6\text{in. from P,}$$

or $62.6 + 8 = 70.6$ inches from C.

In these formulæ the relative μ, which equals μ_2/μ_1, can be found and the calculation then made as if the lower μ were air.

Relationship of F_1 and F_2.—The anterior and posterior focal distances are measured from the refracting surface and they are proportional to the indices of refraction of the two media.

For instance, in the first example

$$\frac{F_2}{F_1} = \frac{\mu_2}{\mu_1} = \frac{24}{16} = \frac{1.5}{1}$$ [41]

and in the second example

$$\frac{F_2}{F_1} = \frac{70.6}{62.6} = \frac{1.5}{1.33}.$$

It will be seen that in a refracting body with a single curved surface, $r = F_2 - F_1$. Subtracting r from F_2 we obtain F_1 or [42]
$F_1 + r = F_2$. This holds good whatever the refractive indices may be.

To find r or μ. —The radius or the refractive index can be found by substituting known values for the symbols given in the above formulæ, and then equating.

Thus, let F be 30in. and the indices of refraction be respectively 1.5 and 1, then to find the radius r we have

$$30 = \frac{1.5\,r}{.5} \quad \text{or} \quad r = \frac{15}{1.5} = 10\text{in.}$$

If $\mu_2 = 1.5$, $r = 8$, and $F_1 = 70.6$, then we can find μ as follows:—

$$70.6 = \frac{8 \times 1.5}{1.5 - \mu_1}; \quad \text{so } 105.9 - 70.6\mu = 12.$$

or $$70.6\mu_1 = 93.9. \quad \text{Therefore } \mu = 1.33.$$

All the formulæ given apply equally when the denser medium has a concave surface, only in this case F will be negative, and care must be taken that the − sign be given to it, in any calculations.

The curved boundary plane between the two media may be regarded either as the convex surface of the one or the concave surface of the other.

Since the anterior and the posterior focal distances of a refracting surface are not the same, it is not so usual to express its power in diopters as is done with lenses.

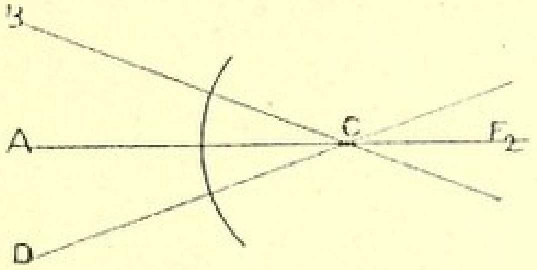


Fig. 86.

Secondary Axes.—The principal axis of a refracting body has been shown to be a line which passes through the centre of curvature and the principal focus (A C F_2, Fig. 86). All other lines such as B C, D C passing through the centre of curvature but not passing through F_2 are termed secondary axes; they correspond to radii of curvature of the surface and are therefore normals. An object point situated on the principal axis always has its image (focus) on the same axis, and it is therefore called the principal focus. Likewise an object point situated on a secondary axis has its image on that same axis, and it is called a secondary focus.

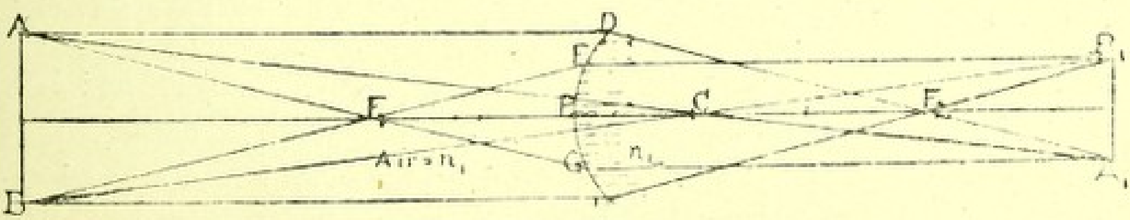


Fig. 87.

Construction of I.—Cx. Surface.—In Fig 87 A B is an object situated in front of the refracting surface D P H. Rays diverging from A and B have their images respectively at A_1 and B_1, therefore B_1 A_1 is the image of the object A B, and it can be constructed in the following way.

There are three rays emanating from any point of the object, the course of which can be easily traced, viz.:

(1) A ray A C directed towards the centre of curvature C. This being normal to the refracting surface passes into the second medium without deviation.

(2) A ray A D which is parallel to the principal axis. This, after refraction, is directed towards the principal focus F_2.

(3) A ray A G passing through the anterior principal focus F_1. This, after refraction, is parallel to the principal axis in the denser medium as G A_1.

The point where these three rays meet at A_1 is common to all the other rays diverging from A and constitutes the image of that point. Similar rays drawn from B form an image at B_1. Any two of the rays mentioned suffice for the construction of the image points A_1 and B_1, and the latter define the position and size of the entire image of the object A B.

The image formed is real and inverted; it is smaller or larger than the object according as the image is nearer to, or further from, the centre of curvature of the refracting surface than the object itself.

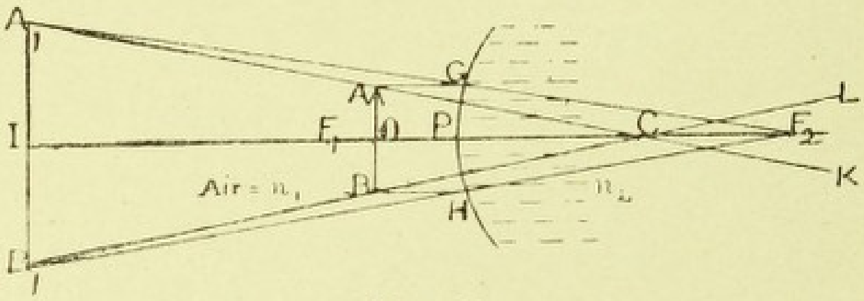


Fig. 88.

When the object is at F_1 the rays, after refraction, are parallel in the denser medium and, although in theory the image is at ∞, in reality no image is formed. If the object is nearer than F_1, as A B in Fig. 88, the rays, after refraction, are still divergent, although less so than before refraction, and cannot meet, therefore no real image is obtained. The rays can, however, be referred back so as to meet in front of the refracting surface as A_1 B_1 and the image, thus obtained, is further away than the object and is virtual (negative), erect and magnified.

The same construction for a virtual as for a real image can be employed. From A draw A C. Now since this passes through C it undergoes no refraction. Draw A G parallel to the axis. This is refracted so as to pass through F_2.

The two lines A C K and G F_2, after refraction, are divergent, and prolonged backwards will meet at A_1. Similarly, B C and B H may be drawn, and produced backwards till they meet at B_1. Thus A_1 B_1 is the virtual image of A B.

Position of Image Point.—The image of a luminous point being on a line drawn from that point through C, the construction consists merely of determining its position on that line. It is on the opposite side of the refracting surface if the rays converge after refraction; and on the same side if, after refraction, they diverge from the axis on which the point is situated. The gr ater the convergence or divergence the sooner do the rays meet and form the image of the object point from which they originally diverged.

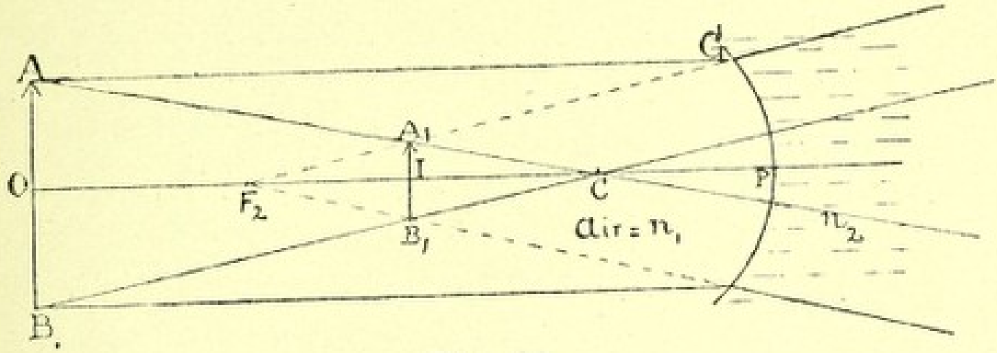


Fig. 89.

Construction of I.—Cc. Surface—If the curved surface is concave the construction is as follows:—Draw from A the ray A G; this, after refraction, is directed as if proceeding from F_2. Draw A C through the centre of curvature; this is unchanged in direction by refraction. Now A C and A G are very divergent, in the denser medium n_2, and when projected backwards meet at A_1, which is the virtual image of A. Similar rays drawn from B show their image to be at B_1. Consequently A_1 B_1 is the virtual image of the object A B.

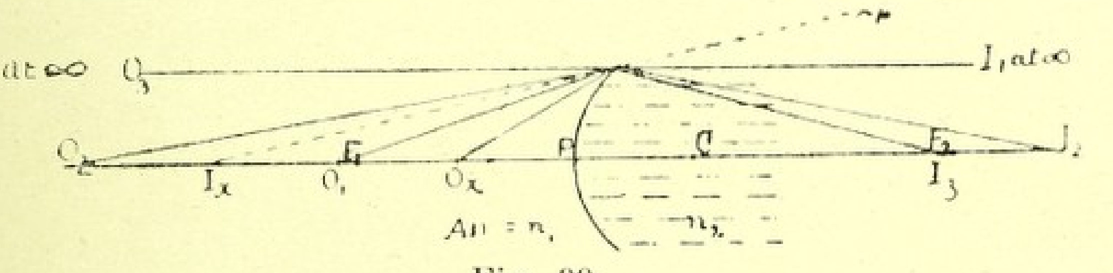


Fig. 90.

Conjugate Foci.—Cx. Surface.—If in Fig. 90 a luminous point is situated at ∞, as shown by O_3, the light proceeding from it is parallel and, after refraction, meets at F_2. This is the nearest point to the refracting surfaces at which a focus can be obtained. If a luminous point is situated at F_1 the rays, after refraction, are parallel to each other, and the only image that can be conceived is at ∞; therefore F and ∞ are conjugates of each other.

If rays diverge from a luminous point situated nearer to the refracting surface than ∞, as shown by O_2, some of the converging power of the medium being expended on the divergence of the light there is less residual convergence, and the rays therefore meet at a greater distance behind the refracting surface than if they had been previously parallel to each other; the image in the denser medium is at some point situated between F_2 and ∞, as at I_2. The positions occupied by the object and its image are conjugate foci and are interchangeable, for if the object were at the one position the image would be at the other.

As the object recedes from F_1 the image I approaches F_2, and *vice versa* until when O is at F_1 the image is at ∞, while if O is nearer than F_1 as at O_x the image is at I_x on the same side of the surface.

Conjugate Foci.—Cc. Surface.—When the surface of the dense medium is concave and the object is at ∞ the image is at F_2. This is the most distant point from the surface at which an image can be formed. If the object is within ∞, the original rays being divergent are rendered still more divergent after refraction than if they had been originally parallel; hence the image is formed nearer to the surface, that it, as O approaches the surface so also does I.

Virtual Conjugates.—Virtual conjugate foci, formed by Cx. or Cc. surfaces, are not interchangeable as are real conjugates, but if the light were directed converging towards f_2 the image formed would be at f_1.

Formula for Conjugate Foci.—Let f_1 be an object and f_2 its image. Then if F_1, F_2, and f_1 be known, the position of f_2 is learnt by the formula

$$\frac{F_1}{f_1} + \frac{F_2}{f_2} = 1. \qquad \text{Therefore } f_2 = \frac{f_1\ F_2}{f_1 - F_1}.$$ [43] [44]

Examples.—Suppose the object to be situated 20 inches in front of a Cx. spherical surface, whose anterior and posterior focal lengths are six and nine inches respectively, then its image is situated at $12\frac{6}{7}$in., since

$$f_2 = \frac{20 \times 9}{20 - 6} = \frac{180}{14} = 12\tfrac{6}{7}\text{in.}$$

If f_1 be a luminous point situated within the anterior focus, the position of the virtual image is found by the same formula, but F_1 being greater than f_1 on deducting the greater from the lesser number a negative quantity remains.

Thus, suppose an object to be situated five inches in front of a spherical surface where F_1 = 6in. and F_2 = 9in. Then

$$f_2 = \frac{5 \times 9}{5 - 6} = \frac{45}{-1} = -45\text{in.}$$

The image f_2 is then negative or virtual and is 45 inches distant on the same side of the surface as the luminous point f_1.

If f_1 is situated at F_1 the divisor of the fraction becomes O, so that the image f_2 is at ∞; so also if f_1 is situated at ∞ then f_2 corresponds to F_2.

If the object f_1 is within the denser medium the position of the image f_2 is found by the formula

$$f_2 = \frac{f_1\ F_1}{f_1 - F_2}. \qquad [45]$$

Thus, suppose an object to be situated $12\frac{6}{7}$in. behind a convex refracting surface whose $F_2 = 9$in. and $F_1 = 6$in, then

$$f_1 = \frac{12\frac{6}{7} \times 6}{12\frac{6}{7} - 9} = \frac{77\frac{1}{7}}{3\frac{6}{7}} = 20\text{in.}$$

This example should be compared with the one previously given, where the object is in front of the refracting surface; 20 inches and 12 6/7 inches being conjugate distances for the given refracting medium.

Let an object be situated 20 inches from a Cc. surface whose F_1 and F_2 are respectively —6 and —9in., then the image is virtual at 6 12/13in., since

$$f_2 = \frac{20 \times (-9)}{20 - (-6)} = \frac{-180}{26} = -6\tfrac{12}{13}\text{in.}$$

Universal Formula.—A more universal formula for the expression of conjugate foci of a single surface is

$$x' = \frac{xF}{x - F'}, \qquad [46]$$

where x and x′ are the two conjugates, F is the principal focal distance of the surface in the medium *towards* which the light proceeds and F′ that in the medium *from* which it proceeds. Thus suppose a point is seen one inch behind a curved surface whose anterior F is four inches and posterior F is six inches, then the actual position of the point is

$$\frac{-1 \times 6}{-1 - 4} = \frac{-6}{-5} = 1.2\text{in.}$$

If the object were known to be at 1.2 inches, its image is at

$$\frac{1.2 \times 4}{1.2 - 6} = \frac{4.8}{-4.8} = -1\text{in.}$$

Size of I.— The image formed at the posterior focus of a curved surface is of a size equal to that formed by a lens whose focal distance $= F_1$ and that formed at the anterior focus is the same as that formed by a lens whose focal length $= F_2$, because the axial rays cross at the centre of curvature. Whatever may be the distance of the object, its size and that of its image are to each other as their respective distances from the centre of curvature, where the axial rays cross each other. This is shown in Figs. 86 to 89, and whether the image be real or virtual it can be seen that object and image always subtend the same angle at C.

Formula for Size of I.—Let the distance of the image from the surface be represented by f_2 and that of the object by f_1, and let r be the distance P C from the surface to the centre of curvature. Let the size of the image be I and that of the object O and their distances from C respectively I C and O C, then the magnification M is

$$M = \frac{I}{O} = \frac{IC}{OC} = \frac{f_2 - r}{f_1 + r} = \frac{F_1}{f_1 - F_1} = \frac{f_2 - F_2}{F_2}. \quad [47]$$

Taking the figures of the example previously given, $r = 3$ins. $f_1 = 20$ins., and $f_2 = 12\frac{6}{7}$ins., then

$$\frac{I}{O} = \frac{12\frac{6}{7} - 3}{20 + 3} = \frac{9\frac{6}{7}}{23}.$$

The linear size of I or O is found when the size of the other is known by the formulæ

$$I = \frac{O\ (f_2 - r)}{f_1 + r}; \quad O = \frac{I\ (f_1 + r)}{f_2 - r}. \quad [48] \ [49]$$

If O and f_1 are in the same terms, i.e., inches, cm., etc., then I is expressed in the same terms as f_2.

Thus again taking the above example, if O be five inches long

$$I = \frac{5 \times 9\frac{6}{7}}{23} = \frac{15}{7} = 2\tfrac{1}{7}\text{in.}$$

Further Examples.—To again illustrate the various formulæ given, suppose an object $\frac{1}{3}$ inch long is situated 20 inches in front of a refracting surface whose radius of curvature is two inches and $\mu = 1.5$, then

$$F_1 = \frac{2}{(1.5 - 1)} = 4\text{in.} \quad F_2 = \frac{2 \times 1.5}{1.5 - 1} = 6\text{in.}$$

Therefore $f_2 = \dfrac{20 \times 6}{20 - 4} = \dfrac{120}{16} = 7.5$ ins.

$$I = \frac{7.5 - 2}{20 + 2} \times \frac{1}{3} = \frac{1}{12} \text{ in.}$$

Unit Magnification.—When the object and its real image are situated at equal distances from C, and on opposite sides of the refracting surface, they are equal in size and situated in the planes of unit magnification. In this case O is at twice the *anterior* focal distance and I is at twice the *posterior* focal distance from the surface, or O is at $2F_1 + r$ and I is at $2F_2 - r$ from C.

That is

$M = 1$ when $O + r = I - r$ or when $\mu_1 I = \mu_2 O$, [50]

and the two conjugates are consequently at $2F_1$ and $2F_2$ respectively from the surface.

If the image is virtual, with a convex or concave surface, unit magnification can only occur when O and I are both at the refracting surface itself and, of course, therefore equi-distant from C.
Also if the O is very small, unit magnification occurs at C itself.

Fundamental Formulæ.—When F_1 and F_2 are not known, the formula, previously proved, for calculating conjugate foci is

$$\frac{\mu_1}{f_1} + \frac{\mu_2}{f_2} = \frac{\mu_2 - \mu_1}{r}. \qquad [51]$$

Where f_1 and f_2 are the two conjugates, μ_2 is the refractive index of the denser and μ_1 that of the rarer medium. If the object is situated within the denser medium, f_2 must be regarded as the object distance and f_1 the image distance. Thus either μ may be that pertaining to the object, the other μ being that of the medium *towards* which the light proceeds but which may or may not be that in which the image is actually situated, since this may be either real or virtual.

Examples.—Let $r = 10$mm., $\mu_2 = 1.5$, $\mu_1 = 1$, and f_1 be in μ_1 at 50 mm. from the surface; then

$$\frac{1.5 - 1}{10} - \frac{1}{50} = \frac{1.5}{f_2}$$

that is $\dfrac{1.5}{50} = \dfrac{1.5}{f_2}$ and $f_2 = 50$.

Let r = 10mm., μ_2 = 1.5, μ_1 = 1, and f_2 in the denser medium be at 15mm. from the surface; then

$$\frac{1.5 - 1}{10} - \frac{1.5}{15} = \frac{1}{f_1}$$

that is $$\frac{-1.5}{30} = \frac{1}{f_1} \text{ and } f_1 = -20.$$

The – sign shows that the focus f_1 is virtual and in the same medium as f_2.

Let r = 8mm., μ_2 = 1.333, μ_1 = 1, and the object be at 3.6mm. behind the surface and 2mm. in size; then

$$\frac{1}{f_1} = \frac{1.333 - 1}{8} - \frac{1.333}{3\ 6} = \frac{-1}{3.05}.$$

The image is virtual at 3.05mm. behind the surface, and its size is

$$\frac{2 \times (8 - 3.05)}{8 - 3.6} = \frac{9.9}{4.4} = 2.25\text{mm.}$$

That is to say, the pupil of the eye, if 2mm. in diameter, and 3.6 mm. from the cornea, appears to be 2.25mm. in diameter and about 3mm. behind the cornea.

Fundamental Formula.—Cc. Surface.—When the surface of the denser medium is concave r is negative and is given the – sign in the formula given before.

When two media are separated by a spherical surface, this can be considered either the Cx. surface of the denser medium or the Cc. surface of the rarer medium, the radius in the latter case being negative. Again, a dense medium having a Cc. surface in contact with air, the latter constitutes a Cx. air surface and r is positive. But when the air surface is considered to give the impression to the light, the $\mu_2 - \mu_1$ of the original formula becomes $\mu_1 - \mu_2$.

Tabulated Conjugate Focal Distances.—If r = 10 and μ_2 = 1.5, then F_1 = 20 and F_2 = 30.

If O is in front of the surface, in the rarer medium

when O is at	100	60	50	40	30	20	10	5
then I is at	375	45	50	60	90	∞	– 30	– 10

If O is behind the surface, in the denser medium

when O is at	100	60	50	40	30	20	10	5
then I is at	285	40	50	80	∞	– 40	– 10	– 4

As will be seen from the above tables the conjugate focus is shorter when light passes towards the rarer medium. The planes of unit magnification are in this case 40 in the rare and 60 in the dense medium, since 40 : 60 as 1 : 1.5. The second conjugate is at ∞ when the first is at F of the medium in which O is situated.

A virtual conjugate in the rarer medium is always more distant than the object, but it should be noted that a virtual conjugate in the denser medium is nearer or further away from the surface than the object, as the object is respectively nearer to or further from the surface than C. When O is at C the virtual I is also there situated. Thus in Fig. 91 if the object is at O_1 then the image is at I_1, if the object is at C then the image is at C, and if

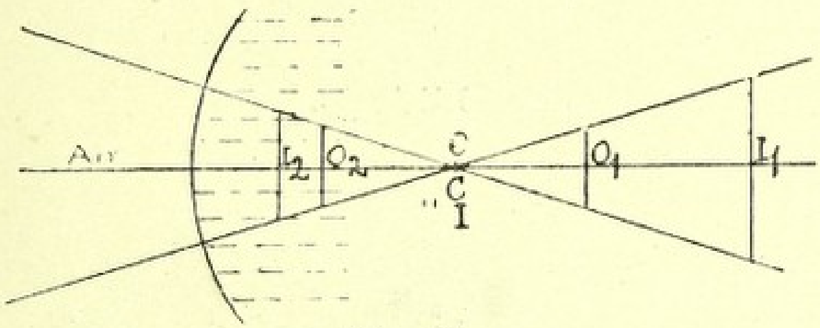


Fig. 91.

the object is at O_2 then the image is at I_2. When light diverges from, or converges to, the centre of curvature it is unaltered in its course by the refraction at the surface of the medium, and a very small object placed there would appear of natural size. If light diverges from or converges to a point beyond C it becomes less divergent or more convergent by refraction at the surface.

Other Formulæ.—It may be noted that if A and B be respectively the distance of O from F_1 and of I from F_2, then

$$A\,B = F_1\,F_2 \text{ and } \frac{I}{O} = \frac{F_1}{A} = \frac{B}{F_2}. \qquad [52]$$

Dioptral Formulæ for a Single Refracting Surface.—Since F and D are reciprocals of each other, in terms of a metre, the numerator of the following formulæ is multiplied by 100, in order to convert distances, expressed in cm., into dioptric powers D_A and D_P.

$$\frac{100\ (\mu_2 - \mu_1)}{r\ \mu_1} = D_A. \qquad \frac{100\ (\mu_2 - \mu_1)}{r\ \mu_2} = D_P. \qquad [53]\ [54]$$

Where μ_2 is refractive index of the denser medium,
,, μ_1 ,, ,, ,, ,, ,, rarer medium,
,, r ,, the radius of curvature of the surface in cm.
,, D_A ,, the dioptric power corresponding to F_1.
,, D_P ,, ,, ,, ,, ,, to F_2.

D_A : D_P as μ_2 : μ_1.

Example.—Find the power of a surface of radius 8mm. and $\mu = 1.333$ in air.

$$D_A = \frac{100 \times (1.333 - 1)}{.8 \times 1} = 41.66$$

$$D_P = \frac{100 \times (1.333 - 1)}{.8 \times 1.33} = 31.25$$

31.25 : 41.66 as 1 : 1.333.

LENSES.

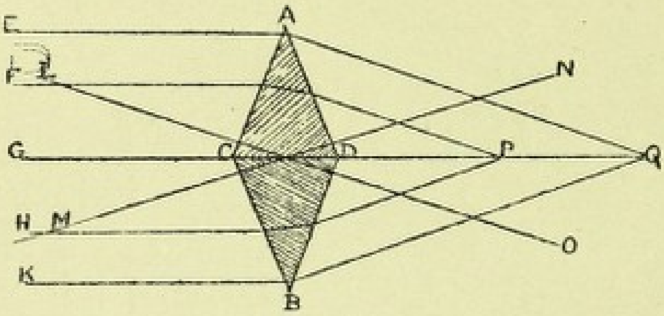


Fig. 92.

Prismatic Formation.—If two similar prisms A C D and B C D be placed base to base as in Fig. 92 incident rays of light E and F are bent towards the base of the prism A C D and rays H and K are bent towards the base of the prism B C D, so that those refracted by the one prism meet those refracted by the other. One ray, viz., G C D P is not refracted, since it passes through the base of both prisms.

Rays of light as L and M may be considered incident perpendicular to the two refracting surfaces, and are therefore also not deviated.

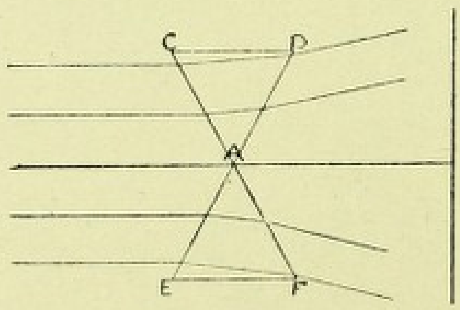


Fig. 93.

If two prisms C A D, E A F, as in Fig. 93, be joined edge to edge all rays incident on them being refracted towards the bases are therefore diverging from each other, except the central ray incident at the junction of the two edges.

What is true of two prisms is also true of any number of prisms placed with their bases or edges together and a convex or concave lens may be considered as formed of prisms whose bases or apices respectively are joined at a common centre.

Definition of Lens.—A lens is a transparent body bounded by one curved and one plane surface or by any two curved surfaces. This definition, therefore, covers all forms of convex and concave sphericals as well as cylindrical and parabolic lenses.

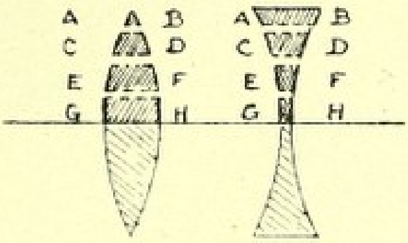


Fig. 94.

Graduated Prismatic Formation.—But not only is a lens to be considered as a multitude of prisms whose bases and edges meet at a common centre, but every meridian of it must also be considered as if formed of a series of prisms of different angles of inclination.

Any two point areas A and B (Fig. 94) opposite to each other constitute a portion of a prism whose base is the principal axis of the lens. The two areas A and B near the periphery of the lens are more inclined towards each other than C and D, situated nearer to the axis, and the inclination between the surfaces decreases gradually until those on the principal axis are parallel. Since the angle formed by A and B is greater than that formed by C and D, a ray passing through A B is bent to a greater extent than one passing through C D, while the ray which passes along the axis is not deviated at all.

Each zone of a lens, therefore, whether concave or convex, has a refractive power which becomes greater as its distance from the axis is increased, and it is due to this fact that rays diverging from a point, and incident on the lens, are brought to a common focus as a point.

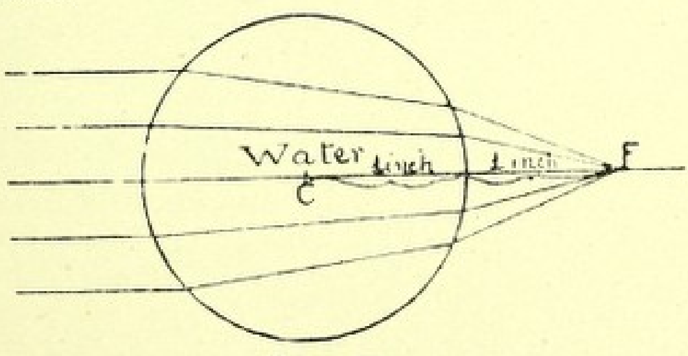


Fig. 95.

The Sphere.—A sphere (Fig. 95) is a body having every point of its surface equi-distant from a common centre C. Parallel rays refracted by a sphere come to a focus F, behind its centre, at a distance dependent on its radius of curvature, and on the index of refraction of the medium of which it is made.

The formula if light passes from air is

$$F = \frac{\mu\, r}{2\,(\mu - 1)}.$$ [55]

Thus with a globe of water with r = 4in. and $\mu = 1.333$

$$F = \frac{1.333 \times 4}{2\,(1.333 - 1)} = \frac{5.332}{.666} = 8\text{in.}$$

from the centre of curvature or four inches from its back surface. If $\mu = 1.5$, F is at $1\frac{1}{2}$ r. If $\mu = 2$, F is at the back surface of the sphere, while if μ is greater than 2, F is within the sphere.

When light passes from one dense medium of μ_1 into another of μ_2 the formula is

$$F = \frac{\mu_2\, r}{2\,(\mu_2 - \mu_1)}.$$ [56]

Thus a globe of glass of $\mu = 1.556$ and r = 4 inches in water of μ 1.333.

$$F = \frac{1.556 \times 4}{2\,(1.556 - 1.333)} = \frac{6.224}{.446} = 13.95 \text{ from C.}$$

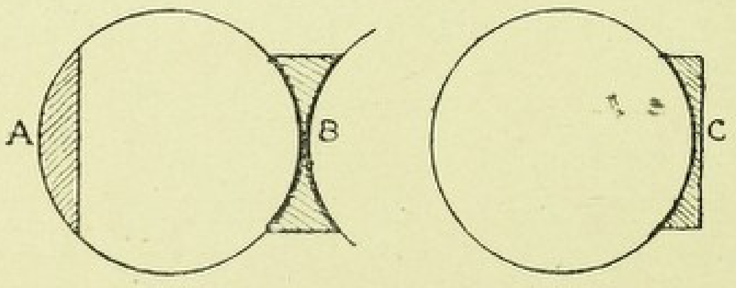
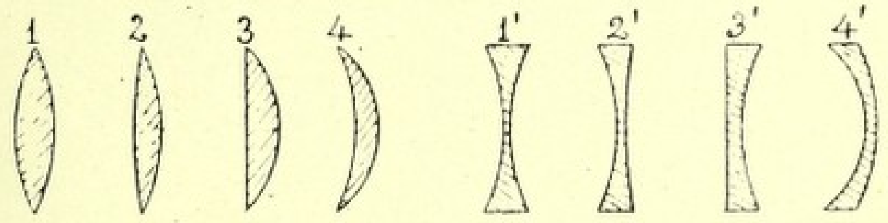


Fig. 96.

Formation by Spheres.—If, from a sphere of glass (Fig. 96) of given radius two portions similar to A be cut off and the two segments brought together, these would form a double convex spherical lens, while one such segment constitutes a plano convex lens.

A double concave spherical lens (Fig. 96 B) is formed by a body hollowed out by a sphere on each side. A plane piece of glass hollowed by a sphere on one side (Fig. 96 C) constitutes a plano concave lens. The focal length of Cx. and Cc. lenses depends as in the case of the sphere, on the radius of curvature and index of refraction of the medium. If the index of refraction = 1.5 the focus of an equi-Cx. and Cc. lens is equal to the radius of curvature and that of planos to the diameter of the sphere which the curves constitute segments.



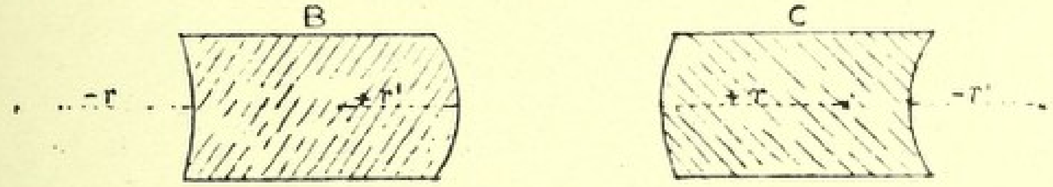


Fig. 97.

Forms of Lenses.—There are four forms of convex and four of concave spherical lenses:—

1. Equi-convex. Two convex surfaces of equal curvature.

1′. Equi-concave. ,, concave ,, ,,

2. Bi-convex. Two convex surfaces of unequal curvature.

2′. Bi-concave. ,, concave ,, ,, ,,

3. Plano-convex. One side convex the other plane.

3′. Plano-concave ,, concave ,, ,,

4. Positive meniscus or periscopic convex. Convex on one side and concave on the other, the concave having the longer radius, i.e., the weaker power.

4′. Negative meniscus or periscopic concave. Concave on one side and convex on the other, the convex having the longer radius, i.e., the weaker power.

Apparent variations of the above are sometimes made by greatly increasing the interval between the two surfaces (Fig. 97 B & C), by which their powers may be profoundly modified (a Steinheil cone is an example of this form), but they are in any case only exaggerated forms of the one or the other of the above types.

Curvature.—In each of the diagrams in Fig. 98 the radius of curvature is a line drawn from the centre of each sphere to the corresponding surface of the lens.

In the equi-convex and bi-convex (Figs. 1 and 2) the centres are on opposite sides of the lens generated by the two spheres. In the plano-convex (3) the curvature of the plano surface may be considered to be of infinite radius, the centre then being at infinity, can be considered to be on either side.

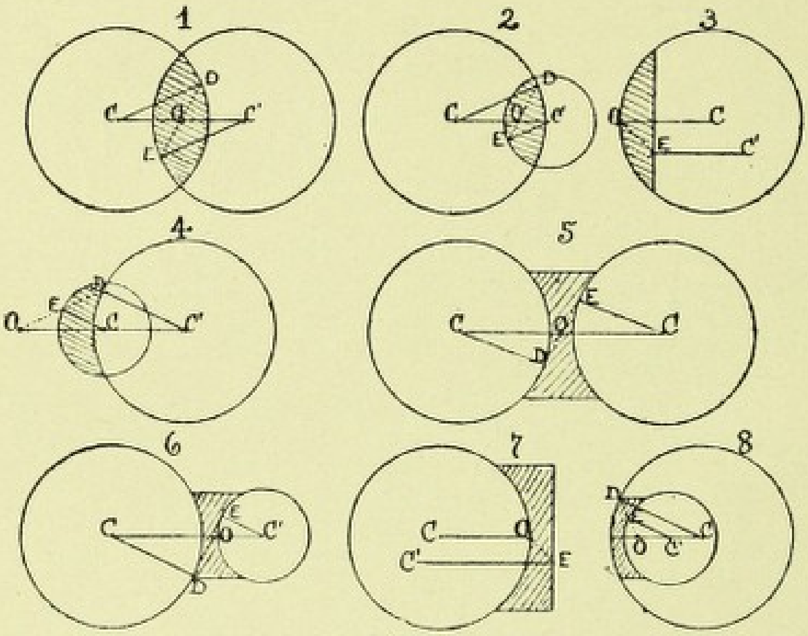


Fig. 98.

In the periscopic convex (4) the centres are on the same side.

A lens consisting of a complete sphere has the centres of its opposite surfaces co-incident.

In the equi-concave and bi-concave (5 and 6) the centres are on opposite sides.

In the plano-concave (7) the centre of the plano surface can be considered to be on either side.

In the periscopic concave (8) the centres are on the same side.

Properties of Cx Lens.—A convex lens is thicker at the centre than at the periphery; it has positive refracting power and, therefore, can form a real focus and a real image; it renders parallel rays convergent and divergent rays less divergent, parallel or convergent as the case may be.

Properties of Cc Lens.—A concave lens is thinner at the centre than at the edge; it has negative refracting power and, therefore, can only form a virtual or negative focus or image; it renders parallel rays divergent and divergent rays more divergent.

General Property.—The general effect of every spherical (and cylindrical) lens is, as with a prism, to bend every incident ray of light towards the thickest part.

Thin Lenses.—For the consideration of the properties of lenses used in visual optics, the thickness of the glass is disregarded, it being negligible in comparison with the focal lengths of such lenses. Yet it may be necessary to locate, what is called, the optical centre, which is situated at a point on the principal axis at a distance from each surface directly proportionate to their respective radii of curvature.

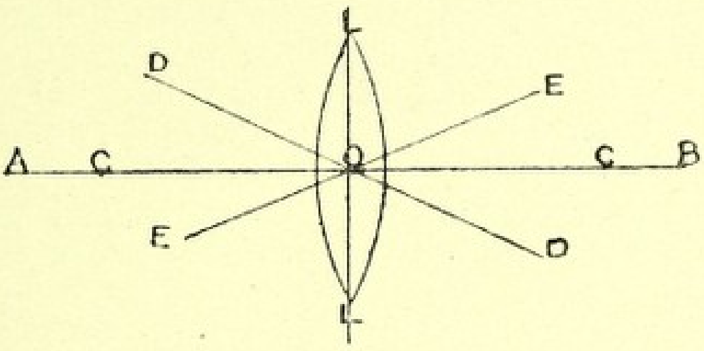


Fig. 99.

Terms of a Lens.—In Fig. 99 L L is an equi-convex lens; C C are the centres of curvature, O the optical centre. The line A O B passing through the two centres of curvature, and the optical centre, is the principal axis. It is perpendicular to both surfaces of the lens and equi-distant from its extremities L and L. The plane L O L passing through O, perpendicular to A B, is the refracting plane. On it all the refraction effected by both surfaces of a thin lens is presumed to take place. Any lines as D D, E E directed to O, are termed secondary axes. They pass, obliquely to the principal axis, through the lens, and when the latter is thin they are *presumed* to suffer no deviation.

The Optical Centre.—The optical centre may be defined as that point from which the focal length of the lens is measured, it being in the plane of refraction; also it is that point where the secondary axes cut the principal axis. Yet for practical purposes, the thickness of a thin lens may be so totally disregarded that the two surfaces themselves are taken as coinciding at the refracting plane.

Position of Optical Centre.—By calculation, the optical centre of a Cx. and Cc. lens is found by dividing the thickness of the glass, on the principal axis, in the ratio of the two radii of curvature; so that if the two surfaces are equal O is equally distant from each, but it is nearer to the more curved surface if the two are unequal. Let r and r′ be the radii of the two surfaces, t the thickness of the lens; and O the optical centre, then

$$\frac{t\,r}{r + r'} = r\,O; \quad \text{and} \quad \frac{t\,r'}{r + r'} = r'\,O. \qquad [57]$$

Thus in a bi-convex lens where t = .2 inch and r and r′ are respectively 6 and 10 inches.

$$r\,O = \frac{.2 \times 6}{6 + 10} = .075\text{in.} \quad \text{and} \quad r'\,O = \frac{.2 \times 10}{6 + 10} = .125\text{in.}$$

The thickness is divided into 6 + 10 = 16 parts, and O lies on the axis six of these parts from the pole of the shorter curve, or ten parts from the pole of the longer curve. In lenses whose surfaces are both convex or both concave O lies within the lens, but in periscopic lenses O lies outside the lens on the side of the surface of greater power.

In a periscopic convex in which t = .2in. r of the convex surface being 9in. and r′ of the concave − 12in., then

$$r\,O = \frac{.2 \times 9}{9 + (-12)} = \frac{1.8}{-3} = -.6\text{in.}$$

$$r\,O = \frac{.2 \times -12}{-12 + 9} = \frac{-2.4}{-3} = +.8\text{in.}$$

The distance from the convex surface being negative must be reckoned *away from* it, and the two distances coincide .6in. from the convex surface. If the lens were periscopic concave O would be on the Cc. side.

With a plano lens the one surface having r = 9in. the other r′ = ∞ and if t = .2in., then

$$r\,O = \frac{.2 \times 9}{9 + \infty} = 0$$

since any positive number divided by ∞ = 0.

The O therefore lies on the curved surface in plano Cx. and Cc. lenses.

Construction of Optical Centre.—The method of finding the optical centre of any form of lens is shown in Fig. 98. From the centre of curvature C, in any of the diagrams, draw a radius C D to the curved surface, of which C is the centre. From C′ draw a radius C′ E to its corresponding surface, and parallel to C D. Connect the extremities of the two radii by the line D E and where it cuts the principal axis at O, is the optical centre of the lens.

In (3) and (7) C′ being at ∞, the only radius that can be drawn from C, parallel to C′ E corresponds to the principal axis itself.

In (4) and (8) the line connecting D and E has to be produced in order to cut the principal axis.

The Focus.—A real focus, formed by a lens, is that point at which rays diverging from a point meet, after refraction.

A virtual focus is that point where rays diverging from a point meet when produced backwards or from whence they appear to diverge, they being still divergent after refraction.

Principal Focus and Focal Distance.—A principal focus is one formed on the principal axis by the convergence or divergence of originally parallel rays. A secondary focus is one formed on a secondary axis.

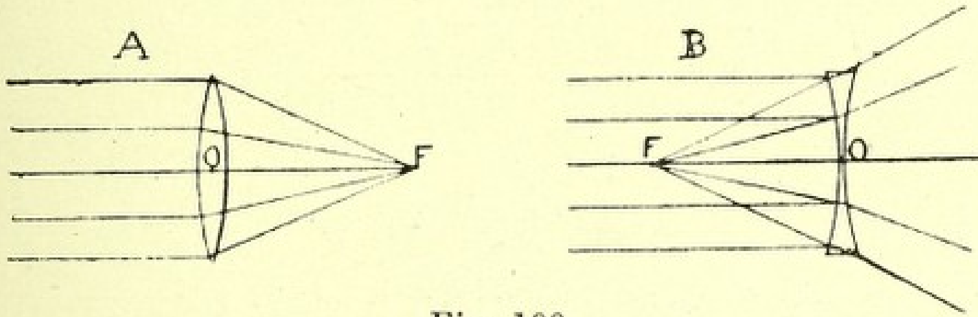


Fig. 100.

The principal focus of a convex lens is positive and is situated on the principal axis on the opposite side of the lens from the source of light.

Natural rays signify those rays which proceed from a source of light and whose course is not altered by a lens or mirror; they may be parallel or divergent, but never convergent.

The distance O F, between the optical centre and F the principal focus, is the principal focal distance of a thin convex lens (Fig. 100 A), F being that point at which, after refraction parallel rays meet. It is the *nearest* point to a convex lens at which a focus of natural rays can be obtained. The parallel rays in the figure are presumed to diverge from a single point at ∞. An equi-convex lens is said to have two principal foci, situated on opposite sides of the lens and equally distant from O.

The principal focus of a concave lens is negative, and is situated on the principal axis on the same side of the lens as the source of light.

In Fig. 100 B, the distance between O and F, is the principal focal distance of a concave lens. The principal focus being the point from which, after refraction, parallel rays appear to diverge, it is the *furthest* point from a concave lens at which a focus can be obtained for natural rays.

Whether the one side or the other of a thin equi-convex or equi-concave lens is exposed to the light, F is at the same distance from the back surface of the lens since O is situated equally distant from each surface; but this is not the case with other forms of spherical lenses.

Distance of Principal Focus.—In Fig. 101 (1) the principal focal distance O F of a bi-convex lens being measured from O, it follows that the distance of F behind the posterior surface of the lens depends on whether the less curved surface A, or the more curved surface B, is exposed to the light. If A is thus exposed, F lies further from B than it does from A when B faces the light. The same applies to the bi-concave. With the periscopic convex, as shown in Fig. 101 (2), and the periscopic concave, the difference in the distance of F as measured to the right from B or to the left from A is very marked.

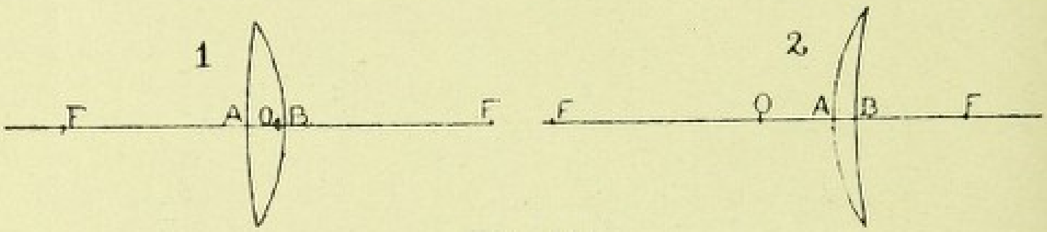


Fig. 101.

With the plano-Cx. and plano-Cc. lens, when the plane side is exposed to the light, O F is the focal distance, but if the curved surface is thus exposed the point from which the focal distance is measured changes, as will be explained later on.

Refraction and Reflection.—For comparison, the following figures show the difference between the focal lengths when an incident beam of light is reflected from or refracted by the surface of a thin plano Cx. or Cc. glass lens.

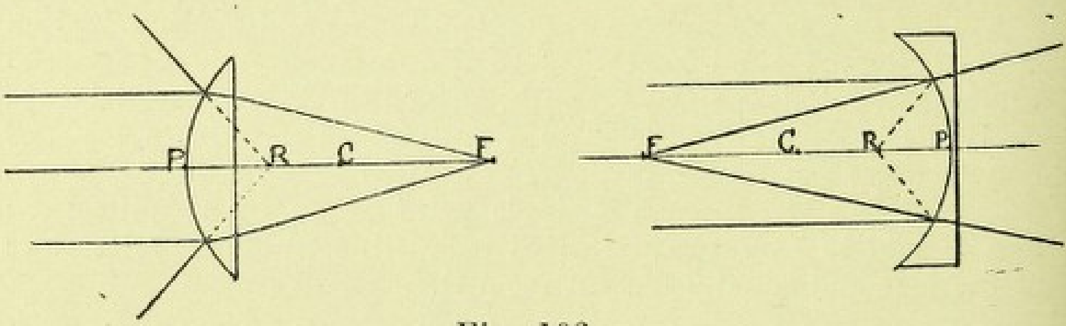


Fig. 102.

In both diagrams, C is the centre of curvature. Rays of light parallel to the axis, if reflected, meet at R, which is half the distance of C from the pole P; if refracted they meet at F, which is twice the distance of C from the surface. The thick lines represent the course of the refracted rays and the dotted lines that of the reflected rays. If F by refraction were within the glass P F = 3 P C.

The virtual focus of a Cx. mirror and the real focus of a Cx. surface are on the opposite side of the surface from the source of light, while the real focus of a Cc. mirror and the virtual focus of a Cc. surface are on the same side.

THE FOCAL LENGTH OF THIN LENSES.

Formulæ and Calculations.—The refractive power of a lens is directly proportionate to its curvature and refractive index, and is simply the sum of the anterior focal powers of the two surfaces. If we know its refractive index and the radii of curvature r and r′,

the power of the first surface is $\frac{\mu - 1}{r}$

and that of the second is $\frac{\mu - 1}{r'}$.

The total refractive power $\frac{1}{F}$ is therefore $\frac{\mu - 1}{r} + \frac{\mu - 1}{r^1} =$

$$\frac{1}{F} = \left(\frac{1}{r} + \frac{1}{r'}\right)(\mu - 1), \text{ or } \left(\frac{r + r'}{r\,r'}\right)(\mu - 1) \qquad [58]$$

Or it may be written

$$F = \frac{rr'}{(r + r')\,(\mu - 1)} \qquad [59]$$

We have to consider the three following conditions—

(a) If both surfaces are of the same nature.

Example.—A lens of μ 1.54 and having surfaces of radii of 8in. and 5in.

$$F = \frac{8 \times 5}{(8 + 5)\,(1.54 - 1)} = \frac{40}{7.02} = 5.7\text{in.}$$

The focus is here positive.

If the surfaces are concave the negative sign must be prefixed to each; thus

$$\frac{-\,8 \times (-5)}{(-8\ -5) \times (1.54 - 1)} = \frac{40}{-\,7.02} = -\,5.7\text{ in.}$$

The focus being negative.

If both surfaces have the same radius, i.e., r = r′, as in an equi-convex or equi-concave lens, the formula becomes simplified, for

$$F = \frac{r\,r'}{(r + r')\,(\mu - 1)} = \frac{r^2}{2\,r\,(\mu - 1)} = \frac{r}{2\,(\mu - 1)} \qquad [60]$$

Thus if r and r′ = 5 and μ = 1.54

$$F = \frac{5}{.54 \times 2} = \frac{5}{1.08} = 4.63 \text{ in.}$$

If the surfaces are concave

$$F = \frac{-5}{.54 \times 2} = -4.63 \text{ in.}$$

If $\mu = 1.5$ then $\mu - 1 = \frac{1}{2}$ and the formula is reduced to F = r

$$\text{for } \frac{r}{2(\mu - 1)} = \frac{r}{2 \times \frac{1}{2}} = \frac{r}{1} = r.$$

So that in equi-convex or equi-concave lens the focal length may be considered as equal to the radius.

(b) If one surface is plane, then $r' = \infty$ and $1/r' = 1/\infty = 0$, so that it may be ignored and only the curved surface considered, and the original formula simplifies to

$$F = \frac{r}{\mu - 1}$$ [61

If $\mu = 1.5$, then $\mu - 1 = \frac{1}{2}$, and F = 2 r. So that in a plano Cx. or Cc. lens, the focus is twice the radius.

(c) If one surface is positive and the other negative, the focus will be positive or negative according as the positive curvature is the greater or lesser power.

Example.—Let the two surfaces be respectively − 8in. and + 4in. and μ = 1.6. Here

$$F = \frac{-8 \times 4}{(-8 + 4) \times .6} = \frac{-32}{-2.4} = +13.3 \text{ in.}$$

The focus is positive and the lens is periscopic convex.

If the surface were + 8in. and − 4in. respectively

$$F = \frac{8 \times (-4)}{(8 - 4) \times .6} = \frac{-32}{2.4} = -13.3 \text{ in.}$$

The focus is negative and the lens is periscopic concave.

To find r.—To calculate the curvature of one of the surfaces r or r′ when that of the other as well as μ and F are known, it is only necessary to substitute the values of the known quantities and then equate as in the following examples.

What curvature r should be given to the second surface of a lens so that F = 6in. r′ = 8in. and μ 1.5?

$$F = \frac{r\ r'}{(r + r')\ (\mu - 1)};\quad \text{or } 6 = \frac{8\ r}{(8 + r) \times .5}$$

then $$6 = \frac{8\ r}{4 + .5\ r};\quad \text{or } 24 + 3\ r = 8\ r$$

$$\therefore\quad 24 = 5\ r \quad \text{or} \quad r = +\ 4.8$$

What should be the curvature of the concave surface when that of the convex of a meniscus is 5in., F being 12in. and $\mu = 1.6$?

Then $$12 = \frac{5r}{(5 + r)\ .6};\quad \text{or } 5\ r = 12 \times (3 + .6\ r)$$

and $$r = \frac{36}{-2.2} = -\ 16.36 \text{ in.}$$

To find μ.—Similarly by substitution μ can be calculated, the other values being known. For example, a periscopic lens has F = 24 cm. and the radii of curvature are respectively + 6 and − 12 cm., then to find μ

$$24 = \frac{6 \times -12}{(6 - 12)\ (\mu - 1)}$$

or $$24 = \frac{-\ 72}{-\ 6\ \mu + 6}$$

$$\therefore\quad -\ 72 = -\ 144\ \mu + 144$$

that is $$\mu = 1.5$$

Calculations when μ **is relative.**—Unless otherwise stated, a lens is always presumed to have the same medium on both sides of it, and in the foregoing, the lens is presumed to be surrounded by air whose $\mu = 1$. When the first and last media are not air—that is to say, if the lens is situated in a dense medium—the following formula is required:—

$$F. = \frac{r\ r'\ \mu_1}{(r + r')\ (\mu_2 - \mu_1)} \quad [62]$$

μ_2 being the index of the lens and μ_1 that of the medium in which it is immersed. Thus, suppose a double convex glass lens having an index of 1.54 and whose surfaces are each of 8 cm. radius placed in water, its focal length in that medium is

$$F = \frac{8 \times 8 \times 1.33}{(8 + 8)\ (1.54 - 1.33)} = \frac{85.12}{3.35} = 25.33 \text{ cm.}$$

Or the relative index μ_r may be found by dividing μ_2 by μ_1, and the formula then becomes as with thin lenses in air

$$F = \frac{r\ r'}{(r + r')\ (\mu_r - 1)} \qquad \text{Now } \frac{1.54}{1.33} = 1.1579.$$

Therefore $$F = \frac{8 \times 8}{(8 + 8) \times .1579} = 25.33 \text{ cm.}$$

If the lens be less dense than the adjacent media the formula is the same. For instance, let a similar lens, but of $\mu = 1.33$, be placed in cedar oil which has an index of 1.54, then

$$F = \frac{8 \times 8 \times 1.54}{(8 + 8)\ (1.33 - 1.54)} = \frac{98.66}{-\ 3\ 36} = -29.33 \text{ cm.}$$

Here the lens acts with a negative effect, and it shows us that an air lens in water must have a concave curvature in order that it may have a positive refracting power. Dr. Dudgeon constructed such a lens to enable divers, without helmets on, to see under water. It consisted of two small watch-glasses of very deep curvature cemented into each end of a vulcanite ring, the convex surfaces facing each other inside the ring. The lens had no magnifying power out of water, as it only contained air. In water, however, the concavity of the lens produces a convexity of the water in contact with it on each side, and this convexity produces the required refractive power.

Let a Cc. air lens be of 10 inch radius on both surfaces. What will its focus be in water?

$$F = \frac{-\ 10 \times -\ 10 \times 1.33}{(-10 - 10)\ (1 - 1.33)} = \frac{100 \times 1.33}{-\ 20 \times -.33} = \frac{133}{6.66} = +\ 20.$$

Since a Cx water lens of the same radius in air has F = 15in., it will be noticed that the effect is not the same when the conditions are reversed. This arises from a similar cause to that which produces a difference in the anterior and posterior foci of a single refracting surface. If light passes finally into a rare medium the focal distance is shorter than when it thus passes finally into a dense medium.

Case of three different Media.—When a thin lens of μ_2 separates two media of μ_1 and μ_3—that is, when there are three different media separated by two curved surfaces—the following formula, which is universal for all conditions, serves for finding the focal length,

$$F = \frac{r_1 \, r_2 \, \mu_3}{r_1 \, (\mu_2 - \mu_3) + r_2 \, (\mu_2 - \mu_1)}$$ [63]

Case of four different Media.—Suppose light to pass from air to a Cx surface of μ_2, then to another Cx surface of μ_3, and finally, by a third Cx surface, into air. Such a combination exists if a bi-focal is made by the insertion of a deeply-curved bi-convex segment of high μ into a space made for it in a larger lens of low μ. Or such a combination is formed by the contact of, say, a double C C lens of $\mu = 1.5$ with a double Cx lens of $\mu = 1.6$, the two being of equal curvature. The focal power can be found by calculating for each lens separately and then adding them together, or by calculating for each surface separately. In this case we have four media and three surfaces, and we can express the focal power in terms of the following formula:—

$$\frac{1}{F} = \frac{\mu_2 - \mu_1}{r_1} + \frac{\mu_3 - \mu_2}{r_2} + \frac{\mu_3 - \mu_4}{r_3}.$$ [64]

It may be observed that the focal length of the surface separating μ_2 from μ_3 is not calculated by the formula for the posterior focal length of a single surface, for the reason that the focus is finally formed in the air, and not in the medium bounded by that surface, the thickness of the medium of μ_3 also being neglected.

Recapitulation of Formulæ.—The following is a recapitulation of the formula for finding the focal length of the various spherical refracting bodies when the light passes from air. Where numerical examples are appended they are in each case for r = 6in. and $\mu = 1.5$.

			Approx. value in r.
Posterior F of a single surface	$= \frac{\mu r}{\mu - 1}$	= 18in.	= 3 r
Anterior F of a single surface	$= \frac{r}{\mu - 1}$	= 12in.	= 2 r
F of a thin plano lens	$= \frac{r}{\mu - 1}$	= 12in.	= 2 r
F of a sphere	$= \frac{\mu r}{2 (\mu - 1)}$	= 9in.	= $1\frac{1}{2}$ r
F of a thin equi-lens	$= \frac{r}{2 (\mu - 1)}$	= 6in.	= r
F of all forms of thin lens	$= \frac{r \, r'}{(r + r') (\mu - 1)}$		

DIOPTRAL FORMULÆ.

Lens in Air.—To find the dioptric number of a Cx. or Cc. lens when μ, r, and r′ are known, the following is the formula, r and r′ being in cm. :—

$$D = \frac{100\ (\mu - 1)\ (r + r')}{r\ r'} \qquad [65]$$

or

$$D = \left(\frac{100}{r} + \frac{100}{r'}\right)\ (\mu - 1) \qquad [66]$$

which for a double Cx. or Cc., since r = r′, becomes

$$D = \frac{2 \times 100\ (\mu - 1)}{r} \qquad [67]$$

and for a plano Cx. or Cc.

$$D = \frac{100\ (\mu - 1)}{r}. \qquad [68]$$

Thus, what is the power of a lens whose radii of curvature are 10 cm. positive and 40 cm. negative, μ being 1.54? Then

$$D = \frac{(100 \times .54) \times (10 - 40)}{10 \times (-40)} = +\ 4.05\ D.$$

What is the dioptric number of a lens whose radii of curvature are each 6 cm., μ being 1.62?

$$\left(\frac{100}{6} + \frac{100}{6}\right) \times .62 = (16.66 + 16.66) \times .62 = +\ 20.66\ D.$$

Substitution and equation of the formulæ given can be employed for finding μ or r when the other quantities are known. Thus r = 4 cm., D = 10, and μ = 1.54 to find r′—

$$10 = \left(\frac{100}{4} + \frac{100}{r'}\right) \times .54$$

that is

$$10 = 13.5 + \frac{.54}{r'} \qquad \therefore\ r' = -\ 15.43\ cm.$$

Similarly if μ has to be found; for example, let D = 8, r = 25 cm., and r′ = 10 cm., then

$$+ 8 = \left(\frac{100}{25} + \frac{100}{10}\right) (\mu - 1)$$

so that $8 = 14\mu - 14$ and $\mu = 1.577$

Lens in a medium denser than Air.—The following formula gives the power of a lens when in a medium other than air, as if immersed in a liquid; μ_2 pertaining to the lens and μ_1 to the medium in which it is placed.

$$D = \left(\frac{100}{r} + \frac{100}{r'}\right) \left(\frac{\mu_2 - \mu_1}{\mu_1}\right). \qquad [69]$$

Thus, find the power of a lens of radii − 10 cm. and − 10 cm. of $\mu_2 = 1$ immersed in a medium of $\mu_1 = 1.33$.

$$D = \left(\frac{100}{-10} + \frac{100}{-10}\right) \times \frac{1 - 1.33}{1.33} = - 20 \times \frac{-.33}{1.33} = + 5$$

By the same formula the power of a double Cx. lens of 10 cm. radius and $\mu = 1.33$ in air is

$$D = 20 \times \frac{1.33 - 1}{1} = + 6.66 \text{ D.}$$

CHAPTER VI.

LENSES.

F and D.—The focal length of a lens and its refractive power are reciprocals of each other. As the one is increased the other is proportionately diminished. If F represents the focal length and D the refractive power of a lens, then

$$F = \frac{1}{D} \qquad \text{and} \qquad D = \frac{1}{F}$$

Lenses are numbered by two principal systems, namely, the inch and the dioptric.

The Inch System.—The inch system is based on the measurement of the focal length of a lens, and the unit of the system is a lens of one inch focus, which is extremely short compared with the focal length of ordinary spectacle lenses.

A lens which brings parallel rays to a focus at 10 inches or at 20 inches has respectively 1/10 or 1/20 of the power of the unit; while one whose focal length is 1/2 inch has twice the power. The abbreviations Cx. for convex and Cc. for concave are commonly employed in conjunction with the focal notation of lenses.

Old Curvature Numeratives.—Originally the inch system of numeration was based on the radius of curvature. No. 10 implies a double convex or concave lens whose radius of curvature is 10 inches on each surface.

Addition of Lenses.—The combined strength of the two thin lenses, whose values are indicated by their focal lengths, is obtained by the addition of their refractive powers, thus:—

$$1/F = 1/F_1 + 1/F_2$$

where F_1 is the focal length of the one lens, F_2 that of the other, and F that of the two combined.

If the two lenses be, say, 24 inch Cx. and 10 inch Cx., their powers are respectively 1/24 and 1/10; the combined power is

$$+ 1/24 + 1/10 = 10/240 + 24/240 = 34/240 = 1/7 \text{ approx.}$$

The two equal a 1/7 Cx., or a lens of seven inch focus. It is evident that the focus of the combination must be shorter than that of either lens alone.

If the two lenses are concave, say 5 and 8, they equal

$$-1/5 + (-1/8) = -13/40 = -1/3 \text{ approx.}$$

When the one lens is convex and the other concave, the algebraical sum of the two is similarly obtained, 1/F being positive or negative according as F_1 or F_2 is the stronger. The two neutralise each other more or less, and the residual power of the stronger is the power of the combination. Thus, a 15 Cx. and a 12 Cc. when combined make a lens of 60 inch negative focus. For

$$1/15 + (-1/12) = 12/180 - 15/180 = -3/180 = -1/60$$

In the same way a 20 Cc. and a 10 Cx. equal a 20 inch Cx. lens. For $1/10 + (-1/20) = +1/20$ or 20 Cx.

The summing up of three or four lenses is achieved in a similar manner, thus:—

10 Cx., 16 Cx., 7 Cx., and 5 Cc. make together

$$1/10 + 1/16 + 1/7 - 1/5 = 59/560, \text{ that is } 9\tfrac{1}{2} \text{ Cx. approx.}$$

When both lenses are Cx. or both Cc. their united power can be mentally obtained by dividing the product of the two original numbers by their sum. Thus, an 8 and a 6 Cx. equal

$$\frac{8 \times 6}{8 + 6} = \frac{48}{14} = 3\tfrac{1}{4}\text{in. approx.}$$

If the one is Cx. and the other Cc. the multiple must be divided by the difference (or the algebraical sum) of the numbers, the result being Cx. or Cc. according to whichever is the stronger of the two original lenses; thus, 8 Cx. and 10 Cc. equal

$$\frac{10 \times 8}{10 - 8} = \frac{80}{2} = 40 \text{ Cx.}$$

Disadvantages of the F System.—There are certain disadvantages connected with the inch system of lens notation. For instance, the inch in various countries differs in value, so that a lens of given focal length in one country may not be the same as a lens of similar number in another. There are 37 French inches but 39.37 English inches in the metre, so that a lens of 18 French (Paris) inches focal length is about equivalent to one of 20 English or American inches.

Again, the intervals between the lenses, although regular as to their focal lengths, are irregular as to their refractive power; thus there is far greater difference between the power of a 5 and a 6 inch than between a 15 and a 16 inch. Lastly, the unit being a very strong lens, and the lenses mostly required being weak ones, calculations involve the use of vulgar fractions.

Dioptric System.—The dioptric system is based on the refractive power of lenses, and the unit is the *diopter*, which is that degree of refractive power that brings parallel rays to a focus at a distance of 1 metre. The diopter of refraction is a measure of converging or diverging power, and is not, strictly speaking, synonymous with the metre, which is a unit of lineal measurement; nevertheless, it is often convenient to express distances in dioptric measure. The symbols + and — are always used with this system.

The dioptric system of lens notation is much more simple than the inch and presents several advantages. It is universally recognised in all countries. The unit lens being weak, the power of most others is expressed by whole numbers, while if fractions are involved they are expressed as decimals. Also the intervals between the lenses are uniform as regards their refracting powers. Finally, the power of each lens being directly expressed, addition of two or more lenses consists of simple algebraical addition of their numbers.

If a 1 D lens has a focus of 1 metre (written 1 M) a 4 D lens (which has four times as much refracting power) has a focus of $\frac{1}{4}$ M; that is to say, parallel rays of light will be made to meet at $\frac{1}{4}$ M.

But since the metre can be sub-divided into 100 cm. or 1000 mm. the focal length of a 4 D is more conveniently expressed as 100/4 = 25 cm.

A 10 D lens has ten times the power of the unit, therefore F = 100/10 = 10 cm.

A 0.50 D has half the power of the unit, consequently its F = 100/.5 = 200 cm.

Addition of Lenses.—The strength of the combined dioptral lenses is obtained by adding them together algebraically,

$$D_1 + D_2 = D$$

D_1 being the power of the one, D_2 that of the other lens, and D that of the two combined. For example:—

+ 2 D and + 4 D = + 6 D,

+ 4 D and − 3 D = + 1 D,

− 5.25 D and − 2.50 D = − 7.75 D,

+ 3 D and − 3 D = 0, i.e., they neutralise each other.

+ 7 D + 4.50 D + 1.75 D and − 6.50 D = + 6.75 D.

Value of the M.—The metre (or 100 centimetres) = 39.37 English inches, but for all practical purposes it may be regarded as equivalent to 40 (or 39) inches.

Conversion.—Since the + 1 D lens refracts parallel rays to a focus at 1 metre or 40 inches, it is equal to the No. 40 Cx. lens of the inch system, and a 40 D lens is the same as a 1 inch lens.

Since a focal length of 1 inch = 40 D, in order to convert from the inch scale into the dioptric system, we multiply the power of the lens by 40; thus,

No. 5in. = 1/5 × 40 = 40/5 = 8 D.

And since 1 D = 1/40 (in.), in order to turn from the dioptric into the inch scale, we multiply the power by 1/40; thus,

10 D = 10 × 1/40 = 10/40 = 1/4 or a No. 4.

Or for conversion from either scale into the other divide 40 or 39 (whichever is the more convenient) by the known number. For instance,

2.5 D = 40/2.5 = 16in.,
13 D = 39/13 = 3in.,
2in. = 40/2 = 20 D,
13in. = 39/13 = 3 D.

In making the division there is often a small fraction left over, as many numbers will not divide evenly into 40 or 39. For practical purposes these fractions need not be considered beyond the $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ in the lower inch numbers, and .25, .50, and .75 in the dioptral numbers. Some numbers of both scales have no exact equivalent in spectacle lenses, numbered according to the other, and the nearest approximate must be taken as the equivalent number. For instance, it is considered that

3.50 D = No. 11; 3.25 D = No. 12; 4.50 D = No. 9, etc.

Fractions.—Vulgar fractions are usually employed with the inch system, as say, $2\frac{1}{2}$; while fractions of diopters are invariably expressed in decimals, as say, 6.50 D.

To Find F or D.—Dividing 40 or 100 or 1000 by the dioptral number gives F in inches, in cm., or in mm. respectively. Thus, a 5 D lens has F = 40/5 = 8in., 100/5 = 20 cm., or 1000/5 = 200 mm.

If F is known in cm., mm., or inches the dioptral number is found by dividing respectively into 100 or 1000 or 40; thus, if F is 200 mm., then D = 1000/200 = 5; if F = 40 cm., D = 100/40 = 2.5; if F = 160in., D = 40/160 = .25.

The employment of the dioptric system greatly facilitates the comprehension of refractive errors of the eye and their correction by means of lenses, while for rapid calculations it is much superior to the focal length system.

Old Cc. System.—In England concave sphericals were formerly numbered by an arbitrary system commencing at 0000—the weakest—and terminating with No. 20—the strongest. The values of these numbers in the inch and dioptric scales are to be found in the appendix, but this system is now obsolete.

Cyls.—The numeration of cylindrical lenses is the same as that of sphericals.

CONJUGATE FOCI OF THIN LENSES AND THE RELATIVE SIZES OF OBJECT AND IMAGE.

In the following rules and examples let O and I represent object and image respectively; let L be the lens, Oc the optical centre of the lens, f_1 the distance of object from Oc, f_2 the distance of image from Oc, F_1 and F_2 the first and second principal focal distances of the lens (often represented by F), and let D be the refractive power of the lens.

The focal distance of a thin lens is the distance from the optical centre of that lens to the point at which parallel rays meet, and it is that distance from which light must diverge in order to be parallel after refraction. The reciprocal of the focal distance, or its value expressed in diopters, indicates the power of the lens. A + 5 D lens has a focal length of 20 cm., and consequently light diverging from 20 cm. is rendered parallel by a + 5 D lens, the converging power of the Cx. lens just neutralises the divergence of the light from 20 cm. Similarly light from ∞ is brought to a focus at 20 cm. by a + 5 D lens.

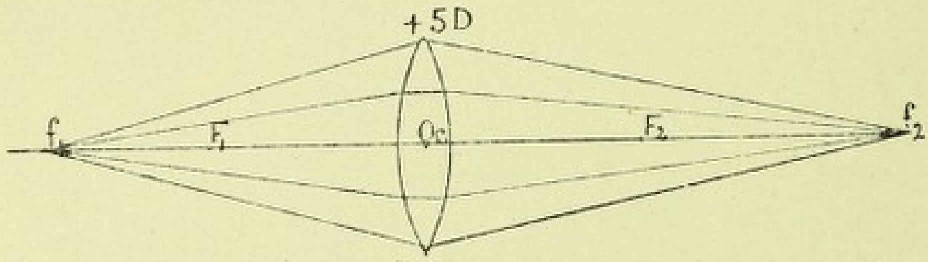


Fig. 103.

The power of the lens is the same whether acting on parallel rays or on rays diverging from any other point, so that if the rays proceed from f_1, which is a point nearer than ∞, the divergence must be overcome by the refracting power of the lens in order to render the rays parallel before it can converge them. Some of the power of L is thus used, and there is less left for convergence. Therefore the focus must be further away from the lens than F.

Suppose we have a 5 D convex lens and let f_1 be 100 cm. distant from it. The divergence from 100 cm. needs a power of 100/100 = 1 D to neutralise that divergence and to render the light parallel. Thus 1 D of the total 5 D which L possesses is exerted on the side of the divergent rays, and there is left for convergence 5 − 1 = 4 D, so that f_2 is 100/4 = 25 cm. beyond the lens, that is, at the distance at which parallel rays would be brought to a focus by a + 4 D lens.

The lens has a converging power of 5 D, the light has a divergence, expressed in diopters, as 1 D. Consequently after refraction the light has a convergence of 4 D.

In Fig. 104 the + 5 D is shown as if split into two lenses, the + 1 D being used for rendering the rays diverging from f_1 parallel, while the + 4 D brings the parallel rays to a focus at 25 cm.

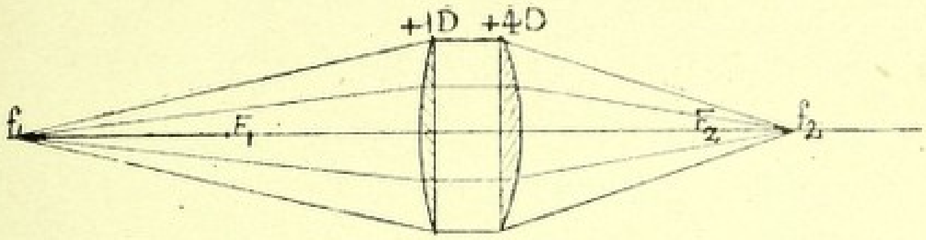


Fig. 104.

Conjugate Distances.—f_1 and f_2 are conjugate foci and are interchangeable in the sense that if O is at the one place, I is at the other. The power of a lens is equal to the sum of the two conjugates f_1 and f_2 expressed in diopters as d_1 and d_2, so that

$$D = d_1 + d_2. \qquad [70]$$

The formula for finding a conjugate focus by the dioptric system is therefore

$$D - d_1 = d_2 \quad \text{or} \quad D - d_2 = d_1 \qquad [71] [72]$$

In the illustration given the two distances 100 cm. and 25 cm. expressed in diopters are +1 and +4 respectively. We may therefore write—

$1 + 4 = 5$ D = the power of the lens.

$5 - 1 = 4$ D = the position of I.

$5 - 4 = 1$ D = the position of O.

Examples, Cx. Lens.—Suppose the object to be placed 50 cm. in front of a lens having its image 12.5 cm. behind it, then to find the power of the lens

$$d_1 = 100/50 = 2, \qquad d_2 = 100/12.5 = 8;$$

therefore $D = 2 + 8 = 10.$

Suppose an object is 200 cm. in front of a 7 D lens, where will the image be?

Here $d_1 = 100/200 = .5, \qquad d_2 = 7 - .5 = 6.5;$

therefore $f_2 = 100/6.5 = 15$ cm.

An image is 22 cm. behind an 8 D lens, where is the object? This is merely the reverse of the last question.

We have $d_2 = 100/22 = 4.5, \qquad d_1 = 8 - 4.5 = 3.5;$

therefore $f_1 = 100/3.5 = 30$ cm.

If O is at ∞, then $d_1 = 100/\infty = 0;$

so that $D - 0 = D$ and $100/D = F,$

consequently I is at F.

If O is at F, then $d_1 = 100/F = D$,

and $D - D = 0$ and $100/0 = \infty$,

consequently I is at ∞.

Thus ∞ and F are conjugate focal distances, and image and object are interchangeable.

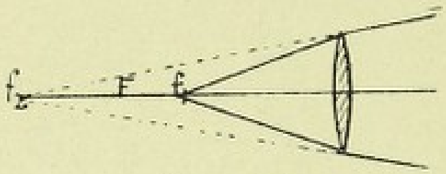


Fig. 105.

Virtual Conjugates.—When O is at F the power of the lens is just sufficient to render incident rays parallel; if, therefore, f_1 (Fig. 105) is situated nearer than F, the power of the lens would be insufficient to render them even parallel, and they emerge divergent after refraction, although less so than before refraction. No real focus is obtained, but if the rays are projected backwards they meet in front of the lens (on the same side as the object) and form a negative focus and virtual image at f_2. Whereas the light diverged originally from f_1 it appears after refraction to diverge from f_2.

Since f_1 expressed as a power is greater than that of F, on deducting the former from the latter the result is a negative quantity, the sign indicating its position with regard to the lens.

Examples.—Let the power of the lens be + 5 D and f_1 be at 14 cm.,

then d_1 $100/14 = 7$; $d_2 = 5 - 7 = -2$, and $100/-2 = -50$, so that f_2 is at 50 cm. in front of the lens.

In this case, − 50 cm. and + 14 cm. are conjugate foci in respect to a + 5 D lens, for + 7 D + (− 2 D) = + 5 D. That is to say, if rays diverge from 14 cm. to a + 7 D lens, they appear after refraction to diverge from 50 cm., and if rays converge to 50 cm. behind a + 7 D lens, they are after refraction convergent to 14 cm.

In other words, while the lens has a converging power of 5 D, the light has a divergence which may be expressed as 7 D. Therefore, after refraction, there is a residual divergence of 2 D.

Position of Conjugates of a Cx. Lens.—A convex lens renders rays convergent, parallel, or less divergent, according as the point of divergence is respectively beyond, at, or within F. The converging property of the lens is decreased, neutralised, or exceeded by the divergence of the light due to the nearness of the object. And since any approach of the object to a Cx. lens causes the light to be less convergent after refraction, it follows that any conjugate focus is more distant than F, and F is the nearest point to L at which a real image can be formed.

Conjugate Foci of Cc. Lens.—A concave lens refracts light divergently so that parallel rays from ∞, after refraction, appear to diverge from F (Fig. 106, 1). If the power of the lens is $-$ 5 D, the virtual F will be at $100/5 = 20$ cm. or 8ins. Its power is the same no matter where the object may be.

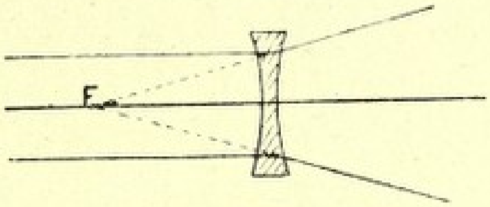


Fig. 106 (1).

Therefore, when f_1 is nearer than ∞ (Fig. 106, 2) the rays incident on the lens, being divergent before refraction, become still more divergent. The divergence of the lens is augmented by that caused by the nearness of O, consequently the conjugate focus is nearer than F. Here again f_1 and $-f_2$ are conjugates, just as is the case of the virtual focus obtained with a convex lens, for the sum of their powers $d_1 + (-d_2) = D$. In applying the formula for calculating f_1 or f_2 it must be remembered that both F and f_2 are negative.

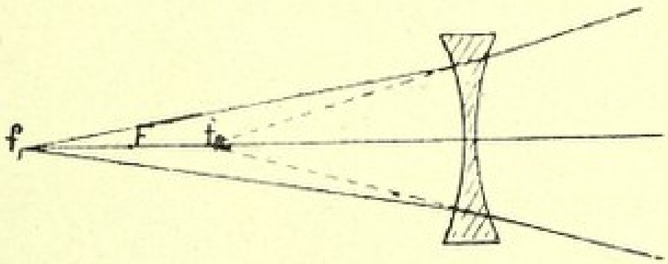


Fig. 106 (2).

Example.—Let L be $-$ 5 D and f_1 at 100 cm., then

$d_2 = -5\text{ D} - 1\text{ D} = -6\text{ D}$, and $100/6 = -16.66$ cm.;

f_2 is therefore virtual and 16.66 cm. in front of the lens.

If the rays diverge from 100 cm. to a $-$ 5 D lens they are after refraction divergent as if from 16.66 cm. If they are convergent to a point 16.66 cm. behind a $-$ 5 D lens they are, after refraction, convergent to 100 cm.

Position of Conjugates of a Cc. Lens.—A concave lens always renders rays divergent, and since any distance of O, nearer than ∞, causes I to be nearer than F, it follows that the most distant conjugate focus of a Cc. lens is F.

Formulæ by Inch System.—For calculating conjugate foci by the inch system, the following formula is used:—

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$ [73]

and since $$\frac{F}{F} = \frac{F}{f_1} + \frac{F}{f_2}$$

we may write the formula $$1 = \frac{F}{f_1} + \frac{F}{f_2}.$$ [74]

By transposing we get

$$1/F - 1/f_1 = 1/f_2, \quad \text{or} \quad 1/F - 1/f_2 = 1/f_1;$$ [75] [76]

or by turning the equation upside down

$$F = \frac{f_1 \, f_2}{f_1 + f_2}$$

therefore $$f_2 = \frac{f_1 \, F}{f_1 - F} \quad \text{and} \quad f_1 = \frac{f_2 \, F}{f_2 - F},$$ [77] [78]

a variation of the formula which is sometimes more convenient.

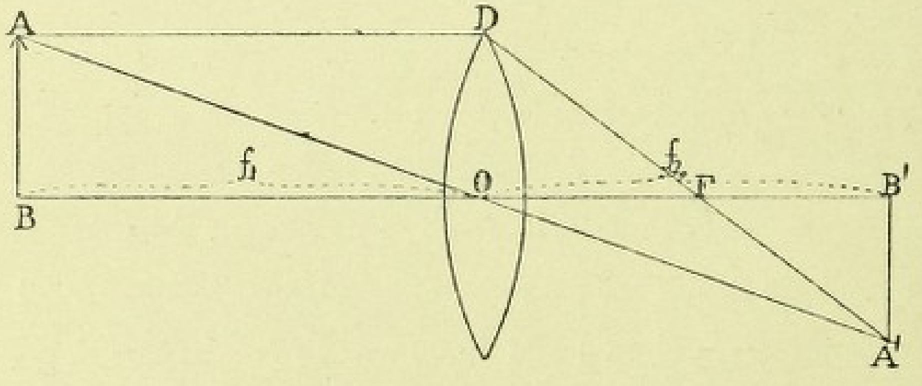


Fig. 107.

Mathematical Proof.—In the above diagram A B = object, A′ B′ = image, O = centre of lens, O F = focal length, F = principal focus; then, as the two triangles A D A′, O F A′ have equal angles,

$$\frac{A\,D}{O\,F} = \frac{A\,A'}{O\,A'} = \frac{B\,B'}{O\,B'}$$

Now $AD = f_1$ and $OF = F$

therefore $\frac{f_1}{F} = \frac{f_1 + f_2}{f_2}$ and $\frac{1}{F} = \frac{f_1}{f_1 f_2} + \frac{f_2}{f_1 f_2}$

that is $$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

Examples and Proofs.—If a lens has $F = 8$ inches and f_1 is at 40 inches, then f_2 will be at 10 inches, for

$$1/f_2 = 1/8 - 1/40 = 4/40 = 1/10.$$

This is proved by $\frac{1}{10} + \frac{1}{40} = \frac{1}{8}$, and $\frac{8}{40} + \frac{8}{10} = 1.$

Worked by the other method we should get

$$f_2 = \frac{40 \times 8}{40 - 8} = \frac{320}{32} = 10\text{in.}$$

An image is 16 inches behind a 7 inch Cx., at what distance is the object in front of the lens?

$1/f_1 = 1/7 - 1/16 = 9/112$ the object is at $12\frac{4}{9}$in.

By the other formula $f_1 = \frac{16 \times 7}{16 - 7} = \frac{112}{9} = 12\frac{4}{9}\text{in.}$

If O is at ∞,

then $1/F - 1/\infty = 1/F - 0 = 1/F$, so that I is at F.

If O is at F,

then $1/F - 1/F = 0/F$, so that I is at ∞.

Let the object be 6in. from an 8in. Cx. lens,

then $1/f_2 = 1/8 - 1/6 = -1/24.$

f_2 is virtual or negative at 24 inches on the same side as f_1,

or $$f_2 = \frac{6 \times 8}{6 - 8} = \frac{48}{-2} = -24\text{in.}$$

Care must be taken when the lens is Cc. that the $-$ sign be prefixed where necessary. As an example let the lens be 1/10 Cc and f_1 at 40in.,

then $1/f_2 = -1/10 - 1/40 = -5/40 = -1/8$

the image is at 8in. negative or virtual.

or $$f_2 = \frac{40 \times (-10)}{40 - (-10)} = \frac{-400}{50} = -8\text{in.}$$

This is proved by the power of the lens being equal to

$$1/F = -1/8 + 1/40 = -1/10;$$

that is $$F = \frac{40 \times (-8)}{40 + (-8)} = \frac{-320}{32} = -10\text{in.}$$

As before stated, the power of an Cx or Cc lens is equal to the sum of the powers of *any* pair of its conjugate foci, whether the image be real or virtual.

If the conjugates are 20 and 50 cm. the lens has a power of $5 + 2 = +7$ D.

If they are 20 cm. and -50 cm. the lens is $+5 + (-2) = +3$ D.

If they are -20 cm. and $+50$ cm. the lens is $-5 + 2 = -3$ D.

In the same way if the conjugates are 5 and 10in. the lens is $1/5 + 1/10 = 3/10$ or $3\frac{1}{3}$in. Cx.

If the conjugates are 5 and -10in. the lens in $1/5 - 1/10 = +1/10$ or 10 Cx.

If they are -5 and 10in. the lens is $-1/5 + 1/10 = -1/10$ or 10in. Cc.

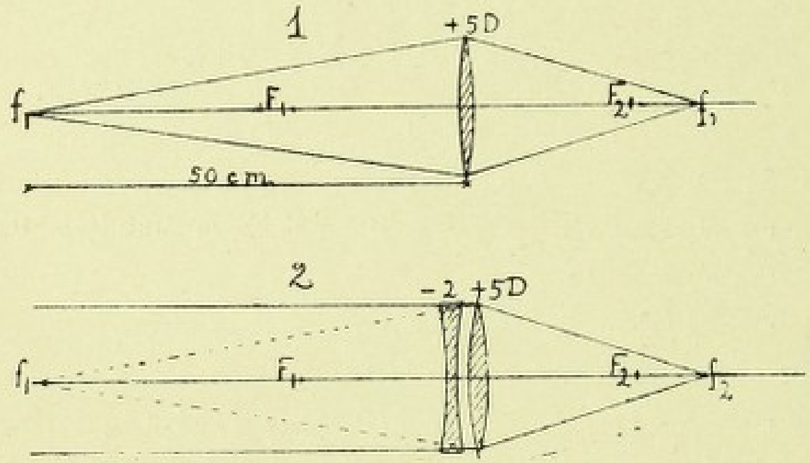


Fig. 108.

Light Divergent.—Whether rays actually diverge from some point nearer than ∞, say 50 cm. (Fig. 108 — 1) or whether parallel rays are rendered divergent by an added -2 D lens (Fig. 108 — 2), the converging effect of the convex is equally reduced and in both cases f_2 is $+5 - 2 = +3$ D at 33 cm. behind the lens.

If f_1 were 14 cm. (7 D) in front of a $+5$ D or if -7 D were added to a $+5$ D the effect in both cases would be the same; the light, after refraction, would diverge as if proceeding from 50 cm.

Similarly with a concave lens, whether rays (Fig. 109 — 1) diverge from f_1 50 cm. (2 D) in front of a − 5 D lens, or whether (Fig. 109 — 2) a − 2 D be added to the − 5 D and the two combined act on parallel rays, the focus f_2 in either case is − 5 − 2 = − 7 D or 14 cm. negative.

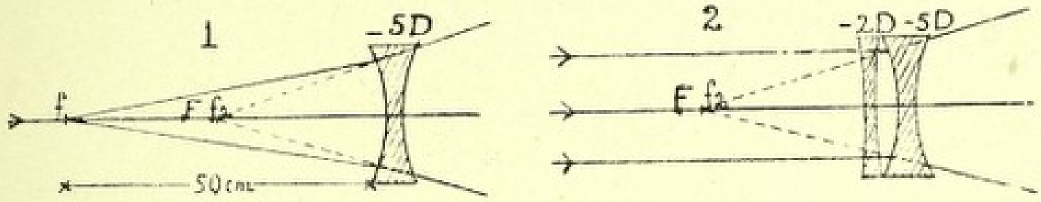


Fig. 109.

Relative Distances of O and I.—With a Cx. lens as O approaches F from ∞ so I recedes from F until when O is at F then I is at ∞. When O is within F, I becomes negative and as O approaches L so also does I until when O is at L, I is at L also.

With a Cc. lens as O approaches from ∞ so I also recedes from F towards L until when O is at F, I is at F/2 and when O is at L, so also is I.

The following diagrams illustrate the positions of f_1 and f_2 for every possible case with thin Cx. and Cc. lenses.

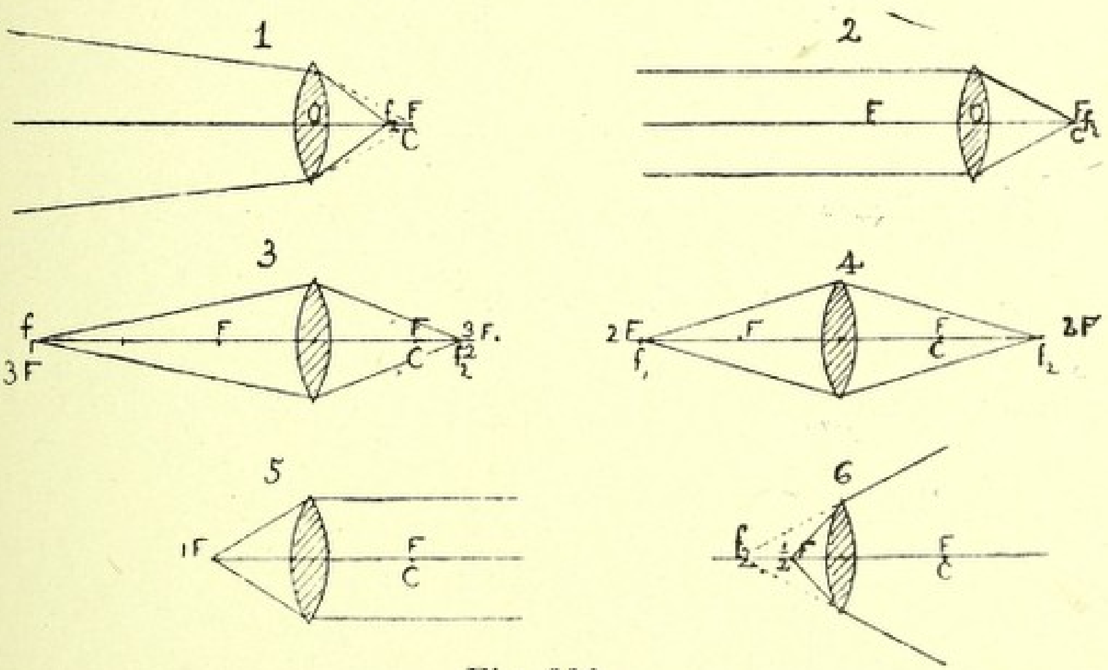


Fig. 110.

In Fig. 110 No. 1 shows convergent rays incident on a Cx. lens, in which case f_2 is within F.

No. 2 shows parallel rays, with f_2 coinciding with F.

No. 3 shows f_1 within ∞ but beyond F, and f_2 is then beyond F.

No. 4 shows f_1 at 2 F, and here f_2 is also at 2 F on the other side of the lens (2 F being the planes of unit magnification for real images).

No. 5 shows f_1 at F and f_2 is at ∞.

No. 6 shows f_1 within F and f_2 negative.

In Fig. 111 No. 1 shows rays rendered convergent, and incident on a Cc. lens; in such case f_2 is beyond F.

No. 2 shows parallel rays, and f_2 coincides with F.

Nos. 3 and 4 shows f_1 within ∞ but beyond F, so that f_2 is within F.

No. 5 shows f_1 at F and f_2 at F/2.

No. 6 shows f_1 within F and f_2 still more so.

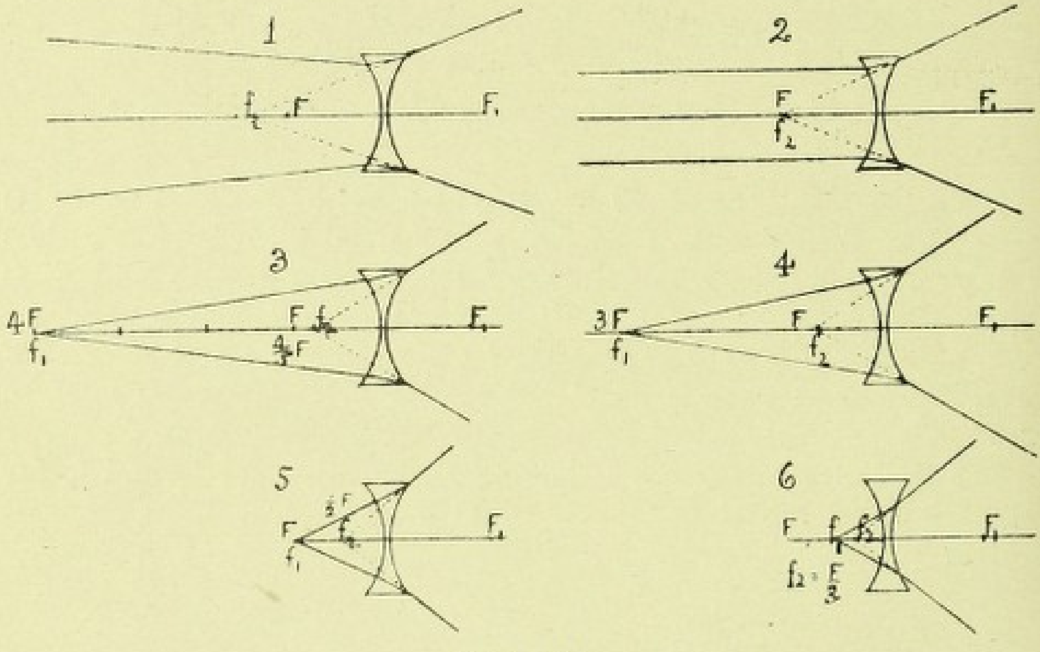


Fig. 111.

Magnification or Relative Sizes of O. and I.—In Fig. 112, the object O and the image I subtend equal angles at N, the optical centre of the lens. In the triangles A N O and A′ N I the angles at N are equal and the angles at O and I are both right angles; therefore the remaining angles at A and A′ are also equal. But in equi-angular triangles the sides containing equal angles are proportionals.

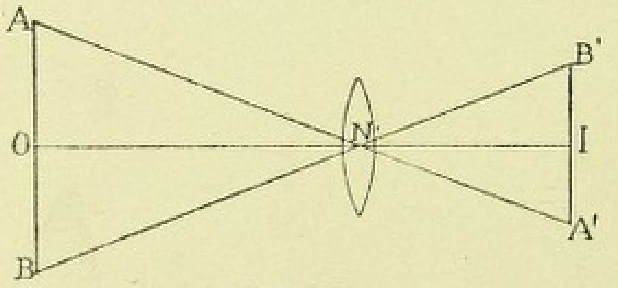


Fig. 112.

Therefore
$$\frac{ON}{NI} = \frac{AO}{A'I} = \frac{AB}{A'B'}.$$

The ratio B′A′/AB is the *magnification* and denotes the linear *increase* or *decrease* in the size of the image with respect to the object. Superficial magnification applies to area and is the linear magnification squared.

That is to say the relative sizes of O and I are proportional to their respective distances from the centre of the lens, and this holds equally true for virtual images of both Cx. and Cc. lenses.

So long as O is beyond 2 F the image must be smaller than O, since it is nearer to the lens. When O is at 2 F the size of I is the same as that of O, because both are at the same distance in what are termed the symmetrical conjugate focal planes. When O is within 2 F, I is larger, because it is further from the lens than O.

To calculate the size of I or of O the following formulæ are applicable to all cases, whether the lens be Cx. or Cc. or the image real or virtual.

$$\frac{h_1}{f_1} = \frac{h_2}{f_2} \quad \text{that is } h_2 = \frac{h_1 f_2}{f_1}, \quad \text{and } h_1 = \frac{h_2 f_1}{f_2}$$ [79] [80]

where f_1 and f_2 are the distances of O and I respectively from the lens, h_1 is the linear size of O and h_2 that of I.

h_1 and f_1 in the first formula must be in similar terms, but not necessarily that of f_2; and then h_2 will be in the same terms as f_2 whether inches, cm., etc. In the second formula h_2 and f_2 must be in the same terms; and h_1 will be in that of f_1.

For example let O be at 2 M, and I .625 cm. long at 25 cm. distance from the lens; then

$$h_1 = \frac{.625 \times 2 \times 100}{25} = 5 \text{ cm. in length}$$

O is eight times the size of I. If O were at 25 cm. and I at 2 M, then I would be eight times the size of O.

If O, 2 inches long, is at 10 feet and I at 10 inches, then

$$h_2 = \frac{2 \times 10}{10 \times 12} = \frac{1}{6} \text{ in.}$$

Let O, 4 yards long, be $\frac{1}{4}$ mile distant from a + 5 D lens; then f_2 = 20 cm. and

$$h_2 = \frac{4 \times 20}{440} = .18 \text{ cm.}$$

The answer here is in cm., showing that O and I need not be in the same terms.

The formula which is perhaps most useful for calculating the magnification of the image is

$$M = \frac{F}{f_1 - F}.$$ [81]

Important Consideration.—It is most essential that students should differentiate between the direction of axial rays and that of the rays from the various points on an object with reference to their axes.

From each point of the object a pencil of rays diverges and each pencil has an axis, which is the axial ray of that pencil.

Axial rays *always converge* to the optical centre of the lens, and their convergence governs the *size of the angle subtended by the object*.

The rays themselves *always diverge* from the luminous point to the lens, and their divergence governs the *position of the image*, the rays after refraction being more or less divergent or convergent, according to the degree of original divergence and the diverging or converging power of the lens.

Parallel light is merely light having a negligible degree of divergence.

These *most important considerations*, for students who are apt to confuse the conditions, should be carefully noted. Thus, in a diagram which shows light parallel to the axis, and incident on various parts of the lens surface, these various rays are presumed to originate, not in various points, but in one single point on the axis. Also these considerations apply to all lenses, all curved mirrors, and all positions of the object.

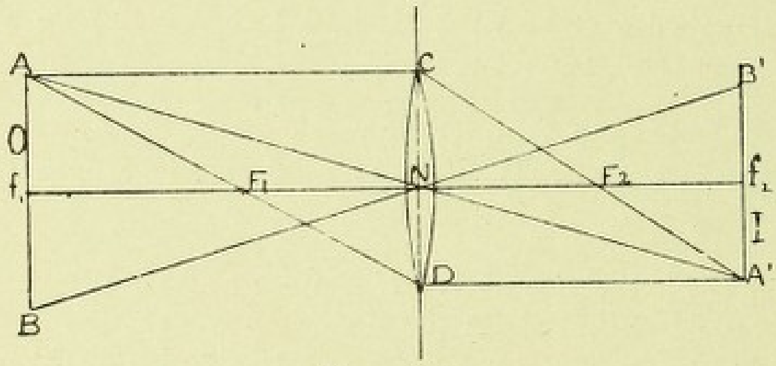


Fig. 113.

Construction. — In Fig. 113 the relationship between O and the real I of a Cx. lens is shown by construction. Tracing the rays A C, A N and A D they meet, after refraction, at A′. If A B and B′ A′ were measured they would be found to be proportional to f_1 N and f_2 N.

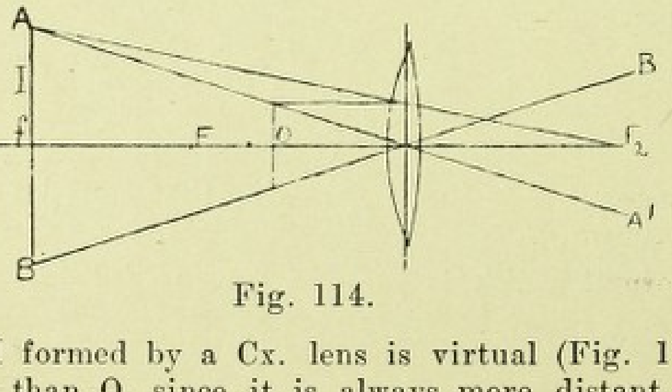


Fig. 114.

When the I formed by a Cx. lens is virtual (Fig. 114), it is always larger than O, since it is always more distant from the lens.

Example.—Let O be at 20 cm. and I at 50 cm. and let O be 5 cm. long, then

$$h_2 = \frac{5 \times 50}{20} = 250/20 = 12.5 \text{ cm.}$$

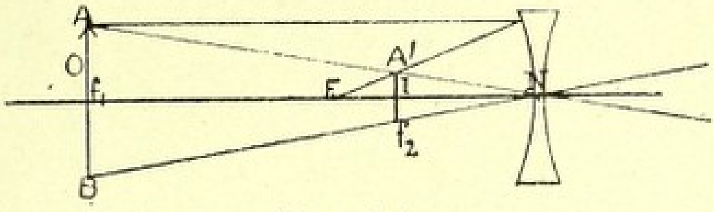


Fig. 115.

With a concave lens (Fig. 115) the virtual I formed is always smaller than O, since it is always nearer to the lens.

Example.—If O is two inches long and at 40 inches, while I is at $6\frac{2}{3}$ inches.

$$h_2 = \frac{2 \times 6\frac{2}{3}}{40} = \frac{1}{3} \text{ in.}$$

The relative size of the object to the real and the virtual images formed by a given Cx. lens is the same when O is as far beyond F in the first case as it is within F in the second case. Thus, suppose O situated at 14in. and at 6in. respectively in front of a 10in. Cx. lens, it being in either position 4in. from F, then the size of the image in each case is $2\frac{1}{2}$ times that of the object.

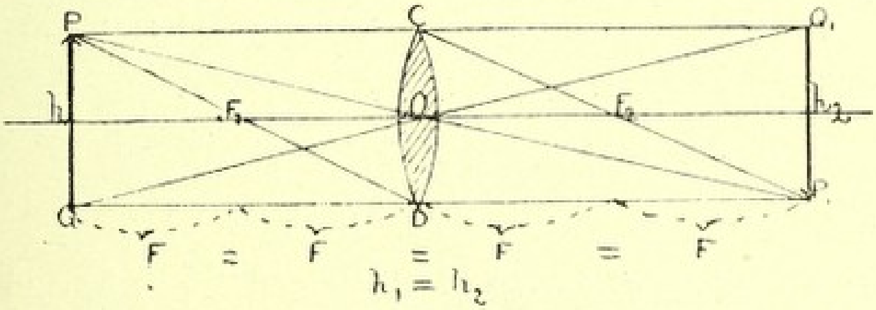


Fig. 116.

Planes of Unit Magnification.—In order that O and I be equal in size they must be equally distant from the lens, i.e., they must be situated in the planes of unit magnification which, for real images, are the symmetrical planes P Q and Q′ P′ (Fig. 116), which cut the axis at twice the principal focal distance. It can be there seen that $h_1 = h_2$.

For a virtual I to be equal in size to O, it must be in contact with the lens. This is true for both Cx. and Cc. lenses, so that the planes of unit magnification for virtual images is zero. It may be remarked that both planes of unit magnification are distant from F a distance equal to F.

Reciprocity of Conjugate Distances from F.—If the distance of the two conjugates f_1 and f_2 of a Cx. lens be measured respectively from F_1 and F_2 they are reciprocals of each other in terms of F. If f_1 is at a distance N F beyond F_1, then f_2 is $(1/N)$ F or F/N beyond F_2. Thus, for instance, if the distance $f_1 F_1$, in Fig. 117, is twice F, then the distance $f_2 F_2$ is one-half of F.

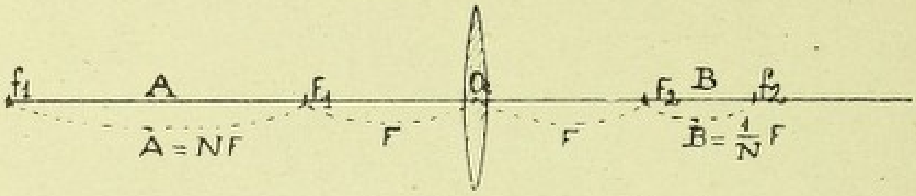


Fig. 117.

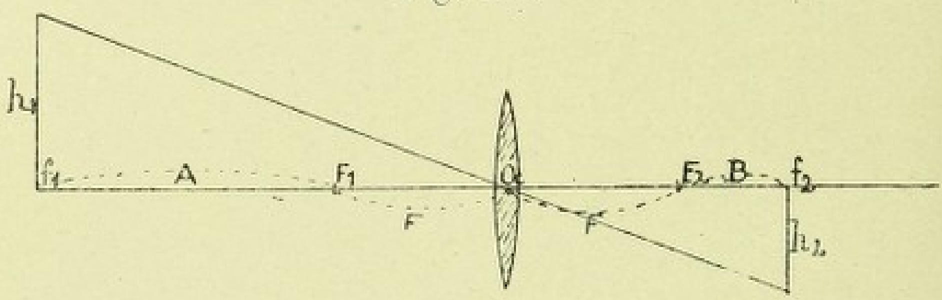


Fig. 118.

From this we see that for M (magnification) = 1 the one conjugate must be at F + F.

For M = 2 it must be at F + 2 F, the other conjugate being at F + F/2.

For M = 3 it must be at F + 3 F, the other conjugate being at F + F/3, and so on.

Let the distance $f_1 F_1$ (Fig. 117) be called A and $f_2 F_2$ be called B, and since

$$N F \times F/N = 1$$

we obtain the formula for conjugate foci of

$$A B = F^2.$$ [82

The ratio between the sizes of object and image h_1 and h_2 is, as shown in Fig. 118

$$\frac{h_2}{h_1} = \frac{F}{A} = \frac{B}{F}$$ [83

Since with a given lens, F^2 is a constant, the value of A B is also a constant, and the multiple of the distances beyond F of any pair of conjugates is the same.

When employing these formulæ it is most essential to remember that positive quantities are measured forwards from F_1 and backwards from F_2. Also that in Cc. lenses F_1 is on the remote side of the lens and F_2 on the object side. Also that A is always reckoned from F_1 and B from F_2. To obtain f_1 or f_2 the value of F must be added to A or B respectively.

These points make these otherwise valuable formulæ difficult of application.

Examples.—Thus, suppose f_1 to be 50 cm. in front of a Cx. lens of 10 cm. focus, then

$$A = 50 - 10 = 40 \text{ cm. or } 40/10 = 4 \text{ F},$$
$$B = 1/4 \text{ F} = 10/4 = 2.5 \text{ cm.},$$

and f_2 is at $10 + 2.5 = 12.5$ cm. from the lens.

Worked by the formula $A\ B = F^2$ we get $40\ B = 10^2 = 100$,

then $$B = 100/40 = 2.5,$$

and $f_2 = 2.5 + 10 = 12.5$ cm. as above.

If O in the above example is 5 cm. high, we have

$$h_2/5 = 10/40, \text{ so that } 40\ h_2 = 50, \text{ or } h_2 = 1.25 \text{ cm.}$$

The image is positive at 12.5 cm. and 1.25 cm. in height.

Let an object 5 cm. high be placed 8 cm. in front of a lens of 10 cm. F, then

$$A = 8 - 10 = -2, \text{ and } -2\ B = 10^2 = 100$$

so $$B = 100/-2 = -50 \text{ and } f_2 = -50 + 10 = -40 \text{ cm.}$$
$$h_2/5 = 10/2 \text{ so that } 2\ h_2 = 50, \text{ or } h_2 = 25 \text{ cm.}$$

The image is negative at 40 cm. and is 25 cm. high.

Let an object 5 cm. high be placed 50 cm. in front of a Cc. lens, whose $F = 10$ cm., then

$$A = 50 - (-10) = 60, \text{ and } 60\ B = 10^2 = 100$$

then $$B = 100/60 = 1.66$$

and $$f_2 = 1.66 + (-10) = -8.33 \text{ cm.}$$
$$h_2/5 = 10/60 \text{ so that } 60\ h_2 = 50, \text{ or } h_2 = .833 \text{ cm.}$$

The image is negative at 8.33 cm. and is .833 cm. high.

Tabulated Conjugates in Terms of F.—The relative positions of O and I, with respect to the lens itself, are, in terms of F, as follows:—

With Cx. lenses if O is at N times F, then I is at $\frac{N\ F}{N-1}$;

with Cc. lenses the latter becomes $\frac{N\ F}{N+1}$

Cx. Lenses.

O at 10 F	8	6	5	4	3	2	1.75	1.5	1.25	1	.75	.5	.25
I at 1.11 F	1.14	1.2	1.25	1.33	1.5	2	2.33	3	5	0	-3	-1	-.33

Cc. Lenses.

O at 4F	3	2	1.5	1	.5	.25
I at .8F	.75	.66	.6	.5	.33	.2

FURTHER CALCULATIONS ON THE POSITION AND SIZE OF I.

Removal of I.—To move the image from f_2 to some other position x more distant, or y nearer, there must be added to the lens another Cc. or Cx. whose power is the difference between

$$\frac{1}{x} \text{ or } \frac{1}{y} \text{ and } \frac{1}{f_2}.$$

Thus, supposing f_2 to be at 20 cm. and x to be 25 cm,

since $f_2 = 5$ D and $x = 4$ D, then $4 - 5 = -1$ D;

the required lens is Cc. because x is more distant than f_2.

If it is required to place the image at y, 16in. behind the lens instead of at f_2, which is 20in., then as

$$\frac{1}{16} - \frac{1}{20} = \frac{1}{80}$$

the added lens must be positive of 80 inches focus or + 0.5 D.

An object is placed 20 inches in front of a 6in. Cx. lens; the image is required to be 10 inches. What lens must be added?

Here $1/6 - 1/20 = 7/60$ and $1/10 - 7/60 = -1/60$,

so that a negative lens of 60in focus (or − 0.66 D) must be added to the 6in. convex lens.

Position of Lens for given M.—Another useful problem is to find where a given lens should be placed so that the image be a certain number of times larger or smaller. For example, suppose the lens is a 6in. Cx., the object 2 inches long, and it is required that the real image should measure 18 inches. In this case if x is the one conjugate it follows that 18/2 or 9 x must be the other, so that

$$\frac{1}{6} = \frac{1}{x} + \frac{1}{9x} = \frac{10}{9x}$$

then $9x = 60$in. and $x = 6\frac{2}{3}$in.

The lens, therefore, must be placed $6\frac{2}{3}$in. from the object and the image will be at $6\frac{2}{3} \times 9 = 60$in. from the lens.

If a virtual image is required to measure 18in., then $1/9x$ is negative and the calculation becomes

$$\frac{1}{6} = \frac{1}{x} - \frac{1}{9x} = \frac{8}{9x} \text{ whence } x = 5\tfrac{1}{3}$$

The negative sign must also be prefixed before $1/F$ when the lens is Cc.

Such calculations as the above can also be readily solved by the formula

$$f_2 = F\,(M + 1) \qquad \text{and} \qquad f_1 = f_2/M$$ [84]

when the image is real, and by

$$f_2 = F\,(M - 1) \qquad \text{and} \qquad f_1 = f_2/M$$ [85]

when the image is virtual, with either a Cx. or Cc. lens.

In both cases M is expressed by a fraction when diminution is required.

Position of Lens for given distance between O. and I.—The calculation of the position of a given lens between two given points so that O be at the one and I at the other necessitates finding two conjugate distances such that their reciprocals are equal to the power of the lens. Let d be the distance between object and image, let x represent the one conjugate, then the calculation becomes

$$\frac{1}{F} = \frac{1}{x} + \frac{1}{d - x}$$

and for its solution a quadratic equation is required. The above, however, may be transposed into the somewhat simpler form of

$$x^2 - d\,x = -\,d\,F.$$ [86]

When d is greater than 4 F, the image is real and may be at either conjugate, and there are two positions for the convex lens, between object and image, which will fulfil the given conditions. In every case d = the sum of the two solutions.

With a Cx. lens when d is less than 4 F, the image is virtual and is negative. The shorter conjugate is positive and is the distance of the object; the greater is negative and is that of the image, d is then a negative quantity.

When the lens is concave, the image is also virtual, d is positive but F is negative. The greater conjugate is positive and is the distance of the object, while the smaller is negative and is that of the image.

Let F = 7in. and the distance between object and image be 36in, then

$$x^2 - 36\,x = -\,252$$

To find x^2 we must add to each side of the equation the square of half one of the factors, viz. 36, that is $18^2 = 324$. Then

$$x^2 - 36\,x + 324 = -\,252 + 324 = 72$$

now $\sqrt{x^2 - 36\,x + 324} = x - 18$ and $\sqrt{72} = \pm\,8.5$

$$\therefore \qquad x - 18 = \pm\,8.5$$

and $\quad x = +\,8.5 + 18 = 26.5$

or $\quad x = -\,8.5 + 18 = 9.5$

The lens may be placed either 9.5in. or 26.5in. in front of the object.

Let F = 5in. and the distance between object and image be 16in. then d is negative, so—

$$x^2 + 16\,x = +\ 80$$

$$x^2 + 16\,x + 64 = 80 + 64 = 144$$

extracting the square roots, we get

$$x + 8 = \pm\ 12$$

$$x = +\ 12 - 8 = +\ 4 \text{ or } -12 - 8 = -\ 20$$

The lens is 4in. beyond the object and 20in. from the virtual image.

Let F be 5in. Cc. and d be, as before, 16in.

$$x^2 - 16\,x = 80$$

$$x^2 - 16\,x + 64 = 80 + 64 = 144$$

$$x - 8 = \pm\ 12$$

$$x = +\ 12 + 8 = +\ 20, \text{ or } -\ 12 + 8 = -\ 4.$$

Therefore the lens is 20in. from the object and the image is 4in. on the negative side of the lens.

If the strength of the lens is expressed in diopters it is better to convert it into focal length for this calculation, but the two distances A and B can also be calculated by the following method, in which two numbers, whose sum and multiple are known have to be found

$$A + B = d, \qquad \text{and} \qquad A\,B = \frac{100\ d}{D}$$ [87

THE CONSTRUCTION OF IMAGES FORMED BY THIN LENSES.

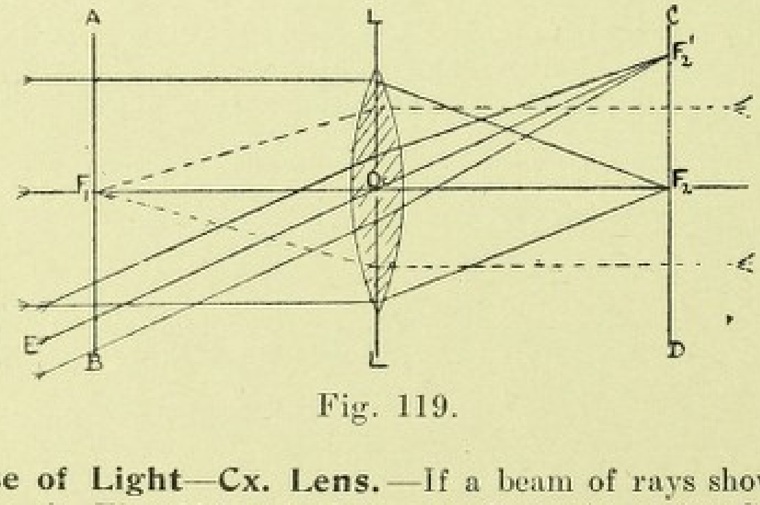


Fig. 119.

Course of Light—Cx. Lens.—If a beam of rays shown by the thick lines in Fig. 119 is incident on the surface of a Cx. lens in a direction parallel to the principal axis F_1 F_2 they are refracted to meet at the point F_2, the principal focus or second focal point, situated on the axis. A line C D drawn through this point perpendicular to the axis is the second focal plane. The distance from O, the optical centre of the lens, to F_1 or F_2 is the focal length of the lens. In the same way parallel rays which are incident on the other surface of the lens (shown by the dotted line) meet in a point at F_1, the first focal point. A line A B drawn through it perpendicular to the axis is the first focal plane. The distance O F_1 is equal to O F_2.

We have, therefore, a point and a plane on either side of the lens equi-distant from O. The plane L O L is the refracting plane of the lens.

Whatever course a ray takes in passing through a lens (or any number of lenses) if the light retraces its course, it follows the same path. It is clear, therefore, that if the source of light be at F_1 or F_2 the rays, after refraction, pass out of the lens parallel to the principal axis. All rays which diverge from a luminous point are refracted on passing through a lens, with the exception of the axial ray, which passes along the principal axis; this undergoes no refraction.

If, instead of the object point being on the principal axis, it is situated on a secondary axis E F'_2, as in Fig. 119, the rays are similarly bent so as to meet in a focus at F'_2, and any ray passing through O obliquely to F_1 F_2 is presumed to be not deviated by the lens.

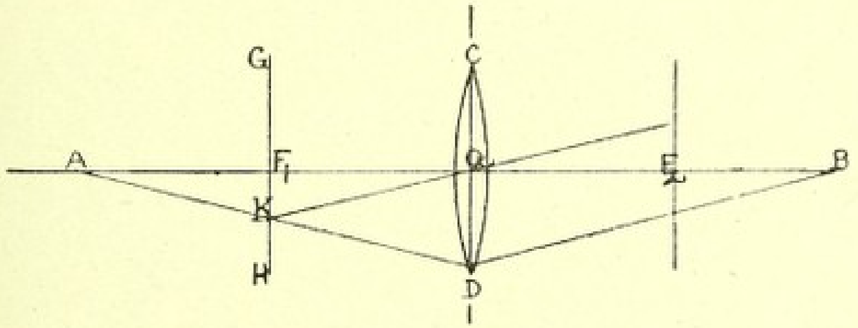


Fig. 120.

I of Point on the Axis—Cx. Lens.—To construct the image produced by a convex lens the object being a point A on the axis. (Fig. 120), the procedure is as follows:—

Draw the axis A B, and through F_1 draw the focal plane G H. From A draw any line A K D, cutting the first focal plane at K and the refracting plane of the lens at D. From K draw a line through the optical centre O and from D draw D B parallel to K O, this refracted ray D B cuts the principal axis at B, which is the image of the point A. This construction holds good because rays diverging from a point in the focal plane are parallel to each other after refraction.

Construction of I for Cx. Lens.—In order to construct the image of an object formed by a Cx. lens we have three rays diverging from any point, whose course after refraction it is easy to follow, viz. :—

(a) The ray which is parallel to the principal axis and which, after refraction, passes through F_2.

(b) The ray which passes through F_1 and which, after refraction, is parallel to the principal axis.

(c) The ray which passes through Oc, the optical centre, and whose course is not altered by refraction.

It is necessary to draw only two of these rays so as to locate the I of a point, since where any two rays diverging from a point meet, all other rays diverging from that same point also meet.

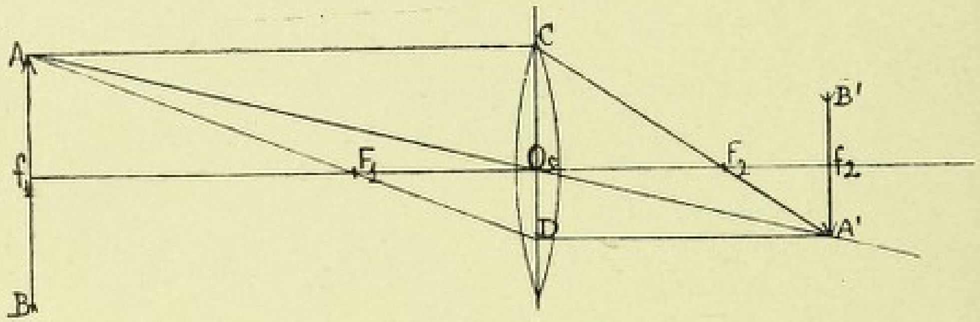


Fig. 121.

Real I.—In order to construct the complete I of an O, the images A′ and B′ of the two extreme points A and B should be located, and these suffice to show the location and size of the image (Fig. 121).

The construction is as follows:—

Draw the principal axis f_1 f_2 through Oc.

Draw from A the line A C parallel to the axis. This line when refracted passes through F_2.

Draw A D passing through F_1. This, after refraction, is parallel to the axis as D A′.

Lastly, draw A A′ passing straight through Oc.

These three lines meet at A′, the image of A. In the same way B′, the image of B, can be constructed. The images of all intermediate points between A and B could be constructed, but are not necessary, for B′ A′ shows the position and size of the real inverted image of the object A B.

Virtual I.—When the object is nearer the lens than F (Fig. 122) the construction is as follows:—

Draw the principal axis C D.

From A draw A E parallel to C D, and E F_2 is the course of the refracted ray.

Draw A Oc passing through the optical centre.

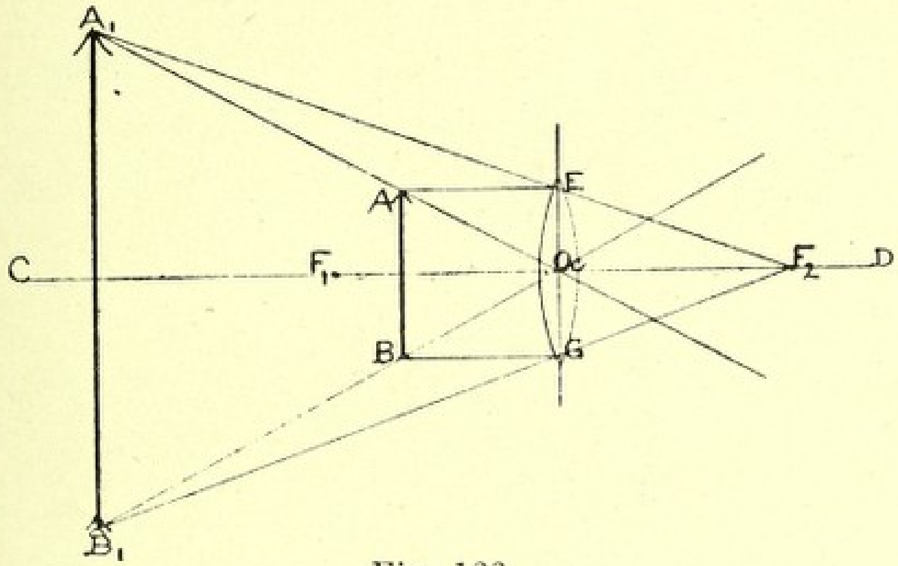


Fig. 122.

Since these rays are divergent, after refraction, no real image can be obtained, but by producing them backwards they are made to meet at A′, which is the virtual image of A. Similar lines drawn from B locates its image as B′ and A′ B′ is the complete virtual image of the object A B.

I AT ∞.—When the object is at F, the rays, after refraction, are parallel to their axes, and, therefore, no image can be constructed, since it lies at infinity.

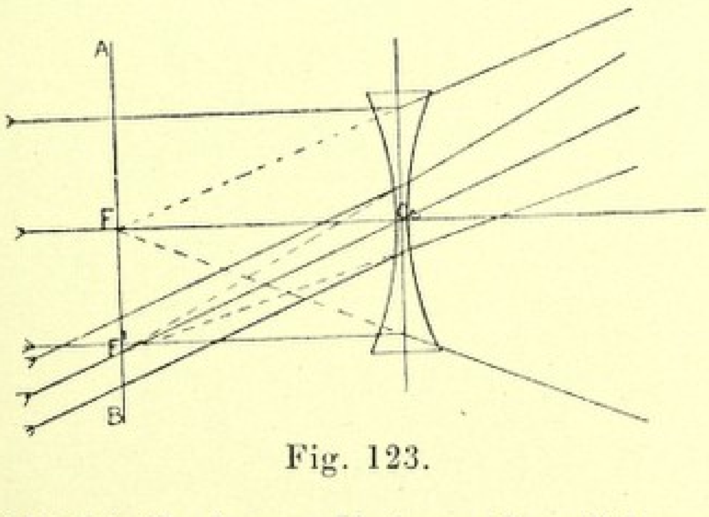


Fig. 123.

Course of Light—Cc. Lens.—If a beam of parallel rays (Fig. 123) is incident on the surface of a Cc. lens they apparently diverge, after refraction, from F, and a plane A B perpendicular to the axis passing through F is the focal plane. Oc F is the focal distance.

Every ray passing through the lens is refracted, except that passing along the principal axis or a secondary axis. A point on any axis has its image on that same axis.

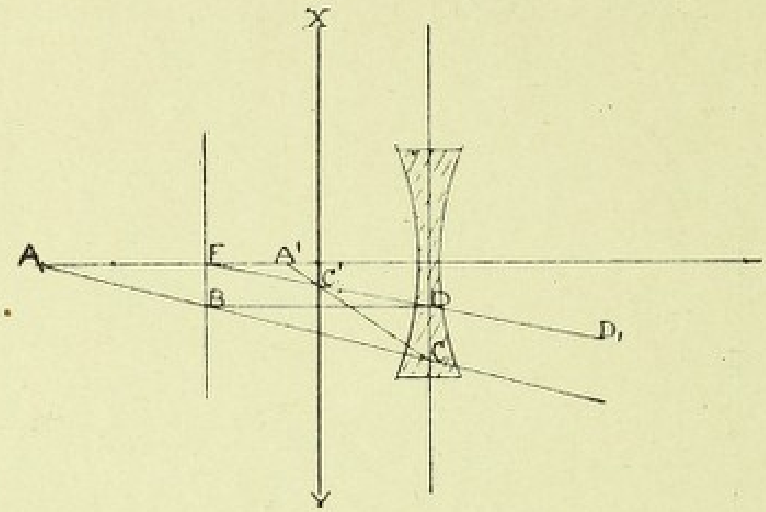


Fig. 124.

I of Point on the Axis—Cc. Lens.—To construct the image produced by a concave lens, the object being a point A on the axis.

Draw any ray A B C cutting the focal plane in B and the refracting plane in C (Fig. 124). From B draw B D parallel to the principal axis, such a ray, after refraction, diverges as F C′ D D, from F. Mark X Y, a plane midway between the focal and refracting planes. Then draw C C′ A′ cutting F D at C′ in the plane X Y and where this line cuts the principal axis at A′ is the image of the point A. This construction holds good because if there were rays B C, B D diverging from B in the focal plane, they would meet after refraction at C′ in the plane X Y midway between the focal and refracting planes.

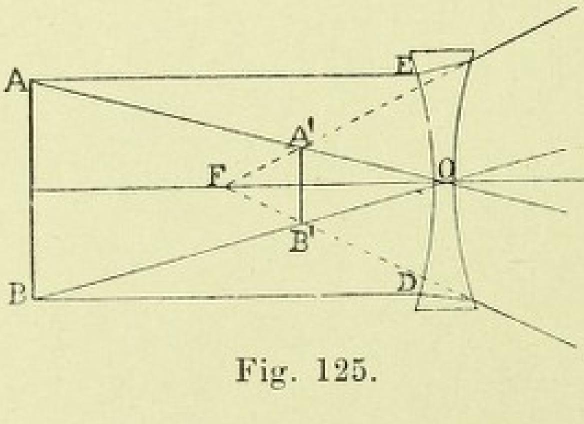


Fig. 125.

Construction of I for Cc. Lens.—The construction of the image formed by a concave lens is the same wherever the object is situated, since the image is always formed on the same side as the object, and between the principal focus and the lens. In Fig. 125, let A B be an object placed in front of a concave lens of which F is the principal focus.

From A trace A O, the axial ray.

Draw A E parallel to the axis before refraction and diverging as if from F, after refraction.

These rays, being divergent, can only meet by being prolonged back when they meet at A′. Similar rays from B meet at B′ its image. The complete image of A B is A′ B′.

MAGNIFYING POWER OF LENSES.

The magnification of an object may be considered in two ways. Firstly, its absolute magnification, and secondly, its apparent or relative magnification.

The former is expressed by the ratio between the angle subtended by the image at the eye and the angle subtended by the object at the eye. Thus, if the former angle is θ_1, and the latter θ_2 then the magnification $= \theta_1/\theta_2$.

It may also be defined as the ratio between the size of the image and the size of the object when both are compared at the same distance from the eye. This is the true magnification applicable to all optical instruments and is independent of the distance of the near point of vision of the observer's eye.

The apparent magnification is the size of the object compared with that of the image observed at the point of most distinct vision. It is this latter with which this article has to deal.

Magnification, as before mentioned, is expressed by increase in diameter, the superficial area which is the true magnification is found by squaring the linear. When X3 is written, it implies that the length of image is three times the original object. While if X$\frac{1}{3}$ it means that the length of the image is one-third of the object.

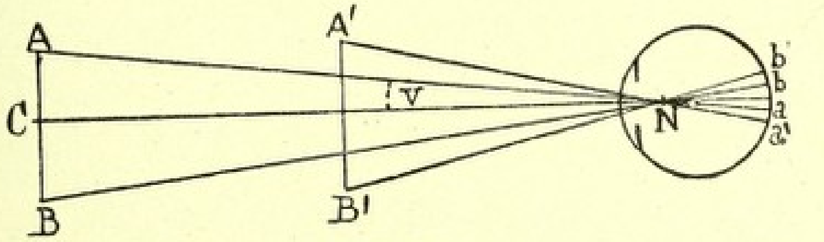


Fig. 126.

The nearer the object is to the eye the larger is the visual magnitude, since the secondary axial rays from the extremities of an object enter the eye under a larger angle and crossing at N (Fig. 126), the nodal point of the eye, cover a larger area on the retina. Thus, if the object A B be approached to half the distance, as at A′ B′, the angles A′ N B′ will be double the angle A N B, and consequently the diameter of the image on the retina a′ b′ will be approximately double that of a b. This is the apparent size of the object and is determined by tan V, where V is the angle subtended by half the object A C at N. Since tan V = AC/CN it is obvious that V is increased by reducing the distance C N.

This is illustrated again in Fig. 127, where the object A B is shown in three positions with its corresponding images.

For the same reason a convex lens causes magnification because it allows the object to be seen at a nearer distance to the eye than can be done without the lens. A pinhole held close to the eye also acts as a magnifier and the smaller the hole, the nearer the object can be approached, without losing definition, and consequently the higher the magnification obtained. This is not, however, a substitute for a lens, since a small hole cuts off an immense amount of light, and causes blurring by obstruction of the light-waves at its margin.

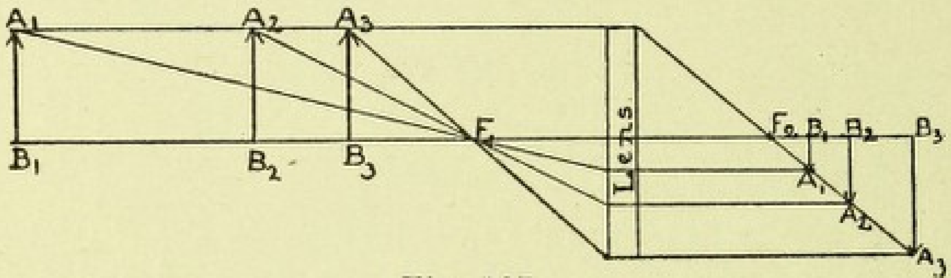


Fig. 127.

If a watchmaker fixes a 2-inch lens in front of his eye he sees an object five times as large as he would without it. If he were able to see distinctly at two inches, without a lens, the object would appear almost as large. Light from an object at two inches is, however, so divergent that it cannot be focussed on the retina by the unaided eye.

It may be here noted that the size of the image is governed simply by the distance, from the axis, of that ray which, before refraction, passes through F_1 and, after refraction, is parallel to the axis.

Magnification is rather an elastic term and more difficult to estimate in relation to the eye than it is when the image is thrown on to a screen. In the latter case the image can be directly measured, and divided by the actual length of the object, the result being the true linear magnification. But in the case of the eye the solution is not so simple. Firstly, the nearest distance at which objects can be distinctly seen, by the naked eye, varies greatly with different people, and at different ages in the same person; the retinal image of the object also varies both with accommodation and with the shape of the eyeball. In the case of a myope, the image being formed in front of the retina, much diffusion results, which, although it renders the object indistinct, makes it appear larger, and this apparent size increases with the brightness of the object.

Again, as will be shown, the magnification varies with the distance of the lens both from the eye and from the object.

And lastly, when looking at any near object, either with the naked eye or through a spectacle lens, there is a mental effect which, although it does not modify the actual size of the image on the retina, does its apparent relative size to the mind. The images of what is seen are virtual (aerial) images of objects projected into space and, as a rule, coinciding in position and magnitude with the objects seen. Thus, if several persons were asked to look at the moon with the naked eye or through a telescope, and then requested to draw its apparent size, they would probably all draw a different sized figure.

When an object is to be seen to the best advantage through a loupe or magnifying lens, the object should be placed a little within its focus. Firstly, if the object be so held, the image is better defined, since if the lens is a strong one everything is out of focus except that part of the object which lies close to the principal axis.

Secondly, it is almost impossible to view a near object without accommodation, therefore if the light is slightly divergent, when incident at the eye, the accommodation insensibly exerted, forms the image on the retina.

If the distance between the object and the lens is short, magnification is small and if the distance = 0, that is if the lens touches the object, M = 1, the object and image being equal in size. The greatest magnification results when the distance is nearly equal to F, but the actual distance at which the object is placed is found by the formula

$$1/f_1 = 1/F - (-1/d) \qquad [88]$$

where 1/F represents the power of the lens and d is the distance to which the image is projected, and this (and therefore also f, which is the distance of the object), varies with the observing eye.

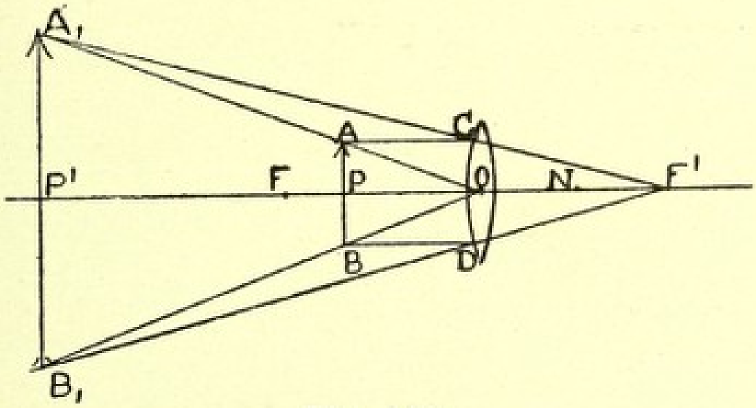


Fig. 128.

To find the magnification of a simple magnifying glass (loupe), let the principal focal distance of the lens be O F′ or O F (Fig. 128) A′ B′ is the image plane since it is the plane of most distinct vision with the unaided eye. Let A B be the object plane. The magnifying power M will then be the ratio

$$\frac{I}{O} = \frac{A'B'}{AB} = \frac{P'O}{PO} = \frac{f_2}{f_1}$$

f_1 and f_2 being the two conjugates ; and

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

Since the image is virtual f_2 is negative and is the distance of most distinct vision. It may be expressed by d; therefore f_1 the distance of the object A B, is

$$\frac{1}{F} - \left(-\frac{1}{d}\right) = \frac{1}{f_1} \quad \text{or} \quad \frac{F\,d}{F + d} = f_1$$

and

$$M = \frac{I}{O} = \frac{d}{f_1} = \frac{d}{Fd/(F+d)} = \frac{d\,(F + d)}{F\,d}$$

or

$$M = \frac{F + d}{F} \quad \text{or} \quad \frac{d}{F} + 1.$$ [8

This formula is that usually accepted to express the lens magnifying power when the lens is close to the eye or at least when the interval is neglected. And since 10 inches is the distance commonly adopted as that of most distinct vision, for the average normal eye, the formula is generally written

$$M = 1 + 10/F.$$ [9

This formula becomes, for lenses expressed in diopters,

$$M = 1 + D/4.$$ [9

Thus with a 2-inch Convex lens $M = 1 + 10/2 = 6$

Or with a + 20 D lens $M = 1 + 20/4 = 6$

When a lens is very strong the formula may be simplified to

$$M = 10/F.$$ [92

Thus with a $\frac{1}{4}$-inch lens $M = 10/\frac{1}{4} = 40$ times instead of $10/\frac{1}{4} + 1 = 41$ times.

Magnification is the ratio between a b and A′ B′ (Fig. 129), a b being the object as it would be seen in the plane of most distinct vision by the eye unaided, and A′ B′ being the image of A B, projected to that plane. The position of the object is the distance which is the conjugate of d. But since d is taken from the eye itself the distance between the lens and the eye must also be taken into account.

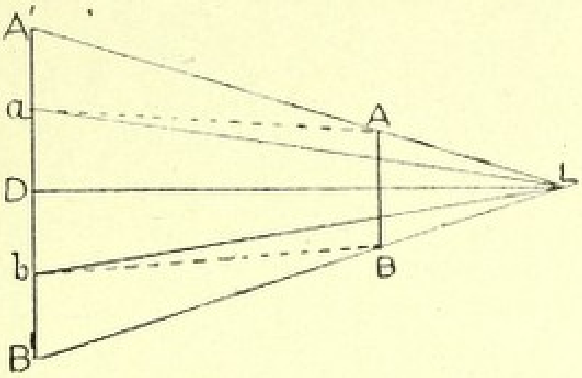


Fig. 129.

Let d be the distance of distinct vision and e the distance of the lens from the nodal point of the eye; then

$$M = \frac{F + d - e}{F} \quad \text{or} \quad \frac{d - e}{F} + 1$$ [93]

Suppose the eye be two inches from a 2in. Cx. lens; then

$M = 1 + (10-2)/2 = 5$ times, instead of $M = 1 + 10/2 = 6$ times.

It is evident, therefore, that as e increases, the magnification decreases, and the magnification must consequently be at a maximum when the eye is close up to the lens, i.e., when $e = 0$.

If d be 10 inches and the lens of F = 2in. be held close to the eye, the object is at $1/2 - (-1/10) = 6/10$ at 1.666in. from the lens and also from the eye,

so $M = 10/1.666 = 6$ times.

But if the lens is 2in. from the eye, the conjugate is at 10in. from the eye, or 8in. from the lens, so that the object must be $1/2 - (-1/8) = 5/8$ at 1.6in. from the lens,

and $M = 8/1.6 = 5$ times.

Magnification being a ratio between the distance at which an object is placed and the distance of most distinct vision, it follows that the magnifying power of any given lens is smaller for a myopic eye, whose point of distinct vision is shorter than 10 inches, while it is greater for the hypermetropic eye, whose position of most acute vision is greater than 10in. The magnifying power of a 2in. lens is:—

For an emmetrope, where d = 10 inches, $M = 1 + 10/2 = 6$.

For a hypermetrope, where d = 16 inches, $M = 1 + 16/2 = 9$.

For a myope, where d = 6 inches, $M = 1 + 6/2 = 4$.

Yet if the distance of most acute vision varies, so also does the distance at which the object is held, in front of the magnifying lens, in order to be most distinctly seen. For the myopic eye the object would be held nearer, so that the light may be more divergent after refraction, while for the hypermetropic eye it would be held further away, so that the light may be less divergent, or the object may be even placed beyond the fcoal distance of the lens in order that the light be convergent after refraction. To the myopic eye a given convex lens acts as a stronger one, while to the hypermetropic eye it acts as a weaker one would to an emmetropic eye. Thus the magnifying power of a lens is, for all eyes, more nearly equal than would appear from the previous paragraph.

If the object is placed exactly in the focal plane of the lens, the light after refraction is parallel and is focussed at the retina by a hypermetropic or emmetropic eye. If the image is projected to the distance of most distinct vision, magnification obtains. In this case there would be no change in M, as e is increased or decreased beyond that which results from the altered distance of the object from the eye.

When a lens is in front of the eye, and not used as a magnifier, it has another effect which is perhaps best demonstrated when the lens is fairly weak. When a Cx. lens is within F_1 (the anterior focus of the eye) it causes diminution of the retinal image; at F_1 it causes neither M nor diminution; beyond F_1 it causes increase which is greater as its distance is greater, until when the lens is midway between F_1 and the object, M is at a maximum; beyond this point the M becomes less until, when the lens touches the object, it is 0. This M is independent of the clearness of the image and explains some of the phenomena of spectacle lenses. With a Cc. lens the retinal image increases when the lens is within F_1; there is no effect at F_1, while beyond F_1 the diminution of the retinal image becomes greater, to reach a maximum at the midway distance; after this the diminution becomes less, and it is 0 when lens and object are in contact.

Another factor which operates in causing magnification by a Cx. lens is that when the latter is in front of the eye the united nodal point is further forward, so that the size of the retinal image is increased. This is the reason that magnification occurs with a Cx. lens, although a person can see quite distinctly without it at a certain distance. Also since the Cx. lens suppresses accommodation the object is conceived to be more distant, and therefore for a given sized retinal image to be larger in size. Finally, even if the object be in the plane of most distinct vision, the interposition of a Cx. lens between it and the eye causes the light to be rendered less divergent, as if proceeding from a more distant and larger object.

CYLINDRICAL LENSES.

The Cylinder.—A cylinder is a body (Fig. 130) which is generated by a revolution of a rectangle about one of its sides as an axis. Such a body consists of two flat circular ends and an intermediate convex surface.

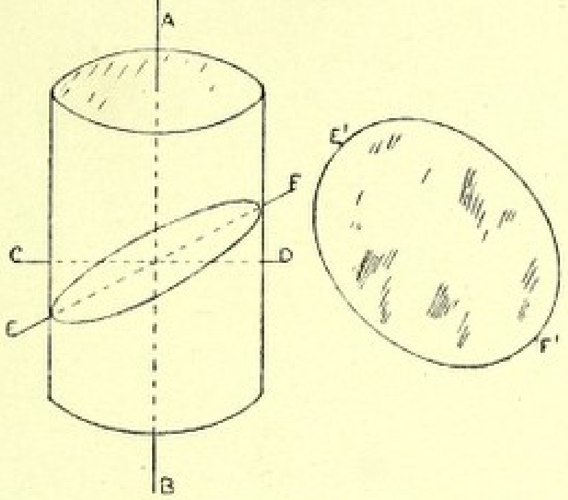


Fig. 130.

Any section of the cylinder taken at right angles to its axis is a circle whose centre coincides with the centre of the axis of the cylinder.

The cylinder possesses no curvature in any line parallel to the axis A B. At right angles to the axis, in any line parallel to the direction C D, the curvature of the cylinder has its maximum value. In any other direction, as E F, the boundary is an elipse, of which E′ F′ is an example. The curvature is always less than that of the circle C D, diminishing as the direction departs from C D and approaches that of A B.

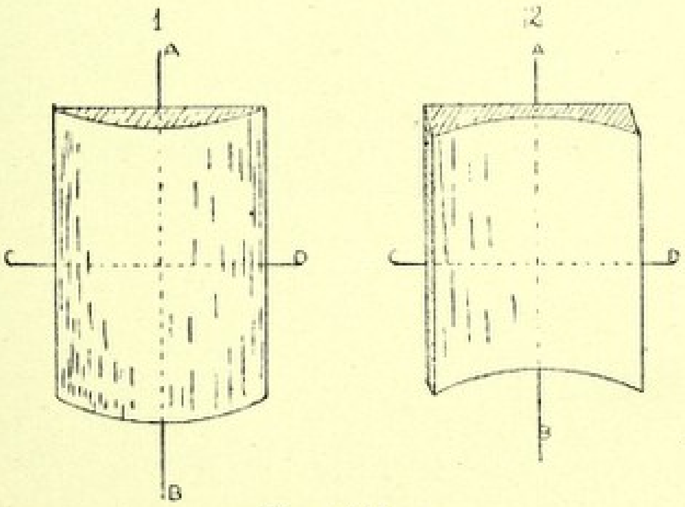


Fig. 131.

Fig. 131 (1) shows a convex cylindrical lens. It is a segment of a cylinder on the one side and has a plane surface on the other; it is formed by a cylinder and a plane which intersect each other.

The concave cylindrical lens (Fig. 131, 2) has a hollow surface on one side. It is formed by a cylinder and a plane which do not intersect each other.

A convex cylindrical lens may be conceived as formed of a series of prisms whose bases meet along a central line and whose apices are outwards. In the same way a concave cylindrical may be considered to be formed of prisms whose apices meet along a central line and whose bases are outwards.

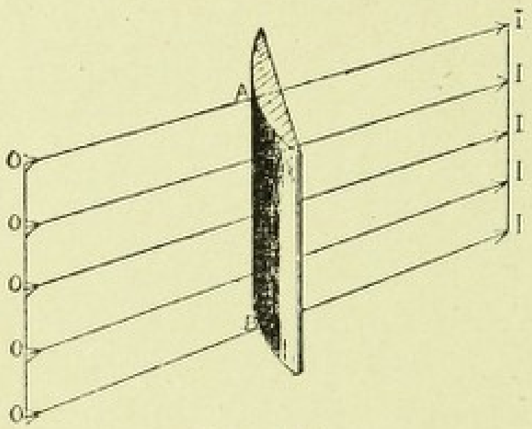


Fig. 132.

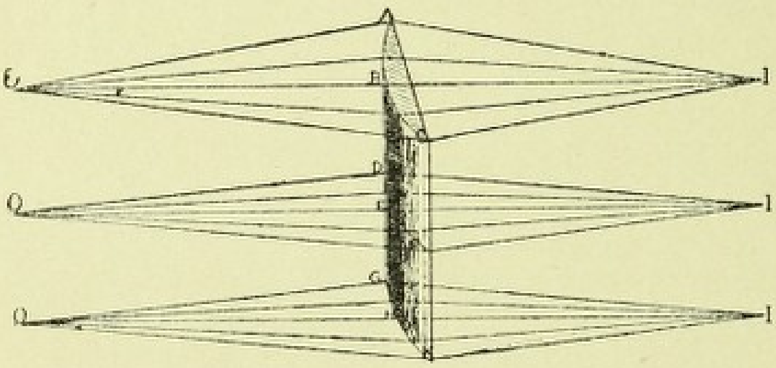


Fig. 133.

Curvature and Power.—Since in the direction of its axis A B (Fig. 132) a cylindrical lens has no curvature, it has in that direction no refractive powers. Fig. 133 shows that such a lens has its greatest power in the meridian of greatest curvature A B C, D E F, G H K, at right angles to the axis, and the dioptral number of the lens is that which results from the curvature of the meridian of greatest refraction.

Meridian.—The term meridian in connection with lenses signifies a plane passing through the geometrical centre of a lens, as shown in Fig. 134.

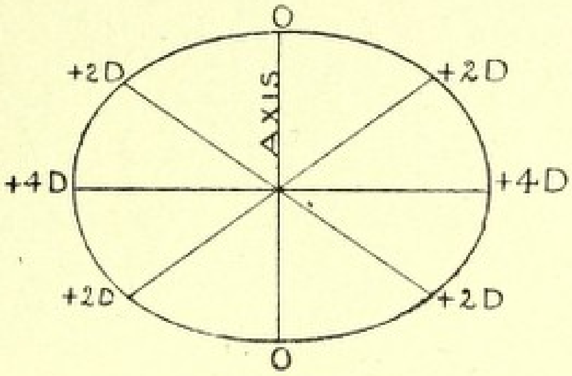


Fig. 134.

Spherical and Cylindrical.—A spherical lens has equal curvature and therefore similar refractivity in every meridian, but in a cylindrical it is otherwise. For example, a + 4 D cylindrical lens has 4 D positive power in the meridian at right angles to the axis, no power in the meridian corresponding to its axis, and in the intermediate meridians a power which varies between 0 and + 4. The meridian of no refraction—i.e., the axis—and the meridian of greatest refraction at right angles to the axis are termed the two *principal meridians*, and these alone need be considered in any cylindrical lens. The position of the cylindrical is indicated by the direction in which its axis is placed, while its power is expressed by the maximum refractivity found in the meridian at right angles to the axis. By means of these two meridians the path of all rays passing through the cylindrical may be traced. This is the reason why they are called the principal meridians.

The Focal Lines.—It is only the meridian at right angles to the axis which can form a focus, for a ray proceeding from a point and meeting the surface in an intermediate meridian cannot meet the other rays which pass through that same meridian. All the light, from an object point, refracted by a cylindrical passes through two focal lines, one of which is at the focal distance of the meridian of the greatest refraction and the other is at ∞. The cylindrical lens has therefore two focal distances, and the image of a point is not a point, as with the spherical, but two lines. But since the one focus is at ∞, it need not be considered so that we can say that the image of a point is a line.

The refraction of a cylindrical lens is shown in Fig. 133, where A B C, D E F, G H K represent arcs of greatest curvature corresponding to one of the principal meridians, and B E H represents the axis of the lens.

None of the rays diverging from O passing through the planes A B C, etc., will be deviated up or down, but form flat planes of light which come to a focus, as with an ordinary spherical lens of the same power, at I, I, I, which are conjugates to O, O, O. And what is true for the three planes shown is also true for any number of planes parallel to A B C, etc. Thus if the object be a point of light the image will be a row of focal points along the vertical line I, I, I, which fuse into a thin streak of light parallel to the axis of the cylindrical, and this is called the *focal line*. If the cylindrical be rotated around its central line the streak also will be rotated with it. This focal line is situated at the focal distance of the meridian of greatest power, which in this case is the horizontal meridian. Thus, if the above lens were a + 5 D Cyl. the focal line would be at 20 cm. At any other distance the streak would broaden out into a band of light.

Each luminous point on the object given rise to its focal line, so that the complete image of an object is a narrow band of light, which is commonly referred to as the focal line. The band is narrower, at the focal distance, as the focal length is shorter.

Two Cylindricals.—If another cylindrical of similar power be placed in contact with that shown in Fig. 133, and with its axis in the same direction, the image will be unaltered, but its position will be brought to a plane half way between B and I.

If the second cylindrical be rotated until the axis is at right angles to the first, the result will be equivalent to a spherical lens of the same power as the single cylindrical. In this case the greatest power of the one corresponds with the axis of the other, and in all intermediate meridians any deficiency of power in the one is supplied by the other. So that if the second cylindrical be rotated from axis vertical to axis horizontal the vertical streak will be seen to shrink until, when the two axes are at right angles, it will have dwindled to a point of light.

Instead of placing a second cylindrical in contact, the curvature representing the second may be ground on to the second surface of the first lens and the optical effect is the same.

Sphero-Cylindricals. — If a spherical is combined with a cylindrical, the curvature of the former is ground on the one side of the lens, and that of the latter on the other. Such a combination is called a sphero-cylindrical or compound cylindrical in contra-distinction to a plano-cylindrical or simple cylindrical.

Cross Cylindricals.—If one cylindrical power is ground on the one side and another on the other side the lens is termed a cross-cylindrical. But any two cylindrical powers whether ground on opposite sides of a piece of glass, or whether as two plano-cylindricals placed in contact, are always equivalent to some sphero-cylindrical combination, no matter what may be the inclination of their axes, except only in the case of the axes being parallel, when they constitute a plano-cyl; or when they are at right angles and the two powers are similar they are equal to a spherical.

Powers of Sphero-Cyl.—Since there is no curvature and consequently no refractive power along the axis of the cylindrical, only the power of the spherical exists there, whereas at right angles to the axis there is the united power of the spherical and the cylindrical. As with the plano-cylindrical, these are the two principal meridians of the combination, and which alone need be considered in practice. Thus, with + 2 Sph. ⊂ + 3 Cyl. Ax. 90°, the two principal meridians are 90° and 180°, and the two principal powers are

+ 2 D at 90° and (+ 2 D + 3 D) = + 5 D at 180°.

A combination of — 2.5 Sph. ⊂ — 0.75 Cyl. Ax. 70° has

— 2.5 D power at 70° and — 3.25 D at 160°.

Or one power might be positive and the other negative, thus + 2.25 Sph. ⊂ — 3.50 Cyl. Ax. 135° has

+ 2.25 D at 135° and (+ 2.25 — 3.50 =) — 1.25 D at 45°.

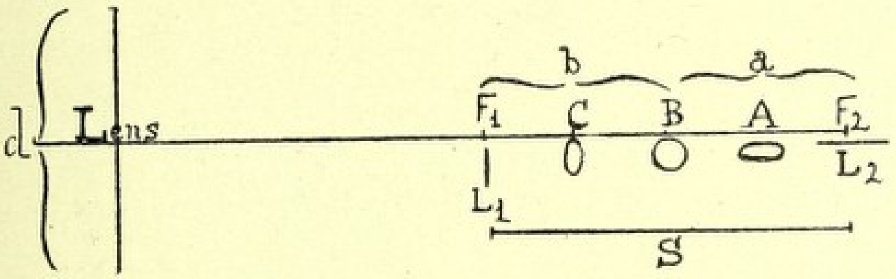


Fig. 135.

Focal Lines.—If a Cx. Sph.-Cyl., say, + 4 Sph. ⊂ + 4 Cyl. (Fig. 135), having its axis vertical, be held in front of a screen at the distance 25 cm., which is equal to F of the spherical, a horizontal line of light is formed at F_2; as the screen is slowly approached to the lens this line develops gradually into a horizontal oblate oval at A, an almost perfect circle at B, a vertical prolate oval at C, and finally into a bright vertical line at F_1. The lens is then at 12.5 cm., which is F of the combined spherical and cylindrical powers. The space between the two principal focal distances F_1 and F_2, represented by the two sharp lines, is termed the *interval of Sturm.*

The diffusion patches formed in the interval of Sturm are due to the different refracting powers of the various meridians of the lens, but all the light refracted by the lens passes through the two so-called focal lines.

The cone of light emergent from the lens is more or less elliptical, except at B, where the horizontal line of light formed at F_2 and the vertical line formed at F_1 are broadened out to equal extents, and the elliptical cones of light, emerging from the lens, are more nearly of equal lengths than at any other distance, so that a circle is there formed.

Interval of Sturm.—The two focal lines are at the focal distances of the two principal meridians, and their lengths are proportional to the diameter of the lens and to their distances from the lens. If the lengths are L_1 and L_2 and the focal distances F_1 and F_2

$$L_1 F_2 = L_2 F_1$$

If d is the diameter of the lens and S the length of interval of Sturm, that is, the distances between F_1 and F_2 (Fig. 135),

$$L_1 = \frac{d\ S}{F_2} \text{ and } L_2 = \frac{d\ S}{F_1}$$

The circular disc of confusion at B divides the interval of Sturm at a point which is distant from L_1 and L_2 proportional to their distances from the lens, so that if the two parts be called a and b,

$$S = a + b \text{ and } \frac{a}{b} = \frac{L_1}{L_2} = \frac{F_1}{F_2}$$

Then the circular disc of confusion is distant from L_1 and L_2,

$$a = \frac{S\ F_1}{F_2 + F_1} \text{ and } b = \frac{S\ F_2}{F_2 + F_1}$$

The size of the circular disc of confusion B depends on the diameter of the lens, and is

$$B = \frac{a\ L_2}{S} = \frac{b\ L_1}{S}$$

Example.—Thus, with + 4 Sph. ⊂⊃ + 2 Cyl. Ax. 90 the two principal powers are + 4 D and + 6 D, and let d = 5 cm. Then

$$F_1 = 16.66 \text{ cm. and } F_2 = 25 \text{ cm.}$$

and $$S = 25 - 16.66 = 8.33 \text{ cm.}$$

Length of the 1st focal line $L_1 = \frac{5 \times 8.33}{25} = 1.66$ cm.

Length of the 2nd focal line $L_2 = \frac{5 \times 8.33}{16.66} = 2.5$ cm.

$$L_1 F_2 = L_2 F_1 = 1.66 \times 25 = 2.5 \times 16.66.$$

$$\frac{a}{b} = \frac{1.66}{2.5} = \frac{16.66}{25} = \frac{3.33}{5}$$

Distance of B from $L_1 = a = \frac{8.33 \times 16.6}{25 + 16.66} = 3.33$ cm.

$$\text{Distance of B from } L_2 = b = \frac{8.33 \times 25}{25 + 16.66} = 5 \text{ cm.}$$

$$\text{Length of S} = a + b = 3.33 + 5 = 8.33 \text{ cm.}$$

$$\text{Size of B} = \frac{3.33 \times 2.5}{8.33} = \frac{5 \times 1.66}{8.33} = 1 \text{ cm.}$$

THE CYLINDRICAL EFFECT OF OBLIQUE SPHERICALS.

Since a spherical lens acts with an astigmatic effect on an oblique pencil of light, a spherical held obliquely to the incident light acts as if it were a sphero-cylindrical lens. Suppose a spherical lens be held at its focal distance from a screen and parallel to it, the light from a flat luminous object forms on the screen a complete image, but if the lens be held obliquely the image is confused and drawn out as if a cylindrical had been added to the spherical. Two bright focal lines are formed on the screen when the lens is held at the proper distance for each.

The effect produced by obliquity of a spherical is that of a slightly stronger spherical combined with a cylindrical whose axis corresponds to the axis of rotation. The refraction is therefore increased in both meridians, but mostly in that at right angles to the axis of rotation. It is this increase of power which enables some people to see better by looking obliquely through their glasses.

Rotation of a cylindrical lens around its axis causes similar increased effect in the meridian of greatest power.

When a spherical lens is held upright and parallel to a screen a luminous point on a level with the axis will, when the lens is rotated round a vertical axis, no longer form a focus on the screen, but will form two ill-defined astigmatic lines, one horizontal, a little within the original focus, and a second, vertical, considerably nearer the lens.

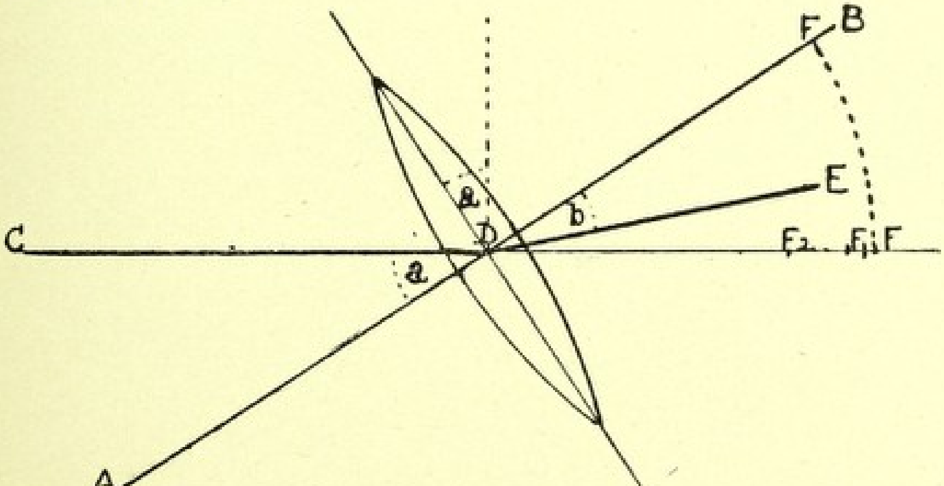


Fig. 136.

If a lens be tilted round a horizontal axis a similar effect is produced, which will be calculated. In Fig. 136 let a represent the angle of rotation of the lens, and therefore also the angle of incidence of the central ray of a beam of light parallel to the principal axis before the lens was rotated. The increase of power in the meridian of the rotation is owing, as can be seen in the figure, to the fact that the light has to pass through a greater thickness of the lens when the latter is oblique than when it is placed normally.

b is the angle of refraction at the refracting plane (the lens being considered thin).

Let F represent the focal length and D the dioptric power of the lens. Let F_1 and F_2 represent the focal lengths, and D_1 and D_2 the dioptric powers of the meridians of least and greatest effect respectively.

A B is therefore the principal axis of the lens. C D is the central ray of an incident beam of light which, had the lens been normally placed, would have been the principal axial ray. D E is the ray after refraction. A D C = a, the angle of incidence, and B D E = b, the angle of refraction. Now b is found from the equation $\mu \sin b = \sin a$. Then

$$F_1 = \frac{F(\mu - 1)}{(\mu \cos b) - \cos a} \qquad [9$$

$$F_2 = \frac{F(\mu - 1)\cos^2 a}{(\mu \cos b) - \cos a} = F_1 \cos^2 a \qquad [9$$

Also
$$\frac{1}{F} = D_1 = \frac{D\left\{(\mu \cos b) - \cos a\right\}}{\mu - 1} \qquad [9$$

And
$$\frac{1}{F_2} = D_2 = \frac{D\left\{(\mu \cos b) - \cos a\right\}}{(\mu - 1)\cos^2 a} = \frac{D_1}{\cos^2 a} \qquad [9$$

Hence
$$\frac{F_2}{F_1} \text{ or } \frac{D_1}{D_2} = \cos^2 a$$

Since $\dfrac{\mu - 1}{\mu \cos b - \cos a}$ does not differ greatly from

$$\frac{2\mu - \sin^2 a}{2\mu}$$

the following simplified formulæ can be used, in place of the foregoing, for approximate calculations:—

$$F_1 = F \frac{2\mu - \sin^2 a}{2\,\mu} \qquad \text{and } F_2 = F_1 \cos^2 a \qquad \begin{matrix}[98] \\ [99]\end{matrix}$$

$$D_1 = \frac{D\,2\mu}{2\,\mu - \sin^2 a} \qquad \text{and } D_2 = \frac{D_1}{\cos^2 a} \qquad \begin{matrix}[100] \\ [101]\end{matrix}$$

The effective power of a cylindrical rotated around its axis is found for the same formulæ as for D_2.

Example.—Let a spherical lens of $\mu = 1.5$ and 100 cm. focal length be rotated 30°. Then $a = 30°$, $\sin^2 a = .25$ and $\cos^2 a = .75$.

$$F_1 = \frac{100 \times (3 - .25)}{3} = 91.66 \text{ cm.}$$

$$F_2 = 91.66 \times .75 = 68.74 \text{ cm.}$$

The lens has now two focal distances of about 92 and 69 cm.

Let an inclination of 25°, around its horizontal meridian, be given to a + 5 D Sph. of $\mu = 1.5$,

$$\sin^2 25° = .1784 \text{ and } \cos^2 a = .8216,$$

$$D_1 = \frac{5 \times 3}{3 - .1784} = 5.32 \text{ D},$$

$$D_2 = \frac{5.32}{.8216} = 6.46 \text{ D}.$$

or + 5.32 Sph. ⊂ + 1.14 Cyl. Ax. 180.

If the source of light is distant and the two focal distances be measured, the angle of rotation of the lens can be found from the equations

$$F_2/F_1 \text{ or } D_1/D_2 = \cos^2 a. \qquad [102]$$

Since for a rough approximation D_1 does not vary greatly from D, the increased or cylindrical effect produced by obliquity of a spherical lens is

$$C = \frac{D}{\cos^2 a} - D \text{ or } D \tan^2 a. \qquad [103]$$

Thus in the above example where $a = 25°$ and $D = 5$,

$$\frac{5}{.82138} = 6.09 - 5 = 1.09 \text{ D}.$$

Table of Cylindrical Effect of Oblique Sphericals.

The following table gives the approximate cylindrical effect obtained by rotating a 1 D lens. The effect on other lenses is proportionate. The rotation is supposed to be around a vertical axis. The calculations are made from the simplified formulæ for D_1 and D_2 in the last page.

Degree of obliquity.	Horizontal focus in cm.	Vertical focus in cm.	The Power of the 1 D becomes in diopters.
5°	99	100	1.00 S ◯ 0.01 C
10°	96	99	1.01 S ◯ 0.03 C
15°	91	98	1.02 S ◯ 0.07 C
20°	83	96	1.04 S ◯ 0.16 C
25°	77	94	1.06 S ◯ 0.23 C
30°	70	91	1.09 S ◯ 0.34 C
35°	59	88	1.13 S ◯ 0.57 C
40°	50	86	1.16 S ◯ 0.84 C
45°	42	83	1.20 S ◯ 1.20 C

The effect increases rapidly with a greater obliquity.

A 1 D cylindrical rotated around its axis becomes

Degree of obliquity.	Diopters.
5°	1.01 C
10°	1.04 C
15°	1.09 C
20°	1.20 C
25°	1.29 C
30°	1.43 C
35°	1.70 C
40°	2.00 C
45°	2.40 C

CHAPTER VII.

THE ANALYSIS AND NEUTRALISATION OF LENSES.

Characteristics of a convex or positive lens.

(a) It is thicker at the centre than the edge.

(b) It gives a magnified image of an object held within the focus.

(c) If held at the proper distance in front of a screen it forms an inverted real image of a sufficiently luminous object as a candle flame or window.

(d) If an object distant a few feet be looked at through the lens, and the latter moved vertically or horizontally, the object appears to move in the *contrary direction*.

Characteristics of a concave or negative lens.

(a) It is thinner at the centre than at the edge.

(b) It diminishes the apparent size of an object seen through it.

(c) No image can be projected by it on a screen.

(d) When moved vertically or horizontally, an object seen through it appears to move in the *same direction*.

Definitions.—The analysis of a lens consists of determining its nature; neutralisation is the determination of its refracting power or focal length by means of a lens of opposite nature, which converts the former into a lens of no power. When analysing or neutralising a lens, care should be taken that it is held so that the front is facing the observer. The front is that surface which is the more remote from the wearer's eye, and is the less concave or more convex of the two surfaces.

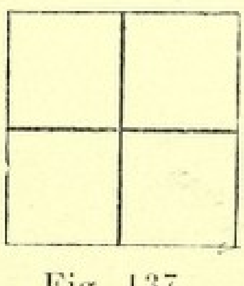


Fig. 137.

Analysing Card.—The work is generally facilitated by the use of an analysing card, as shown in Fig. 137, although in its absence any clearly-defined straight vertical or horizontal line, as the sash of a window, serves the purpose.

The card should be 18 or 20 inches square, with two crossed black lines about ¼ inch in width, running vertically and horizontally, and for most work should be fixed at a distance of not less than three or four feet.

The first step in testing a spectacle lens is to learn whether or not it contains a cylindrical element.

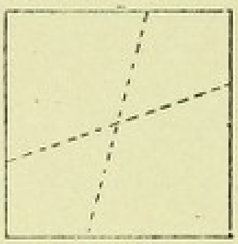


Fig. 138.

Determination of Cylindrical element.—To distinguish between a lens having a spherical power only from one having a cylindrical element, the analysing card is viewed through the lens and the latter turned around its geometrical centre in a plane parallel to the card. If the lens is spherical the lines remain unmoved because its refractive power is alike in all meridians. If the lens has a cylindrical element the lines become oblique, as shown in Fig. 138, where the dotted lines represent the black lines of the card, as seen when the lens is rotated. This obliquity occurs because the refractive power of the lens varies in its different meridians.

Determination of Nature of power.—If the lens has only spherical power, the next necessary step is to learn whether it is convex or concave. This is done by moving the lens horizontally while observing the vertical line, or vertically while observing the horizontal line.

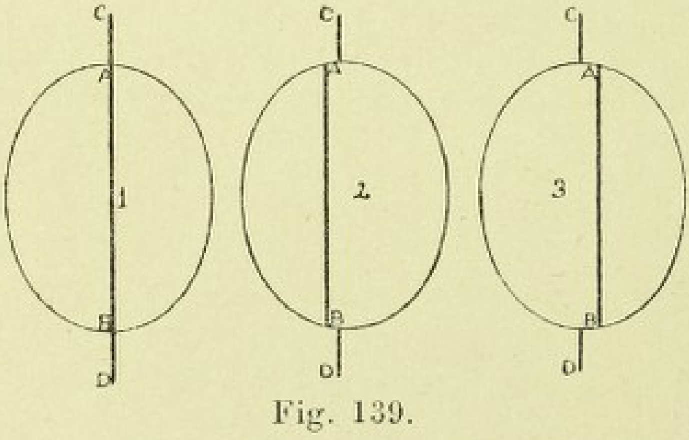


Fig. 139.

When the vertical line is first viewed through the centre of the lens the part A B seen through the glass is continuous with the parts C and D seen beyond its edges (1). If the lens is moved, say, to the *right*, A B becomes broken away from C and D. It is to the left of C and D (2) if the lens is convex, and to the right (3) if it is concave; that is to say, if the line appears to move in the opposite direction to the lens the latter is convex, if in the same direction the lens is concave. When making this test the lens should be moved slowly in a certain direction, and not rapidly from side to side or up and down. If the lens is held too close to the eyes the line C and D beyond the edges cannot be seen, so that the best distance is about 8 or 10 inches. If, however, the lens is a strong convex it must be held nearer the observer's eyes or nothing can be seen through it owing to the convergence of the light, but the nature of such a lens can be at once recognised both from its form and from the fact that the lines, seen through it, are indistinct.

If the glass is a plano, that part A B of the vertical line seen through the glass remains continuous with the parts C and D on either side; that is to say, no displacement occurs on moving the glass.

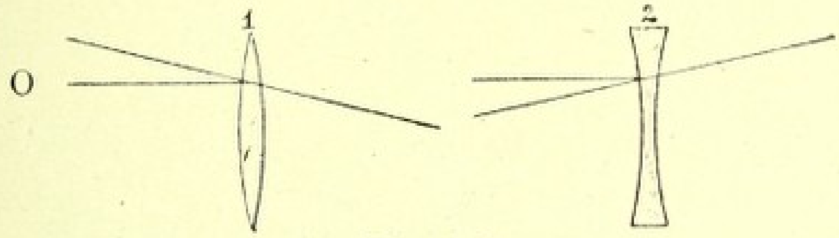


Fig. 140.

If the lens is displaced downwards, a horizontal line, not shown in Fig. 140, but which is in the direction O, is seen through the peripheral portion of the lens, which is of greater refracting power than the centre, and the line appears deviated in the direction of the apices of the prisms of which the lens is formed, that is, towards the edge (1) in a convex, and towards the centre (2) in a concave lens. The degree of deviation and the rapidity of movement of the line is proportional to the strength of the lens; also the deviation is greater, as the part of the lens looked through is near the periphery. The apparent *motion* of the object, viewed as the lens is moved, is due to the fact that the lens increases *gradually* in prismatic or deviating power from centre to periphery.

If the line be first viewed through the, say, bottom of lens and this then moved downwards the motion of the image is continuously *with* or *against* throughout the journey.

If, instead of the lens, the head is moved, an object observed goes with the head if the lens is convex, and in the opposite direction if it is concave; for if the head is moved, say, to the right it produces the same effect as if the lens had been moved to the left.

If a convex lens is looked through when held at a distance greater than its focal length, for instance, if a 3 or 4in. Cx. lens be held 10 inches in front of the eye, the refracted rays, having crossed in the air, enter the eye divergent, and the apparent movement of the object when the lens is moved is the same as with a concave lens, and moreover the apparent size is diminished. What is really observed is an inverted ærial image of the object. As the analysing card is, however, the same in all directions, the inversion may not be noticeable, but the mistake as to the nature of a lens in such a case should be impossible, its central thickness indicating its nature without any special test, and only very strong convex lenses can, when held a few inches from the eye, form an ærial image sufficiently far away to be seen.

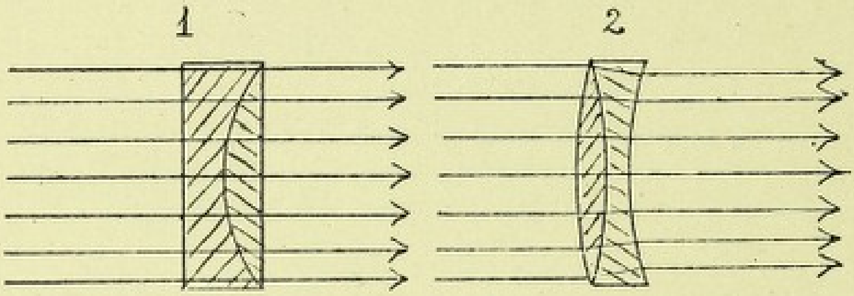


Fig. 141.

Neutralisation.—Neutralisation consists of finding that lens of opposite refraction and of known power (from the test case) which annuls the movement caused by the lens to be measured.

A convex and a concave lens (Fig. 141, 1 and 2) of the same power when placed in contact have practically no refracting power, the convergence of the convex being counteracted by the divergence of the concave, and incident parallel rays emerge parallel as through a plane glass.

If the unknown lens is convex, a concave is selected from the trial case, as near the power as can be judged from the rapidity of the movement, and then the two held together are again moved. If the movement is still that of a convex the power of the neutralising concave is insufficient and a stronger one must be tried. If with the first neutralising lens the movement of the two combined is that of a concave, the neutralising lens is too strong and a weaker one must be tried.

A few trials will enable one to find a lens which, when placed in contact with the unknown lens and moved, causes no displacement of the line. The number of the neutralised convex equals that of the neutralising concave. To find the number of an unknown concave lens a neutralising convex must of course be used. Practice will soon enable one to see by the degree of movement the approximate neutralising power needed, as well as to appreciate such slight movements as occur when neutralisation is nearly, but not quite, effected.

When neutralising, the lenses must be actually in contact, because if separated the convex acts with greater effect than does a concave of the same power. If the convex is in advance it more than neutralises the concave; if the concave is in advance it only partially neutralises the convex.

Neutralising Strong Lenses.—But even when the two are in actual contact, if the lens is strong, say, over 8 D or 10 D, it is difficult to get absolute neutralisation, there being always some slight movement in the peripheral portion of the lenses, although near the centre there may be practically none when the two lenses are of the same dioptral power.

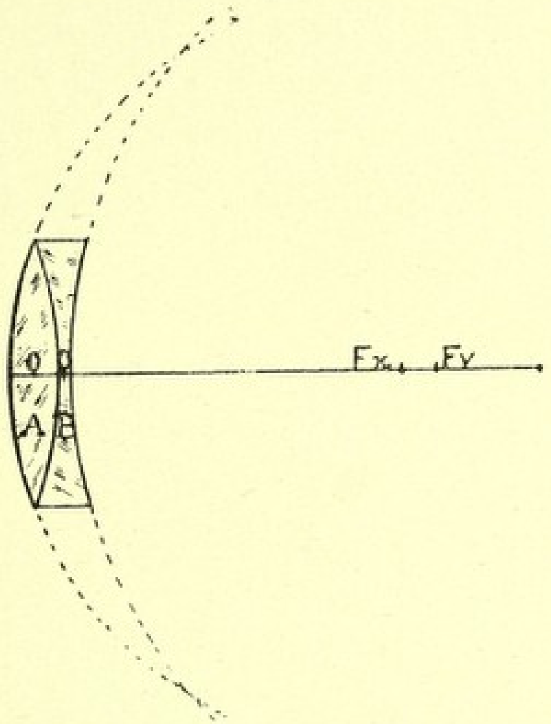


Fig. 142.

The failure to obtain a perfect neutralisation with strong lenses is due to the thickness of the convex. As shown in Fig. 142 by the dotted lines, the two lenses actually constitute a convex meniscus, for with the same radius of curvature the total lens is one formed of two intersecting circles.

The thickness of a concave lens in the centre, no matter how strong it be, can be ignored, but this is not the case with a strong convex. If the focal length of the convex is equal to that of the concave, it is clear that F_v of the concave is further from the central point of the combination than is F_x of the convex. If the rays of light parallel to the axis are incident on B, they are rendered divergent as if proceeding from F_v, a point outside F_x, and are therefore rendered by A slightly convergent after refraction. Similarly if parallel rays are incident on A they are converged to meet at F_x, a point nearer than F_v, and the diverging power of the concave being insufficient to render them parallel, they are slightly convergent after refraction. Thus a strong convex and a strong concave lens of similar μ and radius do not actually neutralise each other.

The concave being thin at its axis, its required radius would be calculated by the formula where thickness is neglected, while that of the convex would be calculated with its thickness considered.

The + 20 D from a trial case, being of large diameter, is about .75 cm. thick in the centre, and its radius would need to be shorter than that of the − 20 D to have equal power. Giving the same radius to each, the convex is weaker than the concave, but at the same time the latter fails to neutralise the former owing to the interval between their optical centres. In order that two strong opposite lenses should neutralise, the concave must be stronger, the focal length of the convex being approximately one-third its thickness longer than that of the concave, which, however, is not the case when the radii of curvature of the two are equal. In short, although a thick convex has a longer focal length than a concave of similar radius and μ, it is not sufficiently so for the convex to be neutralised by the concave. For a − 20 D whose $F = -5$ cm. to exactly neutralise a convex having a thickness of .75 cm., the latter would need have $F = 5.25$ cm., and if $\mu = 1.50$ would require a curvature of 5.125 cm.

If, therefore, a convex and a concave neutralise each other, the latter is stronger, but the difference is quite inappreciable in weak lenses and not of importance in spectacle lenses, even if strong.

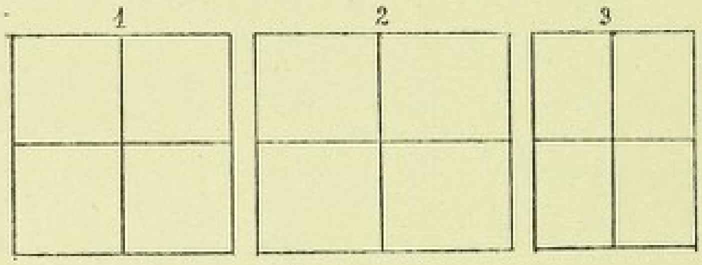


Fig. 143.

Neutralisation of Cylindricals.—A square card (Fig. 143, 1) viewed through a spherical is slightly increased or decreased in size equally in every direction and (disregarding distortion) remains a true square. If viewed through a convex cylindrical having its axis, say, vertical, the square is apparently increased in size horizontally (2), and through a concave it is diminished horizontally (3).

Since the size is not altered in the direction of the axis the square appears oblong in both cases, and the diminution caused by a concave, and the magnification caused by a convex, disappear when the two of equal power are placed together.

On rotating a cylindrical lens in a plane parallel to the analysing chart the lines on the latter appear to move obliquely either *with* or *against* the lens, and if the rotation be continued appear to move back again. The amount of "dipping" being dependent on the strength of the cylindrical. The line appears bent towards the meridian of greatest *positive*, or least *negative*, refraction, therefore it moves towards the axis of a concave or away from the axis of a convex cylindrical. And while the one line is thus declining, the other line moves in the opposite direction, so that the two incline towards each other, as shown in Fig. 138. Thus the vertical or the horizontal line bends the one way or the other, when the lens is rotated, according as it is convex or concave and according as the axis is being moved towards or away from the line viewed.

The dipping occurs on account of the prismatic formation of the lens, the apparent displacement being towards the edges of the virtual prisms contained in the lens. Thus, if the upper part of a convex cylindrical, axis vertical, is rotated to the right downwards, the lower part is at the same time rotated to the left, upwards; then the vertical line inclines to the left above, and to the right below, while the horizontal line inclines upwards to the left and downwards to the right, each moving towards the thin parts of the lens. And as the rotation is continued and the axis approaches the horizontal and the thin part approaches the vertical, both lines again move back towards their real positions. Similar oblique displacements occur when the maximum meridian of a concave is rotated to the right, away from the vertical above. The contrary takes place when the axis of a concave or the maximum meridian of a convex is thus rotated.

Therefore the direction in which the viewed line inclines as a cylindrical is rotated does not indicate whether it is convex or concave, and to attempt to neutralise by "stopping" the apparent inclinations might result simply in selecting for that purpose another cylindrical of similar power and nature, the two together making a spherical lens.

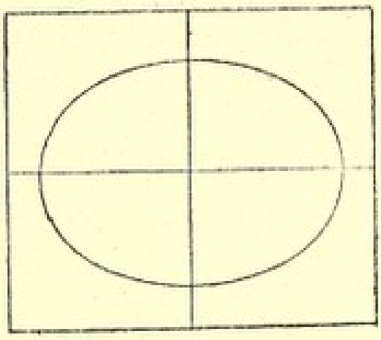


Fig. 144.

Locating the Principal Meridians.—The cross lines of the analysing card are seen continuous within and beyond the edges of the lens as in Fig. 144, when it is in such a position that its axis is horizontal or vertical, that is to say, when the axis is parallel to the one line and the meridian of greatest power is parallel to the other. The two principal meridians are in the same planes as the two lines of the chart.

Such a position for a cylindrical must be found in order (a) to learn whether it is a plano or a sphero-cylindrical, (b) to determine whether it is convex or concave, and (c) to neutralise it.

This position being found, the lens is first moved vertically and then horizontally. If no movement is observed in the one direction it is a plano-cylindrical; if there is movement in both directions it is a sphero-cylindrical or its equivalent, a cross-cylindrical.

The axis of a plano-cylindrical lies in the meridian in which there is no movement. The axis of the cylindrical in a sphero-cylindrical combination which has two positive or two negative powers is in that principal meridian which causes the *lesser* movement.

When they are both + and − powers in a combination the axis of the cylindrical is also *presumed* to be in the principal meridian of *lesser* movement. But in all cases the axis of the *actual* cylindrical might be in the meridian of greater movement, owing to the fact that the same principal powers can be obtained in lenses of various forms (see transposing).

To neutralise a plano-cylindrical the procedure is the same as with a spherical only that cylindricals of opposite nature are employed. Care must be taken that the cylindrical axis is always exactly vertical or horizontal, and that the axis of the neutraliser precisely corresponds to it. In order that this may be the case, continuity of the crossed lines at the edges of the lens must be looked for and constantly maintained during the process of neutralisation.

In the sphero-cylindrical the lesser movement is that caused by the spherical lens alone, while the greater movement is caused by the united powers of the spherical and the cylindrical.

The lens being held with its axis, say, vertical, that spherical of opposite refraction is found which neutralises, in the vertical meridian, the movement of a horizontal line. This having been achieved the lens and the neutralising spherical are held together, and the cylindrical element is then neutralised with a cylindrical axis vertical, of opposite refraction in the same manner as if the lens were a plano-cylindrical. The rapidity and exactitude of the neutralisation depends, as with a plano-cylindrical, on the care exercised in keeping the principal meridians exactly corresponding to the two lines of the card, and the axes of the two cylindricals exactly in line with each other.

Neutralisation of a sphero-cylindrical can also be effected by neutralising each principal meridian separately with a spherical or with a cylindrical whose axis is placed at right angles to the meridian that is being neutralised, the two powers thus found being transposed into a sphero-cylindrical combination. This method is, however, not so exact, especially for beginners.

A cross-cylindrical is neutralised in the same way as a sphero-cylindrical either by a spherical and a cylindrical, by two sphericals, or by two cylindricals whose axes are crossed at right angles; but the first method is the best.

Obliquely Crossed Cylindricals.—Since the two principal powers of a compound lens are always at right angles to each other, a combination consisting of two cylindricals whose axis are obliquely inclined can be neutralised by two suitable cylindricals whose axis are at right angles, or by a spherical and a cylindrical. Provided always that the exact principal powers are such as have equivalents in the trial case, which, however, is not often the case.

Mixed Cylindricals.—A mixed cylindrical is one which has a convex power in the one principal meridian and concave in the other. It might be either a sphero-cylindrical or a cross-cylindrical, and it is neutralised in the same way as other forms of compound lenses.

Toric Lenses.—A toric or toroidal lens is a special form of sphero-cylindrical, and is analysed and neutralised in a similar manner.

Expressing Sphero-Cylindricals.—Since any lens which has two principal meridians can be put up in various forms, a neutralising combination may be found which, while correctly indicating the refractive powers of the lens, may not represent the exact form in which it is made. If, however, transposing be understood, it is an easy matter to consider other forms for the same combination. It is always correct to express a compound combination as a *sphero-cylindrical with a spherical of the lower power.*

The true form of a combination can be learnt by using a lens measure or spherometer, or by using a straight-edge, or even by inspection, the difference between the curvature of a concave and of a convex cylindrical being easily noted.

Order of Neutralisation—A lens which possesses spherical, cylindrical, and prismatic elements should be neutralised in that order, but it is necessary to guard against the common error of supposing a prismatic element to exist, when it is produced by holding the neutralising lens out of centre with the lens which is being tested. It is essential that the geometrical centres of all the lenses should exactly coincide when neutralising.

To Learn to Neutralise.—Practice is necessary in order to neutralise rapidly and correctly, and it is well to commence with simple sphericals and then proceed to plano-cylindricals and compound lenses of known powers.

Focalisation.—The power of an unknown convex lens can also be obtained by focalisation, that is, by measuring the distance between the lens and its principal focus.

A bright object, such as a flame or a window, at a distance of 20 feet or more, may be taken as the object, the rays proceeding from which, being parallel, are by a convex lens brought to a focus at F. By moving the lens backwards and forwards a position for it is found where, on a white screen placed behind it, the sharpest obtainable image of the luminous object is formed. The measured distance between the screen and the lens is its principal focal distance in inches or centimetres. The brighter the luminant and the darker the place where the white screen is located, the easier it is to focalise. The distance should be read off from the scale whose zero end is at the screen. The image is quite sharp only at the exact principal focal distance of the lens. At any other distance the image is indistinct.

As the focal distance is shorter, the image is smaller but sharper and brighter, so that this method serves very well for fairly strong convex sphericals. If, however, the convex lens be weak and therefore of long focal length, the image on the screen is large and indistinct, and the exact principal focal distance is difficult to determine. If F is very short, the exact distance also becomes hard to determine with accuracy.

To focalise the periscopic convex lens the distance of the optical centre from the lens should be considered. The distance from the lens to the screen should be taken first with the one face and then with the other, turned towards the source of light. The mean of the two measured distances is the true focal length. Or, if the symmetrical planes be located, the distance between them divided by four gives the focal length of a periscopic lens.

With ordinary periscopic spectacle lenses, the distance of F, from the lens itself, is sufficiently exact in practice.

Focalising Cx. Plano-Cyls.—If a convex plano-cylindrical be at its principal focal distance in front of a screen, parallel light, refracted by it, forms on the screen a bright line which corresponds to the direction of the axis of the cylindrical. By finding the distance at which the line is narrowest and brightest the focal length of the lens may be directly determined.

Focalising Cx. Sphero-Cyls.—If a convex sphero-cylindrical be held in front of a screen, parallel light, refracted by it, forms on the screen a line at the focal distance of the spherical and another at the focal distance of the united powers of the spherical and the cylindrical. The first is at right angles to the cylindrical axis, and the latter corresponds to it. By finding these two places, and measuring the distance between the lens and the screen for each, the focal length and powers of the two principal meridians of the lens can be learnt. Thus, suppose the two distances are 50 and 33 cm., then the combination is + 2 D and + 3 D, or + 2 D Sph. ⌒ + 1 D Cyl. If the focal distances are 10 and 8 inches the lens is 1/10 Cx. Sph. ⌒ 1/20 Cx. Cyl., since 1/8 — 1/10 = 1/20.

If the combination consists of a strong spherical and a very weak cylindrical it is, however, almost impossible to obtain the exact focal distances of each principal meridian, the difference between them being so small; thus, with a + 10 D Sph. ⌒ + .5 D Cyl. the two distances are 9.5 cm. and 10 cm. Such a lens is difficult to distinguish from a spherical by focalisation.

Focalising Strong Cx. Sphs.—To focalise a very strong Cx lens it is better to combine it with a Cc lens of sufficient power to lengthen the focal distance to a reasonable extent. For instance, it is difficult to determine whether a lens has F of 2in. or $2\frac{1}{4}$in.

But if this lens be focalised together with a, say, 3in. Cc the difference between the one and the other is much more marked, it being then 3in. The focus of the unknown lens is obtained by the formula

$$\frac{1}{F_2} = \frac{1}{F} - \frac{1}{F_1} \qquad \text{or } D_2 = D - D_1$$

where F and D are, respectively, the focal distance and the power of the two lenses combined, F_1 and D_1 those of the added Cc. and F_2 and D_2 are those of the unknown Cx.

Thus if F = 9in. and $F_1 = -3$in.

$$\frac{1}{F_2} = \frac{1}{9} - \left(-\frac{1}{3}\right) = \frac{4}{9} \quad \text{The lens is } 2\tfrac{1}{4}\text{in. Cx.}$$

or $\qquad D_2 = 4.5 - (-13) = 17.5$ D or say + 18 D.

If F = 6in. and $F_1 = -3$in. then

$$\frac{1}{F_2} = \frac{1}{6} - \left(-\frac{1}{3}\right) = \frac{3}{6} \quad \text{The lens is 2in. Cx.}$$

or $\qquad D_2 = 6.5 - (-13) = 19.5$ D or say + 20 D.

When — 8 D is added to + 10 Sph. ⊃ + .5 Cyl. there is an interval of 10 cm. between the two focal lines, instead of .5 cm. as in the original lens.

Focalising Cc. and Weak Cx. Sph's.—To focalise a weak Cx. or any Cc. lens a sufficiently strong convex spherical, say, + 5 D, should be combined with it. Then the focal distance (or distances) of the two lenses is found in the manner above indicated and that of the unknown lens is calculated, as when a Cc. is added, by the formula

$$\frac{1}{F_2} = \frac{1}{F} - \frac{1}{F_1} \qquad \text{or } D_2 = D - D_1$$

Thus, if F = 6in. and F_1 = 7in. the unknown lens is 1/6 − 1/7 = 1/42 Cx.

If D = 6 and $D_1 = 5$, then $D_2 = 6 - 5 = +1$.

Calculating Compound Lenses by Focalisation.—If there are two focal distances at 15 and 33 cm. and D = + 5, the actual powers a and b of the combination are

(a) 100/15 = 6.5 − 5 = + 1.5

(b) 100/33 = 3 − 5 = − 2

The lens is + 1.5 Cyl. ⊃ — 2 Cyl. or a sphero-Cyl. possessing similar powers.

Or the last form of calculation may be solved in another way. The sphero-cylindrical combination of the two lenses can be calculated, and the power of the added spherical deducted from that of the spherical of the combination. Thus, if the added lens be + 5 D and the two focal distances are found to be 25 and 16 cm. the powers are

+ 4 D and + 6.25 D

and the united combination is + 4 D Sph. ○ + 2.25 D Cyl.

Now, deducting the + 5 D from the spherical, the unknown lens is found to be

− 1 Sph. ○ + 2.25 Cyl.

If F = 6in. Cx. and F_1 = 8in. Cx., then 1/6 − 1/8 = 1/24 Cx.

If F = 10in. Cx. and F_1 = 8in. Cx., then 1/10 − 1/8 = 1/40 Cc.

If two foci are obtained at, say, 10 and 6 inches, when F_1 = 8 Cx., the united combination is

1/10 Cx. Sph. ○ (1/6 − 1/10) = 1/15 Cx. Cyl.

Deducting the 1/8 from the 1/10 Sph. we get

1/10 − 1/8 = − 1/40

The unknown lens, therefore, is − 1/40 Sph. ○ + 1/15 Cyl.

Similar procedure is followed when both the focal distances are found to be either longer or shorter than that of the added spherical alone.

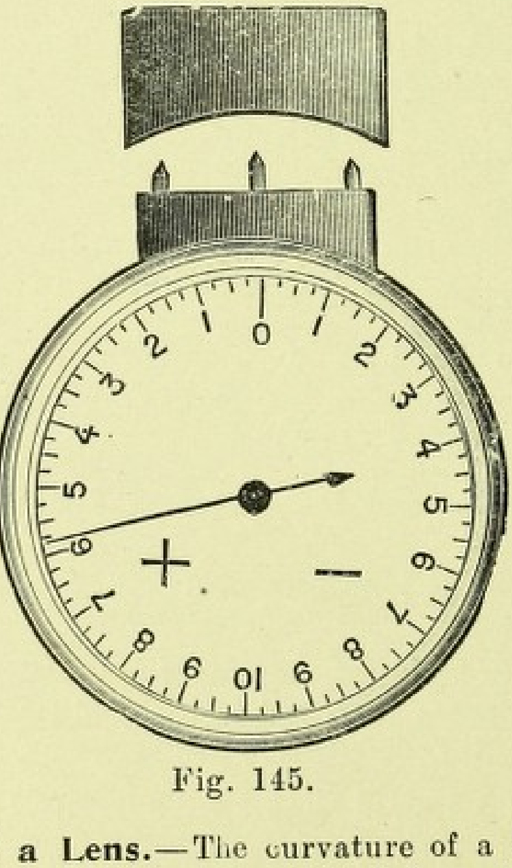


Fig. 145.

Curvature of a Lens.—The curvature of a lens is indicated by a spherometer, or lens measure, and from the curvature its power can be calculated if μ be known; but since the index of refraction of glass used for spectacle lenses varies but little and since the lenses themselves are weak, such an instrument can be directly scaled for diopters or focal lengths. Although the exact power of a very strong spectacle lens may not be indicated by this instrument, it is sufficiently near for all practical purposes. See Fig. 145.

The Spherometer.—The spherometer used in the optical trade consists of a small nickel-plated box resembling a pocket-aneroid. It contains a spring connected with a hand or pointer which indicates the dioptral number or focal length of the lens on a dial. Projecting from the top of the box are three metal pins, the centre one of which projects beyond the other two, and it alone is moveable, and is connected with the spring.

When the surface of a lens is pressed on to the pins until arrested by the two side ones, the central pin becomes depressed and causes the pointer to revolve and indicate the power of the lens (as represented by its curvature) in diopters. Care must be taken that the plane of the lens is at right angles to the plane of the pins. It is also important to see that the pointer indicates zero when a plane glass is applied to the instrument.

The surface is spherical if, on rotating the lens, while pressed against the pins, the index remains stationary, and it is a plane if 0 is then indicated by the pointer.

If the index moves to different positions, when the lens is rotated, it indicates a cylindrical surface, the maximum power being shown by the highest number attained. The axis of the cylindrical is indicated when the index points at 0.

If the lens be a Sph.-Cyl. or cross-Cyl. the power of each surface is distinct from the other. But when both surfaces are spherical, the power of the one must be added to that of the other to obtain the dioptral number of the lens. Thus with − 3 D on each surface, the lens is — 6 D. If the one surface is + 2.75 and the other — 1, the lens is + 1.75 D Sph.

The Four Leg Spherometer.—This is an instrument for ascertaining the radius of a sphere from the curvature of a portion of its surface. The most accurate form consists of three legs uniting in a common centre, so that their points describe an equilateral triangle. A fourth leg moves up and down in the centre, by means of a fine screw, the head of which supports a round horizontal plate. The latter has its edge almost touching a vertical scale divided into mm. or .5 mm. as the case may be, and the plate itself is usually divided into 100 parts. The elevation or depression, therefore, of the centre leg above the plane of the other three can be read with considerable accuracy.

By a well-known proposition of Euclid it can be proved that if two chords of a circle intersect at right angles the product of their respective parts are equal. Thus in Fig. 145a, if A B and C D are at right angles,

$$S \times a = d \times d = d^2$$

But

$$a = 2\,r - S$$

$\therefore$

$$d^2 = S\,(2\,r - S)$$

Whence

$$r = \frac{d^2 + S^2}{2\,S}$$

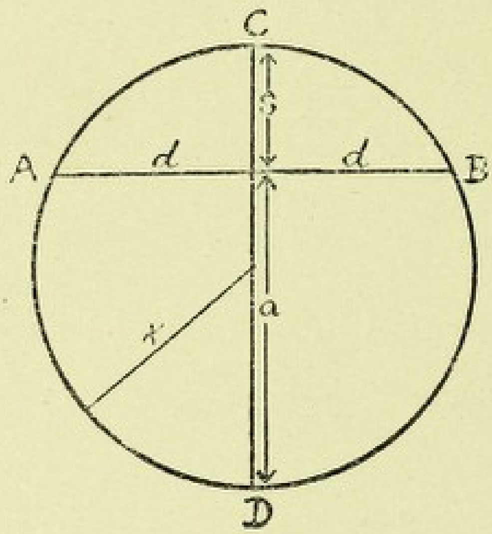


Fig. 145a.

Now S, the sagitta, is measured by the centre leg of the spherometer, and d by the distance between the centre leg and an outside leg. Therefore r, the radius of curvature, is easily found from the formula.

If, however, the distance E between any two fixed legs is taken, the formula becomes

$$r = \frac{E^2 + 3\,S^2}{6\,S}$$

because $E = d\sqrt{3}$

When the sagitta S is very small compared with r (as is nearly always the case), the quantity involving S^2 in the formulæ may be neglected, and they become respectively

$$r = \frac{d^2}{2\,S} \quad \text{and} \quad \frac{E^2}{6\,S}$$

The ordinary dial spherometer, described before, is of course merely a mechanical instrument, graduated in diopters from known curves. To be absolutely accurate, a correction for the variations of the refractive index should be made in every case, but for spectacle work the error is negligible. The ~~three~~ four-leg form is essentially used in scientific work, for the determination of radii alone, the focal power being subsequently determined from a knowledge of the refractive index.

Methods of Determining the Power of a Lens.—We see then that the focal length or power of a spectacle lens can be found by

A. Neutralisation with a lens of opposite refractivity.

B. Direct focalisation. Or indirect focalisation, if concave or weak convex.

C. The spherometer or lens measure.

These are the practical methods and are sufficiently accurate. A is the most exact, while C is the quickest, and can, moreover, be employed for convex or concave mirrors. When greater precision is required there are a number of other methods, some of which are given under the article on thick lenses.

In addition to the above, other methods, such as the following, are suitable for determining the power of a thin lens and are sometimes useful.

D. The method of symmetrical planes.

E. Measurement of any pair of conjugate foci.

F. Magnification method.

G. Focalisation of the reflected image from the surface of a concave lens. (Reflection).

H. Measurement of the distance at which a luminous area of given diameter is projected by a concave lens. (Luminous area).

I. Measurement of the focal length of a concave lens by experiment.

Method D.—Symmetrical Planes.—(Donders.)—The method of symmetrical planes is rapid and accurate to within 0.25 D for Cx. sphericals. It depends on the principal that when image and object are identical in size, the distance of O and I from the lens is 2 F, and the total distance between them is four times the focal length of a thin lens.

This method is best applied on an optical bench, but the latter is not absolutely necessary. The lens is placed in a clip midway between the two screens. The one is opaque with an aperture behind which a lamp is placed; the other consists of frosted glass. The two screens are moved equally towards or away from the lens until the image on the frosted glass screen is sharp and of equal size to the aperture of the other. The experiment is made more accurate if the frosted glass is scaled, and equal movements of the two screens is facilitated if the carriers are connected by a band suitably arranged for moving them equally. If the lens is weak Cx., or if it is Cc., it should be placed between a pair of strong Cx. lenses so as to obtain the symmetrical conjugate foci. The calculation then required is the same as given for focalisation.

Method E.—Conjugate Foci.—If the distance of the object from a convex lens be f_1 and the distance of its image on the opposite side be f_2, then the power of the lens is

$$1/F = 1/f_1 + 1/f_2$$

Suppose a candle be 10 inches from a lens and its image 15 inches, the lens will be

$$1/10 + 1/15 = 1/6 \text{ or } 6\text{in. F.}$$

It is easier to convert each distance into diopters, and calculate by

$$D = D_1 + D_2$$

If the object is at 25 cm. and the image at 20 cm. the lens is

$$\frac{100}{25} + \frac{100}{20} = 4 + 5 = 9 \text{ D.}$$

This method only serves for fairly strong convex spherical lenses, but can be applied to concave and weak convex lenses by adding a convex of sufficient strength, calculating the power of the two combined and then deducting that of the added lens, as shown for focalisation. Method D is a special case of the general method E.

Method F.—Magnification.—The magnification is the ratio between the sizes of I and O, which are proportional to their respective distances from the centre of a thin lens. Let M represent the magnification and f_1 and f_2 respectively, the distance of object and image, then

$$\frac{1}{F} = \frac{M + 1}{f_2} \quad \text{or} \quad F = \frac{f_2}{M + 1} \qquad [108]$$

If the total distance L between f_1 and f_2 be taken and the sizes of O and I be measured, F can be found from the following formula where M is the magnification—

$$F = \frac{L\,M}{(M + 1)^2} \qquad [109]$$

Thus, let O = 12in. high, and I = 4in. when L is 24in.; then M = 3,

and $$F = \frac{24 \times 3}{(3 + 1)^2} = 4\tfrac{1}{2}\text{in.}$$ [110]

When an object is so situated that a lens placed one meter from a screen forms a sharp image on it, then the dioptric power of the lens D = M + 1. This method was given by Prof. S. P. Thompson.

Since the I is fixed at 1 M, the O must be at a shorter distance if the lens is more than + 2 D, while it is at 1 M if the lens is + 2 D when M = 1. If the lens is + 1 D the object must be at ∞, and M may be said to be zero. Therefore, if the lens is Cx. but less than 1 D, or if Cc., in order to obtain an image a convex lens of adequate strength must be added, the power of the latter being deducted from that of the two combined.

The source of light is preferably a square diaphragm of 1 cm. diameter placed in front of a lamp. The lens being 1 M. from the screen a sharp image is obtained by adjusting the position of the light. The size of the image is determined by actual measurement or by a scale on the screen. If, for instance, the image is 4 cm. long, M = 4 and D = 4 + 1 = 5.

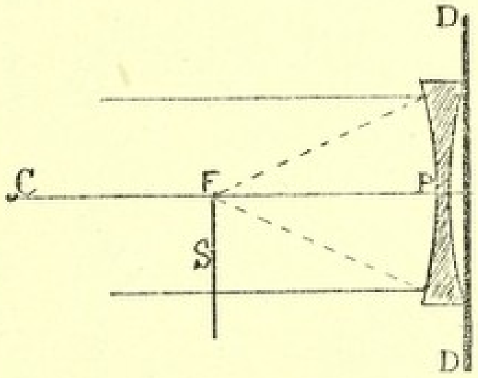


Fig. 146.

Method G.—Reflection.—The focal length of a double concave lens can be learnt by using it as a mirror. Let the light from C, a distant flame (Fig. 146), fall on the lens, the back surface of which is covered by a dark card D D. In front of the lens a small screen of cardboard S is held, so that, if necessary, by inclining the lens very slightly an image of the flame is obtained on the upper part of S, the distance of S being increased or diminished until the image is sharp. The distance between P and S is the principal focal distance of the concave reflector and is equal to half the radius of curvature. The focal length of a double Cc. crown glass lens is very nearly equal to the radius of curvature; consequently if the lens is double concave lens, twice the distance P S = F, or if the lens is plano-concave, four times P S = F. A concave cylindrical can be similarly measured, the reflected image being then a line of light.

Method H.—Luminous Area.—Parallel rays s s are allowed to pass through the lens A B (Fig. 147) on to a white screen M N, forming there a bright ring surrounding a dark central area.

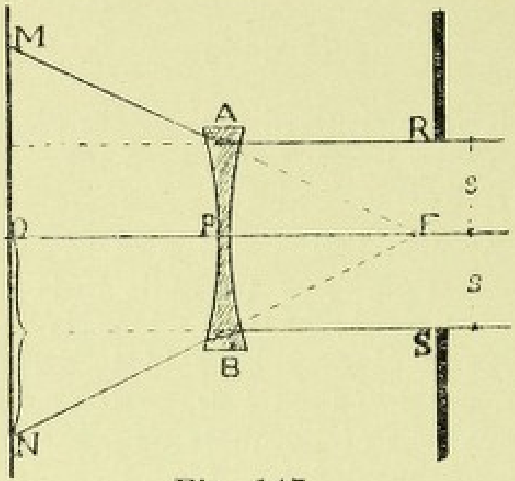


Fig. 147.

The concave lens renders light from a distant object divergent, as if proceeding from F, and when the disc of light on the screen is twice the diameter of the lens, the focal length of the lens P F is equal to the distance P O of the lens to the screen. A space equal to twice the diameter of the lens should be marked on the screen, or an aperture R S may be placed in the path of the light before it falls on the lens and a space equal to twice R S marked on M N.

The lens is moved to and fro until the luminous disc fills the marked space. This method is also applicable to simple and compound cylindricals by measuring the lens and the disc in the two principal meridians.

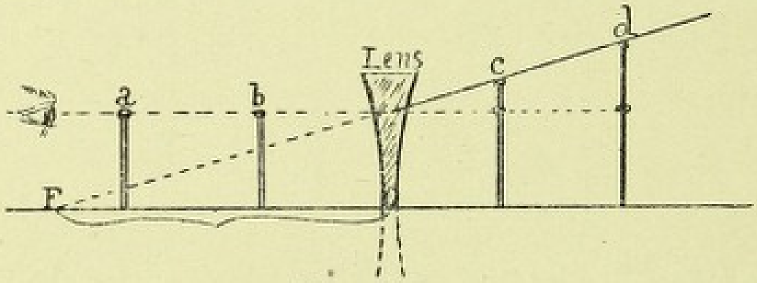


Fig. 148.

Method I.—Experiment.—The distance of the virtual image in front of a concave lens can be determined experimentally. Draw a line on a board (Fig. 148) to represent the axis, and cut a slot so as just to allow the lens being inserted as far as its centre.

Place two pins, a and b, in the marked axis with their heads at exactly the same height above the board. Place a third pin, c, on the axis beyond the lens, and while looking along the tops of a and b push the pin into the board until the head of c appears to be in line with a and b. Place another rather longer pin, d, a little distance beyond c and push it into the board until all four heads appear in a straight line. Remove the lens and lay a straight edge on the top of d and c. The point where the ruler touches the board is F of the lens and O f is the focal length.

A position is found for a long pin so that the image of a distant object, seen through a Cc. lens, coincides with the pin seen above the lens. On moving the head there is no parallax between image and pin when the distance of the pin equals F.

Methods of Testing a Plane Surface.—**Movement.**—A plano spectacle glass can be determined with sufficient accuracy by observing an object (preferably test types or ruled lines) through it while moving the glass to and fro. If the glass has no power due to curvature the image will appear stationary.

If the surface be a true plane no distortion or irregular movements can be detected.

If the glass be held obliquely to the eye, so that the direction of vision forms a small angle with the surface, any unevenness of the surface becomes much more apparent.

Contact.—If one surface be a plane, this can be determined by applying to it a straight edge, or another plano-glass, and observing whether there is contact throughout when holding the applied surfaces against a bright background. Real contact between two surfaces is also quite easily felt.

Reflection.—A plane surface can also be roughly distinguished from a curved one by viewing the reflected image from a bright source of light. If a plane, it acts precisely as a plane mirror, while if a spherical or cylindrical, the image is altered in size or distorted.

Spherometer.—A true plane surface may also be tested by the spherometer (q.v.).

The following are more accurate methods.

Whitworth Plane.— By contact with a Whitworth true plane surface, which has been smeared with some red putty powder, and observing whether any portion has or has not taken an impression.

Telescope. — By optical means, which is the most accurate method of all, but requires more skill, as given in the next paragraph.

Lens data by Telescopic Method.—A telescope is adjusted for infinity. A beam of light rendered parallel by a collimator placed at as small angle as possible, with the telescope, so that this angle may be negligible, or better still, an auto-collimating telescope may be used. The light from the collimator is then allowed to fall on the surface to be tested, and is, after reflection, received in the telescope.

If now, on looking through the telescope, the image seen of the source of light is sharp, the surface is a plane. If the surface is convex, the eyepiece of the telescope must be racked out, and if concave, racked in, in order to get a sharp image. If the surface is irregular, a sharp image cannot be obtained at that spot.

The focal length F of the reflecting surface if curved or the radius of curvature r, is obtained from the following formulæ

$$2\ r = F = \frac{f^2 - d\ (f - a)}{d}. \qquad [11$$

Where f is the focal length of the telescope object glass, a is the distance between the objective and the reflecting surface, and d is the distance which the ocular has to be racked in or out, in order to obtain a clear image. If the ocular has to be racked out, as occurs when the surface is convex, d must be reckoned as a negative quantity.

The distance a must be less than f, and if, as is advisable, a be made equal to f, the formulæ simplify to

$$2\ r = F = \frac{f^2}{d}. \qquad [11$$

The focal length of a lens can also be obtained from the same formulæ and operation.

The telescope adjusted for ∞, is turned so as to face the collimator, from which parallel light emerges. When the lens is interposed the eyepiece must be racked in, to obtain a sharp image, if the lens is convex, and out if the lens is concave, when d is considered of negative value.

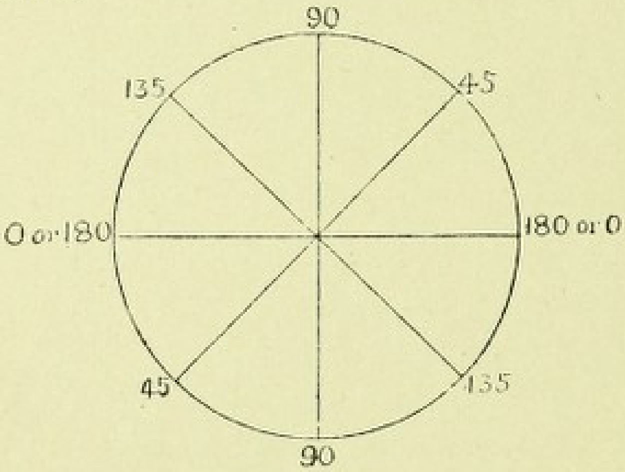


Fig. 149.

ANGLE NOTATION.

Standard Notation.—The Standard angle notation for the location of the various meridians of a lens (Fig. 149), refers to both the right and left eyes. The numeration commences on the right-hand of the imaginary horizontal line drawn through the lens when looked at from the front, the front of the lens being that face of it which is remote from the eye of the wearer.

This notation corresponds with the trigonometrical division of the circle into 360 degrees. The upper right quadrant contains the angles between 0° and 90° and the upper left those between

90° and 180°. The notation need not be carried beyond 180° (the half-circle), since a meridian corresponds to a diameter, i.e., to two continuous radii, for instance, 45° is the same meridian as 225°; 10° the same as 190°, etc. The vertical meridian is 90° and the horizontal is 0° or 180°, but is preferably indicated as 180°.

Locating the Axis of a Cyl.—The position of the axis of a cylindrical lens is, when vertical or horizontal, at once recognised. When it is oblique, its angular position can be estimated by that of the marked axis of the neutralising cylindrical, or its position may be determined by holding the lens against the neutralising lenses when the latter are in a trial frame, the long diameter of the neutralised lens being in the horizontal line of the trial frame. The axis of the neutralised cylindrical corresponds to that of the trial lens, which, being marked by a scratch, can be read off from the angle notation of the trial frame.

There are several forms of inclinometers or axis-finders—that of Dr. Maddox, for instance, is a most excellent one—designed for the purpose of aiding in the location of the axis of an unknown cylindrical spectacle lens. A quick and fairly accurate method of locating the axis is by means of the protractor on the "Orthos" rule.

To learn the exact meridian of the axis of an oblique cylindrical the axis should be marked by a line with pigment or a grease pencil. If the lens be then placed on a protractor, with the (optical) centre at the centre of the circle, the exact angle which the axis forms with the vertical or horizontal can be read off.

A grease pencil is one which contains a rod of colouring matter which adheres to glass.

When the axis is oblique and the lens is not in a frame, consideration must be given as to which of the two faces of the lens is supposed to be directed outwards, since the location of the axis varies accordingly. The rule is that the less convex or the more concave surface of a lens is, as before stated, placed next to the eye.

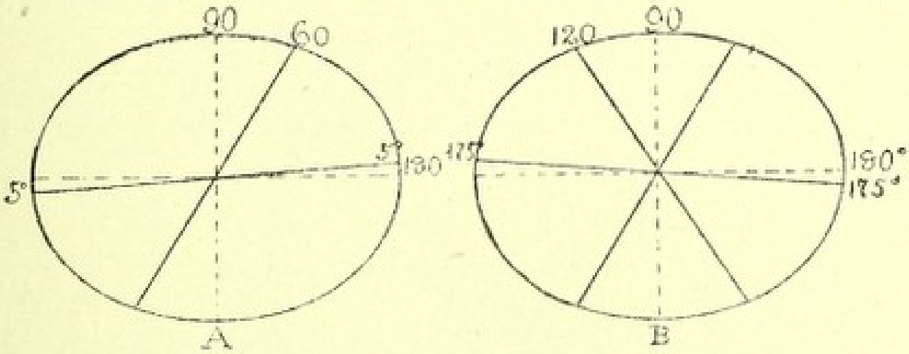


Fig. 150.

Reversion of a Cyl.—If a cylindrical (Fig. 150 A), having its axis at, say, 60° when the one face is to the front, is turned over so that the other face becomes the front, the axis is then at 120° (Fig. 150 B). If the one position were 5°, the other would be 175°. It is only when the axis is vertical or horizontal that no change occurs on turning the lens over. When the one inclination is 45° or at 135°, turning the lens over brings the axis to a position at right angles to the former one. The change in the numerical position of the axis caused by turning an oblique cylindrical, is calculated as so many degrees above or below the horizontal or to the right or to the left of the vertical, and assigning its position accordingly. Or it is done by simply deducting the numerical position of the axis from 180°. Thus, suppose the axis is at 60°, this is 30° to the right of the vertical; on turning the lens the axis is at 90° + 30° = 120°, i.e., 30° to the left of the vertical, or more simply by 180° − 60° = 120°. The corresponding positions of the axis as the one face or the other is in front are shown in the following table:—

1st Position.	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
When reversed.	180°	170°	160°	150°	140°	130°	120°	110°	100°	90°

Other Angle Notations.—Some trial frames and prescription forms are marked differently from that shown in Fig. 149, and it frequently occurs that the optician has to transfer from one notation to another. There are many different methods of *notating* the *two* eyes, but it is hardly necessary to attempt to detail them all here. The most commonly met with are the *bi-nasal* and the *bi-temporal* methods, in which the zero is placed at, respectively, the two nasal and the two temporal extremities of the horizontal line of the eye.

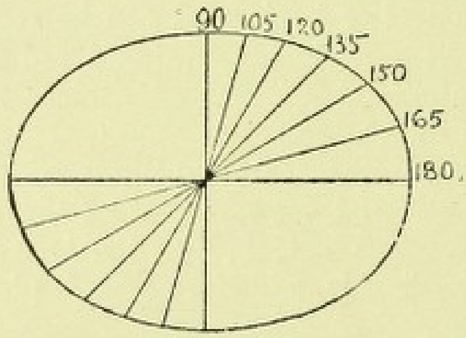


Fig. 151.

Translating Axis Notations.—Suppose a prescription be written with the indicated cylindrical axis at 125° according to the notation of Fig. 151. To translate this to standard notation, it must be considered how many degrees the required position is from the horizontal or the vertical. In this case 125°, in Fig. 151, is 35° from the vertical on the right and, therefore, corresponds to 55° of Fig. 149. If the location of the axis is 40° from the horizontal on the right, it would be 40° in Fig. 149 and 140° in Fig. 151. The same mode of calculating applies if the cylindrical axis is indicated as so many degrees with a stroke to show the direction of inclination.

This last-mentioned method of axis indication is unfortunately used by many medical men, thus making the reading of their prescriptions difficult to the optician.

Many oculists also do not use the ⊂ sign, but write the combination with a dividing line, thus:—

$$\frac{+\ 4\ \text{Sph.}}{+\ 2.50\ \text{Cyl. Axis}\ 70}$$

In all these methods it is, however, understood that the direction indicated refers to the front of the lens or the surface away from the wearer's eye.

CHAPTER VIII.

TRANSPOSING.

THE POWERS OF CYLINDRICAL AND TORIC LENSES.

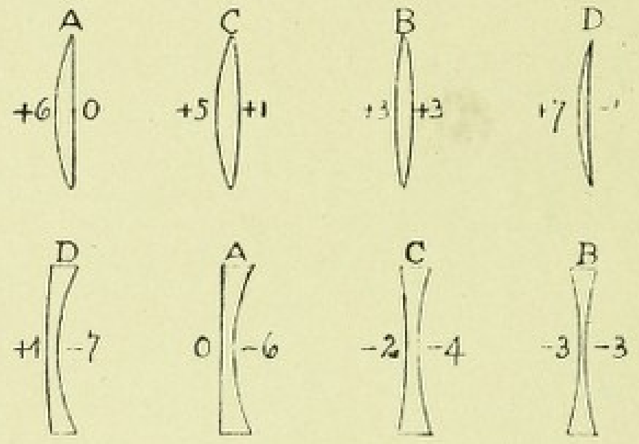


Fig. 152.

NOTE.—Throughout this chapter the abbreviated expressions of Sph. and Cyl. are employed; also the D to represent diopters is omitted for the sake of brevity. The symbol ⊂⊃ represents "combined with."

Transposition of Sph. Lenses.—As shown in Fig. 152 a + spherical lens, say, + 6 D, can be made in the form A of a plano-convex, in which all the power is on the one side; as B equi-convex, in which the power is equally divided between the two surfaces; as C, a bi-convex, in which the powers are unequally divided between the two surfaces; or as D, a periscopic-convex, in which the convex power on the one side is more than 6 D, but the total is reduced to that quantity by the concave curvature of the other surafce. The change from one form to another, without altering the refractive power of a lens, is called a transposition.

Similarly, a concave spherical can be made in the various forms as indicated above and as shown in the diagram.

The power of the one surface increases proportionately as that of the other decreases, so that the number of possible forms for a given power is endless.

The calculations for the needed curvature and the position of the optical centre, which varies with the form of the lens, are given elsewhere.

Transposition of Cyl. Lenses.—Lenses which contain a cylindrical element are susceptible of only two or three changes of form, and it is to such a change which does not alter the refractive powers of the two principal meridians, that the term "transposition" is generally applied.

When the two powers of a combined lens have the same sign, they are said to be of *like* or similar nature, or congeneric; when they are of opposite signs (the one + and the other −) they are said to be of *unlike* or dissimilar nature, or contrageneric.

A *plano* (or simple) cyl. is one possessing no sph. element.

A sph.-cyl. is formed of a sph. and a cyl. and may be either a *compound* cyl., having + or − powers in both principal meridians, or a *mixed* cyl., having a + power in the one and − power in the other.

The plano-cylindrical may, however, be regarded as a special form of sph.-cyl., the curvature of whose sph. element is of infinitely great radius, and it will be so treated in this article.

A *cross* cyl. is one formed of two similar or two dissimilar cyls. crossed at right angles.

Powers and Principal Meridians.—The one principal meridan of a sph.-cyl. corresponds to the axis of the cyl., and its power is that of the sph. alone, the other is at right angles to the axis of the cyl. and its power is the algebraical sum of that of the sph. and that of the cyl. Thus

+ 3 Sph. ⌢ + 2 Cyl. Ax. 70°. The two powers are + 3 at 70° and + 5 at 160°.

+ 3 Sph. ⌢ − 1 Cyl. Ax. 110°. The two powers are + 3 at 110° and + 2 at 20°.

+ 3 Sph. ⌢ − 3 Cyl. Ax. 5°. The two powers are + 3 at 5° and 0 at 95°.

+ 3 Sph. ⌢ − 5 Cyl. Ax. 120°. The two powers are + 3 at 120° and − 2 at 30°.

In the cross-cyl. the two principal powers are those of the cyls. themselves, each being in the meridian corresponding to the axis of the other. Thus—

+ 2 Cyl. Ax. 40° ⌢ + 5 Cyl. Ax. 130°. The two powers are + 2 at 130° and + 5 at 40°.

+ 2 Cyl. Ax. 70° ⌢ − 4 Cyl. Ax. 160°. The two powers are + 2 at 160° and − 4 at 70°.

Possible Combinations.—A cyl. combination may consist of two powers of similar nature but of different numbers,

say, + 2 and + 5, or − 3 and − 7,

or of two powers of dissimilar nature and of the same number,

as + 2 and − 2,

or of different numbers like + 3 and − 4.

Such combinations can be put in three forms, viz., a cross-cyl. and two forms of sph.-cyl..

If the one power is 0 it can be made only as a plano-cyl. and in one form of sph.-cyl.

If there are two equal powers of similar nature the possible forms are only those of a cross-cyl. and of a sph.

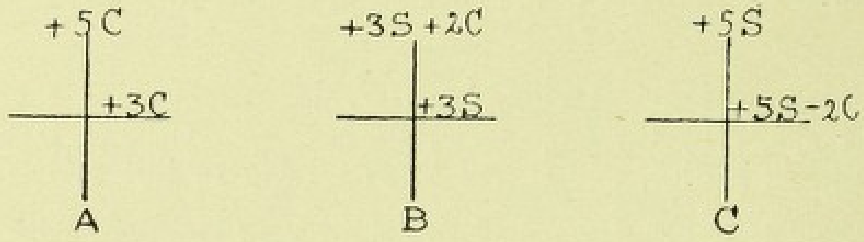


Fig. 153.

The Various Forms of a Lens with Cyl. Element.—Where two unequal powers in the two principal meridians are required, as + 3 at 180° and + 5 at 90°.

A. The + 3 needed at 180° can be obtained from + 3 Cyl. Ax. 90°, and the + 5 at 90° from + 5 Cyl. Ax. 180°, the axis of each Cyl. being at right angles to the direction in which the power is required.

B. The 3 needed at 180° can be obtained from + 3 Sph., which also supplies 3 of the + 5 D needed at 90°, the balance of the latter is obtained from + 2 Cyl. Ax. 180°, which gives + 2 at 90° and 0 at 180°.

C. The + 5 needed at 90° can be obtained from + 5 Sph., but this not only supplies the + 3 needed for 180° but is 2 D too strong. To reduce the latter to + 3 D a − 2 Cyl. Ax. 90° is required, this giving − 2 at 180° and 0 at 90°.

If the two parts of any of the forms A. B. C. be placed over one another, the total combination is, in each case, +5 at 90° and + 3 at 180°.

The rules for the three forms are:—

A. A cyl. of each of the two powers, the axis of each being at right angles to the meridian where the power is needed.

B. A sph. of the lower power and a cyl. of the difference between the two powers, the axis corresponding to the meridian of least power. If the lower power is 0, the sph. is also O.

C. A sph. of the higher power and a cyl. of the difference between the two, the axis being in the meridian of greater power.

Whether the two powers are of like or unlike nature the number of the cyl. is obtained by the algebraical subtraction of the power taken as the sph. from that of the other principal power.

Thus in the example the powers are + 3 and + 5, so that if the sph. is + 3, the cyl. is + 2; if the sph. is + 5 the cyl. is − 2.

If the two powers are + 2 and − 3, then, if the sph. is + 2, the cyl. is − 5; if the sph. is − 3, the cyl. is + 5.

If there is required − 4 at 60° and − 7 at 150°, the three possible forms are:

A. − 4 Cyl. Ax. 150° ⊂⊃ − 7 Cyl. Ax. 60°.
B. − 4 Sph. ⊂⊃ − 3 Cyl. Ax. 60°.
C. − 7 Sph. ⊂⊃ + 3 Cyl. Ax. 150°.

Should there be required − 1 at 45° and + 5 at 135° they are found in:

A. − 1 Cyl. Ax. 135° ⊂⊃ + 5 Cyl. Ax. 45°.
B. − 1 Sph. ⊂⊃ + 6 Cyl. Ax. 45°.
C. + 5 Sph. ⊂⊃ − 6 Cyl. Ax. 135°.

If the powers needed are + 3 at 120° and 0 at 30° the possible forms are:

A. 0 Sph, ⊂⊃ + 3 Cyl. Ax. 30°.
B. + 3 Sph. ⊂⊃ − 3 Cyl. Ax. 120°.

Rules for Transposing.

(1) *To transpose a sph.-cyl. or plano-cyl. into another form of sph.-cyl. or plano-cyl.*

The following apply to all cases, but when the original or the transposed form is a plano-cyl. the one power being 0 the sph. may also be 0.

(a) The new sph. is found by adding algebraically the power of the sph. to that of the cyl.

(b) The new cyl. has the same power as the original cyl, but its sign is changed and its axis is reversed.

(2) *To transpose a sph.-cyl. into a cross-cyl.*

(a) The one cyl. of the new form has the same number and sign as that of the original sph. and its axis is at right angles to that of the original cyl.

(b) The other cyl. has its axis in the same meridian as that of the original cyl. and a sign and number which results from the algebraical addition of the powers of the original sph. and the original cyl.

(3) *To transpose a cross-cyl. into a sph.-cyl.*

(a) The sph. of the new form has the number and sign of the first original cyl.

(b) The new cyl. has its axis corresponding to that of the second original cyl. and a sign and number which results from the algebraical subtraction of the first from that of the second original cyl.

Since either original cyl. may be taken as the first, there are two forms of sph.-cyls. into which a cross-cyl. can be transposed.

Equal Cyls. of *like* power are equivalent to a spherical of the same sign and power.

(4) *To transpose a sph. into a cross-cyl.*

(a) Give the number and sign of the original sph. to both cyls. whose axes may be in any pair of meridians at right angles to each other.

Examples.—The above rules can be better appreciated by studying the example at the same time. In all the following examples, the first combination is the original and those following are the forms into which it can be transposed.

These examples illustrate all possible combinations.

(1) + 4 Sph. ⊂ + 2 Cyl. Ax. 20° =
+ 6 Sph. ⊂ − 2 Cyl. Ax. 110°
+ 4 Cyl. Ax. 110° ⊂ + 6 Cyl. Ax. 20°

(2) − 2.50 Sph. ⊂ − 1.50 Cyl. Ax. 175° =
− 4.00 Sph. ⊂ + 1.50 Cyl. Ax. 85°
− 2.50 Cyl. Ax. 85° ⊂ − 4.00 Cyl. Ax. 175°

(3) + 3.50 Sph. ⊂ − 2.50 Cyl. Ax. 45° =
+ 1.00 Sph. ⊂ + 2.50 Cyl. Ax. 135°
+ 1.00 Cyl. Ax. 45° ⊂ + 3.50 Cyl. Ax. 135°

(4) + 3 Sph. ⊂ − 3. Cyl. Ax. 105° =
+ 3 Cyl. Ax. 15°

(5) + 2.50 Sph. ⊂ − 4.50 Cyl. Ax. 115° =
− 2.00 Sph. ⊂ + 4.50 Cyl. Ax. 25°
+ 2.50 Cyl. Ax. 25° ⊂ − 2.00 Cyl. Ax. 115°

(6) − 1.25 Sph. ⊂ + 1.75 Cyl. Ax. 160° =
+ 0.50 Sph. ⊂ − 1.75 Cyl. Ax. 70°
− 1.25 Cyl. Ax. 70° ⊂ + 0.50 Cyl. Ax. 160°

(7) + 2.75 Cyl. Ax. 95° =
+ 2.75 Sph. ⊂ − 2.75 Cyl. Ax. 5°

(8) + 2 Cyl. Ax. 80° ⊂ + 3 Cyl. Ax. 170° =
+ 2 Sph. ⊂ + 1 Cyl. Ax. 170°
+ 3 Sph. ⊂ − 1 Cyl. Ax. 80°

(9) − 5.50 Cyl. Ax. 155° ⊂ − 2.50 Cyl. Ax. 65° =
− 2.50 Sph. ⊂ − 3 Cyl. Ax. 155°
− 5.50 Sph. ⊂ + 3 Cyl. Ax. 65°

(10) + 2.25 Cyl. Ax. 75° ⊂ − 2.25 Cyl. Ax. 165° =
+ 2.25 Sph. ⊂ − 4.50 Cyl. Ax. 165°
− 2.25 Sph. ⊂ + 4.50 Cyl. Ax. 75°

(11) + 3.50 Cyl. Ax. 120° ⊂ − 0.75 Cyl. Ax. 30° =
+ 3.50 Sph. ⊂ − 4.25 Cyl. Ax. 30°
− 0.75 Sph. ⊂ + 4.25 Cyl. Ax. 120°

(12) − 10.00 Cyl. Ax. 180° ⊂ + 2 Cyl. Ax. 90° =
+ 2 Sph. ⊂ − 12 Cyl. Ax. 180°
− 10 Sph. ⊂ + 12 Cyl. Ax. 90°

(13) + 3.50 Cyl. Ax. 90 ⊂ + 3.50 Cyl. Ax. 180°
+ 3.50 Sph.

(14) − 4 Sph. =
− 4 Cyl. ⊂ − 4 Cyl. with axes at right angles.

Comparison of Original and Transposed Forms.—The two principal powers and meridians of the original form of a combination can be extracted and compared with those of the transposed form, and they must be alike if the transposition is correct.

Thus, suppose − 3 Sph. ⊃ + 4 Cyl. Ax. 90°.

The two principal powers are − 3 at 90° and + 1 at 180° as in Fig. 154 C.

The power of the − 3 Sph. is in both principal meridians, while that of the + 4 Cyl. Ax. 90° is only at 180° its axis being at 90° it contributes no refractive power to that meridian.

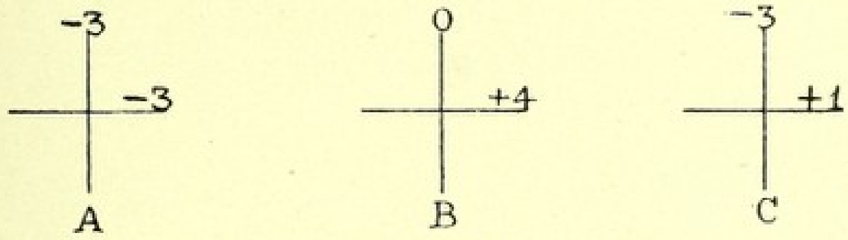


Fig. 154.

The two components separated are represented by A and B of Fig. 154. When combined they are represented by C.

Fig. 155.

The two forms into which − 3 Sph. ⊃ + 4 Cyl. Ax. 90° can be transposed are

(a) + 1 Sph. ⊃ − 4 Cyl. Ax. 180°
(b) + 1 Cyl. Ax. 90° ⊃ − 3 Cyl. Ax. 180°

whose powers are shown diagrammatically in Fig. 155 as D and E respectively.

Proof by Neutralisation.—Since a transposition simply assigns the needed powers in a different way, as regards the two surfaces of a lens, and does not change the refractive power of the combination, that combination which will neutralise the original form will also neutralise the transposed forms. Thus—

A. + 1 Sph. ⊃ − 4 Cyl. Ax. 180° transposed into
B. − 3 Sph. ⊃ + 4 Cyl. Ax. 90°.

A is neutralised by − 1 Sph. ⊃ + 4 Cyl. Ax. 180°, and these also neutralise B as can be seen by adding them together thus—

− 3 Sph. ⊃ + 4 Cyl. Ax. 90° added to
− 1 Sph. ⊃ + 4 Cyl. Ax. 180°
the 2 Sphs. = − 4 Sph., and the 2 Cyls. = + 4 Sph.,
and − 4 Sph. and + 4 Sph., of course, neutralise each other.

It is never required in practice to give crossed cylinders for any combination, since the effect can be equally well obtained by combining a Sph. and a Cyl., and at much less cost. The best form to employ is usually a + Sph. ◯ − Cyl., or a − Sph. ◯ + Cyl., since from these we obtain a certain periscopic effect without additional expense.

TORIC OR TOROIDAL LENSES.

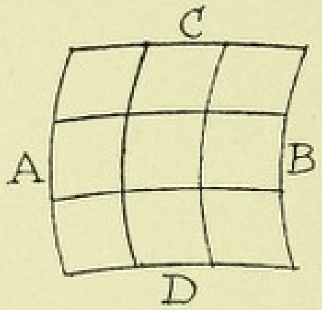


Fig. 156.

A toric lens is one having two principal powers worked on the same surface with their axes at right angles to each other, as shown in Fig. 156. The curvature of the lens along A B is, say, + 3 D, while along C D it is, say, + 5 D. It is, therefore, equal to + 3 Sph. ◯ + 2 Cyl. and has the same optical effects. The name is derived from the tore or arched moulding used in the construction of crypts and pillars. It can be illustrated by a bent tube or rod. Any portion of the area of an egg or bowl of a spoon resembles a toric curvature.

Since the possible combinations in the forms of various toric lenses may be infinitely great, it is usual to employ tools of a given base curve, and often an assortment of plano-toric lenses is kept, on the plane side of which any spherical curve can be ground. These tools or the plano-toric lenses are usually made on a base curve of either 3 D or 6 D or sometimes 9 D, i.e., a lens equal to a spherical of 3 D or 6 D combined with a cylindrical power. Thus, if + 1 D Sph. ◯ + 1.5 Cyl. were required a plano-toric having + 6 D Cyl. power in one principal meridian and + 7.5 D Cyl. in the other would be selected, and a concave surface corresponding to 5 D would be ground on a plane surface, the resulting lens being equivalent to a sphero-cylindrical of + 1 D Sph. ◯ + 1.5 Cyl. In the same way a concave toric tool or lens may be employed.

Advantages of the Toric Lens.—The utility of the toric form is that by its means the refracting power of a lens can more nearly be divided between the two surfaces. Thus if + 10 D Sph. ◯ + I D Cyl. be required, instead of there being + 10 Sph. on the one surface and + 1 Cyl. on the other it can be made with + 5 D Sph. on the one surface and + 5 Cyl. ◯ + 6 Cyl. on the other. Or it can be made with any other convex spherical power, the virtual cylindricals of the toric surface being accordingly stronger or weaker, but always having 1 D difference between them.

Thus, a strong lens as needed in aphakia or high myopia can be made less thick and more nearly resembling a Dcx. or Dcc.

Another and perhaps greater advantage of the toric lens is that with it a Sph.-Cyl. can be made periscopic, if so needed, to any extent, as is shown in the following example.

Required + .50 Sph. ⊂ + .25 Cyl. Ax. 90°. As a toric lens, periscopic 3 D, the combination would be

$$\frac{-\ 3\ \text{Sph.}}{+\ 3.50\ \text{Cyl. Ax. } 180^\circ \subset +\ 3.75\ \text{Cyl. Ax. } 90^\circ}$$

If the toric were of a base curve of 6 D, the sph. would be − 5.50 and the cyls. + 6 D and + 6.25 D.

To convert a Sph.-Cyl. combination into a toric form it is merely necessary to select the spherical and then change the equivalent cylindricals of the original form into others as much stronger or weaker as may be necessary to neutralise or supplement the spherical selected.

The term "toric" is often misapplied to deep meniscus spherical lenses.

Advantages of Menisci.—Of course the advantages derived from the use of periscopic sph. lenses apply also to toric lenses, which are merely deep menisci possessing a cyl. element.

A periscopic Cx. or Cc. Sph. is preferable to a Dcx. or Dcc. A + sph. ⊂ − cyl. is better than a + sph. ⊂ + cyl. A concave surface near to the eye prevents side reflections of light, allows of the lens and frame being placed nearer to the eye, and increases the effect of a Cx., while it decreases that of a Cc. These last result because the optical centre lies outside the lens of a meniscus form on the Cx. side of the Cx. and on the Cc. side of the Cc., and the distance is greater as the form of the lens is more periscopic.

But since the distance of the optical centre depends also on the thickness of the lens, any resultant difference of effect resulting from the toric or meniscus form of the lens is negligible if the lens is *thin*, as is usually the case with spectacle lenses.

THE REFRACTIVE POWERS OF A SINGLE CYLINDRICAL.

Note :—

Tables of natural sines and cosines and of $\sin^2$ and $\cos^2$ are to be found in the appendix. The abbreviations Hor. Ver. are used in this chapter to represent horizontal and vertical. H. and V. represent respectively the Horizontal and Vertical refractive powers and D is the dioptral power of the cylindrical.

The two principal meridians of a single cylindrical are

(a) The meridian which corresponds to the axis, in which the refractive power is zero; and

(b) The meridian at right angles to the axis in which the refractive power is greatest; this may be termed the *maximum* meridian.

The power of the latter indicates the dioptral number of the lens and may be represented by D. Every other meridian has a different refractive power which varies between 0 and D.

The power of any intermediate meridian of a cylindrical applies, however, only to a line of curvature. If any surface area of the lens be disclosed, the maximum curvature of the lens, which is oblique to that meridian, is the effective curvature and governs the refractive power and focal length.

Since all the light refracted by a cyl. passes through a line at the focal distance of the maximum meridian, the focus of light passing through any meridian is always the same as that of the maximum meridian, for we cannot reduce the effective width of the opening to that of a geometrical line.

Let D be the maximum refractive power of a cylindrical, D_1 the refractive power in a given meridian, and D_2 the power of the meridian at right angles to that of D_1. Let a be the angle between the axis and the given meridian. Then the power of a cylindrical in any given meridian is found by the equation

$$D_1 = D \sin^2 a, \text{ and } D_2 = D \cos^2 a. \quad [11$$

Thus wanted the refractive power at 20° of a + 3 D Cyl. Ax. 180°, then

$$3 \times .3420^2 = 3 \times .11694 = +0.35.$$

Since the refractive power along the axis is 0 and at right angles it is D, the total power of this pair of opposite meridians is

$$D + 0 = D.$$

And the sum of the powers of any other pair of opposite meridians of a single cylindrical is always equal to D, for

$$\sin^2 a + \cos^2 a = 1.$$

Then $$D \sin^2 a + D \cos^2 a = D_1 + D_2 = D. \quad [11$$

Thus with a 4 D Cyl. Ax. 90° the sum of the powers at 90° and 180° is 4 D; similarly those at 30° and 120° or those at 60° and 150° = 4 D.

V. and H. of an oblique cylindrical are found by the same formulæ.

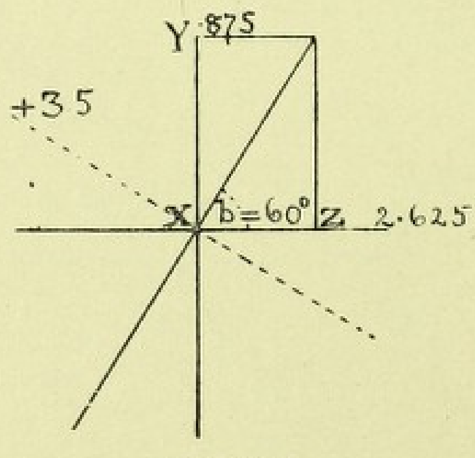


Fig. 157.

Let X Y and X Z (Fig. 157) represent the forces exerted, respectively, in the vertical and horizontal meridians by, say, 3.5 D Cyl.

Ax. 60°. Then X Y = sin 60° and X Z = cos 60°. Let h be the angle between the axis and the horizontal, then

$$H = D \sin^2 h \text{ and } V = D \cos^2 h. \qquad [115]$$

In this case $H = 3.5 \times .75 = 2.625$ D

$V = 3.5 \times .25 = .875$ D

and $2.625 + .875 = 3.5$ D.

If the power of any meridian be known, that of its opposite meridian can be obtained by subtraction, for

since $D_1 + D_2 = D$, then $D - D_1 = D_2$,

so that in the above example, having found V = .875, we know H = 3.5 − .875 = 2.625 or *vice versa*.

The following table gives the proportionate powers of unit cyl. in every meridian at intervals of 5°. The calculations have been carried only to the second decimal place.

Degrees.	Proportionate power.	Degrees.	Proportionate power.
0	.00	90	1.00
5	.01	95	.99
10	.03	100	.97
15	.07	105	.93
20	.12	110	.88
25	.18	115	.82
30	.25	120	.75
35	.33	125	.67
40	.42	130	.58
45	.50	135	.50
50	.58	140	.42
55	.67	145	.33
60	.75	150	.25
65	.82	155	.18
70	.88	160	.12
75	.93	165	.07
80	.97	170	.03
85	.99	175	.01
90	1.00	180	.00

To find the power, in any meridian, of a given cyl. the decimal corresponding to the angle between the axis and the meridian in question must be multiplied by the dioptral number of the lens.

Thus the power of a + 4.50 D Cyl. 25° from the axis is 4.5 x .18 = .81 D.

The hor. power of + 3.50 D Cyl. Ax. 60° = 3.5 x .75 = 2.625 D, and since the ver. meridian is 30° from the axis, V = 3.5 x .25 = .875 D.

TWO OBLIQUELY CROSSED EQUAL CYLINDRICALS OF LIKE NATURE.

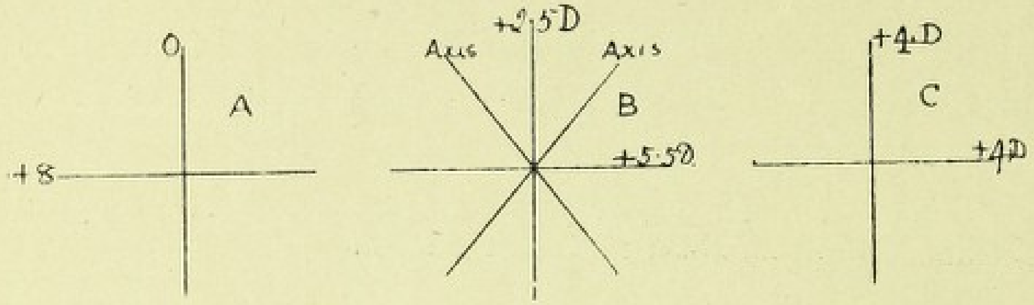


Fig. 158.

Note :—

A table of $\sin^2$ and $\cos^2$ is to be found in the appendix.

If two cyls. (Fig. 158) of the same number and sign, say + 4 D, are placed (A) with their axes corresponding in the vertical meridian, their combined Ver power = 0 and Hor = + 8 D. If the one cyl. be rotated to the right and the other equally to the left (B) they are equal to a combination of some two other principal powers, that of the Ver meridian increasing and that of the Hor decreasing with every increase of rotation. In any position the sum of the two principal powers D_1 and D_2 is always equal to the sum of the individual maximum powers D and D′, that is,

$$D_1 + D_2 = D + D'.$$ [1

When the two axes are at right angles (C) the two principal powers are each + 4 D, i.e., the combination is equivalent to a + 4 D Sph. If the rotation be continued until the two axes are horizontal, V = + 8 and H = 0.

With any obliquity of the axes the two cyls. are always equivalent to some other cross-cyl. whose axes are at right angles, and are therefore also equivalent to a sph.-cyl.

Let V be the Ver and H the Hor effects of two similar cylindricals whose axes are equally distant from the Ver., then from [**113**]

$$V = 2\ D \sin^2 a \text{ and } H = 2\ D \cos^2 a.$$ [11

Thus if two 4 D cylindricals be rotated 30° from the Ver in opposite directions,

$$V = 2 \times 4 \times .25 = 2\ D, \text{ and } H \times 2 \times 4 \times .75 = 6\ D,$$

and $$6 + 2 = 8\ D.$$

Let b be the angle between the axes of two combined cyls., D and D′ their maximum powers, D_1 the resultant power of the one principal meridian, and D_2 that of the other principal meridian. When D and D′ are equal

$$D_1 = 2\ D \sin^2 a \text{ and } D_2 = 2\ D \cos^2 a,$$

but since $$2\ a = b,$$

therefore $$D_1 = 2\ D \sin^2 b/2 \text{ and } D_2 = 2\ D \cos^2 b/2$$ [11

Thus, suppose the resultant combination of 4 Cyl. axis 40° ⊃ 4 Cyl. Ax. 80° be required, b = 40°, and therefore b/2 = 20°, then $D_1 = 8 \times .3420^2 = .96$ and $D_2 = 8 \times .9397^2 = 7.04$,

or 0.96 Sph. ⊃ 6.08 Cyl. Ax. 60°.

The resultant axis is midway between those of the original Cyls.

Now it can be shown that

$$2 \sin^2 b/2 = 1 - \cos b \text{ and } 2 \cos^2 b/2 = 1 + \cos b;$$

therefore the former formulæ may be written more simply,

$$D_1 = D - D \cos b \text{ and } D_2 = D + D \cos b. \qquad [119]$$

As an example, suppose 4 Cyl. Ax. 10° ⊃ 4 Cyl. Ax. 60°

$$b = 50° \text{ and } \cos 50° = .6428\text{; then}$$

$$D_1 = 4 - (4 \times .6428) = 4 - 2.5712 = 1.4288$$

$$D_2 = 4 + (4 \times .6428) = 4 + 2.5712 = 6.5712$$

or 1.4288 Sph. ⊃ 5.1424 Cyl. Ax. 35°,

or say 1.5 Sph. ⊃ 5 Cyl. Ax. 35°.

Since $D_1 + D_2 = D + D'$, it is only necessary to calculate for, say, D_1, then D_2 is obtained by subtraction. If $D + D' = 8$ and $D_1 = 1.4288$, then $D_2 = 8 - 1.4288 = 6.5712$.

Not only the total of the principal meridians of two similar cyls. $= D + D'$ (and this does not vary as they are rotated), but the total power of *any two meridians* at right angles to each other similarly $= D + D'$. If each cyl. is + 4 D, for any inclination of the axes, the total power of any pair of meridians at right angles to each other = + 8 D. Rotation of the axes of one or both cylindricals merely locates the refraction in varying quantities as regards each of any pair of opposite meridians.

Table of the powers resulting from the rotation of two similar unit cyls:—

Rotation of each cyl. from the ver.	Total angle between the axes.	Ver. effect.	Hor. effect.	Sphero-Cylindrical equivalent.								
0°	0°	.0	2.00	.00	D.	Sph.	⊃	2.00	D.	Cyl.	Ax.	90°
5	10	.02	1.98	.02	,,	,,	,,	1.96	,,	,,	,,	90
10	20	.06	1.94	.06	,,	,,	,,	1.88	,,	,,	,,	90
15	30	.14	1.86	.14	,,	,,	,,	1.72	,,	,,	,,	90
20	40	.24	1.76	.24	,,	,,	,,	1.52	,,	,,	,,	90
25	50	.36	1.64	.36	,,	,,	,,	1.28	,,	,,	,,	90
30	60	.50	1.50	.50	,,	,,	,,	1.00	,,	,,	,,	90
35	70	.66	1.34	.66	,,	,,	,,	.68	,,	,,	,,	90
40	80	.84	1.16	.84	,,	,,	,,	.32	,,	,,	,,	90
45	90	1.00	1.00	1.00	,,	,,	,,	.00				
50	100	1.16	.84	.84	,,	,,	,,	.32	,,	,,	,,	180
55	110	1.34	.66	.66	,,	,,	,,	.68	,,	,,	,,	180
60	120	1.50	.50	.50	,,	,,	,,	1.00	,,	,,	,,	180
65	130	1.64	.36	.36	,,	,,	,,	1.28	,,	,,	,,	180
70	140	1.76	.24	.24	,,	,,	,,	1.52	,,	,,	,,	180
75	150	1.86	.14	.14	,,	,,	,,	1.72	,,	,,	,,	180
80	160	1.94	.06	.06	,,	,,	,,	1.88	,,	,,	,,	180
85	170	1.98	.02	.02	,,	,,	,,	1.96	,,	,,	,,	180
90	180	2.00	.0	.0	,,	,,	,,	2.00	,,	,,	,,	180

In order to learn the Ver and Hor powers of the two equal cyls. whose axes have been located, multiply the number of one of them by the figures in the third and fourth columns of the table. Thus, if each cylindrical is 4 D and they are rotated each 20°, then

$$V = 4 \times .24 = .96 \text{ D and } H = 4 \times 1.76 = 7.04 \text{ D}.$$

The resultant sph.-cyl. would be .96 Sph. ⌒ 6.08 Ax. Ver.

That is, a sph. of the lower power and a cyl. of the differential power.

From the table the powers of the two principal meridians can be calculated, or the resultant sph.-cyl. of any two equal cyls. whose axes are inclined at any angle, the axis of the sph.-cyl. being midway between those of the two original cyls. Thus, suppose

4 Cyl. Ax. 10° ⌒ 4 Cyl. Ax. 70°,

the angle of separation is 60°, and from the table we get

.50 Sph. ⌒ 1.00 Cyl., which, multiplied by 4, gives

2.00 Sph. ⌒ 4 Cyl. Ax. 40°,

40° being midway between 10 and 70°.

The following table gives the rotation necessary to produce certain Ver. and Hor. effects from two combined cyls. Both cyls. are 4 D and their primary position of axes is ver.; the rotation of the one is to the right and the other to the left.

Table of the powers resulting from the rotation from the vertical of two similar 4 D Cyls.:—

Rotation of each Cyl.	Total angle between the two axes.	Ver. effect.	Hor. effect.
° ′	° ′	D	D
0.00	0.00	0.00	8.00
7.12	14.24	0.125	7.875
10.10	20.20	0.25	7.75
14.30	29.00	0.50	7.50
17.50	35.40	0.75	7.25
20.41	41.22	1.00	7.00
23.16	46.32	1.25	6.75
25.40	51.20	1.50	6.50
27.53	55.46	1.75	6.25
30.00	60.00	2.00	6.00
32.00	64.00	2.25	5.75
34.00	68.00	2.50	5.50
35.55	71.50	2.75	5.25
37.45	75.30	3.00	5.00
39.35	79.10	3.25	4.75
41.25	82.50	3.50	4.50
43.13	86.26	3.75	4.25
45.00	90.00	4.00	4.00
46.47	86.26	4.25	3.75

Rotation of each Cyl.	Total angle between the two axes.	Ver. effect.	Hor. effect.
48.35	82.50	4.50	3.50
50.25	79.10	4.75	3.25
52.15	75.30	5.00	3.00
54.50	71.50	5.25	2.75
56.00	68.00	5.50	2.50
58.00	64.00	5.75	2.25
60.00	60.00	6.00	2.00
62.70	55.46	6.25	1.75
64.20	51.20	6.50	1.50
66.44	46.32	6.75	1.25
69.19	41.22	7.00	1.00
72.10	35.40	7.25	0.75
75.30	29.00	7.50	0.50
79.50	20.20	7.75	0.25
90.00	0.00	8.00	0.00

For other cyls. weaker or stronger than 4 D the Ver. and Hor. effects are proportional for the given rotations.

TWO OBLIQUELY CROSSED EQUAL CYLINDRICALS OF UNLIKE NATURE.

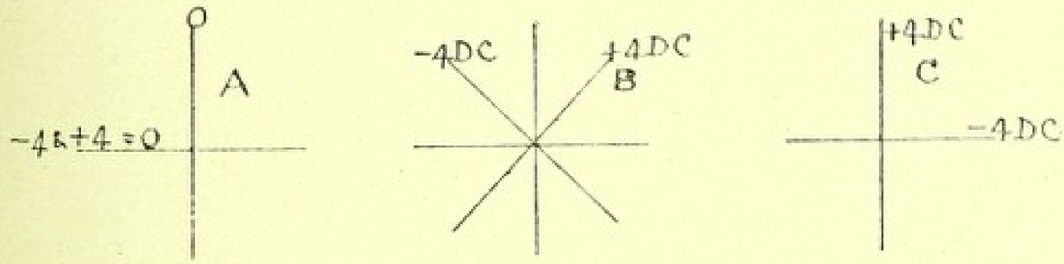


Fig. 159.

If two cyls. of opposite refraction and equal power, say 4 D (Fig. 159), are placed (A) with their axes corresponding they neutralise each other, since they act like a plate of glass. With their axes at right angles (C) they constitute a cross cyl. and $D + D' = 0$. With their axes at any inclination to each other (B) they form a cross cyl. of certain powers D_1 and D_2, which vary with the angle between the two axes. With a slight degree of inclination their dioptric powers are small, and they reach their maximum when the axes are at right angles to each other.

Now $D_1 + D_2 = D + D'$, but $D + D' = 0$, so also $D_1 + D_2 = 0$.

Also $$D_1\, D_2 = D\, D' \sin^2 b$$

but since numerically $D = D'$ and $D_1 = D_2$

we have $$D_1\, D_2 = D^2 \sin^2 b = D_1^2 \text{ or } D_2^2.$$

therefore $$D_1 \text{ or } D_2 = D \sin b$$ [120]

Example: + 4 Cyl. Ax. 60° ⌢ − 4 Cyl. Ax. 120°. Here b = 60°; therefore

$$D_1 \text{ or } D_2 = 4 \times .8660 = 3.464,$$

that is, we obtain + 3.464 Cyl. ⌢ − 3.464 Cyl. with the axes at right angles.

Two cyls. of opposite refraction and equal power inclined at an angle have their two principal powers D_1 and D_2 equal, the one being + and the other −, also the sum of the powers of any pair of opposite meridians = 0.

The position of c_1, the axis of the resultant + cyl., is midway between g_1, the axis of the original + cyl., and m_2, the maximum meridian of the original − cyl. The axis of the resultant − cyl. c_2 similarly is midway between the axis g_2 of the original − cyl., and m_1 the maximum meridian of the original + cyl. That is,

$$c_1 = \frac{g_1 + m_2}{2}, \qquad \text{and} \qquad c_2 = \frac{g_2 + m_1}{2}.$$

In the foregoing example the axis and the maximum meridian of the original + cyl. are respectively 60° and 150°. Those of the original − cyl. are respectively 120° and 30°. Then

$$c_1 = \frac{60 + 30}{2} = 45^\circ, \quad c_2 = \frac{120 + 150}{2} = 135^\circ$$

Therefore + 4 Cyl. Ax. 60° ⌢ − 4 Cyl. Ax. 120° =
+ 3.464 Cyl. Ax. 45° ⌢ − 3.464 Cyl. Ax. 135°.

Table of the powers resulting from the rotation of two equal unit cyls. of unlike nature.

Rotation of each Cyl. from 45°	Total angle between the two axes.	Resultant Cyls.	
0	0	+ .0000 Cyl. Ax. 90	− .0000 Cyl. Ax. 180
5	10	+ .1736 ,, ,, ,,	− .1736 ,, ,, ,,
10	20	+ .3420 ,, ,, ,,	− .3420 ,, ,, ,,
15	30	+ .5000 ,, ,, ,,	− .5000 ,, ,, ,,
20	40	+ .6428 ,, ,, ,,	− .6428 ,, ,, ,,
25	50	+ .7660 ,, ,, ,,	− .7660 ,, ,, ,,
30	60	+ .8660 ,, ,, ,,	− .8660 ,, ,, ,,
35	70	+ .9397 ,, ,, ,,	− .9397 ,, ,,
40	80	+ .9848 ,, ,, ,,	− .9848 ,, ,, ,,
45	90	+1.0000 ,, ,, ,,	−1.0000 ,, ,, ,,
50	100	+ .9848 ,, ,, ,,	− .9848 ,, ,, ,
55	110	+ .9397 ,, ,, ,,	− .9397 ,, ,, ,,
60	120	+ .8660 ,, ,, ,,	− .8660 ,, ,, ,,
65	130	+ .7660 ,, ,, ,,	− .7660 ,, ,, ,,
70	140	+ .6428 ,, ,, ,,	− .6428 ,, ,, ,,
75	150	+ .5000 ,, ,, ,,	− .5000 ,, ,, ,,
80	160	+ .3420 ,, ,, ,,	− .3420 ,, ,, ,,
85	170	+ .1735 ,, ,, ,,	− .1736 ,, ,, ,,
90	180	+ .0000 ,, ,, ,,	− .0000 ,, ,, ,,

For other cyls. the above resultant cyls. must be multiplied or divided accordingly. It should be observed that after a total rotation of 90° the resultant powers diminish in the same ratio as they previously increased. Thus either + 1 Cyl. Ax. 60° ⌒ − 1 Cyl. Ax. 30°, or + 1 Cyl. Ax. 120° ⌒ − 1 Cyl. Ax. 150° are equivalent to + .5 Cyl. Ax. 90° ⌒ − .5 Cyl. Ax. 180°.

Table of powers resulting from rotating + 4 Cyl. from 45° towards the Ver. and − 4 Cyl. towards the Hor.:—

Rotation of each Cylindrical ° ′	Total angle between the two axes. ° ′	+ Cyl. Ax. 90°.	− Cyl. Ax. 180°.
0	0	0	0
.53	1.46	0.125	0.125
1.45	3.30	0.25	0.25
3.35	7.10	0.50	0.50
5.25	10.50	0.75	0.75
7.15	14.30	1.00	1.00
9.6	18.12	1.25	1.25
11.	22.	1.50	1.50
12.55	25.50	1.75	1.75
15.	30.	2.00	2.00
17.6	34.12	2.25	2.25
19.20	38.40	2.50	2.50
21.43	42.26	2.75	2.75
24.18	48.36	3.00	3.00
27.10	54.20	3.25	3.25
30.30	61.	3.50	3.50
34.50	69.40	3.75	3.75
45.	90.	4.00	4.00

For other cyls. stronger or weaker than 4 D the + and − effects for the given rotation are proportional.

TWO OBLIQUELY CROSSED UNLIKE CYLINDRICALS.

To calculate D_1 and D_2 the resultant powers of any two cyls. D and D′ crossed at any angle b, we know that

$$D_1 + D_2 = D + D' \quad \text{and that} \quad D_1\, D_2 = D\, D' \sin^2 b. \qquad [122]$$

From these data the unknown quantities can be calculated—since their sum and their multiple are known. Or they can be worked out by the following formulæ, which represent the extraction of two such quantities:—

$$D_1 = \frac{D + D' - \sqrt{(D + D')^2 - 4\,(D\, D' \sin^2 b)}}{2} \qquad [123]$$

$$D_2 = \frac{D + D' + \sqrt{(D + D')^2 - 4\,(D\, D' \sin^2 b)}}{2}$$

or more simply $\quad D_2 = D + D' - D_1$

Thus, as an example, + 3 Cyl. Ax. 70° $\bigcirc$ + 2 Cyl. Ax. 20°. Here b = 50° and $\sin^2 50° = .5868$. Then

$$D_1 = \frac{+3 + 2 - \sqrt{(3+2)^2 - 4\,(3 \times 2 \times .5868)}}{2}$$

$$= \frac{5 - \sqrt{25 - 14.08}}{2} = .85$$

$$D_2 = \frac{+3 + 2 + \sqrt{(3+2)^2 - 4\,(3 \times 2 \times .5868)}}{2}$$

$$= \frac{5 + \sqrt{25 - 14.08}}{2} = 4.15$$

Or $\qquad D_2 = +3 + 2 - .85 = +4.15.$

The two principal powers are + .85 and + 4.15, so that the combination is

+ .85 D Cyl. $\bigcirc$ + 4.15 D Cyl. or .85 D Sph. $\bigcirc$ 3.30 Cyl.

It will be seen that

$$D\,D' \sin^2 b = D_1\,D_2, \text{ for}$$

$$2 \times 3 \times .5868 = .85 \times 4.15 = 3.52.$$

The sum of the total maximum powers of the two original cyls., in this example 5 D, is not changed by altering the positions of the two axes, for the sum of the two principal meridians of the resultant cylindricals is similarly 5 D.

Working Formulæ.—From the foregoing formulæ we can then deduce the following working formulæ for finding directly S the resultant sph., C the cyl., and c the angular position of the axis of the resultant cyl. of any two obliquely crossed cyls.:—

The cyl. being the difference between the two resultant powers is $D_1 - D_2 = C$, which is obtained from the following formula:—

$$C = \sqrt{(D + D')^2 - 4\,(D\,D' \sin^2 b)}$$

The sph. being the lower of the two resultant powers, is then

$$S = \frac{D + D' - C}{2}$$

The position of the axis is

$$\tan 2c = \frac{D \sin 2b}{D' + (D \cos 2b)}$$

c being the angle between the axis of the resultant cyl. and that of D′, or a similar position with reference to D is found by substituting in the above formula D′ for D and D for D′.

Thus, taking the former example of + 3 Cyl. Ax. 70° ⌒ + 2 Cyl. Ax. 20°. Here b = 50°, so 2 b = 100°. Now sin 100° = .9848 and cos 100° = − .1736, therefore

$$\tan 2\,c = \frac{3 \times .9648}{2 + [3 \times (-\ .1736)]} = \frac{2.9544}{1.4792} = 2 = \tan 63^\circ\ 24'$$

half of which is 31° 42′; this measured from 20° locates the angle of the resultant cyl. at 51° 42′, or

$$\tan 2\,c = \frac{2 \times .9848}{3 + [2 \times (-\ .1736)]} = \frac{1.9696}{2.6528} = .7421 = \tan 36^\circ\ 36'.$$

half of which is 18° 18′; this measured from 70° gives similarly 51° 42′.

It must be remembered that when b is greater than 45° and less than 135° cos 2b is negative; also if b is greater than 90° (and does not exceed 180°) sin 2b is negative. In such cases the negative sign must be prefixed.

Approximate Calculations for the Axis.—The formula for finding the axis of the resultant cyl. being complicated, it may be approximately located by dividing the angle between the two original cyls., if they be of like nature, proportionately to their two powers. The error is not great where the two cyls. do not vary much in strength. In this case the interval between the axes of the two original cyls. is 70 − 20 = 50° and 50/5 = 10,

and 10 × 3 = 30° from the weaker original cyl.,

and 10 × 2 = 20° from the stronger.

Thus the axis of the resultant cyl. would be located at 50° instead of 51° 42′.

Unlike Cyls.—When the cyls. are of unlike nature care must be taken with the signs, and in locating the axis of the resultant cyls. The angular distance of the axis of the resultant + cyl. must be reckoned away from the axis of the original + cyl. and towards the meridian of greatest power of the − cyl. Similarly, if the resultant cyl. is − the axis lies between the axis of the original − cyl. and the meridian of greatest power of the + cyl.

As with cyls. of like nature, the total powers of obliquely crossed unlike cyls. are not altered; the sum of the two resultant principal powers is the sum of the maximum refractive powers of the two original cyls.

Thus, as an example, + 4 Cyl. Ax. 20° ⌒ − 2.75 Cyl. Ax. 65°, b = 45° and $\sin^2 45^\circ = .5$.

Now $D_1 + D_2 = D + D' = 4 - 2.75 = +\ 1.25$,

and $D_1\ D_2 = 4 \times (-\ 2.75) \times .5 = -\ 5.5$.

Using the formulæ before given we have

$$D_1 = \frac{+4 - 2.75 - \sqrt{(+4 - 2.75)^2 - 4\,(4 \times -2.75 \times .5)}}{2} = -1.80$$

$$D_2 = \frac{+4 - 2.75 + \sqrt{(+4 - 2.75)^2 - 4\,(4 \times -2.75 \times .5)}}{2} = +3.05$$

$$D_1 + D_2 = -1.80 + 3.05 = +1.25$$
$$D_1\,D_2 = -1.80 \times 3.05 = -5.5.$$

Using the other (working) formulæ, we should obtain:—

$$C = \sqrt{(+4 - 2.75)^2 - 4\,(4 \times -2.75 \times .5)} = +4.85$$

$$S. = \frac{+4 - 2.75 - 4.85}{2} = -1.80$$

$b = 45^\circ$ and $2\,b = 90^\circ$; now $\sin 90^\circ = 1$ and $\cos 90^\circ = 0$; then

$$\tan 2\,c = \frac{-2.75 \times 1}{4 + (2.75 \times 0)} = \frac{-2.75}{4} = -.6875 = \tan 34^\circ\ 30',$$

half of which is 17° 15′, so that the axis of the resultant + cyl. lies at 20° – 17° 15′ = 2° 45′.

The combination is – 1.80 Sph. ⌒ + 4.85 Cyl. Ax. 2° 45′,
or it may be + 3.05 Sph. ⌒ – 4.85 Cyl. Ax. 92° 45′,
for we could calculate

$$\tan 2\,c = \frac{4 \times 1}{-2.75 + (4 \times 0)} = \frac{4}{-2.75} = -1.4545 = \tan 55^\circ\ 30',$$

half of which is 27° 45′, so that the axis of the resultant – cyl. is at 65° + 27° 45′ = 92 45′.

CHAPTER IX.

PRISMS AND PRISMATIC EFFECT OF LENSES.

CENTERING OF LENSES.

Optical Centre. — The optical centre of a spherical lens lies, as mentioned previously, on the principal axis at a distance from each surface proportional to its radius of curvature. It is situated therefore on the line passing through the thickest part of a convex and the thinnest part of a concave lens, and is that point through which rays of light pass without refraction.

Geometrical Centre.—The geometrical centre is that point of the lens which is equi-distant from the opposite edges. It can be located by inspection, or, more exactly, by drawing a horizontal line across the lens, connecting the two extremities of the long diameter, and a vertical line connecting the highest and lowest points; where the two lines cut each other is the geometrical centre.

Centred and Decentred Lenses.—A lens is said to be centred when its optical and geometrical centres coincide, and is said to be decentered when they do not. When an object is viewed through the geometrical centre of a decentered lens the effect is precisely the same as if the lens were combined with a prism. Similarly, if a centered lens is looked through at a point which is not in line with the optical and geometrical centre the effect is the same as if a sphero-prism were substituted.

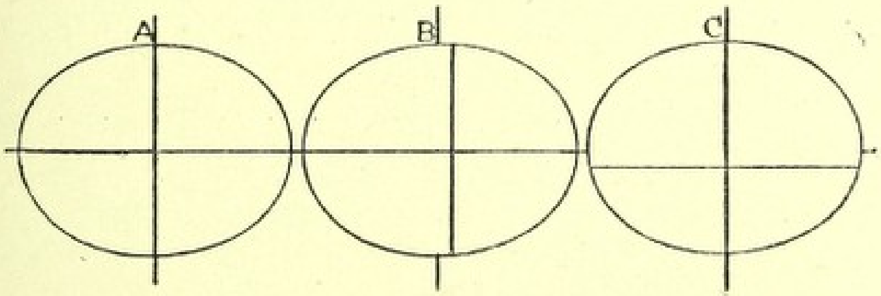


Fig. 160.

To learn whether a spherical lens is truly centred it must be held parallel to the analysing card and viewed through its geometrical centre. If centred (Fig. 160) the junction of the two lines of the card is seen in line with the exact centre of the lens, they being continuous as seen through the lens and beyond the edges. If decentered the junction of the two lines is seen not to coincide with the exact centre of the lens and the vertical line in (B) or the horizontal line in (C) is broken at the edges of the lens. The lines are seen as in (A) through a centred lens.

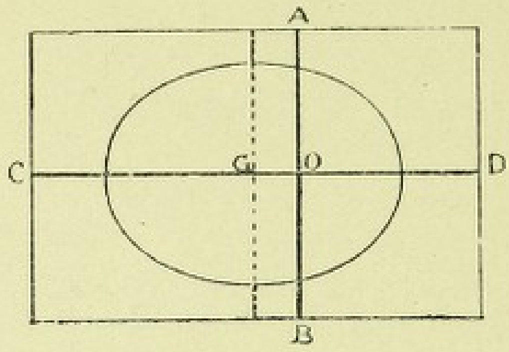


Fig. 161.

Locating the O. C.—To locate the optical centre the lens must be moved about until the cross lines seen through it are continuous with the parts of the lines seen beyond the edges, as in Fig. 161, where G is the geometrical centre of the lens. The optical centre O coincides with that point of the lens opposite to the intersection of the cross lines, and can be, if necessary, marked by a dot.

For greater accuracy the test should be made with fine cross lines drawn on a small card placed on the table, the lens being held steadily a short distance above the card and in a plane parallel to it. This method is preferable for strong lenses, but the analysing card at a reasonable distance is better for a very weak lens.

Positions of No Prism Effect. — In a spherical there is only one point in the refracting plane of the lens, the optical centre, where there is no prismatic effect. This point lies on a principal axis. In a plano-cylindrical there is a line without prismatic effect along the axis.

In Fig. 162 let the lens be a + cylindrical whose axis A X is at 45°, B C is a vertical and D E a horizontal line. On looking through the lens the points F G H on the vertical line B C are seen deflected by the prismatic action of the cylindrical to F′ G′ H′, upwards and to the left, the virtual prisms being base down and to the right. The points K L M on the horizontal line D E are seen deflected to K′ L′ M′, also upwards and to the left, the virtual prisms being base down and to the right. On the other side of the axis the virtual prisms are base up and to the left, and the deflections are downwards and to the right. Thus a convex cylindrical axis at, say, 45° causes a vertical line B C to appear as B′ C′ and a horizontal line D E to appear as the dotted line D′ E′ in Fig. 162.

If another equal + cylindrical were placed axis at right angles to the first, the horizontal deviation of the vertical line, and the vertical deviation of the horizontal are neutralised, but the vertical effect in the vertical meridian and the horizontal effect in the horizontal are doubled, the combination being equivalent to a spherical lens in which the prismatic effects are equal in every meridian.

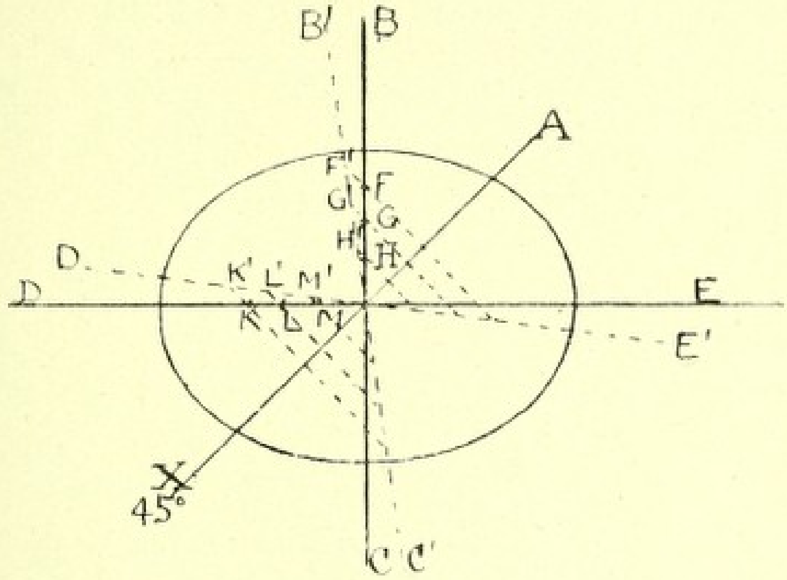


Fig. 162.

With a concave cylindrical the edges of the virtual prisms are towards the axis, and if a —Cyl. Ax. 45° be looked through (Fig. 163) a vertical line B C appears as B′ C′ and a horizontal line D E appears as D′ E′, the deviation of these lines being towards the axis of the lens.

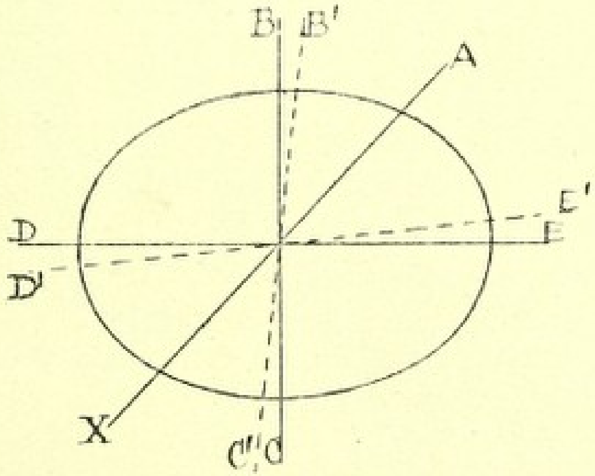


Fig. 163.

In a sphero-cylindrical lens there is (as in the case of a spherical) a point of no prismatic effect. This is where the axis of the cylindrical cuts that of the spherical, and it is therefore at the geometrical centre of a centered lens.

In Fig. 164 let the lens be a + sphero-cylindrical whose axis A X is at 45°. Let B be a point situated between the vertical and the axis. There is at this point, derived from the spherical, the effect O B of a prism base down to the left. The cylindrical contributes a prismatic effect P B, the base of the virtual prism being down to the right. Thus there are two vertical effects both directed upwards, and two horizontal, the one directed to the left and the other to the right. These latter neutralise each other at some point B, and similarly at every point on the line E F.

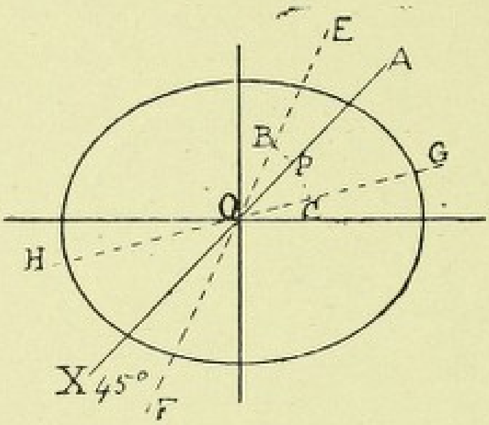


Fig. 164.

Between the axis and the horizontal, at some point C, there is the effect O C of a prism base down and to the left derived from the spherical, and from the cylindrical there is the effect P C of a prism base up and to the left. There are thus two horizontal effects both directed to the left, and two vertical, the one up and the other down. At some point C the opposing vertical effects neutralise each other, and similarly we have a neutralising effect all along the line G H.

In a Cc sphero-cylindrical there are similar prismatic effects, but directed in the opposite directions.

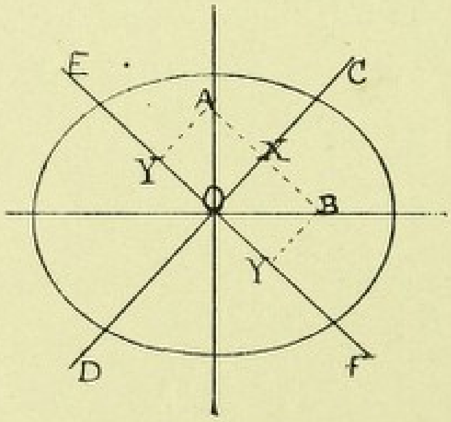


Fig. 165.

Let Fig. 165 be a combination of + Cyl. Ax. 45° ⌢ − Cyl. Ax. 135°, the two of equal power. C D is the axis of the convex cylindrical, and E F that of a concave. At some point A the convex cylindrical has an effect X A of a prism base down and to the right, the concave has an effect Y A of one base up and to the right. The up and down vertical effects neutralise each other and there is a combined lateral effect. At the point B the convex acts with an effect X B base up and to the left, and the concave with an effect Y B base up and to the right. The right and left horizontal effects neutralising each other, the combined deviation is vertical.

Thus the point A is deviated to the left, and B is deviated downwards. The vertical line is seen inclined to the left above, and to the right below. The horizontal line is inclined downwards on the right, and upwards on the left.

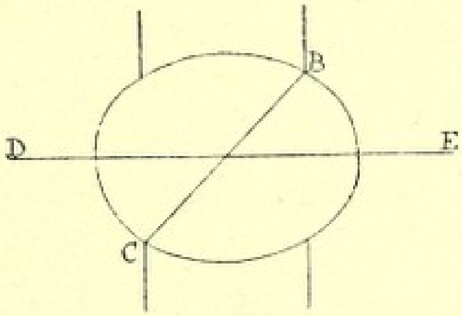


Fig. 166.

Locating the Lines of No Prism Effect.—An oblique sphero-cylindrical is moved horizontally until the oblique image of a vertical line is seen in contact, at B (Fig. 166), at the upper edge of the lens, with the line itself, seen above the lens. Similar contact is then obtained at the lower edge of the lens, say at C. Connecting these two contact points indicates the line of no horizontal prismatic effect.

Similarly the points where the vertical prismatic effects are neutralised are those where, by moving the lens vertically upwards and downwards, a horizontal bar is at each side, in contact with its image. The line connecting them indicates the line of no vertical prismatic effect.

The peculiar prismatic effects of the cyl. and sph.-cyl. explain the appearance of objects seen through them. Thus a square body observed through an oblique sphero-cylindrical appears distorted, the various parts of its image being more deflected in certain directions than in others. If such a body is seen through a concave sphero-cylindrical axis vertical the deflection is greater horizontally than it is vertically, and between the vertical and the horizontal the deflection gradually decreases; hence the square appears to be a rectangle with its short axis horizontal.

THE MEASUREMENT AND NUMERICAL NOTATION OF PRISMS.

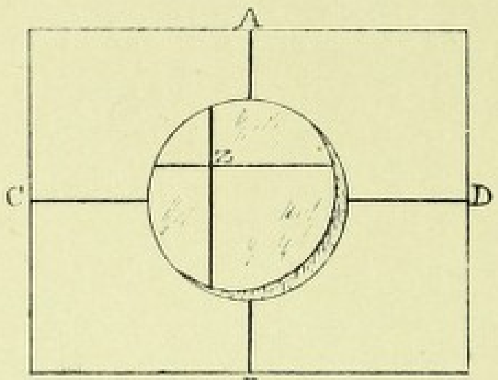


Fig. 167.

The Deviation caused by Prisms.—When a glass which possesses a prismatic element is rotated around its geometrical centre the base apex plane and the edge of the prism are of course rotated also. Consequently if the cross lines of the chart A B C D (Fig. 167) be observed they, being deviated towards the edge of the prism, move around with the latter, the junction Z of the cross lines being always deflected towards the edge of the prism. As the glass is rotated the vertical line moves horizontally and the horizontal line moves vertically, but the two always remain at right angles to each other and do not become oblique as when a cylindrical is rotated. The movement of the lines is the same, whether the prismatic element is derived from a prism or from decentration.

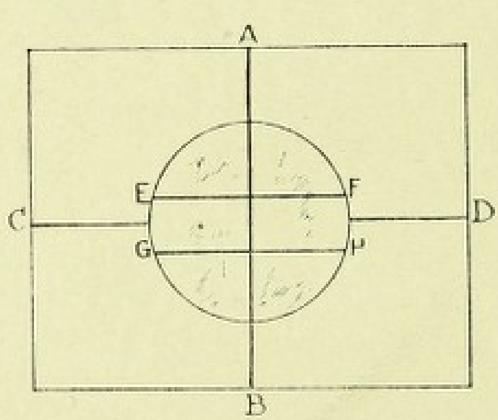


Fig. 168.

Locating the Base-Apex Plane.—As a prism, a sphero-prism, or a decentered spherical is rotated in front of the analyser it is found that at a certain position there is a continuity of one of the lines within and beyond the edges of the glass, as in Fig. 168, where the vertical line A B is continuous. The direction of this line indicates that of the base-apex plane of the prism, or of the virtual prism contained in a decentered spherical.

There is little difficulty in locating by inspection the base and edge of a square prism, but if circular or oval the base-apex line is found by thus rotating the prism until a, say, vertical line is seen through it unbroken and continuous above and below the prism, as in Fig. 168.

If the horizontal line C D is deflected upwards as to E F, the apex is then pointing upwards towards A and the base is down towards B. If the deflection of C D is downwards towards G H the edge of the prism is pointing downwards.

Circular trial prisms should be tested as to their indicated base-apex lines. If properly marked the scratches lie over A B when that line appears unbroken by the prism.

False Images of a Prism.—On looking at a candle flame through a prism a second fainter image can be seen which is often a source of annoyance to the wearer. This image is formed by internal reflection of some of the rays incident on the prism from the flame, and is projected along a line parallel to the base-apex line under an angle about five or six times the deviating angle of the prism, so that in a strong prism it lies too far outside the prism to be observed unless specially sought for. Its position also varies slightly with the refractive index of the glass. In a weak prism Dr. E. E. Maddox indicates, in his work "The Clinical Use of Prisms," that it can be utilised for the exact horizontal or vertical adjustment of the base apex line by noting that the direct and the reflected image are in the same horizontal or vertical plane.

THE VARIOUS METHODS OF NOTATING PRISMS.

Refracting Angle.—The numeration of prisms according to the refracting angle, that is by the physical form, is similar to the numeration of lenses by the radius of curvature. Slight differences in the angle cannot be easily recognised, and the true optical effect is not taken into consideration. If there be two prisms of, say, 3°, the one of glass $\mu = 1.50$ and the other μ 1.54, they are both prisms of 3°, but their prismatic effects are by no means identical.

Angle of Deviation.—The angle of deviation indicates the optical property of the prism, and is the combined result of the angle of inclination of the two refracting surfaces and of the refracting power of the medium, with both of which it varies directly, thus the true optical effect is indicated; but this system has a drawback in that the angle itself cannot be measured. The unit is, of course, 1° deviation (marked 1 °d).

Relationship of the ° and the °d.—The number of degrees in the deviating angle of a prism being about half that of its principal angle, the former unit is nearly double the value of the latter. Therefore, if two prisms of the same strength be numbered respectively in degrees of deviation and degrees of inclination, the number of the former would be about half that of the latter. But in the following paragraphs, μ is taken as 1.52, and, therefore, the relative values are less than two to one.

Prism Diopter.—The prism notation introduced by Mr. Charles Prentice of New York, being based on the deviation itself, presents many advantages. The unit is the prism diopter (indicated by the sign △ or P. D.) which is the strength of a prism that causes a deviation of 1 cm. (on a tangent) at a distance of 1 metre. The deviation is, therefore, 1 in 100, and

$$\frac{\text{N. prism diopters}}{100} = \text{tangent of the angle of deviation.}$$

Separate prisms, numbered in prism diopters, when placed together are not exactly equal to the sum of their powers. Thus 1△ is equal to 34′ 22½′ and 10△ = 5° 45′, but if 10 single 1△ prisms were placed one in front of the other the total angular deviation caused by them would be 34′ 22½″ × 10 = 5° 43′ 45″. The difference is, however, so very inconsiderable, especially in the weak prisms needed in spectacle work, as to be of no practical importance.

The prism diopter is nearly equal to the ordinary prism degree when the glass has an index of refraction of 1.54. The 1° has then .54° deviation or 32′ 12″ and the tangent included by such an angle at 1 M is .94 cm. That of the prism diopter being one centimetre, there is a difference only of about 6 per cent. The principal angle, when μ = 1.54, required to produce 1△ is 1° 3′. If μ = 1.575 the prism degree is just equal to the prism diopter, for .575° = 34′ 30″, the tangent of which is .01. When μ = 1.52 the ° = .9△, and this is the refractive index of the glass usually employed. It must, however, be remembered that these values can only be considered true for small angles such as occur in the optics of spectacle work.

Centrad.— Another prism unit is the centrad which causes a deviation of 1 cm., on the arc of the circle, at one metre. The deviation is again 1 in 100 and the difference between the arc and the tangent of small angles being negligible, the centrad and △ may be considered equal. A given prism numbered in △ would be of fractionally higher number than if numbered in centrads. The centrad more nearly agrees with the metre angle (which is measured by the sine of the angle) than the prism diopter, because there is less difference in value between the sine and the arc than between the sine and the tangent. It is, however, very much more inconvenient to measure on a curved than on a flat surface, and the centrad has never come into general use.

$$\frac{\text{N Centrads}}{100} = \text{arc of the angle of deviation.}$$

The Metran.—Another unit prism suggested by L. Laurance is the metran. This unit is a prism which causes a deviation of 3 cm. when placed in front of the eye at one meter from the scale. It has, therefore, about 1.75° (or 1° .45′) deviation and is the same as the metre angle for the average interpupillary distance of $2\frac{3}{8}$in. or 60mm. The symbol is thus 4∧.

Conversion of Prismatic Expression.—For conversion from one system of prism notation to another, the following rules apply:

To convert refracting degrees into degrees of deviation, multiply by .52; (or divide by 1.9), thus

$$4.5^\circ = 4.5 \times .52 = 2.34^\circ d.$$

To convert degrees of deviation into refracting degrees, divide by .52; (or multiply by 1.9), thus

$$7^\circ d = \frac{7}{.52} = 13.5^\circ.$$

To convert degrees into prism diopters, multiply by .9; (or divide by 1.1), thus

$$6^\circ = 6 \times .9 = 5.4^\triangle$$

To convert prism diopters into degrees, divide by .9 (or multiply by 1.1) thus

$$4^\triangle = \frac{4}{.9} = 4.44^\circ$$

To convert degrees of deviation into prism diopters multiply by 1.75—more exactly by 1.745; (or divide by .575) thus

$$3 \times 1.75 = 5.25^\triangle$$

To convert prism diopters into degrees of deviation, divide by 1.75; (or multiply by .575) thus

$$3.5^\triangle = \frac{3.5}{1.75} = 2^\circ d.$$

The calculation for centrads can be taken as the same as that for prism diopters.

To convert metrans into prism degrees, multiply by 3.5.

To convert prism degrees into metrans, divide by 3.5.

To convert metrans into prism diopters, multiply by 3.

To convert prism diopters into metrans, divide by 3.

To convert metrans into degrees of deviation, multiply by 1.75.

To convert degrees of deviation into metrans, divide by 1.75.

Tabular Relative Values.—The following table gives for comparison the angles of deviation of the various unit prisms and the approximate displacement caused by them at certain distances.

Unit.	Angle of deviation.		Actual deviation in cms. at 1 M.	3 M.	6 M.
1°d	1°		1.745	5.25	10.50
1° (μ = 1.50)	30′	or .5°	.872	2.62	5.15
1° (μ = 1.52)	31′ 12″	or .52°	.9	2.7	5.4
1° (μ = 1.54)	32′ 12″	or .54°	.94	2.82	5.64
1° (μ = 1.575)	34′ 30″	or .575	1.	3.	6.
1△	34′ 30″	or .575°	1.	3	6
1 centrad	34′ 12″	or .57°	1.	3	6
1 metran	1°45′	or 1.75°	3.	9	18

The relative values are:

The degree of deviation being 1. the value of the refracting degree is .52°d and that of the prism diopter is .575°d.

The degree being 1 the value of the degree of deviation is 1.9° and that of the prism diopter is 1.1°.

The prism diopter being 1 the value of the degree of deviation is 1.745△ (say 1.75 or $1\frac{3}{4}$) and the degree is .9△.

Therefore 1°.d = 1.9° = 1.745△ (1.75).
31′ 12″ or .52°d = 1° = .9△
34′ 30″ or .575°d = 1.1° = 1.0△

Or the following simplified scale can be used:

1.75△ = 1°d.
1.75°d or 3△ = 1 Λ or 1 M A (the unit of convergence).

The following table gives the approximate equivalent values of prisms in the three systems of notation :—

Refracting Degrees or Principal Angle in Degrees and decimals.	Degrees of Deviation or Deviating Angle in Degrees and minutes.	Prism Diopters and decimals.
.25	8′	.22
27	9′	25
.50	15′	.45
.55	17′	50
.75	23′	.68
.82	26′	75
.95	30′	.87
1.00	31′	.90
1.10	35′	1.00
1.25	39′	1.12
1.39	43′	1.25
1.43	45′	1.30

Refracting Degrees or Principal Angle in Degrees and decimals.	Degrees of Deviation or Deviating Angle in Degrees and minutes.	Prism Diopters and decimals.
1.50	47′	1.45
1.65	52′	1.50
1.75	54′	1.57
1.90	1°	1.75
2.00	1°.2′	1.80
2.20	1°.9′	2.00
2.35	1°.15′	2.18
2.50	1°.17′	2.25
2.75	1°.26′	2.50
2.85	1°.30′	2.62
3.00	1°.34′	2.70
3.30	1°.45′	3.00
3.50	1°.50′	3.15
3.85	2°	3.50
4.00	2°.4′	3.60
4.50	2°.18′	4.00
4.80	2°.30′	4.37
5.00	2°.34′	4.50
5.50	2°.52′	5.00
5.70	3°	5.25
6.00	3°.8′	5.40
6.60	3°.30′	6.00
7.00	3°.40′	6.30
7.70	4°	7.00
8.00	4°.8′	7.20
8.65	4°.30′	7.87
8.80	4°.36′	8.00
9.00	4°.40′	8.10
9.50	5°	8.75
10.00	5°.10′	9.00
10.50	5°.30′	9.62
11.00	5°.45′	10.00
11.50	6°	10.50
12.00	6°.18′	11.00
12.50	6°.30′	11.37
13.00	7°	12.00
14.00	7°.30′	13.00
15.00	8°	14.00
16.00	8°.30′	15.00
17.00	9°	16.00
18.00	9°.30′	17.00
19.00	10°	17.50
20.00	10°.30′	18.00
21.00	11°	19.00
22.00	11°.30′	20.00

Measurement of Prisms.—The measurement of the refracting angle of a prism is termed *goniometry*, while that of its deviation is termed *prismetry*.

Neutralisation of Prisms.—The strength of a prism can be learnt by neutralisation. The base apex line being located, the displacement of a bar of the analyser can be neutralised by selecting one prism after another from the test case and placing it in opposition to the unknown prism; that is, placing the base of the former over the edge of the latter, until that test prism is found which causes the bar to be seen continuous beyond and through the two prisms. The number of the known prism which neutralises the deviation of the unknown prism, indicates the value of the latter. By this method, however, the deviating angle is really neutralised although the neutraliser may be numbered according to its refracting angle.

Determining the Principal Angle.—The *principal* angle of a prism can be roughly measured by enclosing it between the legs of a pair of compasses and measuring the angle so obtained on a protractor or by any instrument made for the purpose. Also the *deviating* angle can be measured and the principal angle calculated, provided the index of refraction be known.

The principal angle can be determined also by a goniometer consisting of a pivotted arm, at one end of which there are two legs which rest on the face of the prism; the other end indicates the angle on a scale.

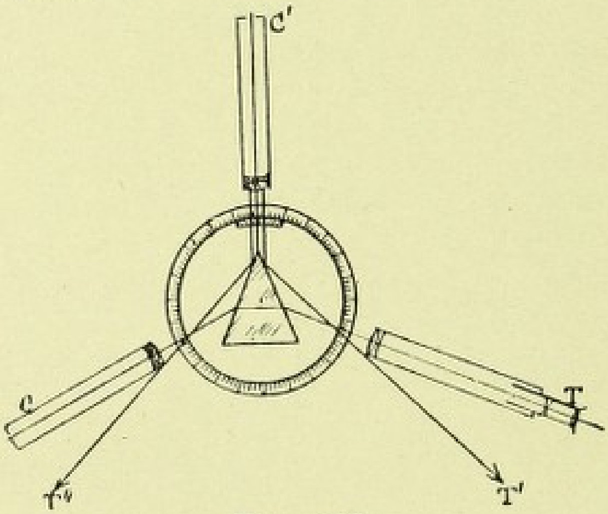


Fig. 169.

More accurately the refracting angle is determined by directing towards its apex a small beam of parallel light from a collimator C′ of the goniometer, described in the next paragraph, and observing through the telescope T light reflected first from one surface and then from the other when the telescope is respectively at, say, T′ and T″. The angle through which the telescope has been turned from T′ to T″ is twice the refracting angle.

Determination of the Degree of Deviation.—The deviating angle of a prism is accurately determined by the goniometer (Fig. 169), which consists of a horizontal circle marked in degrees suitably mounted on a stand on which two telescopes, C and T, can be rotated. One of these, C, is a collimator, from which light emerges, as a parallel beam, through a narrow slit. The other T is an observing telescope. In the centre of the circle there is a small table on which the prism is placed.

In order to measure the deviating angle of a prism the tubes C and T are brought into line so that the image of the slit appears in the centre of the field of view. A reading is taken on the circle. The prism is then placed in position and the telescope must be rotated until the image of the slit can be seen. The angular distance through which T is moved is the deviating angle of the prism, care being taken that the deviation is a minimum. This can be done by slightly rotating the prism backwards and forwards until a position is found when the slightest movement in *either* direction *increases* the deviation.

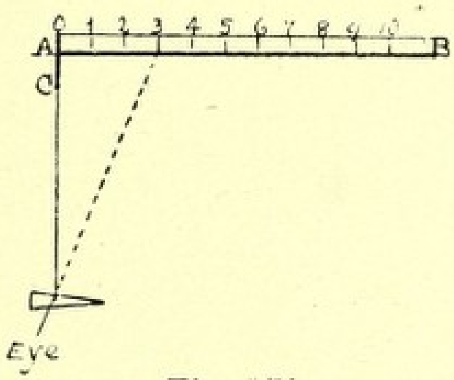


Fig. 170.

The Tangent Scale.—A tangent scale, shown in Fig. 170, constitutes the most convenient method of measuring prisms. It consists of a card, say, 12 inches wide and 30 inches long, scaled so that the intervals between the divisions represent the tangent measurement of the angle of deviation. These intervals, therefore, vary in size with the distance at which the card is used.

The line A C (Fig. 170) is looked at through the prism, which is held sufficiently low for the figures on the card to be seen over it, the base being directed towards A while the edge points to B.

If the line A B is displaced upwards or downwards the prism must be rotated in a plane parallel to the card until A B is continuous and seen unbroken through the prism. The base apex line is then horizontal, and the horizontal deviation is then greater than with any other position of the prism in this plane.

The number towards which the deviated part of A C points indicates the prismatic power of the prism in degrees, degrees of deviation or prism diopters, according to the arrangement of the tangent scale.

The tangent scale was originally designed by Dr. Maddox.

Minimum and other Deviations.—But the deviation caused by the prism varies also if its position departs from that of the minimum deviation. Consequently, when B is unbroken, the prism must be rotated on its axis in order to find the minimum deviation, this being the numerical strength of the prism. Thus, in Fig. 170 the prism is presumed to be in the position of minimum deviation and the indicated number is 3, but if the edge of the prism were turned either towards the scale, or away from it, the indicated deviation would be greater than 3.

If the prism is combined with spherical or cylindrical powers, these must be neutralised before the prismatic power can be measured on a tangent scale, care being taken that the geometrical centres of neutralising and neutralised lenses exactly coincide, otherwise a false measure of the prismatic power is obtained.

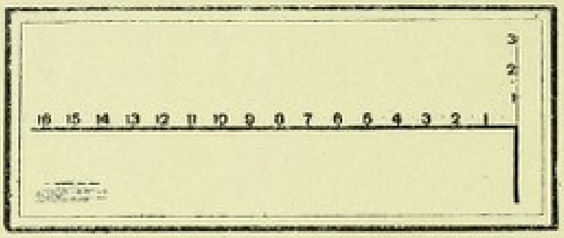


Fig. 170A.

A tangent scale arranged for one system could be utilised for others by holding the prism at the proper distance. Thus, the intervals of the "Orthops" scale (Fig. 170A) are 3.5 cm., so that this used at 2 M. indicates degrees of deviation and at 3.5 M. indicates prism diopters. If used at 4 M. it would serve for ordinary degrees.

The deviation of a prism can be measured by the following modification of the ordinary tangent scale.

Another Tangent Measurement.—Parallel light is passed through a suitable Cx. cylindrical lens and is brought to a sharp focus, as a vertical line at the zero of a tangent scale. The prism is then introduced near to the cylindrical with its base at right angles to the zero line and the sharply-focussed line of light is deviated to some number on the scale which indicates the value of the prism. This method is mentioned by Dr. E. E. Maddox in his work on prisms.

Distance for Tangent Scales.—In theory six metres is the proper distance for measuring prisms on a tangent scale, since from that distance the light has no appreciable divergence, but in practice shorter distances, say, three or four metres, are preferable.

Calculations in Prism Measurements.—Calculations concerning the deviation caused by prisms at the given distances can be made from the following formula, but for degrees and degrees of deviation, while sufficiently accurate for practical purposes, they are not exact since tangents of angle are used in place of the angles themselves.

Let P represent the power of the prism, M its distance in metres from the object viewed, C the deviation in centimetres, and K a constant for each system of prism notation. Then

$$C = P\ M\ K.$$

For the refracting angle, $C = P \times M \times .9$
For the deviating angle, $C = P \times M \times 1.75$
For prism diopters, $C = P \times M$.

Thus at 3 metres, the deviation caused by a 4°, a 4° d, and a 4△ respectively is

$$4^\circ \times 3 \times .9 = 10.8 \text{ cm.}$$
$$4^\circ d \times 3 \times 1.75 = 21 \text{ cm.}$$
$$4^\triangle \times 3 = 12 \text{ cm.}$$

If the deviation caused by a prism at four metres is 5 cm,. the prism is

$$\frac{5}{4 \times .9} = 1.4^\circ \quad \text{or} \quad \frac{5}{4 \times 1.75} = .7^\circ d \quad \text{or} \quad \frac{5}{4} = 1.25^\triangle.$$

Example.—At what distance will a prism of 5°, one of 5°d, and one of 5△ respectively cause a deviation of 15 cm.

$$\frac{15}{5^\circ \times .9} = 3.33 \text{ M.} \qquad \frac{15}{5^\circ d \times 1.75} = 1.75 \text{ M.} \qquad \frac{15}{5^\triangle} = 3 \text{ M.}$$

The following table shows the deviation caused by prisms at various distances. The figures, in the 4 M. and 6 M. columns, are only approximately true in the decimals. For other distances the deviations may be considered proportional.

	Table of Deviation in Cm. at			
Prism.	2 Metres.	3 Metres.	4 Metres.	6 Metres.
1°d	3.49	5.24	7.00	10.50
2°d	6.98	10.48	14.00	21.00
3°d	10.48	15.73	21.00	31.50
4°d	13.98	20.97	28.00	42.00
5°d	17.49	26.24	35.00	52.50
6°d	21.02	31.54	42.00	63.00
7°d	24.55	36.83	49.00	73.75
8°d	28.10	42.16	56.25	84.25
9°d	31.67	47.52	63.50	95.00
10°d	35.26	52.90	70.50	106.00
11°d	38.87	58.32	77.75	116.50
12°d	42.51	63.77	85.00	127.50
13°d	46.17	69.26	92.50	138.50
14°d	49.86	74.80	99.75	149.50
15°d	53.58	80.37	107.00	160.75
16°d	57.34	86.01	114.75	172.00
17°d	61.14	91.71	122.25	183.50
18°d	64.98	97.47	130.00	195.00
19°d	68.86	103.30	137.75	206.50
20°d	72.80	109.20	145.50	218.50

A table for △s is hardly necessary since the calculation $C = P\ M$ is so very easy.

For degrees the distances in the above table can be halved without much error arising.

Table showing the number of a prism which causes a given deviation at 1 M.

Deviation in cm.	Angle of deviation.	Degrees of the refracting angle. ($\mu = 1.52$)	Prism diopters.
.25	9′	.27	.25
.50	17′	.55	.50
.75	26′	.82	.75
1.00	35′	1.10	1.00
1.25	43′	1.37	1.25
1.50	52′	1.66	1.50
1.75	1°	2.00	1.75
2.00	1°.9′	2.25	2.00
2.25	1°.17′	2.50	2.25
2.50	1°.26′	2.75	2.50
2.75	1°.35′	3.00	2.75
3.00	1°.43′	3.33	3.00
3.25	1°.52′	3.60	3.25
3.50	2°	3.85	3.50
3.75	2°.9′	4.00	3.75
4.00	2°.18′	4.40	4.00
4.50	2°.35′	5.00	4.50
5.00	2°.52′	5.50	5.00
5.50	3°.9′	6.00	5.50
6.00	3°.26′	6.60	6.00
6.50	3°.43′	7.15	6.50
7.00	4°	7.70	7.00
7.50	4°.18′	8.25	7.50
8.00	4°.35′	8.80	8.00
8.50	4°.52′	9.35	8.50
9.00	5°.9′	9.90	9.00
9.50	5°.26′	10.45	9.50
10.	5°.43′	11.00	10.
11.	6°.17′	12.10	11.
12.	6°.50′	13.20	12.
13.	7°.24′	14.30	13.
14.	7°.58′	15.40	14.
15.	8°.32′	16.50	15.
16.	9°.6′	17.60	16.
17.	9°.39′	18.70	17.
18.	10°.12′	19.80	18.
19.	10°.46′	20.90	19.
20.	11°.19′	22.00	20.

A tangent scale divided into spaces of 1 cm. and used with the last table would indicate the power which produces the deviation caused by a certain prism at a certain distance. Thus if a prism at three metres caused 6.75 cm. deviation on such a scale, it would at 1 M. cause $6.75/3 = 2.25$ cm., so that the prism is 2.25△ or 1° 17′ d or 2.50°.

Prism Nomenclature.—A prism placed with its base towards the nose is termed + or base in, while a prism placed with its base towards the temple is termed — or base out. A prism is called horizontal, or vertical, according as the base apex line is horizontal or vertical respectively.

OBLIQUE PRISMS.

Direction of Deviation.—A prism so changes the direction of light that an object viewed through it appears in a different position from that which it really occupies. The deviation is parallel to the base apex line and towards the edge of the prism.

If a cross bar is viewed through a prism held with base apex line horizontal, the vertical bar is displaced horizontally to an extent dependent on the strength of the prism, and there is no vertical displacement of the horizontal bar. If, now, the prism be rotated a few degrees in a plane parallel to the card, so that the base apex line is oblique to both bars the horizontal deviation becomes less, and a vertical deviation is occasioned as in Fig. 167. If the rotation be continued the horizontal deviation continues to decrease and the vertical to increase, until when the base apex line is vertical all the deviation is vertical and there is none in the horizontal plane.

The maximum effect d of the prism (Fig. 167) is always in the plane of the base apex line, and when the latter is oblique, its effect can be divided into V, a vertical, and H, a horizontal component, which are equal when the base apex line is at 45°. In every position of the prism the two bars of the card are perpendicular to each other.

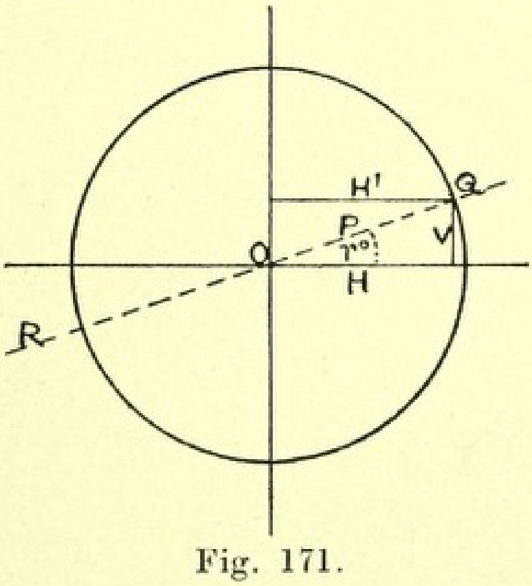


Fig. 171.

Indirect Effects.—(Fig. 171).—Suppose V to represent the vertical and H = H′ the horizontal forces of a rotated prism. Let P = O Q represent the power of the prism, and r the angular rotation of its base apex line from the horizontal.

Then, since $\sin r = \dfrac{V}{P}$ and $\cos r = \dfrac{H}{P}$

$$V = P \sin r \quad \text{and} \quad H = P \cos r. \qquad [12$$

Thus: let the base apex line of a 5°d prism be at 20° from the horizontal, then:

$$V = 5 \times .3420 = 1.71^\circ \text{ and } H = 5 \times .9397 = 4.698^\circ.$$

If the base apex line is at 45°, a 6△ has

$$V = 6 \times .7071 = 4.24^\triangle \text{ and } H = 6 \times .7071 = 4.24^\triangle.$$

The vertical or horizontal effects of an oblique prism, or the effect in any oblique meridian of a vertical or horizontal prism can also be obtained by direct neutralisation in the meridian, the power of which has to be learnt.

The rotation r from the horizontal needed to obtain a required vertical or horizontal effect is respectively,

$$\sin r = \frac{V}{P} \quad \text{and} \quad \cos r = \frac{H}{P} \qquad [12$$

Example.—Given a 4° d prism, at what position should the base apex line be placed so that its vertical effect be 1° d? Here

$$\sin r = \frac{1}{4} = .25 = \sin 14^\circ\ 29'$$

The base apex line must be inclined 14° 29′ to the horizontal.

Then $V = 4 \times .25 = 1^\circ$ and $H = 4 \times .9681 = 3.872^\circ$.

If with a 6△ a horizontal effect of 3△ is needed, then

$$\cos r = \frac{3}{6} = .5 = \cos 60^\circ$$

the base-apex line must be at 60° and $V = 6 \times .866 = 5.2^\triangle$.

If instead of the angular distance, of the base apex line, from the horizontal, its distance from the vertical is considered, the sine would apply to the horizontal, and the cosine to the vertical meridian in these calculations.

Let P′ represent the effect of a prism in a given meridian, P the power of the prism and r the angle between the given meridian and the base apex line; then the effect in the given meridian is

$$P' = P \cos r. \qquad [130]$$

Example.—Find the effect at 40° of a 4° prism whose base apex line is vertical. Now the meridian 40° is 50° from the vertical so that r = 50° and cos 50° = .6427, therefore

$$P' = 4 \times .6427 = 2.57°.$$

The maximum effect of a prism is in the direction of its base apex line, while at right angles to this the effect is zero. Let the prism be 6° base apex line horizontal.

The effect at 180° is $H = 6 \times \cos 0° = 6 \times 1 = 6°$,
The effect at 90° is $V = 6 \times \cos 90° = 6 \times 0 = 0°$.

TABLE OF THE EFFECT OF A UNIT PRISM IN VARIOUS MERIDIANS.

The angle between the given meridian and the base apex line.	The power of unit prism in that meridian.
0°	1.0000
5°	.9962
10°	.9848
15°	.9659
20°	.9397
25°	.9063
30°	.8660
35°	.8192
40°	.7660
45°	.7071
50°	.6428
55°	.5736
60°	.5000
65°	.4426
70°	.3420
75°	.2588
80°	.1736
85°	.0872
90°	.0000

For other prisms multiply the number obtained from column 2.

TABLE OF THE HORIZONTAL AND VERTICAL EFFECTS OF OBLIQUE UNIT PRISM.

Base-apex line at 0° or 180°	Horizontal effect. 1.	Vertical effect. 0
5°.45′	.9950	.1
7°.10′	.9922	.125
10°.00′	.9848	.1736
11°.30′	.9799	.2
14°.30′	.9681	.25
17°.30′	.9537	.3
19°.30′	.9426	.333
20°.00′	.9397	.342
22°.00′	.9272	.375
23°.33′	.9165	.4
25°.50′	.9	.4358
29°.00′	.875	.4848
30°.00′	.866	.5
36°.50′	.8	.6
38°.40′	.7808	.625
40°.00′	.766	.6428
41°.25′	.75	.6615
41°.50′	.7451	.666
44°.25′	.7143	.7
45°.00′	.7071	.7071
45°.35′	.7	.7143
48°10′	.666	.7451
48°.35′	.6615	.75
50°.00′	.6428	.766
51°.20′	.625	.7808
53°.10′	.6	.8
60°.00′	.5	.866
61°.00′	.4848	.875
64°.10′	.4358	.9
66°.25′	.4	.9165
68°.00′	.375	.9272
70°.00′	.342	.9397
70°.30′	.333	.9426
72°.30′	.3	.9537
75°.30′	.25	.9681
78°.30′	.2	.9799
80°.00′	.1736	.9848
82°.50′	.125	.9922
84°.15′	.1	.9950
90°.00′	0	1

For other prisms multiply the figures in the 2nd and 3rd columns.

RESULTANT PRISMS.

Calculation.—A resultant prism is the combined result of two other prisms whose base-apex lines are at right angles to each other. If vertical and horizontal prismatic effects are needed the two can be obtained from a single oblique prism.

To calculate a resultant prism the formula is

$$P = \sqrt{V^2 + H^2} \qquad [131]$$

Where P represents the power of the resultant prism and V and H respectively the required vertical and horizontal effects.

If r represents the angle which the base-apex line of the resultant prism makes with the horizontal, then

$$\tan r = \frac{V}{H} \qquad [132]$$

Or, with a reasonable degree of accuracy the resultant base-apex line may be found by dividing 90° by the sum of V and H and multiplying the result by the weaker of the two original figures. This gives the angular distance from the stronger of the original prisms.

Thus, suppose a 3°d base-apex line horizontal and 2° base-apex line vertical be required, then

$$P = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13} = 3.6^\circ d$$

$$\tan r = \frac{V}{H} = \frac{2}{3} = .666 = \tan 33^\circ\ 40'$$

Therefore $r = 33^\circ\ 40'$

The resultant prism needed is 3.6°d (3° 36′) with its base-apex line inclined 33° 40′ from the horizontal, or 56° 20′ from the vertical; that is, say, 3.5° base-apex line at 35°.

Or by the simplified method the resultant base-apex line would have been found thus:—

$$\frac{90}{3 + 2} \times 2 = 36^\circ \text{ from the original } 3^\circ \text{ prism}$$

or

$$\frac{90}{3 + 2} \times 3 = 54^\circ \text{ from the original } 2^\circ \text{ prism.}$$

Construction. — In a right-angled triangle, a b c (Fig. 172), the square of the hypothenuse P is equal to the sum of the squares of the other two sides.

$$P^2 = V^2 + H^2; \text{ or } P = \sqrt{V^2 + H^2}$$

So a resultant can be constructed as follows:—

Draw a straight vertical line V as many inches (or centimetres) long as there are units (degrees, etc.) in the vertical prism and a horizontal line H as many inches (or cm.) long as there are degrees in the horizontal prism. The ends of these two lines being connected by a third line, P, the number of inches (or cm.) in P = the number of degrees in the resultant prism; and the inclination

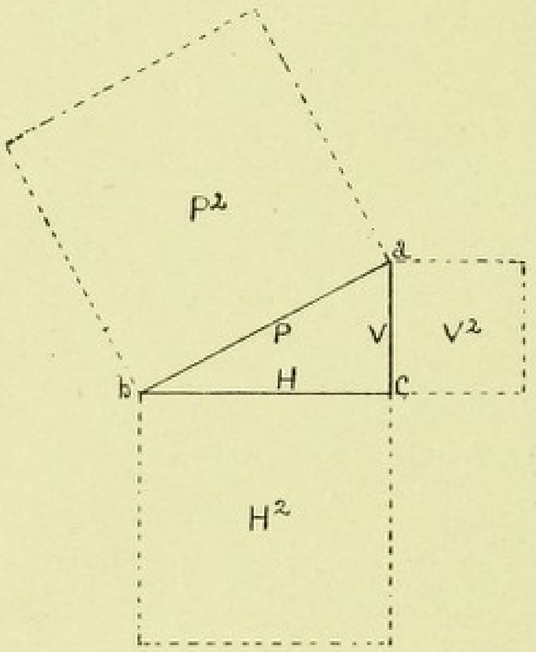


Fig. 172.

of P with respect to H and V, measured by a protractor, is the inclination of the base-apex line of the resultant prism in its relation to the horizontal and vertical. Thus, if the horizontal prism is 3° and the vertical prism 2° then the two lines H and V are respectively three and two inches long, their connecting line, P, is about $3\frac{1}{2}$ inches long and its inclination is about 35° up from the horizontal.

This method is given by Dr. Maddox in his work on prisms.

Neutralisation. — A resultant prism can be found by holding the horizontal and vertical prisms together and finding on a tangent scale the maximum oblique deviation, that is, the number of the resultant prism. Or the two original prisms can be put into a trial frame and neutralised by a single prism from the trial case, the power of the neutraliser is that of the resultant prism and its inclination is at once indicated.

TABLE OF RESULTANT PRISMS.

The two original prisms.		The resultant prism. (Approximate).	Angular distance in degrees of the base apex line from the stronger of the original prisms (approximate).
1	1	1.4	45
1	2	2.2	27
1	3	3.2	18
1	4	4.1	14
1	5	5.1	12
1	6	6.1	10
1	7	7.1	9
1	8	8	7
1	9	9	6
1	10	10	6
2	2	2.8	45
2	3	3.6	34
2	4	4.5	27
2	5	5.4	22
2	6	6.3	18
2	7	7.3	16
2	8	8.2	14
2	9	9.2	13
2	10	10.2	12
3	3	4.2	45
3	4	5	37
3	5	5.8	31
3	6	6.7	27
3	7	7.6	23
3	8	8.5	21
3	9	9.5	18
3	10	10.4	17
4	4	5.7	45
4	5	6.4	39
4	6	7.2	34
4	7	8	30
4	8	8.9	27
4	9	9.8	24
4	10	10.8	22
5	5	7	45
5	6	7.8	40
5	7	8.6	36
5	8	9.4	32
5	9	10.3	30
5	10	11.2	27

TABLE OF RESULTANT PRISMS—*continued.*

The two original prisms.		The resultant prism, (Approximate).	Angular distance in degrees of the base apex line from the stronger of the original prisms (approximate).
6	6	8.4	45
6	7	9.2	40
6	8	10	37
6	9	10.8	34
6	10	11.7	31
7	7	9.9	45
7	8	10.6	41
7	9	11.4	38
7	10	12.3	35
8	8	11.3	45
8	9	12.1	42
8	10	12.8	39
9	9	12.8	45
9	10	13.5	42
10	10	14.1	45

Obliquely Crossed Prisms.—To find the resultant prism of two obliquely crossed prisms whose powers respectively are P_1 and P_2 and whose base-apex lines cross each other at an angle a, let the resultant prism be P and let r be the angle that its base-apex line makes with the base-apex line of P_1, then

$$P = \sqrt{P_1^2 + P_2^2 + (2P_1P_2 \cos a)} \qquad [13$$

and

$$\tan r = \frac{P_2 \sin a}{P_1 + (P_2 \cos a)}. \qquad [13$$

As an example, suppose two prisms of 6° and 8° respectively whose base-apex lines are 30° apart, we find

$$P = \sqrt{6^2 + 8^2 + (2 \times 6 \times 8 \times .866)}$$

$$= \sqrt{36 + 64 + 83.136} = \sqrt{183.136} = 13.53°.$$

and

$$\tan r = \frac{8 \times .5}{6 + (8 \times .866)} = .3091 = \tan 17° \ 11'$$

The resultant prism is 13.53° and its base-apex line is 17° 11′ from that of the 6° prism,

$$\text{or} \qquad \tan r = \frac{6 \times .5}{8 + (6 \times .866)} = .2274 = \tan 12° \, 49'$$

that is 12° 49′ from that of the 8° prism.

These formulæ are not often needed since the two prisms are usually vertical and horizontal when the more simple formulæ previously given can be employed.

When $a = 90°$, then $\sin a = 1$ and $\cos a = 0$, so that the above formulæ simplify to those previously given for resultant prisms. When $a = 0$, that is when P_1 and P_2 coincide, $\sin a = 0$ and $\cos a = 1$, and the formulæ simplify to

$$P = P_1 + P_2, \qquad \text{and } \tan r = 0.$$

When $P_1 = P_2$ the formulæ also simplify to

$$P = (P_1 + P_2) \cos \frac{a}{2} \qquad \text{and } r = \frac{a}{2}.$$ [135] [136]

Rotary Prism.—A rotary prism consists of two vertical prisms of equal power conveniently mounted. In the primary position the base of the one coincides with the edge of the other, so that the effect is 0. From this position they are rotated towards the horizontal, so that their bases approach each other; thus a gradually increasing horizontal effect is obtained while the vertical effect always remains 0. The maximum effect is obtained when the two bases coincide in the horizontal meridian. If the primary position is horizontal a similar vertical effect is obtained by rotation.

TABLE FOR A ROTARY PRISM.

Rotation of the bases of each of 2 unit prisms from the vertical towards each other.	Effect in the horizontal meridian.
1°.45′	.0625
2°.50′	.1
3°.35′	.125
5°.20′	.1875
6°.45′	.2
7°.10′	.25
8°.40′	.3
9°.35′	.333
10°.50′	.375
11°.30′	.4
14°.30′	.5
17°.30′	.6

TABLE FOR A ROTARY PRISM—*continued.*

Rotation of the bases of each of 2 unit prisms from the vertical towards each other.	Effect in the horizontal meridian.
18°.10′	.625
19°.30′	.666
20°.30′	.7
22°.00′	.75
23°.35′	.8
26°.00′	.875
26°.45′	.9
30°.00′	1.
33°.20′	1.1
34°.10′	1.125
36°.50′	1.2
38°.40′	1.25
40°.30′	1.3
41°.50′	1.333
43°.25′	1.375
44°.25′	1.4
48°.35′	1.5
53°.10′	1.6
54°.20′	1.625
56°.30′	1.666
58°.10′	1.7
61°.00′	1.75
64°.10′	1.8
69°.40′	1.875
71°.50′	1.9
90°.00′	2.

The effect obtained by rotation of *any two* equal prisms is found by multiplying the number in the second column by the power of *one* of the prisms.

The rotation necessary to obtain a certain horizontal effect is also found by this table. Suppose the two prisms are each of 8°d and a prism is required of 6°d in the horizontal, then 6/8 = .75, and the corresponding rotation for .75 is 22°; so each prism must be rotated 22° from the vertical. The table can also be used for vertical effects, the rotation being from the horizontal.

THE DECENTERING OF LENSES.

Virtual Prism.—Prismatic effect can be obtained by decentering a lens as well as by combining a prism with it. The prismatic effect thus obtained by decentration is called a *virtual* prism.

How to Decenter.—Decentering is achieved by so cutting the unedged glass disc that the optical centre is nearer than the geometrical centre to the one part of the edge of the finished lens. Thus in Fig. 173, g is the geometrical centre of the finished lens, and o, the optical centre, lies nearer the left edge. In a centered lens o and g coincide. The decentering obtainable is limited by the size of the unedged disc. In Fig. 173 the utmost horizontal decentering for the finished lens is shown.

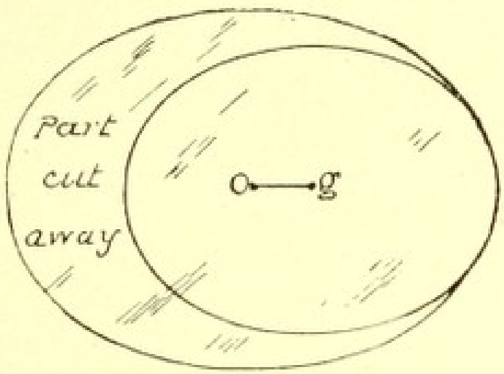


Fig. 173.

Sphero-Prism and Decentered Lenses.—When a prism is combined with a spherical, the curved surfaces are inclined towards each other at an angle (Fig. 174—2) just as if the lens had been divided and the prism inserted. If from a large lens (Fig. 174—1) one part be cut away, the effect is the same; in both these figures the principal axis of the spherical lens is shown by the vertical straight line.

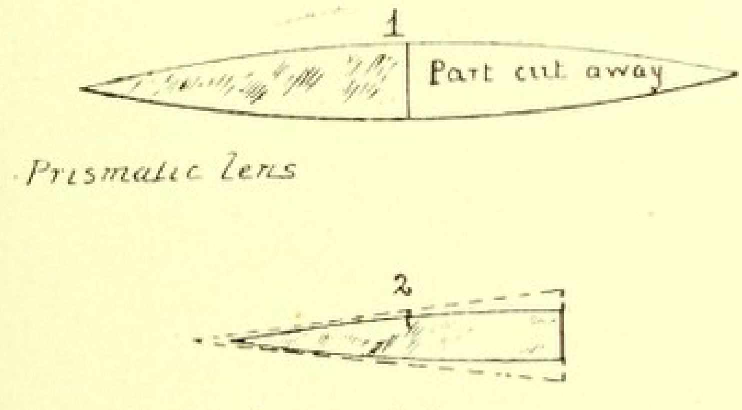


Fig. 174.

The effect produced on a ray of light by the peripheral portion of a lens is the same as if a prism of given power were used, the prismatic power being greater as the part of the lens, through which the ray passes, is more distant from the axis.

A ray of light, A B (Fig. 175), passes through the optical centre O of a lens L, and through the prism P; it is undeviated by the lens, but is bent by the prism towards its base in a direction to the right in the diagram.

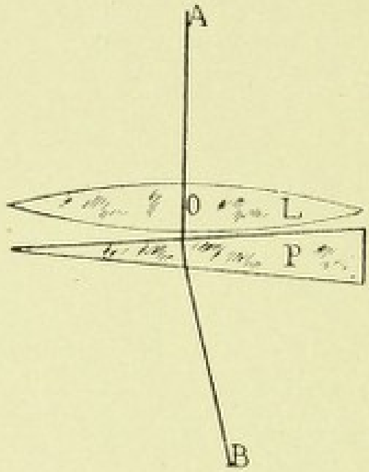
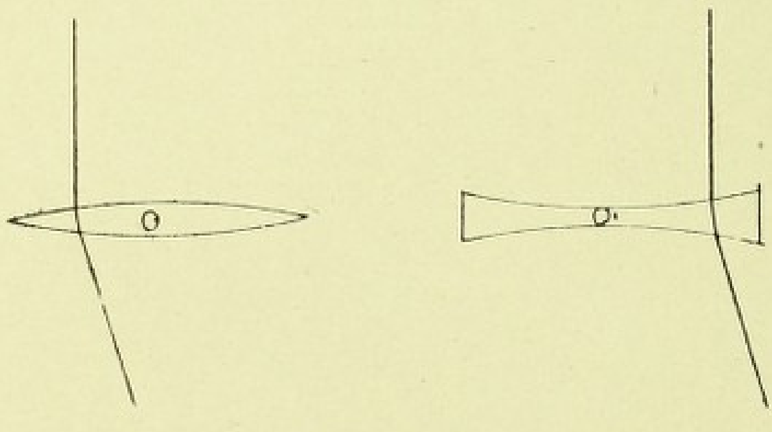


Fig. 175.

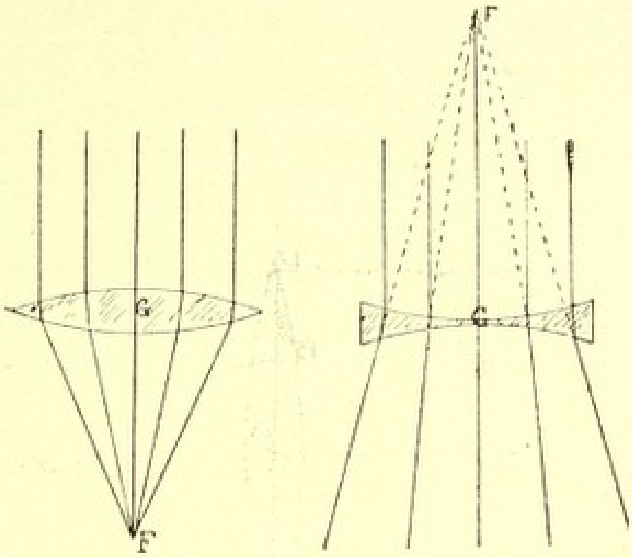
If the prism is removed (Fig. 176—1) and the convex lens decentered to the right, the ray A B is bent in the same way as if it had passed through O and a prism. If a concave lens (Fig. 176—2) is decentered to the left a similar effect is obtained.



1 Fig. 176. 2

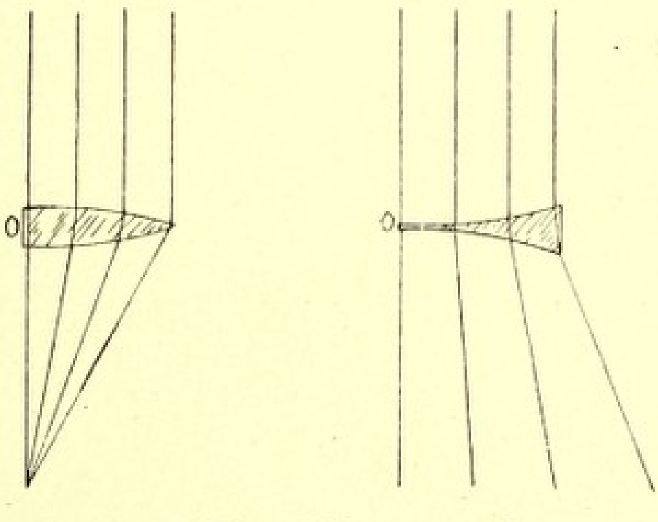
All the rays contained in a beam of light, parallel to the axis, and refracted by a spherical lens are bent towards, or away from, the axis to an extent dependent on the distance from the axis of that part of the lens through which each ray passes. Thus all the rays parallel to the axis before refraction meet, after refraction (disregarding aberration) on the axis at a single point.

If the principal axis passes through the *geometrical* centre of the lens the rays, after refraction, converge towards (Fig. 177—1) or diverge from (Fig. 177—2) a point (F) on a straight line drawn from the luminous point through g, the geometrical centre of the lens.



1 Fig. 177. 2

But if O, the optical centre of the lens is displaced, the rays are not only rendered convergent or divergent but are also bent towards the displaced axis (Fig. 178) in the same manner as they would be, if a prism had been added to the lens.



1 Fig. 178. 2

Direction of Decentration.—To produce the effect of a prism with its base in a certain direction, a convex lens must be decentered in *that same direction* and a concave in the *opposite direction.*

Effect Produced by Decentering.—The prismatic power obtained by decentering is directly proportional to the amount of decentering and to the strength of the lens. So that the decentration necessary to obtain a desired prismatic effect is directly proportional to the effect required and inversely proportional to the power of the lens. The calculation may be made in any of the systems of prism notation, but the prismatic effect of decentering lenses can best be illustrated in connection with prism diopters.

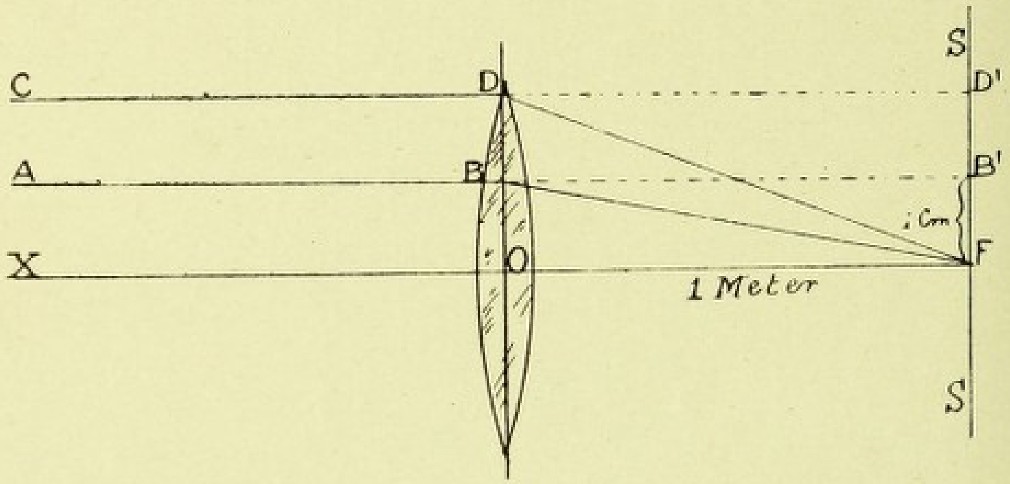


Fig. 179.

In Fig. 179 let S S be a screen situated at the focal distance of a + 1 D lens. The distance O F is, therefore, one metre. All rays of light as A B, C D, parallel to the axis are bent so as to meet at F. The ray A B incident at B situated, say, 1cm. from the axis, instead of falling on the screen at B′ as it would if it were unrefracted, cuts the axis at F. Consequently, the ray is deviated at 1 M, the distance B′ F = B O = 1cm. The ray incident, say, 2cm. from the axis is bent at 1 M over the distance D′ F = D O = 2cm. since it also meets the axis at F.

The effect of placing the geometrical centre of the lens at B or at D would, therefore, be the same as having the lens normally centered and combined with a prism of 1△ or 2△ respectively, since these prisms also have the effect of deviating a ray 1cm. or 2cm. respectively at a distance of one metre.

In Fig. 180 the lens is a + 2 D and S S, the screen, is 1 M from it, a ray A B, parallel to the axis, and incident at B, 1cm. from the axis, meets the latter at F, 50cm. from the lens and the screen at G instead of at B″. The ray is deviated a distance B′ F = 1cm. at 50cm. and B″ G = 2cm. at 1 M, so that the prismatic effect is the same as that of a 2△ acting on a ray unrefracted by the lens.

If the lens were + 4 D, A B would meet the axis at 25cm. from the lens and would be there deviated 1cm., while at 1 M, it would be deviated B″ H = 4cm. and have the effect of 4△. The + 2 D at a point 2cm. from the axis has the effect of 4△, the 4 D at the same point has the effect of an added 8△.

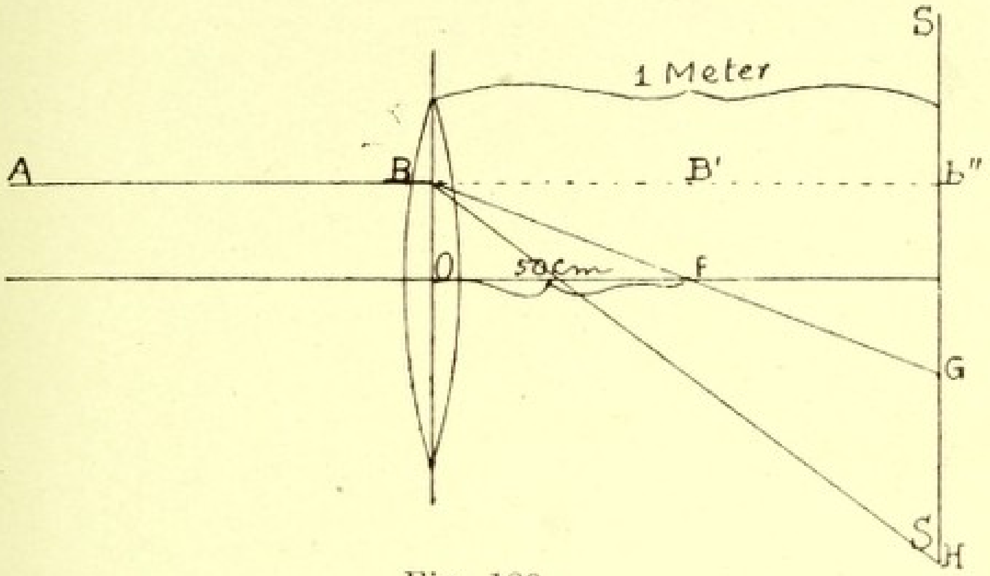


Fig. 180.

Formulæ for Decentering.—Since a +1 D lens decentered 1 cm. has the effect of 1△, the formulæ for calculating decentrations and their effect are extremely simple when the prismatic power is expressed in prism diopters.

Let P represent the prismatic effect needed, D the dioptral number of the lens, and C the decentration *in centimetres*, then

$$P = D\,C \quad \text{and} \quad C = \frac{P}{D} \qquad [137]$$

Thus if, on a 4.5 D lens, the effect of 2△ is required, the lens must be decentered.

$$C = \frac{2}{4.5} = .444\text{cm.}$$

If a 4 D lens is decentered .75cm. the prismatic effect in △ is

$$P = .75 \times 4 = 3^{\triangle}.$$

If the optical centre of the lens is found to be .75cm. from the geometrical centre, and the prismatic effects, as measured on a tangent scale, is 3△, the lens is

$$D = \frac{3}{.75} = 4\text{ D.}$$

By introducing the necessary constant K, the formulæ for prism diopters apply also for degrees whose constant is .94 when $\mu = 1.54$, is .9 when $\mu = 1.52$ and .87 when $\mu = 1.5$. For degrees of deviation the constant is 1.745; or with a sufficient degree of accuracy 1.75, so that

$$P = \frac{D\ C}{K}$$ [1

Suppose the effect of 2°d on an 8 D lens is required, the decentration will be

$$C = \frac{2 \times 1.75}{8} = .44\text{cm.}$$

If on a 5 D lens the effect of 3.5° is required, it must be decentered

$$C = \frac{3.5 \times .9}{5} = .63\text{cm.}$$

A 4 D is decentered .75cm., the prismatic effect will then be

$$P = \frac{.75 \times 4}{1.75} = 1.75^{\circ}\text{d}; \quad \text{or} \ \frac{.75 \times 4}{.9} = 3.3^{\circ}.$$

In using these formulæ, parts of degrees should be expressed as decimals, and not as minutes and seconds, and the decentration in cm. and decimals thereof.

Formulæ Involving F.—Where F or 1/F is given, it is easier, for calculating decentrations, to convert F into diopters, or the calculations can be made by the following formula, where both F and the decentration are expressed in inches, K being the constant—

$$\text{Decentration in inches} = \frac{P \times F \times K}{100}$$ [1

Thus how much should a 4in. lens be decentered for 1°? Then

$$\frac{1 \times 4 \times .9}{100} = .036\text{in.}$$

Decentering for given Prism Angle. — Since prisms numbered by the refracting angle have an indefinite effect, which varies with μ, and since great exactitude is hardly required with weak prisms, the formulæ for prism diopters can be applied for ordinary degrees without sensible error.

Also these rules, while sufficiently accurate for practical spectacle work, especially as no lens can be decentered to a very great extent, are not exact since the variation of angles have been taken as equivalent to that of their tangents.

More Exact Formulæ.—More accurate formulæ for decentration for degrees of deviation are as follows where F and D have the usual significations—

$$\tan P = \frac{C}{F}; \qquad \text{or} \qquad C = F \tan P; \qquad [140]$$

and

$$\tan P = \frac{C\,D}{100}; \qquad \text{or} \qquad C. = \frac{100 \tan P}{D} \qquad [141]$$

These formulæ are illustrated in Fig. 179, where D O is the tangent of the angle of deviation of the ray C D.

Limitations to Decentering.—The smaller the size of the lens required and the larger the disc from which it is cut, the greater is the extent of decentering possible with the ordinary disc used in the trade. If the edged lens were nearly as large as the unedged disc no decentering would be possible. Since the usual lens is longer in the horizontal than in the vertical diameter, a greater vertical than horizontal decentering is possible. Thus a lens which must be of No. 1 eye size when edged can be decentered about .4cm. horizontally and about .7cm. vertically; for 2 or 3 eye lenses the extents are greater, while for 0 and 00 eyes they are smaller.

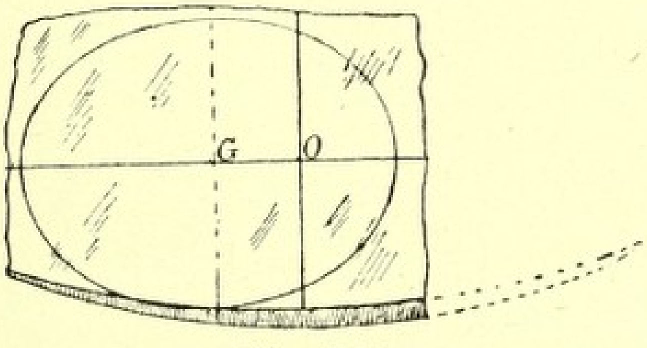


Fig. 181.

How to Decenter.—To decenter a lens, the optical centre O is located, as previously described, by means of a card having two fine cross lines. O is marked by a dot (Fig. 181), the amount of decentering is measured off and the geometrical centre of the edged lens is marked as G. The lens is then cut out so that G is the geometrical centre and the distance O G is the decentration of the lens.

To Measure Decentration.—To measure the decentration of a lens, the geometrical centre must be marked with a fine inkdot, and the optical centre found and similarly marked; the distance between them is the decentration. This distance can be measured by placing the lens on a metric rule.

Yellow chalk pencils are sold by Faber, which bite admirably on glass and are very convenient for marking axes, dots, and outlines on lenses and prisms.

TABLE OF DECENTERING FOR PRISM DIOPTERS.

Lens in diopters.	Decentration needed in Cm.									
	1Δ	2Δ	3Δ	4Δ	5Δ	6Δ	7Δ	8Δ	9Δ	10Δ
1	1.00									
2	.50	1.00								
3	.33	.66	1.00							
4	.25	.50	.75	1.00						
5	.20	.40	.60	.80	1.00					
6	.16	.33	.50	.66	.83	1.00				
7	.14	.28	.43	.57	.71	.86	1.00			
8	.12	.25	.37	.50	.62	.75	.87	1.00		
9	.11	.22	.33	.44	.55	.66	.77	.88	1.00	
10	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
11	.09	.18	.27	.36	.45	.54	.63	.72	.81	.90
12	.08	.16	.25	.33	41	.50	.58	.66	.75	.83
13	.075	.15	.23	.30	.38	.46	.54	.61	.69	.77
14	.07	.14	.21	.28	.35	.43	.50	.57	.64	.71
15	.065	.13	.20	.26	.33	.40	.46	.53	.60	.66
16	.062	.12	.186	.24	.31	.37	.43	.50	.56	.62
17	.06	.118	.177	.236	.29	.35	.41	.47	.53	.58
18	.055	.111	.165	.222	.275	.33	.39	.44	.50	.55
19	.052	.105	.157	.210	.260	.31	.37	.42	.47	.52
20	.05	.100	.150	.200	.250	.30	.35	.40	.45	.50

TABLE OF DECENTERING FOR DEGREES OF DEVIATION.

Lens in diopters	Decentration needed in Cm.										
	30′d.	1°d.	2°d.	3°d.	4°d.	5°d.	6°d.	7°d	8°d.	9°d.	10°d
1	.87										
2	.43	.82									
3	.29	.58									
4	.22	.43	.87								
5	.18	.35	.70								
6	.15	.29	.58	87							
7	.12	.25	.50	.75	1.00						
8	.11	.22	.43	.65	.86						
9	.10	.19	.38	.58	.77	.97					
10	.087	.17	.35	.52	.70	87	1.05				
11	.080	.15	.32	.48	.64	.80	.95				
12	.072	.14	.29	.45	.58	.73	.87	1.02			
13	.067	.13	.27	.40	.54	.67	.80	.94			
14	.062	.124	.25	.37	.50	.62	.75	.88	1.00		
15	.058	.116	.23	.35	.46	.58	.70	.82	.94		
16	.054	.108	.22	.33	.43	.54	.66	.77	.88	1.00	
17	.051	.102	.20	.31	.41	.51	.62	.72	.82	.93	1.03
18	.048	.096	.19	.29	.39	.48	.58	.68	.78	.88	.98
19	.045	.090	.18	.27	.37	.46	.55	.64	.74	.83	.93
20	.043	.087	.17	.26	.35	.44	.52	.61	.70	.79	.88

TABLE OF DECENTERING FOR PRISM DEGREES.

Lens in diopters.	Decentration needed in Cm.									
	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°
1	.9									
2	.45	.9								
3	.30	.6	.9							
4	.22	.45	.66	.9						
5	.18	.36	.54	.72	.9					
6	.15	.30	.45	.6	.75	.9				
7	.13	.26	.40	.52	.66	.80	.9			
8	.11	.22	.33	.44	.55	.66	.77	.9		
9	.10	.20	.30	.40	.50	.60	.70	.80	.9	
10	.09	.18	.27	.36	.45	.54	.63	.77	.81	.9
11	.08	.16	.24	.32	.40	.48	.56	.65	.72	.80
12	.075	.15	.22	.30	.37	.44	.52	.60	.67	.74
13	.07	.14	.21	.28	.35	.42	.50	.56	.63	.70
14	.065	.13	.20	.26	.33	.40	.46	.52	.60	.66
15	.06	.12	.18	.24	.30	.36	.43	.48	.54	.60
16	.056	.11	.17	.22	.28	.34	.40	.44	.50	.56
17	.053	.106	.16	.21	.27	.32	.37	.42	.48	.54
18	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
19	.047	.095	.14	.19	.24	.28	.33	.38	.43	.48
20	.045	.09	.13	.18	.23	.96	.31	.36	.40	.46

RESULTANT DECENTRATIONS.

The value of a prism in any meridian is P cos r where r is the angle between the base-apex line and the meridian in question. Similarly the prismatic value of a decentration is at its maximum along the line of decentration, and its value at any other meridian is D C cos r, D being the dioptral number of the lens and C the decentration in cm. Therefore, if a lens had to be decentered for both vertical and horizontal prismatic effects, each may be separately calculated or the two obtained by a single oblique decentration, or put in another way a lens decentered obliquely causes a vertical and horizontal prismatic effect equal to D cos r C, where r is the angular distance between the direction of decentration and the horizontal or vertical.

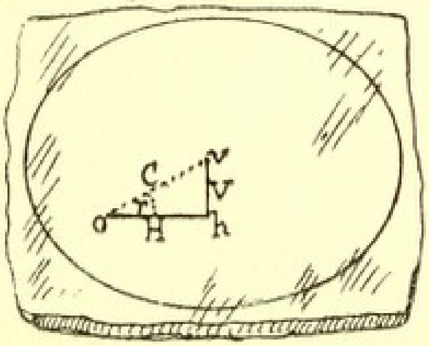


Fig. 182.

In Fig. 182 let o be the optical centre. If the geometrical centre is moved horizontally from o to h and vertically to v, the true displacement is along o v.

Let H = o h V = h v and C = o v.

Let r be the angle v o h, at which C, the resultant displacement, is inclined to the horizontal, and let V and H, respectively, be the required vertical and horizontal decenterings. Then

$$C = \sqrt{H^2 + V^2}$$

$$\tan r = \frac{V}{H}.$$

Suppose a +5 D lens has to be decentered for a horizontal effect of 2△ and a vertical effect of 1.5△, then

$$H = \frac{2}{5} = .4\text{cm.} \qquad V = \frac{1.5}{5} = .3\text{cm.}$$

$$C = \sqrt{.4^2 + .3^2} = .5\text{cm.}$$

$$\tan r = \frac{.3}{.4} = .75 = \tan 36^\circ\ 52'.$$

The two needed prismatic effects are obtained by decentering the lens .5cm. along the meridian of, say, 37°.

Instead of finding tan r, the direction of the resultant decentering can be obtained without serious error, for small values, by dividing 90° proportionately to the two needed decenterings, in this case 4 and 3. We should find

$$\frac{90}{4 + 3} = 12.8,$$ which, multiplied by 3, gives 38.5°

from the greater of the two original decenterings, that is, about 38° from the horizontal.

Since $\frac{H}{C} = \cos r$ and $\frac{V}{C} = \sin r$,

the horizontal effect H, and the vertical effect V of an oblique decentration P are found by the equations

$$H = P \cos r \qquad \text{and} \qquad V = P \sin r.$$

But P = D C where D is the dioptral number of the lens and C the decentration in cm.; therefore

$$H = D\ C \cos r \qquad \text{and} \qquad V = D\ C \sin r.$$

Thus, if a +7 D Sph. be decentered .6cm. at 30°.

$$H = 7 \times .6 \times .866 = 3.637^\triangle$$

$$V = 7 \times .6 \times .5 = 2.1^\triangle$$

and $P = 7 \times .6 = 4.2^\triangle$ at 30°.

THE DECENTRATION OF CYLINDRICALS.

Decentering Hor. and Ver. Cyls.—The effect of decentering a cylindrical across its axis is the same as that of decentering a spherical in that direction; along the axis there is, of course, no effect, since there is no refractive power. Thus a cylindrical, axis vertical, can be decentered horizontally but not vertically; a cylindrical, axis horizontal, can only be decentered vertically.

Decentering a sphero-cylindrical across the axis of the cylindrical has the same effect as decentering a spherical whose power is that of the two powers combined; while in the direction of the axis it is the same as decentering the spherical alone.

Examples.—If +4 Cyl. Ax. 90° requires decentration for a horizontal prismatic effect of 2$^\Delta$, then

$$C = \frac{2}{4} = .5\text{cm.}$$

If +3 D Sph. $\bigcirc$ +2 D Cyl. Ax. 90° has to be decentered for 2$^\Delta$ horizontally, the power in the horizontal meridian is 3 + 2 = 5 D, therefore the amount of decentering is

$$C = \frac{2}{5} = .4\text{cm.}$$

If +6 D Sph. $\bigcirc$ +1.5 Cyl. Ax. 90° needs to be decentered for 1.5$^\Delta$ vertically, the power in the vertical meridian is 6 D, so

$$C = \frac{1.5}{6} = .25\text{cm.}$$

A lens which possesses a cylindrical element should not be decentered except in its principal meridians, that is to say, *vertical and horizontal cylindricals ought never to be decentered obliquely nor should oblique cylindricals be decentered horizontally or vertically.*

As will be shown later on, oblique decentrations of such cylindricals can be made, but the results are difficult to calculate owing to the fact that the virtual prisms have their base-apex lines at right angles to the axis.

No matter what oblique decentrations may be made to such lenses, the result is that of a decentration made in the meridian at right angles to the axis. The same applies to sphero-cylindricals, but here the result is modified by the presence of the spherical, the effect being that of a prism whose base-apex line corresponds neither to that of the decentration nor that of the maximum power of the sphero-cylindrical, but which lies somewhere between them.

Decentering Oblique Cyls.—If an oblique cylindrical is decentered along the axis, there is no prism effect produced by it. If an oblique sphero-cylindrical is decentered along the axis of the cylindrical, there are, besides that in the principal meridian, horizontal and vertical effects as well, due to the oblique decentering of the spherical.

Suppose +3 Sph. ⌒ +2 Cyl. Ax. 30° is decentered .2cm. at 30°,

the effect P at 30° = .2 × 3 = .6^.

Now H = D C cos r and V = D C sin r,

therefore in this case

$$H = 3 \times .2 \times .866 = .52^{\triangle},$$

$$\text{and } V = 3 \times .2 \times .5 = .3^{\triangle}.$$

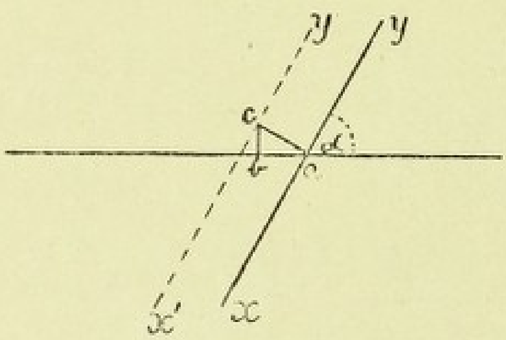


Fig. 183.

The effect of decentering an oblique cylindrical or sphero-cylindrical across the axis is, in the principal meridians, the same as with the vertical and horizontal decentering of such lenses whose axes are horizontal and vertical respectively. Thus if 4 Cyl. Ax. 60° (Fig. 183) is decentered .4cm. at 150°, the principal effect P along 150° = D C, that is

$$P = 4 \times .4 = 1.6^{\triangle}.$$

And there are horizontal and vertical effects as shown in Fig. 183 where x y is the axis of the cylindrical and x′ y′ is its position when decentered; ac represents the displacement in the meridian of 150°; ab is the horizontal and bc is the vertical displacement which occurs in consequence.

Precisely similar prismatic effects are obtained if the lens is decentered horizontally to b and vertically to c, but the maximum effect remains in the direction ac and can be considered to be the composition of the horizontal and vertical components ab and bc, or conversely the components ab and bc can be considered to be resolved from the single displacement ac.

If a is the angle between the axis and the horizontal meridian

$$ab = ac \sin a$$

$$bc = ac \cos a$$

and let C represent the distance ac.

Therefore when an oblique cylindrical is decentered across its axis

$$H = D \sin a \; C \qquad [146]$$

and $$V = D \cos a \; C \qquad [147]$$

In the example given above, let $a = 60°$.

Then $$H = 4 \times .866 \times .4 = 1.386^{\triangle}$$

and $$V = 4 \times .5 \times .4 = .8^{\triangle}$$

$$ab = .4 \times .866 = .3464\text{cm.}$$

$$bc = .4 \times .5 = .2\text{cm.}$$

With sphero-cylindricals both the spherical and cylindrical powers must, of course, be considered. Thus, suppose +3 Sph. ⊂⊃ +2 Cyl. Ax. 60° be decentered .4cm. at 150°. The effect at 150° is

$$P = 3 + 2 = 5 \times .4 = 2^{\triangle}.$$

The effect in the horizontal and vertical meridians, where D is the power of the spherical and D′ that of the cylindrical, is equal to

$$H = (D + D') \sin a \; C \qquad [148]$$

and $$V = (D + D') \cos a \; C, \qquad [149]$$

so that $$H = (3 + 2) \times .866 \times .4 = 1.732^{\triangle}$$

and $$V = (3 + 2) \times .5 \times .4 = 1^{\triangle}.$$

The horizontal or vertical displacement of an oblique cylindrical has the effect of a prism, the base-apex line of which is at right angles to the axis of the cylindrical.

Maximum Effect.—The greatest prismatic effect is always in the meridian across the axis, there being also, in every case, vertical and horizontal prismatic components. Referring to Fig. 183, if the geometrical centre is moved from a to b, the greatest prismatic effect is in the direction ac, with lesser effects horizontally along ab and vertically along bc.

Decentering Obliquely to Axis. — When the decentration of an oblique cylindrical is horizontal, the effects can be shown by the formulæ

$$H = D \sin^2 a \; C.$$

$$V = D \cos a \sin a \; C.$$

$$P = D \sin a \; C.$$

If the displacement of an oblique cylindrical is vertical

$$V = D \cos^2 a \; C.$$

$$H = D \sin a \cos a \; C.$$

$$P = D \cos a \; C.$$

For example if + 4 Cyl. Ax. 60° is displaced .462cm. horizontally, we have

$$H = 4 \times .75 \times .462 = 1.386^{\triangle}.$$

$$V = 4 \times .5 \times .866 \times .462 = .8^{\triangle}.$$

$$P = 4 \times .866 \times .462 = 1.6^{\triangle}.$$

Let + 4 Cyl. Ax. 60° be decentered .8cm. vertically; here

$$V = 4 \times .25 \times .8 = .8^{\triangle}.$$

$$H = 4 \times .866 \times .5 \times .8 = 1.386^{\triangle}.$$

$$P = 4 \times .5 \times .8 = 1.6^{\triangle}.$$

These last two examples should be compared with those given of the prismatic effects of an oblique cylindrical decentered across the axis, and it should be observed that the same effects are produced by decentering the + 4 D mentioned .462cm. horizontally or .8cm. vertically or .4cm. across the axis.

The maximum vertical effect of the horizontal displacement, or the maximum horizontal effect of the vertical displacement of an oblique cylindrical is obtained when the axis is at 45°.

Since the sine and the cosine of an angle of 45° are the same, the effect of a horizontal displacement, when the axis is at 45°, is the same in the vertical as in the horizontal meridian and could be obtained by an equal displacement vertically, as shown in the following examples.

Suppose a + 4 Cyl. Ax. 45° is displaced .33cm. horizontally

$$H = 4 \times .5 \times .33 = .66^{\triangle}.$$

$$V = 4 \times .7071 \times .7071 \times .33 = .66^{\triangle}.$$

$$P = 4 \times .7071 \times .33 = .933^{\triangle}.$$

It may occur that when a displacement of an oblique cylindrical is made horizontally, the prismatic effect is greater vertically and *vice versa*. Thus, suppose +4 Cyl. Ax. 30° is displaced .33cm. horizontally, the effects are

$$H = 4 \times .25 \times .33 = .33^{\triangle}.$$
$$V = 4 \times .866 \times .5 \times .33 = .572^{\triangle}.$$
$$P = 4 \times .5 \times .33 = .66^{\triangle}.$$

The vertical effect of a horizontal decentering of a cylindrical whose axis is at, say, 30° is the same as when the axis is at 60°. This occurs because although the distance ad is less in the first case, the power of the lens vertically is greater. The horizontal effect is, however, different. Thus, let a +4 Cyl. Ax. 60° be displaced .33cm. horizontally; here

$$H = 4 \times .75 \times .33 = .99^{\triangle}.$$
$$V = 4 \times .5 \times .866 \times .33 = .572^{\triangle}.$$
$$P = 4 \times .866 \times .33 = 1.152.$$

Hor. Decentering of Oblique Sphero-Cyls.—If an oblique sphero-cylindrical is decentered horizontally, the spherical causes no vertical effect, but the effects produced by the cylindrical are the same as with a simple cylindrical; and

$$H = (D + D' \sin^2 a)\ C. \quad [156]$$
$$V = D' \cos a \sin a\ C. \quad [157]$$
$$P = (D + D') \sin a\ C. \quad [158]$$

Ver. Decentering of Oblique Sphero-Cyls.—If an oblique sphero-cylindrical is decentered vertically, the spherical causes no horizontal effect, but the cylindrical acts as when not combined with a spherical; and

$$V = (D + D' \cos^2 a)\ C. \quad [159]$$
$$H = D' \sin a \cos a\ C. \quad [160]$$
$$P = (D + D') \cos a\ C. \quad [161]$$

Hor. and Ver. Decentering Compared.—Suppose a +3 Sph. ◯ +2 Cyl. Ax. 30° is decentered .4cm. horizontally

$$H = (3 + 2 \times .25) \times .4 = 1.4^{\triangle}.$$
$$V = 2 \times .866 \times .5 \times .4 = .35^{\triangle}.$$
$$P = (3 + 2) \times .5 \times .4 = 1^{\triangle}.$$

If the same lens is decentered vertically to the same extent

$$V = (3 + 2 \times .75) \times .4 = 1.8^{\triangle}.$$
$$H = 2 \times .5 \times .866 \times .4 = .35^{\triangle}.$$
$$P = (3 + 2) \times .866 \times .4 = 1.732.$$

Actual Prismatic Effects Obtained.—An oblique cylindrical decentered horizontally or vertically always has its greatest prismatic effect across the axis, but with an oblique sphero-cylindrical, the effect across the axis might, or might not, be greater than in that meridian in which the displacement is made, this depending on the power of the spherical. In the last example P is less than H in the one case and less than V in the other, and *the actual resultant prismatic power lies between* P *and the meridian of decentration.*

The horizontal and vertical effects of decentering an oblique sphero-cylindrical being known, the actual resultant prism effect can be calculated by the method elsewhere shown.

When the combination is decentered horizontally, the vertical effect is the same as the horizontal effect, when the decentration is vertical.

Oblique Prismatic Effects for ° and °d.—The necessary constants can be introduced into these formulæ when the prismatic effects of decentered oblique cylindricals are required in degrees or degrees of deviation.

Thus, if a + 7 D spherical is decentered .6cm. at 30°

$$H = \frac{7 \times .6 \times .866}{1.745} = \frac{3.6372}{1.745} = 2.088°d.$$

$$V = \frac{7 \times .6 \times .5}{1.745} = \frac{2.1}{1.745} = 1.2°d.$$

$$\text{The effect at } 30° \; P = \frac{7 \times .6}{1.745} = 2.4°d.$$

If + 2.50 Cyl. Ax. 45° is displaced .33cm. horizontally

$$H = \frac{2.5 \times .5 \times .33}{1.745} = .237°d.$$

$$V = \frac{2.5 \times .7071 \times .7071 \times .33}{1.745} = .237°d.$$

$$P = \frac{2.5 \times .7071 \times .33}{1.745} = .334°d.$$

CHAPTER X.

EFFECTIVITY AND EQUIVALENCE OF THIN LENSES.

Effect of Altered Position of a Cx. Lens.—We know that the power of a lens is 1/F or the reciprocal of its principal focal distance, and that both the power and the focal distance of a lens are fixed quantities. Nevertheless the effectivity of a lens in relation to a given plane behind it, varies with its distance from that plane.

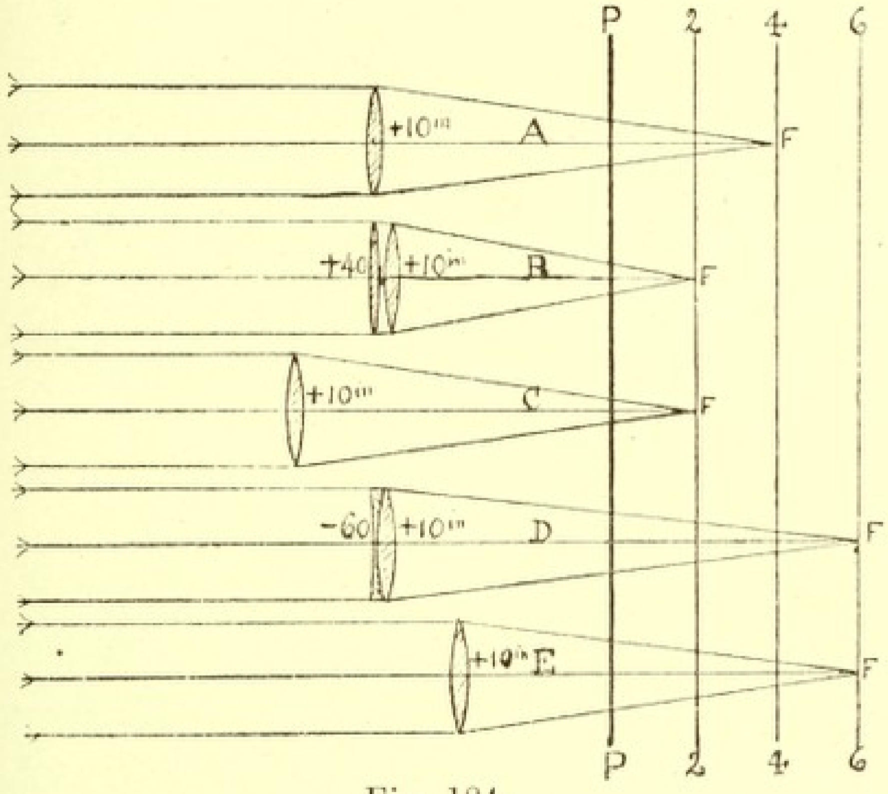


Fig. 184.

Thus, if a 10in. Cx. lens (Fig. 184 A) be placed 6in. in front of the plane P P, F is 4in. behind P P. If now we place a 40in. Cx. lens in contact with it (B), the two combined will have a focus of

$$\frac{1}{10} + \frac{1}{40} = \frac{1}{8} \text{ i.e., 8in.,}$$

so that the combined focus will be 2in. behind P P. The same effect would be produced if we moved the 10in. lens two inches further forward (C). Therefore, a convex lens moved away from a plane acts with increased effectivity, that is to say, it acts like a lens of shorter focus. If we place a 60in. Cc. lens in contact with the 10in. Cx. (D) the two combined will have a focus of 1/10 − 1/60 = 1/12, i.e., 12in., so that F will be 6in. behind P P. The same effect is produced if (E), the 10in. Cx., were moved 2in. nearer to P P. Therefore, a convex lens acts with a lessened effect as regards the plane when brought nearer to it.

The effectivity of a Cx. lens when moved through a given distance d in the direction of the light, is, for parallel rays,

$$\frac{1}{F - d}$$ [162]

If d is equal to F, then $\frac{1}{F - d} = \frac{1}{0} = \infty$,

in other words the converging effect will be infinite at the plane, when the lens is placed at its focal length in front of it.

If the lens be moved beyond its focal length, since d is then greater than F, the effectivity will be negative. Thus, if a + 10in. lens is 12in. from a screen, its effect there is

$$\frac{1}{10 - 12} = -\frac{1}{2}$$

or that of a 2in. Cc., since the light diverges from 2in. in front of it.

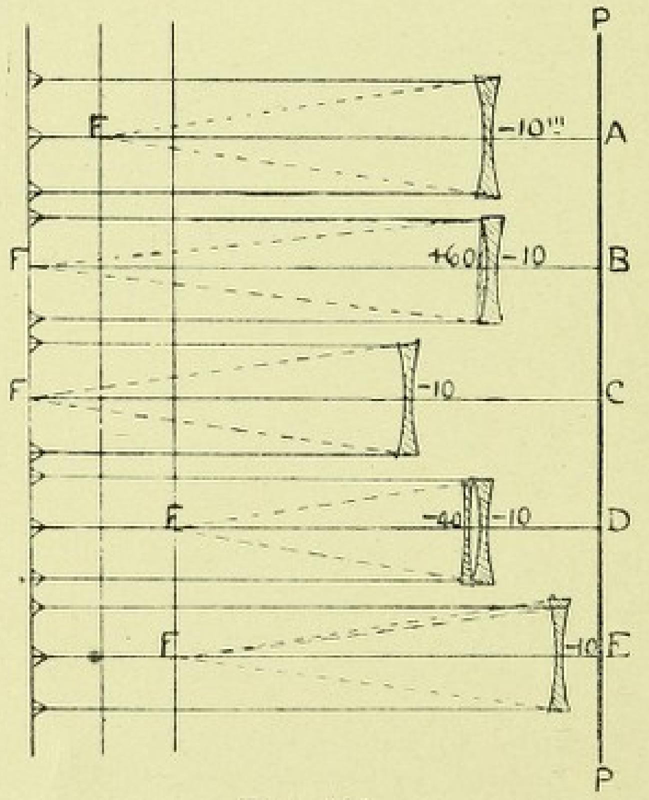


Fig. 185.

Effect of Altered Position of a Cc. Lens.—The effect produced by similarly moving a concave lens are opposite in character. Let P P be the plane, and a 10in. Cc. lens placed 3in. in front of it (Fig. 185 A). F will then be 13in. in front of

P P. If a +60in. lens be placed in contact with the first lens the two combined (B) will have a focus at

$$-\frac{1}{10}+\frac{1}{60}=-\frac{1}{12}, \text{ or } -12-3=-15\text{ in.}$$

The same effect is produced if the lens (C) be carried 2in. further from P P. Thus, the effectivity of a Cc. is decreased by increasing the distance between it and a given plane behind it, the lens acting as one of longer focus.

If a 40in. Cc. is added (D), F lies at

$$-\frac{1}{10}-\frac{1}{40}=-\frac{1}{8} \text{ at } -8-3=11\text{in. in front of P P,}$$

the same as if (E), the 10in. Cc., were carried 2in. nearer to P P, so that a Cc. lens acts with an increased effect when brought nearer to a given plane.

The effect of a Cc. lens when moved in the direction of the light, through a given distance d, is for parallel rays

$$\frac{1}{-F-d}=\frac{1}{-(F+d)}. \qquad [163]$$

Numerical Expression.—The change of value of a lens when moved from one position to another in front of a plane, or in front of another lens is the difference in its effectivity in its original and in its new position; thus, if a 5in. Cx. lens is moved from 1in. to 2in. away from a plane, the value of d is

$$\frac{1}{5-2}-\frac{1}{5-1}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

an increase of effect equal to that of a 1/12 Cx. A 5in. Cc. similarly moved causes a decrease of power just as if a 1/42 convex had been added to the concave, as is proved by

$$\frac{1}{-5-2}-\frac{1}{-5-1}=-\frac{1}{7}-(-\frac{1}{6})=+\frac{1}{42}.$$

Variation of Effectivity for Near Objects.—In Fig. 186 A let a 5in. Cx. lens be placed 5in. in front of P P, and if the light diverges from f_1 at 12in., f_2 is at $8\frac{4}{7}$in. behind the lens or $8\frac{4}{7}$in. $-$ 5 $=$ $3\frac{4}{7}$in. behind P P.

If a lens be carried outwards 2in. from its original position to (B) it is distant 10in. from f_1 and 7in. from P P, and f_2 is now at 10in. or 3in. behind P P, the focus being shortened from $3\frac{4}{7}$in. to 3in. behind P P. Removal of a Cx. lens towards the source of

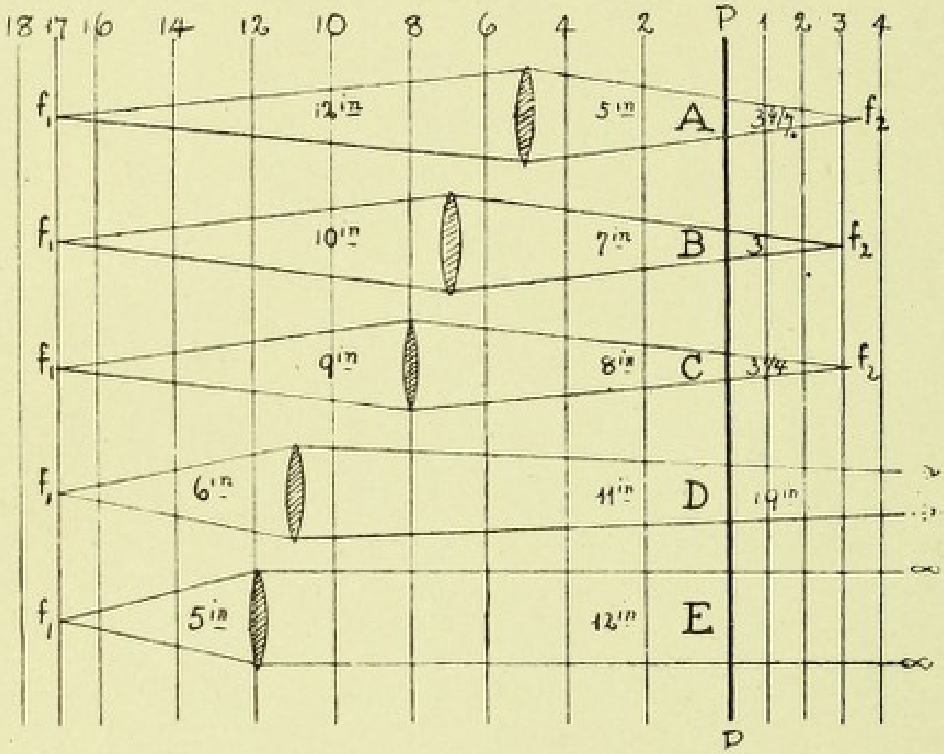


Fig. 186.

light causes increased effectivity so long as the distance between the Cx. lens and the object f_1 is not less than 2 F. At this distance the lens has the highest possible effectivity, with respect to P P, which is reduced by any further withdrawal of the lens outwards. Thus, if the lens (C) is 9in. from f_1 then f_2 will be at $11\frac{1}{4}$ $-$ 8 $=$ $3\frac{1}{4}$in. behind P P, the focus being lengthened $\frac{1}{4}$in. as compared with the position when the lens is 10in. from f_1 and 7in. from P P; there is a lessened effect of 1/39 at the screen.

Further withdrawal of the lens results in a rapid increase in the distance of f_2 behind P P, and a decrease of effectivity; so if the lens (D) is 11in. from P P, f_2 is 19in. behind it, and when the lens is removed to (E) 5in. from f_1 the light, after refraction, is parallel, and f_2 is at infinity; still further removal of the lens from P P towards f_1 renders the light divergent after refraction. If the + lens is in contact with f_1 all effect vanishes.

In Fig. 187 A, a 5in. Cc. lens is placed at P P, and if the light proceeds from f_1 12in. distant, f_2 is at $3\frac{9}{17}$in. in front of P P.

If the lens be withdrawn to B, 4in. from P P and 8in. from f_1 then f_2 lies $3\frac{1}{13}$in. in front of the lens and $7\frac{1}{13}$in. in front of P P. If the lens (C) is 1in. from f_1 then f_2 is about 11in from P P. When the − lens reaches f_1 its effect is zero.

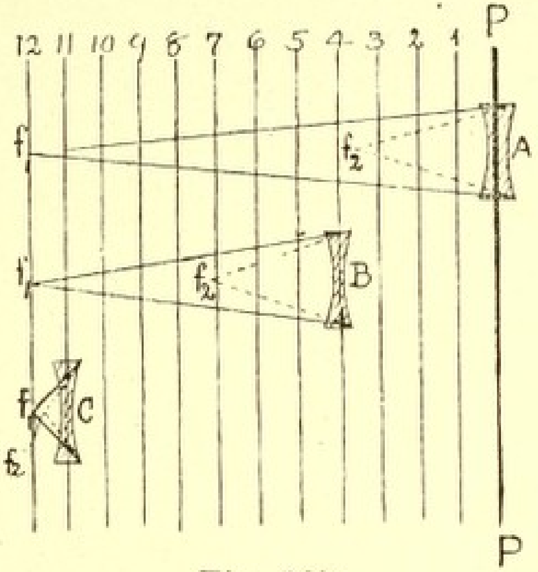


Fig. 187.

These facts prove that if incident rays are divergent, increased or decreased effect, as regards a plane situated behind it, may result from increasing or decreasing the distance between a convex lens and that plane, but with a concave the resultant effect is always that of decreased power, but the change is smaller as the distance between object and lens is less. Thus the value of d changes not only with the strength of the lens but also with the increased or decreased divergence of the light. When either a + or − lens is in contact with the object, the light diverges as if the lens were not there at all.

Dioptral Expression.—If the power of the lens be expressed in diopters, its effective power D_B in a new position, becomes

$$D_B = \frac{1000}{F - d}, \qquad [164$$

F and d being expressed in mm.

Or by

$$D_B = \frac{D}{1 - Dd}, \qquad [165$$

d being expressed in terms of a metre.

Thus, suppose a +8 D lens is moved from a given position to a new one 10mm. further forward, i.e., towards the source of light, which is at ∞, to what lens in the first position is it equal?

Then since F is $1000/8 = 125$mm.; $D_B = \frac{1000}{125 - 10} = 8.7$.

The effectivity of the lens is increased +.7 D.

If a + 10 D lens be moved from 15 to 20mm. in front of a given plane, the altered value for parallel rays can readily be found, for since F = 100mm.

$$\text{at 15mm.} \quad D_B = \frac{1000}{100 - 15} = 11.77;$$

$$\text{at 20mm.} \quad D_B = \frac{1000}{100 - 20} = 12.5$$

so the effectivity is increased by 12.5 − 11.77 = .73 D.

Similarly, moving the lens back from 20 to 15mm. decreases the effectivity to a like extent.

Effectivity of Two Cx. Lenses.—It has been shown that the combined power of two thin lenses, placed together, is equal to the sum of their individual powers, the formulæ being

$$\frac{1}{F} = \frac{1}{F_1} + \frac{1}{F_2} = \frac{F_1 + F_2}{F_1 F_2} \quad \text{and} \quad F = \frac{F_1 F_2}{F_1 + F_2}.$$

But if the two thin convex lenses are separated by an interval d the resultant effect is not the same as if the two were in juxtaposition. The distance of F behind the back lens, that is the back surface or effective focal distance F_B is found by

$$F_B = \frac{(F_1 - d)\, F_2}{F_1 + F_2 - d}$$ [166]

where F_1 is the front and F_2 the back lens and d the distance between them. Or the effective value or power is

$$\frac{1}{F_B} = \frac{1}{F_1 - d} + \frac{1}{F_2}$$ [167]

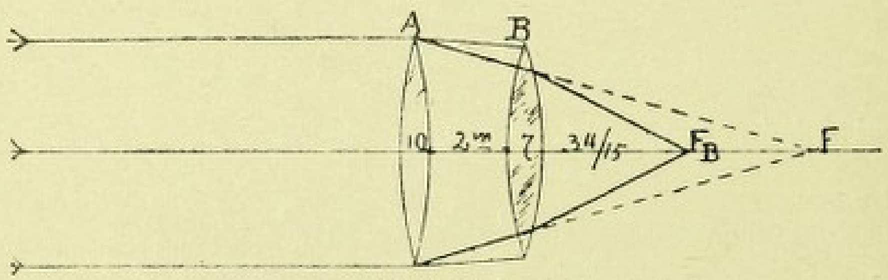


Fig. 188.

In Fig. 188 let A and B be two thin lenses of 10 and 7in. foci respectively, separated by 2in., then

$$F_B = \frac{(10 - 2) \times 7}{10 + 7 - 2} = \frac{56}{15} = 3\frac{11}{15} \text{ in.}$$

Parallel rays incident on A are converged to a point F, 10in. behind it, but on their way they meet, at 2in. from A, the 7in. Cx. lens B, and converge towards a point 10 − 2 = 8in. behind it. The effectivity of A in the plane of B is that of $\frac{1}{8}$, or the effect is the same as if an 8in. lens were in contact with B and the common focus F_B is at $3\frac{11}{15}$in. instead of $4\frac{2}{17}$in., where it would be if A were close to B. The separation of the lenses increases the distance of F from A, while it decreases its distance from B, so that the effect of carrying A out from B is to increase the effectivity of the combination with respect to a plane behind it.

The distance of the back focus differs considerably when the two lenses are of different powers according as the one or the other lens faces the light. Thus, if the combination were reversed so that the 7in. Cx. faced the light and the 10in. Cx. were 2in. behind, $F_B = 3\frac{1}{3}$in. instead of $3\frac{11}{15}$in.

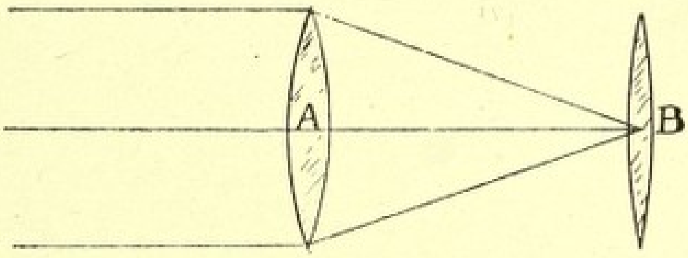


Fig. 189.

Effect when d = F_1.—If a Cx. lens B (Fig. 189), is placed at the focus of the front lens A, the former has no effect whatever. Thus B has no back focus in combination with A when d = F_1.

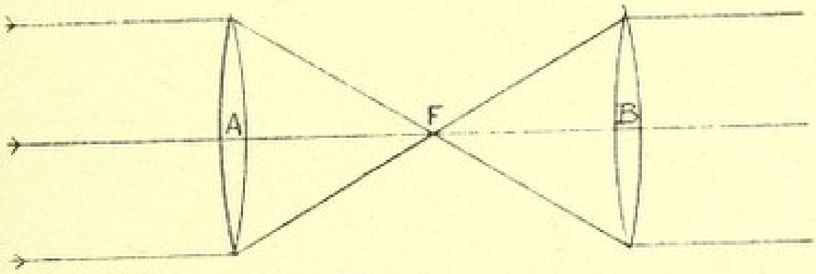


Fig. 190.

Erecting Eye-Piece.—If d = $F_1 + F_2$, i.e., if the interval between two similar Cx. lenses (Fig. 190) is equal to the sum of their focal distance, parallel rays refracted by A, meet at F, which is also the anterior principal focal distance of B. Therefore, as the rays from F are incident on B diverging from its principal focal distance thése rays, after refraction by B, emerge parallel as they were before refraction, but the position of the rays are reversed. *This is the principle of the erecting eye-piece.*

Telescope and Microscope.—In like manner if two *unequal* Cx. lenses (Fig. 191) are separated by a distance equal to the sum of their focal lengths, so that $d = F_1 + F_2$, parallel light, after refraction, emerges parallel and reversed. If the lens of greater focal length A is to the front, magnification of an object viewed results when the two are looked through from a point on the axis just behind B. *This is the principle of the ordinary astronomical telescope.* If the lens of shorter focal length be turned to the light an object appears proportionally reduced. If the object is just outside F of the lens of shorter focal length, *we have the principle involved in the microscope.*

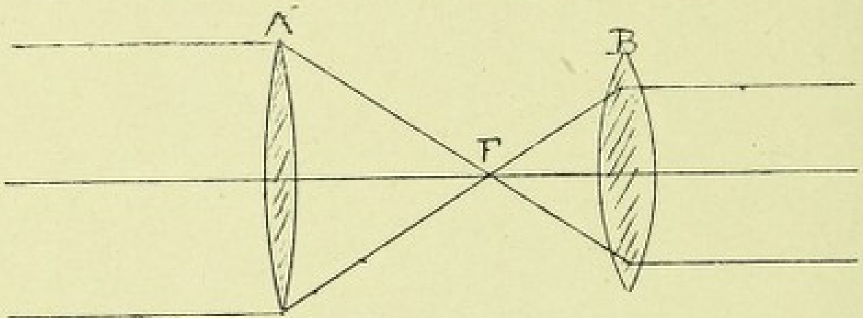


Fig. 191.

Ramsden Eye-Piece.—If two plano convex lenses A and B (Fig. 192) of, say, 4in. focus each, are separated by a distance equal to $\frac{2}{3}$ the focal distance of either (that is $2\frac{2}{3}$in. in this case) parallel rays of light are brought to a focus at $\frac{1}{4}$ F (or 1in.) behind B, as can be proved. Thus

$$\frac{1}{4 - 2\frac{2}{3}} + \frac{1}{4} = \frac{3}{4} + \frac{1}{4} = \frac{1}{1} = 1\text{in.}$$

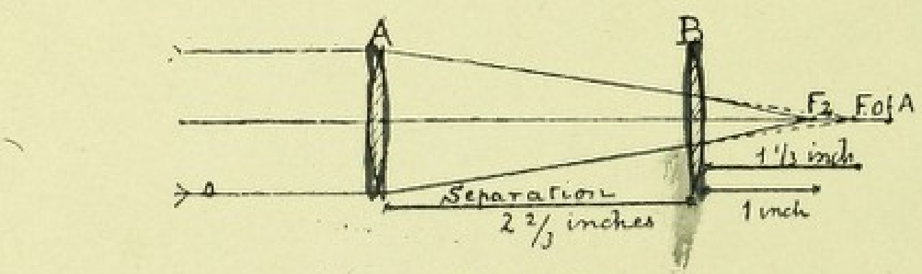


Fig. 192.

This is the recognised form of the Ramsden eye-piece. If the rays diverge from 1in. ($\frac{1}{4}$ F) in front of B they will, after refraction by the two lenses, emerge parallel, for $\frac{1}{4} - \frac{4}{4} = -\frac{3}{4}$. The virtual image formed by B is $1\frac{1}{3} + 2\frac{2}{3} = 4$in. in front of A, and $\frac{1}{4} - \frac{1}{4} = 0$. The lens B makes light from an object placed at 1in. in front of it diverge as if from F of A.

Huyghen's Eye-Piece.—Let, as in Fig. 193, two unequal plano Cx. lenses A and B (where $F_1 = 3\,F_2$) of, say, respectively 6in. and 2in. focal length, be separated by a distance equal to the difference between F_1 and F_2 or half the sum of their focal lengths $(F_1+F_2)/2$. Then d is in this case 4in. and parallel light is brought to a focus at a distance behind B equal to $F_2/2$, for F_B of the combination is

$$\frac{1}{6-4} = \frac{1}{2} + \frac{1}{2} = \frac{1}{1} = 1\text{in.}$$

This is the principle of the Huyghen eye-piece.

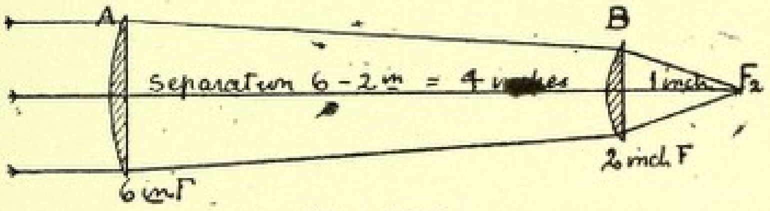


Fig. 193.

Effectivity of Two Cc. Lenses.—In Fig. 194 let A and B be two thin Cc. lenses of, say, 10in. and 7in. focus respectively; when close together F lies $4\frac{2}{17}$in. in front of them. If A is advanced 2in. F is in front of B at

$$F_B = \frac{(-10-2)\times -7}{-10-7-2} = \frac{+84}{-19} = -4\frac{8}{19}\text{in.}$$

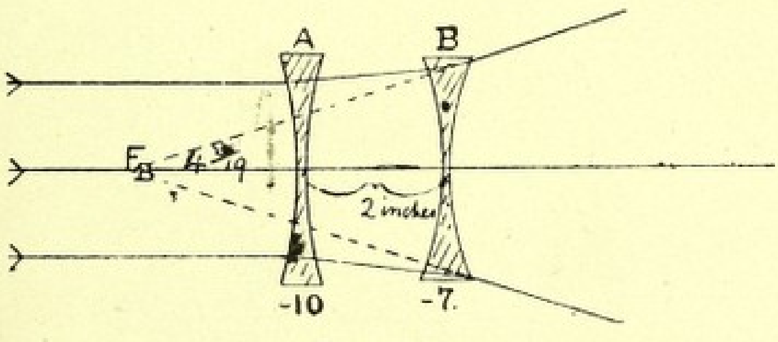


Fig. 194.

As F_B is longer, the effectivity is decreased for any plane behind the lenses, although F_B is nearer to the front lens. As with Cx. lenses, the distance of F from the back lens of a combination of two unequal Cc. lenses, separated by an interval, varies as the one or the other lens faces the light. Thus, if the 7in. Cc. were 2in. in front of the 10in. $F_B = 4\frac{14}{19}$in.

Effectivity of a Cx. and a Cc. Lens.—Advancing a Cx. towards the source of light results in greater increase of effect than retiring a concave of similar power, so that if there be an interval between a Cx. and a Cc. of equal focal length, the combination is Cx. If, however, d exceeds F_1 the light refracted by the Cx. meet in front of the Cc., to which they diverge, so that F is negative and the image is virtual.

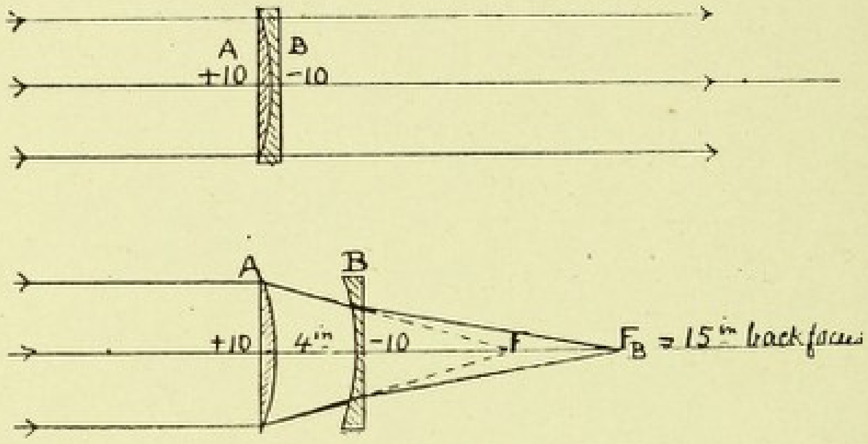


Fig. 195.

Thus, in Fig. 195 let A be a 10in. plano-Cx. and B a 10in. Cc. When placed in contact parallel rays pass through unaltered, but if separated by an interval less than their focal length, say 4in., then parallel light incident on the Cx. are converged to 10in. behind it, but meeting the Cc. at 4in. distance, the rays are converged to

$$\frac{(10 - 4) \times (-10)}{10 - 10 - 4} = \frac{-60}{-4} = 15\text{in. behind B.}$$

Thus the rays are rendered less convergent and form a real focus at F. *This is the principle of the Unofocal photographic lens.*

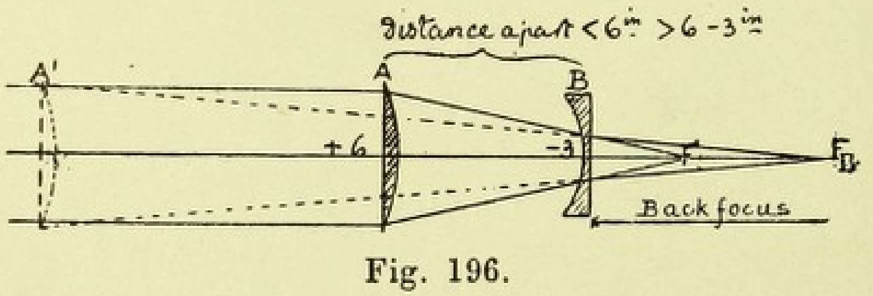


Fig. 196.

Telephoto Lens.—Let the Cc. lens have a shorter focal length than the Cx. and be placed in contact. The result will be an excess of negative power and no real image can be formed. If the Cx. be removed towards the source of light, it gains in effect, but the total effect is still negative until, when the separation is equal to the sum of their focal lengths, the back focal length is infinite. Any further increase of separation will give the two lenses a positive focal length, which diminishes as the separation continues, until the separation = F of the convex, when the focal length is at a minimum. Still further separation produces a virtual image of the real image formed at F_1. *This is the principle of the telephoto lens,* when d is less than F of the Cx., but greater than $F_1 + F_2$, and is shown in Fig. 196.

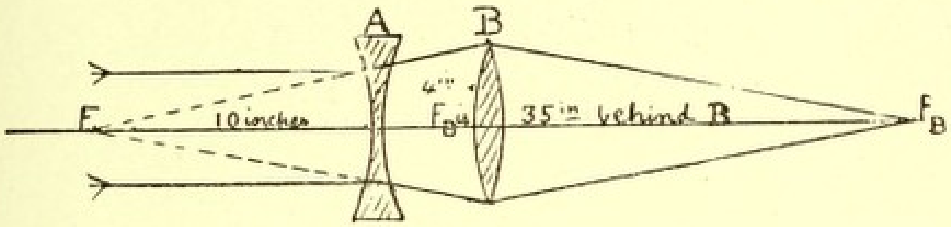


Fig. 197.

"~~Adon" Lens~~.—But, with the same separation of a Cx. and a Cc. lens, the effect is different, as the light is incident first on the one or the other.

If $F_1 = +10$ and $F_2 = -10$ and d = 4in. then $F_E = 15$. Now, if the concave faces the light as in Fig. 197, parallel rays passing through A are diverged as if from F_1 10in. in front of it or 14in. in front of B, so that A acts with the effect of

$$\frac{1}{-10-4} = -\frac{1}{14} \text{ in the plane of B}$$

and

$$\frac{1}{F_B} = \frac{1}{10} - \frac{1}{14} = \frac{1}{35}$$

that is, F_B is 35in. behind B.

~~*This is the principle of the "Adon" telephoto lens*~~.

Thus, although in both cases there is an excess of Cx. power, F_B is nearer to the back lens when the Cx. faces the light than when the Cc. does so.

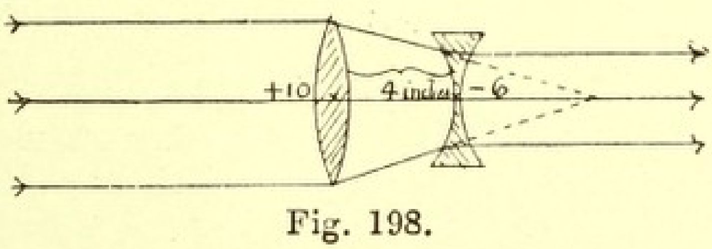


Fig. 198.

Opera Glass.—In order that a Cx. and Cc. should neutralise each other and parallel rays emerge from the second lens parallel, d must be equal to the sum of their focal lengths. Thus, if F_1 = 6in. and F_2 = − 6in., then 6 + (− 6) = 0; in other words the two lenses must be in contact. But if $F_1 = +6$ and $F_2 = -4$ the two lenses must be separated $6 + (-4) = 2$in. in order to neutralise each other.

In this last case the emergent rays are parallel to their axes after refraction, whether the entering rays are first incident on the Cx. or on the Cc.

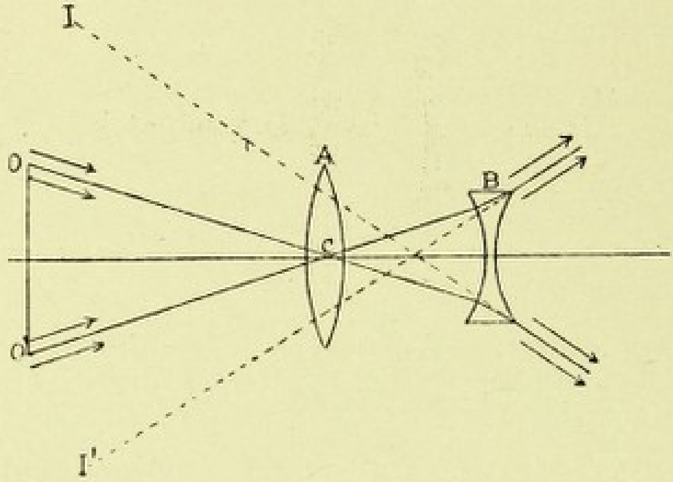


Fig. 199.

When the Cx. is to the front (Fig. 199), the secondary axial rays of A as O C, O′ C are more divergent after refraction by B, than when incident on A; hence they are referred back by an eye, situated behind B, as if they proceeded from a larger object, as shown by the dotted lines, and magnification, which equals F_1/F_2, is thus obtained. *This is the principle of the opera-glass.*

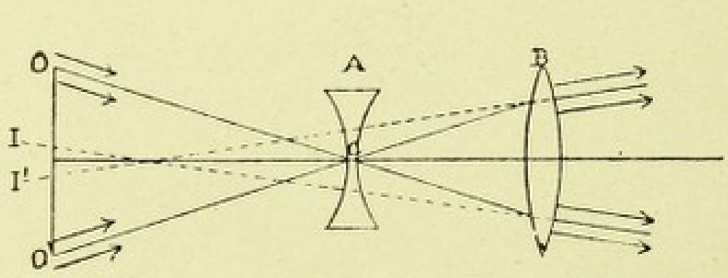


Fig. 200.

When the Cc. is to the front (Fig. 200) the secondary axial rays of A are less divergent after refraction by B. They appear as if they proceeded from a smaller object, so that diminution occurs when an object is viewed through such a combination as an opera-glass, when the Cx. lenses are next to the eyes. The diminution is F_2/F_1 and is equal to the magnification obtained when the convex is to the front.

Dioptral Formulæ for Effectivity.—The working formula for finding the effective dioptral lens D_B of two separated lenses D_1 and D_2 is

$$D_B = \frac{D_1}{1 - (D_1 d)} + D_2 \qquad [168]$$

or

$$D_B = D_1 + D_2 + \frac{D_1^2 d}{100 - D_1 d} \qquad [169]$$

D_1 being the power of the front and D_2 that of the back lens, d the interval being expressed in terms of a metre in the first and in centimetres in the second of the above formulæ.

Thus, if a + 6 D is 3cm. in front of a + 4 D, then

$$D_B = 6 + 4 + \frac{36 \times 3}{100 - (6 \times 3)} = 10 + \frac{108}{82} = 11.3 \text{ D.}$$

Let a − 7.50 D be separated 25mm. from a − 4.5 D, then

$$D_B = \frac{-7.5}{1 - (-7.5 \times .025)} = \frac{-7.5}{1 + .1875} = -6.31 + (-4.5) = -10.81 \text{ D.}$$

If a +3 D be 83mm. in front of a −4,

$$D = +3 - 4 + \frac{3^2 \times 8.3}{100 - (3 \times 8.3)} = -1 + \frac{75}{75} = 0,$$

they neutralise each other. If the Cc. is to the front they similarly neutralise.

The distance d (in cms.) which a lens must be in advance of a given plane in order that it have a given effectivity at that plane, is found by the formula

$$d = \frac{100}{D} - \frac{100}{D_B} \qquad [170]$$

where D is the power of the lens and D_B is that of its required effect.

Opera-Glass Adjustment.—Suppose an opera-glass, formed of a + 10 D objective and a − 20 D ocular has to be adjusted for the vision of a myope of 4 D, the distance between the lenses must be such that the + 10 D has an effectivity of + 16 D in the plane of the − 20 D, so that

$$d = \frac{100}{10} - \frac{100}{16} = 10 - 6.25 = 3.75\text{cm.}$$

Or the separation d which must be given to two lenses in order to produce D_B or F_B of certain powers, can be found by substitution and equation of the formulæ given for finding D_B or F_B respectively, as shown below.

Effectivity when Light is Divergent.—When incident light is divergent the conjugate focus must be found before the value of d can be applied to the front lens of a combination. The conditions for neutralising and for obtaining certain effectivities with separated lenses differ for divergent and parallel light.

An object is 20in. in front of an 8in. Cx. lens behind which at 2in. a 13in. Cc. lens is placed, where is the image? By our formula

$$\frac{1}{f_2} = \frac{1}{8} - \frac{1}{20} = \frac{1}{13\frac{1}{3}}.$$

Now $\dfrac{1}{13\frac{1}{3} - 2} - \dfrac{1}{13} = \dfrac{1}{88\frac{2}{5}}$. Therefore I is at $88\frac{2}{5}$in.

An object is 40in. in front of a 7in. Cx. lens, where should a 5in. Cc. be placed so that the rays may be rendered parallel? Now

$$f_2 = \frac{40 \times 7}{40 - 7} = 8\frac{16}{33}$$

the image is $8\frac{16}{33}$in. behind the Cx., so that the Cc. must be placed $8\frac{16}{33} - 5 = 3\frac{16}{33}$in. behind the Cx.

Separation for Given Effectivity.—In order to adjust the distance between the two lenses so that the effect is that of a given focal length the calculation is as follows. Let the lenses be 5 Cx. and 2 Cc. and the effect required that of a 20 Cx., then

$$+20 = \frac{(5 - d) \times (-2)}{5 - 2 - d} = \frac{-10 + 2d}{3 - d}$$

then $$60 - 20d = -10 + 2d.$$

so $$-22d = -70 \text{ and } d = 3\tfrac{2}{11}\text{in.}$$

If the effect required with the same lenses is that of 20 Cc. then

$$-20 = \frac{(5 - d) \times (-2)}{5 - 2 - d} = \frac{-10 + 2d}{3 - d}$$

then $$-60 + 20d = -10 + 2d$$

so $$18d = 50 \text{ and } d = 2\tfrac{7}{9}\text{in.}$$

These calculations serve for opera-glass adjustments.

An object is 40in. in front of a 7in. Cx. and a 5in. Cc. The image must be 20in. behind the back lens; how much must the lenses be separated?

f_2 is $8\frac{16}{33}$in. behind the Cx., which must have the effect of

$$\frac{1}{5} + \frac{1}{20} = \frac{1}{4} \text{ at the plane of the Cc.}$$

The interval between them, then must be $8\frac{16}{33} - 4 = 4\frac{16}{33}$.

Or
$$\frac{40 \times 7}{40 - 7} - \frac{5 \times 20}{5 + 20} = 4\frac{16}{33}.$$

An object is 30in. in front of a 15in. Cx. and 20in. Cx. At what distance should the latter be placed behind the former so that the image be 5in. behind the back lens? Then

$$\frac{30 \times 15}{30 - 15} = 30$$

the rays converge to 30in., but should converge to

$$\frac{20 \times 5}{20 - 5} = 6\tfrac{2}{3}$$

behind the second lens. Therefore the lenses must be $30 - 6\frac{2}{3} = 23\frac{1}{3}$in. apart.

Suppose a +3 D lens is placed 20cm. in front of a screen, where must a similar lens be placed so that the image of an object 50cm. in front of the first lens be in focus on the screen?

After refraction by the first lens the light is converging to 100 cm., but it must converge to 20cm. behind the second lens, and the effectivity needed is + 5 D. Now, therefore, the front lens must act as $5 - 3 = +2$ D in the plane of the second, and the interval between them must be $100 - 50 = 50$cm.

EQUIVALENT THIN LENSES.

Equivalence.—Any two or more lenses, whether in contact or separated, can always be replaced by a single equivalent lens which will produce the same refracting effect as the component lenses. Thus, as shown in Fig. 201, E will refract a parallel beam to the same degree as L_1 and L_2 separated by the interval d.

Back and Equivalent F's.—Separation of two convex lenses tends to shorten the back focal length F_B, while separation of two concave ones lengthens F_B. On the other hand the actual (equivalent) focal length F_E of convex lenses is increased, and that of the concave lenses diminished, by separation.

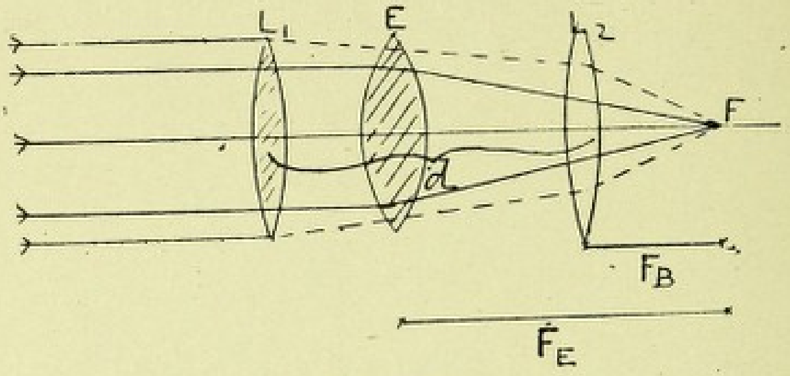


Fig. 201.

If two thin lenses are placed in contact, the resultant focal length is the same as that of a single lens of equal power. If D_1 and D_2 represent the powers of two lenses they may be replaced by a lens having a power of D. The combined power and focal length may be expressed by writing

$$\frac{1}{F} = \frac{1}{F_1} + \frac{1}{F_2} \quad \text{and} \quad F = \frac{F_1 F_2}{F_1 + F_2}.$$

If the lenses F_1 and F_2 be separated by a distance d the effective power and back surface focal distance are determined by

$$\frac{1}{F_B} = \frac{1}{F_1 - d} + \frac{1}{F_2}, \quad \text{and } F_B = \frac{(F_1 - d)\,F_2}{F_1 + F_2 - d}$$

The power of the equivalent lens $1/F_E$ and the equivalent focal length, F_E are found by the equations.

$$\frac{1}{F_E} = \frac{1}{F_1} + \frac{1}{F_2} - \frac{d}{F_1 F_2} \quad \text{and } F_E = \frac{F_1 F_2}{F_1 + F_2 - d}$$ [171

The position behind the first lens F_1 of the equivalent lens E, from which the equivalent focal distance is measured is found by the formula

$$E = \frac{(F_1 - F_E)\,d}{F_1}.$$ [172

Equivalent Points.—In every combination of lenses there are two points E_1 and E_2 called the equivalent points, which lie in two imaginary planes called equivalent planes. E_1 is measured backwards from the front lens and E_2 is measured forwards, i.e., towards the source of the light, from the back lens.

E_1 and E_2 are shown in Fig. 202 with the distance t between them. Sometimes these points are crossed as shown in the figure. If E_1 or E_2 when calculated is a negative quantity it is measured in the opposite direction to that given above, i.e., forwards from L_1 and backwards from L_2.

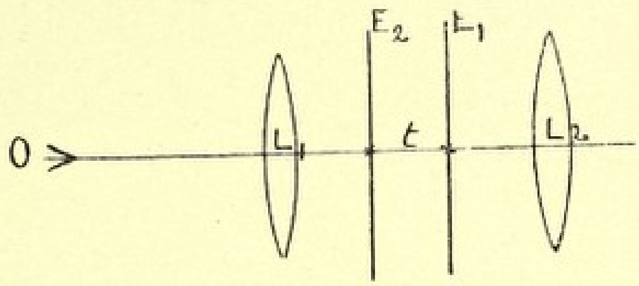


Fig. 202.

If F_1 and F_2 are known we can find their positions from the following formulæ

$$E_1 = \frac{F_1\ d}{F_1 + F_2 - d} \qquad [173]$$

and

$$E_2 = \frac{F_2\ d}{F_1 + F_2 - d} \qquad [174]$$

In the previously given formula E is the distance of E_2 from the front lens, and the last formula gives its distance from the back lens.

The distance t, between these equivalent points, is the optical interval or equivalent thickness. It is obtained from

$$t = \frac{d^2}{F_1 + F_2 - d}, \qquad \text{or} \qquad t = d - (E_1 + E_2). \qquad [175]$$

The last equation shows whether the equivalent points are crossed or not.

Measurement of F_E.—The focal length of the combination is always measured from that equivalent point which is further from the source of light, i.e., from E_2 if L_1 is to the front, and from E_1 if L_2 faces the light.

The equivalent focal length is the same whichever lens is to the front, but the position from which it is measured varies accordingly.

Example.—Suppose a 5in. Cx. lens is placed 2in. from a 10in. convex lens. Then

$$F_E = \frac{5 \times 10}{10 + 5 - 2} = \frac{50}{13} = 3\frac{11}{13} \text{ in.}$$

$$E_1 = \frac{5 \times 2}{13} = \frac{10}{13} \text{ in. behind the front lens L}$$

$$E_2 = \frac{10 \times 2}{13} = 1\frac{7}{13} \text{ in. in front of the back lens } L_2$$

or

$$2 - 1\frac{7}{13} = \frac{6}{13} \text{ in. behind } L_1$$

$$t = \frac{2^2}{13} = \frac{4}{13} \quad \text{or } t = 2 - \left(\frac{20}{13} + \frac{10}{13}\right) = -\frac{4}{13}$$

If the 10 in. lens be in front, the two equivalent points change places. F_E is as before.

Crossed Equivalent Points.—Since the separation = 2in., and E_1 is 10/13in. behind the first lens while E_2 is 1 7/13in. in front of the second lens, the two equivalent planes are (in this case) crossed by a distance equal to 4/13in.

Special Cases.—The following special cases in regard to convex lenses should be considered.

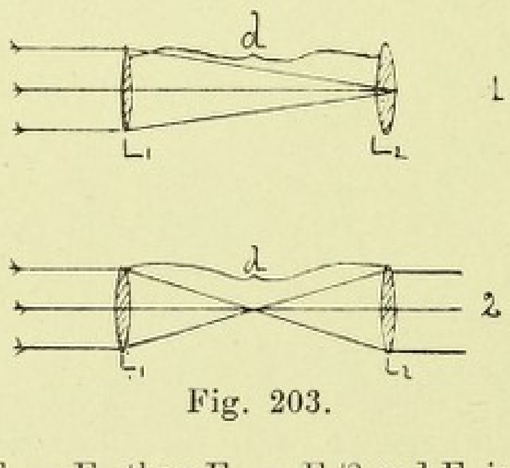


Fig. 203.

(1.) If $d = F_1 - F_2$, then $F_E = F_1/2$ and E_2 is midway between the two lenses.

(2.) If $d = F_1 + F_2$, then $F_E = \infty$ (Fig. 203—2).

(3.) If $d > F_1$ but $< F_1 + F_2$, then F_E is positive and E_1 and E_2 may be one or both beyond the lenses and crossed (Fig. 204).

(4.) If $d > F_1 + F_2$, then F_E is negative, and E_1 and E_2 are also negative (Fig. 205).

(5.) If $d = F_1$, then $F_E = F_1$ (Fig. 203—1).

(6). If $d = F_2$, then $F_E = F_2$.

(7.) If $F_1 = F_2$, then $F_E + F_E = F_1$ or F_2.

If d is $> F_1$ (cases 3 and 4), then the rays from F_1, after refraction, are divergent on F_2 in the same way as if a single lens were placed further than its focus from a given plane.

Thus, suppose $F_1 = 7$in. $F_2 = 16$in. and $d = 9$in. Then $F_E = 8$in. E_1 is $4\frac{1}{2}$in. from F_1 and E_2 is $10\frac{2}{7}$in. from F_2. The effect is the same as if an 8in. lens were placed $10\frac{2}{7}$in. in front of the plane of F_2. Light, refracted by it, is converged to 8in. and then diverges as if from a point $2\frac{2}{7}$in. in front of F_2.

No. 5 (shown in Fig. 203, 1) illustrates the formation of the Kellner eye-piece.

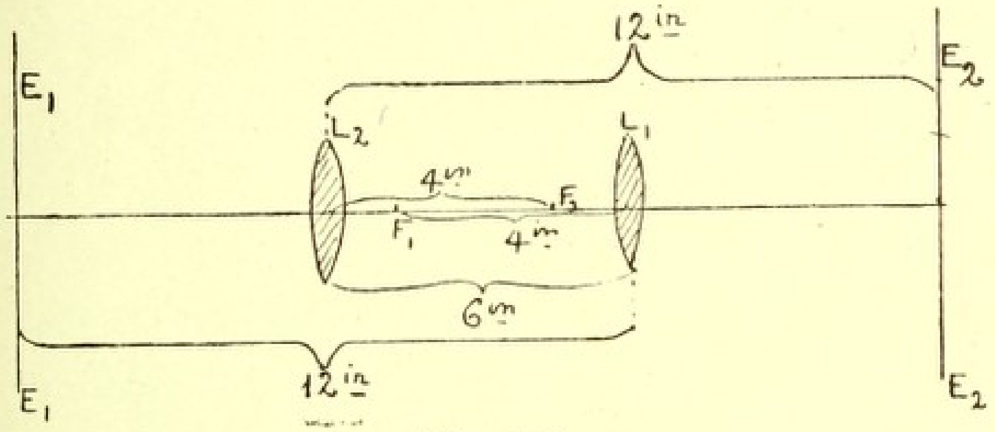


Fig. 204.

Let $F_1 = 4$in., $F_2 = 4$in., and $d = 6$in. Then, as in Fig. 204, $d > F_1$, and

$$F_E = \frac{4 \times 4}{4 + 4 - 6} = \frac{16}{2} = 8\text{in.}$$

$$E_1 = \frac{4 \times 6}{2} = 12\text{in.} \qquad E_2 = \frac{4 \times 6}{2} = 12\text{in.}$$

Parallel rays incident on L_1 come to a focus at 4in., whence they diverge to L_2 and have their virtual focus, after refraction, at 4in. in front of L_2 or 8in. behind E_2.

If d is greater than $F_1 + F_2$ (Fig. 205), as when two 4in. lenses are 20in. apart, we get

$$F_E = \frac{4 \times 4}{4 + 4 - 20} = \frac{16}{-12} = -1\frac{1}{3}\text{in.}$$

$$E_1 = \frac{4 \times 20}{-12} = -6\frac{2}{3}. \qquad E_2 = \frac{4 \times 20}{-12} = -6\frac{2}{3}.$$

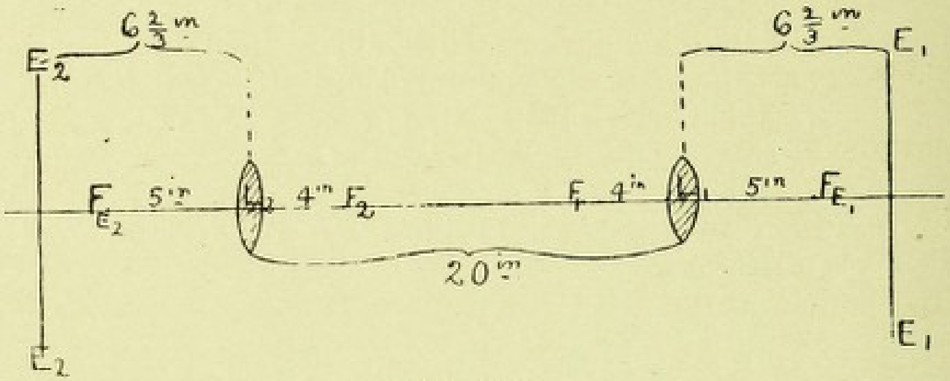


Fig. 205.

Here the equivalent points, being negative, are measured outwards instead of inwards, and F_E lies behind L_2 but $1\frac{1}{3}$in. in front of E_2.

Ramsden Eye-Piece.—In the case of a Ramsden eye-piece, let F_1 and F_2 be each of 2in. focal length, d being $\frac{2}{3} F_1 = 1\frac{1}{3}$in.

Here $$F_E = \frac{2 \times 2}{2 + 2 - 1\frac{1}{3}} = \frac{4}{2\frac{2}{3}} = 1\frac{1}{2}.$$

and $$E_2 = \frac{2 \times 1\frac{1}{3}}{2\frac{2}{3}} = 1\text{in. in front of the back lens.}$$

Thus, $F_E = \frac{3}{4} F_1$ and E_2 is $\frac{1}{2} F_1$ from the back lens or $\frac{2}{3} F_1 - \frac{1}{2} F_1 = \frac{1}{6} F_1$ from the front lens. E_1 is 1in. behind the front lens.

Huyghen's Eye-Piece.—In the case of a Huyghen eye-piece let $F_1 = 3$in., $F_2 = 1$in., and d (which is the difference between F_1 and F_2) = 2in.

Here $$F_E = \frac{3 \times 1}{3 + 1 - 2} = 1\frac{1}{2}\text{in.}$$

E_2 is 1in. in front of the back lens and E_1 being 3in. behind the front lens, is 1in. behind the back lens. Thus, one equivalent point is between and the other is outside the lens system. Here, $F_E = \frac{1}{2} F_1$ or $1\frac{1}{2} F_2$ and E_2 is in front of the stronger back lens at a distance $= F_2$ or $\frac{1}{3} F_1$, behind the front lens, i.e., midway between the two lenses.

Here $d > F_2$ but $< F_1$, and if the lens of shorter focus is facing the light $F = -1\frac{1}{2}$in.; that is, the light diverges from $1\frac{1}{2}$in. in front of the back lens.

Conjugate Foci.—The equivalent focal length of two separated lenses is that of the lens substituted for them, so that the formulæ for calculating conjugate foci hold good, including that of $A B = F^2$, A and B being the respective distances of any pair of conjugate foci beyond the two principal foci of a Cx. lens, or combination of lenses, and $F_E = \sqrt{A B}$

Equivalence of Two Cc. Lenses.—If F_1 and F_2 are both negative, then

$$F_E = \frac{-F_1 (-F_2)}{-F_1 - F_2 - d}.$$

For example, let $F_1 = -8$in., $F_2 = -10$in., and $d = 2$in., then

$$F_E = \frac{-8 \times (-10)}{-8 - 10 - 2} = \frac{80}{-20} = -4\text{in.}$$

$$E_1 = \frac{-8 \times 2}{-20} = \frac{4}{5}\text{in. behind } L_1$$

$$E_2 = \frac{-10 \times 2}{-20} = 1\text{in. in front of } L_2$$

$$t = 2 - (1 + \tfrac{4}{5}) = \tfrac{1}{5}\text{in.}$$

Special Cases.—If $d = F_1 - F_2$ (both being concave), then

$F_E = F_2/2$, and E_1 is midway between the two lenses.

If $F_1 = F_2$, then $F_B + F_E = F_1$ or F_2.

Equivalence of a Cx. and a Cc. Lens.—Further, there is the condition where the one lens is positive and the other negative.

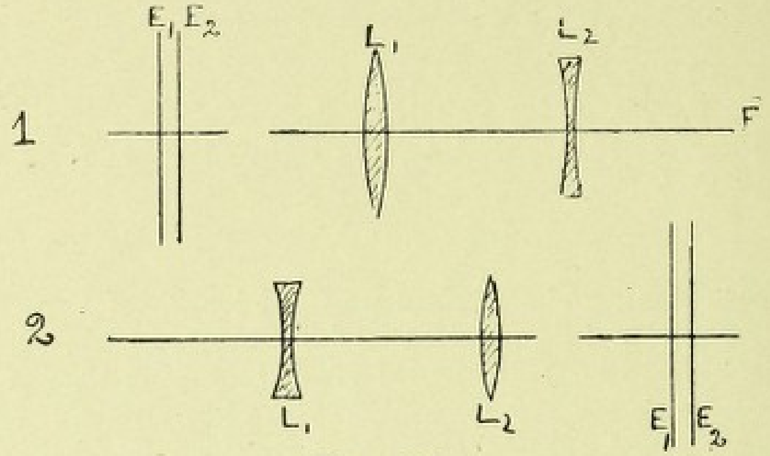


Fig. 206.

Suppose $F_1 = 10$cm., $F_2 = -15$cm., and d = 2cm. (Fig. 206—1). Then

$$F_E = \frac{10 \times (-15)}{10 - 15 - 2} = \frac{-150}{-7} = 21\frac{3}{7} \text{ cm.}$$

$$E_1 = \frac{10 \times 2}{-7} = -2\frac{6}{7}$$

i.e. $2\frac{6}{7}$ cm. in front of L_1 since the position is negative.

$$E_2 = \frac{-15 \times 2}{-7} = 4\frac{2}{7} \text{ cm. in front of } L_2$$

or $$4\frac{2}{7} - 2 = 2\frac{2}{7} \text{ cm. in front of } L_1$$

$$t = 2 - (-2\frac{6}{7} + 4\frac{2}{7}) = \frac{4}{7} \text{ cm.}$$

If the negative lens is in front, the second equivalent point is $2\frac{6}{7}$cm. behind the convex or $-2\frac{6}{7} - 2 = -4\frac{6}{7}$cm. behind the concave lens (Fig. 206—2). In the first case F_E lies $17\frac{1}{7}$cm. behind the back lens and in the second case $24\frac{2}{7}$cm. behind it. This condition is similar to that of a positive miniscus in which the optical centre lies outside the Cx. surface.

The Opera Glass.—If the two lenses are separated by $F_1 + F_2$ (the sum of their focal lengths), the negative being of the shorter focus, then $F_E = \infty$, and the lenses neutralise each other. This is the case in an opera glass.

Thus, with a 4in. Cc. and a 9in. Cc. separated by 5in.

$$F_E = \frac{9 \times (-4)}{9 - 4 - 5} = \frac{-36}{0} = \infty.$$

When $F_1 = F_2$.—If the two lenses have equal focal lengths, $F_B - F_E = F_1$ or F_2, and the formula for finding F_E (which is positive) becomes simplified to $F_E = F^2/d$. In this case E_1 is negative and both equivalent planes lie beyond the convex lens, and E_1 or $E_2 = d$, also $t = d$.

To Find d for a given F_E.—To find the distance d which should separate two lenses so that they may have a given F_E, the following formula serves

$$d = F_1 + F_2 - \frac{F_1 F_2}{F_E}. \qquad [176]$$

But if d were a negative quantity, the construction would be impossible. If both lenses are similar the formula may be written $d = 2F - F^2/F_E$, and if the one lens is convex and the other concave of equal power, the formula simplifies to $d = F^2/F_E$.

Thus, when F_1 is 10in. Cx. and F_2 is 5in. Cc., in order that F_E be 12in. Cx.

$$d = 10 - 5 - \frac{10 \times (-5)}{12} = 5 - (-4\frac{1}{6}) = 9\frac{1}{6}\text{ in.}$$

So that the equivalent focal length may be 12in. Cc. we find that

$$d = 10 - 5 - \frac{10 \times (-5)}{-12} = 5 - 4\frac{1}{6} = \frac{5}{6}\text{ in.}$$

Change of F_E for Variation in d.

As d increases with two Cx. lenses, F_E varies directly and t varies inversely or becomes negative.

As d increases with two Cc. lenses, F_E varies inversely and t varies directly.

As d increases with one Cx. and the other Cc., the Cx. being the stronger, F_E varies inversely and t varies directly.

As d increases with one Cx. and the other Cc., the Cc. being the stronger and F_E being negative, F_E varies directly and t varies inversely or becomes negative.

As d increases with one Cx. and the other Cc., the Cc. being the stronger and F_E being positive, F_E varies inversely and t varies inversely.

Combination of more than two Lenses.—Where more than two lenses are separated by intervals the method of finding F_E of the whole system, is to first obtain that of the first pair of lenses and then combine this combination with the third lens or another pair of lenses, and so on. It must be remembered that the distance between two combinations is that between the two adjacent equivalent planes, that is between E_2 of the first and E_1 of the second combination; also that the position of the equivalent planes of the whole combination is reckoned respectively from E_1 of the first and E_2 of the second combination.

DIOPTRAL EQUIVALENT THIN LENSES.

With dioptral numbers, the equivalent power and equivalent planes of two separated lenses is found by the formula

$$D_E = D_1 + D_2 - \frac{D_1 D_2 d}{100} \quad [177$$

$$E_1 = \frac{D_1 D_2 d}{D_1\left(D_1 + D_2 - \frac{D_1 D_2 d}{100}\right)} = \frac{D_1 D_2 d}{D_1 D_E} = \frac{D_2 d}{D_E} \quad [178$$

$$E_2 = \frac{D_1 D_2 d}{D_2\left(D_1 + D_2 - \frac{D_1 D_2 d}{100}\right)} = \frac{D_1 D_2 d}{D_2 D_E} = \frac{D_1 d}{D_E} \quad [179$$

$$t = d - (E_1 + E_2)$$

where D_1 and D_2 are the powers of the two lenses, d the interval between them expressed in cm. D_E is the equivalent dioptral lens and E_1 and E_2 respectively are the first and second equivalent planes. t is the distance between E_1 and E_2.

If d be expressed in terms of a metre, then we can write the formula

$$D_E = D_1 + D_2 - D_1 D_2 d. \quad [180$$

Thus + 4 D and + 5 D separated by 2cm. Then

$$D_E = 4 + 5 - \frac{4 \times 5 \times 2}{100} = 9 - .4 = +8.6 \text{ D}$$

$$E_1 = \frac{5 \times 2}{8.6} - 1.16\text{cm. behind the front lens.}$$

$$E = \frac{4 \times 2}{8.6} = .93\text{cm. in front of the back lens}$$

or $\quad 2 - .93 = 1.07$cm. behind the front lens.

If the +5 D were to the front, the second equivalent plane would be 1.16cm. in front of the +4 D.

Let −4 D and −5D have an interval between them of 20mm., then

$$D_E = -4 - 5 - \frac{-4 \times -5 \times 2}{100} = -4 - 5 - .4 = -9.4\ D$$

$$E_1 = \frac{-5 \times 2}{-9.4} = 1.07\text{cm. behind the first lens.}$$

$$E_2 = \frac{-4 \times 2}{-9.4} = .85\text{cm. in front of the second}$$

$$\text{or } 2 - .85 = 1.15\text{cm. behind the first lens.}$$

Suppose + 6 D and − 4 D be 10mm. apart.

$$D_E = 6 - 4 - \frac{6 \times -4 \times 1}{100} = 2 + .24 = +2.24\ D$$

$$E_1 = \frac{-4 \times 1}{+2.24} = -1.8 \text{ or } 1.8\text{cm. in front of the first lens.}$$

$$E_2 = \frac{6 \times 1}{+2.24} = 2.7\text{ cm. in front of the second lens}$$

or $2.7 - 1 = 1.7$cm. in front of the first lens.

If +8D and −10D be 25mm. apart.

$$D_E = 8 - 10 - \frac{8 \times -10 \times 2.5}{100} = -2 + 2 = 0.$$

Since they are separated by the sum of their focal lengths, their united power is 0.

If D_1 is positive and equal in power to D_2, which is negative, then

$$D_E = \frac{D^2\, d}{100} \quad [181]$$

Thus, a +4 D and a −4 D lens separated by 5cm. is equal to

$$D_E = \frac{4^2 \times 5}{100} = +.8\ D.$$

The distance between two dioptral lenses so that they have a certain equivalent dioptral power is found by the formula

$$d = \frac{100\ (D_1 + D_2 - D_E)}{D_1\ D_2} \qquad [182]$$

which, when D_1 and D_2 are equal, simplifies to

$$d = \frac{100\ (2\ D - D_E)}{D^2}. \qquad [183]$$

And if the one lens is positive and the other negative, and of equal powers, the formula becomes

$$d = \frac{100\ D_E}{D^2} \qquad [184]$$

For example, let +10 D and −20 D be separated so that D_E = −5 D, then

$$d = \frac{100\ [+10 + (-20) - (-5)]}{10 \times (-20)} = \frac{-500}{-200} = 2.5\text{cm}.$$

CHAPTER XI.

THICK LENSES AND COMPOUND SYSTEMS.

CARDINAL POINTS AND PLANES OF A REFRACTING SURFACE.

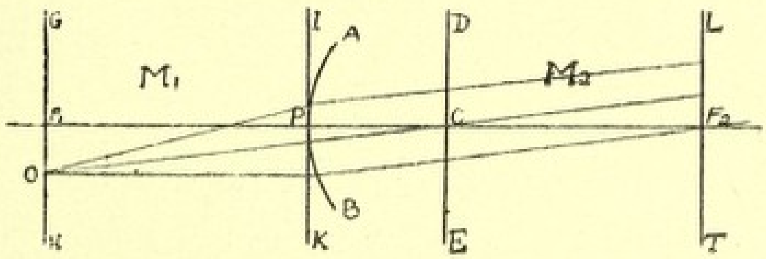


Fig. 207.

Cardinal Points and Planes.—In Fig. 207, A B is a curved surface separating the rarer medium M_1 from the denser M_2 and F_1 F_2 is the principal axis.

In the case of a single refracting surface separating two media, such as the aphakic eye, there are, on the principal axis, four cardinal points, viz., two focal points, one principal point, and one nodal point.

The first principal focal point F_1 is the anterior focus of the refracting surface. All the rays proceeding from it are, after refraction, parallel to the principal axis, and conversely rays parallel to the axis in the second medium M_2, after refraction, meet at this point F_1.

The second principal focal point F_2 is the posterior focus of the refracting surface. All rays proceeding from it are, after refraction, parallel to the principal axis and conversely, rays parallel to the axis in the first medium M_1 meet at this point F_2, after refraction.

The focal points lie in planes G H and L T perpendicular to the principal axis, called respectively, the first and second principal planes.

Rays diverging from any point, as O, in the first focal plane, will, after refraction, take a course parallel to a line O C drawn through that point to the centre of curvature C, i.e., the refracted rays are parallel to the secondary axis O C in medium M_2. Conversely on this plane G H are the foci of those rays which in the medium M_2 are parallel to secondary axes.

In the same way rays diverging from any point on the second principal focal plane L T are, after refraction, parallel to a secondary axis in the medium M_1 and on the plane L T lie the foci of those rays which are, in medium M_1 parallel to secondary axes.

The nodal point C is the centre of curvature of the surface A B. It is that point through which rays pass without deviation, because rays directed to it are radii and, therefore, normal to the refracting surface. Moreover, as every ray directed to the nodal point passes through in a straight line, any line connecting an object point with its image must pass through the nodal point.

The principal point P corresponds to that point at which the principal axis cuts the refracting surface.

If the surface of the refracting medium be small compared with the radius of curvature, the principal or refracting plane I K may be considered as coinciding with the curved surface itself, i.e., I K coincides with A B.

Course of a Ray.—Thus it follows that

(1) A ray parallel to the principal axis in the one medium passes, after refraction, through the principal focus in the other medium.

(2) A ray passing through the principal focus in the one medium is, after refraction, parallel to the principal axis in the other medium.

(3) A ray directed to, or passing through, the nodal point or centre of curvature C passes through the refracting surface without deviation.

(4) Any other ray in the one medium is, after refraction, parallel to a secondary axis in the other medium.

If these rays be traced from an object point, its image is at the point where they meet. The method of tracing the first three mentioned rays has been shown in a previous chapter.

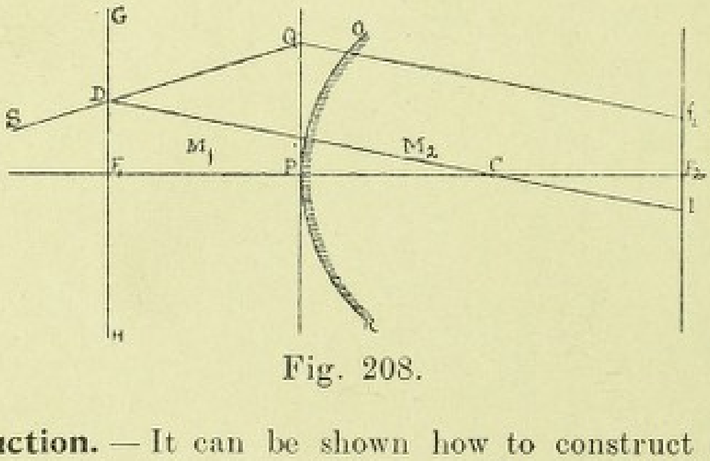


Fig. 208.

Construction.—It can be shown how to construct the course of any ray refracted by a curved surface. The ray S D Q (Fig. 208) incident on the refracting surface, passes through the first focal plane at the point D and through the principal plane at Q. Now the property of the first focal plane is that the rays diverging from any point on it are, after refraction, parallel to each other and to a secondary axis. The ray S D Q C can be regarded as if it proceeded from D on the first focal plane, and its direction, after refraction at Q, will be parallel to D C drawn from D through the nodal point, therefore it takes the direction Q f_2. The construction of the course of such a ray is of importance if it be required to locate the image of a luminous point situated on the principal axis. The distance Q P, between the ray and the principal axis in the refracting plane, is equal to the sum of D F_1 and f_2 F_2, the distances between the ray and the axis in, respectively, the first and second focal planes. This fact gives an alternative construction, the point f_2 can be located by measuring off on the second focal plane F_2 f_2 = Q P − D F_1, and then connecting Q and f_2.

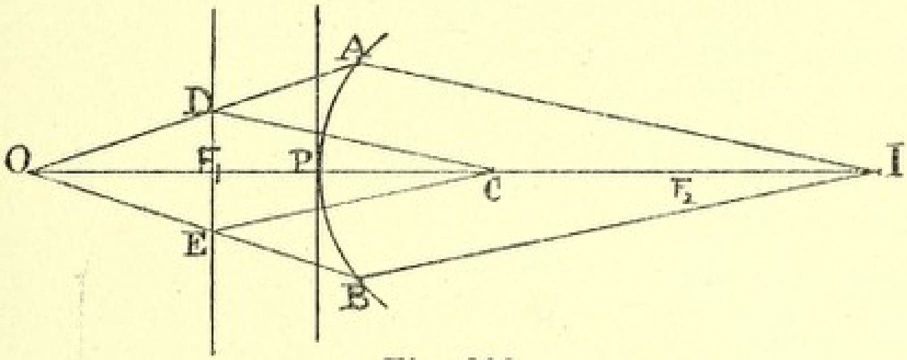


Fig. 209.

In Fig. 209 let O be a luminous point situated on the principal axis, the ray O P passes through C without deviation. The ray O A will take, after refraction, a direction parallel to D C. The ray O B will take a direction parallel to E C, and they meet each other at I, where the image of O is formed.

With a single refracting surface F_1 is to F_2 as μ_1, the refractive index of M_1, is to μ_2 the refractive index of M_2.

$$\frac{F_1}{\mu_1} = \frac{F_2}{\mu_2}, \quad \text{or} \quad \frac{F_1}{F_2} = \frac{\mu_1}{\mu_2}, \quad \text{or } F_1 \mu_2 = F_2 \mu_1.$$

Again, $F_2 - F_1$ = the distance P C = the radius of curvature of the surface.

CARDINAL POINTS AND PLANES OF COMPOUND SYSTEMS AND THICK LENSES.

Cardinal Points and Planes.—When a refracting body consists of more than one curved surface it has, on its principal axis, six cardinal points, namely, two focal points, two principal points, and two nodal points, each being in its corresponding plane, which is perpendicular to the principal axis. These points are sometimes called the Gauss points, and with their aid the course of a ray can be traced through any compound system of lenses and media.

Let Fig. 210 represent an optical system consisting of three refracting surfaces S_1, S_2, and S_3, and various media, M_1 being the first and M_2 the last medium traversed by the light; F_1 F_2 is the principal axis.

F_1 and F_2 are respectively the first and second focal points, G H and L T are their corresponding planes.

P_1 and P_2 are respectively the first and second principal points, I_1 K_1 and I_2 K_2 are their corresponding planes. N_1 and N_2 are respectively the first and second nodal points. P_1 F_1 is the first and P_2 F_2 the second principal focal distance.

Rays which in M_1 are parallel to the principal axis meet, after refraction, in M_2 at the second principal focal point F_2.

Rays which diverge from F_1, the first principal focal point, are, after refraction, parallel to the principal axis in M_2.

Rays which are parallel to the principal axis in M_2 meet, after refraction, in M_1 at the first principal focal point F_1.

Rays which diverge from F_2, the second principal focal point, are, after refraction, parallel to the principal axis in M_1.

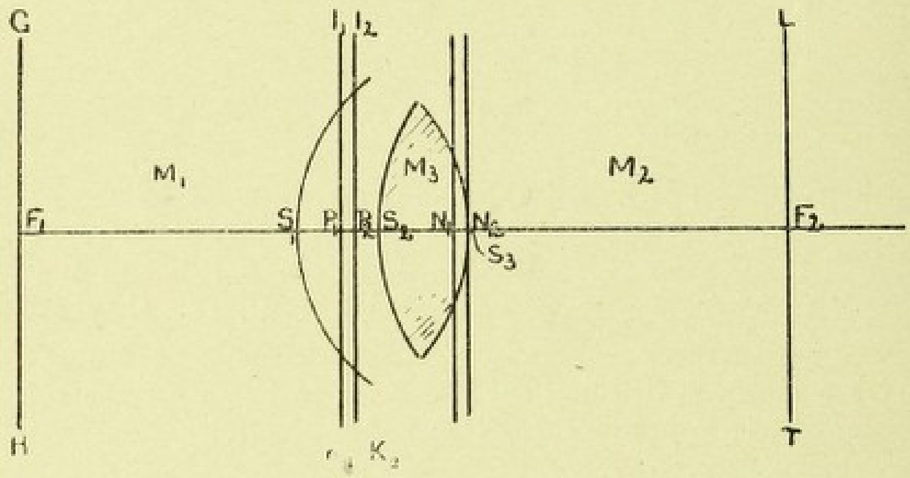


Fig. 210.

A ray directed towards the first principal point appears, after refraction, to proceed from the second, but the direction after refraction is not parallel to its original course.

A ray directed to the second principal point appears, after refraction, to proceed from the first. The two principal points are the images of each other.

A ray directed to the first nodal point, after refraction, appears to come from the second, and its direction is parallel to its original course. A ray directed to the second, appears, after refraction, to come from the first.

In the case of a single refracting surface a ray directed to its nodal point passes through without deviation; but where, in a compound system, there are two nodal points, a ray must be directed to the first in order to appear to come from the second or *vice versa*. The two nodal points are the images of each other.

Rays which, in M_1 are parallel to each other, on any secondary axis are, after refraction, brought to a focus at some point situated on L T, the second focal plane.

Rays which diverge from points on G H, the first focal plane, are, after refraction, parallel to each other in M_2.

Rays which are parallel to each other on any axis in M_2 are, after refraction, brought to a focus at some point on G H, the first focal plane.

Rays which diverge from points on L T, the second focal plane, are, after refraction, parallel to each other in M_1.

A ray directed to any point on I_1 K_1, the first principal plane, appears, after refraction, to proceed from a corresponding point situated on I_2 K_2, the second principal plane. These two points are on the same side of the axis and equally distant from it. A ray directed to a point on I_2 K_2, the second principal plane, after refraction, appears to proceed from a corresponding point on I_1 K_1, the first, equally distant from the axis. Therefore, every point on the one principal plane has its image on the other.

The first principal focal distance of a compound system is P_1 F_1, the distance between the first principal point and the first principal focus. The second principal focal distance P_2 F_2 is that between the second principal point and the second principal focus.

P_1 P_2 = N_1 N_2. The distance which separates the two principal points is equal to that which separates the two nodal points.

N_1 F_1 = P_2 F_2. The distance N_1 F_1 between the first nodal point and the first principal focus is equal to the distance P_2 F_2 between the second principal point and the second principal focus.

N_2 F_2 = P_1 F_1. The distance N_2 F_2 between the second nodal point and the second principal focus is equal to P_1 F_1, the distance between the first principal point and the first principal focus.

$$P_2 F_2 - P_1 F_1 = P_1 N_1 \text{ or } P_2 N_2.$$

P_1 F_1 and P_2 F_2 bear to each other the same relationship as the indices of refraction μ_1 of the first medium M_1 and μ_2 of the last medium M_2.

$$\frac{F_1}{\mu_1} = \frac{F_2}{\mu_2}; \text{ or } \frac{F_1}{F_2} = \frac{\mu_1}{\mu_2}, \text{ or } F_1 \mu_2 = F_2 \mu_1.$$

If the first medium M_1 be air and the last medium M_2 be glass, with an index of refraction of 1.5, then 1 is to 1.5 as P_1 F_1 is to P_2 F_2.

Coincidence of P and N.—And it follows that if the first and the last media through which rays pass, when refracted by a compound dioptric system, are of the same indices of refraction, then P_1 F_1 and P_2 F_2, the two principal focal distances, will be equal.

$N_1 F_1 = P_2 F_2$, therefore if $P_1 F_1 = P_2 F_2$, then $N_1 F_1 = P_1 F_1$ and $N_2 F_2 = P_2 F_2$, so that P_1 and N_1 merge into one as do also P_2 and N_2.

Thus in a dioptric system where the rays of light finally emerge into a medium of the same density as that from which they originally proceeded, the principal and nodal points coincide. This occurs when the refracting body consists of a lens or lenses, the light coming from air and passing out again into air or from and to any other similar media. The *united nodal and principal points* are then termed the *equivalent points.*

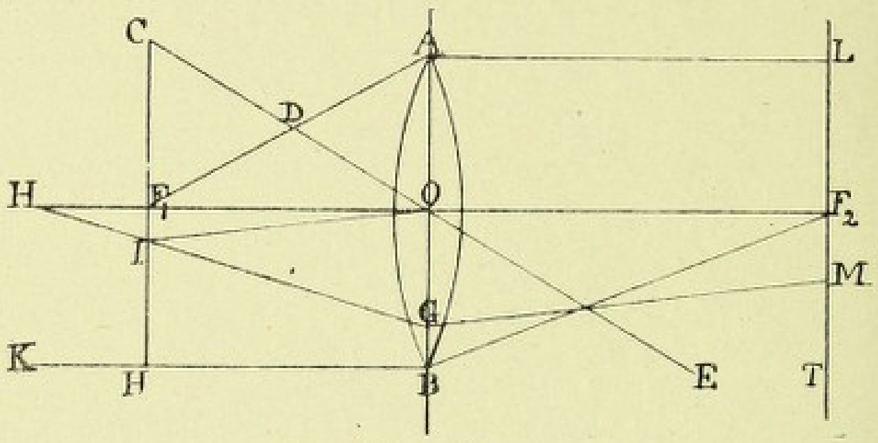


Fig. 211.

Thin Lenses.— Lenses have previously been considered as being of infinite thinness. Not only do the principal and nodal points coincide, but these points being so near to each other, there is substituted for them a single point O, midway between them, which is called the optical centre. Also the two principal and the two nodal planes are united in one single refracting plane A B passing through the optical centre, perpendicular to the principal axis, and even this refracting plane may be considered to coincide with the two surfaces. While this substitution is permissable for thin lenses, it cannot be applied to thick lenses and still less to systems of lenses.

The cardinal points and planes of a thin lens (Fig. 211) are then

O, the optical centre.
F_1, the anterior focal point.
F_2, the posterior focal point.

A B is the corresponding optical or refracting plane, and C H and L T are the corresponding anterior and posterior focal planes drawn perpendicular to the principal axis through F_1 and F_2 respectively.

A ray C D, directed to O, passes in a straight line C D O E through the lens.

A ray K B, parallel to the axis, is refracted in the direction B F_2, so also L A is refracted as A F_1.

A ray F_1 A, diverging from F_1, is refracted in a direction A L parallel to the axis, so also F_2 B is refracted as B K.

Any ray H I G, diverging from a point on the principal axis, is refracted in a direction G M parallel to a line drawn through O from I, the point where H I G cuts the anterior focal plane.

Course of a Ray.—Referring to Fig. 212, there are, in a single thick lens, two equivalent planes, A B, C D, on which all refraction appears to take place. These cut the principal axis perpendicularly and pass respectively through E_1 and E_2, the equivalent points which are, as already shown, in the case of a single lens, identical with the united principal points and nodal points. The distance E_1 E_2 is the equivalent thickness, or optical interval, of the lens.

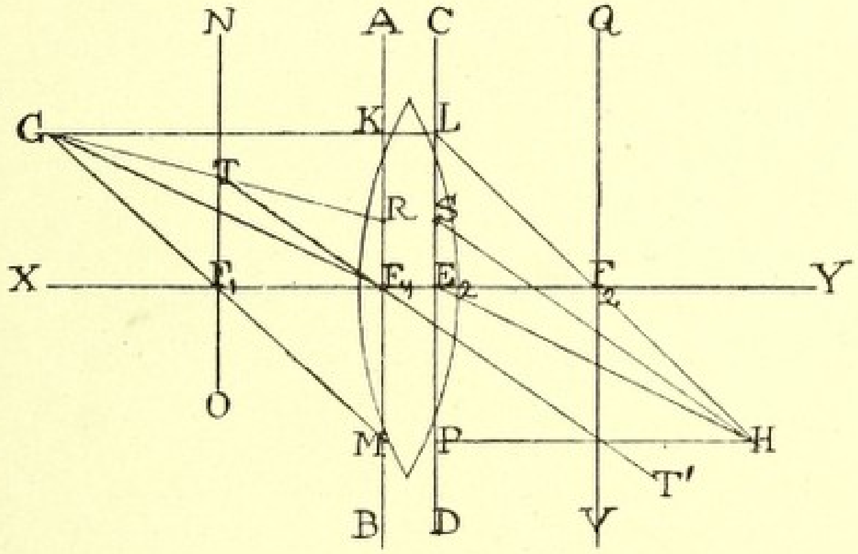


Fig. 212.

A ray directed to a point on the one equivalent plane, after refraction, emerges as if from a point on the other, equally distant from the principal axis. Thus a ray G E_1 directed towards E_1 emerges as E_2 H as if proceeding from E_2. Since this ray is directed towards a point on the principal axis its direction E_2 H, after refraction, is parallel to its original direction G E_1.

A ray G K parallel to the axis, directed towards K on the first equivalent plane, emerges as if from L on the second equivalent plane, the distance from the axis L E_2 being equal to K E_1; this ray passes through F_2, the second principal focus.

A ray G M passing through F_1, the anterior principal focus, directed towards M on the first equivalent plane, emerges parallel to the principal axis as if proceeding from P on the second equivalent plane; M and P being equally distant from the principal axis.

A ray G R directed to R on the first equivalent plane, emerges as if proceeding from S on the second equivalent plane equi-distant from the axis. Its direction S H is parallel to T E_1 T′, a line drawn from T on the anterior focal plane N O—where the ray cuts it—through E_1. Similarly the course of any ray can be traced through a thick lens and the image constructed as shown later in conection with a combination of lenses, the procedure being similar.

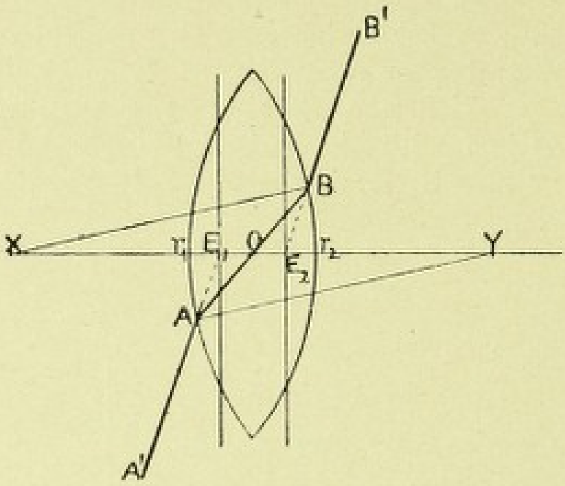


Fig. 213.

The Optical Centre and Equivalent Points.—In Fig. 213, the distance O r_1 is to O r_2 as r_1 is to r_2. The optical centre divides t, the thickness of the lens, into two parts proportionate to each other as the radii of their corresponding surfaces. It lies nearer to the surface of shorter radius, and therefore in the middle of the lens only when the two radii are equal.

$$O\,r_1 = \frac{t\,r_1}{r_1 + r_2}. \qquad O\,r_2 = \frac{t\,r_2}{r_1 + r_2}.$$

Through the point O, let any line A B be drawn. If O B were a ray of light emergent at the point B, it would take some direction B B′ dependent on the curvature of r_2, and the difference in the indices of refraction of the two media. Similarly, if O A were a ray of light emergent at A, it would be refracted in some direction A A′, governed by the radius r_1 and the two μ's. Therefore, a ray A B incident on r_1 at A, being refracted, takes the direction A B, and emerges in the direction B B′. The ray is directed towards the point E_1 on the axis and appears to come from the point E_2. These two points E_1 and E_2 situated on the principal axis being the equivalent points of the lens. They are the points, respectively, towards which a ray is directed and that from which it appears to proceed, when that ray passes through the optical centre of the lens. If O were a luminous point situated in the lens, its image would appear to be at E_1 on the one side and at E_2 on the other. In a bi-convex and bi-concave lens the optical centre O lies midway between the equivalent points E_1 and E_2.

Fig. 214 shows the equivalent points and optical centre of a bi-concave lens.

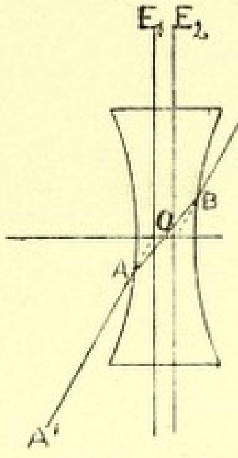


Fig. 214.

In periscopic Cx. or Cc. lenses (Fig. 215) E_1 and E_2 lie outside the lenses; on the Cx. side of the P Cx., and on the Cc. side of the P Cc. Moreover the optical centre O lies outside the equivalent points. A ray directed to E_1 the first equivalent point appears, after refraction, to proceed from E_2, the second equivalent point; its course A B within the lens being on a line connecting the optical centre O, the point of incidence A of the ray at the first surface and the point of emergence B at the second surface.

The position of O is therefore determined by producing A B so as to cut the axis.

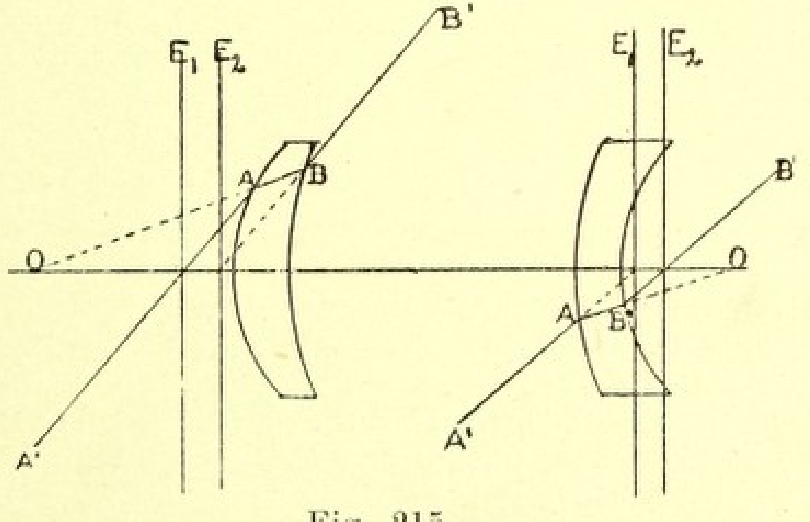


Fig. 215.

In plano-Cx. and Cc. lenses with the curved surface facing the light there are two equivalent points, but when the plane surface faces the light all the refraction takes place at the curved surface and there is but one equivalent point, as with a single refracting surface. Therefore an optical centre cannot properly be assigned to such a body.

Approximate Calculations for E_1 and E_2.—The approximate position of the equivalent points in a bi-convex lens (μ being taken as 1.5), can be found by dividing the thickness t of the lens by 3; that is t/3 gives the distance of E_1, the first equivalent point from the first surface, and of E_2, the second equivalent point, from the second surface. Thus, if the lens is 3 cm. thick, each is 1 cm. from its corresponding surface. The formula

$$E = \frac{t(\mu - 1)}{\mu} \qquad [1$$

gives the distance rather more accurately for other refractive indices than 1.5.

In a bi-convex lens of unequal radius of curvature the positions of E_1 and E_2 are found by dividing $\frac{2}{3}$ of the thickness of the lens into two parts proportional to the two radii of curvature that is

$$\frac{\frac{2}{3}t}{r_1 + r_2}.$$

This quantity multiplied by the two radii gives the distance of the two equivalent points from their respective surfaces.

If the lens is 3cm. thick and if the radii are of 20 and 40cm. the two equivalent points are respectively

$$\frac{3 \times \frac{2}{3} \times 20}{20 + 40} = .666\text{cm. from the surface of 20cm. radius}$$

and $$\frac{3 \times \frac{2}{3} \times 40}{20 + 40} = 1.333\text{cm. from that of 40cm. radius.}$$

In a plano-convex or concave lens when the curved surface faces parallel rays, twice the thickness of the glass divided by 3, that is t 2/3, is the distance of E_2 from the plano surface, or the thickness divided by three, that is, t/3 from the curved surface. E_1 is on the curved surface, and so, when $\mu = 1.5$, E_2 is t/3 behind it. If the lens is 3cm. thick, E_2 is at

$$\frac{3 \times 2}{3} = 2\text{cm. from the plano, or } \frac{3}{3} = 1\text{cm. from the curved surface.}$$

The single equivalent point of a plano-convex or concave lens lies at the apex of the curved surface when the plano surface faces parallel rays.

The calculations for accurately determining the positions of E_1 and E_2 will be found in the following pages. These points, like the optical centre, lie nearer to the surface of greater curvature, and may be, one or both, outside the lens.

THE FOCAL LENGTH AND EQUIVALENT POINTS OF SINGLE THICK LENSES.

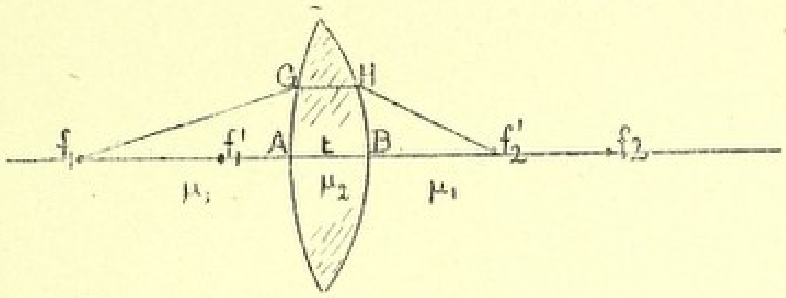


Fig. 216.

The Focal Lengths of the *Surfaces* of a Lens.—In Fig. 216 let μ_1 represent the index of refraction of the media on both sides of the lens and μ_2 the index of refraction of the glass. Let r_1 be the radius of curvature of the first surface A and r_2 that of the second surface B. Let f_1 and f_2 represent respectively the anterior and posterior focal distances of the first surface, and f'_1 and f'_2 represent respectively the anterior and posterior focal distances of the second surface. Let t be the thickness of the lens; then

$$f_1 = \frac{\mu_1 r_1}{\mu_2 - \mu_1} \qquad f_2 = \frac{\mu_2 r_1}{\mu_2 - \mu_1} \qquad [187] \; [188]$$

$$f'_1 = \frac{\mu_2 r_2}{\mu_2 - \mu_1} \qquad f'_2 = \frac{\mu_1 r_2}{\mu_2 - \mu_1} \qquad [189] \; [190]$$

To avoid confusion it should be noted that the anterior and the posterior foci of B are taken to be on the same side as those of A. Usually the anterior focus of a surface is regarded as being that which is in the air, or rarer medium; but here f_1 and f'_1 both point towards the light, while f_2 and f'_2 are pointed in the direction of the light's travel.

The posterior focal distance f_2 of A, is to the anterior focal distance f'_1 of B, as the radius of the first surface r_1 is to that of the second r_2 and a similar relationship exists between the anterior focal distance f_1 of A and the posterior focal distance f'_2 of B. That is

$$\frac{r_1}{r_2} = \frac{f_2}{f'_1} = \frac{f_1}{f'_2}$$

Or otherwise worded the two foci in the air are proportional to the two radii, as are also the two foci in the denser medium.

Let $r_1 = 10\text{cm.}$ and $r_2 = 6\text{cm.}$, $\mu_1 = 1$, $\mu_2 = 1.5$, $t = 3\text{cm}$; then

$$f_1 = \frac{1 \times 10}{1.5 - 1} = 20\text{cm.} \quad f_2 = \frac{1.5 \times 10}{1.5 - 1} = 30\text{cm.}$$

$$f'_1 = \frac{1.5 \times 6}{1.5 - 1} = 18\text{cm.} \quad f'_2 = \frac{1 \times 6}{1.5 - 1} = 12\text{cm.}$$

The relationship is $$\frac{10}{6} = \frac{30}{18} = \frac{20}{12}$$

A ray G H parallel to the axis in the lens (Fig. 216) would, on refraction at the surface A, meet the axis 20cm. in front of it. Refracted at the surface B it would meet the axis 12cm. in front of it.

Combined F_E.—The anterior focal distance F_1 and the posterior F_2 of the lens are found by

$$F_1 = \frac{f_1\, f'_1}{f_2 + f'_1 - t} \qquad F_2 = \frac{f_2\, f'_2}{f_2 + f'_1 - t}$$

In this case $$F_1 = \frac{20 \times 18}{30 + 18 - 3} = 8\text{cm.}$$

and $$F = \frac{12 \times 30}{30 + 18 - 3} = 8\text{cm.}$$

We see that the anterior and posterior principal focal distances are the same, as they are in all lenses, or systems of lenses, but being measured from their respective equivalent points, the distance of each from its corresponding surface may differ considerably.

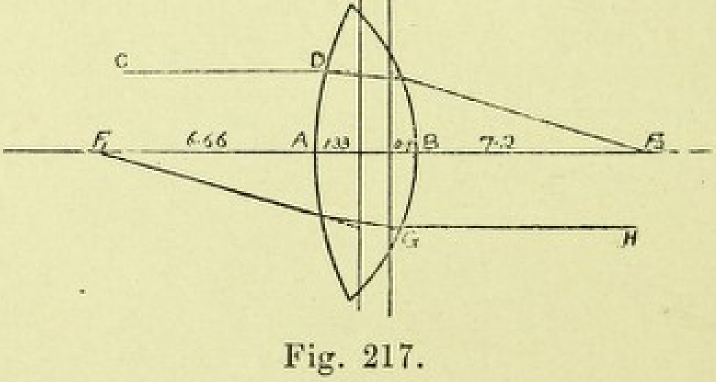


Fig. 217.

A ray C D parallel to the axis (Fig. 217) and incident on the surface A is converged so as to meet the axis at a point 30cm.

behind A and 30 − 3 = 27cm. behind B, where it receives a further convergence, and finally meets the axis at F_2, 7.2cm. behind B.

A ray H G parallel to the axis and incident at the surface B, is converged to 18cm. behind B and meets A, 3cm. distant, converging to 15cm. behind it; then, being again converged, finally meets the axis at F_1, 6.66cm. from A.

Back Surface F.—The formulæ for finding the distance of the principal focus from the surfaces A and B of a thick lens are for F_A the anterior focus.

$$F_A \text{ from A} = \frac{f_1\,(f'_1 - t)}{f_2 + f'_1 - t}$$

and for F_B the posterior focus

$$F_B \text{ from B} = \frac{f'_2\,(f_2 - t)}{f_2 + f_1' - t)}.$$

Equivalent Points.—The distance of the equivalent points E_1 and E_2 from the surfaces of the lens are found by deducting the back surface focal distances, represented by F_A and F_B respectively, from the true focal distances F_1 and F_2. Thus, in the example given

$$F_A = \frac{20 \times (18 - 3)}{30 + 18 - 3} = \frac{300}{45} = 6.66\text{cm.}$$

$$F_B = \frac{12 \times (30 - 3)}{30 + 18 - 3} = \frac{324}{45} = 7.2\text{cm.}$$

$E_1 = 8 - 6.66 = 1.33$cm. from A.
$E_2 = 8 - 7.2 = .8$cm. from B.

Direct Formulæ for a Thick Lens.—But instead of calculating separately the focal distance of each surface of a thick lens, the calculations can be combined in the following formulæ, which serve for all lenses, care being taken with the + and − signs.

F = the equivalent focal length.
E_1 and E_2 = the first and second equivalent points.
T = the distance between E_1 and E_2 (the optical interval).
r_1 and r_2 = the radii of curvature of respectively the first and second surfaces.
A and B = the first and second surfaces at the principal axis.
μ = the index of refraction of the glass.
t = the thickness of the lens on the axis.

The distance of the principal focus F, measured anteriorly from E_1 and posteriorly from E_2, is

$$F = \frac{r_1 \, r_2}{(\mu - 1)\left(r_1 + r_2 - t\,\frac{(\mu - 1)}{\mu}\right)} \qquad [193$$

Now if, as is convenient, we calculate the quantity N which enters into the various formulæ; that is

$$N = r_1 + r_2 - t\,\frac{\mu - 1}{\mu} \qquad [194$$

we have

$$F = \frac{r_1 \, r_2}{(\mu - 1)\,N}. \qquad [195$$

It will be noted that this important formula is the same as for thin lenses, except that the apparent reduction of thickness

$$t\,\frac{\mu - 1}{\mu} \qquad [196$$

is deducted from $r_1 + r_2$ in the formula.

An approximate formula (accurate when $\mu = 1.5$) is

$$F = \frac{r_1 \, r_2}{(\mu - 1)\,(r_1 + r_2 - t/3)} \qquad [197$$

The distance of the first and second equivalent points measured *inwards* from respectively the first and second surface on the principal axis is found by

$$E_1 = \frac{r_1 \, t}{\mu\left(r_1 + r_2 - t\,\frac{(\mu - 1)}{\mu}\right)} = \frac{r_1 \, t}{\mu \, N} \qquad [198$$

$$E_2 = \frac{r_2 \, t}{\mu\left(r_1 + r_2 - t\,\frac{(\mu - 1)}{\mu}\right)} = \frac{r_2 \, t}{\mu \, N} \qquad [199$$

The equivalent thickness T or the distance between the equivalent points is

$$T = t - (E_1 + E_2). \qquad [200$$

The back surface focal distances F_A and F_B are

$$F_A \text{ from } A = \frac{\mu\, r_1\, r_2 - (\mu - 1)\, r_1\, t}{(\mu - 1)\, \mu\, N} \quad [201]$$

$$F_B \text{ from } B = \frac{\mu\, r_1\, r_2 - (\mu - 1)\, r_2\, t}{(\mu - 1)\, \mu\, N} \quad [202]$$

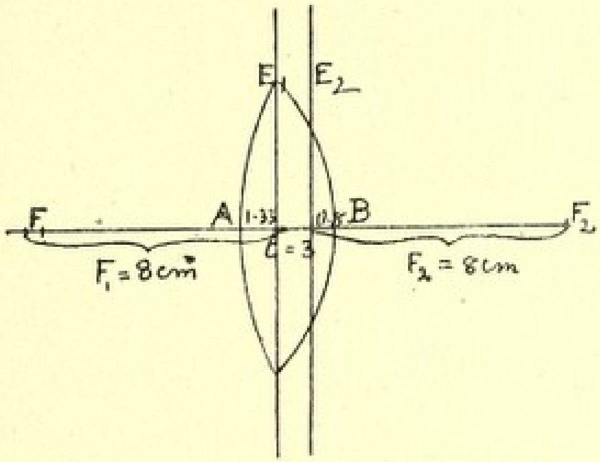


Fig. 218.

Example of a Bi-Cx. Lens.—Taking the same example as previously given where r_1 and r_2 = 10cm. and 6cm. respectively, $\mu = 1.5$, t = 3cm., then

$$F = \frac{10 \times 6}{.5 \times \left(10 + 6 - \frac{3 \times .5}{1.5}\right)} = \frac{60}{.5 \times (16 - 1)} = \frac{60}{7.5} = 8\text{cm.}$$

$$E_1 = \frac{10 \times 3}{1.5 \times (16 - 1)} = \frac{30}{22.5} = 1.33\text{cm.}$$

$$E_2 = \frac{6 \times 3}{1.5 \times (16 - 1)} = \frac{18}{22.5} = .8\text{cm.}$$

$$T = 3 - (1.333 + .8) = .86\text{cm.}$$

F is anteriorly 8 – 1.333 = 6.66cm. from A and posteriorly 8 – .8 = 7.2cm. from B, as shown in Fig. 218.

A thin lens of same radius and μ has

$$F = \frac{10 \times 6}{.5 \times (10 + 6)} = \frac{60}{8} = 7.5\text{cm.}$$

The optical centre is located at

$$\frac{3 \times 6}{10 + 6} = 1.125\text{cm. from B and } 1.875\text{cm. from A.}$$

Thus, F is longer in a thick than in a thin lens, but the focal point is nearer to the back surface in the former. In this case F is 7.2cm. from B, and if the lens were infinitely thin, so that t = 0, then F would be 7.5cm. from B.

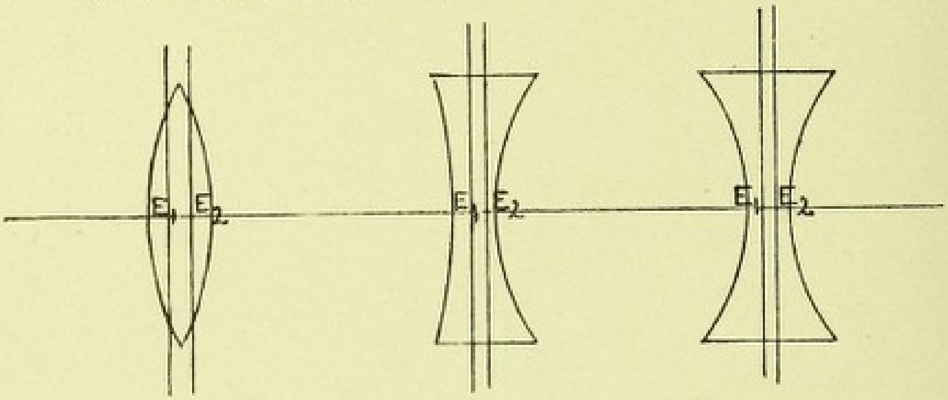


Fig. 219. Fig. 220. Fig. 221.

In a double convex lens (Fig. 219) the equivalent points are equally distant from their respective surfaces, so that when either surface faces the light the principal focus is at the same distance from the back surface, whereas when the two surfaces are of different radii this is not the case.

Example of a Bi-Cc. Lens.—Let Fig. 220, in which r_1 and r_2 = −10cm. and −6cm. respectively, μ = 1.5 and t = 3cm.

$$F = \frac{-10 \times (-6)}{.5 \times \left(-10 - 6 - \frac{3 \times .5}{1.5}\right)} = \frac{60}{.5\,(-16 - 1)} = \frac{60}{-8.5} = -7.06\text{cm.}$$

$$E_1 = \frac{-10 \times 3}{1.5 \times (-16 - 1)} = \frac{-30}{-25.5} = 1.18\text{cm.}$$

$$E_2 = \frac{-6 \times 3}{1.5 \times (-16 - 1)} = \frac{-18}{-25.5} = .7\text{cm.}$$

$$T = 3 - (1.18 + .7) = 1.12\text{cm.}$$

Thus, the distance of F in front of the lens varies as the one or the other surface faces the light, although the true focal length is always the same.

In an equi-concave (Fig. 221), the equivalent points are equally distant from their corresponding surfaces and F is the same distance in front of the lens whichever surface is to the front.

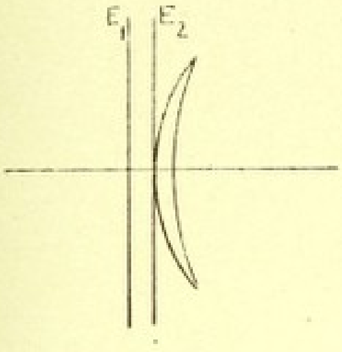


Fig. 222.

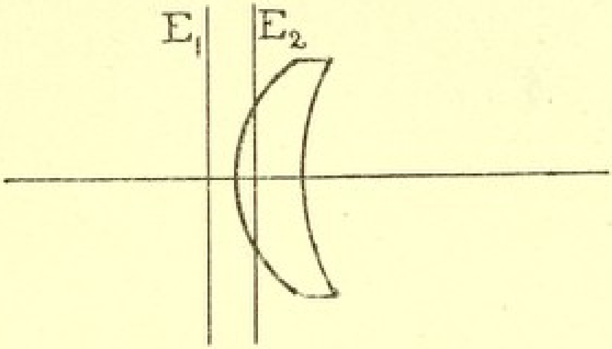


Fig. 223.

Example of a Positive Meniscus.—In a periscopic convex lens (Fig. 222) let r_1 and r_2 of the convex and concave surfaces respectively be $+6$cm. and -10cm., $\mu = 1.5$ and $t = 3$cm.

$$F = \frac{6 \times (-10)}{.5 \times \left(+6 - 10 - \frac{3 \times .5)}{1.5}\right.} = \frac{-60}{.5 \times (-4 - 1)} = \frac{-60}{-2.5} = 24\text{cm.}$$

$$E_1 = \frac{6 \times 3}{1.5 \times (-4 - 1)} = \frac{18}{-7.5} = -2.4\text{cm.}$$

$$E_2 = \frac{-10 \times 3}{1.5 \times (-4 - 1)} = \frac{-30}{-7.5} = 4 \text{ cm.}$$

E_1, being negative, must be reckoned *outwards* instead of inwards, so that the distance of both equivalent points are reckoned the same way, the first outwards from the convex surface, the second inwards from the concave. E_1 is 2.4cm. and E_2 is $4 - 3 = 1$cm. outside the convex surface. In this example

$$T = 3 - (-2.4 + 4) = 3 - 1.6 = 1.4\text{cm.}$$

In some cases, with the periscopic convex lens, the one equivalent point lies within the convex surface as in Fig. 223. The positions of E_1 and E_2 depend on the curvatures of the two surfaces; the more nearly equal the two curvatures, the more are E_1 and E_2 displaced towards the Cx. surface or beyond it. The distance of the focus from the back surface varies very considerably as the one or the other surface is exposed to the light.

Example with a Negative Meniscus.—In a periscopic concave, as in Fig. 224, let r_1 and r_2 of the convex and concave surfaces respectively = + 10cm. and − 6cm. μ = 1.5 and t = 3cm.

$$F = \frac{10 \times (-6)}{.5 \times \left(+10 - 6 - \frac{3 \times .5}{1.5}\right)} = \frac{-60}{.5 \times (+4 - 1)} = \frac{-60}{1.5} = -40\text{cm.}$$

$$E_1 = \frac{10 \times 3}{1.5 \times (+4 - 1)} = \frac{30}{4.5} = 6.66\text{cm.}$$

$$E_2 = \frac{-6 \times 3}{1.5 \times (+4 - 1)} = \frac{-18}{4.5} = -4\text{cm.}$$

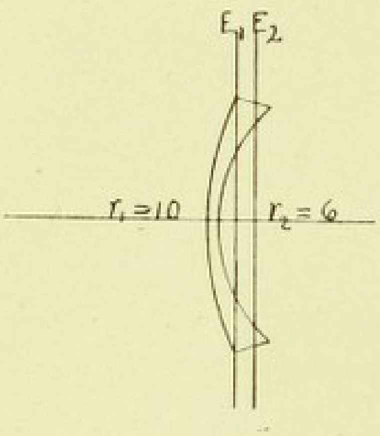


Fig. 224.

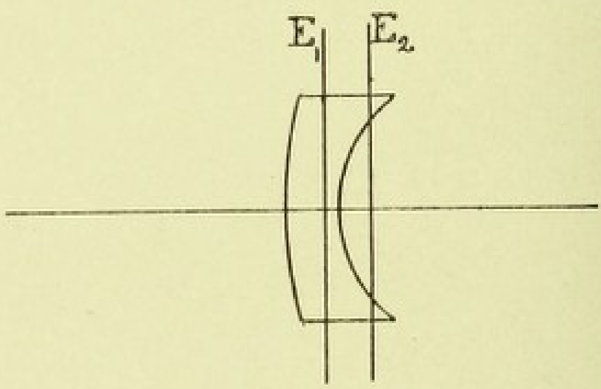


Fig. 225.

That is the distance of both equivalent points are reckoned the same way, E_1 inwards from the convex surface, and E_2, being negative, outwards from the concave. The first is 6.66 − 3 = 3.66cm. outside the concave surface, and the second is 4cm. outside it.

$$T = 3 - (6.66 - 4) = 3 - 2.66 = .33\text{cm.}$$

As with the convex meniscus so also, in some cases, the one equivalent plane of a concave meniscus lies within the concave surface (Fig. 225). This occurs if the concave is much more curved than the convex. Also, as with the periscopic convex, the difference in the distance of the focus in front of the surface which faces the light is very marked as E_1 or E_2 is taken as the first equivalent point.

Special Cases.—Certain special cases of menisci should be considered. Thus, in the meniscus lens when the radius of the convex surface, is longer than that of the concave

$$F = \infty \text{ if } r_1 + r_2 = \frac{t(\mu - 1)}{\mu}$$

that is $$t = \frac{(r_1 + r_2)\,\mu}{\mu - 1} \text{ when } F = \infty.$$ [203]

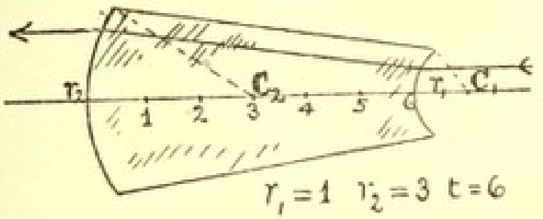


Fig. 226.

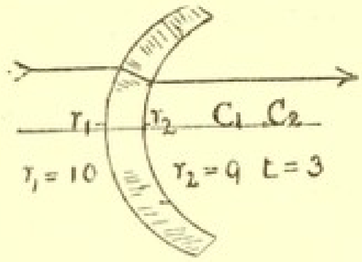


Fig. 227.

Thus, in order that (Fig. 226) $F = \infty$ when $r_1 = -1$, $r_2 = +3$, and $\mu = 1.5$.

$$t = \frac{(-1 + 3) \times 1.5}{.5} = 6\text{cm.}$$

This is the principle of the Steinheil cone, which is practically a fixed focus opera glass.

If $r_1 = +10$ and $t = 3$ when $\mu = 1.5$, then (Fig. 227) r_2, from the foregoing formula, must be -9 in order that $F = \infty$.

$$F \text{ is negative if } r_1 + r_2 \text{ is greater than } \frac{t(\mu - 1)}{\mu}$$

$$F \text{ is positive if } r_1 + r_2 \text{ is less than } \frac{t(\mu - 1)}{\mu}$$

That is to say, when the concave surface has the shorter radius, F is negative or positive according as t is sufficiently great or small respectively; and that when t is of certain value the F of the convex surface neutralises that of the concave.

If $r_1 + r_2 = t$ (r_2 being negative), that is, if $r_1 - t = r_2$ so that the two centres of curvature coincide, F is negative. Thus, let $r_1 = 10$cm., $r_2 = -6$cm., $t = 4$cm., and $\mu = 1.5$. Then

$$F = \frac{10 \times -6}{.5 \times \left(+10 - 6 - \frac{4 \times .5}{1.5}\right)} = \frac{-60}{.5 \times (+2.666)} = \frac{-60}{1.333} = -45\text{cm}$$

$E_1 = r_1 = 10, E_2 = r_2 = -6, T = 4 - (10 - 6) = 0.$
The principal points coincide.

If the glass is thin and the centres coincide, the concave radius is shorter and then we get a slightly concave effect, as is found in the ordinary globular or coquille.

If r_1, the radius of the convex, is shorter than r_2, that of the concave, F is positive; but if t is greater than the posterior focal distance of the first surface, the rays come to a focus within the lens and, after crossing, diverge to the concave surface.

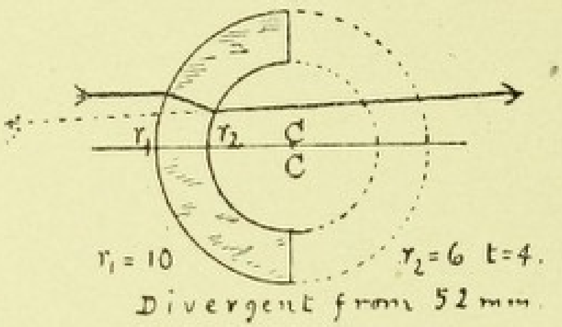


Fig. 228.

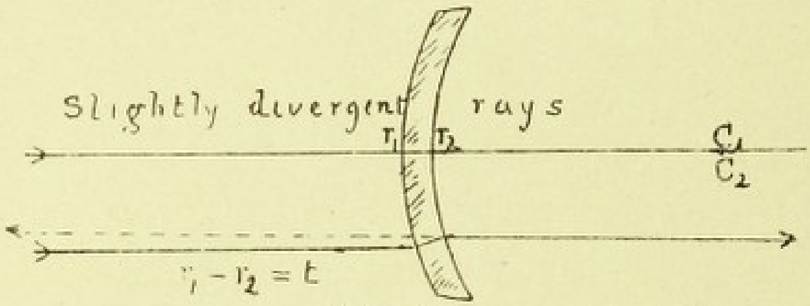


Fig. 229.

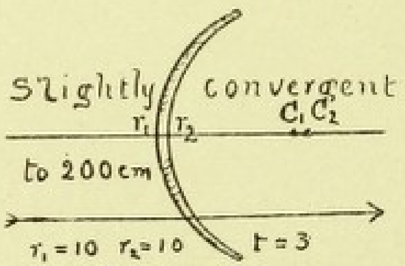


Fig. 230.

If $r_1 = r_2$, F is positive. Thus, let $r_1 = +10$cm., $r_2 = -10$cm., $t = 3$cm., and $\mu = 1.5$. Then

$$F = \frac{10 \times (-10)}{.5 \times \left(+10 - 10 - \frac{3 \times .5}{1.5}\right)} = \frac{-100}{-.5} = 200\text{cm.}$$

$$\text{Here } F = \frac{-\mu\, 1^2}{(\mu - 1)^2\, t} \qquad \text{and } E_1 \text{ or } F_2 = \frac{r}{\mu - 1};$$ [204] [205]

$$\text{that is, in this case } \frac{10}{.5} = 20\text{cm.}$$

But even if $r_1 = r_2$ it might be that t is longer than F of the first surface, and there is consequently a negative effect. If t is very small $F = \infty$.

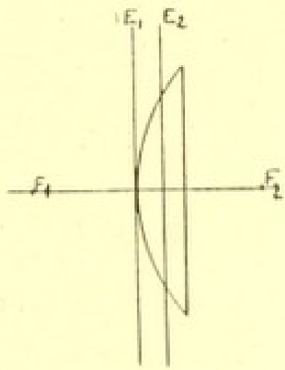


Fig. 231.

Example with a Plano-Convex Lens.—Let r_1 of the curved surface = 6cm.; r_2 of the plano = ∞; $\mu = 1.5$, and t = 3cm.

Then, since $r_2 = \infty$, and this quantity occurs in the upper and lower part of the formula, we can omit it from our calculations as well as the bracket involving this value. The plano surface had no power, either convergent or divergent, and the formula therefore becomes simplified to

$$F = \frac{r_1}{\mu - 1}$$ [206]

That is $$F = \frac{6}{.5} = 12\text{cm.}$$

$$E_1 = \frac{6 \times 3}{1.5\ \infty} = 0 \qquad E_2 = \frac{3}{1.5} = 2\text{cm.}$$

$$t = 3 - 2 = 1\text{cm.}$$

The first equivalent plane is at the curved surface and the second is 2cm. in front of the plane surface. In the above example when the convex surface is exposed to the light, F lies 12 − 2 = 10cm. behind the plane and 13cm. from the curved surface. When the plane surface is so exposed, F lies 12cm. behind the curved and 15cm. behind the plane surface. The plano surface has then no effect on incident rays parallel to the axis and there is only one refracting or equivalent plane.

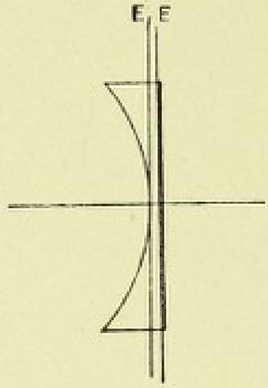


Fig. 232.

Example with a Plano-Concave Lens.—If r_1 of the plane surface $= \infty$, it, as before stated, may be neglected. Let r_2 of the concave $=$ 6cm., $\mu = 1.5$, and $t =$ 3cm.

$$F = \frac{-6}{.5} = -12\text{cm.}$$

$$E_1 = \frac{3}{1.5} = 2\text{cm.}$$

$$E_2 = \frac{6 \times 3}{1.5\ \infty} = 0.$$

$$t = 3 - 2 = 1\text{cm.}$$

The first equivalent plane is 2cm. from the plane surface and the second is at the concave surface. If the curved surface faces the light the focal distance is 12 + 2 = 14cm. in front of the plano surface, and 11cm. from the curved one. When the light is incident on the plane surface, F lies 12cm. from the curved and 9cm. from the plane surface.

Example of a Sphere.—Suppose there be a sphere, the radius of curvature of which is 6cm., then $r_1 = r_2 =$ 6cm. and the diameter $= t =$ 12cm. Let $\mu = 1.5$.

$$F = \frac{6 \times 6}{.5 \times \left(6 + 6 - \frac{12 \times .5}{1.5}\right)} = \frac{36}{.5 \times (12 - 4)} = 9\text{cm.}$$

$$E_1 = \frac{6 \times 12}{1.5 \times (12 - 4)} = \frac{72}{1.5 \times 8} = \frac{72}{12} = 6\text{cm.}$$

$$E_2 = \frac{6 \times 12}{1.5 \times (12 - 4)} = \frac{72}{1.5 \times 8} = \frac{72}{12} = 6\text{cm.}$$

$$t = 12 - (6 + 6) = 0.$$

Therefore, the equivalent planes of a sphere coincide and pass through the centre of curvature C, as in Fig. 233.

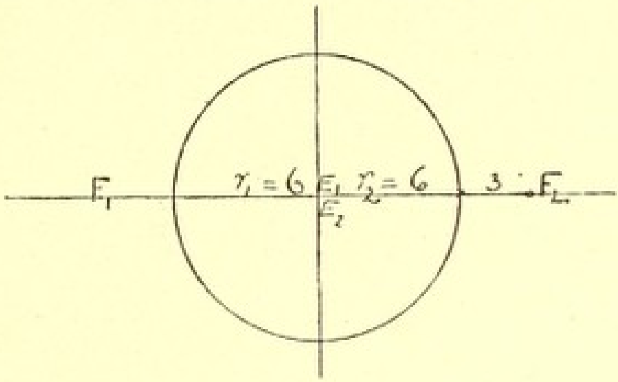


Fig. 233.

It will be seen that the above method of calculation does not differ from the simple formula

$$F = \frac{r\,\mu}{2\,(\mu - 1)}$$

The back surface focal distance of a sphere is found by the same formula as that employed for any thick lens, t being the diameter which is 12cm., we then have

$$f_1 = \frac{6 \times 1}{.5} = 12 \qquad f_2 = \frac{6 \times 1.5}{.5} = 18$$

$$f'_1 = \frac{6 \times 1.5}{.5} = 18 \qquad f_2' = \frac{6 \times 1}{.5} = 12$$

$$F_A \text{ or } F_B = \frac{12 \times (18 - 12)}{18 + 18 - 12} = 3\text{cm.}$$

Or the back surface focal length of a sphere is

$$F_B = \frac{r\,(2 - \mu)}{2\,(\mu - 1)}$$ [207]

which is a much more simple calculation.

When the μ of a sphere is 1.5 the equivalent or true focal length is $1\frac{1}{2}$ times the radius and the back focus is half the radius. Calculations with a sphere are the same as those of any other thick lens, when the object is situated outside the sphere. If, however, the object be within the sphere, the calculations are the same as those connected with a single surface.

The Hemisphere.—With the hemisphere both the equivalent F and the back surface F from the convex surface is

$$F_E \text{ or } F_B = \frac{r}{\mu - 1}, \qquad [20$$

but the back surface F from the plane surface is

$$F_B = \frac{r}{\mu(\mu - 1)}. \qquad [20$$

When $\mu = 1.5$ the back surface focal distances $= 2r$ from the convex and $1\frac{1}{3}r$ from the plane surface.

Other Calculations.—The original formula serves for finding the one radius of a thick lens when the other is known substituting for the symbols the known qualities. For example—

What radius of curvature must be given to a double convex so that F = 5cm. when $\mu = 1.5$ and t = .75cm.

$$F = \frac{r_1\ r_2}{(\mu - 1)\left(r_1 + r_2 - t\,\frac{(\mu - 1)}{\mu}\right)}$$

substituting the known quantities we have

$$5 = \frac{r^2}{.5\left(2r - \frac{.5 \times .75}{1.5}\right)} = \frac{r^2}{.5\,(2r - .25)} = \frac{r^2}{r - .125}$$

then $\quad 5r - .625 = r^2 \quad$ and $\quad - .625 = r^2 - 5r$;

adding to both sides of the equation $\quad \left(\frac{5}{2}\right)^2 = 6.25$

we have $\quad 6.25 - .625 = 5.625 = r^2 - 5r + 6.25$;
extracting the square root gives $\quad 2.37 = r - 2.5$
so that $\quad r = 4.87$

Thus the required curvature is 4.87cm.

If $F = 10in.$, $r_1 = 12in.$, $\mu = 1.5$, and $t = .3in.$, to find r, the radius of the other surface, the procedure is

$$10 = \frac{12r}{.5\left(12 + r - \frac{.5 + .3}{1.5}\right)} = \frac{12r}{.5\,(12 + r - .1)} = \frac{12r}{6 + .5r - .05}$$

that is $$60 + 5r - .5 = 12r$$

and $$59.5 = 7r, \qquad \text{therefore } r = 8.5in.$$

The second surface must have a curvature of 8.5in.

If $F = 24cm.$, $r_1 = 6cm.$, $\mu = 1.5$, and $t = 3cm.$, what is the radius of the other surface?

$$24 = \frac{6r}{.5\,(6 + r - 1)} = \frac{6r}{3 + .5\,r - .5}$$

$$72 + 12r - 12 = 6r \text{ and } 60 = -6r,$$

therefore $$r = -10.$$

The second surface requires to be 10cm. concave.

THE EQUIVALENT FOCI AND POINTS OF A THICK LENS BY THE DIOPTRIC SYSTEM.

Employing the same symbols, let r_1 be the radius of curvature of the first and r_2 that of the second surface, let D be the equivalent dioptric power, and E_1 and E_2 the equivalent points. Then

$$D = \frac{100\,(\mu - 1)\left(r_1 + r_2 - \frac{t\,(\mu - 1)}{\mu}\right)}{r_1\, r_2} = \frac{100\,(\mu - 1)\,N}{r_1\, r_2} \qquad [210]$$

$$E_1 = \frac{r_1\, t}{\mu\, N} \qquad E_2 = \frac{r_2\, t}{\mu\, N} \qquad \begin{matrix}[211]\\ [212]\end{matrix}$$

$$T = t - (E_1 + E_2) \qquad [213]$$

For example, if $r_1 = 10$cm., $r_2 = 6$cm., $t = 3$cm., and $\mu = 1.5$, then

$$D = \frac{100 \times .5 \times \left(10 + 6 - \frac{3 \times .5}{1.5}\right)}{10 \times 6} = \frac{750}{60} = 12.5 \text{ D.}$$

$$E_1 = \frac{10 \times 3}{1.5 \times 15} = \frac{30}{22.5} = 1.33\text{cm.}$$

$$E_2 = \frac{6 \times 3}{1.5 \times 15} = \frac{18}{22.5} = .8\text{cm.}$$

$$T = 3 - (1.333 + .8) = .866\text{cm.}$$

If the distances are expressed in terms of a metre the formula for D can be written

$$D = \frac{(\mu - 1)\left(r_1 + r_2 - t\frac{(\mu - 1)}{\mu}\right)}{r_1 \, r_2}$$

In this case

$$D = \frac{.5 \times \left(.1 + .06 - \frac{.03 \times .5}{1.5}\right)}{.1 \times .06} = \frac{.5 \times \left(.16 - \frac{.015}{1.5}\right)}{.006}$$

$$= \frac{.075}{.006} = 12.5 \text{ D}$$

A Thick Lens Bounded by Different Media.—If light passes from one medium into another and finally into a third, when the thickness of the central medium cannot be ignored, and the bounding surfaces are curved, we have a combination of a thick lens separating different media. Such a combination exists if a glass lens has air on the one side and oil, or water, on the other. Let

r_1 and r_2 be the radii of curvature and μ_1, μ_2, and μ_3, the three refractive indices. Then

$$F_1 = \frac{\mu_1 r_1 r_2}{r_1(\mu_2 - \mu_3) + r_2(\mu_2 - \mu_1) - \dfrac{t(\mu_2 - \mu_1)(\mu_2 - \mu_3)}{\mu_2}} = \frac{\mu_1 r_1 r_2}{N}$$

$$F_2 = \frac{\mu_3 r_1 r_2}{N} \qquad [214]$$

$$E_1 = \frac{\mu_1 r_1 t(\mu_2 - \mu_3)}{\mu_2 N} \qquad [215]$$

$$E_2 = \frac{\mu_3 r_2 t(\mu_2 - \mu_1)}{\mu_2 N} \qquad [216]$$

Here the first and last media being different, F_1 does not equal F_2, but

$$\frac{F_1}{F_2} = \frac{\mu_1}{\mu_3}.$$

The back surface focal distances can be obtained by deducting E_1 from F_1 and E_2 from F_2.

Let $r_1 = 10$, $r_2 = 6$, $\mu_1 = 1$, $\mu_2 = 1.45$, and $\mu_3 = 1.33$

Then $F_1 = 15.92$ and $F_2 = 21.17$

$E_1 = .79$ and $E_2 = 2.36$

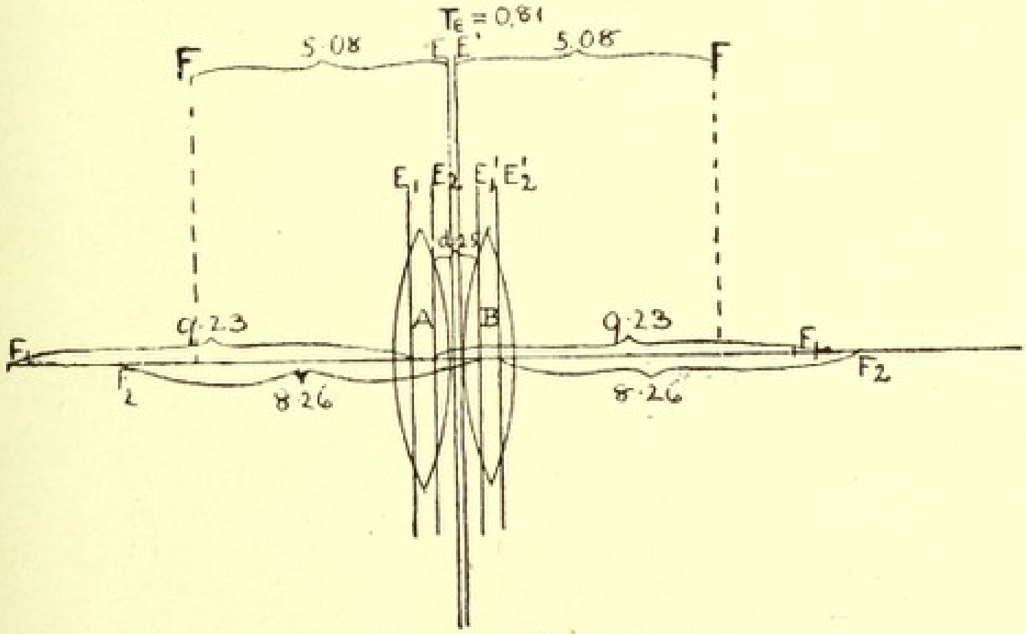


Fig. 234.

Two Thick Lenses in Combination.—Let A be the first and B the second lens (Fig. 234) of a combination of two thick convex lenses separated by an interval.

Let r_1 and r_2 be the radii of curvature of A, and r'_1 and r'_2 those of B.

Let t_1 and t_2 be, respectively, the actual thicknesses of A and B.

Let E_1 and E_2 be, respectively, the first and second equivalent points of A.

Let E'_1 and E'_2 be, respectively, the first and second equivalent points of B.

Let T_1 and T_2 be, respectively, the equivalent thicknesses of A and B.

Let F_1 and F_2 be, respectively, the focal lengths of A and B.

Let d be their distance apart, this being the distance between their most adjacent equivalent points, i.e., the distance between E_2 and E'_1.

Let E and E′ be, respectively, the first and second equivalent points of the combination.

Let F be the equivalent focal distance of the combination.

Let T be the equivalent thickness of the combination.

The equivalent focal distance F of two combined lenses is obtained by the formula

$$F = \frac{F_1 F_2}{F_1 + F_2 - d} = \frac{F_1 F_2}{N} \qquad [217$$

The distance of E, the first equivalent point of the combination, measured from E_1, the first equivalent point of A, is

$$E = \frac{F_1 d}{F_1 + F_2 - d} = \frac{F_1 d}{N} \qquad [218$$

The distance of E′, the second equivalent point of the combination, measured from E'_2, the second equivalent point of B, is

$$E' = \frac{F_2 d}{F_1 + F_2 - d} = \frac{F_2 d}{N}. \qquad [219$$

The distance T = E E′, between the equivalent points of the combination, is determined by the following

$$T = d + T_1 + T_2 - (E + E'). \qquad [220$$

or

$$T = T_1 + T_2 - \frac{d^2}{N}. \qquad [221$$

As an example, let

$r_1 = 10cm.,\ r_2 = 8cm.,$ and $t_1 = 2cm.$

$r'_1 = 9cm.,\ r'_2 = 7cm.,$ and $t_2 = 2cm.$

$\mu = 1.5$ and $d = 2.5cm.$

Then, when calculated, we obtain

$F_1 = 9.23cm.,\ E_1 = .769cm.,\ E_2 = .615cm.,\ T_1 = .616cm.$

$F_2 = 8.26cm.,\ E'_1 = .783cm.,\ E'_2 = .609cm.,\ T_2 = .608cm.$

and for the combination

$$F = \frac{9.23 \times 8.26}{9.23 + 8.26 - 2.5} = \frac{76.2398}{15} = 5.08cm.$$

$$E = \frac{9.23 \times 2.5}{9.23 + 8.26 - 2.5} = \frac{23.075}{15} = 1.538cm.$$

$$E' = \frac{8.26 \times 2.5}{9.23 + 8.25 - 2.5} = \frac{20.65}{15} = 1.377cm.$$

$$T = 2.5 + .616 + .608 - (1.538 + 1.377) = .81cm.$$

$$\text{or } T = .616 + .608 - \frac{2.5^2}{15} = 1.224 - \frac{6.25}{15} = .81cm.$$

The combination is of 5.08cm. focal length and its equivalent planes are .81cm. apart.

Example with a convex and a concave lens.—Let

$F_1 = +12in.;\ F_2 = -10in.;\ d = 5in.;\ T_1 = .5in.;\ T_2 = .2in$

then combined we obtain

$$F = \frac{+12 \times (-10)}{+12 - 10 - 5} = \frac{-120}{-3} = +40in.$$

$$E = \frac{12 \times 5}{-3} = -20in.$$

$$E' = \frac{-10 \times 5}{-3} = 16.66in.$$

$$T = 5 + .5 + .2 - (-20 + 16.66) = 9.03in.$$

$$\text{or } T = .5 + .2 - \frac{5^2}{-3} = .7 - \frac{25}{-3} = .7 - (-8.33) = 9.03in.$$

If, instead of F, the power of the equivalent lens 1/F is required, it is found by

$$\frac{1}{F} = \frac{1}{F_1} + \frac{1}{F_2} - \left(d \frac{1}{F_1} \frac{1}{F_2}\right) = \frac{1}{F_1} + \frac{1}{F_2} - \frac{d}{F_1 F_2}$$

or $$\frac{1}{F} = \frac{F_1 + F_2 - d}{F_1 F_2}.$$ [22

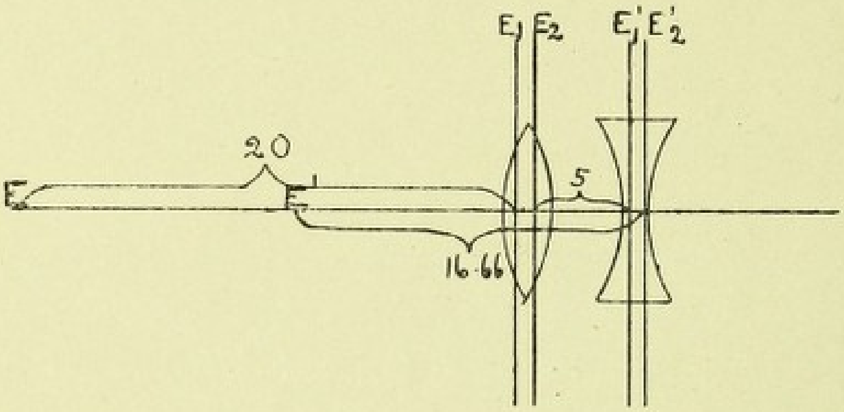


Fig. 235.

Coincidence of E and E'.—In order that E and E' should coincide, d can be found, for two Cx. or two Cc. lenses, by the following formula.

$$d = \frac{\sqrt{(T_1 + T_2)^2 + 4\,(F_1 + F_2)\,(T_1 + T_2)} - (T_1 + T_2)}{2}$$ [22

Taking as an example a combination where $F_1 = 9$in., $F_2 = 8$in., $T_1 = .2$in., and $T_2 = .3$in.

$$d = \frac{\sqrt{(.2 + .3)^2 + 4 \times (9 + 8) \times (.2 + .3)} - (.2 + .3)}{2}$$

$$d = \frac{\sqrt{.25 + 34} - .5}{2} = \frac{5.8524 - .5}{2} = 2\ 6762\text{in.}$$

When the lenses are 2.6762in. apart T = 0.

To prove this

$$F = \frac{9 \times 8}{9 + 8 - 2.6762} = \frac{72}{14.3238} = 5.03\text{in.}$$

$$E = \frac{9 \times 2.6762}{14.3238} = \frac{24.0858}{14.3238} = 1.6815\text{in.}$$

$$E' = \frac{8 \times 2.6762}{14.3238} = \frac{21.4096}{14.3238} = 1.4947\text{in.}$$

$$T = 2.6762 + .2 + .3 - (1.6815 + 1.4947) = 3.1762 - 3.1762 = 0.$$

To Find F_E of more than Two Lenses.—When more than two lenses are in combination the equivalent cardinal points of two of them are determined, and then this combination is again combined with the third lens or with another equivalent lens as the case might be. Thus, if there are four lenses, A B C D, the equivalent of A and B, also of C and D, are found separately, and these two equivalent combinations again merged into a single one.

TO CALCULATE THE EQUIVALENT POWER AND POINTS OF TWO THICK LENSES BY THE DIOPTIC SYSTEM.

Let D_1 and D_2 represent, respectively, the power of the first and second lens and D that of the combination.

Let T_1 and T_2 represent, respectively, the distance between the equivalent planes of the first and second lenses, and let d represent the distance between the adjacent equivalent planes of the combination, d being expressed in centimeters.

$$D = D_1 + D_2 - \frac{D_1 D_2 d}{100}. \qquad [224]$$

If d is expressed in terms of a metre, then

$$D = D_1 + D_2 - D_1 D_2 d. \qquad [225]$$

The first equivalent plane E of the combination is distant from the first equivalent plane of the first lens

$$E = \frac{D_1 D_2 d}{D_1 \left(D_1 + D_2 - \frac{D_1 D_2 d}{100}\right)} = \frac{D_1 D_2 d}{D_1 D} = \frac{D_2 d}{D}. \qquad [226]$$

The second equivalent plane E′ of the combination is distant from the second equivalent plane of the second lens

$$E' = \frac{D_1\ D_2\ d}{D_2\left(D_1 + D_2 - \dfrac{D_1\ D_2\ d}{100}\right)} = \frac{D_1\ D_2\ d}{D_2\ D} = \frac{D_1\ d}{D} \quad [227$$

$$T = d + T_1 + T_2 - (E + E') \quad [228$$

or directly by the formula

$$T. = T_1 + T_2 - \frac{d\ (D_1\ D_2\ d)}{D\ 100} \text{ or } T_1 + T_2 - \frac{d\ (D_1 + D_2 - D)}{D} \quad [229$$

Let +4 be the first lens and +5 the second lens 5cm. apart; each lens having 1.25cm. between their equivalent planes; then

$$D = 4 + 5 - \frac{4 \times 5 \times 5}{100} = 9 - 1 = 8\ D$$

$$E = \frac{5 \times 5}{8} = \frac{25}{8} = 3.125\text{cm.}$$

$$E' = \frac{4 \times 5}{8} = \frac{20}{8} = 2.5\text{cm.}$$

$$T = 1.25 + 1.25 - \frac{5 \times (4 + 5 - 8)}{8} = 1.875\text{cm.}$$

$$\text{or } T = 5 + 1.25 + 1.25 - (3.125 + 2.5) = 1.875\text{cm.}$$

Another example, the one lens being convex and the other concave.

Let D_1 be +6.5 D and D_2 be −5.5 D, d = 3cm., T_1 = 2cm., and T_2 = 2cm., then

$$D = 6.5 - 5.5 - \frac{6.5 \times (-5.5) \times 3}{100} = +1 + 1.0725 = +2.0725$$

$$E = \frac{-5.5 \times 3}{2.0725} = -7.96$$

$$E' = \frac{6.5 \times 3}{2.0725} = +9.41$$

So that E lies 7.96cm. in front of E_1 and E′ lies 9.41cm. in front of E'_2, that is 9.41 − (2 + 3 + 2) = 2.41cm. in front of E_1,

$$T = 3 + 2 + 2 - (-7.96 + 9.41) = 5.55\text{cm.}$$

$$\text{or } T = 2 + 2 - \frac{3 \times (+6.5 - 5.5 - 2.0725)}{2.0725} = 5.55\text{cm.}$$

CONJUGATE FOCI AND THE SIZE OF THE IMAGE OF A THICK LENS OR COMBINATION OF LENSES.

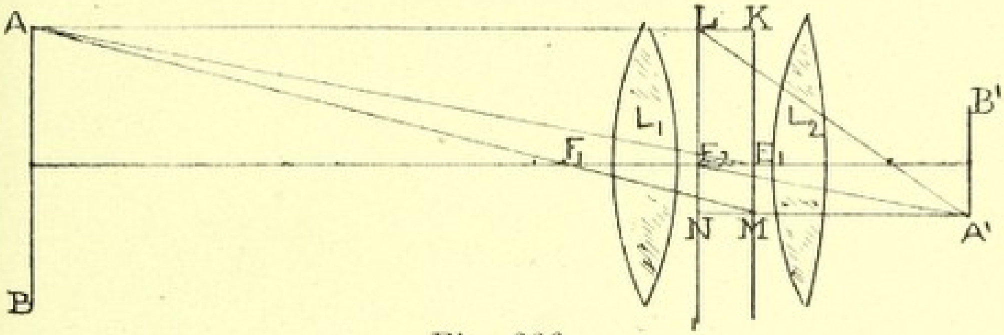


Fig. 236.

The calculations are the same as with thin lenses. Let f_1 represent the distance of the plane of the object from E_1, the first equivalent plane, and let f_2 represent the distance of the plane of the image from E_2, the second equivalent plane. F is the focal distance of the lens or combination.

$$\text{Then} \qquad \frac{1}{F} - \frac{1}{f_1} = \frac{1}{f_2} \qquad \text{or} \qquad \frac{F}{f_1} + \frac{F}{f_2} = 1.$$

With thick lenses, or combinations of lenses h_1 and h_2, the relative sizes of object and image respectively, are, as with thin lenses, proportional to their distances. But these distances are measured from the adjacent equivalent planes, i.e., the object from the first and the image from the second.

$$\frac{h_1}{f_1} = \frac{h}{f_2} \qquad [230]$$

Thus, in Fig. 236, let A B be the object 3cm. long and placed 30cm. in front of a lens combination, whose F = 6cm., then

$$\frac{1}{f_2} = \frac{1}{6} - \frac{1}{30} = \frac{4}{30} \text{ and } f_2 = 7.5\text{cm.}$$

The size of the image is

$$h_2 = \frac{3 \times 7.5}{30} = .75\text{cm.}$$

Construction.—In constructing images formed by the refraction of rays passing through a thick lens or a system of lenses (Fig. 236), the equivalent planes and points must be made use of in place of the single refracting plane and the optical centre of a thin lens. The course of any ray incident on the plane of E_1 is continued from a point on the plane of E_2 equally distant from the principal axis; the rays being presumed to pass over the optical interval E_1 E_2 without further deviation. In the illustrated case the equivalent points are crossed.

From A draw A K parallel to the axis; then from L continue its course, through the second principal focus F_2 to A′; draw A M through F_1 and from N continue its course as N A′ parallel to the axis; draw A E_1 and continue its course from E_2 to E_2 A′. A K, A E_1, and A M meet at A′, which is the image of A. Similar rays drawn from B fix its image at B′. B′ A′ is to A B as f_1 E_1 is to f_2 E_2.

Similar calculations and construction serve for concave and periscopic lenses, also for systems of lenses where the equivalent points are not crossed or where they lie outside the lenses.

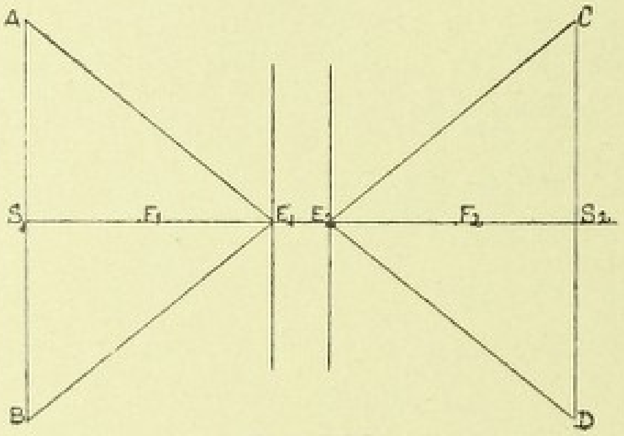


Fig. 237.

The *Symmetrical* Planes.—Referring to Fig. 237, at a distance equal to twice the principal focal distance measured from the first equivalent point E_1 anteriorly, and from the secnod equivalent point E_2 posteriorly, there are two points S_1 and S_2, and their corresponding planes termed the *symmetrical* points and planes, which present the following properties. (1) A luminous point situated at the one symmetrical point on the principal axis, has its image at the other symmetrical point. (2) Any point A or B on the one symmetrical plane has its image C or D, respectively, on the other symmetrical plane at an equal distance from the principal axis.

Thus, when an object A B is situated at the one symmetrical plane its image C D is situated at the other and the two are of equal size; these are the planes of unit magnification for real images.

METHODS OF DETERMINING THE FOCAL LENGTH OF A THICK LENS OR COMBINATION.

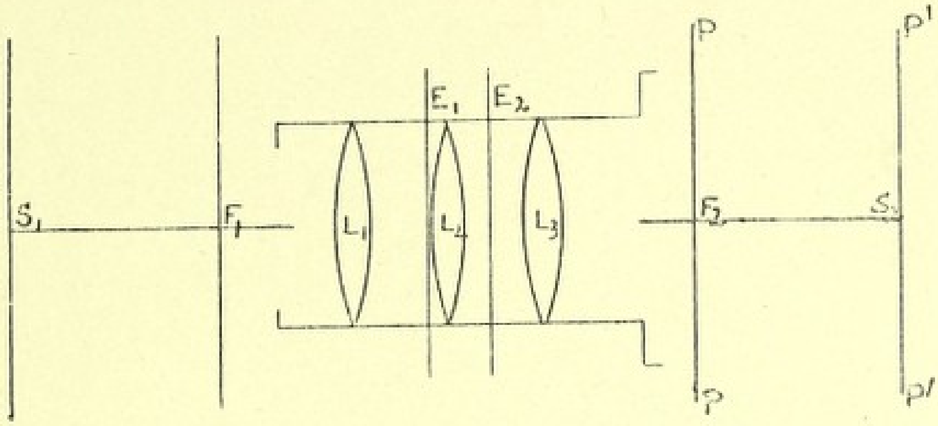


Fig. 238.

Symmetrical Plane Method for a Positive Combination.—To experimentally find the equivalent focal length of a combination it is necessary to locate the equivalent planes since the focal distances are the distances between these planes and the foci.

Let L_1, L_2, and L_3 in Fig. 238 be three lenses forming a combination fixed in a tube.

The light from a distant luminous object is focussed, and a sharp image obtained on a movable screen P P. Now P P is at the posterior principal focal distance, and this must be marked at F_2. Then place a luminous body of definite size, at such a distance S_1 that its image S_2 on the screen, which has been moved back to P′ P′, shall be of equal size to S_1. Now S_1 and S_2 are at the symmetrical planes of the combination and S_2 must be marked.

F_2 is at the principal focal distance and S_2 at twice the principal focal distance; therefore the distance F_2 S_2 is equal to the equivalent focal distance of the combination.

Now, measuring from F_2 towards the lenses a distance equal to F_2 S_2, the second equivalent point E_2, is located.

If the combination is turned around so that L_3 faces the light, the points F_1 and S_1 can be found in a similar manner and E_1 F_1 is equal to E_2 F_2; thus E_1, the first equivalent point, is located. In some combinations E_1 and E_2 are crossed.

Conjugate Foci Method for a Positive Combination.—Since $F^2 = A\ B$, where A and B are the distance of O and I beyond F, respectively, on the one and the other side of the lens system, this enables the focal length to be experimentally determined; thus focus a distant object on the one side and mark F_2, repeat the process on the other side and similarly mark F_1. Then bring the object to a convenient distance f_1 so that its image is, say, f_2; measure $F_2\ f_2 = B$ also the distance $f_1\ F_1 = A$, and calculate as above, that is $F = \sqrt{A\ B}$. The first mentioned method—that of the focal and symmetrical planes—is a special case of this general method, which is illustrated in Fig. 239.

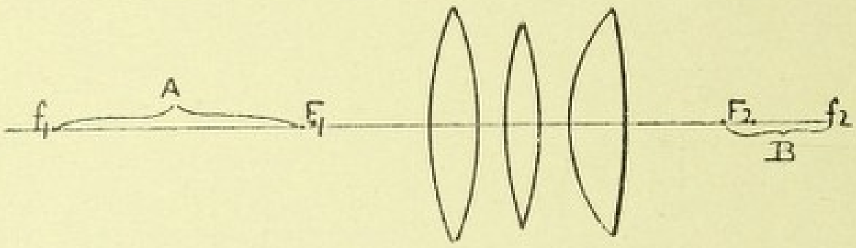


Fig. 239.

L. Laurance's Method for a Positive Combination.—Focus sharply a very distant object on a screen. This latter will then be at the principal focus F.

Move the screen back to f_2 (Fig. 240), which is n inches from F (say $\frac{1}{3}$ of its focal length).

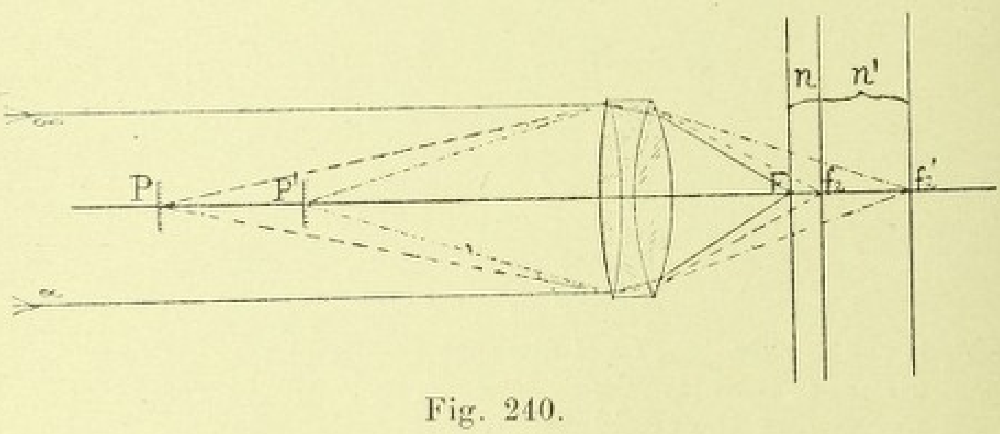


Fig. 240.

Move a scale in front of the lens until its image is sharply focussed on the screen and mark its position P. Again withdraw the screen to f'_2, which is exactly one (or more) inches further back so that it is now n′ inches from F. Shift the scale until it is once more in focus at P′. Measure the distance P P′ through which the scale has been moved. Call it d. Then

$$F = \sqrt{\frac{d\ n\ n'}{n' - n}} \qquad [231]$$

If n′ be exactly 1 in. longer than n, then $n' - n = 1$, and therefore need not be regarded.

Example.—Suppose d = 5 inches, n = 3 inches from F, and n′ 4 inches from F.

$$F = \sqrt{5 \times 3 \times 4} = \sqrt{60} = 7.75 \text{ approx.}$$

If n = 1 and n′ = 2, the calculation simplifies to

$$F = \sqrt{2d}.$$ [232]

This is the true focal length, since it is independent of the position of the equivalent planes, which can be found by measuring the focal distance backwards from the principal focus.

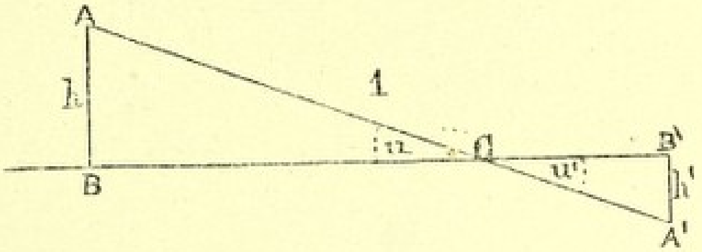


Fig. 241—1.

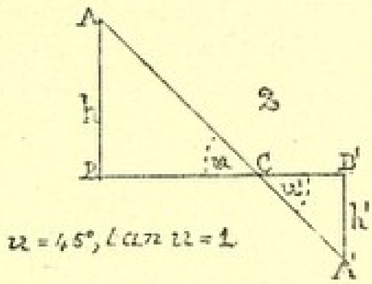


Fig. 241—2.

The Gauss Method for a Positive Combination.—Let u be the angle subtended at the lens by any two distant objects (Fig. 241, 1) A and B, one of which B is situated on the principal axis. This angle can be measured by means of a theodolite, and therefore the angle u′ subtended by the image B′ A′ at the second equivalent point C is also known since it is equal to u. Then

$$\tan u' = \frac{h'}{C\,B'} \quad \text{or} \quad C\,B' = F = \frac{h'}{\tan u'}$$ [233]

The image h′ can be directly measured on the screen. Since this method is independent of the position of the equivalent planes, these are not shown in the figure, C being the 2nd equivalent point.

If u = 45° (Fig. 241, 2), then since tan 45 = 1, F = h′, i.e., the size of the image B′ A′ is exactly equal to the focal length of the lens.

J. R. Dallmeyer's Method for a Negative Combination.—

(1) Take any achromatic positive lens and focus the image of a well-defined object, i.e., a foot rule on a screen. Measure the size of the image formed and let it be m.

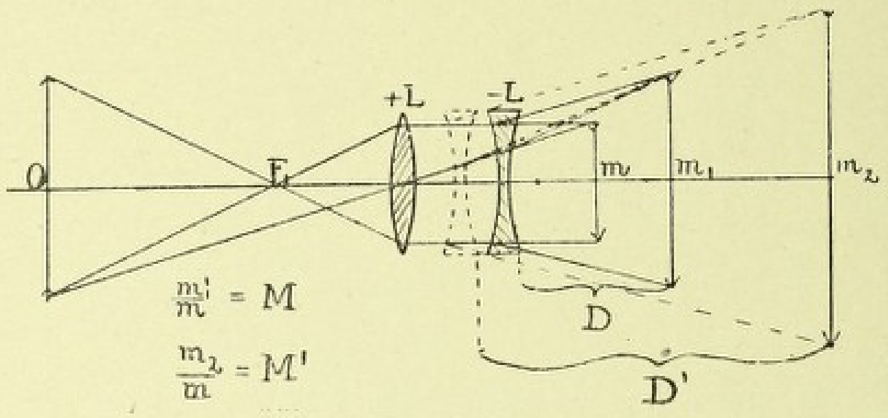


Fig. 242.

(2) Place the negative lens, whose focus is to be found, a short distance within the convergent beam of the positive lens, i.e., between it and the screen. Focus the image formed by the combination by moving the screen to and fro, and measure its distance D from the back surface or flange of the negative lens. Measure the size m_1 of the image formed. The size of m_1 compared with the size of the image produced by the positive lens alone is

$$M = \frac{m_1}{m}$$

(3) Now move the negative lens a little nearer the positive lens (which latter must be kept in a fixed position) and focus a second time on the screen. Measure the distance D′ of the screen from the back of the negative lens or its flange. The size of the image m_2 compared with the size of m is

$$M' = \frac{m_2}{m}$$

The focal length F of the negative lens is

$$F = \frac{D' - D}{M' - M}.$$ [234

This equation is independent of the position of the equivalent planes and, therefore, will hold true for any negative combination of lenses.

Thin Lens Method for Cx. System.—If a single thin lens is found which gives on a screen an image equal in size to that formed by a combination, the focal distance of the former is that of the latter. Also the place at which the single lens is situated determines the second equivalent point of the combination. If the combination is turned so that the original back lens faces the light; the place at which the single thin lens must be held in order to give a similar image to that of the combination fixes the position of the first equivalent point. This is perhaps the best method for finding the focal length of a concave combination, it having been added to a stronger convex of known focal length.

Rotation Method for Cx. or Cc. system.—This is the quickest of all. The combination is so placed that it can be rotated on a vertical axis. A distant object is focussed on a screen. By experiment that point of rotation is found which causes no shifting of the image on the screen. The support around which the lens is rotated is then directly under the second equivalent point, and its distance to the screen = F. If the lens is Cc. the virtual image must be observed through the lens.

TO DETERMINE THE RADIUS OF DEEP CURVES AND THE FOCAL LENGTH OF VERY STRONG LENSES.

Curved Surfaces by Reflection.—If the object be sufficiently distant compared with that of the image, as is the case with mirrors of small radius, when the object is, say, a metre distant, then the radius r of the curved surface bears to the distance of the image from the pole of the mirror, the relationship of r = 2 I, where I is the focal distance and the distance of the image. Let h_1 and h_2 be the sizes of, respectively, the object and the image, and f_1 the distance of the object from the mirror, while f_2 is its focal length.

Then $$f_2 = \frac{f_1\,h_2}{h_1} \quad \text{and } r = 2f_2$$

The radius of curvature, whether convex or concave, of strongly curved lenses and mirrors can be measured by employing an instrument like the Ophthalmometer. The distance between the two objects being known, that between the two images can be measured by a micrometer scale placed in the focus of the eye-piece of the telescope. f_1 is the distance of the objects from the curved surface, h_1 is the distance between them, h_2 is here the distance between the two images, as seen in the micrometer, and F is the distance between the objective and the micrometer.

The relative size of the image formed at f_2 and that formed at the micrometer is as f_1 : F, so that the above formula must be multiplied by f_1/F, and we then obtain

$$r = \frac{2\,f_1^2\,h_2}{h_1\,F}$$ [235]

Curved Surfaces—by Gauges.—The radius of small convex lenses is also determined by accurately made gauges, or more generally by glass cups of known curvature. When the curvature of the lens to be gauged does not correspond to that of the cup, interference rings are exhibited, while these are not shown if the two curves exactly correspond; or they are faint, and of slight brilliancy of colour, if the curves nearly correspond. A total absence of colour is, however, in practice, rarely found.

Curved Surfaces—Dr. C.V. Drysdale's Method.—Dr. Drysdale explains a method of determining the radius of curvature of small surfaces as follows: A microscope has a portion removed from the tube so that light from a distant source placed at the side, enters the aperture and falls on a transparent reflecting surface M inclined at 45° so that part of the light is transmitted down the tube towards, and through, the objective, by which it is brought to a focus at F as in Fig. 243.

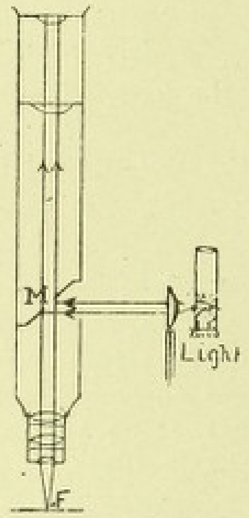
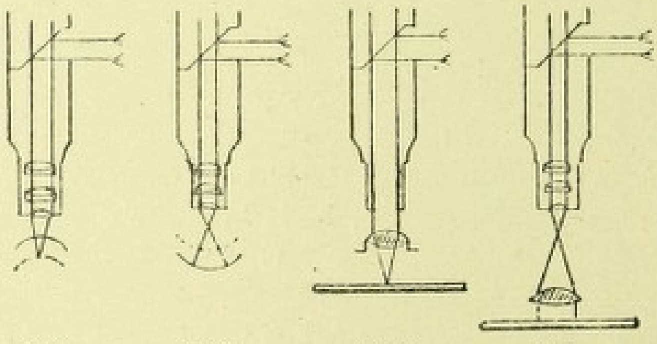


Fig. 243.

If, then, the reflecting surface of a mirror or lens is placed at the focus of the objective, the light is reflected back and seen by the observer, in the field of the eye-piece, as an image of the source.



Figs. 244 245 246 247

This position or distance of the objective from the reflecting surface is then marked on some part of the microscope. Then the microscope tube is racked upwards if the surface to be examined is concave (Fig. 245), or downwards if convex (Fig. 244), until the image can again be clearly seen. The focus of the objective now coincides with the centre of curvature of the reflecting surface, for the light passing through the objective is incident on the reflecting surface normally and is reflected back along its original course. The distance then, between the first and second positions of the microscope objective, when the image is clearly seen, is the radius of curvature. The eye-piece is arranged for parallel light by separation of the components, the adjustment being made by turning the reflector so that the light admitted is reflected towards the eye-piece. The curvature of any zone of the surface can be obtained by using a suitable diaphragm.

A later improvement made by Dr. Drysdale on the arrangement of the instrument used in the above method consists of an illuminator immediately above the microscope objective and a lens above the illuminator, which serves as the objective of the telescope and obviates the necessity of separating the eye-piece lenses.

Focal Length—Dr. C. V. Drysdale's Method.—By a similar use of the microscope the focal length of small lenses can be found. Employing no objective in the microscope and a plane mirror behind the lens to be tested, this mirror is moved to and fro until the image is sharp in the field of the eye-piece. Then the mirror is at the focal length of the lens, the light converged by the lens being reflected back and refracted again as parallel. The lower focal point is thus found, as in Fig. 246.

Replacing the objective (Fig. 247), the lens is moved further back to such a position that it is at its focal length behind the focal point of the objective. Then the light converged by the objective and refracted by the lens is parallel, and falling on the mirror, is again reflected as parallel to be refracted by the lens to meet at the focal point of the objective by which it is again refracted as parallel light. The image is then sharp in the field of the eye-piece and the upper focal point is found as in Fig. 247.

The two focal points being marked, the back surface focal lengths are obtained. If, now, the mirror be moved a given distance downwards and the objective moved upwards until the image is clear, the first-mentioned distance A, being the distance of the one conjugate beyond the anterior F, and the second B, that of the other conjugate beyond the posterior F, we obtain the equivalent focal length by $F_E = \sqrt{A\,B.}$

Dr. Drysdale has also made an experimental microscope in which the lens under examination can be oscillated around its second equivalent point. This enables the focal length to be determined, and further by this means aberrations can be easily detected.

THE GAUSS EQUATION.

In 1840 Gauss, a German mathematician, proposed to overcome the difficulties in ascertaining the constants of thick lenses and lens systems by the use of the imaginary principal or equivalent points and planes, and later, Listing, in studying the optical system of the eye, introduced the nodal points which, in all systems having the first and last media of the same optical density, coincide with the equivalent points. By the aid of these points every system, no matter how many surfaces and media are involved, can be so simplified that all problems of conjugate foci, etc., can be worked by the simple formulæ applicable to single thin lenses. The calculations in the case of more than two surfaces are necessarily long, but they always involve the solution of a continued fraction so that the difficulties are purely arithmetical.

In using the equation, which serves for any number of surfaces, media and thicknesses, the pencils of light are presumed to be axial and small; in other words, *aberration* is neglected. In order to keep the formulæ as symmetrical as possible and avoid a mixture of signs, the following conventions must be observed, namely, (1) all distances measured to the left of the surface in question are negative, and to the right positive; (2) all thicknesses are considered negative, and therefore, on substituting actual values, it will be necessary to use the minus sign.

The following formulæ are deduced from the consideration of the lens having positive radii of curvature according to the above convention, that is, a periscopic with the concave surface turned towards the right. Let μ_1 be the refractive index of the surrounding medium and μ_2 that of the lens, t the axial thickness, r_1 the radius of the first surface, and r_2 that of the second. Let u be the object distance, v_1 the image distance formed by refraction at the first surface, and v the final image distance after refraction at the second.

The fundamental equation connecting u and v_1 is:—

$$\frac{\mu_2}{v_1} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{r_1}$$

but in order to simplify the formulæ $(\mu_2 - \mu_1)/r_1$ is replaced by F_1, while μ_2/v_1 and μ_1/u are replaced by $1/v_1$ and $1/u$ respectively.

These are *reduced* expressions, i.e., the actual distances divided by the μ's of the media to which they pertain. Similarly in the expression connecting v_1 and v, given lower down, $(\mu_1 - \mu_2)/r_2$ and μ_1/v are reduced to F_2 and $1/v$ respectively, while t is also employed reduced, being divided by the μ in which it is measured. Consequently the values subsequently found are reduced and must be multiplied by the μ, in which each occurs, in order that their true values may be arrived at.

The fundamental formula reduced becomes:—

$$\frac{1}{v_1} - \frac{1}{u} = F_1 \quad \text{or} \quad \frac{1}{v_1} = F_1 + \frac{1}{u} \quad \text{where } F_1 = \frac{\mu_2 - \mu_1}{r_1}$$

whence

$$v_1 = \frac{1}{F_1 + \frac{1}{u}} \tag{1}$$

The expression connecting v_1 and v is

$$\frac{\mu_1}{v} - \frac{\mu_2}{v_1 + t} = \frac{\mu_1 - \mu_2}{r_2}$$

Which in reduced terms becomes:—

$$\frac{1}{v} - \frac{1}{v_1 + t} = F_2 \quad \text{or} \quad \frac{1}{v} = F_2 + \frac{1}{v_1 + t} \quad \text{where } F_2 = \frac{\mu_1 - \mu_2}{r_2}$$

whence

$$v = \frac{1}{F_2 + \frac{1}{v_1 + t}} \tag{2}$$

Substituting in (2) the value of v_1, in (1) we have

$$v = \cfrac{1}{F_2 + \cfrac{1}{t + \cfrac{1}{F_1 + \cfrac{1}{u}}}} \tag{3}$$

On working out this continued fraction in (3) we get

$$v = \frac{\mu (F_1 t + 1) + t}{\mu (F_1 F_2 t + F_1 + F_2) + F_2 t + 1} \tag{4}$$

which is usually written for the sake of brevity

$$v = \frac{C u + D}{A u + B} \tag{5}$$

where $A = F_1 F_2 t + F_1 + F_2$; $B = F_2 t + 1$

$C = F_1 t + 1$; $D = t.$

No. (5) connects v and u when both are finite distances. If u is at ∞ the quantities D and B disappear and u cancels, so that the focal length measured from the second surface is

$$v = C/A \tag{6}$$

The value of v in equation (6) *is the back focal distance as measured from the pole of the second surface.*

If v is at ∞, then $A u + B = 0$, so that the focal length measured from the pole of the first surface is

$$u = -B/A \tag{7}$$

Before proceeding further an expression for the total magnification M produced by the lens must be introduced, which in terms of A and B is

$$M = \frac{1}{A u + B.} \quad \text{or} \quad \frac{1}{M.} = A u + B. \tag{8}$$

Now let the magnification be + 1, i.e., let *virtual* image and object be equal in size. Then

$$A u + B = 1$$

whence

$$u = P_1 = (1 - B)/A \tag{9}$$

this distance being measured from the first surface.

On substituting this value of u in (5), the corresponding value of v is

$$v = P_2 = (C - 1)/A \tag{10}$$

this distance being measured from the second surface.

These planes of unit virtual magnification denote the equivalent planes and the points P_1 and P_2 where they cut the axis are the equivalent points. If it were possible to place a small object in the one plane, then its virtual image, identical in all respects to the object, would be situated in the other.

If the magnification is -1, then the corresponding values of u and v will locate the symmetrical planes where object and *real* image are equal in size.

To find, therefore, the equivalent focal distances, the values of (9) and (10) must be added to those of u and v in (5); thus

$$v + \frac{C - 1}{A} = \frac{C\left(u + \dfrac{1 - B}{A}\right) + D}{A\left(u + \dfrac{1 - B}{A}\right) + B}$$

which simplifies to $\quad A = 1/v - 1/u \quad$ (11)

This expression (11) should be compared with that of a simple thin lens for the focal length in terms of u and v.

Then if $u = \infty \qquad v = 1/A \qquad$ (12)

and if $v = \infty \qquad u = -1/A \qquad$ (13)

The principal focal distance given in (12) and (13) are equal when the first and last μ's are of equal optical density. The values are reduced and must be multiplied by the μ in which each occurs, so that when in air they are unchanged.

As a simple example, let $r_1 = 6$, $r_2 = 8$, $\mu = 1.5$, $\mu_1 = 1$ (air), and $t = 1$; then

$$v = \cfrac{1}{F_2 + \cfrac{1}{t + \cfrac{1}{F_1 + \cfrac{1}{u}}}} \qquad v = \cfrac{1}{.625 + \cfrac{1}{-.666 + \cfrac{1}{.0833 + \cfrac{1}{u}}}}$$

which works out to

$$v = \frac{.9445\,u - .666}{.1423\,u + 1.0625}$$

Then, if $u = \infty$

$$v = \frac{C}{A} = \frac{.0833 - .666 + 1}{.0833 \times .0625 \times -.666 + .0833 + .0625} = \frac{.9445}{.1423} = 6.63$$

$$\text{Also } u = \frac{-B}{A} = \frac{-(F_2\,t + 1)}{.1423} = \frac{-.9813}{.1423} = -6.73$$

$$P_1 = \frac{1 - B}{A} = \frac{1 - (.0625 \times -.6666 + 1)}{.1423} = +.29$$

$$P_2 = \frac{C - 1}{A} = \frac{.0833 \times -.666}{.1423} = -.39$$

$$\text{The equivalent focal distance } \frac{1}{A} = \frac{1}{.1423} = 7.02.$$

The Gauss equation may be applied to an optical system having any number of surfaces surrounded by corresponding media of different densities and thicknesses.

The equation

$$v = \frac{C\,u + D}{A\,u + B}$$

is universal, although the various values become more complicated as the number of surfaces is increased, but the problem always takes this form, involving the solution of a continued fraction.

Suppose the case of the eye having three surfaces, F_1, F_3 and F_5 with thicknesses t_2 and t_4 with the following data :—

$r_1 = 8$, $r_3 = 10$, $r_5 = 6$, $t_2 = 3.6$, $t_4 = 3.6$, $\mu_1 = 1$, $\mu_2 = 1.333$, $\mu_3 = 1.45$, $\mu_4 = 1.333$. Then

$$F_1 = \frac{\mu_2 - \mu_1}{r_1} = \frac{1.333 - 1}{8} = .0416$$

$$F_3 = \frac{\mu_3 - \mu_2}{r_3} = \frac{1.45 - 1.333}{10} = .0117$$

$$F_5 = \frac{\mu_4 - \mu_3}{r_5} = \frac{1.333 - 1.45}{-6} = .0195.$$

$$\text{The reduced value of } t_2 = \frac{-3.6}{1.333} = -2.7007$$

$$\text{and that of} \qquad t_4 = \frac{-3.6}{1.45} = -2.4828.$$

Then we have

$$v = \cfrac{1}{F_5 + \cfrac{1}{t_4 + \cfrac{1}{F_3 + \cfrac{1}{t_2 + \cfrac{1}{F_1 + \cfrac{1}{u}}}}}}$$

$$v = \cfrac{1}{.0915 + \cfrac{1}{-2.4828 + \cfrac{1}{.0117 + \cfrac{1}{-2.7007 + \cfrac{1}{.0416 + \cfrac{1}{u}}}}}}$$

which becomes, when worked out,

$$V = \frac{.7586\ u - 5.1050}{.0668\ u + .8689}$$

That is A = .0668, B = .8689, C = .7586, D = −5.1050.

The anterior $F = -\frac{\mu_1}{A} = -\frac{1}{.0668} = -15\text{mm.}$

The posterior $F = \frac{\mu_4}{A} = \frac{1.333}{.0668} = 20\text{mm.}$

$$P_1 = \frac{\mu_1 (1 - B)}{A} = \frac{.1311}{.0668} = 1.96\text{mm. from } r_1$$

$$P_2 = \frac{\mu_4 (C - 1)}{A} = \frac{-.3218}{.0668} = -4.81\text{mm. from } r_2$$

or $7.2 - 4.81 = 2.39$mm. from r_1.

The nodal points N_1 and N_2, found by subtraction, are, respectively, 6.96 and 7.39mm. from r_1.

Neglecting the intervals between P_1 and P_2 and that between N_1 and N_2, we have P at 2.2mm. and N at 7.2mm. from the cornea.

CHAPTER XII.

ABERRATIONS.

CHROMATISM OF A PRISM.

Dispersion.—When a beam of white light is refracted by a prism, its various components are separated, and form a band of colours called the spectrum. The extent of the dispersion varies with the medium of which the prism is formed, also with the angle of incidence of the light, but the dispersion is not at a minimum when the mean ray suffers minimum deviation. The dispersion of a prism can be determined by a goniometer or spectrometer, the difference between the indices of refraction for the C and F lines being the mean dispersion.

The position of the prism must be that of minimum deviation for the yellow light—line D—in order that the deviation of the colours, on either side of the yellow, may be observed when the *mean* deviation of the prism is a minimum.

The wave front of an incident beam of light, refracted by a prism, is retarded sooner at its "base" end than at its "edge" end. Thus the latter overtakes the former, and the beam is deviated towards the base of the prism.

And since the retardation is greater as the wave length is shorter, so a blue wave front is more deviated than a red. The blue rays finally emerge from the prism, more bent towards the base than the red.

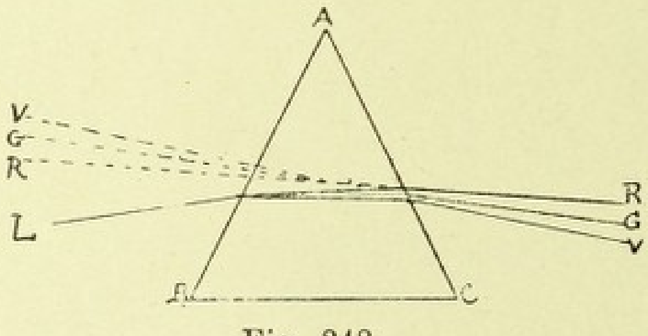


Fig. 248.

Real and Virtual Spectra.—Since a prism refracts the violet ray most and red the least, the spectrum projected on to a screen—a real spectrum—exhibits violet nearest to the base and red nearest to the edge of the prism, as shown in Fig. 248, where L is the source of light. If the rays be referred back by the eye, after refraction, they form a virtual spectrum and the violet is seen nearest to the edge and the red nearest the base. Thus, a disc of light viewed through a prism, base down, exhibits blue above and red below.

When a white body is viewed through a prism, so that apparent displacement occurs, there are a series of separate images of the body, each characterised by a distinctive spectral colour. These overlap in the centre so that a white image is seen, but the ultimate displacements of blue at the one end, and of red at the other, cause a fringe of blue to appear on that border of the body which is nearest the edge of the prism and a red-orange fringe on that nearest the base. If the body is black, the red-orange fringe is towards the edge and the blue fringe towards the base of the prism, there being no light emitted by the black body. These colours result from the dispersion of the light from the space or body surrounding the black.

Angular Dispersion.—The deviating angle of a prism is that of the mean ray (line D), and is expressed (in the case of thin prisms) by $d = P(\mu - 1)$ where P is the refracting and d the deviating angle. Now, since the red ray suffers less and the violet greater refraction than the D line their angular deviations are respectively

$$d_r = P(\mu_r - 1) \quad \text{and} \quad d_v = P(\mu_v - 1)$$

d_r and d_v being the deviating angles and μ_r and μ_v being the indices of refraction for red and violet light respectively.

The angular dispersion of the prism expressed in degrees is

$$P(\mu_v - \mu_r) = P(\mu_v - 1) - P(\mu_r - 1) \qquad [236]$$

Expression for Dispersion.—The ratio of the deviation to the dispersion is expressed by the formula

$$\nu = \frac{P(\mu_D - 1)}{P(\mu_v - \mu_r)} = \frac{\mu_D - 1}{\mu_v - \mu_r}. \qquad [237]$$

Thus if $\mu = 1.6$ $\mu_v = 1.61$ and $\mu = 1.59$ this ratio ν—the efficiency—is

$$\frac{1.6 - 1}{1.61 - 1.59} = \frac{.6}{.02} = 30.$$

Crown and Flint Glass.—Flint glass has only a slightly greater refracting power than crown, but its dispersive power is considerably higher. The proportions for the refraction are approximately as 1.1 to 1 and for the dispersion at 1.4 to 1. A flint prism of 10° and a crown of 11° have each a deviating angle of about 6°, but their spectra are of different lengths; that of the flint glass being about a third longer than the other. If spectra of the same lengths be required, the crown glass prism must have a greater deviation than the flint.

Similar Prisms.—If two similar prisms A and B (Fig. 249) are placed in opposition—base to edge—their angles, refractive indices and dispersions being the same, both the deviation and dispersion are neutralised and all the rays emerge parallel to their original course.

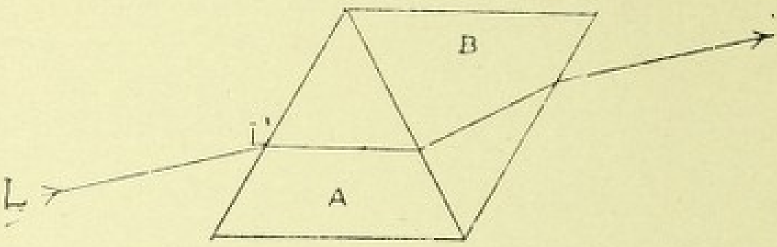


Fig. 249.

Crown and Flint Prisms.—In Fig. 250, let the principal angle of a crown prism ($\mu = 1.54$) be 11.3°, and that of a flint prism ($\mu = 1.61$) be 10°. Their deviating angles are the same, namely, 6.1°.

If in the crown $\mu_r = 1.534$ and $\mu_v = 1.554$

and in the flint $\mu_r = 1.586$ and $\mu_v = 1.62$

their dispersive angles are respectively

$$11.3 \times (1.554 - 1.534) = .226^\circ$$

$$10 \times (1.62 - 1.586) = .34^\circ$$

and $$.34^\circ - .226^\circ = .144^\circ = 6' \, 50''$$

so that while the mean deviation is neutralised, there is still a dispersion of 6′ 50″.

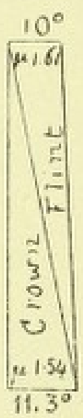


Fig. 250.

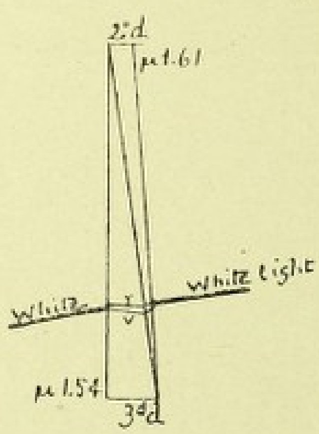


Fig. 251.

Achromatised Prism.—If a crown prism of 3°d and μ 1.54 and a flint of 2°d and μ 1.61, having dispersive efficiencies of 45 and 30 respectively, be placed in opposition, they neutralise each other's dispersion, while there obtains 1° deviation. Such prisms are said to be achromatised, i.e., they constitute an achromatic prism which causes deviation without dispersion.

The crown prism would be of

$$P = \frac{3}{.54} = 5.55^\circ \quad \text{and the flint } P = \frac{2}{.61} = 3.28^\circ$$

Irrationality of Dispersion.—One of the greatest difficulties in optics is to find either prisms or lenses of different glass so that all the lines of the spectrum will nearly coincide. Thus, if we select two kinds of glass so that C and F or D and H lines correspond it will be found that other lines will not coincide. This defect is called the *irrationality of dispersion*, and the spectrum which remains is called "residual" or "secondary." By careful selection it is often possible to combine as many as three spectral lines with two prisms, but it is impossible to construct a formula for this purpose, although it can be done with three prisms, and four lines with four prisms, and so on, but with the better kind of modern glass it suffices, for practical purposes, to unite two lines, say C and F in the luminous part of the spectrum.

δ and ν. — The difference between the indices of refraction of the F and C lines, i.e., $\mu_F - \mu_C$ is represented by the symbol δ (delta) and the ratio of deviation to dispersion, or efficiency by ν (nu), and as shown before

$$\nu = \frac{\mu_D - 1}{\mu_F - \mu_C} = \frac{\mu_D - 1}{\delta}.$$

To Calculate an Achromatic Prism.—In order to calculate the data for an achromatic prism, let d represent its deviating angle and P the refracting angle. Let d_1 and P_1 be those of the crown, and d_2 and P_2 those of the flint components respectively. Then we obtain

$$d_1 = \frac{d\ \nu_1}{\nu_1 - \nu_2} \quad \text{and} \quad d_2 = \frac{d\ \nu_2}{\nu_1 - \nu_2} \qquad [238]$$

$$P_1 = \frac{d_1}{\mu_D - 1} \quad \text{and} \quad P_2 = \frac{d_2}{\mu_D - 1} \qquad [239]$$

P is taken as $d/(\mu - 1)$, which, however, only holds good for weak prisms. If the prisms are strong d_1 and d_2 must be calculated, and P_1 and P_2 found, by the exact formulæ previously given. As an example, an achromatic prism of 5°d is needed, the glasses of the component parts being

Crown	$\mu_D = 1.53$	$\mu_C = 1.527$	$\mu_F = 1.536.$
Flint	$\mu_D = 1.63$	$\mu_C = 1.624$	$\mu_F = 1.644.$

$$\nu_1 = \frac{1.53 - 1}{1.536 - 1.527} = \frac{.53}{.009} = 58.9$$

$$\nu_2 = \frac{1.63 - 1}{1.644 - 1.624} = \frac{.63}{.02} = 31.5$$

$$d_1 = \frac{5 \times 58.9}{58.9 - 31.5} = \frac{294.5}{27.4} = 10.7^\circ\text{d and P} = \frac{10.7}{.53} = 20^\circ$$

$$d_2 = \frac{5 \times 31.5}{58.9 - 31.5} = \frac{157.5}{27.4} = 5.7^\circ\text{d and P}_2 = \frac{5.7}{.63} = 9^\circ$$

$$10.7^\circ - 5.7^\circ = 5^\circ\text{d}.$$

To calculate the power of the flint prism B of ν_2, which will neutralise the dispersion of a given crown A and ν_1, the calculation is made by

$$\nu_1 : \nu_2 : : A : B, \quad \text{or } B = \frac{A\ \nu_2}{\nu_1}. \qquad [240]$$

Thus, let the crown be 10.7°d, $\nu_2 = 31.5$ and $\nu_2 = 58.9$, then

$$B = \frac{10.7 \times 31.5}{58.9} = 5.7^\circ\text{d}.$$

Expression for ω.—These calculations are sometimes based on ω (omega) the dispersive power, which is the reciprocal of ν, and

$$\omega = \frac{\mu_F - \mu_C}{\mu_D - 1}. \qquad [241]$$

PRISMATIC ABERRATION OF FORM.

Small Light Pencils.—Rays parallel before refraction are also parallel after refraction by a prism. If rays are divergent or convergent, the divergence or convergence is not the same after refraction as before, but the difference may be neglected provided the pencil of light is small and the axial ray suffers minimum deviation; such a pencil, if divergent, when referred back, may be considered as meeting at a single point. A similar convergent pencil may be considered equally convergent after refraction by a prism.

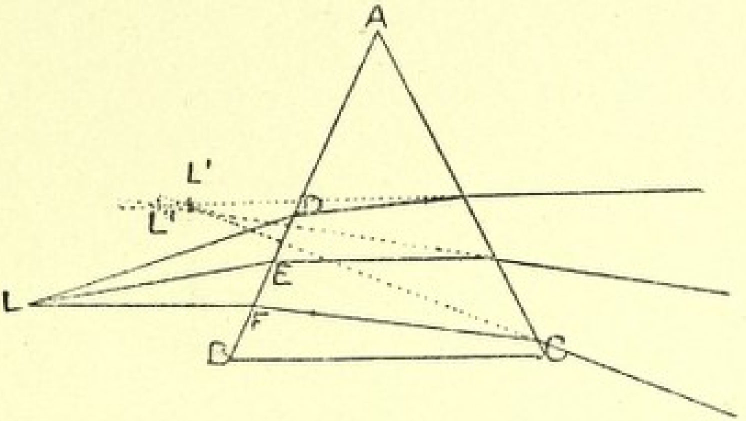


Fig. 252.

Large Light Pencils.—In Fig. 252, let F L D be a cone of light diverging from L, of which E is the central ray, and L F, L D the extreme rays. The rays L D, L E, L F are incident on the surface A B of the prism at different angles, so that they suffer an unequal degree of deviation. If L E, the central ray, suffers minimum deviation, the extreme rays L D, L F cannot do so. After refraction, L D and L F are more bent than L E and cut the line of L E, when referred back, at points nearer to the prism. But L F is still more divergent than L D, and so cuts L E at L′, while L D cuts it at L″.

Astigmatism.—Thus, rays in the pencil do not have a point focus, since there are two focal lines. The circle of least confusion which lies between L′ and L″ may be regarded as the focus. This defect is termed astigmatism, and its degrees and nature varies with the advancement of the edge of the prism towards, or its retirement from, the source of light.

By changing the inclination of the first surface of the prism there is a corresponding change in the ray which suffers minimum deviation.

Apparent Distance of Object.—If the prism is in such a position that either L F or L D suffers minimum deviation, the whole of the pencil is more divergent than before, and the image appears nearer.

Distortion due to Inclination.—If the base-apex line is vertical, the vertical magnitude of a square object appears increased when the edge of the prism is nearer to it than the base. The vertical dimension is lessened, if the base is situated nearer to the object. Thus, an object appears changed, when seen through a prism, according to the inclination of the prism and the direction of the base-apex line. Since the nearness of the object renders light more divergent, prismatic aberration is thereby generally increased.

Distortion due to Thickness.—Another effect of the refraction of a prism is distortion due to the thickness of the glass and the still greater thickness when the incident light from the extremities of an object falls obliquely on the prism. Such rays are more deviated, and therefore appear to come from points higher than those which are nearer to the centre of the body. The effect is similar to the lateral and vertical deviations of a refracting plate. The distortion causes a straight line, at right angles to the base-apex plane, to appear curved with the convexity towards the base, and a square object has its two sides, which are parallel to the axis of the prism, concaved towards the edge of the prism.

Distortion due to Position of Base-apex Line.—If a prism is rotated around its base-apex line—that is if, say, a vertical prism, edge upwards, is rotated horizontally, so that the one side of the prism is nearer the object than the other, the image is drawn out diagonally towards the edge on the side where the prism is nearer the object and it is drawn out considerably more above than it is below. Thus a square object appears drawn out into a distorted parallelogram.

Distortion due to Size of Object.—If a narrow pencil of light from the centre of an object enters the eye through a prism, and suffers minimum deviation and but little aberration, it is clear that pencils from other points cannot do so. Therefore the peripheral portions of a large object are indistinct as compared with the centre.

Distortion due to Plane of Incidence of Light.—Light from points on the body in the same plane as that of the axis, undergoes different degrees of refraction from those in the plane of the base-apex line, but the effects are practically covered by the "inclination" and "thickness" already referred to.

Thus, considerable distortion of the image is produced by strong prisms, although in prisms of small angle, the effects of aberration due to the form of the prism are usually ignored.

CHROMATIC ABERRATION.

Chromatism of a Lens.—The effect of dispersion, when the refracting body is a convex lens, is to bring the more refrangible blue and violet to a focus sooner than the less refrangible red and orange.

This different focalisation of the various colour rays is termed chromatism, and the confusion of the image caused by it, chromatic aberration. The defect, which is made apparent by a fringe of colour on the edge of the real or virtual image projected by the lens, is due to the nature of light, and not to the nature of the lens, although its degree varies with the kind of glass of which the lens is made.

If a horizontal white line (Fig. 253) be observed through the marginal portion of a convex lens, a blue-violet fringe will be seen on the side towards the edge of the lens, and a red-orange on the other, the blue being projected back above the red. Viewed through the periphery of a concave, the colours are reversed. Looking at a black line, the fringes are seen in the opposite order to those on a white line, for the reason given in connection with a prism, in the paragraph "real and virtual spectra."

Lenses being, by construction, prismatic produce the same phenomena as prisms.

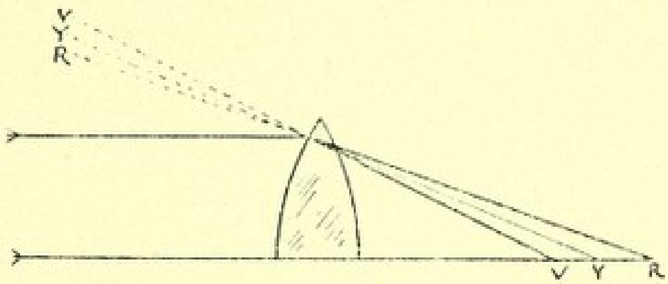


Fig. 253.

Longitudinal and Lateral Aberrations of Colour.—In Fig. 254 a parallel beam of light is refracted by a convex lens; the variously coloured rays meet at different distances behind the lens, the violet focussing at V, the yellow at Y, and the red at R. If the screen is held at V, the diffusion patch has a reddish-yellow fringe; the red and orange rays, being convergent to a more distant point R, impinge on the screen outside the blue and violet. If the screen be placed at R, the fringe becomes blue-violet, since these rays,

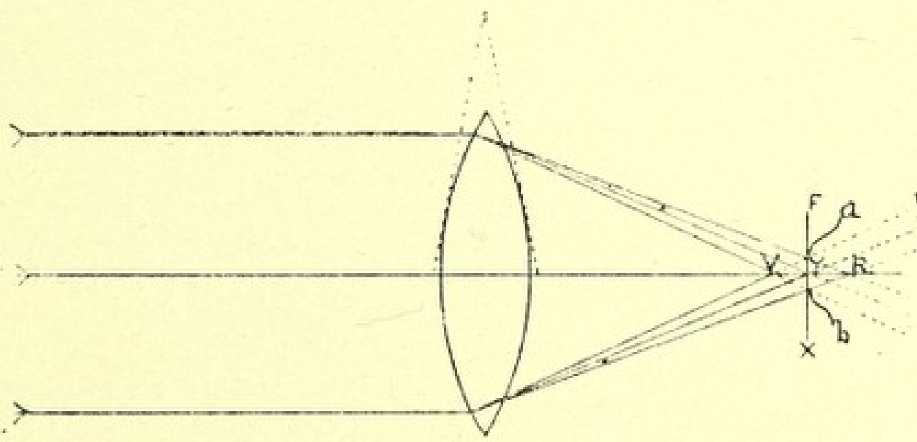


Fig. 254.

having already met at V, and crossed, impinge on the screen outside the red and orange. The distance V R is the longitudinal aberration, and the diameter a b of the disc of confusion in the plane where the extreme violet and red rays cross each other, is the lateral aberration; this plane is very nearly that of F X, where the yellow light is brought to a focus.

Circle of Least Confusion.—At F, the focus of the most luminous rays, the circles of diffusion formed by the red and blue are practically of the same size, and therefore no coloured fringe is appreciable. The centre of the image appears white, because the different colours are superposed, so that only at the extremities, where certain colours are not combined with others, is chromatism apparent.

F of the Various Colours.—The index of refraction of a given medium, refers in a general sense to that of the D (sodium) line which is situated in the yellow or the most luminous part of the spectrum. With such a medium, if $\mu = 1.54$ the index of refraction for the red (line A) might be 1.53 ($\mu_A = 1.53$), while for the violet (line H) might be 1.56 ($\mu = 1.56$). Suppose a thin double convex lens of 10in. radius, then—

$$F = \frac{r\ r'}{(r + r')\ (\mu - 1)}$$

that is,

$$F = \frac{10 \times 10}{(10 + 10) \times .54} = \frac{100}{10.8} = 9.26\text{in.}$$

which is the mean focal length as well as that for yellow light. But instead of $\mu = 1.54$ we must employ $\mu_A = 1.53$ and $\mu_H = 1.56$ in order to find the focal lengths F_A and F_H for red and violet light respectively; thus

$$F_A = \frac{10 \times 10}{(10 + 10) \times .53} = \frac{100}{10.6} = 9.43\text{in.}$$

$$F_H = \frac{10 \times 10}{(10 + 10) \times .56} = \frac{100}{11.2} = 8.93\text{in.}$$

The Chromatic Disc.—The difference in the focal lengths of a lens for red and blue light may be illustrated either by cobalt-blue glass (chromatic disc), which transmits both red and blue light, but absorbs the central part of the spectrum, or by focussing with a convex lens light which is rendered monochromatic by being passed through respectively standard red and blue glass. The difference in the focal distances with these two coloured lights are sufficiently well marked to be appreciated.

Achromatic Lens.—Chromatism can be remedied by making the lens of a combination of two different kinds of glass, so chosen, that, while the dispersion of the positive component is neutralised by that of the negative one, there shall still be some positive converging power left, so that a real image may be formed. Such a combination is termed an achromatic lens, and it usually consists of a flint concave and a crown convex.

Residual Chromatism.—Dispersion cannot be totally abolished by an achromatic combination, there being always a residual chromatism due to the irrationality of dispersion, and the spectrum still obtaining in an achromatised lens, is called the secondary or residual spectrum.

Spectrum Lines Combined.—By an achromatic lens two selected lines of the spectrum, usually the D and F (yellow and blue) or D and H (yellow and violet) are brought to a focus at the same distance, and by uniting it with a third component a third line could also be focussed at the same distance, but for all practical purposes if the C and F lines, which lie in the more central and luminous part of the spectrum, are combined, the combination is one in which chromatism does not cause any appreciable blurring of the image, at least for visual purposes in which critical definition is not essential. In a photographic lens the lines D and G or H are usually selected so as to unite the violet with the yellow, since the former colour lies in the most chemically active part of the spectrum.

Expression for Chromatic Aberration.—Let r and r′ represent the two radii and F the focal length of a thin lens. Then, if F_A and F_H represent the focal lengths, and μ_A and μ_H the indices for red and violet light respectively, the chromatic aberration may be expressed by

$$F_A - F_H = \frac{r\, r'}{(r + r')\,(\mu_A - 1)} - \frac{r\, r'}{(r + r')\,(\mu_H - 1)}$$

$$= \frac{r\, r'\,(\mu_H - \mu_A)}{(r + r')\,(\mu_A - 1)\,(\mu_H - 1)} \qquad [242]$$

If instead of $(\mu_H - 1)\,(\mu_A - 1)$ there be substituted $(\mu - 1)^2$ as may be done without sensible error, then

$$F_A - F_H = \frac{r\, r'\,(\mu_H - \mu_A)}{(r + r')\,(\mu - 1)^2} = \frac{F\,(\mu_H - \mu_A)}{(\mu - 1)} \qquad [243]$$

The expression $\omega = \dfrac{\mu_H - \mu_A}{\mu_D - 1}$ is the dispersive power,

and $\nu = \dfrac{\mu_D - 1}{\mu_H - \mu_A}$ is the efficiency.

Both of these are similar to those found in the case of a prism.

Then the longitudinal aberration is

$$F_A - F_H = F\,\omega = F/\nu. \qquad [244]$$

The same calculations can be used for a thick lens.

As an example let $F = 10$in., $\mu_A = 1.60$; $\mu_D = 1.61$ and $\mu_H = 1.625$, then

$$F_A - F_H = \frac{10 \times (1.625 - 1.60)}{1.61 - 1} = 10 \times \frac{.025}{.61} = .41\text{in}.$$

$$\text{The lateral chromatic aberration of a lens} = \frac{\text{diameter of lens}}{2\,\nu} \qquad [245]$$

Calculation for an Achromatic Combination.—To calculate an achromatic combination for two lenses in contact, let F and C be the two lines of the spectrum which have to be brought to the same focus. Let F be the focal length of the required combination, and F_1 and F_2 the focal lengths of the crown and flint respectively.

Let ν_1 and ν_2 represent the ratios $\dfrac{\mu_D - 1}{\mu_F - \mu_C}$

or the respective efficiencies,

and $$\Delta = \nu_1 - \nu_2. \qquad [246]$$

Then $$F_1\,\nu_1 = -F_2\nu_2 \text{ or } F_1\,\nu_1 + F_2\,\nu_2 = 0 \qquad [247]$$

F_2 being negative.

To have the necessary focal length it must be seen that

$$\frac{1}{F} = \frac{1}{F_1} + \frac{1}{F_2}$$

whence $$\frac{1}{F_1} = \frac{\nu_1}{F\,\Delta} \quad \text{and} \quad \frac{1}{F_2} = \frac{\nu_2}{F\,\Delta} \qquad [248]$$

or $$F_1 = \frac{F\,\Delta}{\nu_1} \quad \text{and} \quad F_2 = \frac{F\,\Delta}{\nu_2}. \qquad [249]$$

Example.—Thus let a positive lens of $6\frac{1}{2}$in. focal length be required; if the indices of refraction for the various lines are—

			μ_C	μ_D	μ_F
Crown	......	......	1.527	1.530	1.536
Flint	......	......	1.630	1.635	1.648

$$\nu_1 = \frac{1.530 - 1}{1.536 - 1.527} = \frac{.530}{.009} = 58.89$$

$$\nu_2 = \frac{1.635 -}{1.648 - 1.630} = \frac{1.635}{.018} = 35.28$$

$$\Delta = 58.89 - 35.28 = 23.61$$

Then $$F_1 = \frac{6.5 \times 23.61}{58.89} = +2.61\text{in.}$$

and $$F_2 = \frac{6.5 \times 23.61}{-35.28} = -4.358\text{in.}$$

therefore $$F = \frac{2.61 \times (-4.358)}{+2.61 - 4.358} = +6\tfrac{1}{2}\text{in.}$$

which is positive and therefore convex.

To find the Achromatising Cc.—If a given lens has to be united with a concave so that the combination may be achromatic, the concave is found by the following formula, F_1 being the focal length of the convex, and F_2 that of the concave lens, ν_1 and ν_2 pertaining to the crown and flint components respectively.

$$F_2 = \frac{F_1\ V_1}{V_2}. \qquad [250]$$

Taking the same figures as in the previous example.

If the convex has F = 2.61in., then—

$$F_2 = \frac{2.61 \times 58.89}{-35.28} = -4.358.$$

Dioptral Formulæ.—With dioptral lenses an achromatic combination is calculated from the following formulæ:—

Let D represent the power of the combination, D_1 and D_2 the powers respectively of the convex and the concave, and ν_1 and ν_2 the respective efficiencies of the crown and flint lenses.

$$\nu_2 D_1 = - \nu_1 D_2 \quad \text{and} \quad \nu_2 D_1 + \nu_1 D_2 = 0 \qquad [251]$$

and since $D = D_1 + D_2$

$$D = \frac{D_1 \nu_1 + D_2 \nu_2}{\nu_1 + \nu_2} \qquad [252]$$

$$\text{whence } D_1 = \frac{D \nu_1}{\nu_1 - \nu_2} \quad \text{and} \quad D_2 = \frac{D \nu_2}{\nu_1 - \nu_2}. \qquad [253]$$

Taking the same glasses as in the previous example.

$$D_1 = \frac{6 \times 58.89}{23.61} = +14.97$$

$$D_2 = \frac{6 \times 35.28}{23.61} = -8.97 \text{ D}$$

$$\text{And } D_1 + D_2 = +14.97 \text{ D} - 8.97 \text{ D} = +6 \text{ D}.$$

Since the powers of the two component lenses are proportional to their efficiencies, if $\nu_2 = 60$ and $\nu_2 = 50$, a +6 D and a −5 D will make an achromatic + 1 D.

If D, the power of the convex, is known, and it is needed to calcul·te the concave required to make it achromatic, the calculation is made thus:—

$$D_2 = \frac{D_1 \nu_2}{\nu_1}. \qquad [254]$$

As in the foregoing example, if the crown is +14.97, the flint concave is

$$D_2 = \frac{14.97 \times -35.28}{58.88} = -8.97 \text{ D}.$$

Illustrating Example.—As an example, given an equi-convex lens of crown glass whose radius of curvature is 10in., there is needed to calculate the radius of curvature of a flint concave so that the two combined make an achromatic combination.

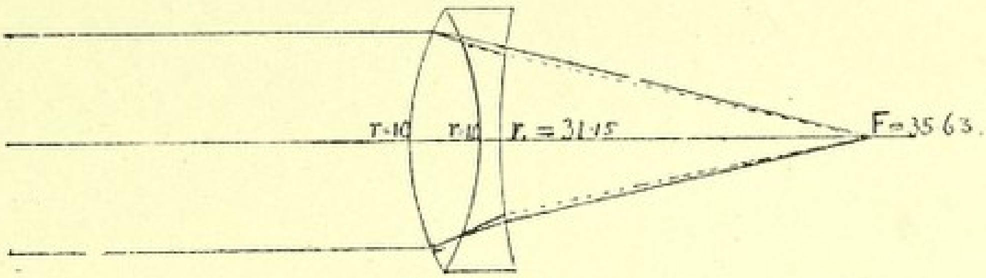


Fig. 255.

If in the crown $\mu_D = 1.5175$ and $\mu_F - \mu_C = .0087$ then

$$\nu_1 = \frac{\mu_D - 1}{\mu_F - \mu_C} = \frac{.5175}{.00877} = 59.$$

If in the flint $\mu_D = 1.571$ and $\mu_F - \mu_C = .01327$ then :—

$$\nu_2 = \frac{\mu_D - 1}{\mu_F - \mu_C} = \frac{.571}{.01327} = 43.$$

Now $$\frac{1}{F_1} = .5175 \times \left(\frac{1}{10} + \frac{1}{10}\right) = \frac{1}{9.662} \text{ for the convex,}$$

and $$\frac{1}{F_2} = \frac{1}{9.662} \times \frac{43}{59} = \frac{43}{570.058} = \frac{1}{13.257} \text{ for the concave,}$$

then $$\frac{1}{F} = \frac{1}{9.662} - \frac{1}{13.257} = \frac{3.595}{128.089} = \frac{1}{35.63} \text{ for the combination,}$$

$$\text{or } F = 35.63\text{ins.}$$

Now, the radii of curvature of the two adjacent surfaces must be equal, that is, 10in. Therefore r, the second radius of the concave, is found from

$$-13.257 = \frac{-10\ r}{(-10 + r)\ (.571)}$$

Whence $$r = -31.15\text{in.}$$

The radius of curvature of the inner surface of the concave is 10in. and that of the outer 31.15in. for

$$\frac{-10 \times (-31.15)}{(-10 - 31.15) \times .571} = 13.257\text{in.}$$

Second Illustrating Example.—Another example; a plano-convex achromatic combination is required of F = 20in. Let the glasses be

	μ_C	μ_D	μ_F
For the crown	1.535	1.54	1.555
For the flint	1.59	1.60	1.63.

Then $\nu_1 = \dfrac{.54}{.02} = 27$, and $\nu_2 = \dfrac{.60}{.04} = 15$, and $\Delta = 12$.

$$\text{Now } F_1 = \frac{20 \times 12}{27} = 8.88 \text{ Cx. and } F_2 = \frac{20 \times 12}{-15} = 16 \text{ Cc.}$$

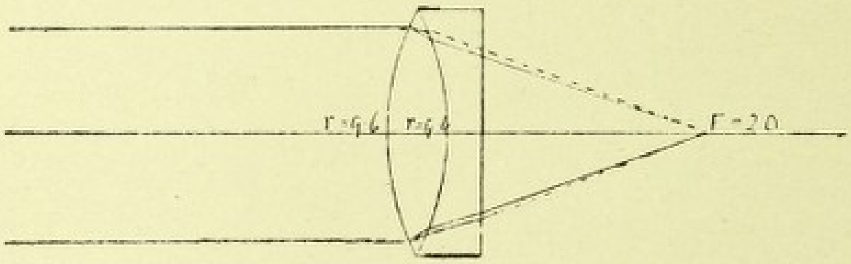


Fig. 256.

The combination will therefore be

$$F = \frac{-16 \times 8.88}{-16 + 8.88} = +20\text{in.}$$

If the one surface of the concave is plano,

$$F_2 = \frac{r}{\mu - 1}$$

so that $$r = 16 \times .6 = 9.6\text{in.}$$

for the curved surface of the negative lens.

Since the Cx. must have one surface of 9.6in. radius the other is

$$8.88 = \frac{r \times 9.6}{.54 \times (r + 9.6)} = \frac{9.6\ r}{.54\ r + 5.184} = 9.6 \text{ (approx.)}.$$

so that the positive lens is a double convex. The combination consisting of a double convex lens and a plano-concave.

Illustrating the Dioptral Formulæ.—To work such a calculation by diopters since 20in. = 2 D, we have

for the crown $$D_1 = \frac{2 \times 27}{12} = 4.5 \text{ D}$$

for the flint $$D_2 = \frac{2 \times -15}{12} = -2.5 \text{ D}.$$

For a plano $$r = \frac{\mu - 1}{D} \text{ in terms of a metre.}$$

that is $$r = \frac{.6}{2.5} = .24 \text{ M or 24cm.}$$

Since $$D = \frac{(\mu - 1)(r + r')}{r\,r'}.$$

substituting we get for the second radius of the crown

$$4.5 = \frac{.54 \times (.24 + r)}{.24\ r} = \frac{.1296 + .54\ r}{.24\ r}$$

That is $$1.08\ r = .1296 + .54\ r$$

Whence $$.54\ r = .1296 \qquad \text{or } r = .24 \text{ M or 24cm.}$$

As shown previously, the lens is a double convex of 24cm. or 9.6in. radius.

Lens Combinations.—A combination of lenses, which only has one achromatised component is not as a rule perfectly achromatic, so in order that the whole combination may be achromatic, the achromatised component must be suitably overcorrected.

Achromatism of a Single Lens.—A single lens cannot be achromatic for a real image. But when it is used as a magnifier, the virtual image is really composed of a series of images formed by every different colour which, being contained within the same visual angle, combine on the retina to form a single impression. This image, however, appears coloured at the edges, owing to the effects of spherical aberration, which is greater for blue than for red. Then, if spherical aberration is eliminated, as in the Huyghen eyepiece, the virtual image is colourless.

Separated Lenses.—If the lenses are not in contact the conditions for achromatism are different. Two lenses made of the same material can be rendered achromatic, for virtual images only, by being separated by a proper distance. This is the case with Huyghen's eye-piece. Those rays which pass through the thin part of the field lens pass through a thicker portion of the eye lens. But the violet is relatively nearer to the axis than the red and so is less refracted and all the components of white light form the same visual angle on emergence. Thus, two convex lenses of equal ν separated by a distance equal to

$$\frac{F_1 + F_2}{2}$$

form an achromatic combination for virtual images.

Further Chromatic Difficulties.—Although a combination of lenses may bring different coloured rays to the same focus, the images are not necessarily of the same size. And, furthermore, a combination achromatic for an axial pencil of light need not be so for oblique pencils. Finally, if a lens is achromatised for rays proceeding from a given plane, it need not be so for light proceeding from other planes.

If a positive and a negative lens neutralise for the D line, the two, being of different dispersions, may not neutralise for red or violet.

Apochromatic Lens.—A combination which unites three lines of the spectrum (as distinct from two) was termed, by Abbé, apochromatic. For such a lens at least three different sorts of glasses must be employed. The residual spectrum still left is so small as to be a negligible quantity.

For such a combination

$$\frac{1}{F} = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} \text{ and } F_1\,\nu_1 + F_2\,\nu_2 + F_3\,\nu_3 = 0.$$

ABERRATIONS OF FORM.

In order that an object located in a plane perpendicular to the axis of a lens combination shall form an image geometrically identical in all respects, it is necessary that the lens shall be corrected for a number of aberrations of form. Von Seidel has* shown that in order that an image formed by a lens, shall be perfect in all respects as regards form, the five aberrations which he denoted by the symbols S_1 S_2 S_3 S_4 S_5 must each = zero. The better the Seidel conditions can be satisfied the more perfect will be the lens. Owing to the nature of light all these aberrations of form are accompanied by chromatic errors which also need correction.

* Prof. S. P. Thompson's, Lummer's Optics.

The *first* error is that of *spherical aberration*, which is due to the fact that rays from a point on the principal axis do not meet at a single point again on the axis.

The *second* error is that of *coma*, which may be defined as spherical aberration for oblique rays. This prevents light from a point on a secondary axis from meeting at a point after refraction, resulting in a disc of confusion which broadens out on one side like a comet or feather.

The *third* condition is that of *radial astigmatism*, which is produced by the unequal refraction of the various meridians of a lens acting on rays which start from a point on a secondary axis, so that two focal lines constitute the image.

The three errors referred to are point aberrations, the two remaining are aberrations of a plane.

The *fourth* error is called *curvature of the field*, because, owing to it, the image of a flat object is curved.

The *fifth* and last error of form is *distortion*, which is due to the contraction or expansion which the image undergoes when the rays pass obliquely through a lens.

Form and Colour Aberrations.—If the form aberrations were corrected, a geometrically perfect image would be produced with monochromatic light, but since different parts of the spectrum are refracted unequally, glasses of different dispersive powers have to be united, so as to bring different coloured rays to the same focus.

Although the dispersion of colours may be largely corrected, still the images for the different colours may not be of the same magnification, so that to produce a theoretically perfect image there are nearly as many corrections to be made for colour as for form. In practice these last requirements need not concern the optician, it being thought sufficient for all purposes if two or three principal lines of the spectrum are brought to a focus, while neglecting the residual chromatic errors.

SPHERICAL ABERRATION.

Since a lens may be regarded as consisting of an infinite number of prisms whose angles of inclination increase with their distance from the axis, it follows that the deviation effected by the various zones of a lens depends on this distance.

In a convergent lens, the varying inclination of the different parts of the two surfaces of each meridian causes rays to meet at a common focus. But, actually, the refraction of a spherical lens is such that the rays which are incident at zones appreciably distant from the axis are not brought to a focus at a single point. Those transmitted by the marginal zones of the lens meet sooner than those transmitted nearer the centre as depicted in Fig. 259.

Each zone of a lens has its own focal length, varying from the principal focus for rays entering the zone surrounding the principal axis to a point nearer the lens than its principal focus for rays passing through the most external zone.

Spherical aberration may also be defined as the inequality in power of the various zones of a lens or surface, and does not necessarily imply that the peripheral portions have the greatest power, although this always occurs in truly spherical curves, with the result that incident waves of light are not truly spherical after refraction, the periphery being insufficiently retarded with respect to the central portions. In photographic lenses it may happen that spherical aberration is completely eliminated for the axis and periphery, while it may still occur in the intermediate zones.

Causes of Spherical Aberration.—The inability to unite in a single point all the rays diverging from an object point on the axis is called *spherical aberration.* It is due, not to the fact that the refracting power is greater towards the periphery, for that is a natural property of a lens, but to the fact that the refracting power *increases too rapidly* towards the periphery.

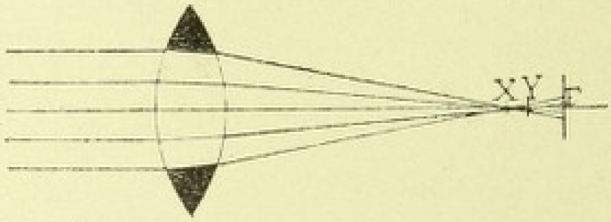


Fig. 257.

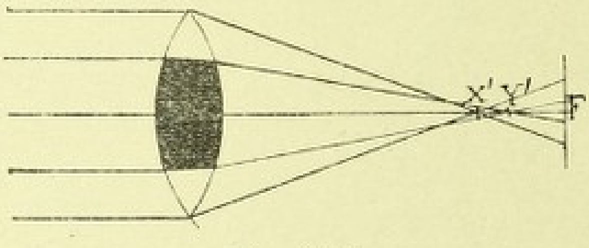


Fig. 258.

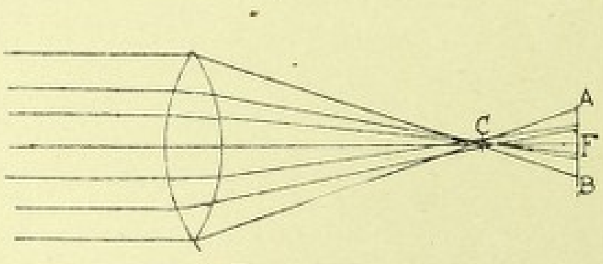


Fig. 259.

Central and Peripheral Refraction.—If we take a circular piece of black paper the full size of the lens, and divide it by cutting out a disc one half of the entire diameter, we shall have a ring and a disc of equal areas. By gumming first the ring and afterwards the disc on to the lens we have a ready means of observing the course of central and peripheral rays separately.

First, let the peripheral part of a lens (Fig. 257) be covered over so that only the inner half of the area of the lens is effective, and parallel rays meet at X Y, distances slightly within F. Next, let the central portion of the lens be covered over (Fig. 258), so that only the periphery acts on the light. Then the rays meet at X′ Y′, distances still further within F. The light refracted by the central zones causes converging circles of confusion, and that refracted by the peripheral zones, diverging circles of confusion at C in Fig. 259.

Circle of Least Confusion.—When the whole lens is exposed to the light (Fig. 259), the two sets of circles of confusion are of about the same mean diameter at C, where the illumination is greatest and the disc of light of minimum size. At any point either nearer or further the disc is larger than at C.

But the greatest concentration of light occurs at F, where the image of a luminous point appears as a bright spot of light surrounded by a faint halo, due to the light from the periphery of the lens having crossed and become divergent.

Distance of the Image.—The difference in the distance of the image from a Cx. lens in the three cases where the periphery only, the centre only, or the whole of the lens is effective, can be shown by experiment; the object being a bright flame placed behind a small aperture, covered by a piece of orange glass, in order to make the light more or less monochromatic.

Longitudinal and Lateral Aberration.—The distance between the extreme foci is called the longitudinal aberration. The diameter of the disc A B, on a screen, caused by the overlapping of the rays refracted by the margin of the lens when the screen is held in the focus of its central portion, is called the lateral aberration.

Influencing Factors.—It is clear that the definition of an image depends on the smallness of the circles of confusion which constitute it, and these circles are dependant on the spherical aberration which is proportional to the incidence of the light, the form, the aperture and thickness of the lens. In proportion as these factors are changed the spherical aberration is increased or decreased.

Aplanatic Lens.—A lens, or lens combination, corrected for spherical aberration is termed aplanatic, but no combination of lenses can be rendered entirely aplanatic for all distances of the object.

Minimum Deviation.—In Fig. 260 the opposite points D and E of the lens constitute a portion of a prism G K H. The ray A D is incident in a direction such that its point of incidence D and its point of emergence E are equi-distant from the edge; A D therefore suffer minimum deviation. The ray A B C is incident, so that its deviation is not minimum and it is more bent from its course than the ray A D.

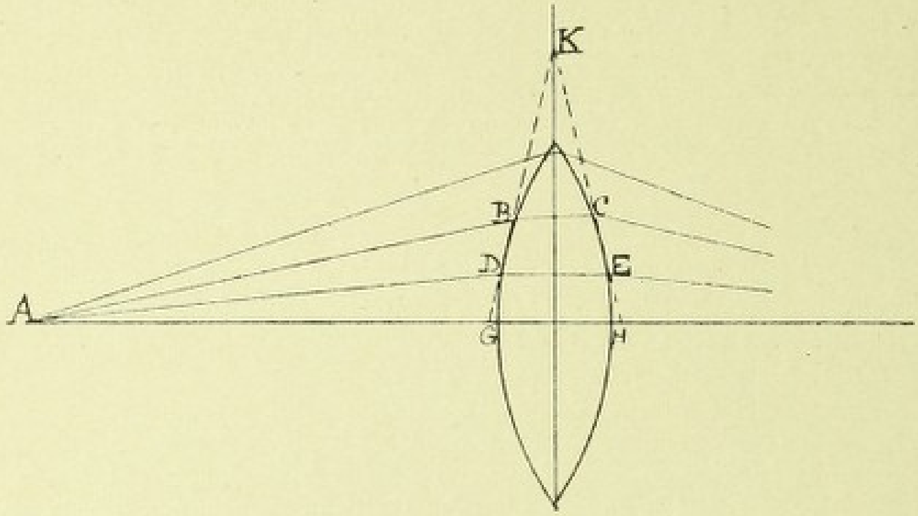


Fig. 260.

Influence of Form.—Thus, the degree of spherical aberration of a lens depends on the direction of the incident rays. It is least when the rays in general are, after refraction at the first surface, most nearly parallel to the bases of the prisms of which the lens is formed, so that the total refraction is approximately divided between the two surfaces, and therefore the angles of incidence and emergence are equal.

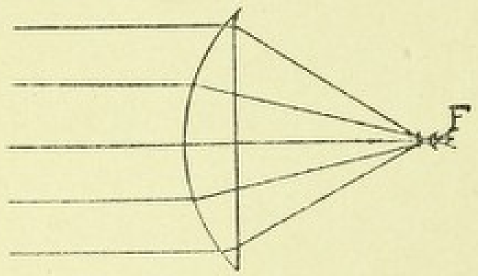


Fig. 261.

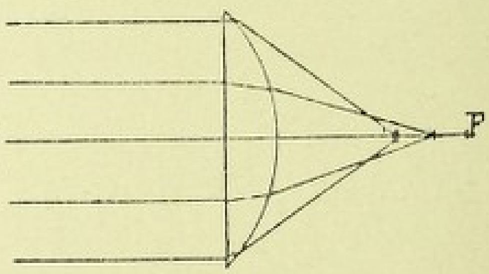


Fig. 262.

As a general rule, with parallel light, the more curved the surface that is exposed to the light and the less curved the back surface of the lens, the smaller is the degree of spherical aberration (Fig. 261). As the object approaches the lens and the rays become divergent the aberration increases, until at length when the rays are very divergent the lens must be turned round and the flatter side presented to the light, to get the least amount of aberration. And since the incidence must vary for any distance of the object spherical aberration depends not only on the form of the lens but also on the distance of the luminous point since both of these factors influence the incidence of the light.

On the other hand a very high degree of spherical aberration results if the less curved surface is exposed to parallel rays, or the more curved surface to rays diverging from an object placed at the focus of the lens, since in these cases a considerable departure from incidence at minimum deviation for peripheral rays occurs (Fig. 262).

Best Form of Lens.—Therefore the curvature needed for a lens having minimum aberration varies with the index of refraction of the glass, the difference between the two radii of curvature increasing directly with μ. It is found that a plano-convex, or better, the crossed lens with its more curved surface turned to the light is the form of single lens which gives the best definition for objects at extreme distances. The same lens turned the other way is the best for very near objects, while the double convex is the best when the incident rays diverge from twice the focal distance, for then object and image being equi-distant from the lens, the incident and emergent rays form equal angles with the two surfaces. If used for all distances, the double Cx. is perhaps the best form of single lens.

The term *crossed* is applied to a lens having unequal radii of curvature, it being usually a bi-Cx. or bi-Cc. whose radii are as 6 : 1 approximately.

Expression for Minimum Aberration.—In order to obtain minimum spherical aberration the radii of the two surfaces of a lens should be in the ratio of $1 + 2\mu$ and $1 - 2\mu + 4/\mu$. These quantities when $\mu = 1.5$ are 6 and 1, which are the relative radii of curvature of the two surfaces.

This ratio holds good when both surfaces are convex or both concave and $1 - 2\mu + 4/\mu$ is a positive quantity, but if the lens is of meniscus form this quantity is negative. Thus, when μ 1.686 the one surface should be plano and if the index is higher the lens must be a meniscus.

Influence of Size of Lens.—If the lens be ground down to a smaller size so that only the central area is left, or what amounts to the same thing, if a stop is placed in contact with the lens, the marginal rays are cut off and spherical aberration is consequently lessened. The employment of a diaphragm or stop at a slight distance from the lens is the practical remedy usually employed for reducing spherical aberration.

Remedy by Cc. Lens.—A Cc. lens unduly diverges the light at the periphery to the same extent as a Cx. converges it, so that a Cx. and a Cc. of similar curvature and in contact, would neutralise each other's spherical aberration, but at the same time would neutralise each other's refractivity, and so act like a piece of plane glass. Since, however, the defect depends on curvature, a combination can be selected, in which the Cx. has more refractive power than the Cc. so that the positive aberration of the first is neutralised by the negative aberration of the second. By proper selection a combination can be made, of given focal length and also achromatic, such that while the curves in contact are similar, the aberration of the front surface of the convex is neutralised by that of the back surface of the concave.

Other Remedies.—Other remedies for spherical aberration may be employed such as two positive lenses, in place of one, by which the curvatures are diminished for the same refractive power; or the positive and negative components of a system separated by an interval by which the positive power of the combination is increased. This last method is sometimes made use of in photographic and microscopic lenses.

Positive and Negative Aberration.—Positive aberration is that which obtains when the marginal rays come to a focus before the central, negative aberration if the central rays come to a focus before the marginal.

Under and over Correction.—A lens combination which partially neutralises the positive aberration is *under-corrected*, and if it more than neutralises the positive aberration, it produces negative instead of positive aberration, and is said to be *over-corrected*.

Expression for Longitudinal Aberration. — A numerical expression is sometimes given to the longitudinal aberration for parallel rays in thin lenses, when $\mu = 1.5$ the values of which are in terms of d^2/F, where d is the semi-diameter of the lens. The value for different forms of lenses varies with each index of refraction. It would also differ for thick lenses and for various distances of the source of light.

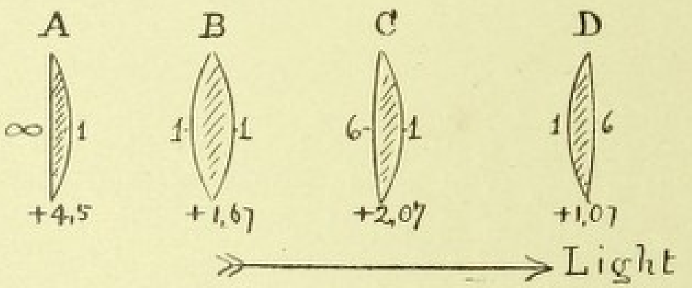


Fig. 263.

The following table gives the relative values for different forms of lenses.

A plano-Cx., with the plane surface to the light	4.5
A plano-Cx., with the curved surface to the light	1.17
A double Cx. (equal radii)	1.67
A bi-Cx., with radii of 1 and 6, with the more curved surface to the light	1.07
The same, with the less-curved surface to the light	2.07

Least Possible Time.—Since light travels in a straight line it takes the least possible time to reach a given point, and this principal of "least possible time" holds good for refraction. Thus, various rays diverging from a point in air and passing into another denser medium must arrive at the same point, at the same time, if a focus is to be obtained. With a lens, disregarding spherical aberration, this occurs because although the distance from A to B, and thence to F, is greater than from A to C and F (Fig. 264—1), yet the distance traversed in the denser medium is greater in the case of A C F. The law of refraction μ_1 sin i = μ_2 sin r, where i and r are, respectively, the angles of incidence and refraction, is in accordance with the principal of least time. If a lens is corrected for spherical aberration all rays diverging from an object point must reach the same image point and in the same time, no matter what course they take.

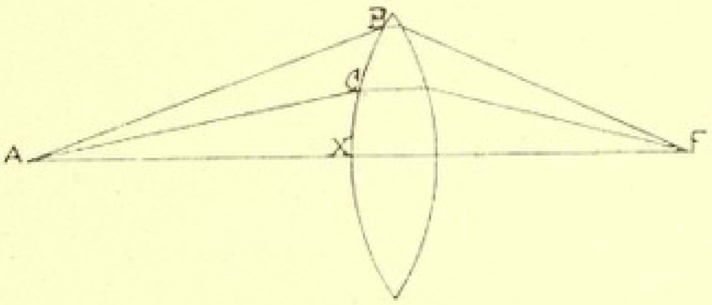


Fig. 264 (1).

Let the distance of A from any point on the refracting surface (Fig. 264—2) be d_1 and the corresponding distance of F be d_2, then $d_1 \mu_1 + d_2 \mu_2$ is the optical length of any ray diverging from A and refracted to F.

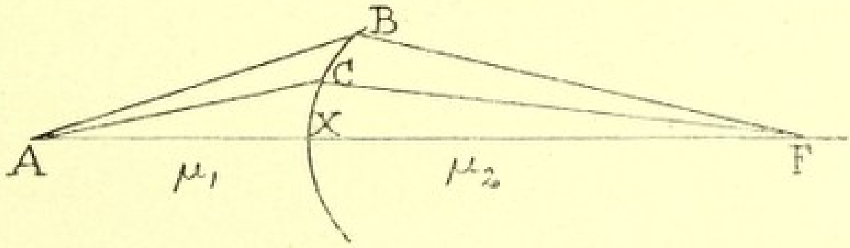


Fig. 264 (2).

So in order that A X, A B and A C shall meet at F it would be necessary that $d_1 \mu_1 + d_2 \mu_2$ be a constant for any incidence of the light, that is to say, A B μ_1 + B F μ_2 = A C μ_1 + C F μ_2 = A X μ_1 + X F μ_2. As this cannot occur with spherical surfaces, spherical aberration may be said to be due to the fact that all the rays diverging from a point cannot reach the same point in a given time, or rather that, within a given time, the rays reach different points of the axis.

In the case of a lens the influence of the two surfaces has to be considered, since each ray travels in three different media. If d_1 be its course in the first medium μ_1, d_2 its course in the second medium μ_2, and d_3 its course in the third medium μ_3, then $d_1 \mu_1 + d_2 \mu_2 + d_3 \mu_3$ must be equal for each ray in order that all rays diverging from an object-point may meet, after refraction, at a single image-point.

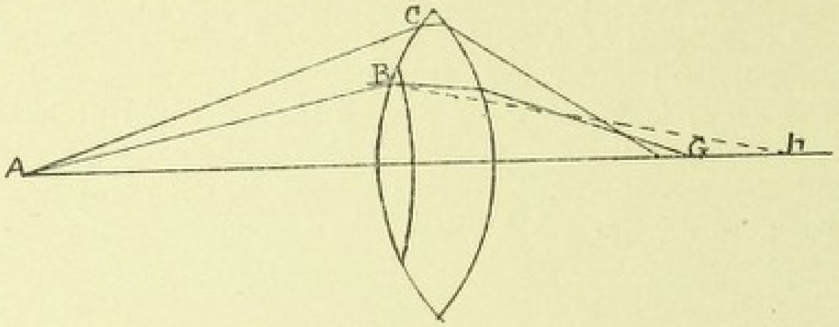


Fig. 265.

Influence of Thickness.—A ray A B traversing a thick lens (Fig. 265), is retarded in the denser medium and thus can only reach, in a given time, a point G on the principal axis which lies nearer to the lens than H, the point reached by a similar ray passing through a thin lens. Thus, spherical aberration increases with the thickness of the lens.

Remedies.— The ray A C can only reach, in a given time, a point on the axis, nearer than that reached by A B, which shows that on the aperture of the lens depends also the degree of aberration. A remedy would be found if the speed of the light could be increased at the periphery. This could be done by a lens made of a medium whose index of refraction decreases, as the distance from the principal axis increases, which occurs in the crystalline. Or it could be remedied by a lens having less curvature at the periphery than at the centre, that is, one having a hyperbolical or ellipsoidal curve.

THE OBLIQUE ABERRATIONS.

Axial Pencils of Light.—A beam of light diverging from a point on the principal axis would, on passing through a lens corrected for central chromatic and spherical aberration, meet again as a sharp point on the principal axis.

Oblique Pencils. — When, however, the luminous point is situated on a secondary axis, this is no longer true since further aberrations are introduced by the oblique incidence of the light. The oblique aberrations are *coma, radial astigmatism, curvature of field, and distortion*, or rather these aberrations result from the oblique incidence of light.

Coma and radial astigmatism are connected with each other, only in as much as they are both oblique point aberrations. If we place a bright point of light obliquely to the axis of a lens and move a white screen behind it, we shall find that the image is blurred. By moving the screen to and fro the spot of light will assume various shapes, triangular, comet-shaped, cup-shaped and pear-shaped figures, which are the result of coma. If now the coma is reduced by placing a fairly small diaphragm in front of the lens and we hold the screen within the focus, and slowly draw it away, the image is seen to form a symmetrical oval gradually narrowing to a horizontal line. As the screen is receded, the horizontal line becomes first an irregular circle, then it broadens into a vertical oval, and this becomes a vertical line which finally broadens out into a blurred patch. These lines are the results of the radial astigmatism.

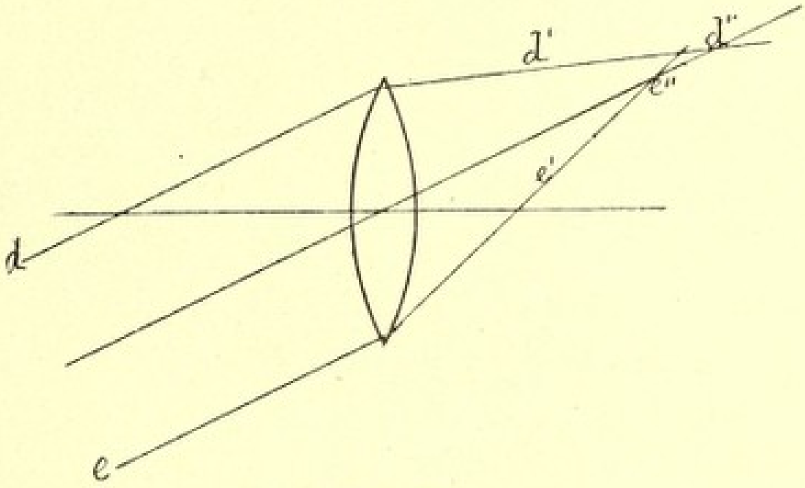


Fig. 266.

Coma.—Let d and e (Fig. 266) be rays proceeding from a distant point on an oblique axis. The ray e meets the surface of the lens sooner than d, and since e departs more from minimum deviation than does d, the ray e e′ cuts the axial ray at e″ sooner than does d at d″, and there results the defect called coma. This may be defined as an aberration produced by the unequal refracting effect, of the different parts of the various meridians of a lens, on a pencil of rays which passes obliquely through it. The result is a blurred halo of confusion partly surrounding and partly extending from a bright point; it is therefore spherical aberration for oblique pencils of light.

Influencing Factors.— Coma is directly proportional to the diameter of the lens aperture and to the obliquity of the incident rays to the principal axis. It can be reduced by the employment of a stop or by using a lens of such a form as will cause the incidence of the rays, passing through any meridian of the lens, to be more alike. Coma is less marked in plano and meniscus lenses than in doubles, for the reason that, in such lenses, less refraction takes place at the second surface.

The confusion disc produced by coma presents various forms, as before stated, but it is usually more or less pear or comet-shaped, the narrow brilliant part being directed towards the principal axis. It is, therefore, non-symmetrical, and in this respect differs from the confusion discs of other aberrations, which are always symmetrical in respect to the axis of the beam of light.

The *Sine* Condition.—In order that coma be eliminated from a lens, the sine of the angle *a* formed by an incident ray with the axis, and the sine of the angle *a'* formed by the refracted ray with the axis, should have a constant ratio; that is

$$\frac{\text{Sin } a}{\text{Sin } a'} = \text{a constant.}$$

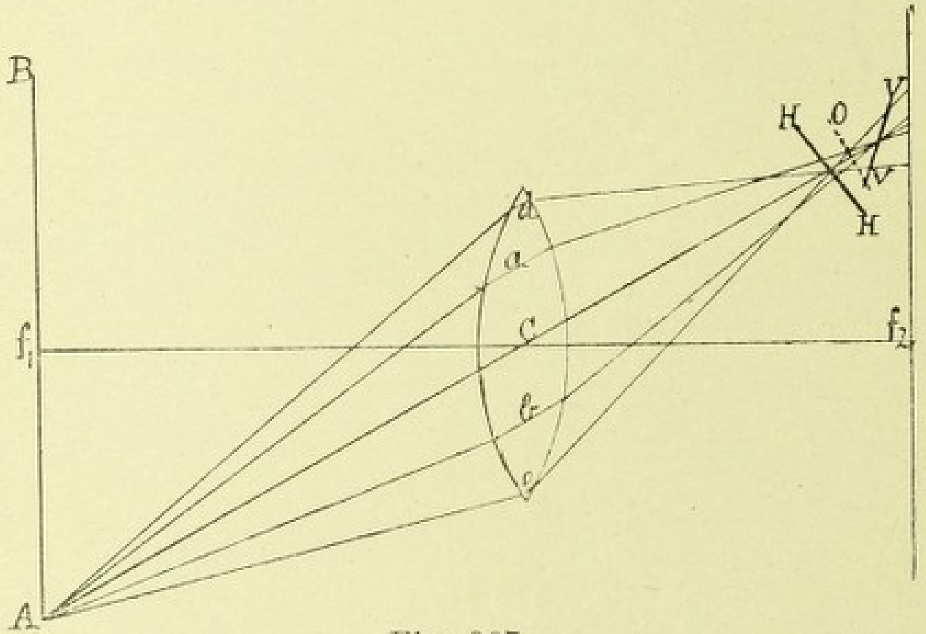


Fig. 267.

RADIAL ASTIGMATISM.

When a beam of light (Fig. 267), proceeding from a point A on the object, falls obliquely on a surface of a lens the rays incident in the different meridians are brought to a succession of imperfectly-formed foci at varying distances behind the lens; of these two groups are clearly defined.

The first group forms a line H H, which is at right angles to A f_1, a line drawn from the luminous point to the principal axis in that same plane; the other group forms a line V V, parallel to A f_1, and both are nearer to the lens than the focal plane f_2.

This defect is called radial astigmatism, and it increases proportionately to the obliquity of the light and the diameter of the aperture of the lens.

The result of astigmatism is that a point situated on a secondary axis is represented in the image, not by a point, but by two short focal lines somewhat similar to those formed by a sphero-cylindrical lens.

But there is this important difference between the astigmatism of a sphero-cylindrical and the radial astigmatism of a spherical lens, inasmuch as the former occurs when the object point lies on the axis and the defect is due to the curvature of the lens not being the same for all meridians. Whereas in the latter the curvature is the same for all meridians and the astigmatism is due to the oblique incidence of the light, the direction of the focal lines varying with the position of the object point.

Sagittal and Meridional Planes.—Thus, turning to the figure, suppose A be a point on the lower edge of an object, then such rays a, b, as are incident in the horizontal plane of the lens meet and form a vertical focal line at V. Those rays d, e, incident in the vertical plane meet as a horizontal line at H H. The distance between H and V is the measure of the astigmatism; a c b is the meridional, and d c e, the sagittal plane of incidence. The sagittal plane is that which contains the line connecting the luminous point and the principal axis (as already defined), while the meridional plane is at right angles to it.

Influencing Factors.—Now when the luminous point is oblique to the principal axis, the effective aperture of the lens is an ellipse in which the sagittal plane of incidence corresponds to the short diameter, and the meridional plane to the long diameter.

In the meridional plane the light has to traverse a greater thickness of the lens and is more oblique than an axial pencil; it is, therefore, rendered more convergent, and has its focus nearer the lens than the focal plane, thus forming the second focal line V.

In the sagittal plane the light has also a greater thickness to traverse than an axial pencil would have, and it is still more oblique than in the meridional plane. It, therefore, has its focus still nearer the lens and forms the first focal line H. Consequently radial astigmatism is due to the increased angles of incidence of the light and the increased effective thickness of the lens.

Tangential and Radial Lines.—The astigmatism of a lens is essentially the distance between the focal lines and produced by the difference in the effective power of the lens in the meridional and sagittal planes of incidence. The first focal line H is called the tangential and pertains to the sagittal plane of the lens. The second focal line V is the radial and pertains to the meridional plane of the lens. Each focal line lies at right angles to the meridian of the lens which produces it.

Between the two focal lines there is a position O, where the circles of confusion are of least diameter. This may be regarded as the mean focus of the oblique pencil of light.

The tangential and radial lines are formed on curved surfaces concave towards the lens, and both are inside the principal focal plane. The curved surfaces touch each other at the principal axis in the focal plane.

Radial astigmatism has been illustrated with the light diverging from a point on the lower edge of an object so that the resulting tangential focal line is horizontal and the radial line is vertical. If the luminous point is to be right or left of the lens the tangential line is then vertical and the radial line is horizontal.

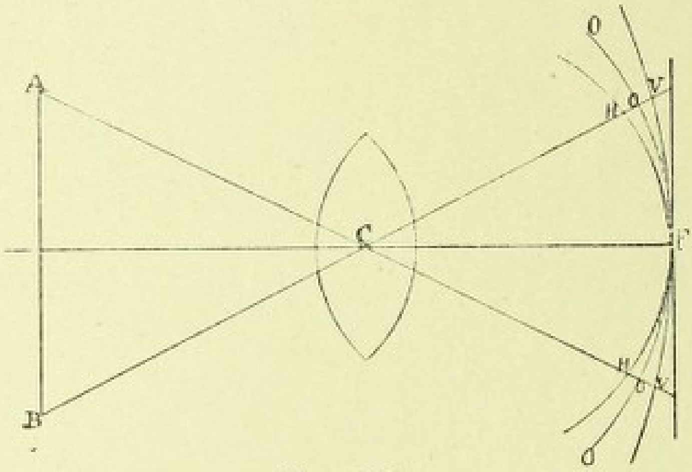


Fig. 268.

Remedies.—A stop cuts off the extreme rays in both planes and can be so placed that, in the sagittal plane, only the less refracted rays are utilised, then the tangential lines lie nearer to the radial. Generally the further the stop is from the lens, the more distant is the tangential line and *vice versa*. Astigmatism may be overcome by making the lens combination of meniscus form and corrected for chromatism by combining glasses, of high refractive power and low dispersion, with those of opposite qualities as is now possible with certain newer glasses, but which was not possible with the old glasses.

CURVATURE OF THE FIELD.

If A be a point on the lower extremity of an object the light diverging from it, after refraction by the lens, forms two focal lines at H and V. Beyond H (Fig. 268), the cone of refracted light, contracting horizontally and extending vertically, forms at some position O about midway between the lines H and V an irregular circle, called the circle of least confusion, which may be regarded as the focus of the rays diverging from A. On the surface containing the circles of least confusion the sharpest representation of the periphery of the object is formed. Since the effective power of a lens is greater as the light is more oblique, this surface forms a portion of a sphere with its concave surface towards the lens. The defect is known as *curvature of field*. The image of a convex object would be still more curved than that of a flat object, but a concave object might be so placed as to neutralise the curvature of field.

Influencing Factors.—While curvature of field is partly due to the same cause which produces radial astigmatism, i.e., the increased power of a lens for oblique light, it is not entirely so, for if H were made to coincide with V there would still be curvature. And even if the peripheral foci were at the same distance from the optical centre (or second equivalent point) as the focus on the principal axis they would form radii of a circle and curvature would remain. Thus, a sphere of glass has equal refracting effect on rays coming from any point and is therefore entirely free from astigmatism, but the field is nevertheless curved.

Positive and Negative.—The curvature is said to be under-corrected, or positive, when the image is concave towards the lens and negative if by over-correction the image becomes convex.

It follows, therefore, that if the image is projected on to a flat screen, either the centre or the periphery may be focussed sharply, but it will be impossible to obtain both defined at the same time, although the difference may be greatly reduced by stopping the lens down.

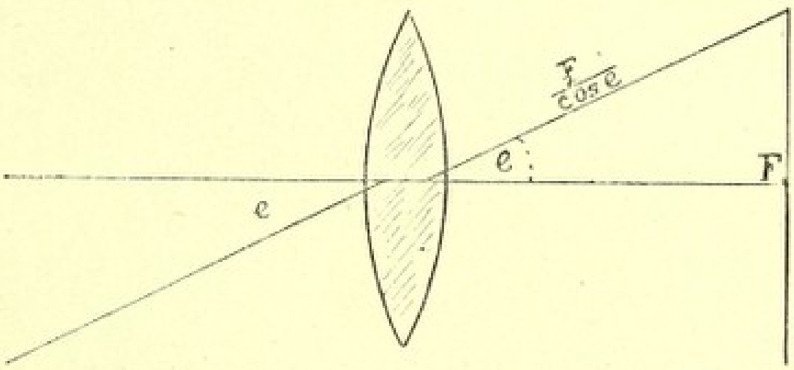


Fig. 269.

The image is flat if the focal length of each oblique pencil is equal to F/cos e (Fig. 269), e being the angle which the oblique axis makes with the principal axis.

Condition for a Flat Field.—In order that an achromatic combination of two lenses may form a flat image, the condition (known as the Petzval condition) which has to be satisfied is that

$$F_1 \mu_1 = -F_2 \mu_2 \qquad \text{or} \qquad F_1 \mu_1 + F_2 \mu_2 = 0$$

where μ_1 and F_1 refer to the crown and μ_2 and F_2 to the flint components respectively. In order that this shall not controvert the condition for achromatism the crown, with less dispersion, must have a higher refractive index than the flint, a condition which has already been referred to in the section on radial astigmatism, and in this case $\nu_1/\nu_2 = \mu_1/\mu_2$.

Remedies. — The field can be flattened by placing a stop in front of the lens. The stop by narrowing the beam determines at what particular point on a secondary axis the focus as represented by the disc of least confusion, shall be formed. A distance dependent on the form of the lens can be found, at which curvature is a minimum.

If a Cx. and a Cc. of equal power be separated, so as to have a convex effect, the distance may be adjusted so as to make the image flat. The oblique rays, after refraction by the convex, meet the concave nearer to the periphery and the diverging effect is thereby increased. Therefore the final convergence is to a point which is further away, for oblique pencils, than would be the case after refraction by a single Cx. lens, whose power is equal to the effective power of the combination.

An almost perfectly flat and undistorted virtual image is obtained with two equal plano-convex lenses placed with their convex surfaces facing each other, or by two plano-convex lenses whose respective focal distances are at 1 and 3, both curved surfaces facing the same way. Such combinations represent, respectively, the Ramsden and Huyghenian eye-pieces.

DISTORTION.

There are several forms of distortion known to photographers, but these need not concern us here, as they are unconnected with the form of the lens. The only kind, dependent on the lens itself, is curvilinear distortion, which causes the peripheral image points to be relatively further from or nearer to the axis than their corresponding object points. A lens which is free from this defect is termed orthoscopic or rectilinear.

Causes of Positive and Negative Distortion. — Distortion of the image is a natural consequence of refraction; even an object seen through a thick plane glass appears distorted. An object viewed through a prism is also distorted, a square appearing to have its two sides, which are parallel to the edge and base, curved with its concavity towards the edge of the prism. Similarly a square object viewed through a convex lens has the sides of its *virtual* image concaved outwards, this being termed *pincushion* or *positive* distortion. If a similar object is viewed through a concave lens it appears convex outwards—*barrel* or *negative* distortion. The *real* image of a square formed by a convex lens of full aperture is *barrel-shaped.* Distortion is an inherent fault of a lens, and is the direct result of the effect of the increased thickness of the lens when the light refracted by it is incident obliquely. Therefore, the degree of distortion varies directly with the thickness of the lens and the obliquity of the light, and is also partly the result of spherical aberration.

Fig. 270 shows that since each of the incident rays leaves the second equivalent plane at a point, at the same distance from the principal axis that it meets the first equivalent plane, the emergent rays will meet, on the secondary axis, at a point nearer the principal axis than they would do were the lens infinitely thin. As the distance of the object point from the principal axis is increased, so the disproportion between the distance of the object and image points from the principal axis is also increased.

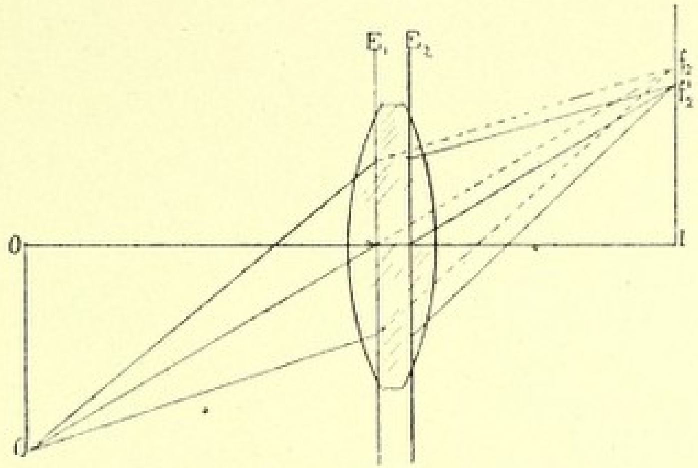


Fig. 270.

Further, any arrangement of the stop, or separation of the components of a lens system, which causes the light forming the image to be refracted by a portion only of the lens, or of one of the component lenses, will produce distortion.

Influence of a Stop.—A diaphragm is used with a single lens, or combination, in order to reduce the size of devergent pencils from the object and diminish spherical aberration, coma, astigmatism and curvature of field. This accentuates and brings into prominence distortion so that rectilinear lines of the object near the margin appear curved in the image.

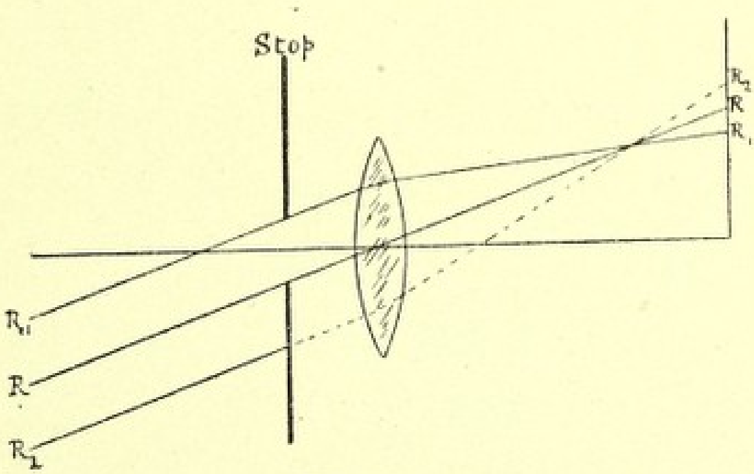


Fig. 271.

When a stop is placed in front, the periphery of the lens is effective only for oblique rays, and these proceed from the periphery of the object situated on the opposite side of the principal axis. This causes barrel-shaped distortion, as shown in Fig. 271, for the rays have their common focus between R and R_1, which is nearer the axis than if the whole lens were effective. The focal points are crowded towards the centre, and the image of an object appears not only smaller than it should be, but also distorted, the extreme points being relatively more drawn in towards the principal axis than those nearer it. A straight line appears curved, with its concavity towards the axis.

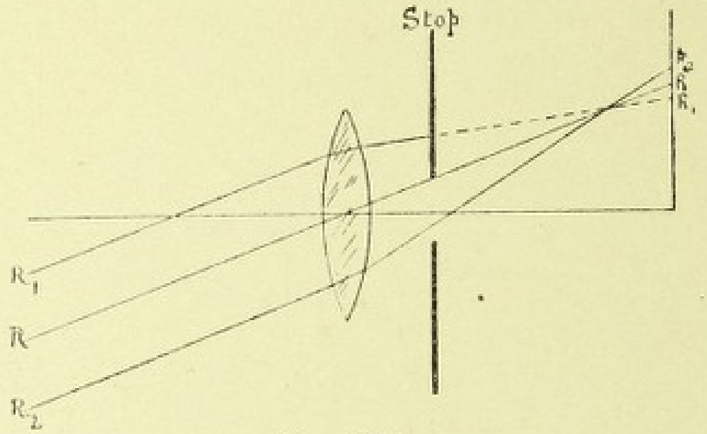


Fig. 272.

When a stop is placed behind the lens the image of a point oblique to the principal axis, as shown in Fig. 272, is situated between R and R_2, more distant from the axis than it would be if there were no stop.

The image is distorted by being drawn outwards, and the more distant the point on the object is from the principal axis the more relatively is the corresponding point on the image drawn away. A straight line appears curved, with its convexity towards the axis, and the area of the image is actually larger than it should be.

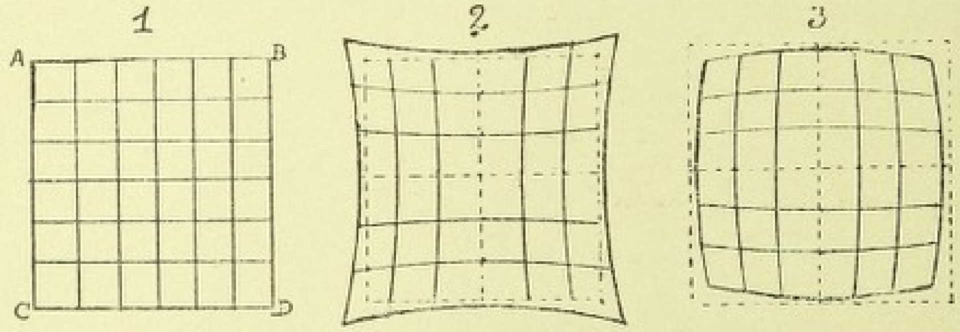


Fig. 273.

Barrel and Pincushion Appearance of a Square.—These two cases are illustrated in Fig 273, where No. 1 is a square object, No. 2 is the cushion distorted image as results when the stop is behind the lens, and No. 3 is the barrel distorted image produced when the stop is in front of the lens.

Although the position of the stop makes distortion more apparent, the latter is due to the inherent nature of the lens and not to the stop, for if we use a combination corrected for distortion we can put the stop in front of the lenses, between the lenses or behind them, or use no stop at all and no distortion will ensue.

Remedies.—Distortion is eliminated by employing a symmertical combination of lenses with the stop placed between its two components. Then all those oblique rays which pass through the right side of the front element must pass through the left side of the back element and *vice versa*. By this means the distorting effect of the front lens is neutralised by that of the back lens.

Separation of the component parts of a lens system can be utilised for the correction of distortion, and in single lenses it may be reduced somewhat by altering the thickness of the lens and its curves.

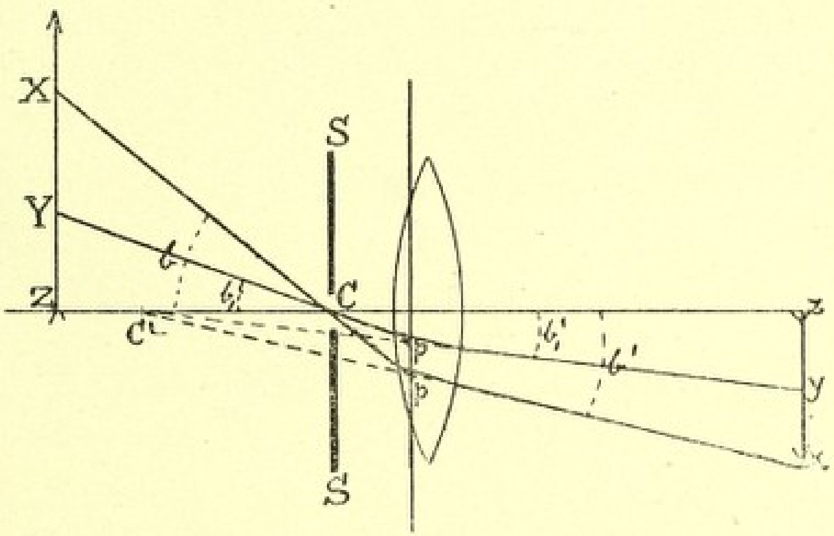


Fig. 274.

The Tangent Condition.—A *chief ray* X C or Y C is one which passes through C, the *centre of the stop*. If it be produced forwards and, after refraction, be produced backwards the point of intersection P is a *chief point*.

When the chief points thus formed all lie in a plane perpendicular to the axis, i.e., the refracting plane, the chief rays when produced back to the axis will meet in a single point C′. The lens is then said to be spherically corrected with regard to the stop.

Each chief ray makes with the principal axis, before refraction, some angle b, and, after refraction, some angle b′, and when the foregoing conditions obtain,

$$\frac{\tan b'}{\tan b} = \text{a constant for every chief ray.}$$

The image will then be uniformly magnified throughout. That is to say, the image will be free from distortion when this *tangent condition* is fulfilled.

CHAPTER XIII.

POLARISATION AND PEBBLES.

Polarised Light. — The beam of light transmitted by a homogeneous medium, such as air or glass, is ordinary in the sense that it consists of waves whose transverse vibrations lie in every direction across the line of travel, whereas the vibrations of polarised light are confined to certain directions only. The polarisation of the light may be plane, circular, or elliptical. The plane of polarisation of plane-polarised light is that plane which contains the ray; the vibrations are at right angles to the plane of polarisation.

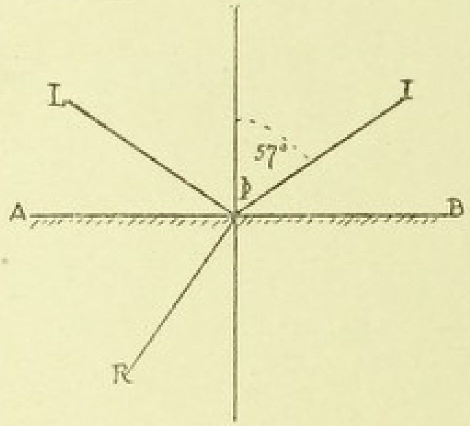


Fig. 275.

Polarisation by Reflection. — Light can be polarised by reflection at a certain angle from a surface A B (Fig. 275), which angle depends on the index of refraction. A reflected beam L and a refracted beam R at right angles to each other result from this incidence. The angle of incidence necessary to obtain polarisation of the reflected beam is found by the equation $\mu = \tan p$, where p is the polarising angle.

Thus the polarising angle of water is 48°, of glass about 57°, of a diamond 68°. The different coloured rays have different polarising angles, so that white light is never completely polarised by reflection, and the polished surfaces of metal have little or no polarising effect. Polarised reflected light can be obtained from a sheet of glass blackened on the further side and, of course, suitably placed with respect to the incidence of the light. The plane of polarisation is perpendicular to the surface and is the same as the plane of incidence. The vibrations are parallel to the reflecting surface.

Polarisation by Refraction.—The light which is incident at the polarising angle on a transparent body, and which is refracted and transmitted at right angles to the reflected beam, is also polarised. The plane of polarisation is at right angles to that of the polarised reflected light. The vibrations of the incident light perpendicular to the surface penetrate it and are transmitted, while those parallel to the surface are reflected. Polarised refracted light can be obtained when a beam of light is transmitted obliquely through a bundle of thin glass plates bound together.

Double Refraction. — Most crystals polarise light owing to double refraction, notably calcite (Iceland spar) and tourmaline. When Iceland spar is used as a Nicol prism for polarising purposes the one set of rays is totally reflected away from the layer of Canada Balsam, which is used to cement the two prisms together. In a tourmaline polariscope one set of rays is absorbed.

A light wave in the air or in any homogeneous body vibrates in every direction across its line of propagation, and its velocity is uniform and inversely proportional to what is termed the optical density of the medium. In a crystal, owing to its molecular structure, the retardation of waves, when incident obliquely to the axis of crystallisation, is greater in one direction than in another, so that the rays are transmitted along two separate paths, the one ray being called *ordinary*, and the other the *extraordinary* ray. Both of these are polarised in planes at right angles to each other and travel at unequal speeds. Most crystals, when viewed through a polariscope, present arrangements of colour which are characteristic of them.

Rock Crystal or Pebble.—Rock crystal or quartz is a pure, usually colourless, crystalline variety of silica, which occurs in nature in the form of a hexagonal (six-sided) prism, terminating in a six-sided pyramid. Its index of refraction ($\mu = 1.54$) is about the same as that of ordinary crown glass, but lower than that of flint glass, its dispersion ($\mu_H - \mu_A$) = .014 being lower than either. When cut into a slab or ground to form a lens, it is more usually styled a pebble. It is much harder than glass, more brittle, and a better conductor of heat, and it transmits more readily than glass the ultra-violet rays which lie outside the visible spectrum. Its density is 2.65, that of glass being from about 2.4 to 3.4.

The relative scarcity and greater difficulty of working pebble makes it comparatively dear. Its low dispersive power and freedom from liability to become scratched seem to be its sole advantages, so that, all being considered, pebble is not superior to good optical glass for spectacle lenses, although perhaps for simple spherical convex lenses which are frequently put on and off, and therefore specially liable to become scratched in the centre, it is sometimes to be preferred. As lenses, the pebble should be quite clear and free from striae, specks, and flaws, and should be axis cut.

Axis-Cut Pebble.—Axis-cut pebble is that which is cut into slabs at right angles to its line of crystallisation, so that when the surfaces receive their spherical curvatures, the axis of the crystal coincides with the principal axis of the lens. Axis-cut is more expensive than non-axis-cut pebble, because in working it there is not so good an opportunity of utilising those parts of the crystal which are free from flaws, as when the slabs are cut without regard to any particular directions.

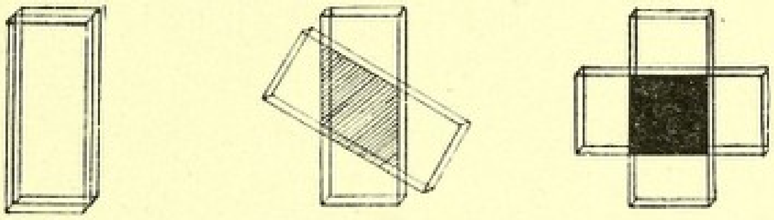
To Recognise Pebble.—Pebble is recognised by (a) feeling colder to the tongue than glass (b) by the fact that on account of its hardness a file makes no impression on it, and (c) by the polariscope test. By the latter the difference between axis-cut and ordinary pebble can also be seen. As supplied to the optical trade pebble is usually quite colourless and when in the form of a lens it has sharper ring than glass.

Double Refraction in Pebble.—Pebble possesses the property of double refraction, which, however, is not noticeable in ordinary thin spectacle lenses, the two images being so very close together. The refractive index of pebble for the ordinary ray is 1.548 and for the extraordinary ray 1.558, and since the index is higher for the extraordinary than the ordinary ray, pebble is described as a positive crystal. It is because the difference in the μ's of the two rays is so small that double refraction by a pebble lens is not appreciable.

Tourmaline.—Tourmaline cut parallel to its axis reduces an incident beam of light to two sets of polarised waves, the one in the plane of the axis of the crystal, the other at right angles to it. The former (the ordinary ray) is extinguished almost immediately, and the latter (the extraordinary ray) only is transmitted, so that all the emergent plane polarised light is travelling in the plane at right angles to the axis, the vibrations being parallel to the axis or at right angles to the direction of the ray. The plane of polarisation of a tourmaline plate can be determined by analysing the light polarised by reflection from a plate of glass. If held at the proper angle, the position of complete or partial extinction is found when the axis of the tourmaline is at right angles to the surface of the glass.

The Pebble Tester.—The simple polariscope consists of two plates of tourmaline cut parallel to their axes and suitably mounted. These plates are sometimes fitted to the ends of a wire spring like a pair of sugar tongs and called a *pincette*.

If the two plates are placed in such a position that their axes are parallel, the plane polarised beam of light transmitted by the first plate will traverse the second, and if a polariscope, so fixed, is looked through, green or brown light—due to the colour of the tourmaline—can be seen. The combination looks much more opaque than would pieces of glass of the same intensity of colour, because half the light received by it is quenched. The outer plate which polarises the light is called the *polariser*, and the second plate —the one near the eye—is called the *analyser*. If now, the analyser is rotated, while still looking through the instrument, the light will be found to become less and less bright, until, when it has been turned through a quarter-circle, the two axes being then at right angles to one another, the plane polarised beam transmitted by the polariser is stopped by the analyser. If the axis of the polariser is, say, horizontal, it can transmit only waves whose vibrations are horizontal, while the analyser can transmit only one whose direction is vertical, consequently all the light is blocked out. So long as the two axes are oblique to one another, some light passes through both plates.



A B C

Fig. 276.

It is in the position of *extinction* of the two plates that the polariscope serves as a pebble tester, so that if required for that purpose, it should be looked through and the one plate rotated until the *darkness* is complete. Unless this is done it is useless for the work, although even if it cannot be made quite *dark* there is an appreciable difference in the quantity of light transmitted by glass and by pebble placed between the plates, as explained in the following paragraph.

In Fig. 276, A shows the two tourmalines with their axes parallel, B with their axes oblique, and C with their axes at right angles.

Recognition of Pebble by Polariscope.—If an ordinary glass lens, being homogeneous in its nature, is placed between the two plates of the polariscope, it has no effect on the plane polarised beam of light transmitted by the polariser, and nothing can be seen through the instrument. A pebble placed in the instrument, by virtue of its double refracting nature, turns the beam, transmitted by the first tourmaline plate so that the light is incident on the second plate in directions other than at right angles to its axis, and part of it is transmitted. Hence with a pebble tester a pebble can be distinguished from glass, since, when a pebble placed between the tourmalines, light is seen, while none is seen when glass is so placed.

If a pebble cut obliquely to, or parallel to, the axis of the crystal (non-axis cut) is placed between the *dark* tourmalines and rotated, there are found two positions in which practically no light passes;

the one is where the axis of the pebble is parallel to, or in the same line with, the axis of the polariser, and the other is where it bears the same relation to the axis of the analyser. In either case, the polarised beam of light received by the pebble cannot be made to vibrate so as to be transmitted by the analyser.

Recognition of Axis-cut Pebble.—A ray of light transmitted by quartz cut perfectly perpendicular to its axis (axis-cut pebble) is not bifurcated. Such pebble possesses the property of rotatory polarisation. On looking through the polariscope, when an axis-cut pebble is between the two plates, the light received from the polariser is turned by the crystal to an extent varying with its refrangibility, so that when a beam is so transmitted, the different colours are separated and are seen as a series of coloured rings surrounding a clear central space. The light is not blocked out on rotating an axis-cut pebble, between tourmaline plates, since its axis is always at right angles to the axes of both polariser and analyser and in an opposite plane to both.

When the pebble is cut nearly, but not quite, perpendicular to its axis, coloured arcs of circles (incomplete rings) are seen; the light also cannot be blocked out, no matter what its position between the plates of tourmaline, because the axis cannot be made parallel to that of either the polariser or analyser. The intensities of the colours and the sizes of the arcs are both dependent on the nearness of the section of the pebble to that of right angles to the axis, i.e., on its nearness to axis-cut.

Advantage of Axis-cut Pebble.—Rock crystal which is *axis-cut* is preferable for lenses to that which is non-axis-cut, because in the former there is no double refraction; the light is propagated along its axis of crystallisation and is refracted in every azimuth symmetrically with respect to it.

Unannealed Glass.—Glass which is unannealed or has been subjected to pressure, strain or twisting, polarises light and therefore acts in the polariscope somewhat similarly to a pebble, in that light is transmitted and colours are seen.

CHAPTER XIV.

THE SIMPLEST PRINCIPLES OF SOME OPTICAL INSTRUMENTS.

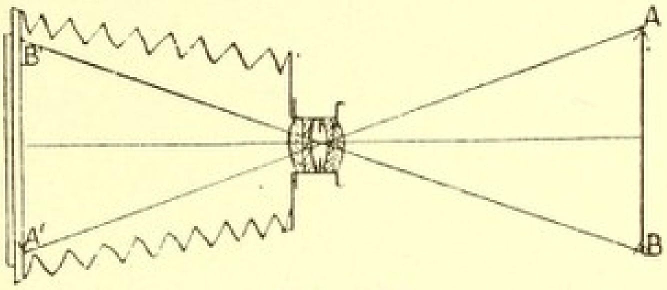


Fig. 277.

The Camera consists essentially of a light tight box, fitted at one end with a corrected convex lens and, at the other, with a ground glass screen on which a real image, B′ A′ of any object A B, is projected and rendered sharp by some form of a rack and pinion attachment. A glass plate coated with a film sensitive to light is substituted for the screen and the action of the light, reinforced by certain chemicals termed developers, forms a negative image of pure silver on the plate by the reduction of the silver compound contained in the gelatine emulsion. From the negative a positive image may be obtained by printing, or by exposing and developing a second plate behind the negative.

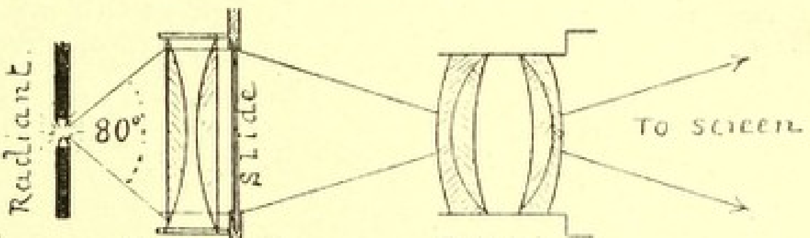


Fig. 278.

The Optical Lantern is an apparatus used for projecting on to a distant screen, a highly-magnified real image of a transparent slide upon which is depicted the subject to be exhibited. A very bright radiant source of light is placed behind a condenser which illuminates an inverted slide placed just beyond the principal focus of the compound objective. A real erect image of the slide is then thrown on to the screen. Any degree of magnification compatible with the intensity of the source and the distance between it and the screen can be obtained by racking the objective towards or away from the slide. Calculations involving the relative positions of slide, objective and screen are dealt with in the chapter on conjugate foci.

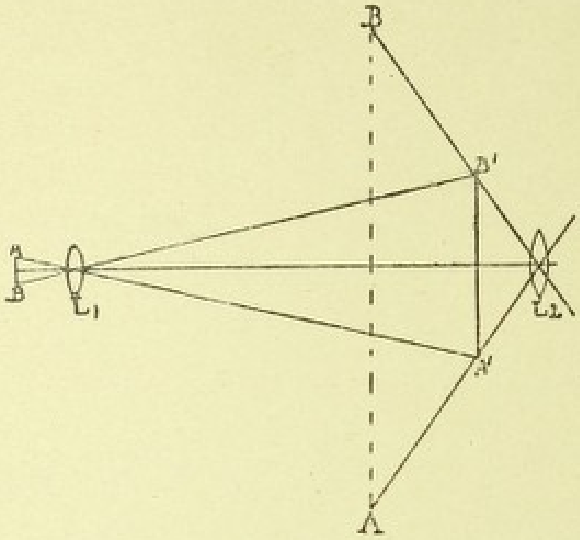


Fig. 279.

The Compound Microscope is used to obtain a magnified view of a small near object and consists essentially of two convex lenses separated by a distance dependent on the available length of the instrument, usually from 6 to 10in. the distance of most distinct vision of the observer being generally 10in. also. The first lens L_1, called the objective, is a short focus combination, highly corrected for aberrations, and the second, called the eye-piece, or ocular, L_2, is also a strong combination, but less so than the other.

A small object A B is placed just beyond F_1 of the objective L_1, so that the latter forms a real, inverted,, magnified image, B′ A′, of the object. This image is formed practically in the focal plane of the eye-piece L_2, and an eye placed behind the latter sees an enlarged virtual image B A of B′ A′ at the distance of most distinct vision. Hence there is magnification due to the objective and to the ocular and is very approximately expressed by

$$\frac{d_1 d_2}{F_1 F_2}.$$

Where d_1 is the distance of most distinct vision, d_2 is the distance from L_1 to L_2, and F_1 and F_2 are the equivalent focal lengths of L_1 and L_2 respectively.

The Opera-Glass consists of a convex lens, L_1 of focal length F_1, placed in front of a concave lens, L_2 of focal length F_2, of higher power, at a distance equal to the algebraical sum of their focal lengths so that they neutralise each other by separation. The principal focus of the convex is behind the concave as far as that of the latter is front of it. Although the rays of each pencil emerge parallel or divergent from a very distant point after refraction by both lenses, yet the pencils themselves are deviated so that an object appears under a larger angle. The axial rays from an object at ∞ subtends an angle a at L_1 and are refracted to form the larger visual angle a′, so that the magnification is in the ratio of a′ to a, which can be expressed as F_1/F_2. Thus, if F_1 = 5in. and

F_2 = 2in. the magnifying power of the opera-glass is $5/2 = 2\frac{1}{2}$. If the combination is reversed so that the concave lens is to the front, a object viewed appears diminished in size in the ratio of F_2/F_1, which, in the case given, is 2/5, or the diminution is $F_1/F_2 = 2\frac{1}{2}$ times.

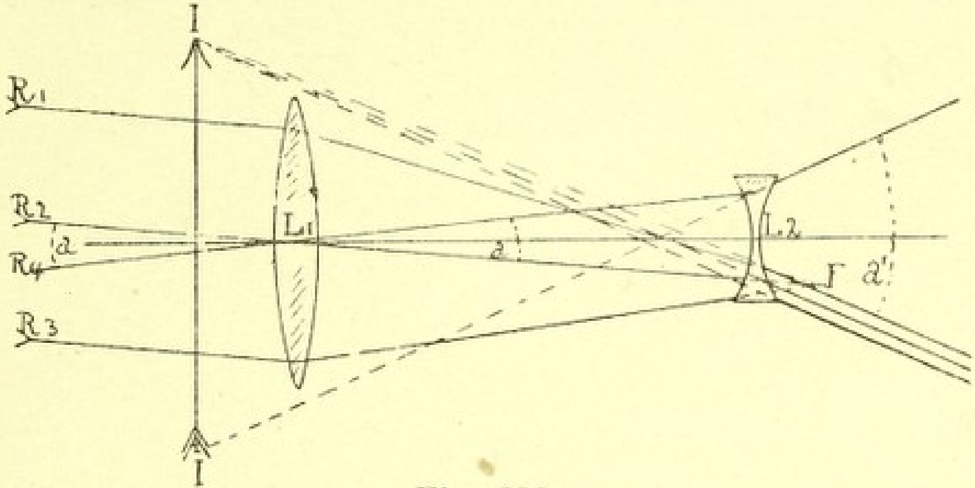


Fig. 280.

The emmetrope adjusts the glasses so that the separation is exactly $F_1 + F_2$. The hypermetrope, requiring convergent light for clear vision, makes the separation greater, while the myope, needing divergent light, makes it shorter.

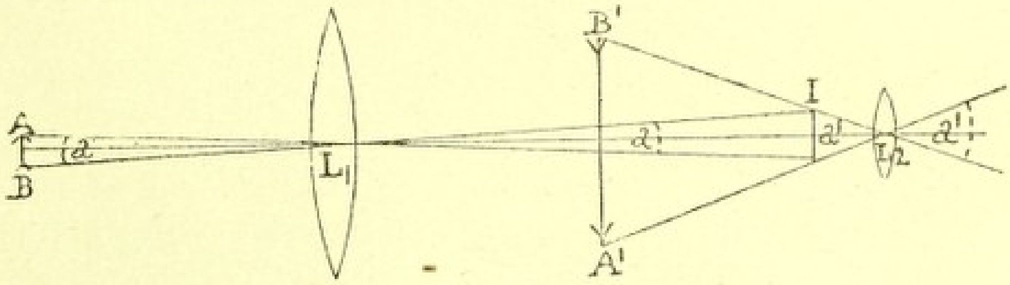


Fig. 281.

The Telescope is used for obtaining an enlarged view of distant objects and consists of an objective L_1 of long focus F_1 and an eye-piece L_2 of short focus F_2, both corrected for spherical and chromatic aberration.

The objective forms a real inverted image I of a distant object A B, subtending an angle a, and this image is viewed through the eye-piece L_2. For an emmetropic eye the distance between F_1 and F_2 is equal to the sum of their focal lengths, so that the light, after refraction by both lenses, may enter the eye in parallel or slightly divergent beams. The magnification depends on the ratio between the angle a, which the object subtends, and a′, which the final image B′ A′ subtends and this may be written F_1/F_2.

A hypermetrope would adjust the telescope so that the distance between the lenses is greater than $F_1 + F_2$, the light then entering the eye convergently, while the myope, in order to obtain divergent light, would make the interval less than $F_1 + F_2$.

The final image is inverted with respect to the object, and for terrestrial purposes this difficulty is overcome by means of an erecting eye-piece which, when suitably placed between the objective and eye-piece, causes a reinversion of the image. For astronomical purposes an erector is not needed since inversion of a heavenly body is of no importance, while, on the other hand, loss of light owing to increase of the refracting surfaces is avoided.

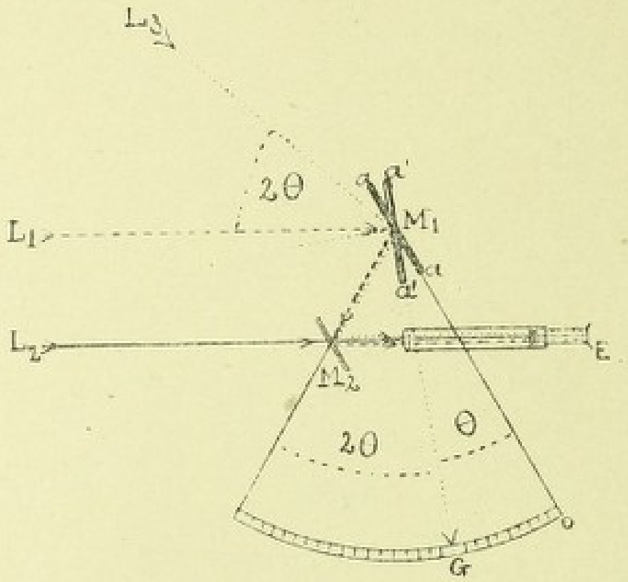


Fig. 282.

The Sextant is used to measure the angle subtended at the eye by the sun and the horizon, from which the position of the sun with respect to the earth can be calculated. It also serves to measure the angle between any two inaccessible objects.

A small mirror M_1 revolves about a horizontal axis to which is attached a pointer G moving over a scale of degrees. M_2 is a small fixed mirror of which one half is silvered and the other half is clear. It is so inclined that when M_1 and M_2 are parallel the pointer indicates zero on the scale. E is a small telescope so directed forwards that it receives at the same time light from the horizon by direct transmission through the clear part of M_2, and by reflection, from the silvered part of M_2, the light which has been reflected to M_2 from M_1.

Let L_3 be a ray emanating from the sun and L_2 a ray from the horizon. Then to an eye at E the image of the sun along the path L_3 will apparently coincide with the image of horizon seen directly along L_2. The angle which L_3 makes with L_1, which is parallel to L_2, is the angular distance between the sun and the horizon, and this is measured on the scale by the rotation of the indicator G from zero. From the laws of reflection the reflected ray L_3 is turned through twice the angle θ, through which the mirror M_1 is turned so that the scale, instead of being divided into degrees, is divided into half-degree spaces, which, however, are numbered as if they were whole degrees so as to get direct readings from the scale to which also a vernier is attached for greater accuracy of reading.

The Theodolite consists of a telescope capable of rotation in the vertical meridian mounted on a plate capable of rotation in the horizontal meridian, fixed to a rigid tripod. Both the vertical arc and the horizontal circle are scaled in degrees which can be read to seconds of arc by means of verniers.

The theodolite is employed in surveying, for measuring the angular distance between points in the horizon or field, and to find the angular interval between objects at different heights. Also sometimes for measuring the altitude of heavenly bodies for finding local time.

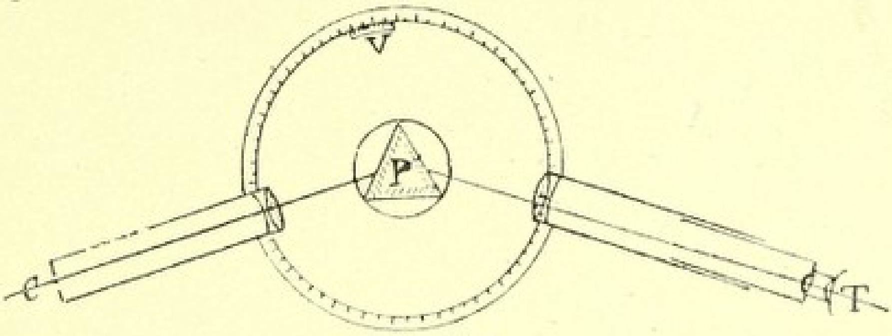


Fig. 283.

The Spectroscope is used for viewing and comparing spectra produced by prisms. The collimator C consists of a tube having at one end a convex lens and at the other a narrow slit parallel to the refracting edge of the prism P. The distance between the slit and the lens is equal to the focal length of the latter so that light, entering the slit, is rendered parallel by the lens before reaching the prism. The light being dispersed by the prism is then received by the objective ef the telescope T. a real spectrum being formed in the focal plane of the eye-piece through which it is viewed. The lines of the spectrum, produced from a given source, can thus be studied and, if necessary, two spectra, from two given sources can, by suitable arrangement, be formed side by side for comparison. Both telescope and collimator can be rotated around a vertical axis beneath the prism P. For spectrum analysis, the prism must be placed in the position of minimum deviation of the line D, so that the different colours may be separated most nearly in the order of their wave lengths.

The Spectrometer is similar in construction to the spectroscope, with the addition of a horizontal scale of degrees on which the position of the moveable telescope can be measured. The deviation of a prism, for any given colour, can be determined by first viewing the slit direct, with the prism removed, and then its image, after refraction by the prism. The difference between the two positions of the telescope gives the angular deviation—minimum or otherwise. The refracting angle of a prism can also be measured by the spectrometer by turning the prism until its edge splits into two halves the beam of light issuing from the collimator. The telescope is rotated until the image of the slit is seen reflected from the one surface and then turned to receive the image from the other surface of the prism. Half the angle through which the telescope has been rotated to view the two images gives the refracting angle of the prism. If the refracting angle and the deviating angle are known, the refractive index of the glass, of which the prism is made, can be calculated.

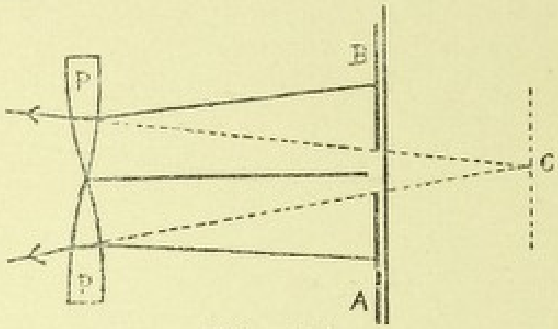


Fig. 284.

The Stereoscope is an instrument used to artificially produce the appearance of solidity and perspective which ordinarily results from binocular vision. It comprises a frame or box fitted at one end with two sphero-prisms P P (which are the transposed halves of one large convex lens), bases out, and at the other with a slide carrying two photographs A and B. The latter are taken from slightly different stand-points, usually about 2½ inches apart, this amount varying with the degree of stereoscopic effect required. The photographs, therefore, are slightly dissimilar, the right A being the view as seen by the right eye, and the left B that as seen by the left eye. The lenses P P cause the apparent position of A and B to be displaced inward, with the result that, without exerting any muscular action, the two are fused to form the single image C, the latter being virtual and situated near the point of most distinct vision of the observer. Thus an effect of solidity and depth is given to the otherwise flat pictures, since each eye obtains a view practically identical with that which they would obtain in viewing directly the object.

The distance between P P and A and B is equal to F of the spherical lenses, so that an emmetrope uses neither accommodation nor convergence. If there is Am. the distance P A, P B has to be altered, and if there is Heterophoria, muscular action results.

The Vernier is an attachment to all instruments where great precision of linear or angular measurement is required, and it obviates the necessity of the division of the main scale into very minute parts. It consists of a short scale which slides along the main scale to which it is attached. Let V be the vernier and S the scale in the following description.

The V occupies the same space as a definite number of divisions on the S, but contains one division more. Thus, if V is divided into 10 parts, these equal nine divisions of S, or if V has 30 divisions they correspond to 29 of S. Then each division of V is smaller than each division of S by a fraction whose denominator is the number of divisions of V, viz., 1/10th or 1/30th less respectively in the examples just quoted. The greater the number of divisions of V, the greater is the accuracy of the reading, but the greater also is the difficulty of its use.

The scale itself may be divided into whole terms of measurements, as mm. or degrees, or more commonly into main fractions of such terms as $\frac{1}{2}$ mm. or $\frac{1}{2}$ degrees. Such whole terms, or main fractions thereof, are read from the S itself, the measurement being the last beyond which the zero of the V has passed.

Then the minute measurement is obtained from the V by finding that division mark of the V corresponding to, or is in exact line with, a division mark of S. Thus, if 10 V = 9 S and the third division mark of V is in line with a division of S the exact measurement is 3/10 more than the whole number indicated by S itself. If V has 60 parts and the 33rd is in line with an S division, the fractional reading is $\frac{33}{60}$ plus the whole divisions indicated on S.

The following figure illustrates a reading on a scale A directly divided to inches and tenths of inches with a vernier B whose 10 parts = 9 parts of the scale, as shown in the left-hand side of the figure. The length of an object X whose one extremity is at 1 is .65in., the fifth division of the vernier coinciding with a division of the scale. The limit of accuracy is 1/100in.

In this example .60in. is read from the scale itself, where the right-hand extremity of X lies between the 6th and 7th division of S; the balance .05in. is read from the V.

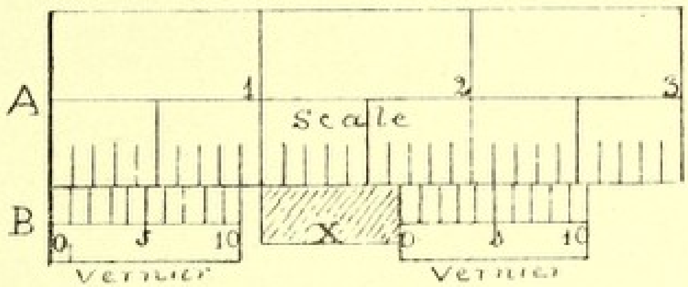


Fig. 285.

As another example, let the scale be divided to inches and tenths of inches and 25 V = 24 S. If then the V zero showed 5in. and six spaces of 1/10in. plus a certain distance when the fourth division of V is in line with a scale mark, the total measurement would be $5 + \frac{6}{10} + \frac{4}{250}$ or 5.616in. The accuracy of the reading is carried to $\frac{1}{250}$in.

Instruments have been made with the V divisions longer than those of S, so that, say, 9 V = 10 S. The V divisions were then on the near side of the zero and were read backwards.

Verniers for fine straight rules are usually made so that 10 V = 9 S, thus measuring to 1/10 mm. For box sextants and small surveying instruments 30 V = 29 S, so that $\frac{1}{2}$° divisions are subdivided to minutes.

For barometers the readings are usually taken to 1/10mm. when 10 V = 9 S, or to 1/250in. when 25 V = 24 S. For marine sextants and theodolites 60 V = 59 S, measurements being taken to 1/10 of 20′ or of 10′, which equal, respectively, 20″ or 10″ in the case of these two instruments.

APPENDIX.

THE MOST ELEMENTARY PRINCIPLES OF SOME SIGHT-TESTING ADJUNCTS.

Fig. 286.

Test Types.—The average healthy emmetropic eye can recognise an object which subtends, at the nodal point of the eye, an angle of 5′ and can visually separate two points, the distance between which similarly subtend an angle of 1′. Test types for distance are based on this principle and copy Snellen's types, which consist of a series of block letters each one of which, when placed at the distance for which it is made, subtends this angle of 5′, while its limbs, which are, in diameter, $\frac{1}{5}$ of the whole letter, subtend the needed angle of 1′. The types are made for distances varying between 60 M or 200ft. and 2 M or 6.6ft. They are employed for determining the visual acuity and, in conjunction with test lenses, for determining the refractive condition of the eye.

For near vision it is not usual to employ block letters, because these are difficult to make sufficiently small and because people are more accustomed to ordinary types. Consequently for this purpose a series of printer's ordinary types, graduated in size as arranged by Jaeger, is commonly used. No. 1 is the smallest and No. 10 or 20 is the largest; they are presumed to subtend, more or less, similar angles to the block letters of Snellen.

The Astigmatic Chart consists, in its usual forms, of either a fan of radiating lines or of a series of lines running at right angles to each other and which can be revolved so as to bring the two sets into those positions which correspond to the two principal meridians of an astigmatic eye. An eye which is not astigmatic sees lines running in any direction, with equal degrees of clearness, whereas an astigmatic eye does not. If the two sets of lines correspond to the two principal meridians of an astigmatic eye, the one set is seen clearly defined, while the other is seen very blurred. This results

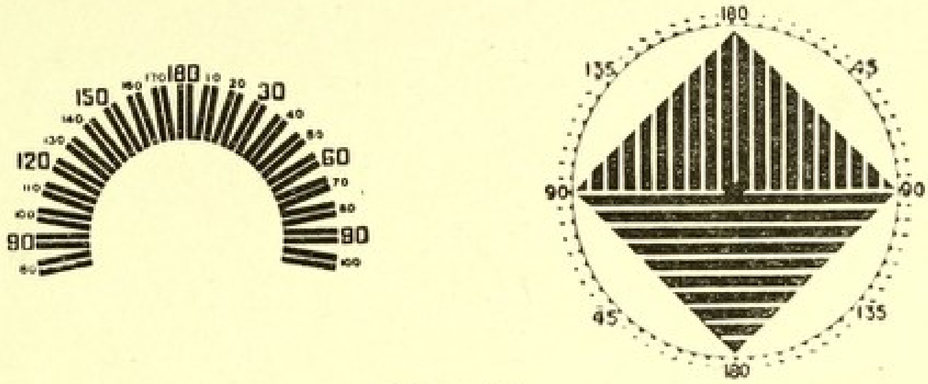


Fig. 287.

because the ellipses of confusion in an astigmatic eye correspond in direction with the image of the lines seen clearly, and cross at right angles the image of those in the opposite meridian.

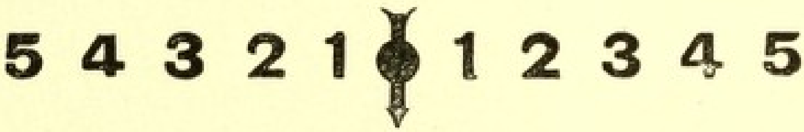


Fig. 288.

The Muscle Chart serves for determining the degree of imbalance of the motor muscles. It usually consists of a series of numbers running each way from a central zero pointer and coloured red on the one side and green on the other. If a vertical diplopia is produced by a vertical prism, the scale is seen double and the horizontal deviation of the eye is indicated by the relative positions of the pointer of the one scale and the red or green figures of the other. More commonly and preferably a flame is placed at the centre of the scale and this, when seen through a Maddox groove or rod placed before the one eye, appears as a streak of light whose position is indicated by the number on the scale, seen by the other eye, over which its image lies.

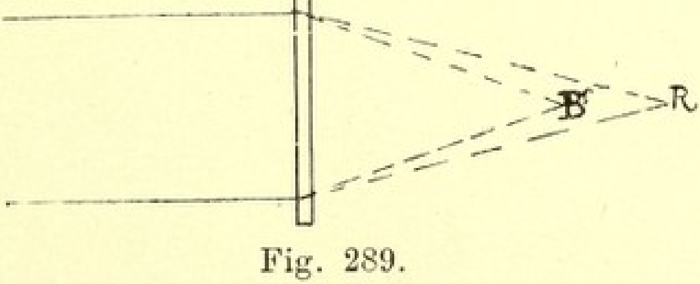


Fig. 289.

The Chromatic Disc consists of a very deep cobalt blue glass, which, excluding the central portion of the spectrum, transmits only red and blue light. The normal or emmetropic eye sees a light, through the disc, as purple in colour. The short hypermetropic eye sees a blue centre with red borders, while the long myopic eye sees a red centre with blue border. The astigmatic eye sees ellipses or streaks of light.

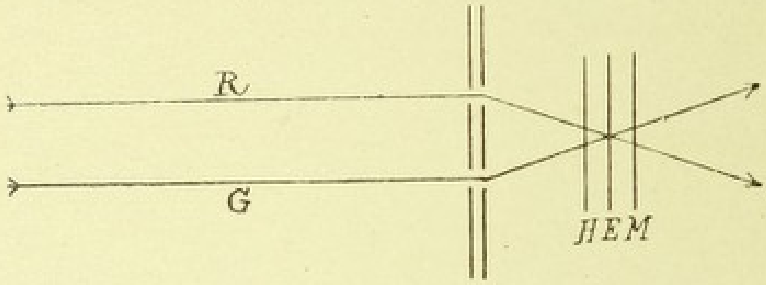


Fig. 290.

The Scheiner Disc is an opaque disc having two small perforations separated by a distance shorter than the diameter of the pupil. The one is covered by a red and the other by a green glass. On viewing a flame, through the disc, the emmetrope sees one flame, the hypermetrope sees two images which are crossed, and the myope sees two images which are uncrossed. This occurs because in hypermetropia the two beams of light, entering the eye, reach the retina before crossing each other and in myopia after doing so.

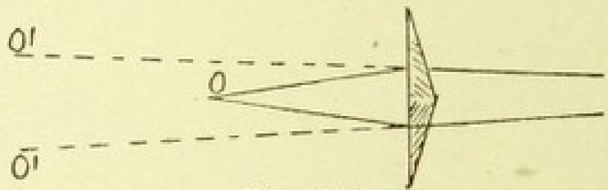


Fig. 291.

The Double Prism consists of two weak prisms joined base to base so that when an object O is seen through it, two images O′ O′ are seen by the eye before which it is placed. The other eye sees a single image of the same object and the relative position of the single and double images serves to indicate muscular imbalance.

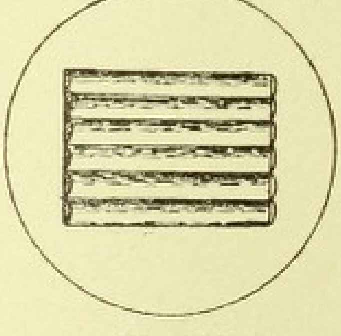


Fig. 292.

The Maddox Groove consists of a disc having one or more deep grooves constituting very strong concave cylindricals.

The Maddox Rod is similar except that the cylindricals are convex. A flame viewed through either is seen as a streak of light, whose position relative to the naked flame, seen by the other eye, indicates the nature and degree of muscular insufficiency.

These appliances, invented by Dr. Ernest E. Maddox, are the best and simplest of tests for muscular imbalance.

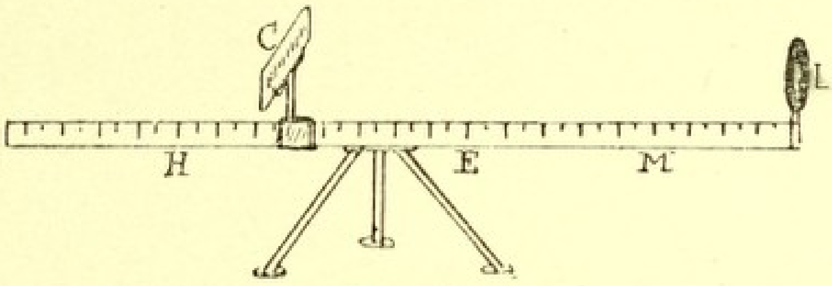


Fig. 293.

The Optometer is an instrument by means of which the refractive condition, the far and near points, and therefore also the range of accommodation, can be measured. Their designs are very numerous, ranging from the simplest form, as in the "Orthos" pointer, to a very complicated and scientific instrument. The employment of complicated instruments of this nature is, however, now very little in vogue for the reason that the sense of nearness and therefore the unconscious exertion of accommodation can never be eliminated when they are used. It is realised that the best form of optometer is that in which the far point is at ∞ in actuality, as is the case when the P R is at the far end of a 20 foot room—that is to say, when test types are used at this distance or near to it.

Optometers, as such, may be classified as based on (1) the employment of a single Cx. lens so as to bring the P. R. to a measurable distance; (2) the employment of two Cx. lenses separated by a distance equal to $F_1 + F_2$, i.e., a telescopic arrangement; (3) the employment of a Cx. and a Cc. lens separated also by a distance equal to $F_1 + F_2$, i.e., a Gallilean telescope or opera-glass combination; (4) the employment of a disc having two small perforations, like the Scheiner disc (*q.v.*) or two slits as in the Young optometer; (5) the employment of a cobalt blue glass admitting only red and blue light, this is termed the Chromatic disc (q v). The common form is that of No. 1. The + lens L makes the eye myopic and the nearest and most distant points of vision are determined by sliding C, the carrier of the types along a graduated scale, to the nearest and most distant points at which this type can be read. The furthest point determines the P R and the (manifest) refractive condition of the eye as indicated by the scale of the instrument. The nearest point of vision gives the P P, and the distance between them in diopters is the amplitude of accommodation. If the eye is emmetropic the P R is at F of the lens of the instrument; if myopic it is nearer than F, and if hypermetropic it is beyond **F**. If an eye is very myopic no Cx. lens is needed since the P R is already at a measurable distance.

For determining and measuring astigmatism by an optometer a small fan of radiating lines is employed with some forms, and a stenopaeic slit with others of those mentioned under the various classifications.

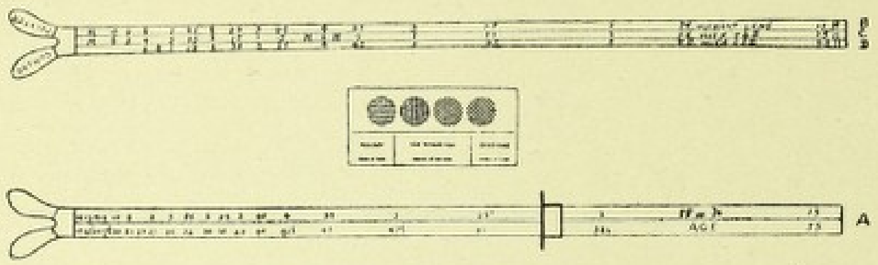


Fig. 294.

The "Orthos" Pointer is scaled on the one side for the near point and gives the estimated minimum amplitudes of accommodation for ages varying between 10 and 55. On the other side there are scales for determining the far point without any lens for high myopia, with a +4D lens for low myopia, emmetropia and low hypermetropia, and with a +8D lens for high hypermetropia; such lenses being placed in an ordinary trial frame. The pointer in conjunction with distance types constitutes, owing to its simplicity, almost the ideal optometer.

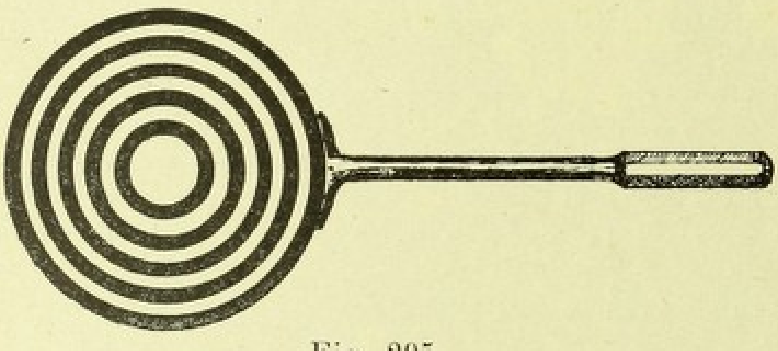


Fig. 295.

The Placido Disc or Keratoscope consists of a disc having alternate black and white rings. In the centre there is an aperture, through which the observed eye is seen, with a convex lens so that the image is magnified. The image seen is that of the white rings of the disc reflected from the cornea, which acts as a Cx. mirror. The placido disc serves for determining corneal astigmatism and its principal meridians. If the cornea is truly spherical, the image of the white rings is quite circular, while if the cornea is astigmatic, i.e., of toroidal curvature, the image is elliptical. The long axis of the ellipse is in the meridian of least refraction and *vice versa*.

High degrees of corneal astigmatism are at once apparent, but it requires some practice to appreciate low degrees, also to determine the angular position of the long and short axes of the elliptical image.

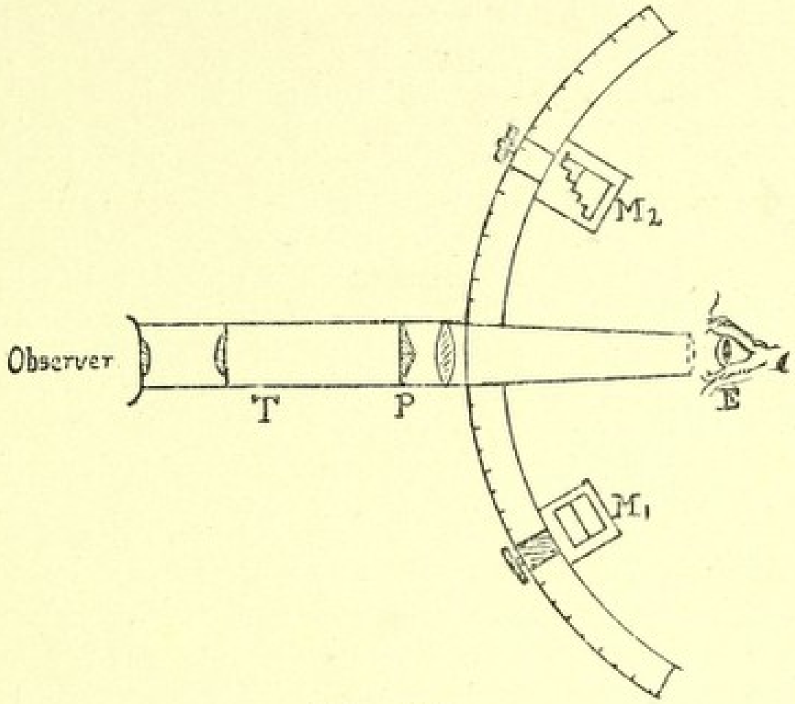


Fig. 296.

The Ophthalmometer is an instrument designed for the measurement of the corneal radius of curvature and, therefore, also of any existent corneal astigmatism resulting from a difference of the curvature of the two principal meridians of the cornea.

The cornea constitutes a Cx. mirror, and from the size of the image formed by it, of an object of known size, its radius and dioptic power can be calculated.

The typical instrument consists of a telescope T with a prismatic arrangement P for doubling the image seen, the observed eye E being at the one conjugate and the observing eye at the other of the lens system of this telescope. The object, whose image is observed, consists of two targets M_1 and M_2 (called mires), which are of different shapes, the one being usually a square and the other a series of steps. These mires are strongly illuminated and are carried on an arc so that they can be brought nearer together or further apart. The arc with the mires can be rotated so as to lie in any meridian. A chin-rest, adjustable as to height, enables the observed eye to be placed in the field of the telescope and in the centre of curvature of the arc which bears the mires. The telescope being focussed on to the eye there is seen reflected from the cornea a doubled image of each of the mires, the two central ones being those of interest. The two images are on the same level, as indicated by continuity of their central lines, only when the position of the arc is such that it corresponds to one of the principal meridians of the eye; in any other position the two images are noticeably oblique to each other.

The doubling, by the prismatic arrangement of the instrument, causes the two images to be crossed. The images are larger as the radius of curvature of the cornea is long, or its dioptric power is low and are smaller as the radius is short or the power high. Although we speak of the two images, actually *the* image is that of both, the distance between them being greater as the cornea is less curved and *vice versa*. Now, if the amount of doubling is exactly equal to the whole image it would follow that the two central parts of that image are in contact. On observing the eye and obtaining a clear view of the images, the telescope is rotated and if the distance between the two central parts, or the amount of overlapping, remains constant in all meridians there is no corneal astigmatism. If the relative position varies, the cornea is astigmatic, and its degree is obtained thus. In the meridian of greatest separation, or least overlapping, the mires are adjusted so that the two central images appear in exact contact. Then the arc being rotated 90° the amount of overlapping is read by the steps of the one mire, each step of overlapping being equal to 1 D of astigmatism or of difference in the refractive power of the eye in its two principal meridians. The first position—that of contact—is the meridian of least refraction and the opposite one that of greatest refraction.

The curvature of the cornea or its dioptric power can be read from a scale attached to the instrument and the numerical position of the principal meridians are indicated by a protractor.

While in some instruments the doubling is a fixed quantity and the size of the object is altered by moving the mires, in others the object is fixed and the doubling is varied. Also the doubling has been obtained in both principal meridians at the same time, thus obviating the necessity of rotating the arc after the one principal meridian has been located. This is found in the latest type of instrument, the "Sutcliffe."

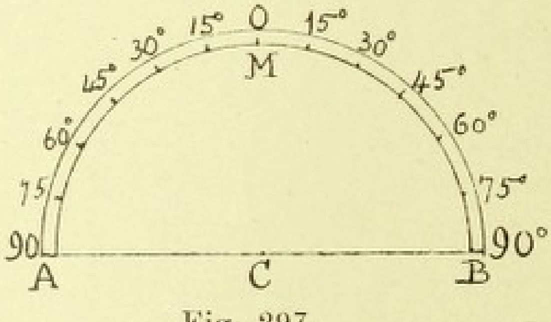


Fig. 297.

The Perimeter.—This consists of a bar of metal A M B forming a half-circle, graduated in degrees from zero to 90° at each extremity. This arc is fixed at its centre M so that it can be revolved into any meridian. The instrument is used for measuring the fields of vision and fixation and for locating and mapping out blind patches in the field; also for determining the angular deviation of a squinting eye. For measuring the field of vision, the eye under test is placed at C, which is the centre of the circle of which A M B is an arc, and an object placed in line with C M is observed. A small white disc—or one of a given colour as may be needed—is moved inwards from 90°, on both sides alternately, towards 0, while the arc is in the horizontal meridian, and where the disc first comes into view indicates the angular limits of the visual field in the horizontal plane. The field is similarly measured in the vertical meridian and at least two intermediate ones, the angular limits being marked on a specially prepared chart. For locating blind patches the points, in various meridians, where the observation disc comes into, and disappears from, view are located and mapped out on the chart. For determining the angular deviation of a squinting eye, the position on the arc is noted at which a small flame, carried along the arc, has its image in the centre of the pupil, such position indicating the angular deviation sought for.

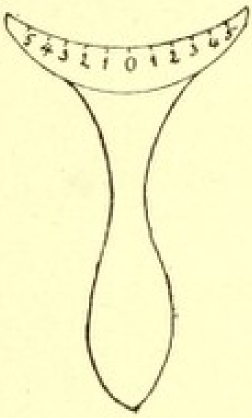


Fig. 298.

The Strabismometer.—This is an instrument which when placed beneath a squinting eye indicates the degree of deviation. When the good eye is occluded and the squinting eye fixes an object in the centre of the field, the centre of the eye is above zero. When the good eye is uncovered and fixes the same object, the centre of the deviating eye is above some other point of the instrument, which point indicates in mm. or degrees the extent of deviation.

The Phorometer or rotary prism serves for measuring the strength of the motor muscles or the degree of imbalance of the muscles. It may be made so that there is a prism before each eye or there may be a single revolving prism or a combination of prisms before the one eye. The principle involved is that two similar prisms in opposition have no prism effect and that when revolved in opposite directions their effect varies, also that a single prism revolved in front of the eye has, as it is thus revolved, a varying effect in the vertical and horizontal meridians. The variation in effect has been fully treated in the chapter on prisms. The most usual form of phorometer, to which the name is generally applied, consisting of a 5° prism before each eye arrange to revolve in opposite directions. When the one is base up and the other base down a vertical diplopia is produced and the degree of horizontal imbalance is indicated on the scale, attached to the instrument, when by revolving the prisms the two images are brought into the same vertical plane. If the base-apex lines are horizontal, a horizontal diplopia is produced and any vertical imbalance is indicated by the amount of rotation of the prisms necessary to bring the two images into the same horizontal plane.

The Mirror is the name given to either the ophthalmoscope or retinoscope and consists of a concave or plane mirror, having in its centre an aperture. Light from a source placed above, or to the side of, the client's head is received by the mirror and reflected therefrom into the observed eye thus illuminating the fundus, as the interior of the back of the eye is called. The light is then reflected from the illuminated fundus and emerges from the eye in parallel beams in the case of emmetropia, while the beams are divergent if the observed eye is hypermetropic and convergent if it is myopic. Some of the emergent light, passing through the aperture of the mirror, is received by the observer's eye so that the details, or at least the general red appearance, of the fundus can be seen, this not being possible, under ordinary conditions, without the aid of the mirror.

The Ophthalmoscope serves for viewing the interior of the eye and determining its refractive condition. There are two methods of Ophthalmoscopy, viz., the *indirect* and the *direct* methods.

For the *indirect* method a large concave mirror of about 10in. focus is held about 20in. from the eye, in front of which a 3in. magnifying-glass is placed. Light from a source placed behind, and to one side of, the client's head after reflection from the mirror crosses and diverges from about 8 or 10in. to the condenser. By the latter it is rendered very convergent, and on entering the eye is brought to a focus in the vitreous, from which focus it again diverges and illuminates a large area of the fundus.

The light reflected from the fundus, on emergence from the eye, is brought to a focus by the condenser and forms an image in the air in front of the lens. The image is real, i.e., it can be formed on a screen, and it is beyond, at, or within F of the condenser according to whether the light emerges from an eye which is respectively hypermetropic, emmetropic, or myopic. This real aerial image of the fundus, as seen by the observer, is inverted and magnified about five times.

For the *direct* method, a smaller and deeply-concaved mirror is generally used; it is tilted so as to reflect light from a source at the side of the client's head directly into the observed eye. The mirror is held as close as possible to the observed eye and the light is brought to a focus in the vitreous, whence diverging it illuminates a portion of the fundus.

The light reflected from the illuminated fundus is brought to a focus at the retina of the observer's eye, placed behind the mirror's aperture by means of a suitable lens which is rotated into position behind the aperture and so between the observed and the observer's eyes. The image seen is virtual and erect and, being projected to the observer's distance of most distinct vision, is magnified some sixteen times.

The Retinoscope is employed for determining the refraction of the eye by what may be appropriately termed neutralisation. Light from a source conveniently placed is reflected from the mirror into the eye so that the fundus, being illuminated, can be seen by the observer through the aperture of the mirror. The *original* source is that from which the light is received, and the *actual* source is that formed by the mirror. The observer is usually one metre from the observed eye. If the light is sent into the eye in a direction parallel to the optic axis the central area of the fundus is illuminated; if it is sent in obliquely, the one side of the fundus is illuminated and appears red, while the other part is not illuminated and appears black. Therefore, if the light is first direct and then gradually made oblique by tilting the mirror, the respectively illuminated and non-illuminated area of the fundus appear to move, and this phenomenon is commonly referred to as *movement of the shadow.*

If the mirror is concave, it is generally of 10in.F and the *actual* source of light is the image formed by it in the air. From this image the light, after crossing, diverges to the eye, and if the mirror is tilted this image (the actual source) is moved in the same direction as the tilting, but since the light crosses at the image, the effective part of the total beam illuminates only the opposite side of the fundus. Then, if the tilting is continued the *shadow movement* is contrary to that of the mirror itself. This always occurs, and it is so seen provided the light from the observed eye enters the eye of the observer before it again crosses, otherwise the movement appears to be reversed. Consequently the shadow movement is contrary to that of the mirror in H, Em, and low M, while it appears to be with that of the mirror in higher M.

With the plane mirror, the *actual* source is the virtual image, of the original source, formed as far behind the mirror as the latter is in front of it. On tilting the mirror the actual source moves in a contrary direction, thus illuminating one side of the fundus so that the shadow movement is the same as that of the mirror. This is true in all refractive conditions of the observed eye, but if, as in higher M, the light comes to a focus before entering the observer's eye, the shadow movement appears to be in the contrary direction.

With the plane mirror, the crossing of light, before incidence at the observed eye, is avoided and the fundus illumination is more brilliant. These advantages are enhanced by the use of the "Orthops" retinoscope, which, being of 1¼ M focal length, causes, when used at 1 M, shadow movements similar to those of the plane mirror but with more brilliant illumination and clearer definition of the boundary between the illuminated and non-illuminated fundus areas.

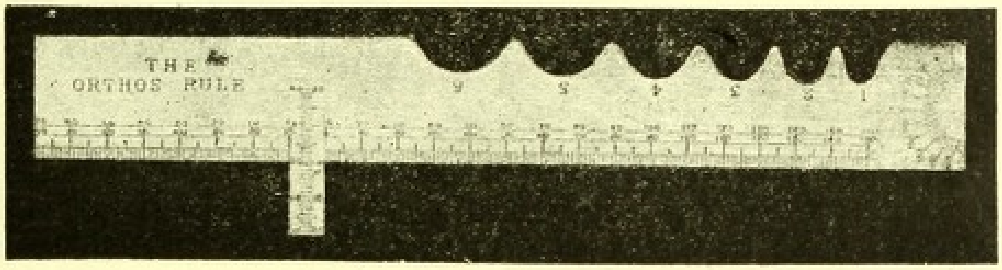
The principle involved in retinoscopy is to make the observed eye, by means of lenses placed in front of it, so myopic that the far point is at the observer's eye. If the observer's position is 1 M, the observed eye is made M 1 D, and the true refractive condition is calculated from the addition of − 1 D sph. to the lenses which produced this artificial condition, or which, in other words, brought the *reversal* point to 1 M.

FACE AND FRAME MEASUREMENTS.

WITH THE "ORTHOS" RULE.

In the interpretation of the following notes it is necessary to remember that right and left of the rule are respectively to the right and left of the Optician. But right and left of the face or frame measured, are respectively to the Optician's opposite hand.

There are two horizontal scales on the rule. The lower scale has its zero at the right side of the projection and is used chiefly for face measurements. The upper scale has its zero to the left of the projection and serves mainly for frame measurements. There are also two vertical scales; the one having its zero at the end of the projection and the other in the middle reading upwards and downwards.



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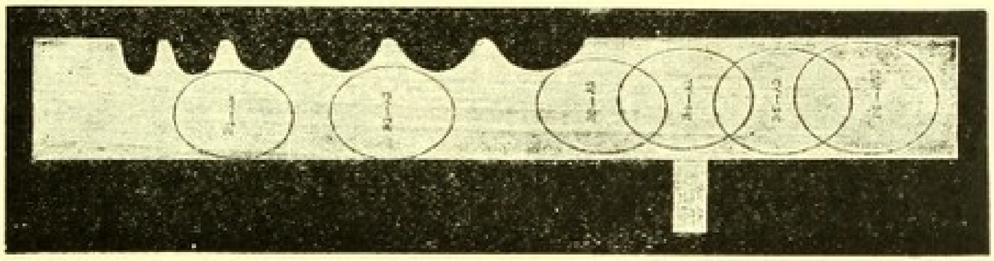

B

Fig. 299.

FACE MEASUREMENTS.

Interpupillary Distance for Distance and Constant Wear.

Being face to face the client looks straight into the Optician's left eye, the rule is held horizontally with the projection pointing downwards and with the right edge of the projection bisecting the right pupil, the distance on the lower scale to the centre of the nose is read, and being doubled gives the interpupillary distance. Or when the centre of the right pupil is located, if the client be directed to look straight into the Optician's left eye, the reading on the scale above the centre of the client's left pupil gives the true interpupillary distance. The full distance between the centres of the pupils when the eyes are directed to ∞ is the distance required. The average interpupillary distance is from 60 to 63 mm.

Unequal Distances of the Two Eyes from the Centre of the Nose.

The distance of the right eye is read as above. For the left, the client looks straight into the Optician's right eye; the right edge of the projection is placed to bisect the left pupil, and a reading is taken to the centre of the nose.

Interpupillary Distance for Presbyopia.

The client looks, with both eyes, at the centre of the Optician's face and the rule is placed and the reading taken as for distance. This is usually about 3 or 4 mm. less than for distance.

Height of Bridge.

The client looks straight in front, the rule held horizontally, with the lower edge resting on the bridge of the nose, and the height is read on the projection downwards to the centre of the pupil for *distance only;* to the bottom of the pupil for *constant wear;* to the bottom of the iris for presbyopia. The height for presbyopia is about 3 or 4 mm. greater than for distance.

Unequal Heights of the Two Eyes.

For unequal heights of the two eyes the distance is read for each separately.

Bridge Below Centres.

The rule is held horizontally with its lower edge bisecting the pupils horizontally, and with the projection pointing downwards at the right side of the nose; the distance is read downwards on the vertical scale to a point level with the bridge of the nose.

Set or Projection of Bridge Out.

The Optician stands to the left of the client's head; the rule is held horizontally, with the lower edge resting against the bridge of the nose and with the projection pointing inward at the left temple; the distance is read on the vertical scale backwards to a point level with the tips of the lashes. A similar measurement should be made for the other side and the frame selected for the more prominent eye, if there be any difference between the two. The lenses should be as near as possible to the tips of the lashes say, 13 mm from cornea. For presbyopia they may be about 25 mm. out, the bridge resting lower on the nose and projection need not be taken. C bridges are practically always 3 mm. out.

Set or Projection of Bridge In.

In the same position as last, the projection is placed against the bridge of the nose and the distance is read on the vertical scale outwards to a point level with the tips of the lashes.

Depth of Bridge.

The rule is held horizontally with the projection at the side of the nose and the distance is read on the vertical scale outwards to a point level with the bridge of the nose.

Facial Width.

The rule is held horizontally with the projection pointing inwards near the left temple, and the distance is read on the upper scale to the middle of nose—this distance being doubled gives the facial width. Facial width is usually about 45 mm. greater than full interpupillary distance.

Width of Nose for Spectacle Bridge or for Distances Between Placquets.

The rule is held horizontally with the right edge of the projection pointing downwards at the right side of the nose, and the distance on the lower scale is read to the middle of the nose—this distance being doubled gives the width. Or the size of bridge can be given from the standard spreads of the rule. For distance and constant use that spread should be selected which fits the thin part of the nose and will not pass over the wider part of the bridge. For presbyopia that spread should be selected which will just freely slide over the wide part of the bridge of the nose.

Curl Side.

The projection is placed at the back of the ear, and the distance on the lower scale is read to a point level with the tips of the eyelashes. The total length of the side is about 50 mm. greater.

Straight Side.

The right edge of the projection is placed at the side of the head where the extremity of the temple should be and the distance is read on the lower scale to a point level with the tips of the eyelashes. The total length is about 50 mm. greater than that from the tips of the lashes to the front of ear.

FRAME MEASUREMENTS.
SPECTACLES.

Height of Bridge.

The frame is placed so that the lower edge of the rule bisects the joints. The height "up" is read on the vertical scale above the edge. Height "down" is read below the edge. The reading is made at the middle of the bridge wire, it being the No. covered by the wire.

Set or Projection of Bridge.

The frame is placed with the backs of the "eyes" against the lower edge of the rule and the bridge astride the projection. "Set-out" is read on the vertical scale below the edge. "Set-back" is read above the edge. The reading is made at the middle of the bridge wire.

Width of Base of Bridge.

The bridge is placed with the inside of the bridge at the left edge of the projection and the distance is read on the upper scale. The reading is made at the narrowest point of contact with the nose.

Interpupillary Distance.

The frame is placed with the inner side of one eye at the left edge of the projection and the distance is read on the upper scale to the corresponding inner extremity of the other eye.

Width of Front.

The frame is placed with one pin at the left edge of the projection and the distance is read on the upper scale to the other pin.

Length of Straight Side, or Total Length of Curl Side.

The frame is placed with one joint at the left edge of the projection and the distance is read on the upper scale to its extremity when flattened.

Length of Curl Side to End of Curl.

The frame is placed with the joint near the end of the projection at the left and the distance is read on the upper scale.

Angle of Joint.

The frame is placed with the *left* eyewire against the right end of the rule with the joint at the centre of the protractor, on which the angle, between the eyes and sides, is then read,

Curve of Straight Side.

The frame is placed with the joint to the left of the projection, and the extremity of the side to the right, both at the lower edge of the rule; the greatest depth of curve, or the sagitta, is then read on the vertical scale.

EYEGLASSES.

Interpupillary Distance.

The frame being opened as when in use, this distance is taken as on a spectacle frame.

Distance Between Placquets.

The frame is placed with the inside of one placquet at the left edge of the projection and the distance is read on the upper scale to the inside of the other placquet.

For oscillating placquets the distance between them is taken at the centre.

For fixed straight placquets the distance is taken at the top and bottom.

For fixed curved placquets the distance is taken at the top, centre and bottom.

These measures are taken when the frame is at rest and not in use.

The lengths of the placquets should also be taken when necessary.

Unequally Placed Placquets.

When the two placquets differ, the distances of each from the median line should be taken separately. The rule is turned so that the frame can be placed across the vertical scale with the middle of the spring or bar bisected by its zero line; the distance is then read on each side from this central line.

BRIDGES OR SPRINGS.

The length is taken on the upper scale between the inside extreme points measured horizontally.

USE OF THE "ORTHOS" RULE WITH LENSES.

To Locate the Axis of a Cylindrical.

Holding the lens and the neutralising cylindrical firmly together and geometrically centered, they are placed against the end of the rule so that its edge exactly coincides with the long diameter of the lens, and its middle point with the centre of the lens. The position of the axis is then indicated on the protractor by the small scratch on the neutralising cylindrical. The lens should not be laid flat, but held at a small angle with the rule, and the outer face of the lens must be upwards.

To Locate the Optical Centre.

The lens is held in a plane parallel to that of the rule, and a few inches above it, over the intersection of the vertical and horizontal scales. The optical centre is opposite to the intersection of the horizontal and vertical edges when these edges are seen continuous within and beyond the periphery of the lens.

To Measure Decentration.

The geometrical centre is marked with a dot of ink, and the optical centre being located as above, the distance on the horizontal scale, between the optical and geometrical centres, is read—this gives the horizontal decentration; or the decentration vertically is read on the vertical scale.

To Measure a Prism.

The rule is placed 1 m. distant from the prism which is to be measured, the base of which is to the left opposite to the projection. The base apex line must be exactly parallel to the horizontal edge of the rules. This edge being viewed through the prism, the projection is seen deflected to the right, and its right edge indicates on the lower scale seen above the prism the amount of displacement in cm. and fractions thereof. The number of cm. indicated equals the number of prism diopters. Care must be taken that the displacement is minimum.

To Calculate Resultant Prisms.

The oblique distance from a point on the horizontal edge, as many cm. from 0 as there are units of horizontal prismatic power, to a point on the vertical edge, as many cm. from 0 as there are units of vertical prismatic power is measured. This distance in cm. and fractions thereof equals the number of units of strength and fractions thereof in the resultant prism; the angle which the oblique line forms with the horizontal is the angle which the base apex line of the resultant prisms also forms with the horizontal.

LENSES.

Bifocal Lenses.

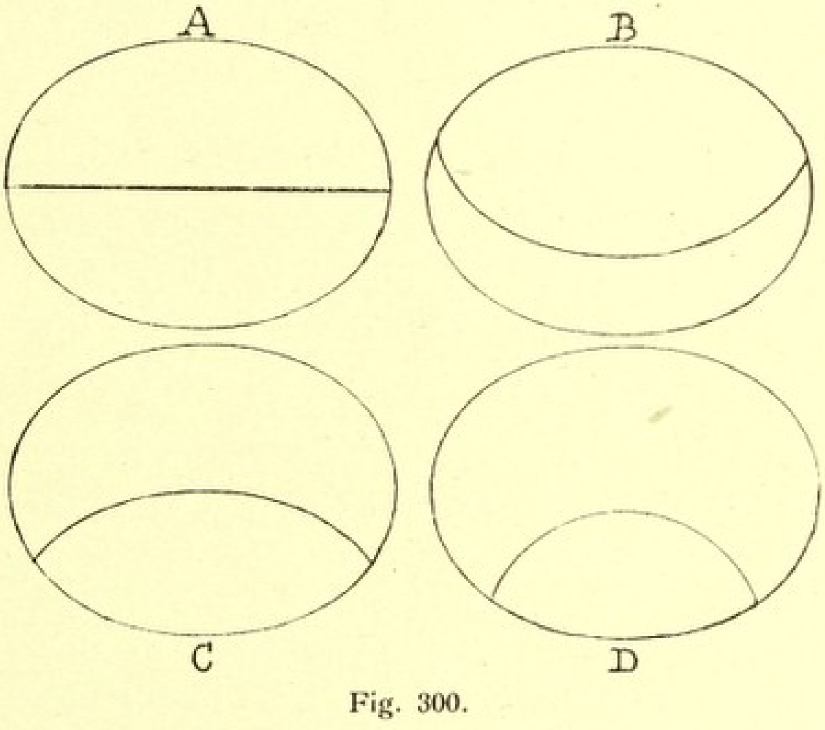

Fig. 300.

The bifocal, as its name indicates, is a lens of two foci, set into the same eye-wire. Of the two powers the upper is used for distance and the lower for reading or close work. In the Cx bifocal, the upper is the weaker and in the Cc, the reverse is the case. A bifocal can also be made to have Cc power for distance and Cx for reading. There are various forms of bifocals, many having fancy names, but the following are the chief styles.

1. The Franklin, or split, shown in Figure A is made of two lenses cut and joined, the juncture being a straight line, which can be at the centre or above or below it.

2. The perfection, which is a modification of the split; the juncture being being curved.

3. The solid, old style or up curve. The stronger lower part is ground on to the weaker upper. Its appearance is shown in Figure B.

4. The solid, new style or down curve. The stronger lower part is ground on to the weaker upper. Its appearance is shown in Figure D.

5. The cement consists of a thin segment, of the required additional power, cemented on to an ordinary lens of the strength required for distance. The segment can be made so thin as to be almost invisible.

6. The Borsch. This has been made by inserting a strongly-curved small lens of high μ between two Cx plano lenses of low μ.

7. The Fused is made by fusing a strongly-curved small lens of high μ on to the weaker main lens of low μ.

8. The Inset. In this a space is ground out from the main lens and a small lens of higher μ set in it.

Figures C and D represent all the forms of bifocals except 1 and 3, but of course the reading part can, in most of the forms, be made of any shape or size.

Spectacles and Lenses.

The thickness of a spectacle lens varies with its power and, if strong, with its size also to some extent. An ordinary concave is about 1 mm. thick in the centre and about 2 or 3 mm. at the periphery. Convex lenses average from to 2 to 3 mm. thick at the centre. The weight of a pair of lenses varies with the size and the thickness; an average weight may be 3 to 4 dwt. An average frame weight rather less, so that the total weight of frame and lenses is about 6 dwt.

Disc for Lenses.

The following formula gives the required thickness of a disc of glass for a given diameter of lens when one side is plano. If both sides are curved each must be separately calculated and the two added together. If focal length or dioptric power are given they must be converted into radius of curvature it being sufficiently accurate to take the radius of a thin plano convex or concave as half the focal length (strictly true only for $\mu = 1.50$). About 1 mm. should be added for the thickness of a concave at the centre, or of a convex at the bevel. Let d be the diameter, r the radius of curvature, and t the required thickness; then from the spherometer formula given previously.

$$t = \frac{\left(\frac{d}{2}\right)^2}{2\,r}$$

Example, required the thickness of the disc for a double Cx lens of 20 D, the diameter being 2 in. The focal length of 20 D = 50 mm. (or 100 mm. for each surface). It the diameter is 2 in. i.e. 50 mm., the semi-diameter is 25 mm; therefore

$$t = \frac{25^2}{2 \times 50} = 6.25 \text{ mm. for each surface}$$

or a total of $6.25 + 6.25 + 1 = 13.5$ mm.

For a double convex or concave of equal curvature on both surfaces the total thickness is

$$t = \frac{\left(\frac{d}{2}\right)^2}{r} + 1 \text{ mm.}$$

Suppose a glass without power be required of a radius of 25 cm. and diameter 30 mm.; then

$$t = \frac{15^2}{200 \times 2} = .56 \text{ mm. for each surface}$$

or a total thickness of say 2.25mm.

STANDARDS.

RELATING TO SPECTACLES AND EYEGLASSES.

Those adopted by the Optical Society (London), in May 1904, are marked O.S.

Angle Notation. The O.S. Standard notation for cylindrical lenses is given in the chapter on neutralisation.

Eyewire. The O.S. Standards are the same as those of the Imperial Wire Gauge, viz.—

O.S. or I.W.G. No.	20	19	18	17	16
Equal those formerly styled	6/-	4/-	3/-	2/6	2/-

Screws. The O.S. Standards for screws are the same as those of the British Association, Nos. 9 to 16 being the most suitable for spectacle work.

Spread of Bridge. The O.S. Standards are 1 cm. deep and a measurement is taken of an upper chord which is 2 mm., and a lower chord which is 7 mm. from the apex of the curve.

	Upper Chord (mm.)	Lower Chord (mm.)
No. 1	6·	9
2	7·5	12
3	9·	15
4	10·5	18
5	12·	21
6	13·5	24

Inclination of Bridge. The O.S. expression of the inclination of a spectacle bridge is so many degrees from the plane of the eyes.

AMERICAN STANDARD EYES.

No.	Long and Short Axis.	Length of wire required to make up the standard eye.
4 Eye	33.8 × 24.5 m/m	93.5 m/m
3 Eye	34 × 26 m/m	95.9 m/m
2 Eye	35 × 25.5 m/m	98.6 m/m
1 Eye	36.5 × 27.5 m/m	103.5 m/m
0 Eye	37.8 × 28.8 m/m	107.5 m/m
00 Eye	39.7 × 30.7 m/m	113.8 m/m
000 Eye	40.9 × 31.9 m/m	117.5 m/m
0000 Eye	44.3 × 36 m/m	128.2 m/m
Jumbo Eye	46 × 38 m/m	134.3 m/m

O.S. Standards Based on the Peripheral Measurement of the Edges of the Lens.

O.S. No.	Length of periphery.	Corresponding to the American No.
1	92.5 mm.	4
2	94.5 „ = 92.5 + 2	3
3	97.5 „ = 94.5 + 3	2
4	101.5 „ = 97.5 + 4	1
5	106.5 „ = 101.5 + 5	0
6	112.5 „ = 106.5 + 6	00

This numeration, on the same basis of measurement applies to all shapes of eyes for spectacles and eye-glasses.

Long Diameters of Lenses based on the above Peripheral Measurements.

O.S. No.	Oval.	Long Oval.	Round Oval.	Pantos.	½ Oval.	Round.
1	33.5	35	31	34	36	29.5
2	34	35.5	31.5	34.5	36.5	30
3	35	36.5	32.5	35.5	37.5	31
4	36.5	38	34	37		32.5
5	38	39.5	35.5	38.5		34
6	40	41.5	37.5	40.5		36

The ratio of the long to the short axis of the *oval lens* being approximately 1.3 to 1, and that of the *long oval* 1.5 to 1.

COLOURED GLASSES FOR SPECTACLE WORK.

These are usually numbered 1 to 6 or A to F. They vary considerably with respect to the quality and quantity of light transmitted, but approximately they transmit light as follows:—

No.	Percentage of light transmitted. Smoke.	Blue.
1 or A	60	80
2 „ B	50	70
3 „ C	30	50
4 „ D	20	25
5 „ E	10	20
6 „ F	2	4

Smoked glass absorbs a certain quantity of all the colours and therefore to some extent reduces the visual acuity. Cobalt blue glass absorbs principally orange light and has little effect in this direction if of a light tint. Johnson's spectrum blue glass cuts off the red and orange on the one side and the violet and ultra violet on the other, leaving the centre of the spectrum, practically unaltered.

The suggested O.S. Standard Colour Glasses for spectacle work are as follows :—

No.	Percentage of light transmitted.
1.	80 p.c.
2	60 p.c.
3	50 p.c.
4	40 p.c.
5	30 p.c.
6	20 p.c.
7	10 p.c.
8	5 p.c.
9	2.5 p.c.
10	1.25 p.c.

The quantity of light absorbed depends directly on the thickness of the glass and consequently no ordinary lens which varies in thickness owing to its curvature can have the same depth of tint all over. When a certain tint is selected, by trial with the coloured glasses of the test case, and lenses are required similarly tinted, a modification is necessary. The lens should be ordered of a lower tint if Cx and of a higher if Cc, the former being thick and the latter thin in the centre. The variation from the No. of the trial glass would necessarily depend on the strength of the lens required.

Equality of tint can be obtained by employing a plano Cx or Cc lens cemented to a plano coloured glass. For a sphero-cylindrical combination equality can be obtained, by cementing a thin plano spherical and a thin plano cylindrical to the two sides of a thin plano coloured glass. Or if one of the components be weak, in comparison with the other, by employing coloured glass for the weaker and white for the stronger, both being planos and cemented together. In this way practical equality of tint can be obtained.

OPTICAL GLASS.

Glass is a hard, generally transparent or translucent substance, made by the fusion of silica with potash, soda, lime, lead and other substances, such as pearlash, arsenic, manganese, saltpetre, chalk, &c. It is brittle, sonorous, ductile when heated, and fusible only at a very high temperature. It is usually not soluble, but is acted on by hydrofluoric acid, and it is a very bad conductor of heat. There are many varieties of glass, and the process of manufacture as regards the ingredients used and the treatment after complete fusion of the various components depends on the nature of the glass produced.

If suddenly cooled glass becomes extremely brittle owing to the state of tension produced by the cooling of the outer portions while the inner are still in a molten condition; annealing tends to reduce brittleness. Glass used for optical purposes must be necessarily homogenous, *i.e.*, of equal density and refractive power throughout and perfectly transparent, and it is therefore most carefully mixed and very gradually cooled. It should also be free from air bubbles, striae and colour for spectacle lenses, although a few air bubbles, if small, may be of little or no consequence in a camera lens. The solid block of glass is usually polished on two sides, so as to allow of the detection of striae, air bubbles, &c., and from it clear discs of appropriate size are cut.

Lenses are made of crown glass which contains lime or of flint glass which contains lead. Flint has generally a higher refractivity and chromativity; the greater percentage of lead in the glass the greater usually are the refractive and dispersive powers. It is denser, heavier, and softer than crown, and is almost perfectly colourless. Crown glass has the advantage of lower dispersion and is harder, so that it does not so easily become scratched, but it is more brittle than flint. It has sometimes a decided greenish tint, due to the presence of iron. The pinkish tint found in some glass results from the admixture of manganese.

According to its component ingredients and manufacture, the indices of refraction of glass vary for the various lines of the spectrum. The mean μ of different kinds of glass made for optical purposes was found to be about 1.574, that of the crown being 1.524, and of the flints 1.624.

The following may be taken as very rough examples of the proportions of the materials entering in the manufacture of glass:—

FLINT GLASS.

Silica	50 parts.
Lead	30 ,,
Potash	10 ,,
Other Ingredients	10 ,,
	100 parts.

CROWN GLASS.

Silica	70 parts.
Soda	10 ,,
Lime	10 ,,
Other ingredients	10 ,,
	100 parts.

OPTICAL GLASS.

Some examples (not actual kinds) given to illustrate the refraction, dispersion, and specific gravity of different kinds of optical glass:—

Description.	μ_D	$\nu = \frac{\mu_D - 1}{\delta\mu}$	Dispersion. Medium. $C - F = \delta\mu$	A – D	D – F	F – G	Specific Gravity.
Very light Crown ...	1.48	66	.0073	.0050	.0055	.0040	2.25
Light ,, ...	1.50	62	.0081	.0055	.0065	.0045	2.50
Ordinary ,, ...	1.52	60	.0087	.0060	.0070	.0050	2.75
Heavy ,, ...	1.56	55	.0102	.0065	.0075	.0055	3
Very heavy ,, ...	1.60	52	.0115	.0070	.0085	.0065	3.5
Very light Flint ...	1.54	48	.0123	.0075	.0090	.0070	3
Light ,, ...	1.58	43	.0135	.0085	.0095	.0080	3.25
Ordinary ,, ...	1.62	40	.0155	.0095	.0115	.0100	3.50
Heavy ,, ...	1.68	35	.0194	.0105	.0130	.0110	4
Very heavy ,, ...	1.85	24	.0354	.0185	.0280	.0250	5.5

TABLE OF REFRACTIVE INDICES.

Air	μ_D	= 1.000
Ice	μ_D	= 1.310
Water (distilled)	μ_D	= 1.336
Seawater	μ_D	= 1.343
Blood	μ_E	= 1.354
Albumen	μ_E	= 1.360
Absolute Alcohol	μ_D	= 1.366
Acetate of Potash	μ_D	= 1.370
Salt Solution	μ_E	= 1.375
Fluor Spar	μ_D	= 1.434
Spermacetti	μ_E	= 1.444
Alum	μ_D	= 1.457
Glycerine	μ_D	= 1.460
Oil of Bergamot	μ_D	= 1.464
Olive Oil	μ_E	= 1.470
Borax	μ_E	= 1.475
Naptha	μ_E	= 1.475
Turpentine	μ_E	= 1.478
Chloride of Tin	μ_D	= 1.503
Oil of Cinnamon	μ_E	= 1.508
Cedar Oil (Lens immersion oil)	μ_D	= 1.512
Gum Arabic	μ_E	= 1.512
Gum Dammar	μ_D	= 1.520
Oil of Cloves	μ_D	= 1.533
Sugar	μ_D	= 1.535
Felspar	μ_E	= 1.764
Canada Balsam (liquid)	μ_D	= 1.520
Canada Balsam (hard)	μ_D	= 1.535
Oil of Fennel	μ_D	= 1.544
Rock Crystal, or Pebble ordinary ray	μ_D	= 1.544
Rock Crystal, (extraordinary ray)	μ_D	= 1.553
Rock Salt	μ_D	= 1.555
Anilin Oil	μ_D	= 1.580
Styrax	μ_D	= 1.582
Benzylanilin	μ_D	= 1.611
Oil of Cassia	μ_E	= 1.618
Tolu Balsam	μ_D	= 1.628
Tourmaline (extraordinary ray)	μ_D	= 1.620
Tourmaline (ordinary ray)	μ_D	= 1.636
Monobromide of Naphthaline	μ_D	= 1.657
Iceland Spar (ordinary ray)	μ_D	= 1.659
Iceland Spar (extraordinary ray)	μ_D	= 1.486
Bisulphide of Carbon	μ_D	= 1.687
Phosphorus	μ_D	= 2.224
Diamond	μ_D	= 2.470
Chromate of Lead	μ_D	= 2.500 to 2.970

TABLE OF THE REFRACTIVE INDICES OF SOME OF THE METALS. (KUNDT.)

	Red Rays.	Yellow Rays (D.)	Blue Rays
Silver	—	0.27	—
Gold	0.38	0.58	1.00
Copper	0.45	0.65	0.95
Platinum	1.76	1.64	1.44
Iron	1.81	1.73	1.54
Nickel	2.17	2.01	1.85
Cobalt	2.61	2.26	2.16

SCALE

Of the approximately relative values of lenses numbered by the Dioptric system and by focal length in mm. and inches.

Diopters. Refractive Power.	Focal length in mm.	Focal length in inches.	No. of Cc. Old English System. (These Nos. have a very uncertain value.)
0.125	8000	320	
0.25	4000	160	0000
0.375	2666	100	
0.50	2000	80	000
0.625	1600	60	00
0.75	1333	52	
0.875	1143	48	0
1.00	1000	40	
1.125	900	36	1
1.25	800	32	
1.375	727	30	
1.50	666	26	
1.625	600	24	2
1.75	550	22	
1.875	533	21	
2.00	500	20	$2\frac{1}{2}$
2.125	470	19	
2.25	444	18	3
2.375	420	17	
2.50	400	16	$3\frac{1}{2}$
2.625	380	15	
2.75	363	14	4
3.00	333	13	$4\frac{1}{2}$
3.25	300	12	5
3.50	275	11	6
4.00	250	10	7
4.50	222	9	8
5.00	200	8	9
5.50	182	7	10
6.00	166	$6\frac{1}{2}$	
6.50	156	6	11
7.00	142	$5\frac{1}{2}$	
7.50	133	$5\frac{1}{4}$	
8.00	125	5	12
8.50	115	$4\frac{3}{4}$	13
9.00	111	$4\frac{1}{2}$	14
9.50	105	$4\frac{1}{4}$	15
10.00	100	4	16
10.50	95	$3\frac{3}{4}$	17
11.00	90	$3\frac{1}{2}$	18
12.00	83	$3\frac{1}{4}$	19
13.00	77	3	20
14.00	71	$2\frac{3}{4}$	21
16.00	62	$2\frac{1}{2}$	22
18.00	55	$2\frac{1}{4}$	23
20.00	50	2	24
22.00	45	$1\frac{3}{4}$	
26.00	38	$1\frac{1}{2}$	
32.00	31	$1\frac{1}{4}$	
40.00	25	1	

METRIC MEASUREMENTS.

1 Kilometre (K) = 1000 metres = $\frac{3}{5}$ mile.

1 Metre (M) = 10 decimetres = $\frac{1}{1000}$ Kilometre = 39,37 inches.

1 Decimetre (Dm) = 10 centimetres = $\frac{1}{10}$ Metre = 3,937 inches.

1 Centimetre (cm) = 10 millimetres = $\frac{1}{100}$ Metre = 0,3937 inches.

1 Millimetre (mm) = 1000 microns = $\frac{1}{1000}$ Metre = 0,03937 inches.

1 Micron (μ) = 1000 micromillimetres = $\frac{1}{1000}$ mm = $\frac{1}{25000}$ inch.

1 Micromillimetre ($\mu\mu$) = 10 Ångstrom units = $\frac{1}{1,000,000}$ mm = $\frac{1}{25,000\ 000}$ inch.

1 Ångstrom unit (Å) = $\frac{1}{10}$ $\mu\mu$ = $\frac{1}{10,000}$ μ = $\frac{1}{10,000,000}$ mm.

The Kilometre = 1 billion $\mu\mu$.

EQUIVALENT MINUTE MEASURES.

Metric.	Inches.	Inches.	Metric.
1 μ	1/25600 or .000039	1/20000	1.27 μ
2 μ	1/12800 „ .000079	1/10000	2.54 μ
3 μ	1/8530 „ .000118	1/5000	5.07 μ
4 μ	1/6400 „ .000157	1/3000	8.46 μ
5 μ	1/5120 „ .000197	1/2000	12.7 μ
6 μ	1/4265 „ .000236	1/1000	25.4 μ
7 μ	1/3650 „ .000276	1/250	101.6 μ
8 μ	1/3200 „ .000315	1/100	.25mm.
9 μ	1/2843 „ .000354	1/50	.5 mm.
10 μ	1/2560 „ .000394	1/25	1.0 mm.
100 μ	1/256 „ .003937	1/10	2.5 mm.
1000 μ	1/25 „ .039373	1/5	5.0 mm.

Exact Equivalents.

To convert mms. into inches multiply by .03937 or divide by 25.4.

To convert cms. into inches multiply by .3937 or divide by 2.54.

To convert metres into inches multiply by 39.37 or divide by .0254.

To convert metres into feet multiply by 3.28 or divide by .3048.

Approximate Equivalents.

To convert feet into metres multiply by 3 and divide by 10.

To convert feet into centimetres multiply by 30.

To convert inches into centimetres multiply by 2 1/2 or multiply by 10 and divide by 4.

To convert inches into millimetres multiply by 25 or multiply by 100 and divide by 4.

To convert metres into feet multiply by 10 and divide by 3.

To convert metres into inches multiply by 40.

To convert centimetres into inches divide by 2 1/2 or multiply by 4 and divide by 10.

To convert millimetres into inches divide by 25 or multiply by 4 and divide by 100.

PRACTICAL IMPERIAL AND METRIC SCALES.

Feet.	Metres.	Inches.	Cm.	Inches.	Mm.	Metres.	Feet.	Cm.	Inches.
200	60	320	800	15/16	23.4	100	333.3	100	40
133	40	240	600	7/8	21.8	60	200.	75	30
100	30	160	400	13/16	20.3	50	166.6	60	24
80	24	100	250	3/4	18.7	40	133.3	50	20
60	18	80	200	11/16	17.2	30	100.	45	18
50	15	60	150	5/8	15.6	24	80.	40	16
40	12	40	100	9/16	14.1	20	66.6	35	14
30	9	36	90	1/2	12.5	18	60.	32.5	13
20	6	30	75	7/16	11.0	16	53.3	30.	12
18	5.4	26	65	3/8	9.5	15	50.	27.5	11
17	5.1	24	60	5/16	8.	14	46.6	25.	10
16	4.8	22	55	1/4	6.4	12	40.	22.5	9
15	4.5	20	50	3/16	4.7	10	33.3	20.	8
14	4.2	18	45	1/8	3.2	9	30.	17.5	7
13	3.9	16	40	1/16	1.5	8	26.6	15.	6
12	3.6	15	37.5			7	23.3	12.5	5
11	3.3	14	35			6	20.	10.	4
10	3.	13	32.5			5	16.6	9.	$3\frac{3}{5}$ or 3.6
9	2.7	12	30			4.5	13.3	8.	$3\frac{1}{5}$ or 3.2
8	2.4	11	27.5			4.	11.6	7.	$2\frac{4}{5}$ or 2.8
7	2.1	10	25			3.5	10.	6.	$2\frac{2}{5}$ or 2.4
6	1.8	9	22.5			3.	8.3	5.	2 or 2.
5	1.5	8	20			2.5	6.6	4.	$1\frac{3}{5}$ or 1.6
4	1.2	7	17.5			2.	5.	3.	$1\frac{1}{5}$ or 1.2
3	.9	6	15			1.5	3.3	2.5	1 or 1.
2	.6	5	12.5			1.		2.	4/5 or .8
1	.3	4	10					1.5	3/5 or .6
		3	7.5					1.	2/5 or .4
		2	5					.5	1/5 or .2
		1	2.5					.2	2/25 or .08
								.1	1/25 or .04

Equivalents of Standards of Measurement.

1 millimetre (mm)	=	.03937 English inches.
,,	=	.03694 Paris inches.
,,	=	.03824 Prussian inches.
,,	=	.03796 Austrian inches.
1 English inch	=	25.4 mm.
1 Paris inch	=	27.07 ,,
1 Prussian inch	=	26.15 ,,
1 Austrian inch	=	26.3 ,,
1 English inch	=	.94 Paris inch.
1 Paris inch	=	1.07 English inch.

COMMERCIAL NOTATION FOR OPERA GLASSES AND TELESCOPES.

The French inch is divided into 12 lignes or lines, in which the diameter of the object glass of ordinary opera glasses and small telescopes is expressed.

Lines.	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
M.M. (approx.).	11	13	15	17	20	22	24	26	29	31	33	36	38	40	43	45	47	49	52	54	56	58	61	63	65	67	70	72	74	76	78	81	83	85	87	90

THE GREEK ALPHABET,

with the English equivalents and the pronunciation of the letters.

Α α ...Alpha ...*a*	Ι ι ...iōta ...*i*	Ρ ρ ...rho ...*r*
Β β ...bēta ...*b*	Κ κ ...kappa ...*k*	Σ σ ς...sigma ...*s*
Γ γ ...gamma *g* (hard)	Λ λ ...lamda ...*l*	Τ τ ...tau ...*t*
Δ δ ...delta ...*d*	Μ μ ...mu ...*m*	Υ υ ...ūpsīlon ...*u* or *y*
Ε ε ...epsīlon ...*e* (short)	Ν ν ...nu ...*n*	Φ φ ...phi ...*ph*
Ζ ζ ...zēta ...*z*	Ξ ξ ...xi ...*x*	Χ χ ...chi ...*ch* (hard)
Η η ...ēta ...*e* (long)	Ο ο ...ŏmicron *o* (short)	Ψ ψ ...psi ...*ps*
Θ θ ϑ ...thēta ...*th*	Π π ...pi ...*p*	Ω ω ...ōmĕga ...*o* (long)

MATHEMATICAL SYMBOLS AND ABBREVIATIONS.

M. ...	...Meters.	r or ρ	...Radius.
Cm. ...	...Centimeters.	r° or ρ°	...Radius in degrees.
Mm....	...Millimeters.	r′ or ρ′	...Radius in minutes.
μ ...	...Microns.	r″ or ρ″	...Radius in seconds.
μμ ...	...Micromillimeters.	θ φ ...	...Any angles.
Ft. ′...	...Foot.	+ ...	...Plus. Addition.
In. ″...	...Inch.	− ...	...Minus, Subtraction.
‴ ...	...Line.	± ...	...Either + or −
° ...	...Degree.	× ...	...Multiplied by.
°d ...	...Degree of deviation.	÷ or :	...Divided by.
′ ...	...Minute.	∼ ...	...The difference between
″ ...	...Second.	$\sqrt{}$	...The square root of.
∞ or 1/0	...Infinity, a number infinitely great.	$\sqrt[3]{}$	...The cube root of.
0 or 1/∞	...Zero, a number infinitely small.	$\sqrt[n]{}$	...The nth. root of.
∠ ...	...Angle.	x^2 ...	...x squared (xx).
: ...	...Is to, the ratio between.	x^3 ...	...x cubed (xxx)
:: ...	...So is, as or equals (used with ratios).	x^n ...	...x raised to the power of a number equal to n.
∴ ...	...Therefore.	$\overline{a + b}$	...Bond or vinculum, showing that the numbers are to be taken together. Is the same as (a + b)
∵ ...	...Because.	= ...	...Equal to
∝ ...	...Variesas, proportional to.	> ...	...Greater than.
⊥ ...	...Perpendicular to.	< ...	...Less than.
∥ ...	...Parallel to.		
∟ ...	...Right angles to.		
π ...	...(Pi) Ratio of circumference to diameter.		

OPTICAL AND SCIENTIFIC SYMBOLS AND ABBREVIATIONS.

R. ...	...Right.
L. ...	...Left.
N. ...	...Nasal.
T. ...	...Temporal.
T. ...	...Tension or hardness of the eyeball.
FV. F.	...Field of vision.
P.L....	...Perception of light.
L.D....	...Light difference.
L.M.	...Light minimum.
Con. C.	...Convergence.
Ac. A.	...Accommodation.
PP. P. N.P.	Punctum proximum. Near point.
PR. R. FP.	Punctum remotum. Far point.
P^C ...	...Near point of convergence.
R^C ...	...Far point of convergence.
A^C ...	...Relative amplitude of convergence.
P^A ...	...Near point of accommodation.
R^A ...	...Far point of accommodation.
C^A ...	...Relative amplitude of accommodation.
Crys.	...Crystalline lens of the eye.
Y.S. or M.L.	Yellow spot. Macula lutea.
F. or P.F.	...Principal focal distance or focus.
F_1 and F_2	...Anterior and posterior focal distances or foci.
f_1 and f_2	...Conjugate focal distances or foci.
O. ...	...Object.
I. ...	...Image.
J. ...	...Jaeger.
S. ...	...Snellen.
Æt. AE.	...Ætatis. Age.
O. ...	...Oculus, the eye.
S. ...	...Sight.
V. Vn. VA.	...Visus. Vision. Acuteness of vision.
V=0...	...V=less than 6/60.
OD. RE.	...Oculus dexter, right eye.
OS. LE.	...Oculus sinister, left eye.
OU, BE, OV, O_2	...Oculi unā. The eyes together, both eyes.
ODV.	...Oculi dextri visus. Vision of the right eye.
OSV.	...Oculi sinistri visus. Vision of the left eye.
R ...	...Recipe, Prescription, Take.
F ...	...Formula.
c. w....	...Cum, with.

Hor. H.	...Horizontal.
Ver. V.	...Vertical.
Mer....	...Meridan.
D. ...	...Diopter.
M.A.	...Meter Angle.
Em. E.	...Emmetropia. Emmetropic.
Am. ...	...Ametropia. Ametropic.
H. Hy.	...Hypermetropia. Hypermetropic.
Hl. ...	...Hypermetropia latent.
Hm....	...Hypermetropia manifest.
Hma. Ha.	...Hypermetropia(manifest) absolute.
Hmf. Hf.	...Hypermetropia (manifest) facultative.
Hmr. Hr.	...Hypermetropia relative.
Ht. ...	...Hypermetropia total.
M. My.	...Myopia. Myopic.
As. ...	...Astigmatism. Astigmatic.
M. As. or AM.	Myopic Astigmatism.
H. As. or AH.	Hyperopic Astigmatism.
Ahm. Amh....	Mixed Astigmatism.
Pr. Pb. Pby.	Presbyopia. Presbyopic.
P.D....	...Pupillary distance.
Ht. ...	...Height of bridge.
Pj. ...	...Projection.
S. Sph.	...Spherical.
C. Cyl.	...Cylindrical.
Pr. ...	...Prism.
Ax. ...	...Axis.
Pc. Peris	...Periscopic.
Pcx. Pcvx.	...Periscopic convex.
Pcc. Pcvc.	...Periscopic concave.
Dcx. Dcvx.	...Double convex.
Dcc. Dcvc.	...Double concave.
+, Cx, Cvx.	...Convex, plus.
−, Cc. Cvc.	Concave, minus.
△ PD.	...Prism diopter.
∇ ...	...Centrad.
∧ ...	...Metran.
∞ ...	...Infinity, a distance infinitely great. (in visual optics 20 ft. or 6 M.).
⊖ ...	...Combined with.
μ. m. n.	...The index of refraction.
M. n. ν.	...A medium.
△ δ ...	...The difference between (applied to lines of the spectrum).
ϖ or ω	...The ratio between the dispersion and refraction of a medium.
ν ...	...The ratio between the refraction and dispersion of a medium.

TABLE OF CONJUGATE FOCI.

Of real images of a convex lens of unit focal length.

Relative sizes of Object and Real Image for magnification.		Relative sizes of Object and Real Image for diminution.		Distance in terms of F. Object and Image or vice-versa.	
1	1	1	1	2	2
1	2	1	·500 or 1/2	3	1·500 or 1 1/2
1	3	1	·333 or 1/3	4	1·333 or 1 1/3
1	4	1	·250 or 1/4	5	1·250 or 1 1/4
1	5	1	·200 or 1/5	6	1·200 or 1 1/8
1	6	1	·166 or 1/6	7	1·166 or 1 1/6
1	7	1	·143 or 1/7	8	1·143 or 1 1/7
1	8	1	·125 or 1/8	9	1·125 or 1 1/8
1	9	1	·111 or 1/9	10	1·111 or 1 1/9
1	10	1	·100 or 1/10	11	1·100 or 1 1/10
1	11	1	·091 or 1/11	12	1·091 or 1 1/11
1	12	1	·083 or 1/12	13	1·083 or 1 1/12
1	13	1	·077 or 1/13	14	1·077 or 1 1/13
1	14	1	·071 or 1/14	15	1·071 or 1 1/14
1	15	1	·066 or 1/15	16	1·066 or 1 1/15
1	16	1	·062 or 1/16	17	1·062 or 1 1/16
1	17	1	·059 or 1/17	18	1·059 or 1 1/17
1	18	1	·055 or 1/18	19	1·055 or 1 1/18
1	19	1	·053 or 1/19	20	1·053 or 1 1/19
1	20	1	·050 or 1/20	21	1·050 or 1 1/20

The above are calculated from A = F (M + 1) and B = A/M where A and B are the two conjugates and M is the magnification or diminution.

TABLE OF CONJUGATE FOCI.

Of virtual images of a convex lens of unit focal length.

Relative sizes of Object & Virtual Image.		Distance in terms of F. Object.	Image.
1	1	0	0
1	1 1/4 or 1·250	1/5 or ·200	1/4 or ·250
1	1 1/3 or 1·333	1/4 or ·250	1/3 or ·333
1	1 1/2 or 1·500	1/3 or ·333	1/2 or ·500
1	1 2/3 or 1·666	2/5 or ·400	2/3 or ·666
1	1 3/4 or 1·750	3/7 or ·428	3/4 or ·750
1	2	1/2 or ·500	1
1	3	2/3 or ·666	2
1	4	3/4 or ·750	3
1	5	4/5 or ·800	4
1	6	5/6 or ·833	5
1	7	6/7 or ·857	6
1	8	7/8 or ·875	7
1	9	8/9 or ·888	8
1	10	9/10 or ·900	9
1	11	10/11 or ·909	10
1	12	11/12 or ·917	11
1	13	12/13 or ·923	12
1	14	13/14 or ·928	13
1	15	14/15 or ·933	14
1	16	15/16 or ·938	15
1	17	16/17 or ·941	16
1	18	17/18 or ·944	17
1	19	18/19 or ·947	18
1	20	19/20 or ·950	19

The above are calculated from A = F (M — 1) and B = A/M where A and B are the two conjugates and M is the magnification.

TABLE OF CONJUGATE FOCI.

Of virtual images of a concave lens of unit focal length.

Relative sizes of Object & Virtual Image.		Distance in terms of F. Object.	Image.
1	1	0	0
1	7/8	1/7 or ·143	1/8 or ·125
1	3/4	1/3 or ·333	1/4 or ·250
1	5/8	3/8 or ·375	3/11 or ·273
1	1/2	1	1/2 or ·500
1	1/3	2	2/3 or ·666
1	1/4	3	3/4 or ·750
1	1/5	4	4/5 or ·800
1	1/6	5	5/6 or ·833
1	1/7	6	6/7 or ·857
1	1/8	7	7/8 or ·875
1	1/9	8	8/9 or ·888
1	1/10	9	9/10 or ·900
1	1/11	10	10/11 or ·909
1	1/12	11	11/12 or ·917
1	1/13	12	12/13 or ·923
1	1/14	13	13/14 or ·928
1	1/15	14	14/15 or ·933
1	1/16	15	15/16 or ·938
1	1/17	16	16/17 or ·941
1	1/18	17	17/18 or ·944
1	1/19	18	18/19 or ·947
1	1/20	19	19/20 or ·950

The above are calculated from A = F (M — 1) and B = A/M where A and B are the two conjugates and M is the diminution.

SIMPLE ALGEBRAICAL CALCULATIONS.

Addition. Two + quantities, added together, make a + quantity, thus:— $+ 2 + (+ 3) = + 2 + 3 = + 5$.

Two − quantities added together make a − quantity, thus: $- 1 + (- 6) = - 1 - 6 = - 7$.

A + and a − quantity, added together, make a + or − quantity according to which is the greater of the originals. The addition is achieved by subtracting the lesser from the greater quantity and prefixing the sign of the greater, thus:

$$+ 4 + (- 3) = + 4 - 3 = + 1$$
$$+ 6 + (- 8) = + 6 - 8 = - 2$$

Subtraction. This is performed by changing the sign of the number to be deducted and then proceeding as in algebraical addition, thus:

$$+ 4 - (+ 2) = + 4 - 2 = + 2$$
$$+ 4 - (- 2) = + 4 + 2 = + 6$$
$$- 3 - (+ 1) = - 3 - 1 = - 4$$
$$- 3 - (- 1) = - 3 + 1 = - 2$$

Multiplication. Two like quantities multiplied together make a + quantity, thus:

$$+ 2 \times (+ 3) = + 6 \qquad - 4 \times (- 5) = + 20.$$

Two unlike quantities multiplied together make a − quantity, thus:

$$+ 4 \times (- 3) = - 12.$$

Division. Two like quantities divided the one by the other gives a + result, thus:

$$\frac{+ 4}{+ 2} = + 2 \qquad \frac{- 12}{- 3} = + 4$$

Two unlike quantities divided the one by the other gives a − result, thus:

$$\frac{+ 12}{- 3} = - 4 \qquad \frac{- 20}{+ 4} = - 5$$

When there are various quantities in an expression, multiplication and division take precedence over addition and subtraction for instance, $8 + 5 \times 7$ is not $13 \times 7 = 91$, but $8 + 35 = 43$. If $8 + 5$ have to be multiplied by 7, these two quantities should be enclosed in a bracket thus $(8 + 5) \times 7$.

In an equation there are always two expressions which equal one another thus:

$$A + B = C \times D.$$

Any quantity added on the one side can be carried to the other and subtracted or vice versa.

Any quantity which multiplies on the one side can be carried to the other as a divisor thus:

$$\text{If } A + B = C \times D.$$

$$\text{Then } A = C \times D - B, \text{ or } \frac{A + B}{D} = C.$$

If $A = 4$, $B = 10$, $C = 7$ and $D = 2$; that is $4 + 10 = 7 \times 2$.

$$\text{Then } 4 = 7 \times 2 - 10; \qquad \text{or } \frac{4 + 10}{7} = 2.$$

Let n be any definite quantity; then

$n + \infty = \infty$	$n - \infty = 0$	$n \times \infty = \infty$	$n \div \infty = 0$
$n + 0 = n$	$n - 0 = n$	$n \times 0 = 0$	$n \div 0 = \infty$
$\infty - n = \infty$	$0 - n = -n$	$\infty \div n = \infty$	$0 \div n = 0$

SIMPLE TRIGONOMETRICAL VALUES.

CIRCLES AND ANGLES.

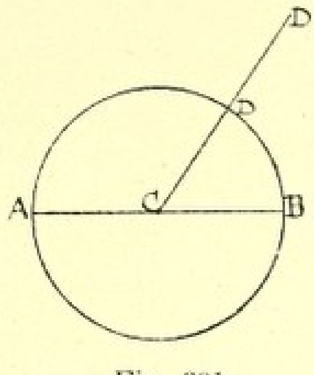

Fig. 301.

A circle is a plane figure, bounded by a continuous curved line—the circumference—every point of which is equi-distant from the centre C. The length of the circumference varies directly with that of the radius. The diameter of a circle is any straight line A B drawn through the centre and touching the circumference on each side. The radius is any straight line C A or C D drawn from the centre to the circumference. The length of a radius C B is half that of the diameter A B and all the radii of a given circle are equal to each other. Any radius C D is perpendicular to the circumference at the point where it cuts it, so also any radius continued to D′ beyond the circumference is perpendicular thereto.

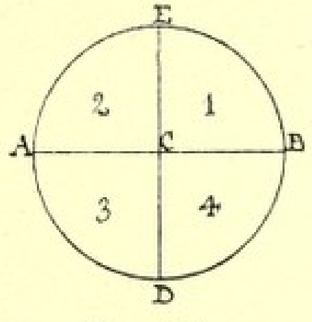

Fig. 302.

As shown in Fig. 302, the intersection of the horizontal diameter A B and the vertical D E divides the circle into four quadrants or right angles, of which E C B is the first, E C A the second, D C A the third and D C B the fourth quadrant.

The circle being divided into 360 degrees, each quadrant contains 90°. The first quadrant E C B contains those between O and 90°, the second 90° to 180°, the third 180° to 270°, the fourth 270° to 360°. The degrees are counted in the opposite direction to the movement of a clock. The value of an angle is expressed by the number of degrees it contains.

ANGULAR MEASURE.

60 seconds (″)...	=	1 minute (′).
60 minutes	=	1 degree (°).
30 degrees	=	1 sign.
60 degrees	=	1 sextant.
90 degrees	=	1 quadrant or right angle.
180 degrees	=	1 semi-circle or two right angles.
360 degrees	=	1 circle or circumference, or 4 right angles.

The circle A E F B (Fig. 303) is formed on a radius of, say, 1 inch, around the centre C; then C B and C F are radii of this circle and they make the angle B C F.

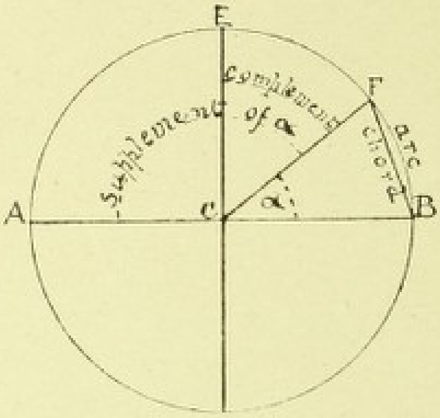

Fig. 303.

The arc is that portion of the circumference which is enclosed between the extremities of the two radii by which an angle is formed, thus B F is the arc of the angle B C F and the angle B C F is said to be subtended by the arc B F.

The ratio of the circumference to the diameter is expressed by the symbol π (pi) that is

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{C}{d}$$

and since d = 2r $$\pi = \frac{C}{2r}$$

Therefore it follows that $2\,\pi\,r = C$ or 360° or 4 right angles; $\pi\,r = \frac{C}{2}$, or 180° or 2 right angles: $\frac{\pi\,r}{2} = \frac{C}{4}$ or 90° or one right angle. The value of π is approximately $\frac{22}{7}$ or 3.1416—that is to say, the circumference of any circle is about 3.14 times as long as its diameter A B or 6.28 times its radius C B.

The unit of *circular measure* is an arc called the radian, whose length is equal to the radius of the circle. Since $2\,\pi\,r = 360°$, or a circle, the radian subtends an angle of:—

$$\frac{360°}{2\,\pi} = \frac{180°}{\pi} = \frac{180°}{3.14159} = 57.3°$$

and since r is taken as unity, $1° = \frac{\pi}{180} = .01745$ and $1' = \frac{\pi}{180 \times 60} = .00029.$

The expression of an angle in circular measure is

$$\frac{\pi \times \text{angle required in degrees.}}{180}$$

The radian is subdivided into 100 centrads; each centrad is therefore rather over 1/2°.

The quadrant is also subdivided into 100 grades, each grade containing 100 minutes and each minute 100 seconds. This is the French method of angle measurement, but it is little used except in France.

An angle may then be considered the divergence of a straight line, revolved about a fixed point, from a stationary line with which it originally coincided. If, in Fig. 304, C F originally coincided with C B and is revolved around C, it has described the angle B C F when it is in the position shewn in the figure; let this angle be called a. The distance passed over by F is the arc F B of the angle a, and the arc B F is said to subtend the angle a or the angle a encloses the arc B F.

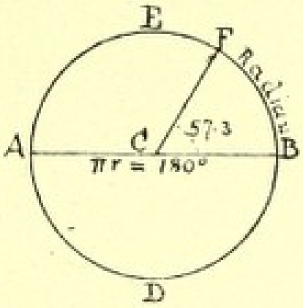

Fig. 304.

A chord is a straight line having its two extremities in the circumference of a circle as the line F B.

The complement of an angle is that angle which completes 90°, thus E C F is the complement of B C F. The complement of $a = 90° - a$.

The supplement of an angle is that angle which completes 180°, thus A C F is the supplement of B C F. The supplement of $a = 180° - a$.

An angle of less than 90°, as B C F is *acute*, one of more than 90°, and less than 180°, as A C F, is *obtuse*.

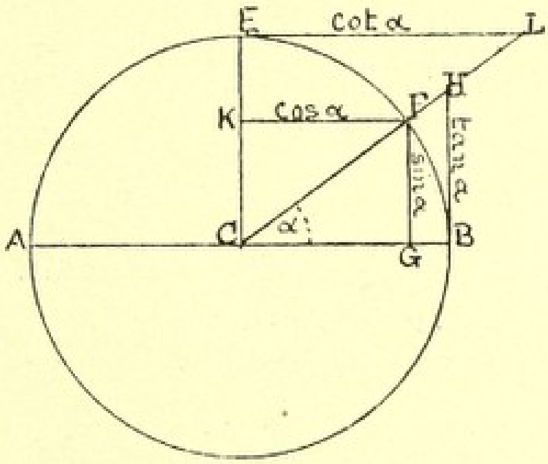

Fig. 305.

If C B is unity, the sine (fig. 305) is represented by a perpendicular line dropped from the extremity of the one radius on to the other, between which an angle is formed; thus F G = sin a.

The tangent is represented by that part of a straight line, touching the circumference, which is perpendicular to a radius and enclosed between it and the other radius (continued beyond the circumference) which forms the angle; thus, H B = tan a.

The secant is represented by a line drawn from the extremity of this tangent to the centre, thus H C = sec. a.

The cosine of an angle is the sine of its complement; thus F K = cos a. It is the distance C G between the centre and the foot of the sine.

The cotangent of an angle is the tangent of its complement; thus E L = cot a.

The cosecant of an angle is the secant of its complement; thus C L = csc a.

The line from the foot of the sine to the circumference is the versed sine of an angle ; thus G B = versin a.

The co-versed sine of an angle is the versed sine of its complement ; thus K E = co-versin a.

The sine is smaller and the tangent is greater than the arc of an angle. The chord is smaller than the arc, but greater than the sine. If the angle is small, that is, under 5°, the sine, chord, arc and tangent may be considered equal. If the angle is very small the sine or tangent can be replaced by the angle itself, and the cos a = 1. Tangents increase proportionately to sines very quickly when angles are large, say over 20°.

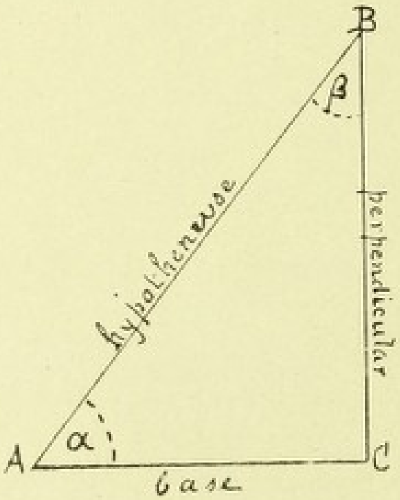

Fig. 306.

A B C (fig. 306) is a right angled triangle. B C is called the perpendicular, A C is the base, and A B is the hypothenuse. The sides of a right angled triangle bear such a relationship to one another that

$$\text{Base}^2 + \text{Per}^2 = \text{Hyp}^2 .$$

In this figure $AC^2 + BC^2 = AB^2$ whence $AB = \sqrt{AC^2 + BC^2}$

or $AC = \sqrt{AB^2 - BC^2}$ or $BC = \sqrt{AB^2 - AC^2}$

Let the angle B A C = a then :—

$$\frac{BC}{AB} = \frac{\text{Per.}}{\text{Hyp.}} = \sin a \qquad \frac{AC}{AB} = \frac{\text{Base}}{\text{Hyp.}} = \cos a$$

$$\frac{BC}{AC} = \frac{\text{Per.}}{\text{Base}} = \tan a \qquad \frac{AC}{BC} = \frac{\text{Base}}{\text{Per.}} = \cot a$$

$$\frac{AB}{AC} = \frac{\text{Hyp.}}{\text{Base}} = \sec a \qquad \frac{AB}{BC} = \frac{\text{Hyp.}}{\text{Per.}} = \csc a$$

The sine, tangent, cosine, etc., are termed functions of an angle, and can be expressed as numbers which are relative to the length of the radius.

Natural sines, tangents, etc., are the ratios of given lines, in the diagram, to the radius when the latter is 1 (unity). They are usually expressed by numbers and decimal fractions as given in the tables.

To find the numerical value of the sine, tangent, etc., of a given angle, multiply the number given in the table of natural sines, etc., by the length of the radius.

The value of the ratios for some of the more common angles are :—

Angle.	0°	30°	45°	60°	90°
Sine	0	.5 or $\frac{1}{2}$	.7071 or $\sqrt{\frac{1}{2}}$	.8660 or $\frac{\sqrt{3}}{2}$	1
Tangent	0	.5774 or $\sqrt{\frac{1}{3}}$	1	1.7321 or $\sqrt{3}$	∞
Secant	1	1.1548 or $\sqrt{\frac{2}{3}}$	1.414 or $\sqrt{2}$	2	∞
Cosine	1	.8660 or $\frac{\sqrt{3}}{2}$	.7071 or $\sqrt{\frac{1}{2}}$	.5 or $\frac{1}{2}$	0
Cotangent	∞	1.7321 or $\sqrt{3}$	1	.5774 or $\sqrt{\frac{1}{3}}$	0
Cosecant	∞	2	1.414 or $\sqrt{2}$	1.1548 or $\sqrt{\frac{2}{3}}$	1

The sine and cosine vary between 0 and 1.

The tangent and cotangent vary between 0 and ∞.

The sine and cosecant, the tangent and the co-tangent, the secant and the cosine form respectively pairs of reciprocals.

The values of the functions of angles greater than 90° are the same as those up to 90° for a similar number of degrees from the horizontal or initial line.

SIGNS OF THE FUNCTIONS.

Quadrant ...	1st	2nd	3rd	4th
Sine...	+	+	−	−
Tangent	+	−	+	−
Secant	+	−	−	+
Cosine	+	−	−	+
Cotangent ...	+	−	+	−
Cosecant ...	+	+	−	−

TRIGONOMETRICAL EQUIVALENTS OF ANGLE *a*

$\sin^2 a + \cos^2 a = 1$

$1 - \sin^2 a = \cos^2 a$

$1 - \cos^2 a = \sin^2 a$

$\sqrt{1 - \sin^2 a} = \cos a$

$\sqrt{1 - \cos^2 a} = \sin a$

$1 + \tan^2 a = \sec^2 a$

$1 + \cot^2 a = \operatorname{cosec}^2 a$

$\sin a \times \operatorname{cosec} a = 1$

$\cos a \times \sec a = 1$

$\tan a \times \cot a = 1$

$\sin a \times \cot a = \cos a$

$\sin a \times \sec a = \tan a$

$\cos a \times \tan a = \sin a$

$\cos a \times \operatorname{cosec} a = \cot a$

$1 - \sin a = \operatorname{coversin} a$

$1 - \cos a = \operatorname{versin} a$

$\frac{1}{\sin a} = \operatorname{cosec} a$

$\frac{1}{\cos a} = \sec a$

$\frac{1}{\tan a} = \cot a$

$\frac{1}{\cot a} = \tan a$

$\frac{1}{\sec a} = \cos a$

$\frac{1}{\operatorname{cosec} a} = \sin a$

$\frac{\sin a}{\tan a} = \cos a$

$\frac{\sin a}{\cos a} = \tan a$

$\frac{\cos a}{\sin a} = \cot a$

$\frac{\cos a}{\cot a} = \sin a$

$\frac{\tan a}{\sin a} = \sec a$

$\frac{\tan a}{\sec a} = \sin a$

$\frac{\cot a}{\cos a} = \operatorname{cosec} a$

$\frac{\cot a}{\operatorname{cosec} a} = \cos a$

The complement of angle $a = 90 - a$.

The supplement of angle $a = 180^\circ - a$.

$\sin(90^\circ - a) = \cos a$
$\cos(90^\circ - a) = \sin a$
$\tan(90^\circ - a) = \cot a$
$\cot(90^\circ - a) = \tan a$
$\sec(90^\circ - a) = \csc a$
$\csc(90^\circ - a) = \sec a$

$\sin(180^\circ - a) = -\sin a$
$\cos(180^\circ - a) = -\cos a$
$\tan(180^\circ - a) = -\tan a$
$\cot(180^\circ - a) = -\cot a$
$\sec(180^\circ - a) = -\sec a$
$\csc(180^\circ - a) = -\csc a$

$\sin(a + \beta) = (\sin a \cos \beta) + (\cos a \sin \beta)$
$\cos(a + \beta) = (\cos a \cos \beta) - (\sin a \sin \beta)$
$\sin(a - \beta) = (\sin a \cos \beta) - (\cos a \sin \beta)$
$\cos(a - \beta) = (\cos a \cos \beta) + (\sin a \sin \beta)$

ELLIPSES.

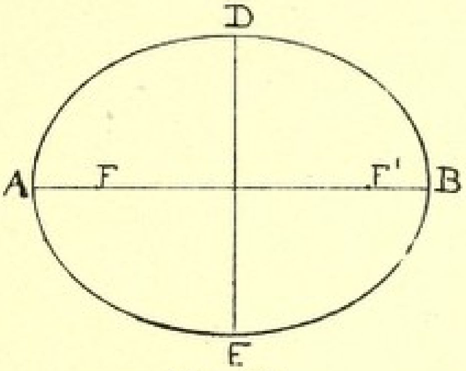

Fig. 307.

An ellipse (Fig. 307) is a plane figure bounded by a curve called the perimeter, which corresponds to an oblique section of the cone. The sum of the distances from any point on its perimeter drawn to two fixed points F F′ (termed the foci) is equal to the line A B drawn through the foci. A B is the major or *traverse* axis, D E is the minor or *conjugate* axis. An ellipse may be constructed by putting stout pins through two points, which become the two foci, and passing a knotted thread of the required length of perimeter over the pins. A pencil held upright and pressed against the thread in the direction outward from the foci, on being moved around, describes the ellipse on paper.

An ellipsoid is a solid body generated by the revolution of an ellipse about one of its axis. It is prolate when formed on the major axis and oblate when formed on the minor axis.

There is no exact method by which the perimeter of an ellipse can be calculated but the following formula gives it with a fair degree of approximation.

Let a and b be respectively the long and short axis, and P be the perimeter then :—

$$P = \pi \sqrt{\frac{a^2 + b^2}{2}}$$

When the length of the one axis a is known, that of the other, b is found by

$$b = \sqrt{2 \left(\frac{P}{\pi}\right)^2 - a^2}$$

If the two axes are equal the figure becomes a circle, so that the perimeter of the ellipse is analagous to the circumference of a circle.

Since $\pi \times$ diameter = circumference of a circle, the diameter of the circle whose circumference is equal to the perimeter of a given ellipse, is

$$\sqrt{\frac{a^2 + b^2}{2}} = \text{diameter}.$$

USEFUL DATA.

The circumference of a circle $= 2\ \pi r$. The diameter of a circle $= \frac{C}{\pi}$

The area of a circle $= \pi r^2$ The surface of a sphere $= \pi d^2$

The volume of a sphere $= \frac{\pi d^3}{6}$ The area of a triangle $= \frac{\text{Per} \times \text{base}}{2}$

The length of an arc $= .01745r \times$ No. of degrees.

The area of a sector of circle $= \frac{\text{area of circle} \times \text{degrees of arc}}{360}$.

The area of a circular ring $= \frac{\pi}{4} \times$ difference of the squares of the diameters of the two circles.

The perimeter of an ellipse $= \pi \sqrt{\frac{a^2 + b^2}{2}}$

The area of an ellipse $= \pi \frac{ab}{4}$

RECIPROCALS.

Reciprocal numbers are those which bear such a relation to each other that their multiple is 1 or unity. The reciprocal of any number is therefore obtained by dividing 1 by that number. 1/4 and 4 are reciprocals, because $1/4 \times 4 = 1$. The reciprocal of 2/7 is 7/2 and vice versa. The reciprocal of 1.5 is .6363, that of .6666 is 1.5.

0 and ∞ representing numbers indefinitely or infinitely small and great respectively may be regarded as reciprocals of each other.

Instead of dividing one number by another, the same result is obtained by multiplying the first by the reciprocal of the second. Dividing a number by the reciprocal of another is the same as multiplying the first by the second, thus a

$$\frac{a}{b} = a \times \frac{1}{b} \qquad ab = \frac{a}{1/b} \qquad \frac{10}{6/5} = \frac{10 \times 5}{6} = 8\ 1/3$$

$$89.54 \times 1/4 = \frac{89.54}{4} = 22.385.$$

TABLE OF NATURAL VALUE OF SINES, &c., IN THE UNIT CIRCLE.

Read downwards from 0° to 45°, and upwards from 45° to 90°

°	′	Sines.	Tangents.	Cotangents.	Cosines.	′	°
0	0	0.0000	0.0000	∞	1.0000	0	90
0	30	0.0087	0.0087	114.5887	1.0000	30	89
1	0	0.0175	0.0175	57.2900	0.9998	0	89
1	30	0.0262	0.0262	38.1885	0.9997	30	88
2	0	0.0349	0.0349	28.6363	0.9994	0	88
2	30	0.0436	0.0437	22.9038	0.9990	30	87
3	0	0.0523	0.0524	19.0811	0.9986	0	87
3	30	0.0610	0.0612	16.3499	0.9981	30	86
4	0	0.0698	0.0699	14.3007	0.9976	0	86
4	30	0.0785	0.0787	12.7062	0.9969	30	85
5	0	0.0872	0.0875	11.4301	0.9962	0	85
5	30	0.0958	0.0963	10.3854	0.9954	30	84
6	0	0.1045	0.1051	9.5144	0.9945	0	84
6	30	0.1132	0.1139	8.7769	0.9936	30	83
7	0	0.1219	0.1228	8.1443	0.9925	0	83
7	30	0.1305	0.1317	7.5958	0.9914	30	82
8	0	0.1392	0.1405	7.1154	0.9903	0	82
8	30	0.1478	0.1495	6.6912	0.9890	30	81
9	0	0.1564	0.1584	6.3138	0.9877	0	81
9	30	0.1650	0.1673	5.9758	0.9863	30	80
10	0	0.1736	0.1763	5.6713	0.9848	0	80
10	30	0.1822	0.1853	5.3955	0.9833	30	79
11	0	0.1908	0.1944	5.1446	0.9816	0	79
11	30	0.1994	0.2035	4.9152	0.9799	30	78
12	0	0.2079	0.2126	4.7046	0.9781	0	78
12	30	0.2164	0.2217	4.5107	0.9763	30	77
13	0	0.2250	0.2309	4.3315	0·9744	0	77
13	30	0.2334	0.2401	4·1653	0.9724	30	76
14	0	0.2419	0.2493	4.0108	0.9703	0	76
14	30	0.2504	0.2586	3.8667	0.9681	30	75
15	0	0.2588	0.2679	3.7321	0.9659	0	75
15	30	0.2672	0.2773	3.6059	0.9636	30	74
16	0	0.2756	0.2867	3.4874	0.9613	0	74
16	30	0.2840	0.2962	3.3759	0.9588	30	73
17	0	0.2924	0.3057	3.2709	0.9563	0	73
17	30	0.3007	0.3153	3.1716	0.9537	30	72
18	0	0.3090	0.3249	3.0777	0.9511	0	72
18	30	0.3173	0.3346	2.9887	0.9483	30	71
19	0	0.3256	0.3443	2.9042	0.9455	0	71
19	30	0.3338	0.3541	2.8239	0.9426	30	70
20	0	0.3420	0.3640	2.7475	0.9397	0	70
20	30	0.3502	0.3739	2.6746	0.9367	30	69
21	0	0.3584	0.3839	2.6051	0.9336	0	69
21	30	0.3665	0.3939	2.5386	0.9304	30	68
22	0	0.3746	0.4040	2.4751	0.9272	0	68
22	30	0.3827	0.4142	2.4142	0.9239	30	67
°	′	Cosines.	Cotangents.	Tangents.	Sines.	′	°

TABLE OF NATURAL SINES, &c.—*Continued.*

°	′	Sines.	Tangents.	Cotangents.	Cosines.	′	°
23	0	0.3907	0.4245	2.3559	0.9205	0	67
23	30	0.3987	0.4348	2.2998	0·9171	30	66
24	0	0.4067	0.4452	2.2460	0·9135	0	66
24	30	0.4147	0.4557	2.1943	0.9100	30	65
25	0	0.4226	0.4663	2.1445	0.9063	0	65
25	30	0.4305	0.4770	2.0965	0.9026	30	64
26	0	0.4384	0.4877	2.0503	0.8988	0	64
26	30	0.4462	0.4986	2.0057	0.8949	30	63
27	0	0.4540	0.5095	1.9626	0.8910	0	63
27	30	0.4617	0.5206	1.9210	0.8870	30	62
28	0	0.4695	0.5317	1.8807	0·8829	0	62
28	30	0.4772	0.5430	1.8418	0·8788	30	61
29	0	0.4848	0.5543	1.8040	0.8746	0	61
29	30	0.4924	0.5658	1.7675	0.8704	30	60
30	0	0.5000	0.5774	1.7321	0.8660	0	60
30	30	0.5075	0.5890	1.6977	0.8616	30	59
31	0	0.5150	0.6000	1.6643	0.8572	0	59
31	30	0.5225	0.6128	1.6319	0.8526	30	58
32	0	0.5299	0.6249	1.6003	0.8480	0	58
32	30	0.5373	0.6371	1.5697	0.8434	30	57
33	0	0.5446	0.6494	1.5399	0·8387	0	57
33	30	0.5519	0.6619	1.5108	0·8339	30	56
34	0	0.5592	0.6745	1.4826	0·8290	0	56
34	30	0.5664	0.6873	1.4550	0·8241	30	55
35	0	0.5736	0.7002	1.4281	0.8192	0	55
35	30	0.5807	0.7133	1.4019	0.8141	30	54
36	0	0.5878	0.7265	1.3764	0.8090	0	54
36	30	0.5948	0.7400	1.3514	0.8039	30	53
37	0	0.6018	0.7536	1.3270	0.7986	0	53
37	30	0.6088	0.7673	1.3032	0.7934	30	52
38	0	0.6157	0.7813	1.2799	0.7880	0	52
38	30	0.6225	0.7954	1.2572	0.7826	30	51
39	0	0.6293	0.8098	1.2349	0.7771	0	51
39	30	0.6361	0.8243	1.2131	0.7716	30	50
40	0	0.6428	0.8391	1.1918	0.7660	0	50
40	30	0.6494	0.8541	1.1708	0.7604	30	49
41	0	0.6561	0.8693	1.1504	0.7547	0	49
41	30	0.6626	0.8847	1.1303	0.7490	30	48
42	0	0.6691	0.9004	1.1106	0.7431	0	48
42	30	0.6756	0.9163	1.0913	0.7373	30	47
43	0	0.6820	0.9325	1.0724	0.7314	0	47
43	30	0.6884	0.9490	1.0538	0.7254	30	46
44	0	0.6947	0.9657	1.0355	0.7193	0	46
44	30	0.7009	0.9827	1.0176	0.7133	30	45
45	0	0.7071	1.0000	1.0000	0.7071	0	45
°	′	Cosines.	Cotangents.	Tangents.	Sines.	′	°

TABLE OF SINE SQUARED AND COSINE SQUARED.

Read downwards from 0° to 45° and upwards from 45° to 90°.

Degrees.	Sin.²	Cos.²	Degrees.
0	000000	1000000	90
1	000306	999694	89
2	001218	998782	88
3	002735	997265	87
4	004872	995128	86
5	007583	992414	85
6	010920	989080	84
7	014859	985141	83
8	019377	980623	82
9	024461	975529	81
10	030137	969863	80
11	036405	963595	79
12	043183	956817	78
13	050625	949375	77
14	058516	941484	76
15	066977	933023	75
16	075995	923005	74
17	085498	914502.	73
18	095481	904519	72
19	106015	893985	71
20	116964	883036	70
21	128450	871550	69
22	140325	859675	68
23	152646	847354	67
24	165405	834595	66
25	178591	821409	65
26	192195	807805	64
27	206116	793884	63
28	220430	779560	62
29	235031	764969	61
30	250000	750000	60
31	265225	734775	59
32	280794	719206	58
33	296589	703411	57
34	312605	687395	56
35	329017	670983	55
36	345499	654501	54
37	362163	637837	53
38	379086	620914	52
39	396018	603982	51
40	413192	586808	50
41	430467	569533	49
42	447695	552305	48
43	465124	534876	47
44	482608	517392	46
45	500000	500000	45
	Cos.²	Sin.²	

DECIMAL EQUIVALENTS OF VULGAR FRACTIONS.

1/2	.5
1/3	.3333
1/4	.25
1/5	.2
1/6	.1666
1/7	.1428
1/8	.125
1/9	.1111
1/10	.1
1/11	.0909
1/12	.0833
1/13	.0769
1/14	.0714
1/15	.0666
1/16	.0625
1/17	.0588
1/18	.0555
1/19	.0526
1/20	.05
1/25	.04
1/30	.0333
1/33	.03
1/40	.025
1/50	.02
1/60	.0166
1/70	.0143
1/80	.0125
1/90	.0111
1/100	.01
1/200	.005
1/300	.0033
1/400	.0025
1/500	.002
1/1000	.0001
15/16	.9375
7/8	.875
13/16	.8125
3/4	.75
11/16	.6875
5/8	.625
9/16	.5625
1/2	.5
7/16	.4375
3/8	.375
5/16	.3125
1/4	.25
3/16	.1875
1/8	.125
1/16	.0625
1/32	.03125
1/64	.015625

TABLE OF RECIPROCALS, SQUARES AND SQUARE ROOTS, CUBES AND CUBE ROOTS.

No.	Reciprocals.	Square	Sq. root.	Cube.	Cube root.	No.	Reciprocals.	Square	Sq. root.	Cube.	Cube root.
1	1.00000	1	1.000	1	1.000	61	.01639	3721	7.810	226981	3.936
2	.50000	4	1.414	8	1.260	62	.01614	3844	7.874	238328	3.958
3	.33333	9	1.732	27	1.442	63	.01589	3969	7.937	250047	4.979
4	.25000	16	2.000	64	1.587	64	.01564	4096	8.000	262144	4.000
5	.20000	25	2.236	125	1.710	65	.01540	4225	8.062	274625	4.021
6	.16667	36	2.449	216	1.817	66	.01517	4356	8.124	287496	4.041
7	.14286	49	2.646	343	1.913	67	.01494	4489	8.185	300763	4.061
8	.12500	64	2.828	512	2.000	68	.01471	4624	8.246	314432	4.082
9	.11111	81	3.000	729	2.080	69	.01450	4761	8.307	328509	4.101
10	.10000	100	3.162	1000	2.154	70	.01429	4900	8.367	343000	4.121
11	.09091	121	3.316	1331	2.224	71	.01408	5041	8.426	357911	4.141
12	.08333	144	3.464	1728	2.290	72	.01390	5184	8.485	373248	4.160
13	.07692	169	3.605	2197	2.351	73	.01372	5329	8.544	389017	4.179
14	.07143	196	3.742	2744	2.410	74	.01354	5476	8.602	405224	4.198
15	.06667	225	3.873	3375	2.466	75	.01336	5625	8.660	421875	4.217
16	.06250	256	4.000	4096	2.520	76	.01318	5776	8·718	438976	4·236
17	.05882	289	4.123	4913	2.571	77	.01301	5929	8·775	456533	4·254
18	.05555	324	4.243	5832	2.621	78	.01284	6084	8·832	474552	4.273
19	.05263	361	4.359	6859	2.669	79	.01267	6241	8.888	493039	4.291
20	.05000	400	4.472	8000	2.714	80	.01250	6400	8.944	512000	4.309
21	.04762	441	4.582	9261	2.759	81	.01234	6561	9.000	531441	4.327
22	.04545	484	4.690	10648	2.802	82	.01220	6724	9.055	551368	4.344
23	.04348	529	4.796	12167	2.844	83	.01206	6889	9.110	571787	4.362
24	.04166	576	4.899	13824	2.884	84	.01192	7056	9.165	592704	4.379
25	.04000	625	5.000	15625	2.924	85	.01178	7225	9.219	614125	4.397
26	.03846	676	5.099	17576	2.962	86	.01164	7396	9.274	636056	4.414
27	.03704	729	5.196	19683	3.000	87	.01150	7569	9.327	658503	4.431
28	.03571	784	5.292	21952	3.036	88	.01137	7744	9.381	681472	4.448
29	.03448	841	5.385	24389	3.072	89	.01124	7921	9.434	704969	4.465
30	.03333	900	5.477	27000	3.107	90	.01111	8100	9.487	729000	4.481
31	.03226	961	5.568	29791	3.141	91	.01099	8281	9.539	753571	4.498
32	.03125	1024	5.657	32768	3.175	92	.01087	8464	9.592	778688	4.514
33	.03030	1089	5.744	35937	3.207	93	.01075	8649	9.644	804357	4.531
34	.02941	1156	5.831	39304	3.240	94	.01064	8836	9.695	830584	4.547
35	.08854	1225	5.916	42875	3.271	95	.01053	9025	9.747	857375	4.563
36	.02777	1296	6.000	46656	3.302	96	.01042	9216	9.798	884736	4.579
37	.02702	1369	6.083	50653	3.332	97	.01031	9409	9.849	912673	4.595
38	.02632	1444	6.164	54872	3.362	98	.01020	9604	9.899	941192	4.610
39	.02564	1521	6.245	59319	3.391	99	.01010	9801	9.950	970299	4.626
40	.02500	1600	6.324	64000	3.420	100	.01000	10000	10.000	1000000	4.642
41	.02435	1681	6.403	68921	3.448	101	.00990	10201	10.050	1030301	4.657
42	.02381	1764	6.481	74088	3.476	102	.00980	10404	10.100	1061208	4.672
43	.02326	1849	6.557	79507	3.503	103	.00971	10609	10.149	1092727	4.687
44	.02274	1936	6.633	85184	3.530	104	.00962	10816	10.199	1124864	4.703
45	.02222	2025	6.708	91125	3.557	105	.00953	11025	10.247	1157625	4.718
46	.02174	2116	6.782	97336	3.583	106	.00944	11236	10.296	1191016	4.733
47	.02128	2209	6.856	103823	3.609	107	.00935	11449	10.344	1225043	4.747
48	.02083	2304	6.928	110592	3.634	108	.00926	11664	10.392	1259712	4.762
49	.02041	2401	7.000	117649	3.660	109	·00917	11881	10.440	1295029	4.777
50	.02000	2500	7.071	125000	3.684	110	·00909	12100	10.488	1331000	4.791
51	.01961	2601	7.141	132651	3.708	111	.00901	12321	10.536	1367631	4.806
52	.01923	2704	7.211	140608	3.732	112	.00893	12544	10.583	1404928	4.820
53	.01885	2809	7.280	148877	3.756	113	.00885	12769	10.630	1442897	4.834
54	.01852	2916	7.348	157464	3.780	114	.00877	12996	10.677	1481544	4.849
55	.01818	3025	7.416	166375	3.803	115	.00869	13225	10.724	1520875	4.863
56	.01786	3136	7.483	175616	3.826	116	.00861	13456	10.770	1560896	4.877
57	.01752	3249	7.550	185193	3.849	117	.00854	13689	10.817	1601613	4.891
58	.01724	3364	7.616	195112	3.871	118	.00847	13924	10.863	1643032	4.905
59	.01694	3481	7.681	205379	3.893	119	.00840	14161	10.909	1685159	4.919
60	.01667	3600	7.746	216000	3.915	120	.00833	14400	10.954	1728000	4.932

TABLE OF RECIPROCALS, SQUARES AND SQUARE ROOTS, CUBES AND CUBE ROOTS.—*Continued.*

No.	Recip-rocals.	Square	Sq. root.	Cube.	Cube root.	No.	Recip-rocals.	Square	Sq. root.	Cube.	Cube root.
121	.00826	14641	11.000	1771561	4.946	181	.00552	32761	13.454	5929741	5.657
122	.00819	14884	11.045	1815848	4.960	182	.00549	33124	13.491	6028568	5.667
123	.00812	15129	11.090	1860867	4.973	183	.00546	33489	13.528	6128487	5.677
124	.00805	15376	11.135	1906624	4.987	184	.00543	33856	13.565	6229504	5.688
125	.00799	15625	11.180	1953125	5.000	185	.00540	34225	13.601	6331625	5.698
126	.00793	15876	11.225	2000376	5.013	186	.00537	34596	13.638	6434856	5.708
127	.00787	16129	11.269	2048383	5.026	187	.00534	34969	13.675	6539203	5.718
128	.00781	16384	11.314	2097152	5.040	188	.00531	35344	13.711	6644672	5.729
129	.00775	16641	11.358	2146689	5.053	189	.00528	35721	13.748	6751269	5.739
130	.00769	16900	11.402	2197000	5.066	190	.00526	36100	13.784	6859000	5.749
131	.00763	17161	11.445	2248091	5.079	191	.00523	36481	13.820	6967871	5.759
132	.00757	17424	11.489	2299968	5.092	192	.00520	36864	13.856	7077888	5.769
133	.00751	17689	11.532	2362637	5.104	193	.00517	37249	13.892	7189057	5.779
134	.00745	17956	11.576	2406104	5.117	194	.00514	37636	13.928	7301384	5.789
135	.00739	18225	11.619	2460375	5.130	195	.00511	38025	13.964	7414875	5.799
136	.00734	18496	11.662	2515456	5.142	196	.00508	38416	14.000	7529536	5.809
137	.00729	18769	11.705	2571353	5.155	197	.00506	38809	14.036	7645373	5.819
138	.00724	19044	11.747	2628072	5.168	198	.00504	39204	14.071	7762392	5.828
139	.00719	19321	11.790	2685619	5.180	199	.00502	39601	14.107	7880599	5.838
140	.00714	19600	11.832	2744000	5.192	200	.00500	40000	14.142	8000000	5.848
141	.00709	19881	11.874	2803221	5.205	201	.00498	40401	14.177	8120601	5.858
142	.00704	20164	11.916	2863288	5.217	202	.00496	40804	14.213	8242408	5.867
143	.00699	20449	11.958	2924207	5.229	203	.00493	41209	14.248	8365427	5.877
144	.00394	20736	12.000	2985984	5.241	204	.00490	41616	14.283	8489664	5.887
145	.00689	21025	12.041	3048625	5.254	205	.00487	42025	14.318	8615125	5.896
146	.00684	21316	12.083	3112136	5.266	206	.00485	42436	14.353	8741816	5.906
147	·00679	21609	12.124	3176523	5.278	207	.00483	42849	14.387	8869743	5.915
148	.00675	21904	12.165	3241792	5.290	208	.00481	43264	14.422	8998912	5.925
149	.00671	22201	12.206	3307949	5.301	209	.00478	43681	14.457	9129329	5.934
150	.00667	22500	12.247	3375000	5.313	210	.00476	44100	14.491	9261000	5.944
151	.00662	22801	12.288	3442951	5.325	211	.00474	44521	14.526	9393931	5.953
152	.00658	23104	12.329	3511808	5.337	212	.00471	44944	14.560	9528128	5.963
153	·00654	23409	12.369	3581577	5.348	213	.00469	45369	14.594	9663597	5.972
154	.00650	23716	12.410	3652264	5.360	214	.00467	45796	14.629	9800344	5.982
155	.00645	24025	12.450	3723875	5.372	215	.00465	46225	14.663	9938375	6.991
156	.00641	24336	12.490	3796416	5.383	216	.00463	46656	14.697	10077696	6.000
157	·00637	24649	12.530	3869893	5.395	217	.00461	47089	14.731	10218313	6.009
158	·00633	24964	12.570	3944312	5.406	218	.00459	47524	14.765	10360232	6.018
159	.00629	25281	12.610	4019679	5.417	219	.00457	47961	14.799	10503459	6.028
160	.00625	25600	12.649	4096000	5.429	220	.00455	48400	14.832	10648000	6.037
161	.00621	25921	12.688	4173281	5.440	221	.00453	48841	14.866	10793861	6.046
162	·00617	26244	12.728	4251528	5.451	222	.00451	49284	14.900	10941048	6.055
163	·00613	26569	12.767	4330747	5.462	223	.00449	49729	14.933	11089567	6.064
164	.00609	26896	12.806	4410944	5.474	224	.00447	50176	14.967	11239424	6.073
165	.00605	27225	12.845	4492125	5.485	225	.00445	50625	15.000	11390625	6.082
166	.00601	27556	12.884	4574296	5.496	226	.00443	51076	15.033	11543176	6.091
167	.00597	27889	12.923	4657463	5.507	227	.00441	51529	15.066	11697083	6.100
168	.00594	28224	12.961	4741632	5.518	228	.00439	51984	15.100	11852352	6.109
169	.00591	28561	13.000	4826809	5.529	229	.00437	52441	15.133	12008989	6.118
170	.00588	28900	13.038	4913000	5.540	230	.00435	52900	15.166	12167000	6.127
171	.00584	29241	13.077	5000211	5.550	231	.00433	53361	15.199	12326391	6.136
172	.00580	29584	13.115	5088448	5.561	232	.00431	53824	15.231	12487168	6.144
173	·00577	29929	13.153	5177717	5.572	233	.00429	54289	15.264	12649337	6.153
174	·00574	30276	13.191	5268024	5.583	234	.00427	54756	15.297	12812904	6.162
175	.00570	30625	13.229	5359375	5.593	235	.00425	55225	15.330	12977875	6.171
176	.00567	30976	13.266	5451776	5.604	236	.00423	55696	15.362	13144256	6.180
177	.00564	31329	13.304	5545233	5.615	237	.00421	56169	15.395	13312053	6.188
178	.00561	31684	13.342	5639752	5.625	238	.00419	56644	15.427	13481272	6.197
179	.00558	32041	13.379	5735339	5.636	239	.00418	57121	15.460	13651919	6.206
180	.00555	32400	13.416	5832000	5.646	240	.00417	57600	15.492	13824000	6.214

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