OPTICS

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A MANUAL FOR STUDENTS

BY

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PREFACE.

THIS book is designed primarily for the use of ophthalmic students. There are many excellent manuals on the diseases of the eye, but in them the subject of Optics is relegated to a single chapter where it is necessarily dealt with in only a cursory way.

The student who refers to the standard works on Optics will probably not find them exactly adapted to meet his requirements. With a great deal of information about rainbows and halos they will not tell him what is the actual size of a retinal image, or what is the effect of a decentred lens on the incident light.

Feeling the importance of a more accurate knowledge of Optical principles than is at present attainable by those who cannot afford to devote much time to the subject, I have endeavoured to place at the reader's disposal within reasonable compass such a knowledge of Optics as would be of use to an ophthalmic surgeon in his daily practice.

I am not without hope that this book may also be of use to mathematical students as an introduction to more advanced works on Geometrical Optics such as that by

Professor Heath. To this work and to Mr Preston's excellent *Theory of Light* I am under deep obligations, while my data for the last chapter have been taken from Dr Landolt's article in the *Traité Complet d'Ophtalmologie* by De Wecker and Landolt.

To illustrate all the points of fundamental importance throughout the book a large number of examples have been worked out. The student, to whom the subject is entirely new, is advised to omit in the first reading the following chapters, which require a slight acquaintance with the Differential Calculus: Chaps. VI, IX, XIV, XV, XVI.

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CONTENTS.

CORRIGENDA.

PAGE

112 l. 6 from bottom. *Read* "$PQRR_1$ represents the cross-section of the reflected pencil in the neighbourhood of the mirror."

113 l. 9 omit "reflecting."

123 l. 7 from bottom. *For* $\text{L}^{\text{t}} \dfrac{SN}{\tan SCN}$ *read* $\text{L}^{\text{t}} \dfrac{SN}{\tan NCS}$.

197 In Fig. 65 the caustic is incorrectly drawn. It should be more acute as represented in Fig. 64. The lower limb of the caustic should touch RT produced at a point about $3\frac{1}{2}$ ins. from P.

269 Question (1). *Read* Shew that incident parallel rays will emerge parallel, that when the convex surface is turned towards a radiant point it will act as a diverging lens, and when turned the reverse way at what finite distance it will act as a converging lens.

Question (3), l. 3. *Read* "as long as the radiant point does not lie at the centre of curvature." Omit latter half of question.

341 l. 20. For $\dfrac{q}{p}$ read $\dfrac{q-r}{p-r}$.

385 l. 8 from bottom.

For anisometropic *read* ametropic and anisometropic.

ADDITIONAL CORRIGENDA.

PAGE

12 l. 15. *For* it vibrate less quickly
Read its vibrations travel less quickly.

193 l. 21. *Read* $v_1 = \mu u \frac{\cos^2 \phi'}{\cos^2 \phi}$.

200 l. 6. *Read* $\phi' = 0,\ p = 0$.

215 l. 8. *Read* $\frac{i}{o} = \frac{bC}{BC}$.

253 l. 31. *For* mutually the image of each other
Read mutually each the image of the other.

355 l. 10 and p. 360 l. 11.

For '*punctum proximum*'
Read '*punctum remotum.*'

398 Chap. XIII. *Read* $p > \frac{tr_1 (\mu r_1 - r_2)}{\mu (r_2^2 - r_1^2)}$.

399 last line. *Read* (14) $r \backsimeq \cdot 0106$ mm.

CHAPTER I.

INTRODUCTION.

WHEN we say that we see the sun or a tree, we really mean that we see the light that comes from them. The sun of course sends out light of its own, whereas the tree merely passes on or reflects the light which it receives from something else.

It is obvious that in neither case do we see the thing itself; we are only conscious of a certain sense-impression derived from it. The nature of this sense-impression, the way in which it is developed from a physical stimulation of the retina, and the question as to which parts of the brain are engaged in this development are problems that are still engaging the attention of physiologists and anatomists.

The science of Optics however deals with none of these questions; it is in fact confined to a study of the nature of this physical stimulus called light, and the investigation of its laws and properties. Now although it is the latter branch of the subject that chiefly demands our attention at present, a brief sketch of the nature of light will not be out of place, as it may serve to render these laws and properties more intelligible.

The phenomena of light can only be explained on the hypothesis that they are due to some sort of wave motion, though the actual kind of wave is not definitely established.

Let us consider first the kind of waves most familiar to us, namely water-waves. If a stone be thrown into the centre of a tranquil pool, the surface of the water becomes mapped out into a series of concentric circles, which steadily increase in size until finally they reach the margin of the pool. The immediate disturbance produced by the falling stone sets up a further disturbance of the water which takes the form of surface waves. Two points are specially worthy of notice:

(1) The circles always remain concentric, with their circumferences separated from each other by the same distance. This shews that the disturbance is propagated in all directions along the surface of the water at the same rate.

(2) It is the wave that moves forwards not the water. If a cork be floating on its surface it will be noticed that as each wave passes it, it merely bobs up and down, and is not carried along with the wave. The particles of water merely rise and fall in succession as the wave passes over them[1], the extent or amplitude of their movement determining the height of the wave.

Waves of course may differ from each other in frequency and wave-length, as well as in height. The wave-length or distance from crest to crest of two successive waves depends both on their frequency, and the velocity with which they are moving. Let us suppose the particles of water occupy one second in rising and one second in falling; after the

[1] The statement in the text is not quite accurate. The motion of an individual particle of water that is subject to waves depends on the depth of the water and other factors. If the particle considered be far removed from the bottom, its motion is circular.

lapse of two seconds any one particle is then in precisely the same phase of its motion as that in which it was at first. A wave crest will then be formed once in every two seconds and if we further imagine these waves to be travelling forwards with a velocity of 10 feet per second, the length of the wave or distance from crest to crest must be 20 feet[1].

But further, waves may differ from each other in a still more essential way. We have been considering waves that are due to the medium heaving up and down: a periodic to and fro movement in a series of particles will produce a wave of quite a different nature; in fact this is the kind of movement that gives rise to sound. There may indeed be no motion at all, and yet, if it be periodic in space and time, the disturbance in the medium is rightly called a wave, for instance if the same conditions of temperature or of electrical state recur regularly at equal intervals of distance at the same time, and also present themselves at equal intervals of time at the same place, a wave of heat or a wave of electrical displacement may be said to pass over the medium. A given medium then may be quite competent to transmit some kinds of waves though absolutely impenetrable to others. Sound-waves for instance will travel readily through air, but not through a vacuum, though this is quite transparent to light-waves.

[1] This relation is expressed by the simple formula

$$VT=\lambda,$$

where λ denotes the wave-length, V the rate at which the waves are moving, and T the time it takes for one complete undulation to pass a given point. This time T is sometimes called the undulation period, for it is the period which a given particle occupies in going through the complete cycle of changes that constitutes one undulation, from the crest of one wave to the crest of the next. The period (T) is therefore the reciprocal of the wave frequency, so the frequency may be denoted by $\frac{1}{T}$.

For our present purpose however it will be sufficient to regard light as due to an up and down motion something like that of the surface of the sea.

We have further to consider the nature of the medium in which these light-waves are formed. It must obviously exist in what is called a vacuum, as well as in all transparent bodies. Indeed from several considerations we are driven to regard it as a continuous substance permeating all space and filling up the interstices of all matter. It is so subtle that it can flow freely through the pores of the most solid objects without resistance, and yet it possesses a certain rigidity, inasmuch as it can transmit transverse vibrations. We may regard it as a kind of continuous jelly or pitch, which, while it is so fluid that it offers no resistance to bodies moving about in it, is yet capable of quivering under the influence of a luminous object. Now this property of quivering we ordinarily associate with rigid solids, for it implies a resistance to change of shape. The two properties are not however absolutely antagonistic. Pitch, for instance, is a semi-fluid, and yet it can be made to vibrate like a tuning-fork, provided that vibrations of sufficient frequency are set up. The nature of the ether, as this luminiferous medium is called, may appear a little more intelligible when it is remembered that the wave frequency of light-vibrations varies from 400 billions to 700 billions a second according to the colour of the light. As far then as the transmission of light is concerned, the ether may be regarded as an extremely fluid kind of pitch capable of being made to quiver billions of times in a second.

Let us now consider a luminous object, or rather, a luminous point in free space. It is a source of energy and may be regarded as a vibrating point causing the ether which fills the space to quiver. When a disturbance is caused in water, such as that due to a falling stone, a series of concentric circular waves appears to be formed on the surface of the water. The light vibrations of ether however are not limited to one plane, but are transmitted equally in all directions throughout its substance, so that a series of concentric spherical waves is formed which spreads out and out through space. They always remain concentric, and always preserve their spherical form as long as the medium is homogeneous and isotropic[1], for all the waves will be moving on at exactly the same rate. The laws of wave-propagation are well known; the rate at which waves travel through a perfectly homogeneous medium depends solely on the nature of the medium and not at all on the properties of the wave, provided of course that the wave is of a kind which can be transmitted by the medium considered; sound waves for instance cannot be transmitted by ether. The velocity of all the waves that can be propagated through ether is equal to the square root of the ratio of its elasticity to its density

$$v = \sqrt{\frac{E}{D}}.$$

The direction in which a wave travels is obviously at right angles to the wave front, so that the light waves that we are considering must travel in the direction of the radii of their spheres, these radii being in fact called rays of light.

When we are merely considering the direction in which light is travelling we may therefore confine our attention to the direction of these rays, and it is chiefly to the consideration of rays under varied conditions, or in other words,

[1] If a medium be not isotropic, *i.e.* if its elasticity in different directions be unequal, waves due to vibrations in the direction in which the elasticity is greater will travel quicker than those due to vibrations in the direction in which it is less. Such conditions obtain in many crystals and produce the phenomenon of double refraction.

to the direction in which light travels when a change of medium is encountered, that this book will be devoted.

Yet for the intelligent comprehension of even such laws as will be here discussed, it is necessary to bear in mind the nature of light, as being a rapid and regular succession of waves of some kind in ether. Now a thorough conception of all that is involved in the expression wave motion is somewhat difficult to grasp, and it is convenient to regard it now from one, now from another, point of view, according to the problem before us. Let us take the case of water waves again.

(1) We may confine our attention to the expanding circle of disturbance, and consider the direction of its movement.

(2) Or we may think of the actual particles of water successively oscillating up and down, neglecting for the moment the form of the wave so produced.

(3) Again since every particle exactly imitates the movement of the particle first excited, it must exert on those adjoining it precisely the same influence that the first particle excited on its neighbours.

This leads us to the great principle of Huygens, which may be briefly stated in the following way:—Every vibrating particle sets up a new and independent wave system, so that innumerable elementary wave systems are simultaneously produced. These by their cooperation or interference give rise to that resultant wave system by which the medium appears at any moment to be moved.

Each elementary wave system acts independently, and superimposes itself upon any others that may be simultaneously disturbing the medium. A familiar illustration of this is afforded by the tiny ripples formed by falling raindrops on the surface of the sea.

The cooperation of two wave systems of equal amplitude that meet each other can be easily observed in any liquid. At all the points where two wave crests meet, the surface of the fluid rises to twice the height of either: where the troughs of the two waves meet, it sinks to double the depth. At those points where the crests of one system of waves coincide with the troughs of the other system, the upheaving and depressing forces are in equilibrium, and the surface of the fluid remains at its original level.

The accompanying diagram (Fig. 1), which is a sectional view of the spherical disturbances produced by light, may

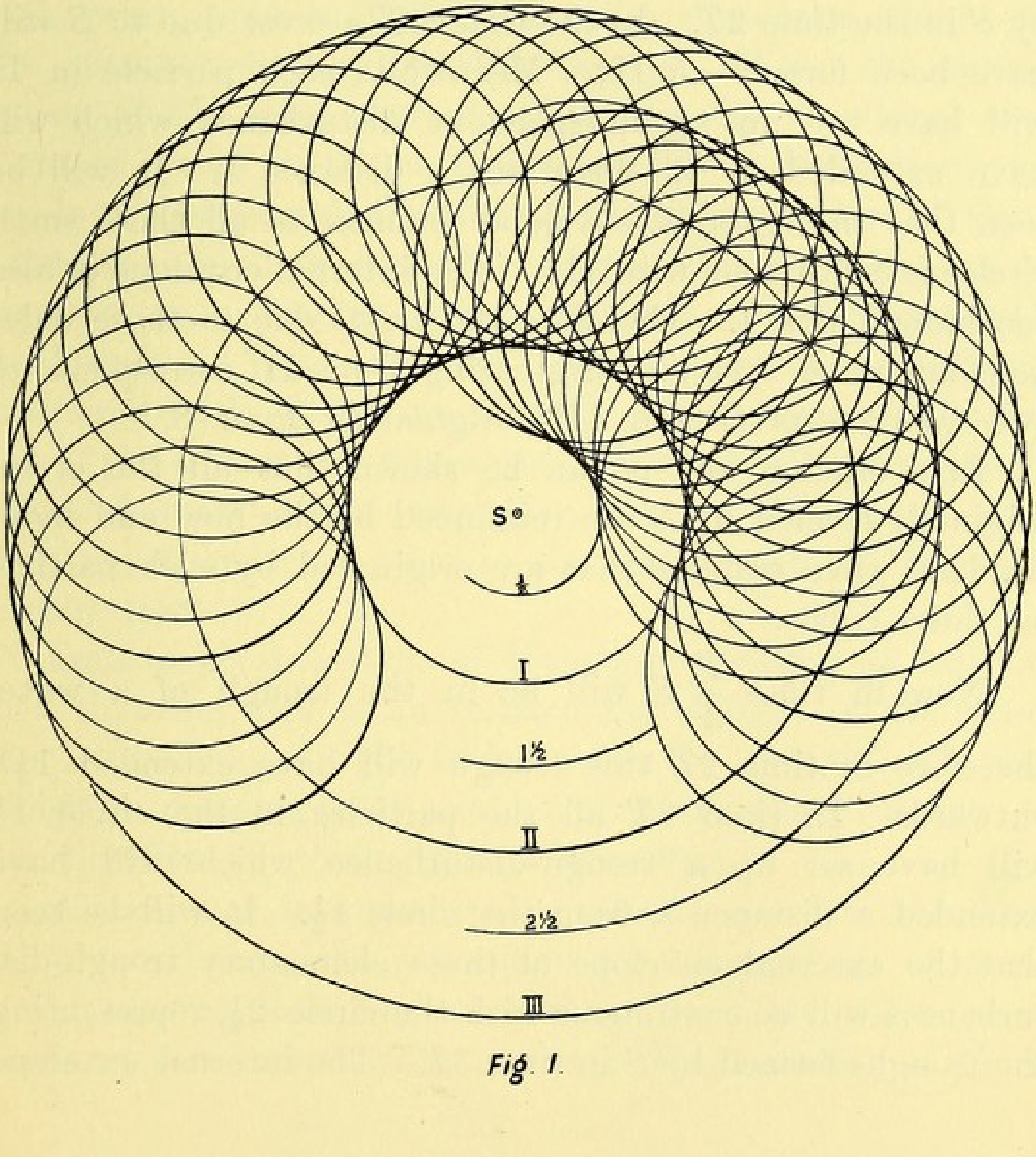

Fig. 1.

make this more clear. Let S represent the source of light emitting light of a wave-length (λ) S I. Then if T be the undulation period, after the lapse of T the phase of S will occur throughout the circle of particles denoted by I. For instance, a crest will be formed at S at periodic intervals of time T. Similarly crests will be formed at periodic intervals of space λ. The circles I, II, III will represent crests.

Now, according to Huygens' principle, each particle in these circles will set up a new and independent wave system.

Let us consider circle II, which will have been formed by S in the time $2T$. In the time $3T$ a crest due to S will have been formed at III. Meanwhile each particle in II will have set up an independent disturbance which will have extended in all directions a distance λ. It will be seen that the external envelope common to all these small circles is continuous with III. The internal envelope is also continuous with I. Therefore the crests due to these independent wave systems originating from II correspond to two wave-crests (I and III) originating from S.

In a similar way it can be shewn that all the crests originally formed by S are reinforced by the new and independent wave systems that are originated by each particle on these crests.

Now in time $\frac{T}{2}$ S will be in the trough of a wave, therefore in time $2T$ this trough will have extended $1\frac{1}{2}\lambda$ outwards. In time $3T$ all the particles in the circle $1\frac{1}{2}$ will have set up a trough-disturbance which will have extended a distance λ from the circle $1\frac{1}{2}$. It will be seen that the external envelope of these elementary trough-disturbances will be continuous with the circle $2\frac{1}{2}$, representing the troughs formed by S in time $3T$. The internal envelope will be continuous with the troughs formed by S in time T, *i.e.* the circle of radius $\frac{\lambda}{2}$.

Mathematical investigation confirms Huygens' theory completely: by taking into account the effect produced by each elementary wave, the form of the principal wave system can be exactly foretold. Finally the phenomena of diffraction shew that this is undoubtedly the real mechanism of the undulatory movement. The simple conception of the direct propagation movement from a single centre outwards is then erroneous. It is the innumerable elementary wave systems which, here reinforcing, there interfering with each other, produce those expanding concentric circles of the resultant wave system that we see. When the propagation of the wave movement is free, the individual elementary waves withdraw themselves from observation, the result of their combination being the only evidence of their existence. In such cases the simple view of wave movement as an expanding sphere or circle of disturbance (1) is allowable. When however the transmission of light is interfered with, should an obstacle occur which suppresses any of the elementary waves, the adjoining ones immediately assert themselves and may give rise to phenomena that were never understood before the appearance of Huygens' famous tract.

From a consideration of Huygens' principle it will be seen that the idea of an isolated ray of light in the sense of a single line of light is inconceivable, and the only intelligible meaning that can be given to the term ray is the direction in which light is travelling.

We are now in a position to consider some of the laws of light and from the conception we have gained of wave movement, we shall find that they inevitably follow as a necessary consequence of its nature. The radiation of light in a homogeneous medium may be considered from the first point of view that we have discussed.

Law 1. In a homogeneous medium light from a luminous point is propagated in every direction in straight lines. As has been shewn above, a luminous point in free space gives rise to a series of concentric spherical waves which spread out and out in the direction of the radii of the spheres. In other words, light travels away from its source in every direction in straight lines.

Law 2. The amount of illumination varies inversely as the square of the distance from the source of light.

The luminous object is obviously to be regarded as a source of energy, inasmuch as it creates a wave disturbance in the surrounding medium. These waves will travel on and on through the medium, and so convey away the energy that instigates them, until they meet with some absorbing obstacle which quenches them, when the energy will re-appear in some other form, such as heat.

Now, since the light waves in free space have a spherical contour, the energy of the source of light must be distributed over the surface of these spheres. As is well known the surface of a sphere is proportional to the square of its radius ($A = 4\pi r^2$); and as the total energy of each sphere is the same, it follows that the part which is distributed over a given area must vary inversely as the square of the radius of the sphere. For instance, if the radius of the sphere be doubled, its surface will be quadrupled; the light then which falls on a given area of the large sphere will be one-fourth of that which falls on an equal area of the small sphere. For example, we find experimentally that the illumination produced by one candle at a given distance is equal to that produced by four candles at twice the distance.

Up to this point we have been considering light travelling in a homogeneous medium so that its velocity remains constant throughout its course. The problems that will now occupy our attention demand some knowledge of the velocity of light and the conditions which modify it[1].

The velocity of light has been measured in several ways: one of the most exact is that of Foucault. A narrow beam of light is allowed to enter a dark room by a small hole and is then received on a rotating plane mirror (R) that is placed in its course (Fig. 2). In one part of the revolution

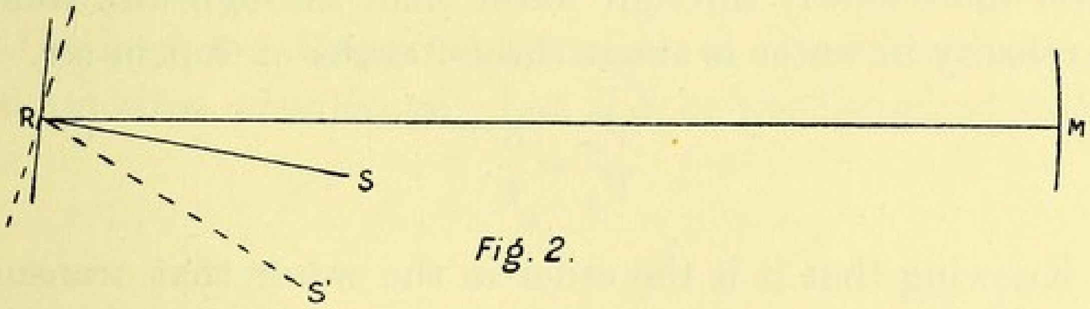

Fig. 2.

the reflected beam is directed towards a concave mirror (M), so placed at the other end of the room that it is reflected back again to the rotating mirror, and thence back to the hole at which it first entered. Now, were the transmission of light absolutely instantaneous, however rapidly the plane mirror rotated, the light would be reflected back to the hole. When however the mirror is made to revolve exceedingly quickly, it turns through a sensible angle during the interval that the light is travelling from it to the fixed mirror and back again. Consequently the reflected beam does not return exactly along its original course to the hole but it is deflected to one side. By noting the extent

[1] Ordinary sunlight consists of waves of very various periods, those of relatively long period giving to the observer the sensation of red light, while those of shorter period excite the sensation of violet light.

In dense media it is found that light waves of short period (*i.e.* of great frequency) travel at a slower rate than those of long period. Thus violet light travels slower in glass than red light, and hence is refracted more. In order to avoid unnecessary complications, this fact has been neglected—so that the term light to the end of the chapter must be understood to refer to monochromatic light, *i.e.* light of definite period.

of this deflection and the number of revolutions of the mirror per second, the time taken in travelling from the rotating to the fixed mirror and back again can be calculated.

The velocity of light in air is thus found to be about 186379 miles per second. If, however, a tube of water be interposed between the two mirrors, the deflection of the light is found to be greater. Light must therefore travel more slowly through water than through air: indeed its velocity in water is about three-fourths of that in air,

$$\frac{V_w}{V_a} = \frac{3}{4}.$$

Knowing that it is the ether in the water that transmits the vibrations of light through it, we are naturally led to enquire what is the matter with the ether in the water that should make it vibrate less quickly. Is it less elastic, or is it more dense? It must be one or the other, for $V = \sqrt{\frac{E}{D}}$ (p. 5). Fresnel's view is that part of the ether in such substances as glass or water is in some way bound to the molecules of the matter, so that it moves about with them and behaves as if it were more dense[1].

Let us now consider what will happen if light travelling

[1] In some substances however the elasticity of the ether appears to be affected also. Thus the ether in doubly refracting crystals behaves as if it were more elastic in some directions than others. Further, although we cannot measure the velocity of light in metallic substances, there are certain considerations, based on their electrical conductivity, which lead us to believe that in metals the elasticity or rigidity of the ether is enormously diminished (see p. 32). Consequently the value of μ (see p. 18) for metals, if it could be determined, would probably be found exceedingly high. This indeed might almost have been anticipated from their characteristic lustre, for it is found that the amount of light reflected from a surface increases with its refractive index, as will be explained more fully in chapter III.

in one medium with a velocity V_1, enters a different medium in which its velocity is greater, V_2.

Let us suppose that the source of light is at a very great distance; the contour surface of the light waves may then be regarded as plane if a relatively small area be under consideration. We may consequently say that all portions of this area of the wave surface are travelling in the same direction, in other words, the rays corresponding to this area are all parallel.

On applying Huygens' principle to this case we easily find out what must occur. Let *MN* be the bounding surface

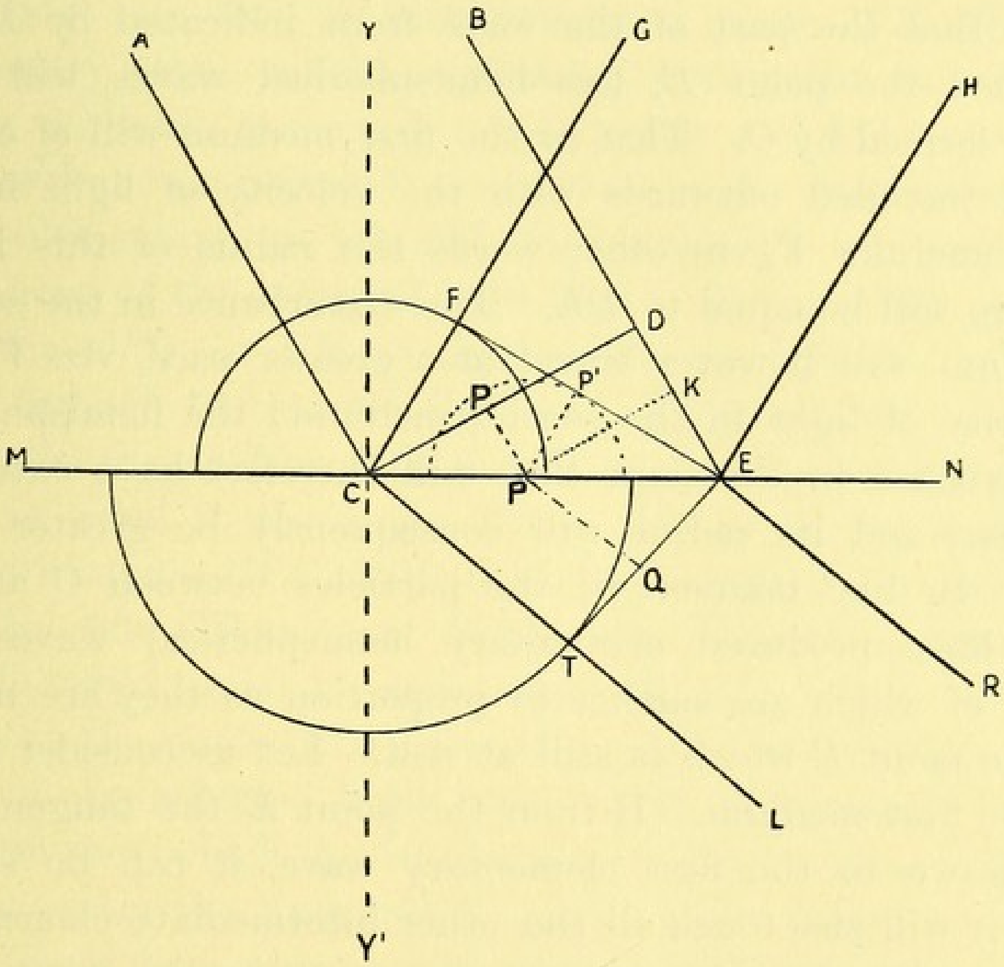

Fig.3.

between the two media, the plane of the surface being at right angles to the plane of the paper, and let *CD* represent the portion of the wave front under consideration. The direction in which it is moving is that of the lines *AC*, *BD*, which are drawn at right angles to the wave front. We may, if we prefer it, regard *AC* and *BD* as representing a beam of parallel rays of light, provided that we carefully bear in mind the meaning that is to be assigned to the term ray. In the figure, part of the wave front, that indicated by the point *C*, has already reached the boundary of the two media. The particle at *C* will then in accordance with Huygens' principle give rise to a disturbance or elementary wave, which will travel outwards in all directions, but with different velocities in the two media. By the time that the part of the wave front indicated by *D* has reached the point *E*, two hemispherical waves will have been formed by *C*. That in the first medium will of course have travelled outwards with the velocity of light in the first medium V_1, in other words the radius of this hemisphere will be equal to *DE*. The disturbance in the second medium will however travel at a greater pace, viz. V_2, the velocity of light in the second medium; the hemispherical disturbance in the same time will therefore have extended further, and its radius will consequently be greater than *DE*. In like manner all the particles between *C* and *E* will have produced elementary hemispherical waves the radii of which are smaller in proportion as they are nearer to the point *E* which is still at rest. Let us consider those in the first medium. If from the point *E* the tangent *EF* be drawn to the first elementary wave, it can be shewn that it will also touch all the other intermediate elementary waves. For let *P* represent any point in the wave front *CD* intermediate between *C* and *D*. Draw *Pp* parallel to *DE*, and *pK* parallel to *CD*, and *pP′* parallel to *CF*. Since $Pp = DK$, the disturbance at *P* will have reached *p* when that from *D* has reached *K*. During the time that the disturbance is passing from K to E, p is originating an elementary hemispherical wave the radius of which is equal to EK when the disturbance at K has arrived at E.

Now $$pP' = CF\frac{Ep}{EC} = ED\frac{EK}{ED} = EK,$$

$\therefore$ pP' is the radius of the hemispherical disturbance when the disturbance at K has reached E.

And since pP' is parallel to CF,

$$\angle pP'E = \angle CFE,$$

$\therefore$ EF is also a tangent to the hemispherical disturbance at P'.

Along this line or rather tangent plane EF, all the elementary waves will be reinforcing each other since they are in the same phase. In other words EF will represent the resultant or principal wave due to the cooperation of these elementary waves; EF must therefore represent the wave front of the reflected light, and it is travelling in the direction CFG, or EH at right angles to EF.

If through the point C the line YCY' be drawn at right angles to the surface MN it will be found that the angle GCY is equal to the angle YCA. For in the two right-angled triangles CFE, EDC, the hypothenuse CE is common, and the side CF is equal to the side ED. The triangles are therefore equal to each other and the angle CEF is equal to the angle ECD, i.e. the angle, which the wave front of the reflected light makes with the surface MN is equal to that which the incident wave front makes with it. Consequently the lines which are at right angles to these wave fronts must make equal angles with the perpendicular to the surface. In other words the ray GFC must make with the normal CY an angle equal to that which the ray AC makes with it,

$$\angle GCY = \angle ACY.$$

Moreover it is found experimentally that both incident and reflected rays lie in the same plane, namely that at right angles to the surface *MN* and in that of *CD*, in fact in the plane of the paper. We can state now the two laws of the reflection of light which have been found by experiment and we have shewn that they agree with our deductions from the undulatory theory of light.

Laws of Reflection.

(1) The incident and reflected rays lie in one and the same plane with the normal to the surface at the point of incidence, and they are on opposite sides of this normal.

(2) The angles which the incident and reflected rays make with the normal are equal to one another.

Let us now return to the consideration of the elementary hemispherical waves in the second medium. If from the point *E* the tangent *ET* be drawn to the first elementary wave, viz. that produced by *C*, it can be shewn that it will touch all the other intermediate elementary waves, that have been formed in the second medium by the disturbance of the series of particles between *C* and *E*. The tangent plane *ET* will therefore represent the resultant wave front of the light that penetrates the second medium, and it is travelling in the direction *CTL* or *ER* at right angles to *ET*. We see then that incident light travelling in the direction *AC* in the first medium is refracted on entering the second medium in such a way that it travels in it in the direction *CL*. It is also obvious that the only cause of this change of direction or refraction of rays is the different rate at which light travels in the second medium. Since the disturbance arising from *C* or the phase of *C* reaches *T* in the same time that the phase of *D* reaches *E*,

$$\frac{CT}{DE} = \frac{V_2}{V_1}.$$

Now the angle CET, which the wave front TE makes with the surface MN of the medium, is equal to the angle $Y'CL$ which the line or ray CL at right angles to the wave front makes with CY' the normal to the surface at C, and for a similar reason the angle ECD is equal to the angle ACY.

$$\therefore \sin Y'CL = \sin CET, \text{ and } \sin ACY = \sin ECD.$$

Now $$\sin ECD = \frac{DE}{CE} \text{ and } \sin CET = \frac{CT}{CE}.$$

Then $$\frac{V_2}{V_1} = \frac{CT}{DE} = \frac{\frac{CT}{CE}}{\frac{DE}{CE}} = \frac{\sin CET}{\sin ECD} = \frac{\sin Y'CL}{\sin ACY}.$$

But $Y'CL$ and ACY are the angles of refraction and incidence respectively, so that whatever the angle of incidence its sine bears a constant and unalterable relation to the sine of the angle of refraction. This relation is simply the ratio of the velocity of light in the first medium to its velocity in the second.

Now the velocity of light in most transparent substances has been either directly or indirectly found by experiment, so that a table can be formed expressing the relation that the velocity in one medium bears to that in any other. Any standard might of course be chosen, but since light travels more quickly in vacuo, *i.e.* in free ether, than in any other transparent medium, it is found convenient to refer its velocities in other media to its velocity in free space. In water for instance the velocity of light is about $\frac{3}{4}$ of that in vacuo; in glass about $\frac{2}{3}$. In vacuo, then, the rate at which light travels is about $1\frac{1}{3}$ times that in water and about $1\frac{1}{2}$ times its rate in glass,

or $$\frac{V_o}{V_w} = \frac{4}{3}; \quad \frac{V_o}{V_g} = \frac{3}{2}.$$

This ratio of the velocity in vacuo (V_o) to that in the substance under consideration is called the absolute *index of refraction* and is usually denoted by the symbol μ*. Thus for water $\mu = 1{\cdot}33$, for glass $\mu = 1{\cdot}5$. Light, then, in passing from a vacuum into glass, as in a Swan lamp, will be refracted in such a way that the ratio which the sine of the angle of incidence bears to the sine of the angle of refraction is 1·5.

For $$\frac{\sin i}{\sin r} = \frac{V_o}{V_g} = \mu_g = \frac{3}{2}.$$

This is true whatever the obliquity of the incident ray, *i.e.* whatever value we choose to assign to i.

When then light encounters a new medium, such as that we have been considering, we are entitled to make the following assertions about the way in which it behaves. While part is reflected at the bounding surface back into the old medium, part enters the new medium and travels through it, but it undergoes a change of direction so that its course in the second medium makes an angle with its previous course. The property of the medium that determines this change of direction and the nature of the change, we have just been studying. It is found by experiment that both the reflected and refracted ray as well as the corresponding incident ray lie in the same plane (in the figure that of the paper), and this is at right angles to the plane of the surface of the second medium. It is customary to call the plane, that contains the incident ray and the normal at the point of incidence, the plane of incidence. It may be as well to repeat that the terms incident and refracted rays mean the lines drawn from the

* The velocity of light in air is very nearly equal to its velocity in vacuo, so that in most calculations regarding the passage of light from air into glass or any other substance, the absolute value of μ for that substance may be employed.

point of incidence, which indicate the respective directions of the incident and refracted light. We are now in a position to state the fundamental laws of refraction in isotropic media.

Laws of Refraction, when light passes from one medium to another.

(1) The refracted ray lies in the plane of incidence.

(2) The sines of the angles of incidence and refraction are in a constant ratio for the same two media.

In anisotropic media, *e.g.* doubly refracting crystals, this statement of the laws of refraction requires some modification, as when expressed in this simple form, these laws do not hold good in such cases. Our attention however throughout these pages will be confined to the consideration of light in isotropic media and it will be sufficient therefore to say that by applying Huygens' principle to the intricate conditions that arise when the medium is not isotropic, we can predict all the various phenomena that have been observed.

Now, although the truth of Huygens' principle is thus completely established, and although we have been by its means able to discover the laws of reflection and of refraction, it is sometimes inconvenient or troublesome to apply it. We may indeed regard the reflection and refraction of light from our first point of view (p. 6), attending rather to the general form of the resultant wave system than to the effect of its elementary components.

The phenomenon of the reflection of waves may be observed in fluids. The accompanying diagram (Fig. 4) shews sufficiently clearly what takes place. Let MN represent an obstacle with a plane surface, *e.g.* a pane of glass if the experiment is performed in a basin of water or mercury. If a drop of the fluid be allowed to fall from a height on to its surface, say at S, a series of concentric ripples is seen. These spread outwards until they meet the obstacle, when

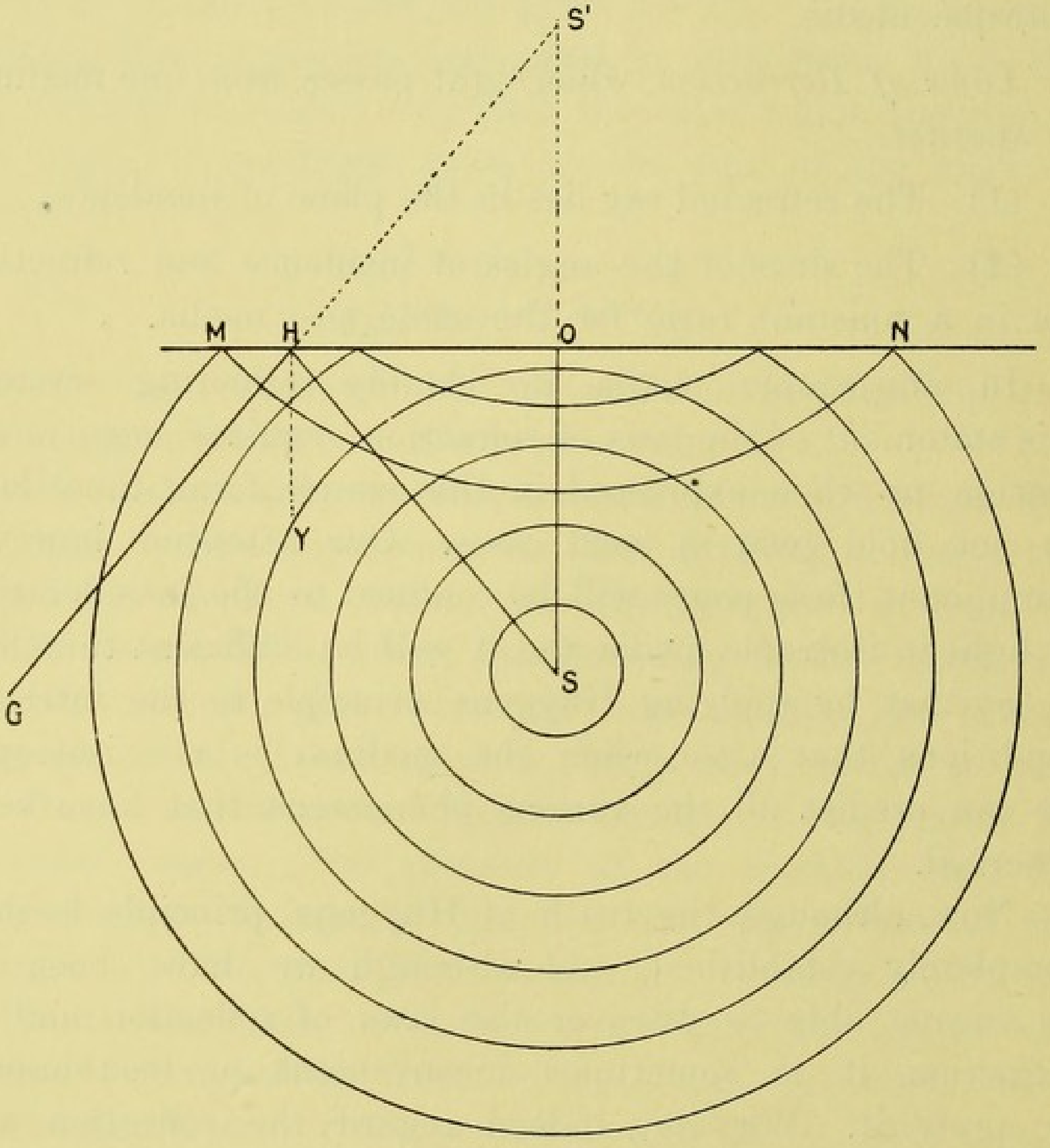

Fig. 4.

they become reflected in such a way that they form a series of concentric circular arcs which travel back from the reflecting surface crossing the ripples that are moving forwards. It will be found that the centre of these concentric circular arcs is situated at a point S' behind the obstacle exactly opposite to the spot where the drop fell. Moreover the distance of this point behind the obstacle is found by experiment to be exactly equal to the distance that the seat of original disturbance was in front of it.

$$S'O = SO.$$

Precisely the same thing occurs in the case of light waves, only their contour is spherical instead of being circular. If the point S represent a candle, and MN a plane mirror, the spherical light waves emitted by S will be reflected by the mirror, just as if they were coming from S'. They will appear therefore to come from S', *i.e.* there will appear to be a candle the same distance behind the mirror that the real candle is in front of it. The formation by reflection of virtual images, as such appearances are called, becomes therefore intelligible from this point of view. The laws of reflection also might be inferred from an inspection of the diagram, provided that the equality of $S'O$ and SO be granted. For if we consider the incident ray or radius SH for instance, which meets the reflecting surface at the point H, we shall find that it is reflected in the direction HG, which makes with the normal HY an angle GHY equal to the angle SHY.

For in the two triangles SOH, $S'OH$,

$SO = S'O$ and OH is common to both,

and the right angle $SOH =$ the right angle $S'OH$;

$\therefore$ the triangles are equal to one another and

$$\angle HS'O = \angle HSO.$$

But since YH is parallel to SS',

$$\angle GHY = \angle HS'O \text{ and } \angle SHY = \angle HSO,$$

$$\therefore \angle GHY = \angle SHY.$$

Figure 5 represents the phenomena of refraction from this point of view. Let the point S represent a luminous object in a medium bounded by the surface MN of a second medium. Further, let us suppose that the velocity of light in the second medium is twice its velocity in the first medium. Spherical waves will be propagated from

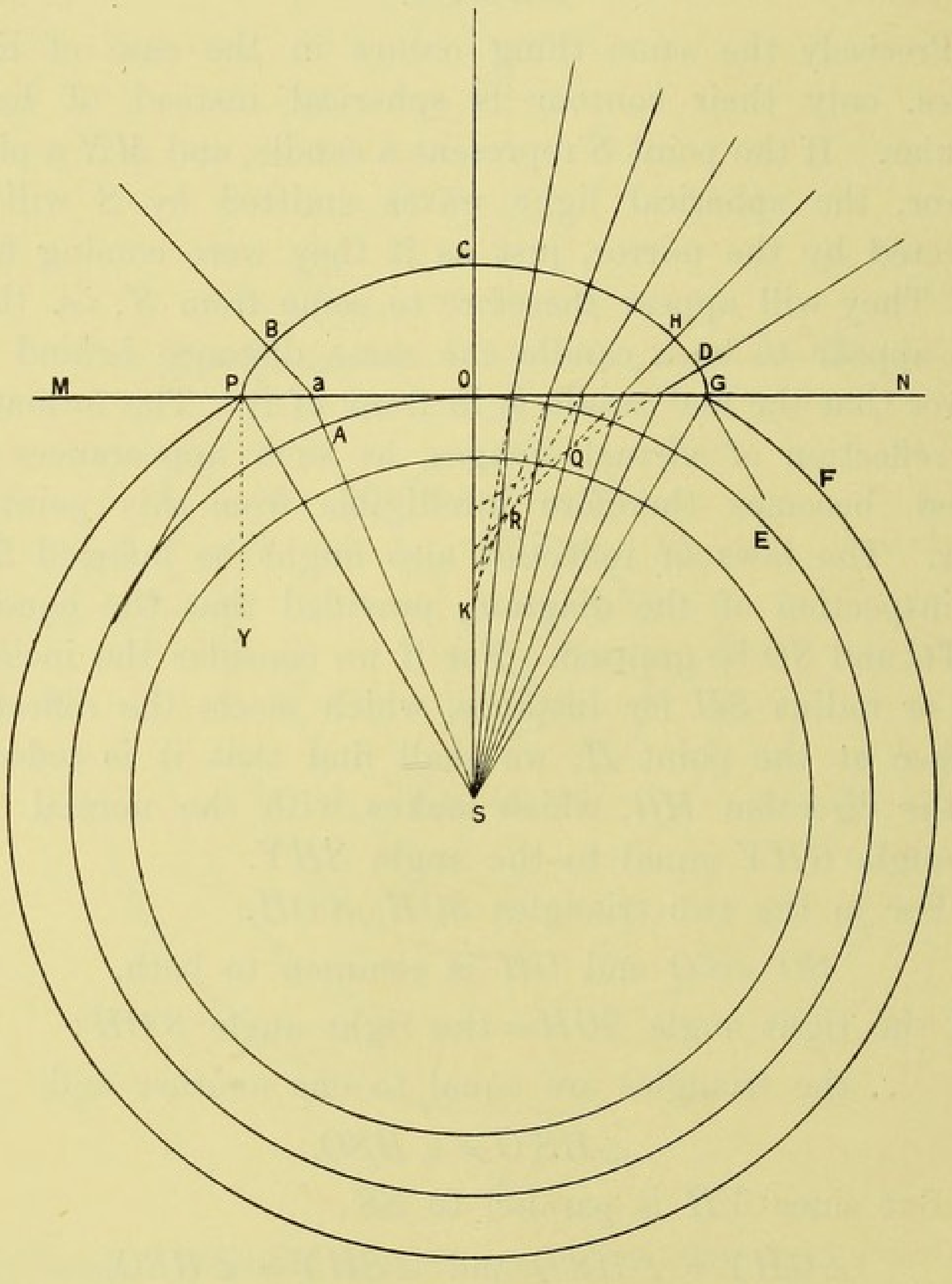

Fig. 5.

the point S which will spread outwards until they meet the surface MN of the new medium at O. They will now however undergo a remarkable alteration in shape, for that part of the wave which has entered the new medium will spread out in it with twice the velocity of that in the old. The spheroidal surface that represents the wave will present a bulging boss corresponding to the part in the second medium.

While the disturbance at E is travelling to F, that at O will be travelling to C. When F is reached, the phase of O will have reached C, the distance OC being twice that between E and F. In half the period the phase at A will have reached a at the surface of the second medium; this disturbance will now travel with twice its previous velocity, so that in the remaining half of the period it will travel a distance aB equal to the distance EF, or twice that of Aa. Adopting Huygens' construction we might represent the elementary wave formed by it by a semicircle whose radius is aB. By taking different time intervals we can then construct in this way a series of semicircles on the line MN. The curved line which touches all of them is represented by the arc BCD. Along this curved line or rather curved surface all the particles of the second medium will be in one and the same phase as those of FG. In other words the arc BCD represents the continuation in the second medium of the wave FG.

Now the arc that represents the wave in the second medium is not an arc of a circle. A small portion however in the neighbourhood of C is approximately coincident with a circular arc described from a centre K, the distance of K from O being half of that of S from O. That portion then of the wave front that is in the neighbourhood of C will travel forwards as if it came from the point K.

If the first medium were water, and the second air in which light travels at $\frac{4}{3}$ of its pace in water, the distance OC' would be $\frac{4}{3}$ of the distance EF, and the part of the wave in the neighbourhood of C' would travel on just as if it came from K', a point whose distance from O would be $\frac{3}{4}$ of that of S' from O. If the eye of an observer be situated immediately above O so as to receive this portion of the wave near C', an object at S would appear to be situated at K'. Consequently the apparent depth of a clear pool of water when viewed vertically is $\frac{3}{4}$ of the real depth, for the light, that comes from the bottom by which it is seen, on emerging into the air behaves in all respects as if it came from a plane at this distance below the surface of the water.

It will be found that the lateral parts of the arc CD are more convex than the central part. If we confine our attention to any given portion of the wave front represented by BCD, *e.g.* that part in the neighbourhood of D, we see that it must be travelling in a direction at right angles to the tangent plane at this point. Similarly, if a series of normals be drawn through the arc CD, they will represent the different directions in which each portion of the wave front represented by CD is moving respectively. In other words they represent the refracted rays of the point S. If, for example, from the point a a normal aB be drawn to the curve, light travelling in the direction SAa, will, on entering the second medium, travel in the direction aB; that is, if SAa be the incident ray under consideration aB will be the corresponding refracted ray. These refracted rays intersect each other in the first medium at different points. The line joining all these points will form a curve QRK with a cusp at K. Such a curve is called a caustic, and since the diagram represents only a sectional view of what actually occurs, we must imagine it as representing the caustic surface that would be generated by making the curved line KRQ revolve round the axis KO. The rays that pass through H and D for instance intersect at the point Q. The small portion of the wave between H and D may then, without introducing any serious error, be regarded as if it were moving away from the point Q. In other words the wave front HD appears to be coming from Q. An eye therefore which is so placed as to receive this portion only of the wave front will see an image of the point S at Q. If the eye receives a larger cone of rays, the image will be blurred, for our assumption that they then come from one and the same point Q will be incorrect. Consideration will shew that the rays received by the eye from S will appear to come from a small area RQ on the caustic surface KRQ. The point S will appear therefore as a small patch RQ; in other words the image of S will be distorted and blurred.

A glance at the diagram will shew that the refracted rays are more oblique than the incident rays. Let us consider the ray SP. On attempting to draw a normal from the point P to the curve representing the wave surface in the second medium, it will be found to coincide with the plane MN. But MN is the surface of the first medium, so it is impossible for light travelling in the direction SP to get out of the first medium.

We must now refer to a point that we have hitherto neglected. We know that when light encounters a fresh medium, only part of it is refracted, for a part is reflected at the bounding surface. The incident light SP must therefore be all reflected, as no part of it can be refracted. The angle of incidence SPY, at which this total reflection occurs, is called the critical angle. In like manner all rays from S that meet the surface MN beyond the point P will undergo total internal reflection, provided that the boundary between the media is a plane surface.

When however light is .passing the reverse way, from a rare to a dense transparent medium, this phenomenon of total reflection is not observed.

Fig. 6 represents the contour of the waves and the course of the rays in such a medium. It will be seen that in this case the refracted rays are less oblique than the incident rays. It is obvious also that a critical angle does not now

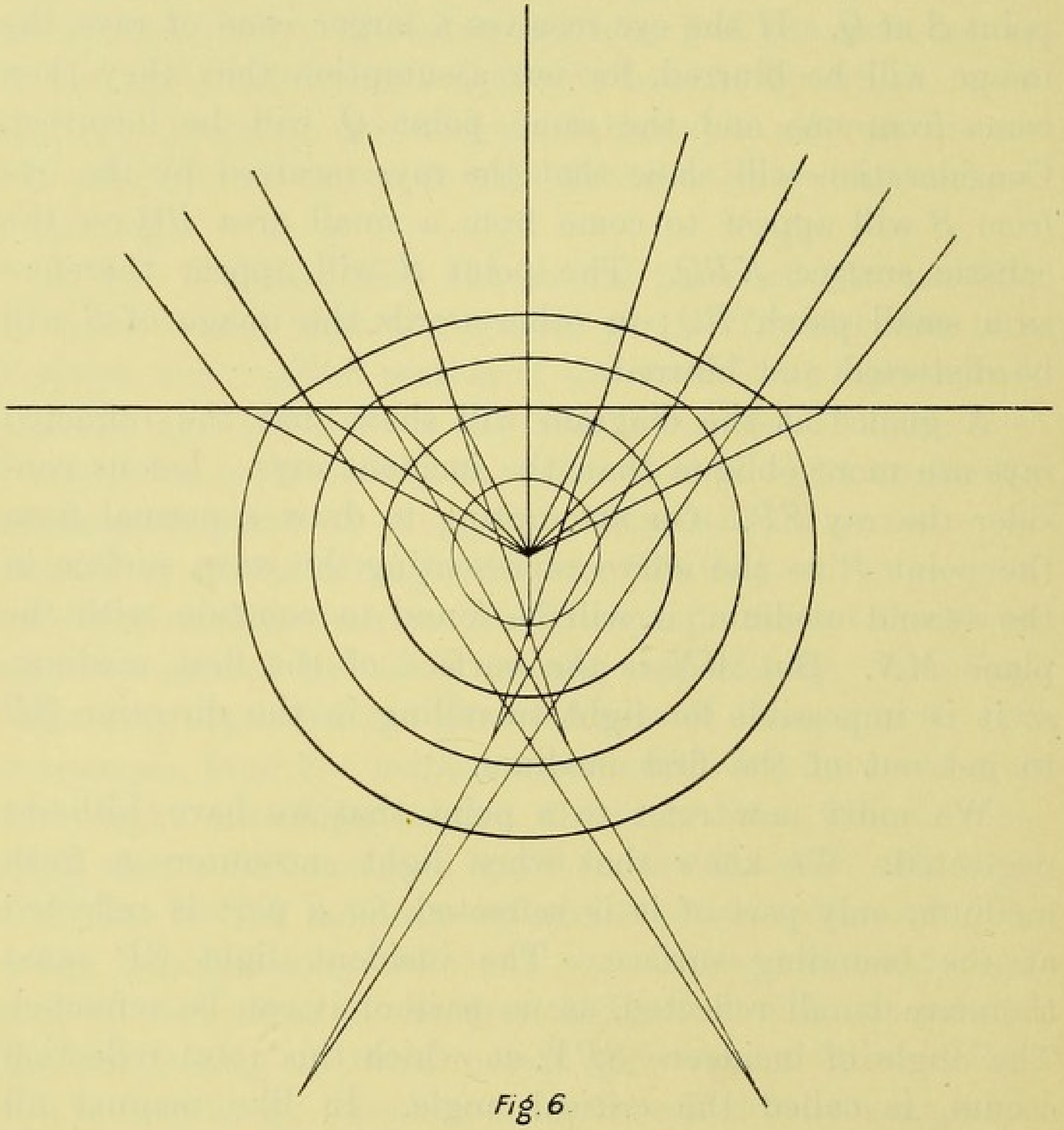

Fig. 6

exist, for however oblique the incident ray, part of the light at any rate will get out. As in the previous case, the refracted rays, on being produced backwards into the first medium, will intersect each other at different points in it, and the surface on which all these points occur forms a caustic surface with its cusp upwards. It is in fact exactly the reverse of what occurred in the previous case. Since we have rarely an opportunity of placing our eyes in a denser medium than air and so viewing objects in air, it will be unnecessary to investigate what occurs in this case more minutely.

We shall now be able to trace the course of a beam of light from a rare medium, such as air, into a dense

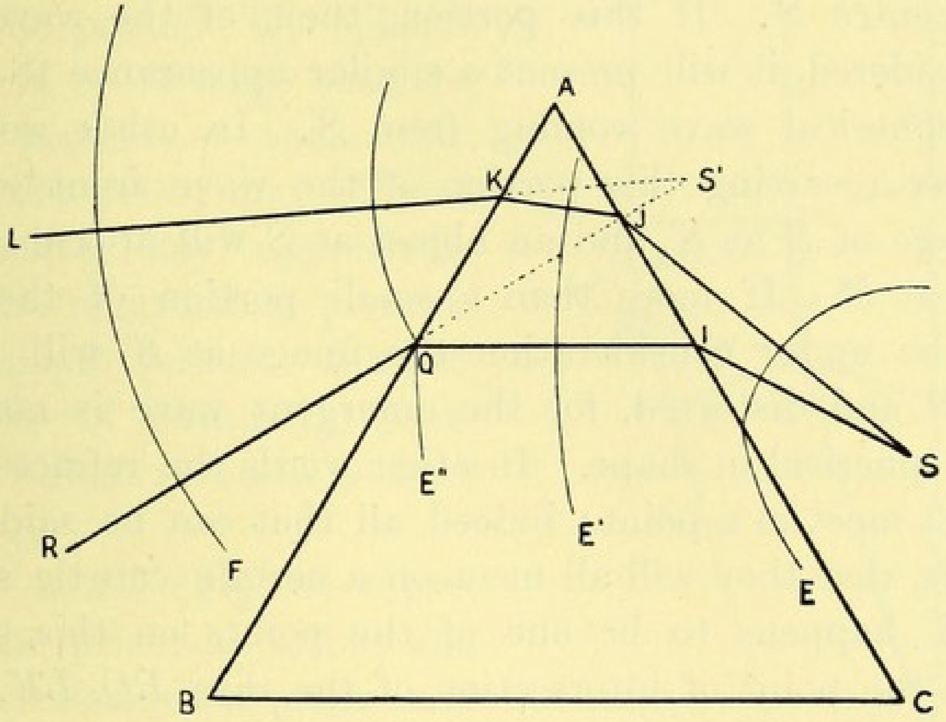

Fig. 7.

medium, such as glass. Let BAC represent a glass prism and let us suppose that the surrounding medium is air, and that the velocity of light in the air is $1\frac{1}{2}$ times greater than in the glass, *i.e.* that the relative index of refraction or the relative value of μ is 1·5. Let S be a luminous point sending out spherical waves in all directions: these on encountering the face of the prism will divide themselves into two sets: one set will be reflected at the surface, the other set will enter the glass, and to this set we will confine our attention. That part of the wave front which enters the glass will become flattened or less convex, since its rate of progression is less than the part which is still in the air. The curved lines E, E', E'', represent the shape of the wave front in the prism in three stages of its progress through it. As soon as the wave front or any portion of it passes the bounding surface of the prism as at Q, it immediately travels forward at its former speed and necessarily becomes more convex as at F. Now the arc F is approximately coincident with the arc of the circle drawn from centre S'. If this portion, then, of the wave front be considered, it will present a similar appearance to a part of a spherical wave coming from S'. In other words an observer receiving this portion of the wave front will see an image of S at S', and an object at S will appear to him to be at S'. If more than a small portion of the wave front be under consideration the image at S' will appear blurred and distorted, for the emergent wave is not accurately spherical in shape. In other words the refracted rays will not meet in a point: indeed, all that can be said about them is, that they will all meet on a certain caustic surface, and S' happens to be one of the points on this surface, and is the point of intersection of the rays RQ, LK, which are being considered.

We see then, that after passing through such a prism, waves which were originally spherical have acquired a new shape. Now, all optical instruments may be regarded from this point of view, and their degree of perfection depends simply on the extent to which they succeed in impressing the desired shape on a given form of light wave. One of the forms which is most frequently required is that in which the surface presented by the travelling wave is plane. Such a wave is called a plane wave, and since the normals drawn to any series of points on its surface are parallel to each other, we may say that all its rays are parallel. It is usual to speak of a portion of any such plane wave as a beam or pencil of parallel rays. Let us now consider how such a pencil of parallel rays will behave on meeting a prism. Should there be any difficulty in conceiving how such a plane wave is produced, we may imagine the point *S* to be removed to an infinite distance from the prism. The spheres will then be so large that the small portion of them that meets the prism may without introducing any appreciable error be regarded as presenting a plane surface. Simpler methods of obtaining plane waves will be given in another chapter.

Let *BAC* (Fig. 8) represent the prism as before and *HI* a portion of the plane wave front that meets the face *AC*. Then by the time that the phase of *H* has reached *J*, the phase of *I* will only have reached a point *M* such that the distance *IM* is two-thirds of the distance *HJ*. The wave front therefore will be deflected so as to present the aspect *JM* or *KN* in the glass. On emerging from the prism it will be again deflected, for the upper part of the wave gets out first and then gains on the

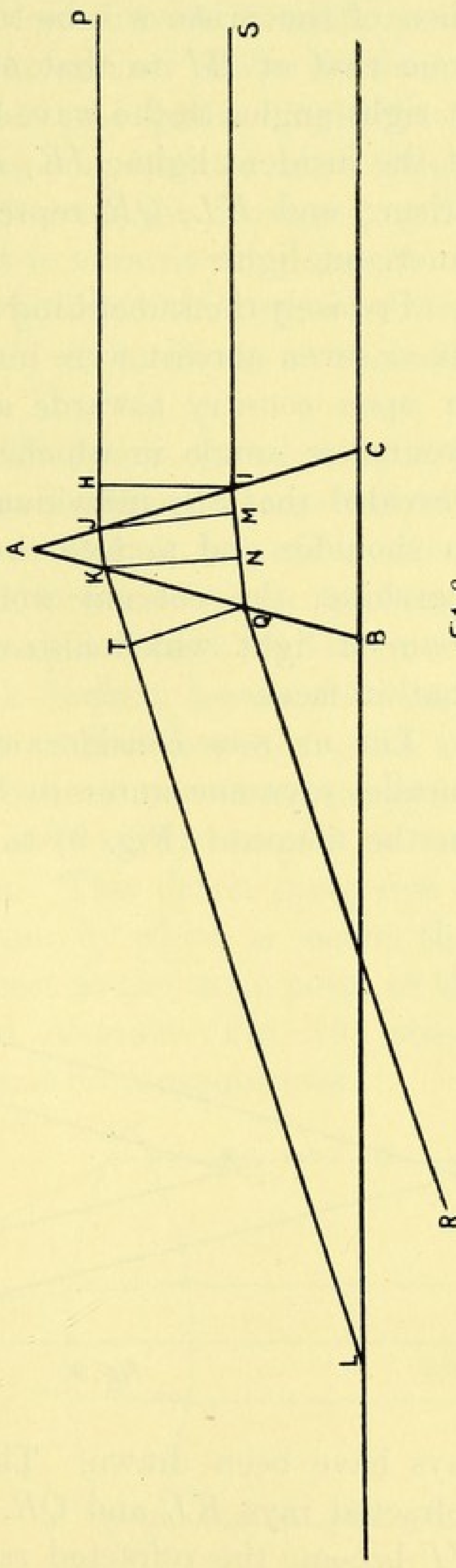

Fig. 8.

part still forcing its way through the glass. The effect then of the prism will be to change the aspect of the wave from that of *HI* to that of *TQ*. The rays *PJ*, *SI* drawn at right angles to the wave front *HI*, represent the direction of the incident light; *JK*, *IQ* its direction while within the prism; and *KL*, *QR* represent the final direction of the emergent light.

Precisely the same thing would occur if a column of soldiers six or seven abreast were marching in the direction *SI* or *PH* in open country towards a wedge-shaped piece of rough ground or jungle in which they had to take shorter steps. Provided that the individual soldiers had to keep shoulder to shoulder and to face the direction in which they were marching, the column would wheel round just like the beam of light which also can only travel in the direction that it faces.

Let us now consider what will happen if a pencil of parallel rays encounter two similar prisms set base to base. In the diagram (Fig. 9) to avoid confusion only four such

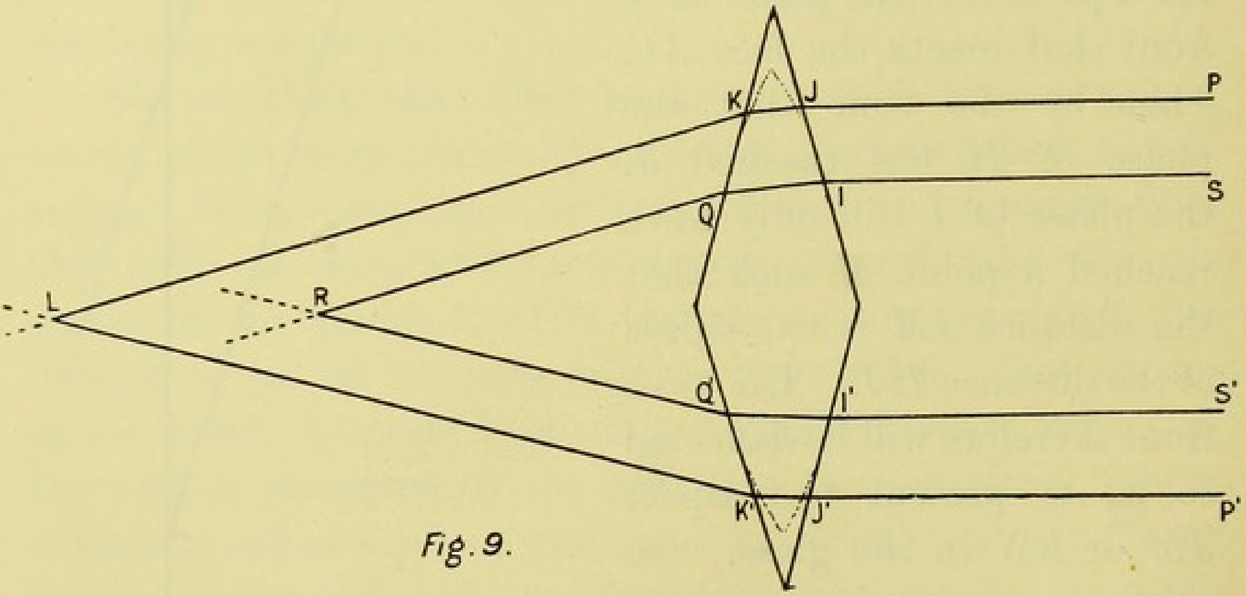

Fig. 9.

rays have been drawn. The rays *PJ* and *SI* become the refracted rays *KL* and *QR*. In like manner the rays *P'J'*, *S'I'* become the refracted rays *K'L*, *Q'R*. It will be noticed that the rays from S and S' cross each other at the point R, and that those from P and P' cross at L. Similarly the intermediate rays will intersect at various points between L and R. Now it is obvious that the prism might be so bevelled at the points J and K as to present a more oblique refracting surface so that its effect on the incident ray PJ could be so increased as to direct it towards the point R. If the lower prism were similarly bevelled the four rays from P, S, S', and P' respectively would all intersect at the same point R.

When the surface of a double prism of this kind is so bevelled that all the parallel rays incident on one face of it intersect in the same point after refraction, it is called a lens. Now it is found that if the double prism be given a spherical surface, this condition is very nearly attained, and since it is an easy matter to grind such a surface, lenses are almost always made in this way. They are not however quite the right shape, for the periphery of the lens refracts too much, in fact the bevelling process has been carried too far, so that there remains an over-correction. This defect gives rise to what is called spherical aberration, by which is meant that the peripheral rays do not intersect at the same point as the more axial rays. The annexed diagram (Fig. 10) shews clearly the nature of the defect and its consequences.

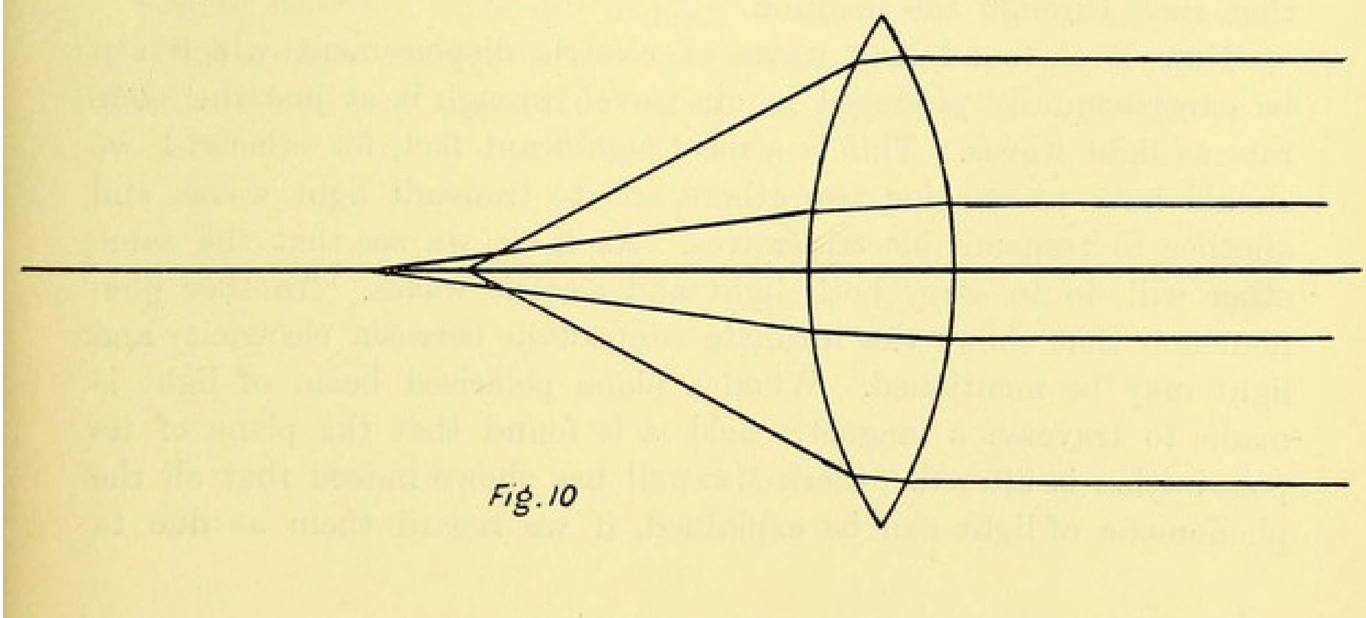

Fig. 10

Note. In the preceding pages we have adopted the simplest conception of wave movement, as it is sufficient to explain the phenomena of light with which we had to deal. There are however certain phenomena which force us to a wider interpretation of the term wave when applied to light. Indeed there will be few readers who will not find a considerable difficulty in understanding how the ether can be so perfectly fluid as to offer no resistance to the movement of bodies in it, and yet be able to transmit transverse vibrations.

The most obvious property of ether is its extreme fluidity. In what way can it so strenuously resist a change of state as to transmit a periodic disturbance with such enormous speed? Let us consider the most familiar fluid—water. Chemical analysis shews that it consists of two constituents, hydrogen and oxygen, and further, that an enormous expenditure of energy is required to decompose it into these constituents. Water then which scarcely resists a change of shape, strenuously opposes this change of state. If we imagine then the ether to be similarly composed of two constituents we shall have no difficulty in realizing that although perfectly fluid, it may yet resist decomposition with the greatest vigour; in other words its rigidity or elasticity with reference to its composition may be enormous.

Now it has already been stated that any disturbance whether of temperature, or indeed of any other state, is rightly called a wave, if it be periodic in space and time.

Our conception then of a light wave will be that of a periodic decomposition and combination of these two ether constituents. What has been loosely called the quivering of ether under the influence of light will now be regarded as a succession of waves of decomposition that rush through the medium.

Now it is found that waves of electric displacement which can be experimentally produced in air travel through it at just the same rate as light waves. This is a most significant fact, for otherwise we should have to imagine two ethers, one to transmit light waves, and another to transmit electric waves. As it is, we see that the same ether will do to carry both light and electric waves. Another phenomenon that shews the intimate connection between electricity and light may be mentioned. When a plane polarised beam of light is made to traverse a magnetic field it is found that the plane of its polarisation is altered. Clerk-Maxwell has shewn indeed that all the phenomena of light can be explained, if we regard them as due to waves of electric displacement of extreme frequency. Although electric waves have lately been produced with a frequency of many millions per second, attempts to produce them of the wave frequency of light (400 billions per second) have hitherto proved unsuccessful.

Professor Lodge has done great service by still further simplifying these conceptions. He regards positive and negative electricity, as the names that are ordinarily applied to the two essential elements that when combined form ether, so that electric energy is simply due to their affinity for each other, or their eagerness, if such an expression may be allowed, to combine when separated. It follows therefore that all solids which are conductors of electricity, *i.e.* bodies in which the ether is readily decomposed, must be opaque to light, for in them the ether has lost its rigidity. Similarly, all substances that are transparent to light must be insulators. In the case of fluids however this inference does not hold good, on account of the part played by electrolysis in the conduction of fluids. I would refer the reader who is anxious to pursue this subject further, to Professor Lodge's *Modern Views of Electricity*.

Mathematicians however object to such a view of the ether being put forward as an *explanation* of the phenomena of light. Since accurate scientific measurements of only space and time can be made, they naturally view with disfavour all but mechanical hypotheses which deal with conditions that can be fully and adequately treated by dynamical considerations. They would welcome any view of electrical stress or chemical affinity which would admit of the mathematical methods of dynamics being applied to it, while they regard such a view as the above as involving the fallacy of explaining the ignotum per ignotius.

Several facts are in favour of such a conception of the ether, and though it cannot be regarded as an explanation of the phenomena of light, it may be very reasonably considered a plausible hypothesis.

QUESTIONS.

(1) If in Fig. (2) the distance RM between the rotating and the fixed mirror be 610 metres, and the distance RS be 10 metres, and when the mirror R is rotating 256 times a second, the image of S is displaced to S' so that $SS' = 130{\cdot}85$ mm., what is the velocity of light?

(2) The velocity of light in air is 186379 miles per second; in vacuo it is 186433·8 miles per second. From these data calculate the value of μ for air to six places of decimals.

(3) What will be the velocity of light in water ($\mu = 1{\cdot}\dot{3}$), and in crown glass ($\mu = 1{\cdot}5$)?

(4) The wave-frequency of yellow light (D line) is about 510 millions of millions per second, and its velocity in air is about 186379 miles per second. Hence deduce the wave-length of this yellow light in air.

(5) "We observe no change of tint when the yellow light of sodium passes from air into water; as its wave-length however in water is $\frac{3}{4}$ of what it was in air, we infer that it is wave-frequency and not wave-length that determines colour." Criticize this statement.

(6) In Fig. (3) shew geometrically that the tangent ET will touch the intermediate hemispherical wave originating from p.

(7) In Fig. (5) if $\dfrac{\sin i}{\sin r} = \frac{1}{2}$, shew from the second law of refraction that $KO = \frac{1}{2} SO$.

CHAPTER II.

SHADOWS. ILLUMINATION. PINHOLES.

MATERIAL bodies may be classified with reference to their behaviour to light in some such way as the following:—

- Luminous
- Not luminous
  - Opaque
  - Not opaque
    - Translucent (*e.g.* oiled paper or ground glass)
    - Transparent

Such a classification however may be somewhat misleading, for we are as yet unacquainted with any body that is absolutely opaque to light, as it is found that all bodies whether inorganic or organic, if examined in sufficiently thin slices, will transmit a certain amount of light. Our knowledge indeed of the intimate structure of rocks as well as of tissues has been almost entirely acquired by examining with the microscope the light that is transmitted by such thin sections. No one however has any doubt as to the meaning of the term opaque in the ordinary acceptation of the word, so that for practical purposes the above classification is found to be convenient.

Shadows. As previously explained, we must regard luminous bodies in space as sending out light in all directions in straight lines. If now an opaque body be placed near the source of light, that side only will be illuminated that is turned towards the light, whereas the opposite surface as well as a space behind it remains in shadow. Let us take first the simplest case, when the source of light is so small or so distant that it may be regarded as a luminous point in space. In the figure (Fig. 11) O represents the source of light, and AB the opaque body which we may imagine to be a disc viewed edgewise. The rays drawn from the luminous point O to the margin of the disc indicate the cone of light that illumines its proximal surface, and the prolongation of this cone behind the disc indicates the shadow cast by it.

This may appear sufficiently simple, but the attentive reader will probably encounter a difficulty, which may be put in the following way. If light is due to some kind of wave movement, why can it not like sound turn round a corner? This is the difficulty that occurred to Newton, and led him to oppose the undulatory theory.

Now on carefully examining the shadow formed by a body with a sharp knife-edge margin it is found that the edge of the shadow is not quite sharp even when the source of light is a luminous point; light in fact is able to some extent to turn round a corner. In accordance with Huygens' principle, on suppressing any of the elementary waves the adjoining ones immediately assert themselves. This diffraction of light, as it is called, may also be seen when light is made to traverse a very narrow slit in an opaque body. For a complete explanation however of diffraction and of the interference phenomena due to it, the reader must refer to more exhaustive works. It is only necessary here to point out that for the occurrence of diffraction phenomena the slit must be small compared with the wave-length of light. It is owing to the extreme shortness of light waves that under ordinary circumstances light appears to travel in straight

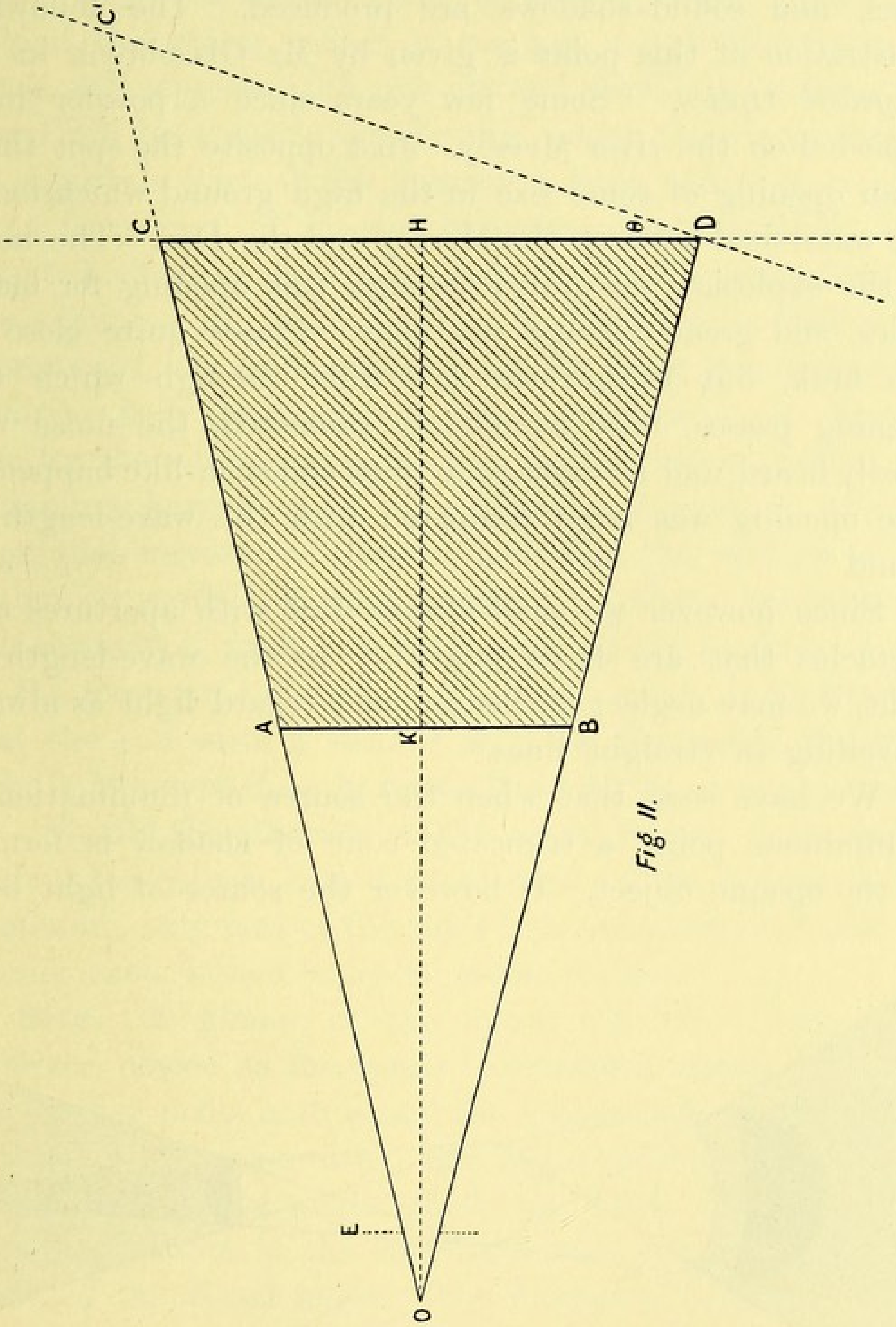

Fig. 11.

lines. Similarly it is owing to the length of sound waves that they can usually turn round a corner. When however sound passes through an aperture large compared with the wave-length of sound, it also appears to travel in straight lines, and sound-shadows are produced. The following illustration of this point is given by Mr Glazebrook in his *Physical Optics.* "Some few years since a powder hulk exploded on the river Mersey. Just opposite the spot there is an opening of some size in the high ground which forms the watershed between the Mersey and the Dee. The noise of the explosion was heard through this opening for many miles, and great damage was done. Places quite close to the hulk, but behind the low hills through which the opening passes, were completely protected, the noise was hardly heard, and no damage to glass and such-like happened. The opening was large compared with the wave-length of sound."

Since however we shall always deal with apertures and obstacles that are large relatively to the wave-length of light, we may neglect diffraction, and regard light as always travelling in straight lines.

We have seen that when the source of illumination is a luminous point, a truncated cone of shadow is formed by the opaque object. If however the source of light be a

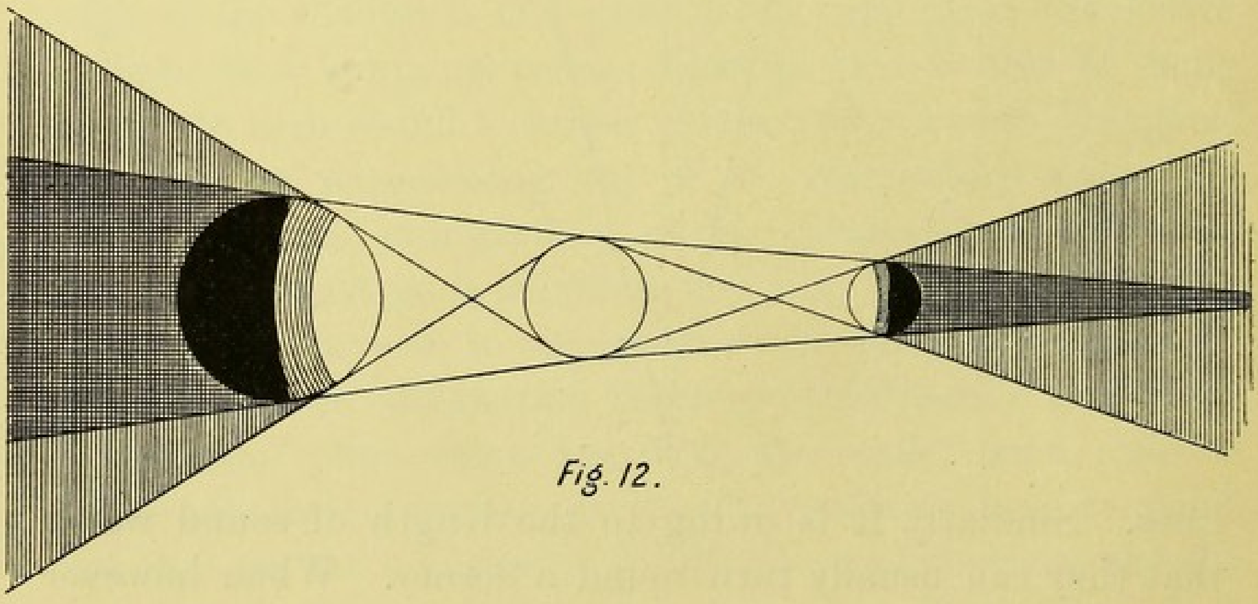

Fig. 12.

luminous body possessing innumerable points, the object will be illuminated by innumerable cones of light, and we must imagine a shadow cone for each of them. The space behind the object, which is common to all these shadow cones, will represent the area of total shadow or umbra; but there will be a space outside this, which is only in shadow as regards a part of the luminous body, while it receives light from part of it, and is consequently partially illuminated. This is the area of half-shadow or penumbra. In the adjoining illustration (Fig. 12) two opaque bodies are represented, one being smaller and the other larger than the luminous body itself which is placed between them. In each case for the sake of clearness the limits between umbra and penumbra have been sharply defined. In reality there is a gradual increase of illumination from the margin of the umbra outwards. A good example of these shadows and half-shadows is furnished by those that are sometimes cast by the moon on the earth. When the moon is so situated that the sun casts a shadow of it on the earth, the sun's light is intercepted by it, partially or wholly as the case may be. Should the observer on the earth happen to be in the umbra, the eclipse is said to be total; if he is only in the penumbra, only part of the sun's light is intercepted and the phenomenon is then termed a partial eclipse.

Size. A glance at the figure will shew that to an observer, placed at the apex of the conical umbra formed by the smaller body, both it and the luminous body will appear of exactly the same size. The apparent size of a body is determined by the angle which the lines, drawn from its outermost points to the eye, make with one another. This is called the visual angle. The further a body is removed from the eye the smaller is the visual angle under which it is seen, and the smaller consequently is its apparent size. The apparent linear dimensions of an object are determined by the tangent of the visual angle that the dimension considered subtends at the (anterior) nodal point of the eye. Thus the apparent height of an object is determined by the ratio $\frac{h}{d}$, where h represents its real height, and d its distance from the nodal point. The apparent area is similarly given by the ratio $\frac{s}{d^2}$, where s denotes the real area.

If we know the real size of an object we can determine its distance from us by the visual angle under which it appears to us; and vice versa, if we know its distance, its real size may be determined by the visual angle or apparent size. Such estimates of size or of distance we make unconsciously every day of our lives; they often indeed turn out to be grossly erroneous from our premises being false. An illustration of this is afforded by the apparent variation in size of the sun or moon when in different parts of the sky. The vault of heaven has the appearance of being flattened, so that the zenith seems to be nearer than any point on the horizon. Consequently when the sun is setting, it appears to be at a greater distance from us than when it is immediately over our heads; and so, since the visual angle that it subtends remains the same, it gives us the impression of being bigger. Two other factors which contribute to this delusion of increased distance might be mentioned, viz. the diminished brightness of the setting sun, and the facility with which its size and distance can be compared with that of known objects on the horizon.

The effect of diminished brightness in giving this delusive appearance of distance has nothing to do with the law of illumination discussed in the previous chapter; it is simply due to the turbidity of the atmosphere near the earth's surface. The light from an object at a distance is in great part lost from reflection or absorption by the smoke dust and germs that float in it; and so we are inclined to associate dimness with distance. In a fog, for instance, objects seem to be larger, as, since we underestimate the opacity of the medium, we imagine from their dimness that they are further off than they really are. As soon as anything causes us to correct this false impression of distance, the object at once seems to become smaller.

Brightness. It can be easily shewn that if the medium be homogeneous and isotropic, so that no light is lost in transmission through it, the brightness of a luminous surface is the same, whatever its distance from the eye, provided that the size of the pupil remain constant. For the brightness of an object is naturally measured by the amount of light it sends to the eye per unit area of its apparent size; in other words, the brightness (R) of an object is directly proportional to the quantity (q) that it sends to the pupil, and inversely proportional to the apparent area (A) of the surface observed, or

$$R = \frac{q}{A}.$$

Since however both q and A are each of them inversely proportional to the square of the distance of the object, the ratio which determines the brightness is independent of this distance.

Illumination. It will however be well to examine in greater detail the distinction between the illumination of a surface and its apparent brightness. In Fig. 11 O is the source of light. AB is the opaque disc exposing, let us say, a surface s to the light and at a distance d from O. CD is the base of the shadow cone formed by the disc on a screen placed at a distance D from O. Let the area of CD be denoted by S. Then a glance at the diagram shews that

$$\frac{AK}{CH} = \frac{OK}{OH} \text{ or } \frac{d}{D}.$$

Now we know that the measure of the area of a circle is πr^2; and AK, CH are obviously the radii of the circles whose areas are s and S respectively,

$$\therefore \frac{s}{S} = \frac{\pi (AK)^2}{\pi (CH)^2} = \frac{d^2}{D^2}.$$

Now if AB were taken away, the area S on the screen would receive precisely that quantity of light which now falls on AB. Consequently an area s on the screen would receive only the fraction $\frac{s}{S}$ of this light; so that if AB were moved up to the screen its illumination would only be $\frac{s}{S}$ of what it was before. If now the light (L) which AB receives when at unit distance from O be taken as the standard for comparison, the light (L') which it receives at distance D from O will be $\frac{L}{D^2}$.

For $$L' = \frac{s}{S}L = \frac{d^2}{D^2}L \text{ or } \frac{L}{D^2} \text{ when } d = 1.$$

The amount of illumination, then, received by a surface varies inversely as the square of its distance from the source of light. This is the law that we have already inferred from theoretical considerations based on the nature of light.

If the rays of light falling on CD are parallel to each other this law of course no longer holds good, but it may be applied whenever the free radiation of light is under consideration. Under these circumstances we are able by its means to determine the relative intensities of two sources of illumination. Several methods are in vogue; one of the simplest is that of Rumford.

A white screen is illuminated by the two sources of light that are to be compared, *e.g.* two lamps. A few inches in front of this is placed a vertical rod so that two shadows of it are cast on the screen. The lights are placed symmetrically with regard to the rod, and in such a way that the two shadows are close to each other. Each shadow is then illuminated by only one of the lights. One of the lights is then moved about until that position (d_1) is found, in which both shadows appear equally dark. Each lamp is then sending an equal amount of light to the screen, and their relative illuminating power is given by the ratio of the squares of their respective distances, for the light which the two shadows receive are respectively $\frac{L_1}{d_1^2}$ and $\frac{L}{d^2}$. Consequently when these are equal $L_1 = \frac{d_1^2}{d^2} L$.

Light sense. The accuracy of the determination depends on the capacity of the observer for estimating small differences of illumination, *i.e.* on his light sense. It is evident that, if the illuminating powers of the two lamps be known, such a photometer may be used to test this light sense. A simple method of applying the test is the following. Two candles (A and B) of equal illuminating power are used which throw two shadows (a and b) on the screen, all other light being excluded. A is now removed to such a distance (d_1) that the shadow (a) that it casts is only just visible. The light $\left(\frac{L}{d^2}\right)$ which illumines a is then almost indistinguishable from the light $\left(\frac{L}{d^2} + \frac{L}{d_1^2}\right)$ which falls on the general surface of the screen. The smallest appreciable difference is therefore $\frac{L}{d_1^2}$.

Schirmer has recently shewn that Weber's law holds good for the light sense between the illuminations of 1 and 1000 standard candles at unit distance, provided that the eye is fully adapted[1]. The ratio $\frac{\frac{L}{d_1^2}}{\frac{L}{d^2}}$ therefore will be constant throughout this range of illumination. It should however be remembered that the power of appreciating small differences of illumination increases with practice. Thus Schirmer found that at first the smallest difference that he could appreciate was $\frac{1}{128}$ of the total illumination, but that after eight days' practice he could recognize a change of $\frac{1}{217}$ or even less.

Below the illumination specified Weber's law ceases to be true. Indeed we have been using as a determinant of the light sense, the ratio of the smallest appreciable increment of light to the original light $\left(\frac{\Delta L}{L}\right)$. Now this expression ceases to have a determinable meaning if the original illumination becomes zero ($L = 0$). Hence it is customary, when investigating the light sense, to determine (1) the minimum appreciable difference between two intensities of illumination, and (2) the minimum stimulus capable of exciting a sensation, *i.e.* the difference between no light and the least amount of light that can be recognized.

It should be noticed that the sense for light, unlike that for form and colour, is almost equally acute throughout the whole retina with the exception of the extreme periphery. The small zone that immediately surrounds

[1] "Adaptation is a process depending upon three factors. First, the principal, is some as yet unknown occurrence in the bacillary layer, which depends upon a normal relation between it and the pigmented epithelium; second, the optical effect of the latter, by which the bacillary layer is to a certain extent shaded in bright illumination; third, the pupillary reaction, which acts more rapidly than the other two."

The quotation as well as the account of Schirmer's work in the text has been taken from the report in the *Ophthalmic Review*, Vol. x. p. 179.

the fovea is moreover found to be more sensitive to light than the fovea itself. Arago, the astronomer, was the first to point this out; he noticed that he could see very faint stars more easily, when he looked slightly to one side of them, than when he looked directly at them. Finally it must be remembered that the peripheral parts of the retina tire far more readily and far sooner than the more central parts, as is shewn by Purkinje's experiment. Hence moving objects in the peripheral part of our field of view create a more vivid impression than fixed objects.

Illumination of an inclined surface. We must return from this digression to the consideration of the illumination of a surface that is not in the plane perpendicular to the incident rays. Referring again to Fig. 11 let us suppose that the screen is inclined at an angle θ to its previous position, so that the light which previously fell on CD would now fall on an area $C'D$ or S'. If the area S were mapped out on the surface $C'D$ it would be found to occupy only a part of it, viz. $\frac{S}{S'}$; and it would consequently only receive $\frac{S}{S'}$ of the light (L) that falls on CD or $C'D$. Then if L' denote the illumination of the inclined surface S,

$$L' = \frac{S}{S'} L;$$

but $S = S' \cos\theta$ when the illumination of $C'D$ is effected by a beam of parallel rays normal to CD.

Then $$L' = \frac{S}{S'} L = \frac{S' \cos\theta}{S'} L \text{ or } L \cos\theta.$$

And since the angle θ is evidently equal to the angle of incidence of the light, the relation may be expressed as a law in the following form.

If a beam of parallel rays be received obliquely by a surface, its illumination will be proportional to the cosine of the angle of incidence.

When the incident light forms, as in the diagram, a divergent cone, the law is only approximately true, for then S is not equal to $S' \cos \theta$.

Brightness. Now let us consider how a luminous surface will impress an observer at O. Suppose the disc AB in the figure to be self-luminous and denote by Q the constant quantity of light radiated by AB every second. This will obviously be proportional to the area s of its surface and to the intensity i of its luminosity. $Q = si$.

Now only a small part (q) of this light will reach the eye of the observer at O. And by our previous considerations it is evident that q must be directly proportional to the area (e) of his pupil, as well as to Q, and it must be inversely proportional to the square of the distance (d_1) between the pupil E and AB.

$$\therefore q = \frac{eQ}{d_1^2} \text{ or } \frac{esi}{d_1^2}.$$

But the brightness (R) must be measured by the ratio $\frac{q}{A}$, where A represents the apparent size of AB. If d denotes OK, the distance of AB from O, the nodal point of the eye, $A = \frac{s}{d^2}$.

Then if we neglect the small difference between d_1 and d

$$R \text{ or } \frac{q}{A} = \frac{qd^2}{s} = \frac{esi}{d^2} \cdot \frac{d^2}{s} \text{ or } ei.$$

Hence the apparent brightness of a surface is independent both of its size and of its distance, and consequently of its inclination to the line of sight. Under ordinary circumstances, if the size of the pupil be constant, it is simply proportional to the intrinsic brightness of the surface.

When an object is observed through a microscope or a telescope a certain amount (*e.g.* 15 %) of light is lost by reflection and the imperfect transparency of the glasses; but if allowance is made for this, the brightness of the image is equal to the brightness of the object. An apparent exception occurs if high powers are used, for then only part of the pupil will be filled with light. This is equivalent of course to reducing the size of the pupil. If then e' be the size of the area filled, the effective brightness will be equal to $e'i$. This explains the diminished brightness of the image when the higher eyepieces of a microscope are used.

It is impossible therefore by any optical arrangement to obtain an image whose brightest part shall exceed the brightest part of the object.

There is one case in which this law does not hold good. If the object subtend an angle less than the minimum visible, it may yet, if excessively bright, succeed in stimulating a retinal element, and so cause a visual impression of a very small bright body. If such an object be magnified until it subtends this minimum angle, there will be no increase of its apparent size, although an increased amount of light from it will be entering the eye. When for instance stars are observed through a telescope, their apparent size is not increased, for they still subtend an angle less than the minimum visible; but all the light that falls on the object glass may by a suitable eyepiece be concentrated on the pupil of the observer's eye, provided that the light lost in transmission through the instrument be neglected. Under these conditions we may then regard the action of the telescope as tantamount to increasing the area of the pupil to that of the object glass. If then α denote the fraction of incident light that is transmitted through the telescope (usually about ·85) and O represent the area of the object glass, e being the size of the pupil as before, the increase of brightness will be $\alpha \frac{O}{e}$. Or if o and p be the diameters of the object glass and pupil respectively, the increase of brightness will be $\alpha \frac{o^2}{p^2}$, for the areas of circles are proportional to the squares of their diameters. If the pupil be regarded as of unit diameter, we get the expression αo^2. This is what astronomers call the *space penetrating power* of a telescope, that is to say, its power of rendering very faint stars visible.

Pinholes. If a pinhole be made in a card, and this be held between a candle and a screen, it will be found that an inverted image of the candle will be formed on the screen. The nearer the screen is brought to the card, the smaller and sharper will be the image of the candle. If the candle be brought nearer the card, the image will be larger but less sharp. Finally, if the hole in the card be made larger, the image will appear brighter but its definition will be again diminished.

The explanation is simple. Every point of the candle is sending out light in all directions, and all that falls on the card is intercepted by it, so that its shadow is thrown on the screen. From each luminous point however of the candle there will be one tiny cone of light that will make its way through the aperture in the card. On the screen the section of this cone will appear as a small bright patch. In this way each point of the candle will be represented on the screen by a corresponding bright patch, so that all these bright areas taken together will represent the whole candle flame, in other words they will form an image of it.

In the diagram (Fig. 13) AB represents a luminous object, the small divergent cone from A forms a bright patch at a and similarly the light from B that traverses the aperture in the card lights up a little area at b, and the intermediate points between A and B light up intermediate little areas between a and b, so that an inverted image of AB is formed at ab. It is easily seen that the size

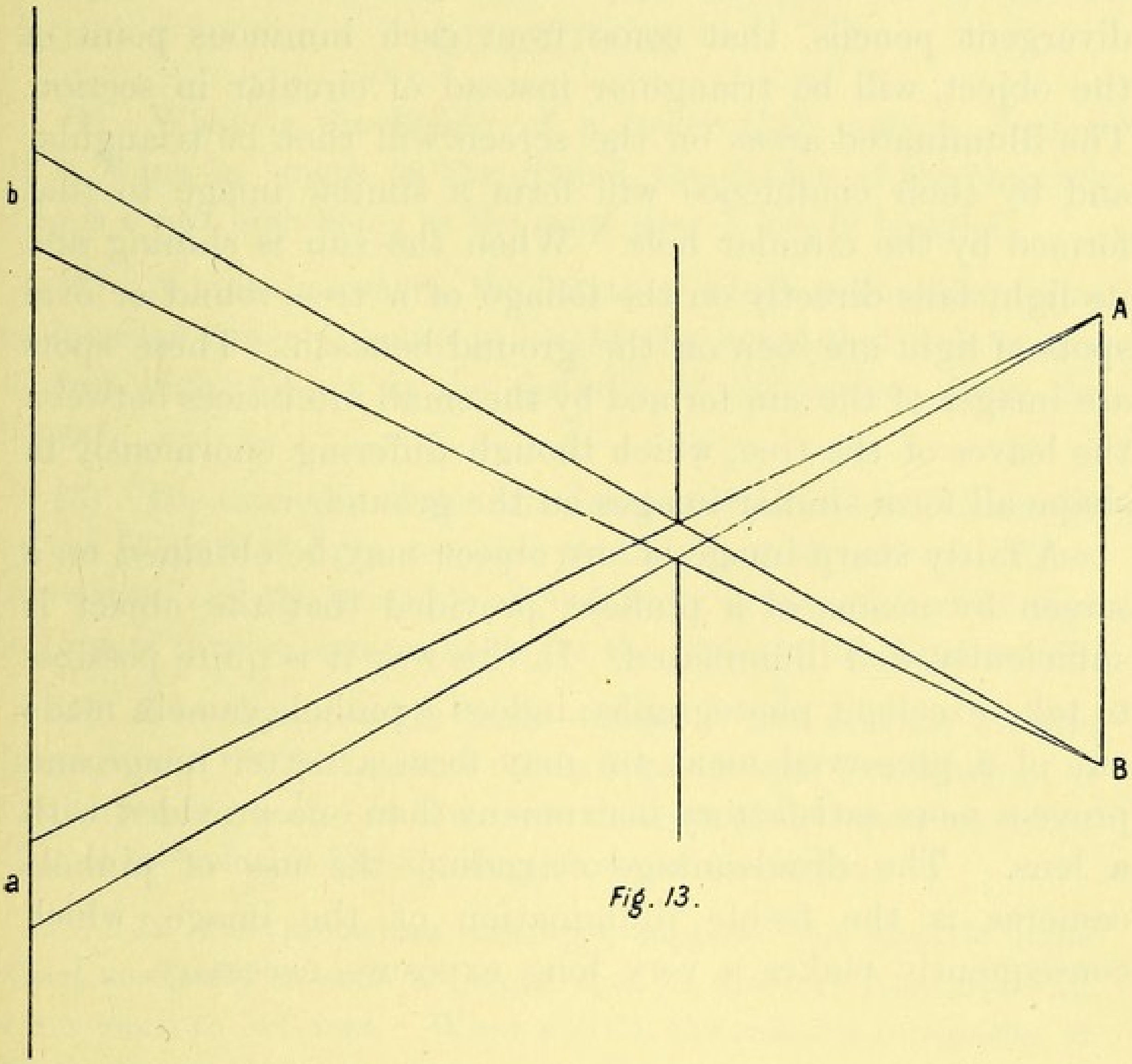

Fig. 13.

of ab is proportional to the distance of the screen from the card; and that if the screen be brought nearer to the card, the bright patches formed by the several cones of light will become smaller, and the image will therefore become more distinct. If the object be brought nearer, each of the conical pencils of rays will include a wider angle, *i.e.* each pencil will form a more divergent cone. The same result will be obtained if the aperture in the card be enlarged. Hence in both these cases the definition of the image will be impaired.

The shape of the hole in the card has no influence on the character of the image formed. If the hole be triangular instead of circular, the only result will be that the small divergent pencils, that come from each luminous point of the object, will be triangular instead of circular in section. The illuminated areas on the screen will then be triangular and by their confluence will form a similar image to that formed by the circular hole. When the sun is shining and its light falls directly on the foliage of a tree, round or oval spots of light are seen on the ground beneath. These spots are images of the sun formed by the small interstices between the leaves of the tree, which though differing enormously in shape all form similar images on the ground.

A fairly sharp image of any object may be obtained on a screen by means of a pinhole, provided that the object is sufficiently well illuminated. In this way it is quite possible to take excellent photographs; indeed a pinhole camera made out of a preserved meat tin may form a better image and prove a more satisfactory instrument than one provided with a lens. The disadvantage attending the use of pinhole cameras is the feeble illumination of the image, which consequently makes a very long exposure necessary.

QUESTIONS.

(1) What is the height of a tower that casts a shadow 52 ft. 6 ins. in length on the ground, the shadow of the observer who is 6 feet high being at the same time 7 feet in length?

(2) A pinhole camera, the length of which is 7 inches, forms an inverted image 4 inches in height of a house that is in reality 40 feet high. What is the distance of the camera from the house?

(3) The intensity of the illumination of a surface is measured by the amount of light received per unit area of surface. From this datum and the principle of rectilinear propagation shew that the law of inverse squares holds for the intensity of illumination.

(4) Two lights, a gas lamp distant 5 feet and an electric light distant 150 feet, throw on an opposite wall two shadows of a neighbouring post. If these two shadows are of equal intensity, what is the relative illuminating power of the lights?

(5) In the preceding example suppose the electric light raised vertically to such a height that its distance from the wall is increased to 300 feet. What will be the relative intensities of the shadows approximately?

If the shadows are equal calculate approximately the relative intensities of the lights.

(6) The moon is observed to subtend a visual angle of 31′. Assuming that the diameter of the moon is 2160 miles, what is its distance from the observer? Given $\tan 31' = \cdot0090178$.

CHAPTER III.

DIFFUSED LIGHT. COLOUR. QUANTITY AND QUALITY OF LIGHT REFLECTED. MECHANICAL MODEL.

EVERY part of an object that is seen must send light to the eye that sees it. If the body is not self-luminous, the light that it sends must have been received from something else; and since its exposed surface can be seen whatever the position of the observer, it follows that every visible point of it must scatter this light in all directions. This scattering of the light that falls upon a surface is due to its minute irregularities, and so is often somewhat loosely called irregular reflection.

Colour. The light that is reflected from the surface of a body is always less in amount than that which falls upon it, for a certain quantity of light enters the substance, to be transmitted or absorbed in proportions that depend on its transparency or opacity. As has been already pointed out, the term opacity is merely relative. A heap of fresh cut leaves may be called opaque, but any one of them if held up against the bright sky allows green light to traverse it. In the same way the most opaque substance will allow some light to penetrate a thin layer of it, and if this penetrating light be chiefly of one particular colour, the object itself will in most cases appear tinged with that colour.

The colour of most substances is the same as that of the light which they most readily transmit. A leaf for instance appears green whether on the ground or held up against a light. The leaf may indeed be regarded as consisting of innumerable thin layers, which all reflect the light that reaches them, and, in proportion to the depth of the layer, the light so reflected is tinged green owing to the fact that the chlorophyll in the leaf absorbs red light. Similarly a crystal of copper sulphate appears blue because it absorbs red and yellow light. The light which is reflected by its first surface is white or almost white, but that which comes from its deeper layers appears blue, as the red and yellow constituents of the white light have been absorbed by transmission through these layers.

There are however some substances, metals for example, which reflect light of one colour more readily than that of another. The light which they transmit is then quite different from that which they reflect. The surface of a crystal of copper sulphate reflects light waves of every period indiscriminately, so that the light reflected from its surface is white. The light however that is reflected from the surface of gold is yellow, whereas that which is transmitted by a thin layer of gold leaf is blue. Many of the aniline dyes present a similar difference between what may be called their surface colour and their body colour; these are usually indeed complementary to each other. Rose aniline for instance transmits rose-coloured light, but the light that it reflects is green.

There are a few bodies whose colour is produced in a different way; but, speaking generally, it may be said that the coloration of a body is due to the absorption or transmission of the constituents of white light in unequal proportions, so that what we see is the remainder left of the white light that it received.

Ordinary diffused daylight consists of light-waves of every period in what may be called normal proportions. If a piece of blue glass be taken which absorbs red light completely, it will be found that any red object viewed through it will appear black. If a red glass be placed behind this blue glass the combination will be opaque to light, for whatever light can be transmitted through the one will be quenched by the other. If however a yellow glass be substituted for the red, green light will be transmitted through the combination, for both glasses transmit green, and this colour which is common to each is consequently seen.

The colours of pigments are similarly impure. Gamboge behaves like powdered yellow glass, it transmits green light as well as yellow light, and hence when mixed with indigo, which transmits both blue and green, the resulting colour is that which is common to each, namely green. If these pigments contained no admixture of green the resulting combination would be grey, for the specific colour of each would be quenched by transmission through the other. If metallic substances formed the basis of pigments the colour of a mixture would be more nearly the sum of the tints of its constituents, instead of being merely the part that is transmitted in common by the constituents.

The result of mixing two coloured lights, *i.e.* the investigation of the colour sensation produced, when they both fall on the same part of the retina, belongs rather to the domain of physiology than to that of optics. It will be sufficient in this place to give one or two examples to indicate the difference between mixing lights and mixing pigments. The addition of red light to blue causes a colour sensation of purple, although the mixture of vermilion with ultra-marine is of a blackish-grey colour. Yellow and violet-blue are complementary colours, *i.e.* the effect of superimposing them on the retina is to cause a sensation of white, though the mixture of pigments of these colours is greyish-green. Similarly red and bluish-green are complementary colours.

Diffused Light. In the previous chapter it was shewn how an opaque body placed in the course of light produces a shadow. The question naturally arises—how is it that, when the sun's light is cut off in this way by an intervening obstacle, a house for example, we are not immersed in total darkness? Evidently the light that reaches our eyes under these circumstances must have come indirectly from the sun, as all the direct light has been intercepted. In fact, what is called diffused daylight must be the light from the sun that has been scattered or irregularly reflected by impinging on something. The observation of a sunbeam in a room suggests what it is that diffuses light in this way. The track of the light is mapped out by the brilliant illumination of the dust or motes that are always present in a room. If a spirit lamp (or even a red-hot poker) be placed in the sun-beam, dark spaces will be seen above the flame. If all the dust were burnt up, the sunbeam would be quite invisible, for there would be no particles to scatter the light, and hence it would all reach its destination on the opposite wall, and the illumination of this would be all that would be apparent to the observer.

Blue Sky. It is impossible to lay too great stress on the fact that we cannot see light unless it reaches our eyes. How is it then that the sky is blue, *i.e.* that the light that comes from the sky to our eyes is blue, although the sun's light is nearly white? The air is not blue, for if it were the sun when low down on the horizon should acquire a bluish tinge, from the transmission of its light through a thicker layer of the medium. The blue light of the sky moreover comes across or even against the direction of the sun's rays. It must therefore be light reflected from small particles suspended in the air. Numerous experiments have been made shewing that if particles suspended in a medium be sufficiently small they will reflect the shorter light-waves more completely than the longer ones. "A small pebble," as Professor Tyndall aptly says, "placed in the way of the ring-ripples produced by heavy rain-drops on a tranquil pond, will throw back a large fraction of each ripple incident upon it, while the fractional part of a larger wave thrown back by the same pebble might be infinitesimal." If the particles are large as compared with the longest waves of light, as the rain-drops in a cloud for instance, all the waves of every length will be reflected in equal proportions, and the light coming from them will be white. The meteoric dust in the air is however so fine that it reflects the shorter blue waves while it offers very little opposition to the transmission of the longer waves. The white light of the sun is consequently in transmission through the air gradually robbed of its shorter waves by these successive reflections and hence appears yellowish. When low down on the horizon the sun appears red, for then its light has to travel through a thicker layer of air, and so through a greater number of scattering particles; this sifting process is then more complete. A difficulty may arise in understanding how this meteoric dust remains suspended in the air. Why should it not fall and settle on the earth? A little consideration will clear up the matter. Let us take the most disadvantageous case for our argument, and imagine the shape of each particle to be spherical. The mass of each particle and consequently the weight, varies as the cube of its radius; but the surface, that it exposes to the resistance of the air when falling, varies only as the square of its radius. It is evident then that, if the particle be only small enough, the resistance of the air may very materially counteract the effect of gravitation upon it. In this connection it is interesting to note that it has been shewn, both by experiment and by mathematical investigations, that the viscosity of a gas is independent of its density[1]. There is then little difficulty in understanding how this meteoric dust remains suspended in the air, even at a height of forty miles or more from the earth, for in spite of its rarity the air even at this height offers nearly the same resistance to a falling body as at the earth's surface.

Quantity of Reflected Light. It is as we have seen the scattered light that comes from a body that renders it visible; hence if this scattered light be diminished by polishing its surface, the body itself may become almost invisible. A mirror for instance may scatter so little light that under certain circumstances it cannot be seen; although the light, that it reflects from illuminated objects near it, is seen with the utmost distinctness. It is however impossible to make a perfect mirror, that is, one that shall reflect all the incident light; a part is always scattered; consequently, if the conditions be favourable the reflecting surface itself can be always seen.

It must not however be inferred that smoothness of surface is the only requisite for a satisfactory mirror. A fluid at rest presents a smoother surface than any that can be obtained by a mechanical process of polishing, yet it usually reflects only a small fraction of the light that falls upon it. Water for example reflects but 1·8 per cent. of the light that falls perpendicularly upon its surface, whereas polished metals reflect a much higher percentage. The best

[1] Graham-Otto's *Lehrbuch der Chemie*, Bd. I. p. 157. The explanation given in the text is somewhat loose. Advanced mathematics are required for the exact investigation of the problem. In reality it is found that a particle in the air eventually falls with a uniform velocity v, where $v \propto \frac{r^2 g}{\nu}$, ν being the kinematic viscosity which diminishes with the temperature.

silvered mirrors reflect nearly 90 per cent. of the incident light, while mercury itself reflects more than 66 per cent.

There is further this striking difference between metallic and non-metallic surfaces. In the case of metals the ratio of the reflected to the incident light is constant, whatever the angle of incidence may be; whereas with non-metals this ratio increases with the obliquity of the incident light. The following table indicates the amount of light reflected from the surfaces of water and from that of the metallic fluid mercury at various angles of incidence.

Angle of incidence	0°	40°	60°	80°	89½°	
Water	1·8	2·2	6·5	33·3	72·1	per cent.
Mercury	66·6	66·6	66·6	66·6	66·6	„

It will be seen that as the angle of incidence approaches its limiting value of 90°, the amount of light reflected by the water rapidly increases, so that when the incident light just grazes its surface, it is almost wholly reflected. Similarly if a piece of note-paper be so placed that the light from a candle just grazes its surface, a considerable amount of this light will undergo regular reflection; indeed a reflected image of the candle may sometimes be distinctly seen under these circumstances from a suitable position.

Lastly, the refractive index of the substance, or rather the difference between its index and that of the surrounding medium, exerts a most important influence on the quantity of light reflected. The diamond owes its brilliancy to its high refractive index ($\mu = 2{\cdot}5$), and most probably the characteristic lustre of metals is largely due to a similar cause. It has already been shewn that there is some ground for believing that the refractive index of metallic substances is exceedingly high; in other words, the velocity of light through the thin layer that it can penetrate must be relatively very slow.

Mechanical Model. A simple mechanical illustration will explain the relation between the reflecting properties of a medium and the speed with which light is transmitted through it. Let us imagine a row of perfectly elastic balls of equal mass suspended by threads from a horizontal beam. If now the end ball be raised and allowed to fall against its neighbour, it will give up the whole of its motion to it, so that the first ball will come to rest, whereas the second ball will begin to move forwards with the original velocity of the first, but on hitting the third ball, it will in its turn give up its motion to it. In this way the motion of the first ball will be passed on through the entire series of balls. Such a model, when the balls are all precisely similar, may be taken to represent the propagation of light in a homogeneous medium; each particle of ether after giving up its motion remains at rest until it again receives a new impulse from behind from the source of light. It is however the transference of the impulse, not the movement of the balls, that represents the propagation of light. If the balls are in contact, the impulse will still travel along the series although the individual members composing it shew no movement. Further, the velocity with which this compression impulse travels will diminish as the density or mass of the balls is increased, just as the velocity of light diminishes when the refractive index or density of the medium is increased. Provided then that the balls are all of equal mass and of equal elasticity, no matter how great their differences in other respects, the transference of the motion will be simple and complete, and, should they be in contact, the speed of the compression impulse will be uniform.

If then a transparent substance be immersed in a medium of the same refractive index as itself, light will be transmitted through it without undergoing any loss by reflection at its surface; for, the velocity of light being the same in the immersed body as in the medium, they will behave as two different systems of equally massive balls. When the difference of velocity or of refractive index is very slight, a very small proportion of the light is reflected, and this may easily escape notice. Thus it requires careful scrutiny even with favourable illumination to detect the presence of the lens in the eye, or of a piece of glass in a solution of Canada balsam, or indeed of ice in water.

A substance of higher refractive index may be represented by a system of more massive balls. If now this set be placed close to the first series so that a continuous row of balls is formed, we have a rough representation of two contiguous media of different density. As before, when the terminal ball of the first series is made to strike its neighbour it will give up its motion entirely to it and come to rest. In this way the motion will be passed on from ball to ball until the second series is reached. Let us suppose each ball in this to be of twice the mass of the constituents of the previous set. The end ball of the first set will, on striking its more massive neighbour, give up its motion and rebound from it with one-third of its previous velocity. A wave of movement in the reverse direction will therefore travel along the first series of balls. If they are very close together, we may regard it as a wave of compression that travels backwards along the system, and the previous phenomenon as a wave of compression that travelled forwards. This reflected compression-wave will however travel back at precisely the same rate that the original impulse travelled forwards, but it will be less intense, for the momentum of each ball is now but one-third of what it was before. Meanwhile the impulse which the first massive ball had received has been traversing the second system, so that an onward-moving compression-wave has resulted in this also. We have then a representation of the partial reflection of the light that falls on a denser medium, and of its partial transmission through it. The reflection will be more complete if the difference of density (*i.e.* of refractive index) of the two media is increased, just as the intensity of the reflected compression-wave in the model is increased by augmenting the difference between the masses of the balls in the two systems.

An impulse given to the second system of balls will produce a wave-movement analogous to the passage of light from a dense to a rare medium. There is however a peculiarity in this case that demands careful attention. Let us imagine that the balls are attached to each other by elastic threads. As before the impulse travelling along the massive system, which we will now call the first system, may be regarded as a forward-moving compression-wave. The end ball however will, after striking the first ball of the second lighter series, still continue its onward motion (though at one-third its previous rate). It will in fact only give up a part of its momentum to its less massive neighbour. This onward movement of the end ball will pull forwards each member of the first series in turn by means of the connecting threads. In fact a wave of extension will travel backwards throughout the system. If the threads do not stretch, this pulling or extension-wave will travel backwards at the same rate that the original compression-wave travelled forwards. Meanwhile a pushing impulse or compression-wave has been travelling onwards along the lighter series of balls.

How are all these facts to be interpreted? We have already regarded the compression of the balls as analogous to some phase of a light wave that travels through the medium. But extension is evidently opposite in phase to compression: if the latter represent the crest of a wave, the former must be analogous to the trough of a wave.

When then light passes from a denser medium to a rarer, we are prepared to find one portion transmitted, and another portion reflected with its phase reversed. This is precisely what is actually observed, the incident wave-crest is reflected as a trough and vice versa. No such reversal however occurs when the reflection takes place at the surface of a dense medium. This peculiarity of the reflection at the surface of a rare medium is of fundamental importance in the comprehension of certain interference phenomena of light, *e.g.* the central dark spot in Newton's rings, etc.

We have seen then that the amount of light reflected from a surface depends to a large extent on the difference between its refractive index and that of the surrounding medium. It is not unusual, for example, to see a well-marked reflex from the surface of the lens in the eyes of old people. This is best seen when the illumination is oblique, for, as we have said, the amount of light reflected from a non-metallic surface increases with the obliquity of its incidence. The appearance indicates that the refraction of the lenticular cortex has increased ; the reflection is not a sign of opacity or cataract, but merely of the difference between the refractive index of the aqueous humour and of the surface of the lens. Similarly particles or striae of higher index in the lens may be suggestive of dots of opacity. The diagnosis is made by seeing if they will transmit light by using the ophthalmoscope.

One word more about the opacity of substances. As we have seen, even a perfectly transparent substance will not transmit all the incident light, unless it be in a medium of the same refractive index as its own. Further a thin layer of material may not transmit light, either because it absorbs it, like black velvet, or because it scatters it like white lead. A mass of snow is opaque although each of its component crystals is transparent, for, such is the difference between the refractive index of these ice crystals and the air which separates them, that a considerable fraction of the incident light is reflected and scattered at each of the innumerable surfaces that it meets.

The translucency of oiled paper or of a moistened towel is due to a similar cause. The refractive index of the constituent fibres is considerably higher than that of air. Hence when this is replaced by a medium whose refractive index more nearly resembles their own, less light is reflected —the object appears darker—and more light is transmitted.

CHAPTER IV.

REFLECTION AT PLANE SURFACES. PRINCIPLE OF LEAST TIME.

We must now consider the manner in which light is reflected by polished surfaces. In the introductory chapter we deduced from the undulatory theory the two laws of reflection. But it must be remembered that these laws have been primarily deduced from direct observation; they are in no way dependent upon the theory of light, though the theory that has been adopted explains them.

The plane of incidence is that plane which contains both the incident ray and the normal to the surface drawn from the point of incidence.

I. The reflected ray lies in the plane of incidence and on the side of the normal opposite to the incident ray.

II. The angles which the incident and reflected rays make with the normal are equal to one another.

These laws hold good whether the reflecting surface be plane or curved. In the latter case we have only to draw a tangent plane at the point of incidence, and consider the ray to be reflected at this plane.

Reflection at Plane Surfaces. Let AB represent a plane reflecting surface, and let S (Fig. 14) represent a

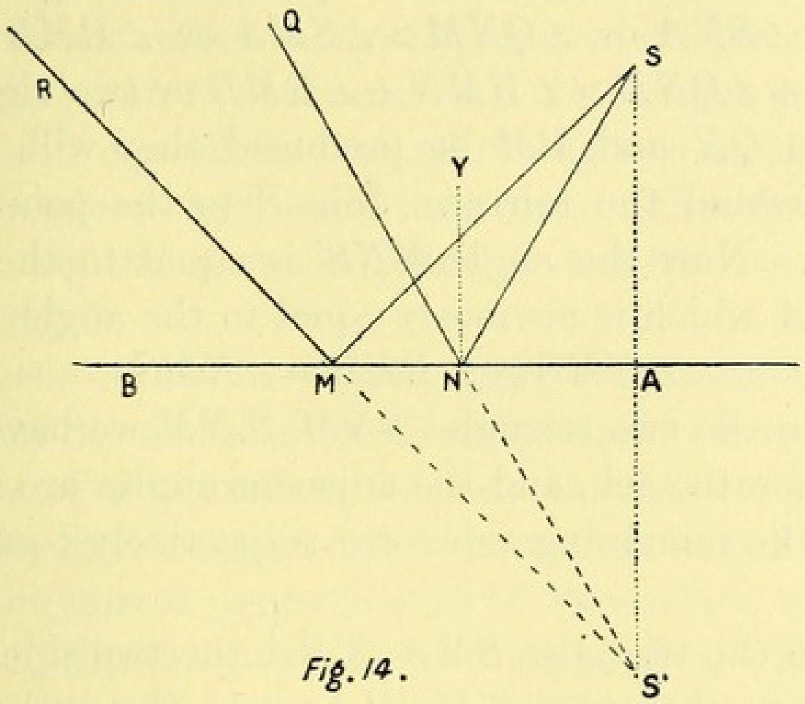

Fig. 14.

luminous point which is sending out light in all directions. Now if SN represent one of these directions, we may call the line SN a ray of light. Further from the laws of reflection we see that, if the normal NY be drawn from the point of incidence, the line NQ which makes an equal angle with the normal will represent the direction of the reflected light, in other words NQ is the reflected ray: for it lies in the plane of incidence, *i.e.* the plane of the paper, and the angles which SN and NQ make with the normal are equal to one another.

It is easy to see that the incident and reflected rays also make equal angles with the surface of the mirror.

If then there be an eye in the neighbourhood of Q which can receive this reflected ray, it will perceive light coming towards it in the direction NQ; but we have not yet found from which point in this line it will appear to have come. To do this we shall have to take another incident ray from S, and discover where the corresponding reflected ray intersects the previous one. Let SM be a contiguous incident ray in the same plane, and let the angle SMA be less than the angle SNA. Then the corresponding reflected ray MR makes the angle RMB equal to the angle SMA,

and $\qquad \angle SNA$ or $\angle QNM > \angle SMA$ or $\angle RMB$;

$\therefore \angle RMN + \angle QNM > \angle RMN + \angle RMB$ or two right angles.

If then QN and RM be produced, they will meet in a point S' behind the mirror. Join S to the point of intersection S'. Now the angle MNS' is equal to the alternate angle QNA which is obviously equal to the angle MNS.

Also $\qquad \angle NMS' = \angle RMB = \angle NMS.$

Then in the two triangles SNM, $S'NM$ we have the side NM common to each, and the adjacent angles are also equal, therefore the remaining sides are respectively equal each to each.

Now in the triangles SMA, $S'MA$ the two sides SM, MA are equal to the sides $S'M$, MA; and the included angle SMA is equal to the included angle $S'MA$, so that the base SA is equal to the base $S'A$, and the angle SAM is equal to the angle $S'AM$.

The line SS' is consequently perpendicular to the surface of the mirror, and it is bisected by the plane of the mirror.

Similarly it may be shewn that any other ray in the same plane will be reflected in such a direction that when produced backwards it will meet SS' in the same point S'.

We see then that every ray in the plane of the paper, that falls upon the surface of the mirror, will be reflected in such a direction that it will appear to come from S'.

Now if we suppose the paper to be revolved about an axis SS', the figure will represent the course of incident and reflected rays in every plane.

Hence it follows that all the rays that fall upon the mirror from S, whatever the plane of their incidence may be, will be so reflected that the prolongations of these reflected rays will intersect at the point S'.

Consequently the light, that is reflected from the mirror, will appear to the eye that receives it, to be coming from S'. In other words, the mirror will form an image of S at S'.

Since all the reflected rays if produced intersect each other at the same point, it is obvious that for the determination of this point, it is sufficient to take only two of the reflected rays and produce them until they meet. This meeting point will then be the common point of intersection of all the reflected rays.

As however we have seen that this common point of intersection lies at the same distance behind the mirror that the source of illumination is in front of it, and that it is also on the line which is drawn from the source through the plane of the mirror perpendicular to its surface, we may still further simplify the geometrical construction. Thus if P be a point in front of a plane mirror (Fig. 15) we can find the position of its image Q by drawing PLQ perpendicular to the

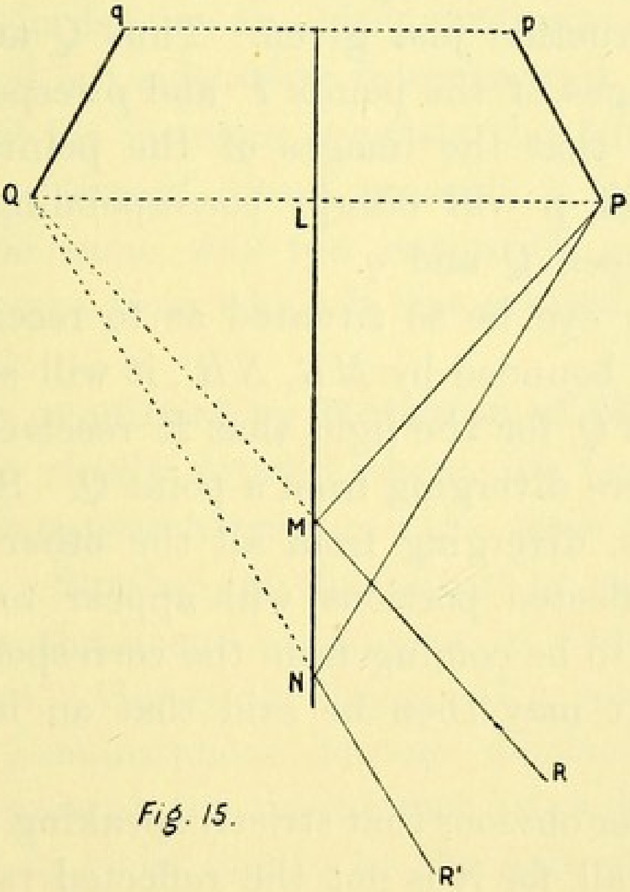

Fig. 15.

plane of the mirror and cutting it at L, and then making LQ equal to PL. The course of any of the rays from P that undergo reflection at the points M, N,... on the mirror is indicated by joining QM, QN,... and producing them to R, R'....

If the positions of the object the image and the observer's eye be given, it is an easy matter to indicate the course of the light that by reflection reaches the observer's eye. Thus if P represent the position of the object, and Q that of the image, while R represents the position of the eye, join QR, cutting the mirror in M. Join PM. Then PMR represents the course of the light that, originating from P to an eye at R, gives rise to a virtual image at Q.

If an object Pp be placed in front of the mirror AB we can find the position of its image in a similar way. For we may consider the object to be composed of innumerable points, each of which is scattering light in all directions. The images of all these points can be found by the geometrical construction just given. Thus Q and q are found to be the images of the points P and p respectively, and it is easy to see that the images of the points intermediate between P and p will occupy corresponding intermediate positions between Q and q.

If then an eye be so situated as to receive the conical pencil of rays bounded by MR, NR', it will see an image of the point P at Q, for the light that it receives will reach it, just as if it were diverging from a point Q. Similarly of the conical pencils, diverging from all the other points of the object, the reflected portions will appear to the eye that receives them to be coming from the corresponding points of the image. It may then be said that an image of Pp is formed at Qq.

It is however obvious that strictly speaking the image Qq is not *formed* at all, for it is not the reflected rays themselves, but only their prolongations backwards that intersect. Such an image is called a *virtual* image: indeed it has no *real* existence, inasmuch as the reflected rays do not actually pass through Qq. It has however a virtual existence with respect to an observer who is suitably placed.

It will be noted that the image Qq is similar and equal to the object Pp in every respect; for the corresponding points of the object and image are similarly situated with respect to the mirror. Thus the image is erect, and the point of the object nearest to the mirror (p) is represented by the point of the image nearest to the mirror (q); and the right and left sides of the image correspond to the right and left sides of the object. Since however the object and image face each other, the observer obtains a view of the opposite aspect of the object, *i.e.* a view of that side of the object which in his position he could not see without turning the object round. Hence he thinks he sees an object that he has turned round, and, to account for the appearance that it presents, he regards it as an object similar to the real object whose right and left sides have interchanged. The image of the right hand for instance suggests the left hand, for the left hand when turned round presents a precisely similar aspect. In the same way the image of a printed letter resembles the type from which it was printed.

Deviation produced by Rotation of Mirror. If now the mirror be slowly rotated about its vertical axis, the image also will appear to rotate in the same direction about the same axis. Similarly if the mirror be rotated about its horizontal axis, the image will appear to undergo a rotation about this axis. Hence the image of a vertical tube may under these circumstances become displaced to such an extent as to present the appearance of a horizontal tube, whose axis is at right angles to the axis of rotation.

The laryngoscope well exemplifies this displacement of the image. The mirror, which is inclined at an angle of 45° to the vertical, when properly placed at the back of the pharynx, forms an image of the larynx with its axis horizontal and its lumen facing the observer. Since the symmetry of the correspondence between the different parts of the object and image relates to their distances from the mirror, and not to their distances from the observer, we find the anterior parts of the larynx (epiglottis, etc.) represented in the upper part of the image, while the posterior structures (arytenoids, etc.) occupy the lower portion of the image.

It appears then that the rotation of a mirror through an angle of 45° causes the image to move through an angle of 90°. The alteration then of the direction of the reflected rays is measured by twice the angle through which the mirror is turned.

Let AB be a mirror (Fig. 16) and S a luminous point. All the incident rays that fall on AB will be reflected in

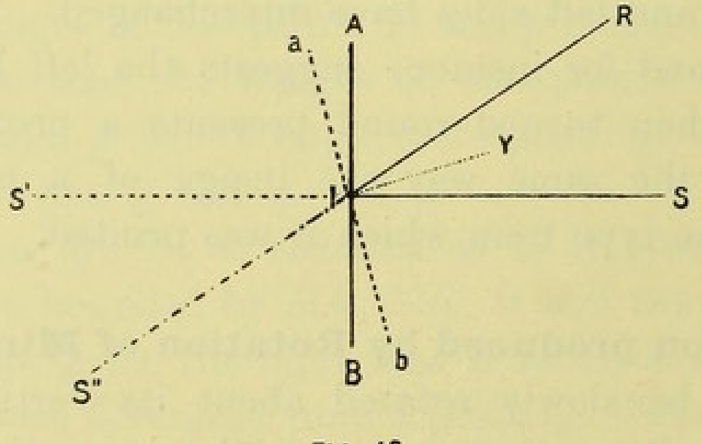

Fig. 16.

such a way that they will appear to be proceeding from a point S', which is at the same distance behind the mirror that S is in front of it. The incident ray SI which is normal to AB will then be reflected back upon its course as IS. If now the mirror be rotated through an angle θ so as to occupy a position *ab*, the normal at the point I will now occupy a position IY, making an angle θ with IS. The incident ray SI therefore will be reflected as IR, making an angle θ with IY or 2θ with IS.

Further the image of the point S will now be at a point S'', that may be found by the method given on p. 67, and all the rays incident on *ab* will appear after reflection to be coming from S''. It is obvious that S' has undergone an angular displacement equal to SIR; we may therefore conclude that on rotating the mirror through an angle θ all the reflected rays undergo a deviation of 2θ.

There are several important applications of this principle in daily use. It will be sufficient to specify two as illustrative examples.

The Galvanometer is used to detect the presence of an electric current and to measure its magnitude. There are many forms of the instrument, but they all essentially consist of a coil of insulated wire and a magnet so suspended as to be easily deflected when the current is led through the coil. In the mirror galvanometer the deflection of the magnetic needle is measured by the alteration in the direction of the light reflected from a small mirror attached to the centre of the needle. But as we have seen the deflection of the reflected light is twice that of the mirror; it is only necessary then to receive the reflected light on a graduated scale, to measure the angle of its deflection, and then to halve the angle obtained. This number will give the angle through which the magnet has been deflected by the current.

The Sextant is an instrument employed by navigators to measure the angle between any two distant objects as seen from the position of the observer. A is a small mirror fixed to the limb BQ of the sextant, in such a position that any light incident in the direction BA will be reflected in

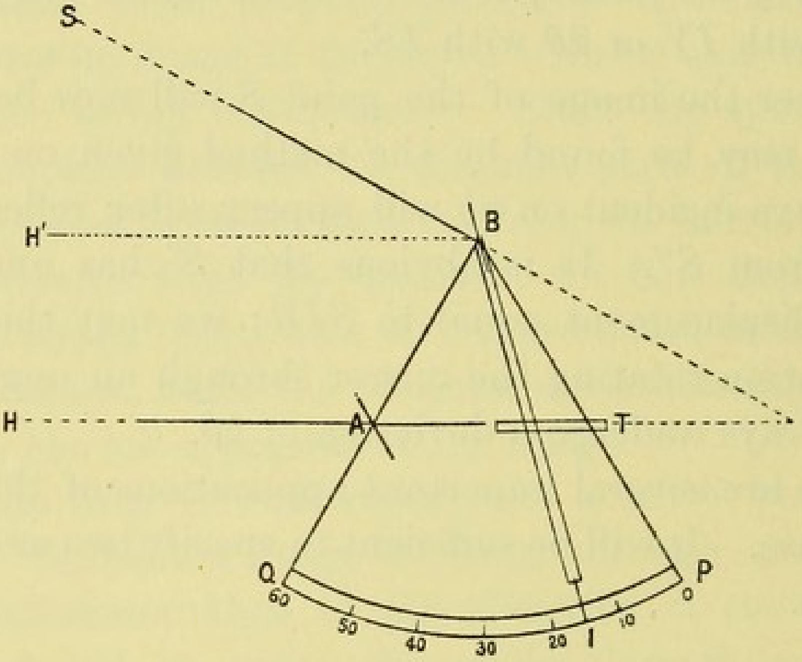

Fig. 17.

the direction AT. B is another mirror fixed to the moveable arm BI. This mirror can be rotated about its centre by means of the arm, and the amount of rotation can be read off on the scale PQ. When I is at the zero division of the scale (P), the mirror B is parallel to the mirror A. Consequently the light which reaches A from B must have been travelling in a direction $H'B$ parallel to AT. If then H and S be the objects to be observed, the telescope T is first directed towards H, and the instrument is maintained in such a position that the object H is kept in view just above the upper edge of the mirror A. If the arm BI is now slowly moved forwards from P, the objects between H and S will in succession appear reflected in the mirror A. Let us suppose that after the mirror B has been rotated through an angle θ, an image of S is formed by the mirror A below H.

In this case the light from S must, after reflection at B, have travelled along BA, as it eventually has obtained the direction AT. But when zero on the scale was indicated, the incident ray on B, that corresponded to BA, was $H'B$. So the angle SBH' must be double of the angle θ through which the mirror was turned, and since $H'B$ is parallel to HA, the angle sought between S and H must also be 2θ.

Repeated Reflection at Inclined Mirrors. When an object is placed between two plane mirrors inclined to one another at an angle, a limited number of virtual images is formed which may be seen by an observer in a suitable position. It will be found that if the angle between the mirrors is an integral divisor of 180°, the object together with its images forms a perfectly symmetrical figure with respect to the reflecting surfaces.

The kaleidoscope invented by Sir David Brewster consists essentially of two plane mirrors AC and BC inclined to one another at an angle of 60° (Fig. 18). If an object such as

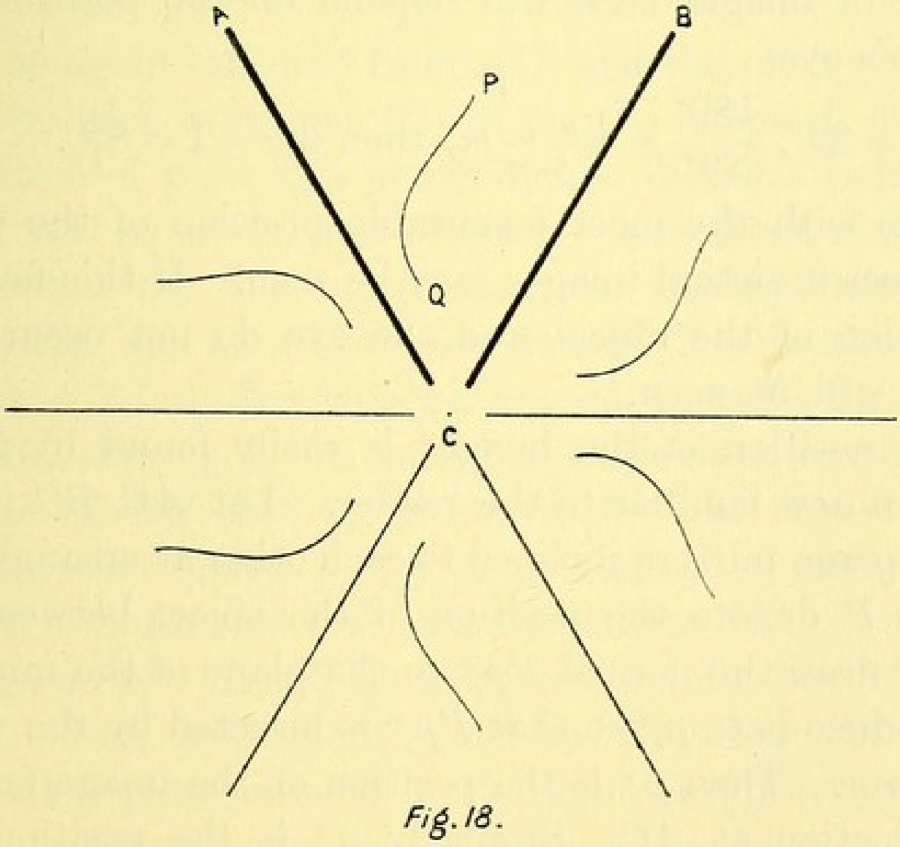

Fig. 18.

PQ be placed between the mirrors, an observer, if in a suitable position, will see 5 virtual images symmetrically arranged about the point of intersection C of the planes of the mirrors.

In order to determine how many images are seen, we first have to discover how many reflections can occur from each mirror, and finally how many of these will reach the eye of the observer. This is not a difficult calculation but it is rather long and laborious, and as it has no direct bearing on the subject of this book it will be sufficient to give the general result.

If α be the angle between the mirrors, and $\frac{\pi}{\alpha} = m$, the number of images seen by an eye that is between the planes of the mirrors is $2m - 1$, if m is an integer. So if $\alpha = 60°$, $m = \frac{180°}{60°} = 3$, $\therefore 2m - 1$ or 5 virtual images are seen; if $\alpha = 45°$, 7 virtual images are seen. If m is not an integer, the number of images seen will depend on the position of the observer's eye.

If $\alpha = 39°$, $\frac{180°}{39°} = 4\frac{8}{13} = m$, then $2m - 1 = 8\frac{3}{13}$.

Then with the most favourable position of the observer and object 9 virtual images may be seen. If this favourable disposition of the object and the eye do not occur, only 8 images will be seen.

The position of the images is easily found by the construction now familiar to the reader. Let AC, BC (Fig. 19) be two plane mirrors inclined to each other at an angle of 90°, and let P denote the position of the object between them. From P draw the normal $Pp_{,}^{a}$ to the plane of the mirror AC, and produce it to $p_{,}^{a}$ so that $Pp_{,}^{a}$ is bisected by the plane of the mirror. Then $p_{,}^{a}$ is the position of the image formed by one reflection at AC. Similarly $p_{,}^{b}$ is the position of the image formed by one reflection at *BC*. Now part of this light which is reflected from *BC*, and appears to come from

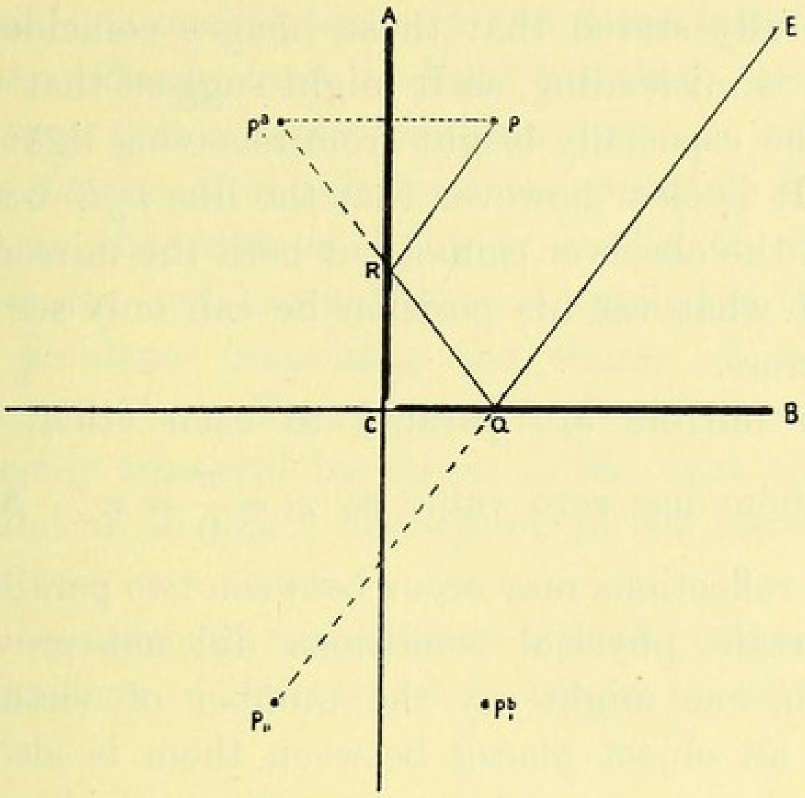

Fig. 19.

$p_{,}^{b}$, will be again reflected from AC; and the observer, if in a suitable position to receive it, will see the image $p_{,,}$ due to it. The position of $p_{,,}$ is at a perpendicular distance behind the plane of AC equal to that which $p_{,}^{b}$ is in front of it. The eye that receives this image must be situated between P and A. An eye at E between P and B will also see an image at $p_{,,}$; but this will be formed by reflection at the surface BC, since $p_{,,}$ is the same perpendicular distance below the plane of BC that $p_{,}^{a}$ is above it. The course of the light from this image $p_{,,}$ to an eye at E is easily found. Join $p_{,,}E$. Since this line cuts the mirror BC, $p_{,,}$ must be the image of $p_{,}^{a}$. Let Q be the point of intersection of $p_{,,}E$ and BC. Join $p_{,}^{a}Q$ cutting AC in R. Join PR. Then $PRQE$ is the path pursued by the light that gives to an eye at E the image $p_{,,}$.

It will be noticed that the image $p_{\prime\prime}$ may be formed either by the reflection of the image $p_{\prime}^{b}$ in the mirror AC, or by the reflection of the image $p_{\prime}^{a}$ in the mirror BC. It is generally stated that these images coincide; but the expression is misleading, as it might suggest that the image $p_{\prime\prime}$ would be especially bright from receiving light from two sources. It is clear however that the line $p_{\prime\prime}E$ between the image and the observer cannot cut both the mirrors AC and BC, and so whatever his position he can only see $p_{\prime\prime}$ in one of the mirrors.

If two mirrors are parallel to each other, the angle between them has zero value, so $m = \frac{\pi}{0} = \infty$. An infinite number of reflections may occur between two parallel mirrors, and if certain physical conditions did not prevent their observation, one might say the number of virtual images formed of an object placed between them is also infinite.

If the image formed by an ordinary looking-glass be carefully observed it will be found to be double. There are in fact two reflecting surfaces, each of which forms an independent mirror. The upper surface of the glass gives rise to a very faint though distinct image, and the lower surface of the glass which is silvered forms the bright image. If the frame in which the looking-glass is mounted be removed so that one can look edgewise through the glass, a considerable number of images of the upper and lower surfaces of the glass can be seen. A knowledge of refraction however is required for the exact determination of the position of the images.

Principle of Least Time. The fact that light radiating in space travels in straight lines might be expressed, if there were any advantage to be gained by it, by saying that it travels to its destination by the shortest possible path.

Before adopting such an expression it is necessary to shew that it is an adequate description of the course of light under different conditions, and further that it is useful. The principle is more frequently and more generally stated in a slightly different form, *i.e.* light travels to its destination by that course that occupies the least time. It will be found that when expressed in this form the principle is generally applicable to the path of light in media of different or varying density, and has hence been applied to the solution of many problems respecting the course of light under different conditions. That the principle of "least time" is not universally true will be shewn in the next chapter, and a correct and an adequate description of the course of light will be given.

It will be sufficient here to shew that the principle applies to the behaviour of light which is subject to reflection at a plane surface.

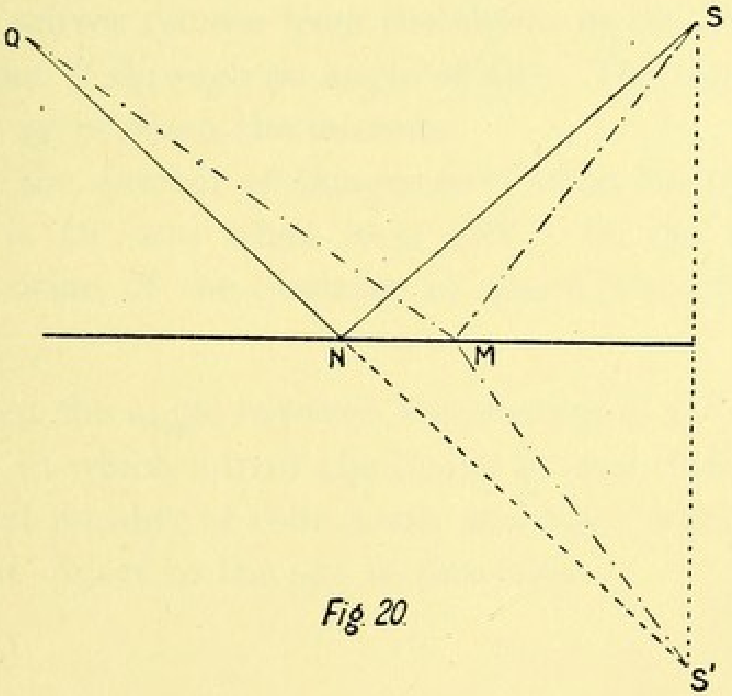

Fig. 20.

If S be a luminous point, and if we consider the light that reaches Q from S after reflection at the surface of the mirror, it will be shewn that the course SN, NQ that the light actually takes is also the course that occupies the least time. For if not, let SM, MQ in the plane of the paper be a shorter path.

Then $$SM, MQ < SN, NQ.$$

Now $$SM = S'M \text{ and } SN = S'N.$$

$$\therefore \quad S'M, MQ < S'N, NQ,$$

which is absurd, for $S'N$, NQ is a straight line.

A precisely similar proof could be applied if the point M were not in the plane of the paper; indeed one has only to imagine the point M to revolve round N in the plane of the reflecting surface to see that the proposition is universally true for plane surfaces. It follows then that the principle of least time embodies both the laws of reflection at plane surfaces, for it does not admit of the slightest difference between the angles of incidence and reflection, or the slightest difference between their planes.

QUESTIONS.

(1) Two parallel plane mirrors face each other at a distance of 3 ft., and a small object is placed between them at a distance of 1 ft. from one of them. Calculate the distances from this mirror of the three nearest images that are seen in it.

(2) Two plane mirrors are placed at right angles to each other, and a small object is placed between the mirrors at an angular distance of 15° from one of them and the line of intersection between them (C). The observer's eye is placed half way between the mirrors.

What is the number of images seen? Say in which mirror the image is seen that corresponds to the greatest number of reflections.

(3) The mirror remote from the object in the above example is turned about C through an angle of 45°. The observer's eye is placed half way between the mirrors.

Calculate the number of images seen when the angle between the mirrors is 45° and when it is 135°. In the latter case is there any position of the observer in which more images would be seen?

(4) When the angle between the mirrors is 45° in the above example, say in which mirror the image is seen, that corresponds to the greatest number of reflections, and trace the course of the light from the object to the eye in this case.

CHAPTER V.

REFLECTION AT CURVED SURFACES. CONJUGATE FOCAL DISTANCES. SIZE OF IMAGE. PRINCIPLE OF SAME PHASE NOT LEAST TIME.

Concave Spherical Mirrors. We have now to consider the manner in which light is reflected by a curved surface. At the point of incidence a tangent plane may be drawn, and the reflection may be considered as taking place at this tangent plane in the way we have already explained.

There are however less tedious methods of dealing with reflection at spherical surfaces, with which we are chiefly concerned, and these we proceed to describe in their simplest form.

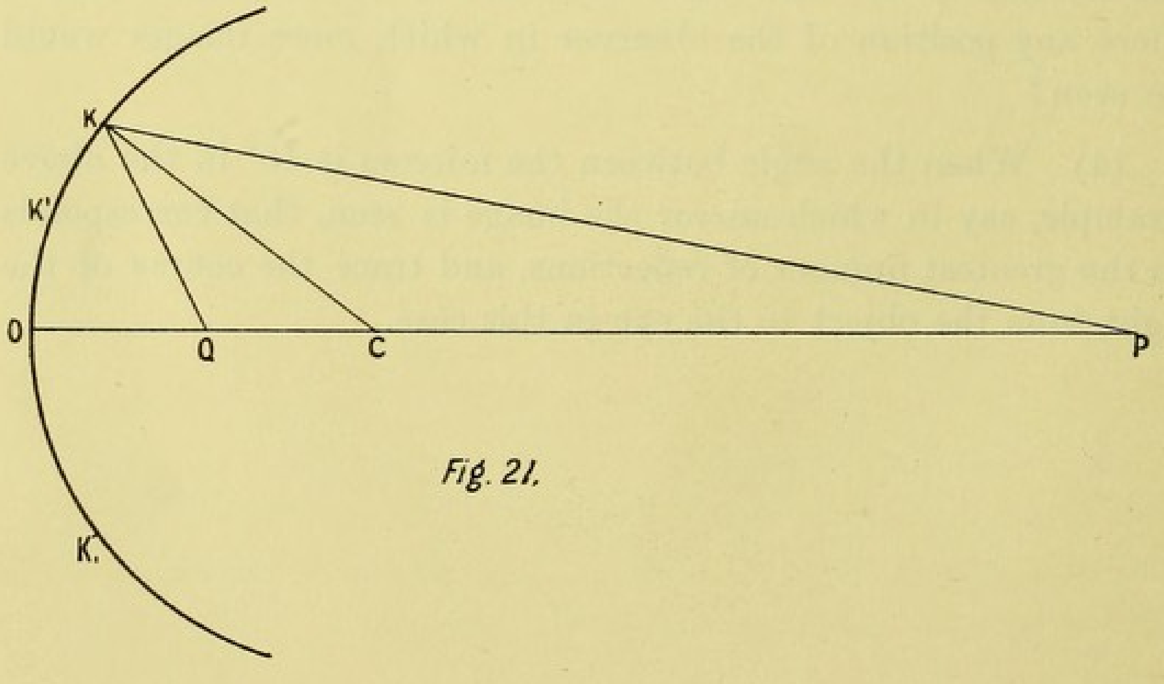

Fig. 21.

Let OK represent a concave spherical reflecting surface, the centre of the sphere of which it is a part being at C.

Any line drawn through C to the mirror is called an axis of the mirror. If the parts of the mirror on either side of the axis are symmetrical, the axis is called the principal axis. Thus in the figure CO is the principal axis and O is the vertex of the mirror.

If any point P be taken on the principal axis and if it be supposed to become self-luminous, it will radiate light in all directions.

Let PK be one of these directions; join CK. Then CK being a radius is the normal to the spherical surface at the point K.

The incident ray PK will therefore be reflected at K in a direction KQ, such that the angle of incidence PKC is equal to the angle of reflection CKQ, and KQ will be in the same plane as PK and KC, *i.e.* KQ will be in the plane of the paper.

Then since in the triangle PKQ the vertical angle PKQ is bisected by the line KC that cuts the base at C,

$$\frac{PC}{CQ}=\frac{PK}{QK}. \qquad \text{(Euc. VI. 3.)}$$

Now if K, the point of incidence considered, be very near to O, the line PK will be very nearly equal to the line PO, and at the same time QK will be very nearly equal to the line QO.

Under these conditions then we may substitute PO and QO for PK and QK, and by this means we shall obtain a much more manageable formula which is approximately correct.

For convenience we shall always consider lines drawn in the direction of the incident light as positive, lines drawn in the reverse direction as negative.

Let the distance of P from O be denoted by p, and the distance of Q from O be denoted by q, and let r be the length of the radius of the sphere (CK or CO).

Then $\quad PC = p - r, \quad CQ = r - q,$

$\quad PK = p, \quad QK = q;$

and the relation $\dfrac{PC}{CQ} = \dfrac{PK}{QK}$ becomes $\dfrac{p-r}{r-q} = \dfrac{p}{q}$;

$$\therefore \; qp - qr = pr - pq, \text{ or } qr + pr = 2pq.$$

On dividing by pqr we obtain the formula

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{r} \quad \text{........................} \quad (1).$$

Now since the surface is of uniform curvature it follows that the nearer K is to O, the closer do the values of PK and QK approach those of PO and QO.

Hence if for the purposes of a given calculation it is allowable to regard the distances from K, namely PK and QK, as equal to PO and QO, we may with still greater justice substitute these values of PO and QO for the corresponding distances PK', QK' from any point K' intermediate between K and O.

It follows then that all the rays from P incident on the arc KO will be reflected in such a way that the formula (1) will apply to each of them;

$$\therefore \; \frac{1}{q} = \frac{2}{r} - \frac{1}{p}.$$

As the right-hand side of this equation is constant under the conditions considered, it follows that $\dfrac{1}{q}$ is also constant. In other words all rays falling on the arc KO will on reflection cut the axis in the same point Q.

If now the figure be rotated about the axis CO, the arc KO will trace out the corresponding segment of the reflecting spherical surface, and it is clear that all rays from

P incident on this small segment, whatever their plane of incidence, will intersect at the same point Q to the assigned degree of approximation.

The point Q is called the conjugate focus of P. The term conjugate implies that if P's position changes, Q's position changes also; it is called a focus because it is a common point of intersection of rays.

It should be observed that any incident light which passes through the centre of curvature (C) will be so reflected as to retrace its previous course. Consideration of the formula $\frac{1}{p}+\frac{1}{q}=\frac{2}{r}$ shews that, if r be constant, as p diminishes, q increases; when $p=q$, p and q are each equal to r. That is to say, when the luminous point is at C, its conjugate focus coincides with it at C. It is obvious that this must be so, for all rays from C are normals to the reflecting surface, and consequently all the incident light is reflected along its original incident path.

This may be more instructively regarded from another point of view. If there be a luminous point at C, spherical waves are radiating from C, and all the elements of a given incident wave-front must reach the reflecting surface at precisely the same moment, *i.e.* they must all be in precisely the same phase. Reflection will occur simultaneously throughout the incident wave-front and consequently the wave-front will return as a contracting spherical segment to its centre C. Here the disturbance produced by each element of the returning wave will be of precisely the same character, for each element is in the same phase. Each element will therefore add to the disturbance of the central particle at C, or increase the amplitude of its vibration, *i.e.* C is a focus; for in the language of physical optics a focus is a point at which a large number of elements of the same wave-front arrive in precisely the same phase.

As p becomes less than r, q becomes greater than r; in fact, the positions of the conjugate foci with respect to C are merely interchanged. If in Fig. 21 Q be regarded as the luminous point, P will be its conjugate focus.

Now let p increase, then q will diminish.

As P is removed further and further away, the rays from P incident on the mirror include a smaller and smaller angle. Finally if P be supposed at an infinite distance its rays become parallel.

Now when $p = \infty$, $\frac{1}{p} = \frac{1}{\infty}$ or 0;

$\therefore$ when $p = \infty$, $\frac{1}{q'} = \frac{2}{r}$, *i.e.* $q' = \frac{r}{2}$.

Consequently when parallel rays fall on the mirror, they will intersect at a point at a distance of half the radius from the vertex of the mirror.

This point (F), at which incident parallel rays intersect after reflection, is called the Principal Focus.

Its distance from the vertex may be conveniently denoted by f, so that $\frac{1}{f} = \frac{2}{r}$.

The value of f depends therefore solely on the curvature of the reflecting surface and so is constant for each mirror. Consequently our previous formula may be written

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

It is clear that if $p = f$, $q = \infty$, *i.e.* if a luminous point be placed at the principal focus, the diverging rays that proceed from it to the mirror will be reflected as parallel rays[1].

[1] When we come to deal with refraction at a spherical surface, we shall find that such a point is called the first principal focus, whereas the point towards which incident parallel rays converge is called the second principal focus. We may say then that when reflection at a spherical surface is under consideration, the two principal foci are coincident.

If p be less than f, the value of q is negative. This means that the distance of Q from O must be measured in the opposite direction; the point Q will therefore be behind the mirror, Fig. 22. The focus Q in that case is virtual.

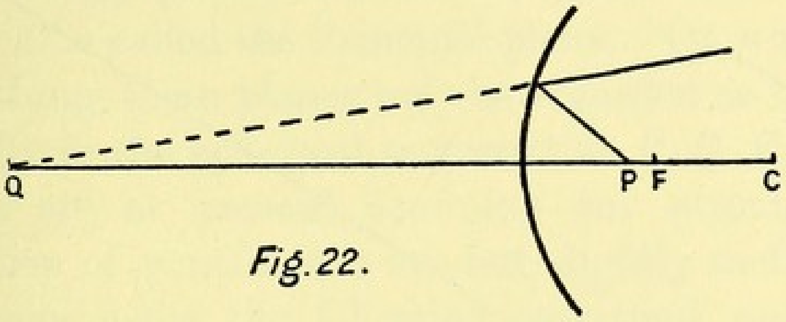

Fig. 22.

Let us consider the matter from another point of view. When p is a little greater than f, the divergent pencil from P is reflected by the mirror as a slightly convergent pencil, the apex of which is at Q, Q being a great distance off. As P approaches F the divergence of the incident pencil increases, and the reflected rays approach parallelism. When P falls within F (so that $p < f$), the divergence of the incident pencil is so great that it is not overcome by reflection at the mirror, and so even after this reflection the rays still diverge.

If now these reflected rays be produced backwards they will intersect at a point Q behind the mirror, which is consequently their virtual focus.

When then p is greater than f, the conjugate foci P and Q are *real*, *i.e.* an actual intersection of rays occurs at these points.

When however p is less than f, Q the conjugate focus of P is *virtual*, for the reflected rays do not actually intersect, but their subsequent course after reflection is that which would be pursued by rays that had intersected at Q. The focus or intersection at Q is therefore called virtual.

Now let the axis PCO be rotated about C into a new position $P'CO'$ (Fig. 23). Any line such as $P'CO'$ which passes through C, but does not pass through the vertex of the reflecting surface, is called a secondary axis. It is evident that the conjugate focus of P' will be formed on the

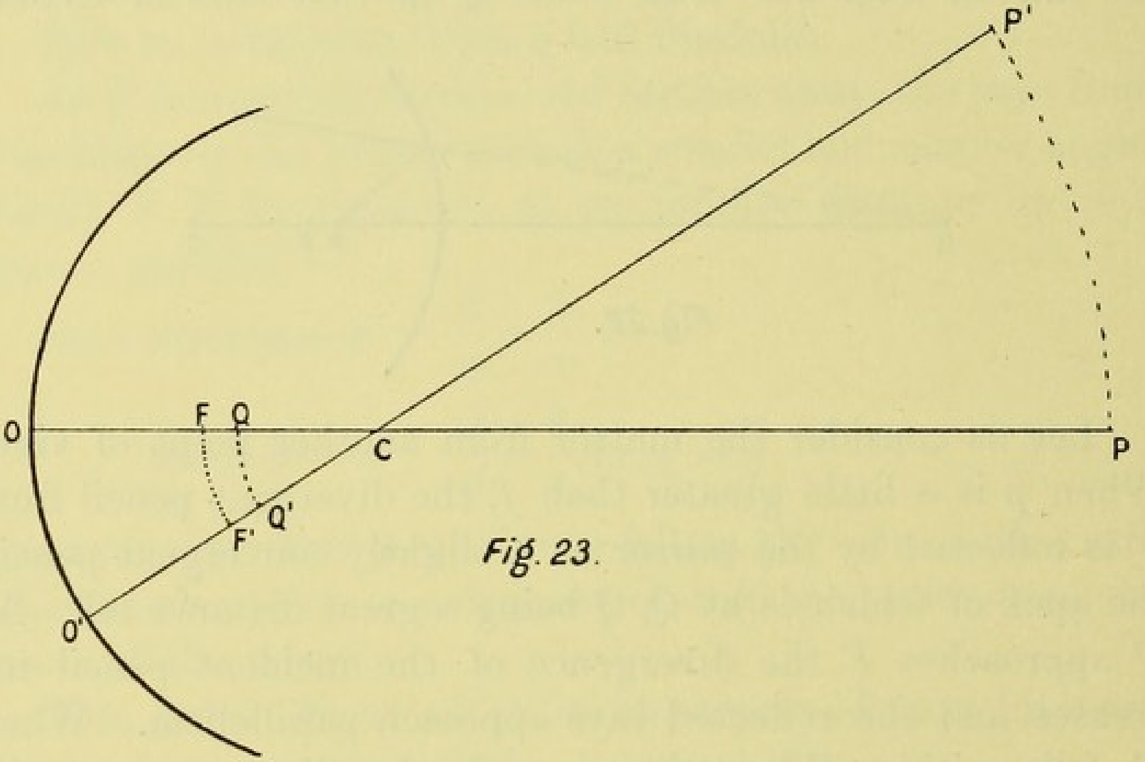

Fig. 23.

secondary axis at the point Q' such that $CQ' = CQ$. In fact the axis PCO in its rotation carries the points Q and F with it. Incident light the rays of which are parallel to $P'O'$ will be reflected to a focus F', *i.e.* the reflected rays will intersect at F'.

If now the axis PCO be turned through a small angle in all directions about C, the point F will trace out a small spherical segment $F'F$. At the same time the points P and Q will trace out spherical segments of a different size. If then an object be curved towards the mirror exactly as PP', its image will be inverted and curved away from the mirror as QQ'. In fact the segment QQ' may be regarded as the image of the segment PP'. It may here be noted that all real images are inverted, whereas all virtual images are erect; the reason of this will become more apparent subsequently.

If the segments traced out by P, Q, F and O on rotating the axis PCO about C are very small, they may be regarded as approximately plane surfaces. The planes so described at P and Q are called the Conjugate planes, the plane at F is called the Focal plane, whereas the plane at the vertex of the mirror O is called the Principal plane. Or, what comes to the same thing, these planes may be regarded as being drawn tangentially to the spherical segments at P, Q, F, and O.

As we are at present confining our attention to the consideration of pencils that are but slightly inclined to the axis, we may make the following assertions regarding the properties of these planes.

The surface at which reflection occurs may be regarded as coincident with the Principal plane.

All pencils of parallel rays that are but slightly inclined to the optic axis, will after reflection intersect in some point on the Focal plane, and moreover light from any luminous point on this plane will after reflection travel in rays parallel to that secondary axis on which the point lies.

Any point on one Conjugate plane will form an image at a corresponding point on the other Conjugate plane.

The relation between the distances of the Conjugate planes from the Focal plane is easily obtained.

Since
$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f},$$

or
$$\frac{f}{p}+\frac{f}{q}=1,$$

on multiplying by pq we get

$$pq-qf-pf=0,$$

and on adding f^2 to each side

$$(p-f)(q-f)=f^2.$$

We have shewn how to find the distance of the image from the vertex of the mirror, when the distance of the object (p) and the radius of curvature (r) are known. From the properties of the centre of curvature, and from those of the planes just described, we are now able to give a geometrical construction for determining the position of the image.

Let AB be the object (Fig. 24), and let C be the centre of curvature. Join BC, and produce it to meet the reflecting

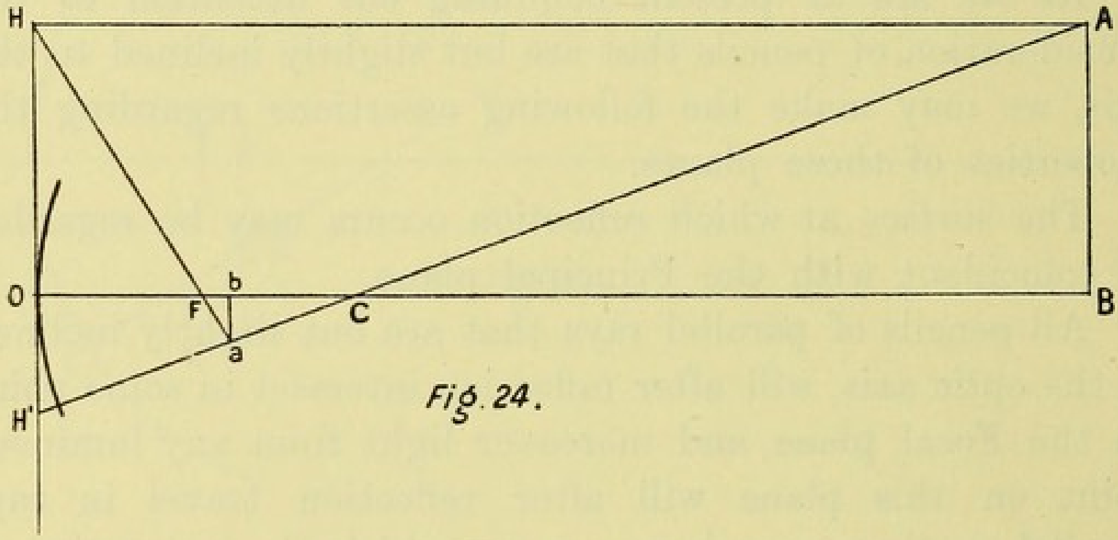

Fig. 24.

surface at O. Then the plane HOH' at right angles to the axis BCO represents the principal plane, and F the mid-point of CO is the principal focus.

To determine the position of the image of a point A that is not on the principal axis, either of the following methods may be adopted.

(1) (Fig. 24.) Draw the ray AH parallel to the axis meeting the principal plane in H. Join HF, and produce it to meet the secondary axis ACH' in a.

Then a is the conjugate focus, or the image of A.

For the incident light which is travelling in the direction AH parallel to the principal axis must after reflection pass through the focus F: and the light which is travelling in the direction ACH', since it passes through the centre (C), must be reflected back along its previous path. If then we regard

AH, AH' as the extreme rays of the incident cone of light that diverges from A, we see that they will be so reflected as to intersect at a. In fact the incident cone HAH' becomes the reflected cone HaH' converging towards a. We shall presently find that when the incident cone is as wide as that represented in the diagram it is not accurately reflected from a spherical mirror to a single point a. For our present purpose however the construction given is sufficiently accurate.

(2) (Fig. 25.) Through A draw the ray AFH'' cutting the principal axis in F and meeting the principal plane in

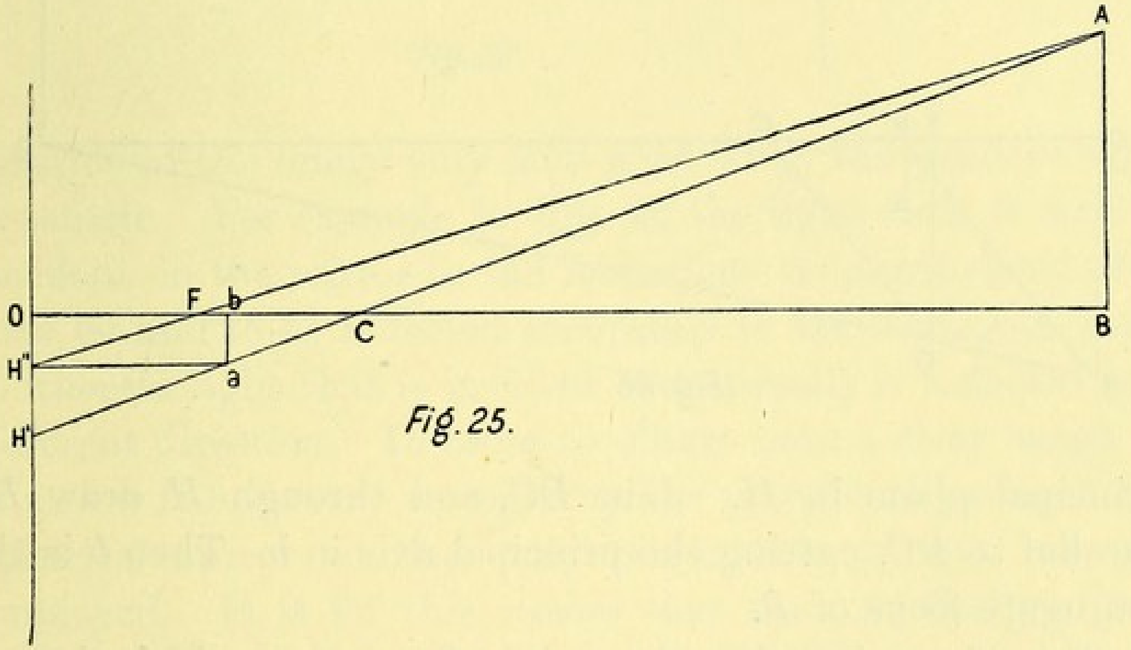

Fig. 25.

H'', and through H'' draw $H''a$ parallel to the principal axis, meeting the secondary axis ACH' in a. Then since the ray AH'' passes through the principal focus it must be reflected parallel to the axis, and a, its point of intersection with the ray drawn through the centre, must be the image of A.

To determine the position of the image of a point B on the principal axis, some special device is necessary. The conjugate b is evidently some point on the principal axis, but its position can only be determined by finding where this line is crossed by some ray of the light that starting from B has undergone reflection.

We may draw an arbitrary secondary axis through C and treat it as the principal axis in the previous construction; or we may adopt the following method, which is preferable.

Through F (Fig. 26) draw FD at right angles to the principal axis, then FD represents the principal focal plane. Take any ray BDH cutting the focal plane in D, and the

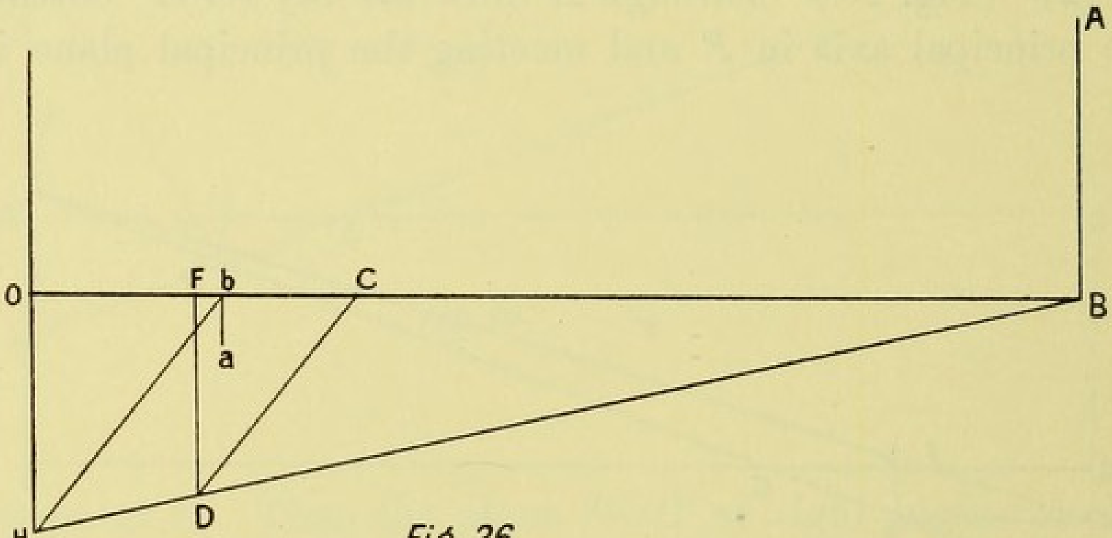

Fig. 26.

principal plane in H. Join DC, and through H draw Hb parallel to DC, cutting the principal axis in b. Then b is the conjugate focus of B.

For if we imagine the point D on the principal focal plane to become self-luminous, the light from D after reflection will travel in a pencil of rays parallel to the secondary axis DC. Now DH may be regarded as a ray of incident light either from D or from B; in either case Hb is the course of this light after reflection, and b, the point where this ray crosses the principal axis, is the conjugate focus of B.

If the distance of the object from the mirror is less than the principal focal distance, its image is virtual, erect and magnified, as is evident from a glance at Fig. 27 where the construction first described has been employed.

At this point it may be well again to warn the reader that these geometrical constructions for determining the

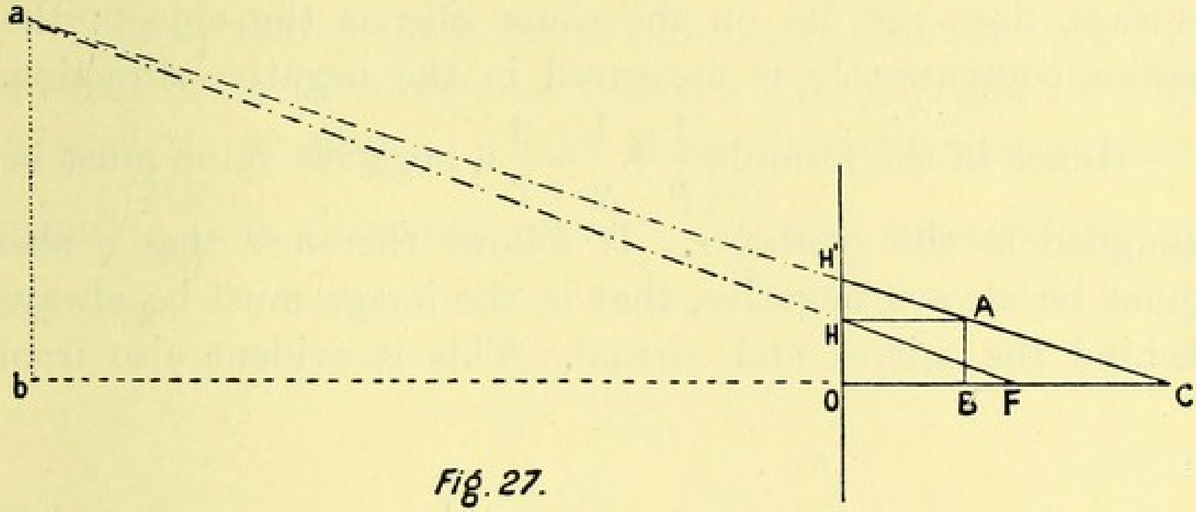

Fig. 27.

position of the image only hold good when the incident light is centric. For example in Fig. 26 the light from *B* that is incident on the mirror in the immediate neighbourhood of *O* may be said to be reflected accurately to the conjugate focus *b*, whereas light that is incident eccentrically is reflected in a different direction. In order to obtain then a clear image of *B* at *b* it is necessary to cover up the peripheral parts of the spherical mirror so that only the centric portion may be employed. It is for this reason that the principal plane is used in the construction, for it is evident that the tangent plane at the vertex is a better representation of this centric portion than the whole curved surface of the mirror would be. The line *BH*, which we have for convenience called the ray *BH*, does not therefore represent any ray that is actually incident on the mirror, for the point of its incidence is so eccentric that it would undergo anomalous reflection. We are however justified in asserting that a small centric pencil from *B* will after reflection come to a focus at the same point as the line *Hb* crosses the axis.

Convex Spherical Mirrors. Reflection from Convex spherical mirrors will offer no difficulty to the reader who has understood the preceding sections.

Since the object (P) is on the convex side of the mirror, the point C, which indicates the centre of its spherical surface, does not lie on the same side as the object; the radius consequently is measured in the negative direction. Hence in the formula $\frac{1}{p}+\frac{1}{q}=\frac{2}{r}$ a negative value must be assigned to the symbol r. It follows therefore that q also must be always negative, that is, the image must be always behind the mirror and virtual. This is evident also from

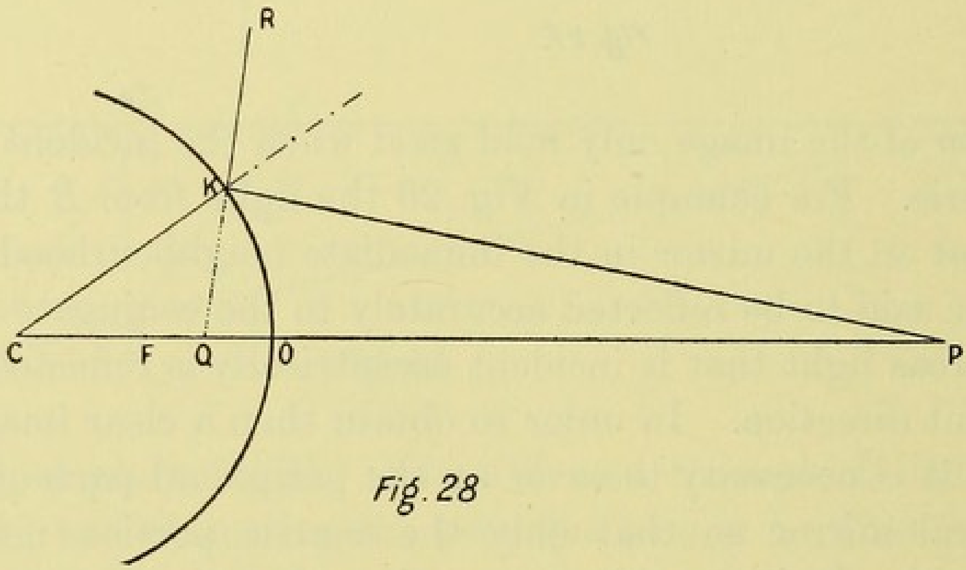

Fig. 28

general considerations; for a convex mirror causes divergence of the incident rays, and these, if coming from a real object, must have been divergent even before reaching the mirror.

The geometrical proof of the formula for the convex mirror is almost identical with that for the concave.

Let POC denote the principal axis, and let PK denote a ray from P incident on the mirror at K, a point very near to the vertex O. Then PK will be reflected as KR which makes an angle with the normal equal to the angle of incidence.

Produce RK to Q cutting the axis in Q.

Then PKQ is a triangle, the exterior angle of which is bisected by a line CK that meets the base produced in C.

$$\therefore \frac{PK}{KQ} = \frac{PC}{QC}. \qquad \text{(Euc. VI. A.)}$$

Now by taking the point K sufficiently near the vertex O we may make the difference between PK, KQ, and PO, OQ as small as we please. Under these circumstances we may regard $\frac{PK}{KQ}$ as ultimately equal to $\frac{PO}{OQ}$ and consequently in the limit

$$\frac{PC}{QC} = \frac{PO}{OQ},$$

or

$$\frac{PO + OC}{OC - OQ} = \frac{PO}{OQ}.$$

Now if we assign to p, q and r the same values as before, $p = PO$, $q = QO$, $r = CO$, then $OC = -r$ and $OQ = -q$.

$$\therefore \frac{p - r}{q - r} = \frac{p}{-q},$$

or

$$qr + pr = 2pq,$$

or

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{r}.$$

It should be noted that p, q and r are regarded as algebraic quantities, *i.e.* as symbols, which in addition to their numerical significance represent direction.

Under these circumstances the old formula $\frac{1}{p} + \frac{1}{q} = \frac{2}{r}$ holds good for both kinds of spherical mirrors.

If the incident rays are parallel $\frac{1}{p}$ becomes $\frac{1}{\infty}$ or 0, and $\frac{1}{q}$, or $\frac{1}{f} = \frac{2}{r}$. Since r is negative f is also negative, *i.e.* incident parallel rays after reflection diverge as if from a point F, situated behind the mirror at half the distance of the centre from the vertex.

Ex. An object is placed 8 ins. in front of a spherical mirror, and a virtual image of it is formed 4·8 ins. behind the mirror. What is the radius of curvature of the mirror, and what is its focal length?

Let us employ the fundamental formula

$$\frac{1}{p}+\frac{1}{q}=\frac{2}{r} \text{ or } \frac{1}{f}.$$

In this case $p=8$ ins. and $q=-4·8$ ins., for the image is behind the mirror; so we get, on substituting for the symbols their numerical values and paying due regard to the signs that they bear,

$$\frac{1}{8}+\frac{1}{-4·8}=\frac{2}{r} \text{ or } \frac{1}{f}, \quad \text{or} \quad \frac{6-10}{48}=-\frac{1}{12}=\frac{2}{r} \text{ or } \frac{1}{f},$$

$$\therefore\ r=-24 \text{ ins., and } f=-12 \text{ ins.}$$

The negative signs shew that the radius and the focal distance are to be measured in the negative direction, *i.e.* both the centre of curvature and the principal focus lie behind the mirror; in other words, the mirror is convex.

The methods for constructing the image that results from reflection at a convex mirror are identical with those that

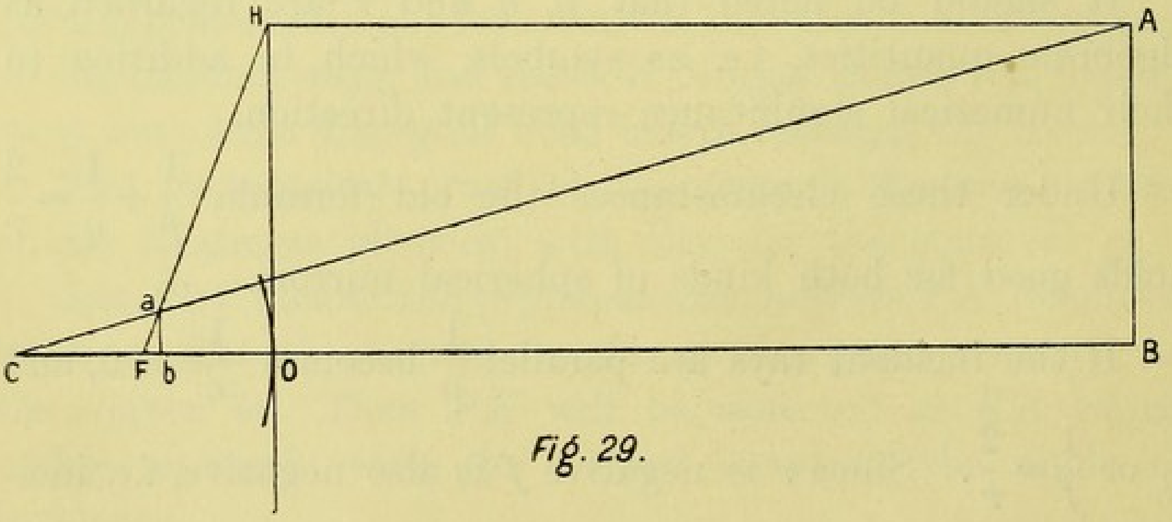

Fig. 29.

were employed for reflection at a concave surface. The diagram (Fig. 29) shews the construction for the image ab of the object AB, according to the method described (p. 88).

Size of the Image. We have now to determine the size of the image. This may be expressed algebraically in several different ways, but in every case the same formula holds good both for concave and convex mirrors, provided that the appropriate signs are prefixed when numerical values are substituted for the symbols.

Let AB represent the object (Fig. 30) and ab the image formed by reflection at a concave surface, and let BCO be the axis. Join AO and Oa.

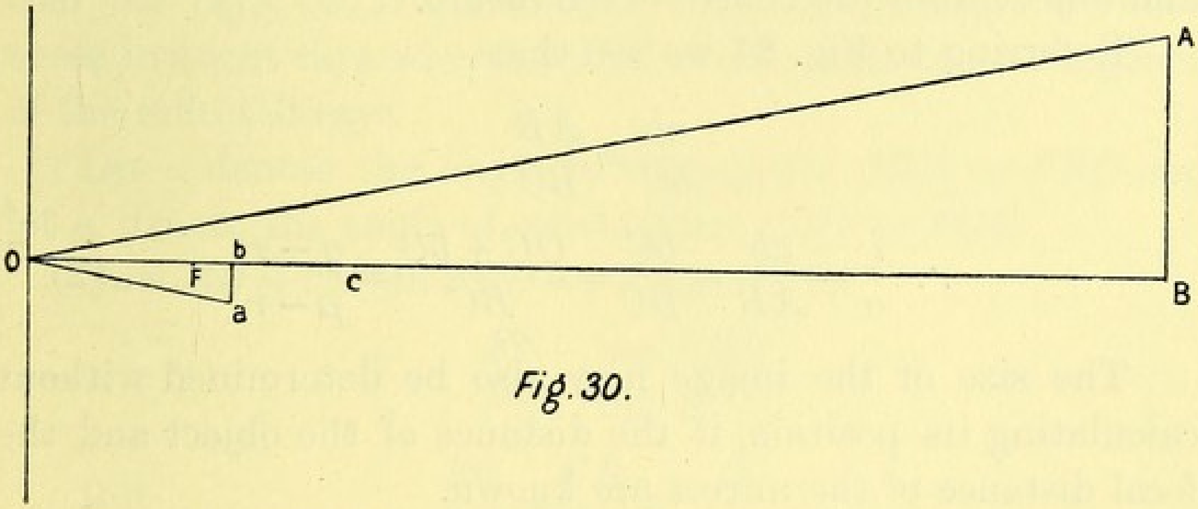

Fig. 30.

Then since CO is normal to the reflecting surface at O it is clear that the angle of incidence AOB is equal to the angle of reflection aOb, and the remaining angles of the triangle AOB are equal to the remaining angles of the triangle aOb.

It should be noted that the angle aOb is measured in the reverse direction to the angle AOB.

$$\therefore \tan aOb = -\tan AOB,$$

or
$$\frac{ab}{bO} = -\frac{AB}{BO};$$

$$\therefore \frac{ab}{AB} = -\frac{bO}{BO} = -\frac{q}{p}.$$

Now if o denote the height of the object and i the height of the image, it is clear that

$$\frac{i}{o} = -\frac{q}{p} \text{ .... (1).}$$

It is evident therefore that all erect images are virtual, for since o and p are always considered positive, i can only be positive (*i.e.* erect) when q is negative; or in other words, when the image is formed behind the mirror. For similar reasons, if i is negative q must be positive, which is equivalent to the statement that all inverted images are real.

The size of the image as compared with that of the object may also be expressed in terms of the distances of the image and object from the centre of curvature.

Referring to Fig. 24 we see that

$$\frac{ab}{bC} = \frac{AB}{BC};$$

$$\therefore \frac{i}{o} = \frac{ab}{AB} = \frac{bC}{BC} = \frac{OC + bO}{BC} = \frac{q - r}{p - r} \text{.........(2).}$$

The size of the image may also be determined without calculating its position, if the distance of the object and the focal distance of the mirror are known.

Thus it is evident from Fig. 25 that $ab = H''O$;

$$\therefore \frac{ab}{OF} = \frac{H''O}{OF} = \frac{AB}{BF},$$

$$\therefore \frac{i}{o} = \frac{ab}{AB} = \frac{OF}{BF} = -\frac{f}{p-f} = \frac{f}{f-p} \text{ .... (3).}$$

Consideration of this formula shews that if f is positive, *i.e.* if the mirror is concave, the image can only be erect and virtual when p is less than f, and that under these circumstances the image is always larger than the object. If the mirror is convex, f is negative and i is consequently always positive and always smaller than the object.

Again, from Fig. 24 we see that $AB = HO$,

$$\therefore \frac{ab}{bF} = \frac{HO}{OF} = \frac{AB}{-FO},$$

$$\therefore \frac{i}{o} \text{ or } \frac{ab}{AB} = \frac{bF}{-FO} = \frac{q-f}{-f} = \frac{f-q}{f} \quad \ldots\ldots (4).$$

Finally, the relation between the size of the image and the size of the object may be expressed in terms of the angle between any two rays that proceed from a given point of the object, and of the angle between these same rays after reflection has occurred.

Thus in Fig. 26 we see that the two rays BO, BH that proceed from the point B of the object become after reflection the rays Ob, Hb, and that OBH is the angle between these incident rays whereas ObH is the angle of convergence of the reflected rays.

Let α denote the angle of divergence OBH or FBD and let α' denote the angle of convergence ObH or FCD.

Now $FD = FC \tan FCD = FB \tan FBD$,

$$\therefore \frac{FC}{FB} = \frac{\tan FBD}{\tan FCD}.$$

But

$$\frac{FC}{FB} = \frac{CF}{BF} = \frac{f}{p-f},$$

$$\therefore \frac{f}{p-f} = \frac{\tan FBD}{\tan FCD} = \frac{\tan \alpha}{\tan \alpha'},$$

$$\therefore \frac{i}{o} \text{ or } \frac{f}{f-p} = -\frac{\tan \alpha}{\tan \alpha'}, \quad \ldots\ldots\ldots\ldots\ldots (5).$$

It also is evident from (1) that

$$\frac{q}{p} = \frac{\tan \alpha}{\tan \alpha'}.$$

This expression we owe to Helmholtz.

A few examples are given to shew the way in which these formulae are used.

Ex. (1). A concave mirror has a radius of curvature of 10 ins. What is the focal length? An object $4\frac{1}{2}$ ins. in height is placed 50 ins. in front of the mirror. What is the height of the image, and where is it formed?

Here $r = 10$, so f or $\frac{r}{2} = 5$ ins.

And since
$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f},$$
$$\frac{1}{50} + \frac{1}{q} = \frac{1}{5},$$
$$\therefore \frac{1}{q} = \frac{1}{5} - \frac{1}{50} \text{ or } \frac{9}{50},$$
$$\therefore q = \tfrac{50}{9} \text{ or } 5\tfrac{5}{9} \text{ ins.}$$

The image is formed $5\frac{5}{9}$ ins. in front of the mirror (since q is positive). The image is therefore real.

And since
$$\frac{i}{o} = \frac{-q}{p},$$
$$\frac{i}{4\frac{1}{2}} = \frac{-\frac{50}{9}}{50},$$
$$\therefore i = -\frac{4\frac{1}{2}}{9} \text{ or } -\tfrac{1}{2} \text{ in.}$$

The negative sign shews that the image is inverted; its height is $\frac{1}{2}$ in.

Ex. (2). The object is now placed 3 ins. in front of the same mirror. What is the height of the image?

It is unnecessary here to determine the situation of the image, so we may apply formula (3) $\frac{i}{o} = \frac{f}{f-p}$,
$$\therefore \frac{i}{4\frac{1}{2}} = \frac{5}{5-3} = 2\tfrac{1}{2},$$
$$\therefore i = 4\tfrac{1}{2} \times 2\tfrac{1}{2} \text{ or } 11\tfrac{1}{4} \text{ ins.}$$

The image is therefore $11\frac{1}{4}$ ins. in height, and the positive sign shews that it is erect.

If it is required to know the position of this image, we employ the formula $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$.

Then $$\frac{1}{3}+\frac{1}{q}=\frac{1}{5},$$

$$\therefore\ \frac{1}{q}=\frac{1}{5}-\frac{1}{3}\text{ or }-\frac{2}{15},$$

and $$q=-7\tfrac{1}{2}\text{ ins.}$$

The negative sign shews that the image is situated $7\frac{1}{2}$ ins. behind the mirror; in other words, the image is virtual.

Ex. (3). An object 6 cm. in height is placed at a distance of 9 cm. from a convex reflecting surface. An erect (virtual) image, 2·4 mm. in height, is formed of it. What is the radius of curvature of the reflecting surface?

Since $$\frac{i}{o}=\frac{f}{f-p},$$

$$f(i-o)=ip\text{ or }f=\frac{ip}{i-o},$$

$$\therefore f=\frac{2\cdot4\times90}{2\cdot4-60}=\frac{216}{-57\cdot6}=-\frac{18}{4\cdot8}=-\frac{15}{4},$$

$$\therefore\ \frac{r}{2}\text{ or }f=-3\cdot75\text{ mm., or }r=-7\cdot5\text{ mm.}$$

As the sign is negative the surface is convex.

It may be noted in passing that this is the basis of the method by which the radius of curvature of the cornea or of the lens of the eye is determined. A special apparatus is used to measure the size of the images reflected from these surfaces, and from this measurement the curvature is calculated precisely as that in the example.

Principle of same phase not least time. We have already alluded to the principle of least time in connection with the path that light takes when it is reflected by a plane surface. It can easily be shewn that when the reflection takes place at a convex surface the same principle holds good, but when the surface is concave we shall find that we must modify the statement that light always travels to its destination by the quickest route.

Convex mirror. Let SP (Fig. 31) represent a ray incident at a point P on the surface of a convex mirror, and

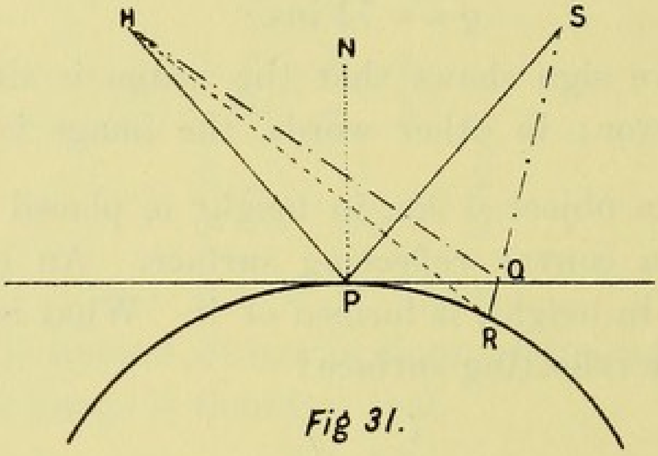

Fig 31.

let PH represent its course after reflection, making NPH equal to SPN.

Through the point P draw a tangent plane to the convex surface.

Then it is obvious that the path SPH is the shortest path from S to H by way of the mirror. For not only would any other path involve penetrating the tangent plane twice to reach and return from the mirror, but the path by any other points in the tangent plane would involve a longer course.

For if possible let SR, RH be a shorter course. Let the incident ray SR cut the tangent plane at Q; join QH.

Then

$$SP + PH < SQ + QH \text{ (p. 77)} < SQ + QR + RH,$$

$$\therefore\ SP + PH < SR + RH.$$

Therefore the path $SP + PH$ is the shortest path from S to H by way of the mirror.

Concave mirror. Let SP (Fig. 32) represent the path of light incident at P on the concave spherical surface PR, and let PH denote its course after reflection.

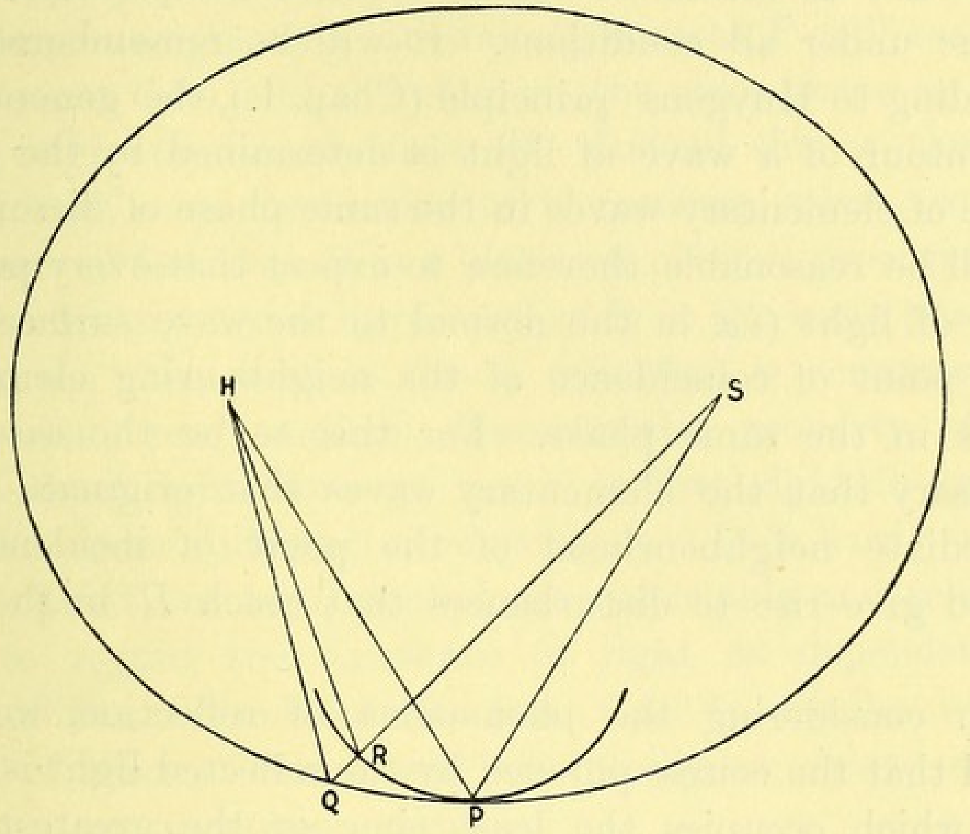

Fig. 32.

Describe an ellipse, with S and H for foci, touching the spherical surface at the point P.

Now if the curvature of the spherical surface be greater than that of the ellipse at P, it will be readily seen that the path actually taken by the light instead of being a minimum is a maximum.

For suppose it took any other course such as SR, RH,

then $$SR + RH < SR + RQ + QH = SQ + QH.$$

But by a well-known property of the ellipse

$$SQ + QH = SP + PH,$$

$$\therefore\ SR + RH < SP + PH.$$

In this case therefore the course actually taken by the light is a maximum, and the time taken to traverse it is the greatest, and not the least.

The principle of *least time* is not therefore universally true. Let us see if we cannot formulate a principle which is true under all conditions. It will be remembered that, according to Huygens' principle (Chap. I.), the general form or contour of a wave of light is determined by the coincidence of elementary waves in the same phase of disturbance. It will be reasonable therefore to expect that every point in a ray of light (*i.e.* in the normal to the wave surface) shall be a point of coincidence of the neighbouring elementary waves in the same phase. For this to be the case it is necessary that the elementary waves that originate in the immediate neighbourhood of the point of incidence (P) should give rise to disturbances that reach H in the same time.

In considering the phenomena of reflection, we have found that the course pursued by the reflected light is either that which occupies the least time or the greatest time. Now it is a property common to all maxima and minima that an indefinitely small variation in the variable makes no difference in the result.

A maximum or a minimum may be regarded as a turning point, and it is a property of all such turning points that in their immediate neighbourhood the variation is exceedingly small. If a train, the speed of which is steadily increasing, be suddenly retarded by putting on a brake, a turning point or maximum of the speed is obtained just before the application of the brake. It is clear that the speed of the train half-a-second before the brake was applied is very nearly identical with the speed half-a-second after its application, and indeed with the maximum speed attained. The smaller the interval of time taken the less error shall we introduce by regarding these three speeds as identical.

Therefore when reflection occurs at P in such a way that the course taken by the light occupies either a maximum or a minimum time, a very small change of path will make an inappreciably small change of time. In fact the elementary disturbances originating indefinitely near to P will reach H in the same time, and therefore in the same phase, as those which originate directly from P. Indeed this is the only condition of non-interference of elementary waves, so that the observed fact that H is a luminous point is proof positive that elementary waves have reached it in the same phase.

We see then that, as far as reflection is concerned, we have found one common characteristic property of the behaviour of light, however much at first sight it appears to differ under different circumstances. It is hardly necessary to point out how much more reasonable and scientific it is to regard the existence of light as dependent on the condition of non-interference, than to ascribe to it a power of discrimination and an intelligent choice of the minimum or maximum path.

QUESTIONS.

(1) An object 10 cm. in height is placed 150 cm. from a concave mirror of focal length 25 cm. Where is the image formed, and what is its height?

(2) The radius of a concave mirror is 16 ins. What is the distance of the image from the mirror when the object is respectively at 12, 4, and 7 ins. distance?

(3) A real image ·5 cm. in height is formed at a distance of 6 cm. from a concave mirror of an object 2·5 cm. in height. What is the radius of curvature of the mirror?

(4) An erect image one-fourth the height of the object is formed by a mirror. If the distance of the object is 9 ins. what is the radius of curvature of the mirror?

(5) An inverted image of a candle is thrown on a screen at a distance of 6 feet from a mirror of focal length 6 ins. Where is the candle placed, and what is the relative size of the image?

(6) What is the diameter of the image of the sun formed by a concave mirror of focal length 1 m.? The apparent diameter of the sun is $31' 9''$. Given $\tan 31' 9'' = \cdot 00906$.

CHAPTER VI.

ECCENTRIC PENCILS. FOCAL LINES. CAUSTICS. CONTOUR OF REFLECTED WAVE-FRONT. APLANATIC REFLECTING SURFACES.

Eccentric Pencils. In the preceding sections we have been considering the reflection of small centric pencils only. Referring to Fig. 21 the point Q is the point of intersection of those rays that indicate the course of the light which, coming from P, has been reflected by the small segment $KOK_{,}$. Light from P incident on peripheral portions of the mirror will not converge to the point Q but will be reflected in such a way that the rays cross the principal axis at points on the mirror side of Q. This is easily seen by constructing a diagram similar to Fig. 21 and taking a point K'' on the peripheral part of the arc that represents the mirror. The reflected ray $K''Q''$, determined by making the angle $CK''Q''$ equal to the angle $PK''C$, crosses PO at a point on the mirror side of Q.

In Fig. 33 light from the luminous point S is supposed to be reflected by a concave hemispherical mirror. To avoid confusion in the diagram the incident rays are omitted with the exception of the small beam incident at PQ. The rays of reflected light are seen to intersect the principal axis at different points on the mirror side of I. The rays near the axis converge very nearly to the point I, the conjugate focus of S, whereas the eccentric rays cut the axis at *progressively* increasing distances from I.

All the reflected rays touch a certain caustic surface which has a cusp at I. The light is most intense in the neighbourhood of the cusp for the reason just given. A

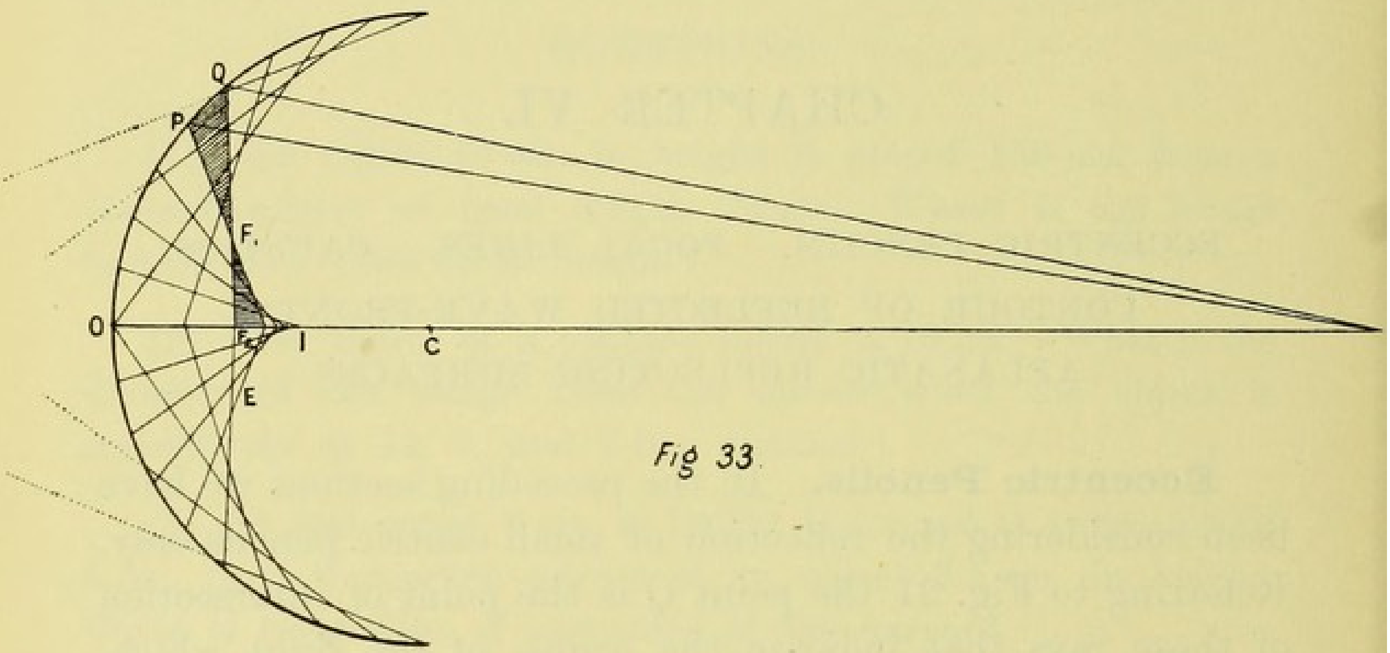

Fig 33

familiar example of such a caustic is that observed on the surface of the fluid in a teacup; this is formed by the light reflected from the sides of the cup. The diagram (Fig. 33) gives a view of a section of the caustic surface; the surface itself would be generated by the revolution of the figure about the axis OCS.

Fig. 34 gives an enlarged view of the caustic $LM'I$ in the neighbourhood of its cusp. The aperture $KK_{,}$ represents the central part of the mirror in the preceding figure. The marginal reflected rays KL, $K_{,}L_{,}$ intersect the principal axis and each other at G. If the light reflected by the mirror $KK_{,}$ were received upon a screen, it would be found that with the screen in the position $LL_{,}$ it would be illuminated by a circular patch of light, the outer edge of the patch being especially bright. As the screen is moved towards I this bright ring will gradually contract. When G is reached a bright spot develops in the centre; when $M'M$ is reached the circular patch of light reaches its smallest dimensions, being brightest in the centre. Beyond this point the circle of light expands again while its central bright part still contracts, until *I* is reached, when an

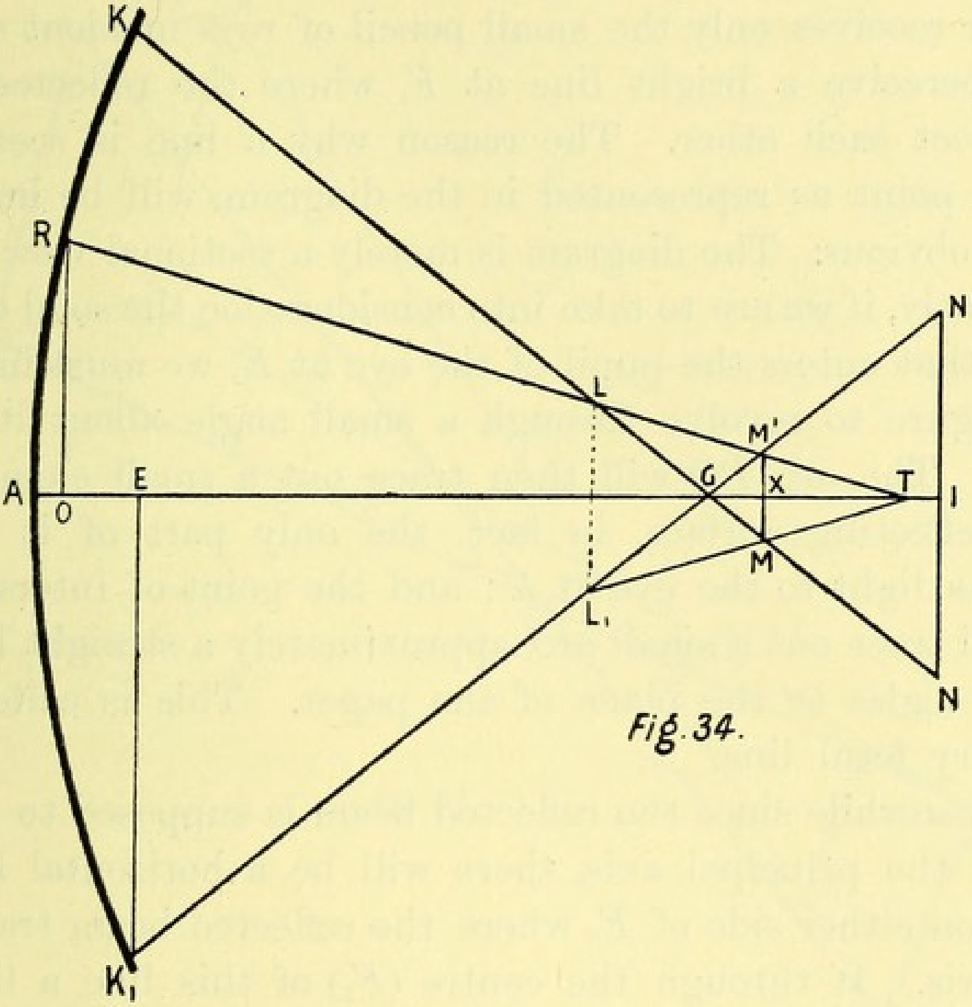

Fig. 34.

intensely bright point of light is seen at *I* surrounded by a dimly illuminated marginal zone extending to *N'N*. The distance *GI* is called the longitudinal aberration, or sometimes simply the aberration of the marginal ray *KG*, while the distance *IN* is called its lateral aberration.

The circle at *MM'* is called the circle of least confusion, and is what may be regarded as the blurred image of the luminous point *S* formed by the mirror $KAK_{\prime}$. If the peripheral parts of the mirror be blocked out by an annular diaphragm so that only a thin centric pencil is reflected at the mirror, a sharply defined image of the point *S* is formed at *I*.

When the light reflected from the mirror is viewed by the eye directly it must be remembered that it is itself provided with an annular diaphragm, the iris, and therefore can only receive a very slender pencil of rays.

Referring again to Fig. 33 we see that an eye at *E*, which receives only the small pencil of rays incident at *PQ*, will perceive a bright line at F_1 where the reflected rays intersect each other. The reason why a line is seen, and not a point as represented in the diagram, will be immediately obvious. The diagram is merely a sectional view; consequently, if we are to take into consideration the solid cone of light that enters the pupil of the eye at *E*, we must imagine the figure to revolve through a small angle about its axis *OCS*. The arc *PQ* will then trace out a small element of the reflecting surface, in fact, the only part of it which reflects light to the eye at *E*; and the point of intersection F_1 will trace out a small arc, approximately a straight line at right angles to the plane of the paper. This is called the primary focal line.

Meanwhile since the reflected beam is supposed to rotate round the principal axis, there will be a horizontal line of light on either side of F_2 where the reflected beam traverses the axis. If through the centre (F_2) of this line a line be drawn parallel to the tangent at *PQ* it will on rotating the figure map out a figure with two slender loops something like a figure of eight, which may be regarded as approximately a straight line. This is called the secondary focal line.

The lines at F_1 and F_2 are in fact the cross-sections of the reflected beam at these two points. It is usual to call the plane that contains the axis of a pencil, and the axis of the surface that reflects or refracts it, the primary plane. We see then that the primary focal line is at right angles to the primary plane, and that the secondary focal line is in this plane.

A glance at Fig. 35, which gives another aspect of the reflected pencil (shaded in Fig. 33) on a greatly enlarged scale, will make this clear.

The figure is somewhat diagrammatic. Thus the portion of the reflecting surface traced out by the above-mentioned revolution is considered to be a rectangle $PQRR_1$ viewed

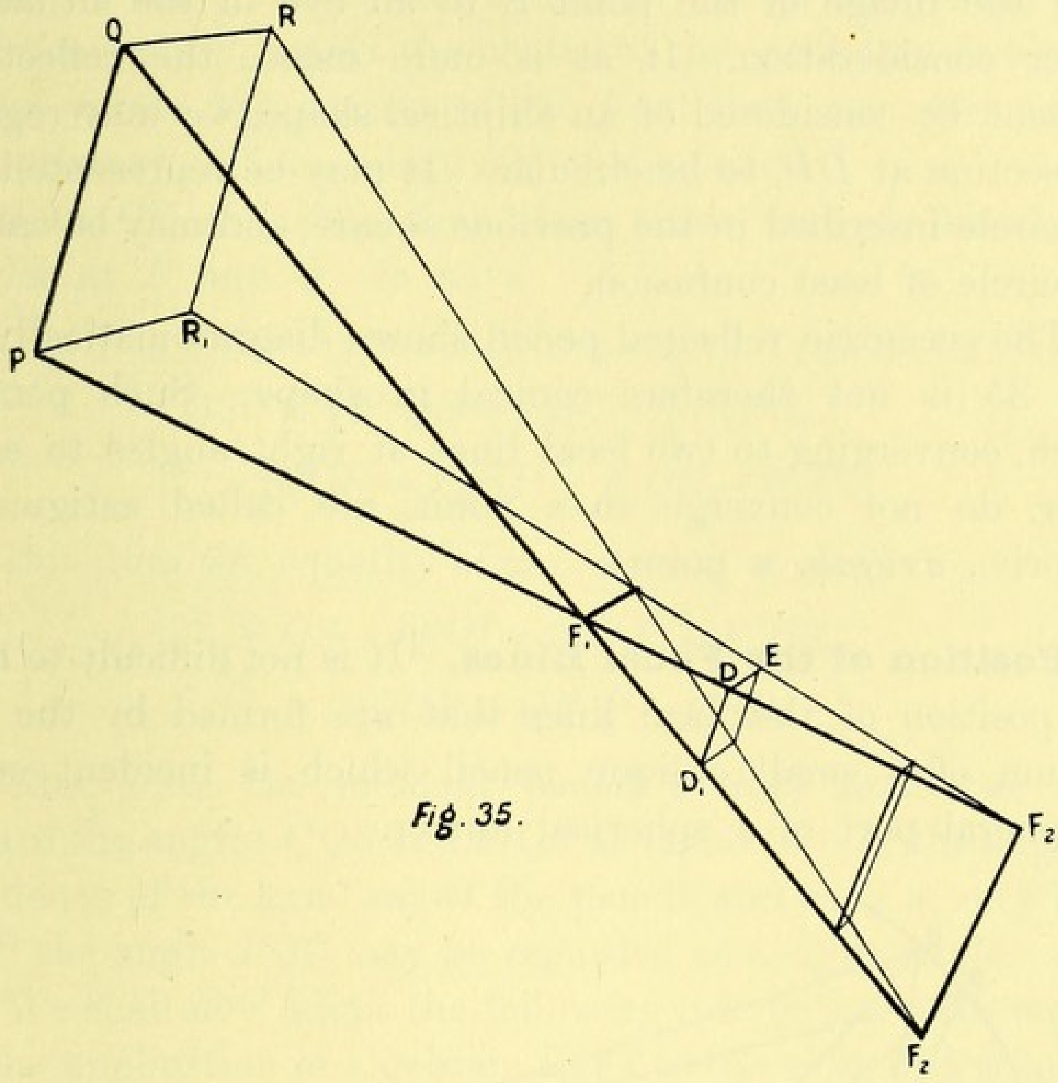

Fig. 35.

obliquely. The light reflected by this portion converges to a primary focal line at F_1 and to a secondary focal line F_2, which is in a plane at right angles to the line at F_1. The shape of the reflected pencil between these two lines is what is termed in crystallography a sphenoid. A cross-section near F_1 would be oblong in shape, the width of the section being much greater than the height. A little further on the section would be nearly square. A section near F_2 would be oblong again, its height being much greater than its width.

It will be seen that an eye which receives this obliquely reflected pencil cannot see a sharply defined image of the point S. The place where the rays are nearest together is denoted by ED, where the cross-section is approximately square. The blurred patch of light at this spot represents then the image of the point S to an eye in the situation under consideration. If, as is more usual, the reflecting element be considered of an elliptical shape, we may regard the section at DD_1 to be circular. It may be represented by the circle inscribed in the previous square, and may be called the circle of least confusion.

The eccentric reflected pencil shewn diagrammatically in Fig. 35 is not therefore conical in shape. Such pencils which, converging to two focal lines at right angles to each other, do not converge to a point, are called astigmatic (*a*, priv., *στίγμα*, a point).

Position of the Focal Lines. It is not difficult to find the position of the focal lines that are formed by the reflection of a small oblique pencil which is incident on a peripheral part of a spherical mirror.

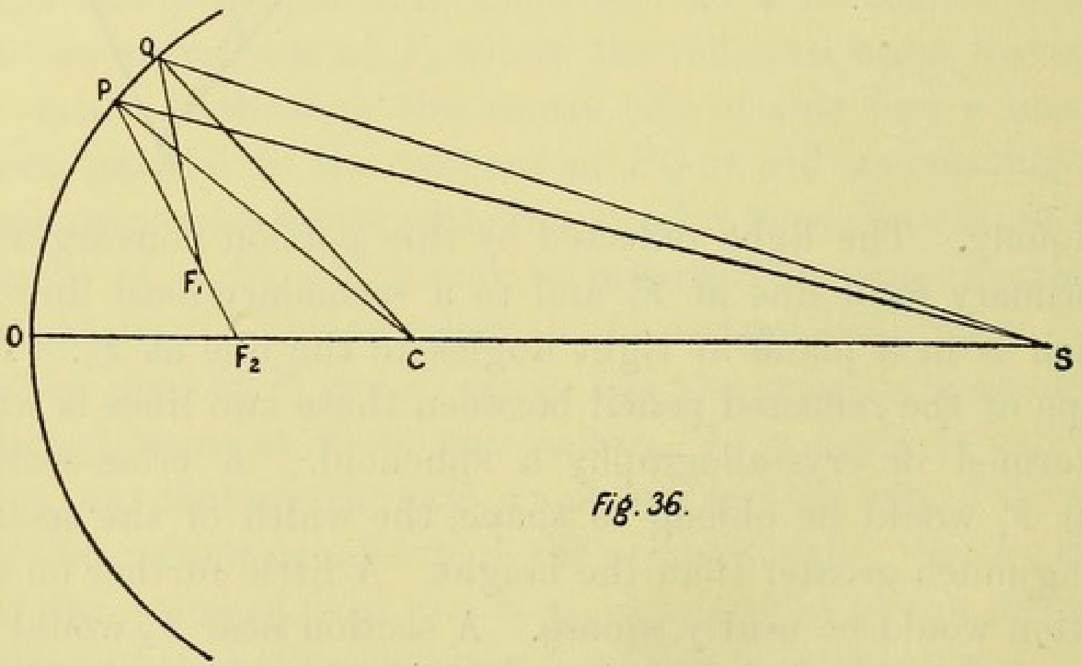

Fig. 36.

Let SP represent the axial ray of the obliquely incident pencil, and let SQ represent an extreme ray of the pencil considered. Join SC and produce it to meet the mirror in O, then OCS is the axis of the system. Let the reflected ray PF_2 cut the axis in F_2, and let the reflected ray QF_1 cut PF_2 in F_1. Then F_1 marks the centre of the primary focal line, and similarly F_2 marks the centre of the secondary focal line. These points, the centres of the focal lines, are often called the primary and secondary foci of the pencil.

Join CP and CQ; then since these normals bisect the angles at P and Q, we have

$$QCP = QCO - PCO = QSC + CQS - (PSC + CPS)$$
$$= QSC + \tfrac{1}{2}F_1QS - PSC - \tfrac{1}{2}F_1PS,$$
$$\therefore\ 2QCP = 2QSP + F_1QS - F_1PS.$$

But from the equality of the vertical angles

$$F_1QS + QSP = QF_1P + F_1PS,$$
$$\therefore\ 2QCP = QF_1P + QSP.$$

Now since the incident pencil is very small, we may regard the angles CQS and F_1QC as equal to $-\phi$, the angle of incidence of the axial ray of the pencil, and as Q is very near to P the angle PQC may be regarded as a right angle.

We shall now adopt the following convention with regard to the application of algebraic signs to the sides of a triangle.

In any triangle such as QSP we shall consider the ratio $\dfrac{\sin QSP}{\sin PQS}$ as bearing the same sign as the ratio $\dfrac{PQ}{SP}$ or $\dfrac{QP}{PS}$ but the opposite sign to $\dfrac{QP}{SP}$.

Then in the triangle QF_1P,

$$\frac{PQ}{F_1P} = \frac{\sin QF_1P}{\sin PQF_1} = \frac{\sin QF_1P}{\sin (90 - F_1QC)} = \frac{\sin QF_1P}{\cos \phi},$$

and in the triangle QSP,

$$\frac{PQ}{SP} = \frac{\sin QSP}{\sin PQS} = \frac{\sin QSP}{\sin (90 + CQS)} = \frac{\sin QSP}{\cos \phi}.$$

On replacing the sines of these small angles by the angles themselves we obtain

$$QF_1P = \frac{PQ\cos\phi}{F_1P} \text{ and } QSP = \frac{PQ\cos\phi}{SP},$$

and the equation $2QCP = QF_1P + QSP$ becomes

$$\frac{2PQ}{CP} = \frac{PQ\cos\phi}{F_1P} + \frac{PQ\cos\phi}{SP}.$$

If as is usual we denote the distances SP by u, F_1P by v_1, F_2P by v_2, and the radius CP of the spherical surface by r, we have

$$\frac{1}{u} + \frac{1}{v_1} = \frac{2}{r\cos\phi}.$$

Again, since the triangle $SPF_2 = SPC + CPF_2$,

or $$\tfrac{1}{2}uv_2\sin 2\phi = \tfrac{1}{2}ur\sin\phi + \tfrac{1}{2}rv_2\sin\phi,$$

$$\frac{\sin 2\phi}{r\sin\phi} = \frac{1}{v_2} + \frac{1}{u},$$

$$\therefore \frac{1}{u} + \frac{1}{v_2} = \frac{2\cos\phi}{r}.$$

Circle of Least Confusion. Now that we have found the position of the focal lines of a small eccentrically reflected pencil, we can determine the situation of the circle of least confusion with reference to them. Let DD_1 (Fig. 35) be that situation, the circle being that inscribed in the square there represented.

Let $DE = DD_1 = k$.

In the figure $PQRR_1$ represents that portion of the mirror which is reflecting the pencil under consideration.

Let QP and PR_1 be denoted by a and b, and let the distances F_1P, F_2P and DP be denoted by v_1, v_2 and x.

Then by similar triangles

$$\frac{DE}{PR_1} = \frac{F_2D}{F_2P} \text{ and } \frac{DD_1}{DF_1} = \frac{PQ}{PF_1} = \frac{QP}{F_1P},$$

$$i.e.\ k = b\frac{F_2D}{v_2} \text{ and } k = a\frac{DF_1}{v_1},$$

$$\therefore\ a\frac{x - v_1}{v_1} = b\frac{v_2 - x}{v_2}.$$

$$\therefore\ x(av_2 + bv_1) = v_1v_2(a + b),$$

or
$$\frac{a}{v_1} + \frac{b}{v_2} = \frac{a + b}{x} \quad \text{.................... (1).}$$

Again since $\dfrac{x - v_1}{k} = \dfrac{v_1}{a}$ and $\dfrac{v_2 - x}{k} = \dfrac{v_2}{b}$

we obtain by addition the value of k the side of the square of least confusion

$$\frac{v_2 - v_1}{k} = \frac{v_1}{a} + \frac{v_2}{b} \quad \text{.................... (2).}$$

If the reflecting surface $PQRR_1$ be square or circular, $a = b$ and the expression (1) becomes

$$\frac{1}{v_1} + \frac{1}{v_2} = \frac{2}{x} \quad \text{.......................(3),}$$

and from (2) we get
$$k = a\frac{v_2 - v_1}{v_2 + v_1} \quad \text{..................... (4).}$$

This determination of the magnitude and position of the circle of least confusion is applicable in every case of an astigmatic pencil whether it be formed by reflection or refraction.

Aberration. The aberration of a direct pencil after reflection at a spherical surface may be treated approximately in the following way.

Let S represent the luminous point on the axis of the spherical mirror KK'. Let KSK' represent the incident cone of light which may be considered to be made up of numerous thin pencils of different degrees of obliquity. Let the centric pencil be so reflected as to converge towards the point I, then I is the conjugate focus of the point S. Let SK represent the axial ray of the marginal pencil that is incident at K, and let the corresponding reflected pencil meet the axis in G. Then GI or a is the aberration.

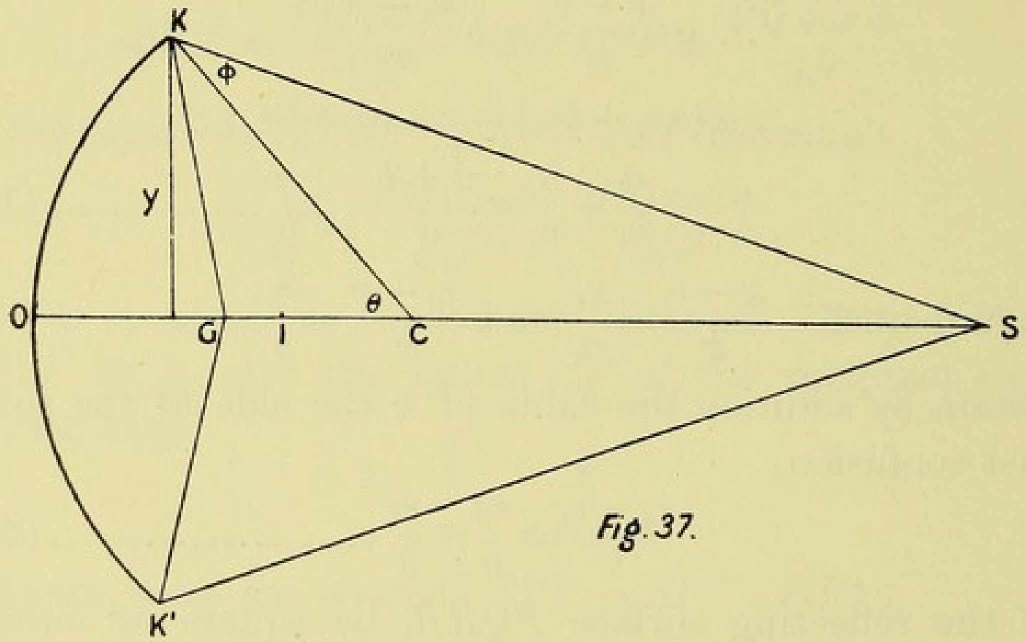

Fig. 37.

Let the angle of incidence of this marginal pencil be denoted by ϕ, and let the angle OCK subtended at the centre by the semiaperture of the mirror be denoted by θ.

Then $$\frac{CK}{SC} = \frac{\sin CSK}{\sin SKC} \text{ or } \frac{r}{SC} = \frac{\sin(\theta - \phi)}{\sin\phi}.$$

And $$\frac{CK}{CG} = \frac{\sin CGK}{\sin CKG} \text{ or } \frac{r}{CG} = \frac{\sin(\theta + \phi)}{\sin\phi}.$$

By subtraction

$$\frac{r}{CG} - \frac{r}{SC} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi - \sin\theta\cos\phi + \cos\theta\sin\phi}{\sin\phi}$$
$$= 2\cos\theta.$$

For the centric pencil, $\theta = 0$ and CG becomes CI,

$$\frac{r}{CI} - \frac{r}{SC} = 2.$$

On subtracting the former of these equations from the latter we obtain

$$\frac{r}{CI} - \frac{r}{CG} = 2(1 - \cos\theta),$$

or $$\frac{r}{r-q} - \frac{r}{r-q-a} = 2(1 - \cos\theta) \text{ .... (1).}$$

If θ be so small that powers of θ above the second may be neglected,

$$\frac{1}{CI} - \frac{1}{CG} \text{ or } \frac{CG - CI}{CI \,.\, CG} = \frac{\theta^2}{r}.$$

$$\therefore\ IG = \frac{\theta^2}{r} CI^2 \text{ approximately or } \frac{\theta^2}{r}(r-q)^2,$$

where q denotes the distance IO of the image from the mirror.

If $\frac{y}{r}$ or $\sin\theta$ be substituted for θ, which is allowable in the order of approximation to which we are now confining ourselves, the longitudinal aberration

$$GI \text{ or } a = -\frac{y^2}{r^3}(r-q)^2 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If the incident rays are parallel,

$$r - q = \frac{r}{2} \text{ or } f,$$

and
$$GI = -\frac{y^2}{4r} \text{ or } -\frac{y^2}{8f}.$$

If r or f is negative, GI is positive. The cusp of the caustic consequently points towards the centre of curvature of the reflecting surface.

When we come to deal with refraction at spherical surfaces we shall find that the aberration varies as y^2, or the square of the semiaperture allowed, when we limit ourselves to the second order of approximations. A convenient expression for the position and magnitude of the circle of least confusion for a direct pencil may be obtained by making use of this theorem.

Circle of Least Confusion. In Fig. 34 KN, $K_{,}N'$ represent the axial rays of the marginal reflected pencils intersecting on the axis at G. Let RT represent the axial ray of another reflected pencil cutting the principal axis at T, and the ray $K_{,}N'$ at M'. Then if RT be that particular ray whose intersection with $K_{,}N'$ is at the greatest possible distance from the axis, the point M' shall be the point at which it touches the caustic surface, and all the reflected rays shall pass through a circular space of which $M'X$ is the radius.

When $M'X$ is a maximum, $M'X$ defines the size and position of the circle of least confusion.

Let the ordinates of $K_{,}$ and R be denoted by $-y$ and y', and let the distance GX be denoted by x and the aberration GI by a.

In Fig. 34, x and a are negative.

Now $$\frac{M'X}{GX} = \frac{K_{,}E}{GE} \text{ or } M'X = \frac{-yx}{GE}.$$

But as $-y$ and GE are invariable, $M'X$ is a maximum when x is a maximum, so that we have only to find when x is a maximum.

Now $$XT = M'X\frac{OT}{RO} = \frac{-yx}{GE}\cdot\frac{OT}{RO}.$$

And when the aberration is very small, the difference between GE and TO is negligible.

$$\therefore\ XT = y\frac{x}{y'}, \text{ approximately.}$$

And $$GT = GX + XT = x + \frac{xy}{y'}.$$

But $$GT = GI - TI.$$

$$\therefore\ \frac{GT}{GI} = 1 - \frac{TI}{GI} = 1 - \frac{\dfrac{-y'^2}{r^3}(r-q)^2}{\dfrac{-y^2}{r^3}(r-q)^2} = \frac{y^2 - y'^2}{y^2}. \text{ (Cf. (2), p. 115.)}$$

On equating these values of GT we get

$$x\left(\frac{y+y'}{y'}\right)=a\,\frac{y^2-y'^2}{y^2},$$

or
$$x=\frac{a}{y^2}(y-y')\,y' \text{ ....(1).}$$

Then $\dfrac{dx}{dy'}=\dfrac{a}{y^2}(y-2y')=0$, when $y'=\dfrac{y}{2}$.

Hence x and consequently $M'X$ is a maximum when

$$y'=\frac{y}{2}.$$

Therefore from (1) $x=\dfrac{a}{4}$ and consequently $XI=\dfrac{3a}{4}$.

And since

$$\frac{M'X}{N'I}=\frac{GX}{GI},\ M'X=N'I\,\frac{x}{a}=\frac{N'I}{4}.$$

The approximate distance of the circle of least confusion from the conjugate focus I is three-fourths of the longitudinal aberration.

The approximate radius of the circle of least confusion is one-fourth of the lateral aberration $N'I$.

These results are approximately true in the case of refraction also, for they hold good whenever the aberration varies as the square of the semiaperture allowed.

It is evident from the figure that

$$\frac{N'I}{GI}=\frac{K_{,}E}{GE}.$$

If the aberration is small, we may take IA or q as a rough approximation to the value of GE.

In that case $$N'I \backsimeq \frac{-y}{q}\,a.$$

In the case of a spherical mirror when the incident rays are parallel, the longitudinal aberration

$$a \backsimeq \frac{-y^2}{8f},$$

the lateral aberration $\quad N'I \backsimeq \frac{y^3}{8f^2},$

the distance of the circle of least confusion

$$XI \backsimeq -\frac{3}{32}\frac{y^2}{f},$$

the radius of the circle of least confusion

$$MX \backsimeq \frac{1}{32}\frac{y^3}{f^2}.$$

The Contour of the reflected Wave-front. A Caustic Surface may be regarded as the envelope of the reflected (or refracted) rays. A glance at Fig. 33 shews that the caustic curve there shewn is the locus of the intersections of the consecutive reflected rays. But these rays are normals to the new wave-front. The source of light at S is sending out spherical waves expanding in all directions. On encountering the spherical reflecting mirror, a new shape is impressed upon the incident wave-front. It is obvious that were the source of light placed at C, the centre of curvature of the mirror, the incident expanding spherical wave-front would be reflected as a contracting spherical wave-front to the centre C again. In fact the focus C would coincide with the source of light, and there would be no aberration and no caustic.

If however the source of light is not situated at C, the reflected wave-front will be no longer spherical in shape. We proceed to discover the contour of the reflected wave-front under these conditions. The caustic surface is clearly the evolute of this reflected wave-front.

Let S (Fig. 38) be the source of light. In order to find the direction of a ray of light SP after reflection at P, we may use the method given on p. 80. At P draw the

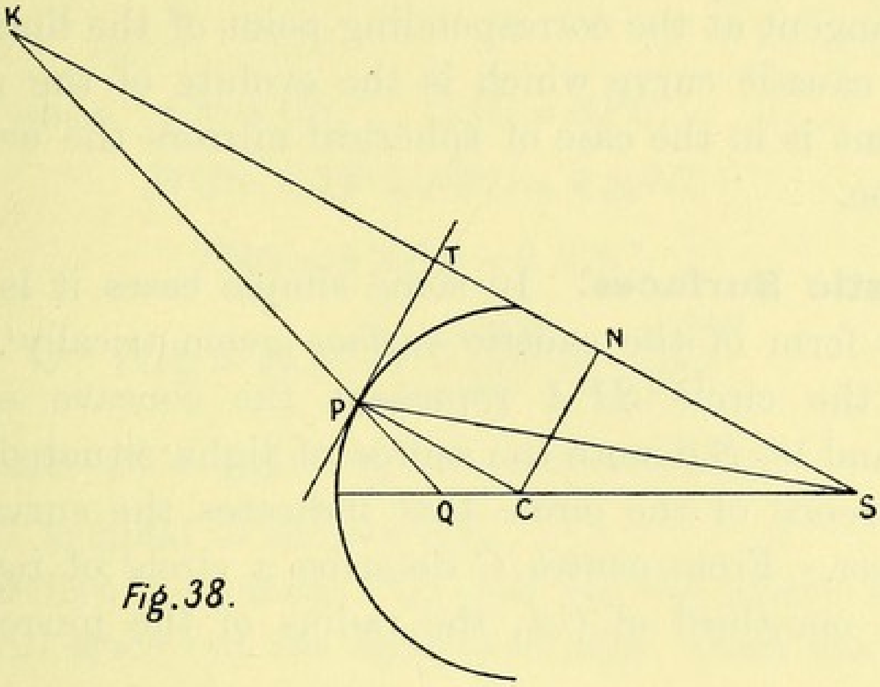

Fig. 38.

tangent PT, and from S let fall the perpendicular STK upon it, making TK equal to ST. Join KP, and produce to Q. Then PQ is the corresponding reflected ray.

The locus of K is easily found.

Draw CN perpendicular to SK.

Then

$$SK = 2ST = 2(SN + NT) = 2(SC \cos CSN + CP).$$

If we denote the radius vector SK by r, the radius CP of the reflecting spherical mirror by c, the distance SC of the source of light from the centre of curvature by a, and the angle CSN by θ, we get

$$r = 2(a \cos \theta + c).$$

The locus of K is therefore a limaçon. When, as in the diagram, $a > c$, a complete limaçon is not formed, for the mirror forms only a part of a reflecting sphere. If $a = c$ the limaçon reduces to a cardioid. Fig. 39.

Now since the reflected rays are normals to the reflected wave-front, it follows that the shape of the reflected wave-front must be a parallel to the above limaçon. That is, the tangent at any point of the reflected wave-front is parallel to the tangent at the corresponding point of the limaçon.

The caustic curve which is the evolute of the reflected wave-front is in the case of spherical mirrors the evolute of a limaçon.

Caustic Surfaces. In some simple cases it is easy to find the form of the caustic surface geometrically[1].

Let the circle SPA represent the concave spherical mirror, and let S denote the source of light, situated on the circumference of the circle that indicates the curvature of the mirror. From centre C describe a circle of radius CF equal to one-third of CA, the radius of the mirror.

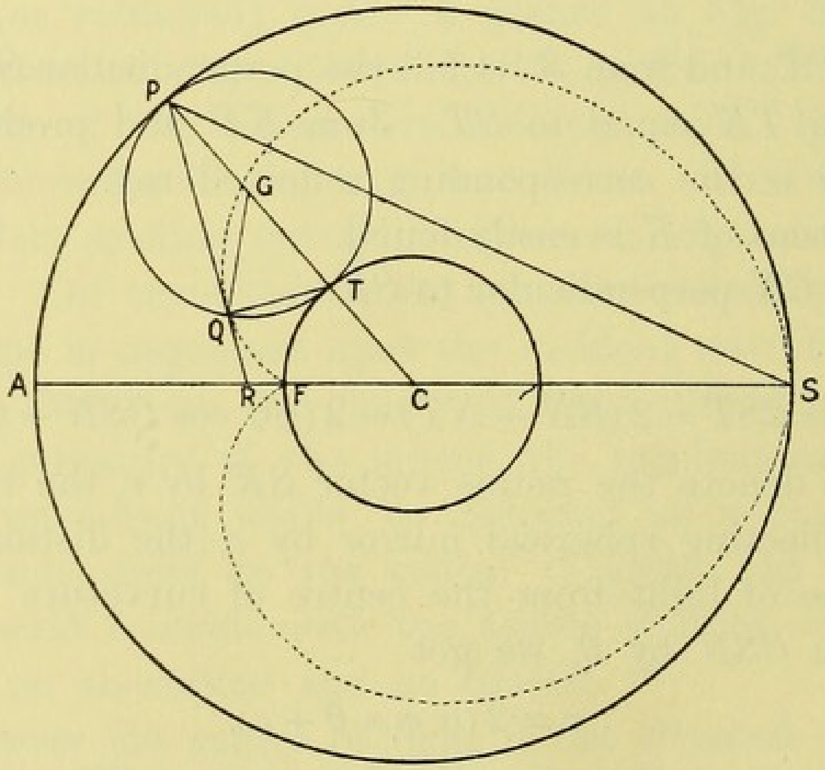

Fig. 39.

[1] The method used in these cases is taken from T. Preston's *Theory of Light*.

Let SP represent one of the incident rays. Join CP, cutting the smaller circle in T. On PT as diameter describe a circle PQT, its centre being denoted by G. Make CPR equal to SPC. Then PQR is the reflected ray corresponding to SP.

Then since $CP = CS$, $CSP = SPC$,

and $$FCT = CSP + SPC = 2\,SPC.$$

But $$TGQ = 2\,TPQ = 2\,SPC,$$

$$\therefore\ TGQ = FCT, \text{ or } \frac{\text{arc } QT}{TG} = \frac{\text{arc } FT}{FC}.$$

And since $FC = \frac{1}{3}AC = \frac{1}{3}PC = GT = -TG$, the arc QT is equal to the arc FT.

If therefore the circle PQT roll on the circle TF, the point Q will trace out the epicycloid SQF, which has a cusp at F.

Now since PQT is a right angle, being the angle of a semicircle, and QT is always normal to the path of Q, as T is the instantaneous centre, the line PQR is a tangent to the epicycloid at Q.

The epicycloid is therefore the envelope of the reflected rays in the plane of the paper. The caustic surface is described by the revolution of the epicycloid round the axis AC.

In the above example, where the rolling circle is equal to the fixed circle, the epicycloid described is a cardioid.

If the distance SC were infinite, so that parallel rays of light were incident upon a hemispherical mirror, a geometrical construction similar to the above would illustrate the case. The only modification in the construction would be that in this case the circle TF described about the centre C would be made of radius $\frac{1}{2}CA$ instead of $\frac{1}{3}CA$. The rolling circle PQT will therefore have a diameter equal to the radius of the fixed circle *TF*. The caustic will be this epicycloid (see Fig. 41).

General Expression for Caustics. The method given above is well suited to a few special cases, but if a general method is sought to enable one to trace any caustic curve, it will be found that the tangential-polar form of equations furnishes the simplest and most convenient way of dealing with the problem.

If *CA* be the initial line, and if ψ be the angle which the tangent to the curve makes with it, and if p denote the perpendicular from C upon the tangent, the relation between p and ψ forms a simple equation to the curve.

Before dealing with any example of a caustic surface, it will be well to prove the following lemma, which enables us to determine the curvature $\left(\frac{1}{\rho}\right)$ at any point of the surface.

Lemma $\rho = p + p''$.

Let *CA* be the initial line, its origin being C. Let $CN = p$, the perpendicular from the origin on the tangent

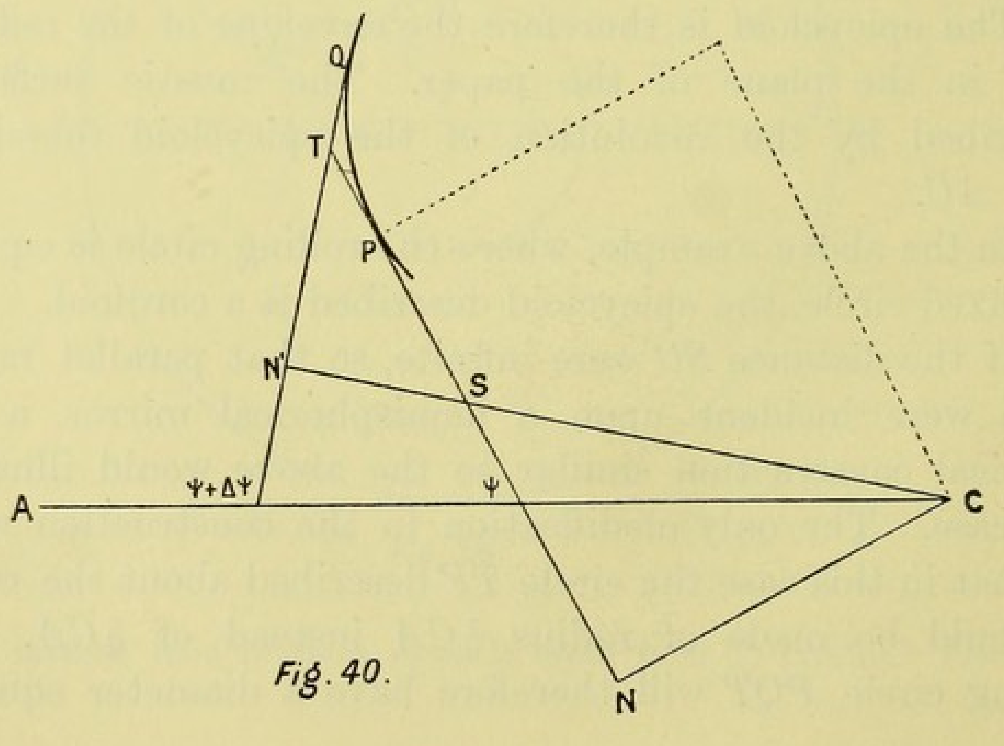

Fig. 40.

to the curve at P. Let $PN = t$, the length of the tangent at P to its intersection with CN. Similarly draw CN', the perpendicular from the origin on the tangent at an adjacent point Q. Let ψ be the angle which the tangent to the curve at P makes with the initial line.

Then $$STN' = NCS = \Delta\psi,$$

$$SN' = TS \sin STN',$$

$$\therefore \text{L}^{\text{t}}\, TS = \text{L}^{\text{t}} \frac{SN'}{\sin STN'} = \text{L}^{\text{t}} \frac{\Delta p}{\Delta\psi}.$$

In the limit when Q coincides with P, $TS = PN$,

$$\therefore PN \text{ or } t = \frac{dp}{d\psi}. \quad \text{Let it be denoted by } p'.$$

N.B. PN or p' is obviously equal to the perpendicular from C upon the radius of curvature at P.

This, as will appear presently, is a very useful point in using tangential-polar equations for determining the position of cusps.

Again $$\rho = \frac{ds}{d\psi} = \text{L}^{\text{t}} \frac{QT + TP}{\Delta\psi},$$

$$\frac{dt}{d\psi} = \text{L}^{\text{t}} \frac{QN' - PN}{\Delta\psi};$$

but $$TP + PN + QT - QN' = TN - TN',$$

$$\therefore \frac{ds}{d\psi} - \frac{dt}{d\psi} = \text{L}^{\text{t}} \frac{TN - TN'}{\Delta\psi} = \text{L}^{\text{t}} \frac{SN}{\Delta\psi} = \text{L}^{\text{t}} \frac{SN}{\tan SCN} = CN = p,$$

$$\therefore \rho = p + \frac{dt}{d\psi} = p + \frac{d^2p}{d\psi^2} = p + p''.$$

Examples.

(1) A plane wave-front of light (*i.e.* light of parallel rays) is incident on a reflecting hemisphere. Find and trace the caustic.

Let SCA be the centric ray passing through C the centre of the hemisphere. Let SP be any incident ray reflected from the mirror at P in the direction PN. Then $SPC = CPN = \phi$ the angle of incidence or the angle of reflection. Also ATP or $\psi = 2\phi$.

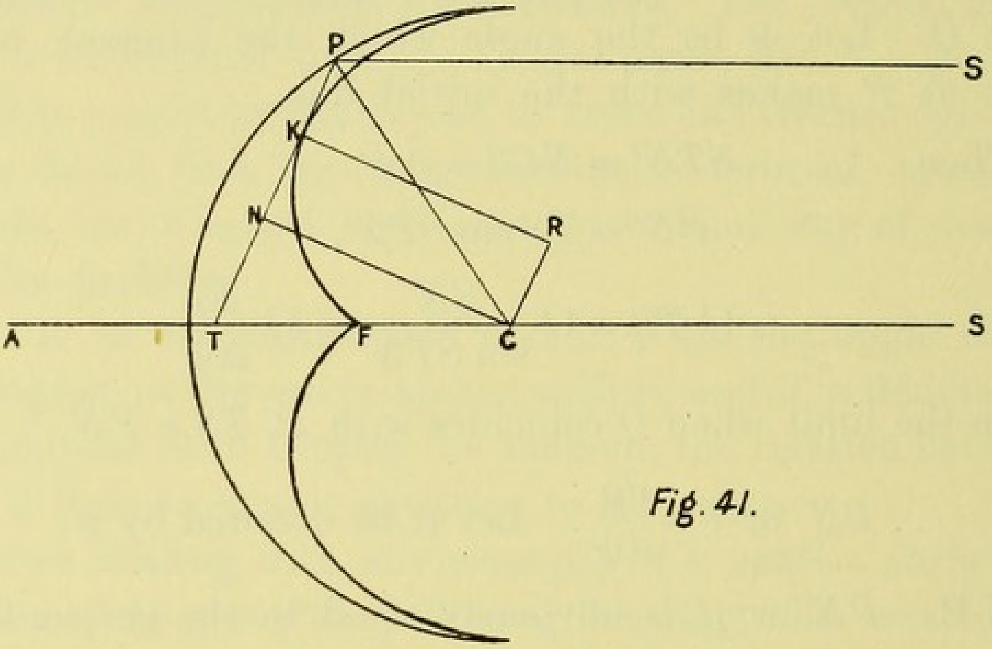

Fig. 41.

Now CN or $p = r\sin\phi = r\sin\frac{\psi}{2}$, where r = the radius of the hemisphere.

$$\therefore \quad p' \text{ or } \frac{dp}{d\psi} = \frac{r}{2}\cos\frac{\psi}{2},$$

$$p'' \text{ or } \frac{d^2p}{d\psi^2} = -\frac{r}{4}\sin\frac{\psi}{2},$$

$$\rho = p + p'' = \frac{3r}{4}\sin\frac{\psi}{2},$$

when $\psi = 0$, $\phi = 0$, $\rho = 0$, $p = 0$, $p' = \frac{r}{2}$.

There is a cusp (F) on the initial line at distance $\frac{r}{2}$ from C. As ψ begins to increase, ρ is positive and increases.

The curve advances from this point (F) on SCA with continually decreasing curvature, until $\psi = \pi$. Here $\rho = \frac{3r}{4}$, and a small arc of the circle of curvature can be drawn.

Clearly the curve is symmetrical with respect to SCA, and we have a cusp at the starting point. When $\psi = 0$, $p' = \frac{r}{2}$, and p' we have seen is the perpendicular from C upon the radius of curvature, or normal. The cusp is midway between C and the reflector, at distance $\frac{r}{2}$ from C.

Any point on the caustic curve can be found at which the tangent makes the angle ψ with the initial line SCA, and the radius of curvature (ρ) at the point determined.

E.g. Draw CR parallel to the tangent (or reflected ray) PN. Make $CR = p'$ or $\frac{r}{2}\cos\frac{\psi}{2}$. Draw RK perpendicular to PN meeting PN in K. Then K is the required point on the curve, and the radius of curvature at this point is ρ or $\frac{3r}{4}\sin\frac{\psi}{2}$.

The caustic surface is given by the revolution of the caustic curve about the initial line SCA.

(2) Light from a luminous point S on the circumference of a circle is reflected once at all points round the circumference. Find the caustic.

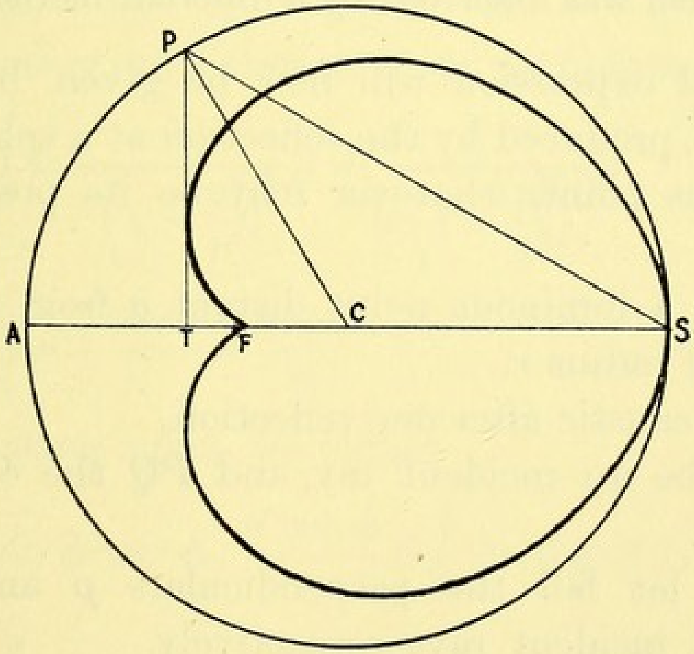

Fig. 42.

Here $SPC = CSP = \phi$,

$$\therefore\ ATP \text{ or } \psi = 3\phi,$$

$$p = r \sin\phi = r \sin\frac{\psi}{3},$$

$$p' = \frac{r}{3}\cos\frac{\psi}{3},$$

$$p'' = -\frac{r}{9}\sin\frac{\psi}{3},$$

$$\rho = p + p'' = \frac{8r}{9}\sin\frac{\psi}{3},$$

when $\psi = 0,\ \phi = 0,\ p = 0,\ p' = \frac{r}{3},\ \rho = 0.$

There is a cusp (F) on the initial line SCA, at distance $\frac{r}{3}$ from C. The curve advances from this point F, gradually reducing its curvature until $\psi = \frac{3\pi}{2}$, where $p = r,\ p' = 0,\ \rho = \frac{8r}{9}$.

Here therefore S is the point on the caustic curve; its radius of curvature at this point being $\frac{8r}{9}$.

The curve represented in Fig. 42 is identical with the cardioid in Fig. 39 which was described by a different method.

A general expression will now be given, by which the caustic curve, produced by the reflection at a spherical mirror of a luminous point, whatever may be its position, can be traced.

Let S be a luminous point distant a from the centre C of a sphere of radius r.

Find the caustic after one reflection.

Let SP be an incident ray, and PQ the corresponding reflected ray.

From C let fall the perpendiculars p and n on the reflected and incident rays respectively.

Then the two triangles NPC, QPC have the angles

NPC, *PCN* equal to the angles *QPC*, *PCQ*, each to each, and the side *PC* common; therefore the base n is equal to the base p.

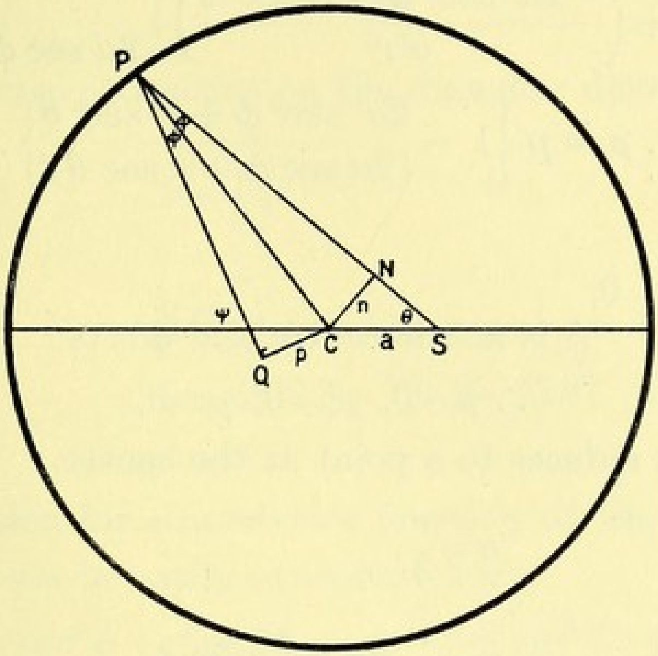

Fig. 43.

Now $$p = r\sin\phi = n = a\sin\theta \text{ .... (A)}$$

$$\psi = 2\phi + \theta = 2\sin^{-1}\frac{p}{r} + \sin^{-1}\frac{p}{a} \text{ ....(B).}$$

Differentiating with respect to ψ, we obtain from (B)

$$1 = \frac{2}{r\left(1 - \frac{p^2}{r^2}\right)^{\frac{1}{2}}}p' + \frac{1}{a\left(1 - \frac{p^2}{a^2}\right)^{\frac{1}{2}}}p',$$

$$\therefore \frac{1}{p'} = \frac{2\sec\phi}{r} + \frac{\sec\theta}{a}.$$

Differentiating again, we obtain

$$\frac{-p''}{p'^2} = \frac{2}{r}\sin\phi\sec^2\phi \,.\, \frac{p'\sec\phi}{r} + \frac{1}{a}\sin\theta\sec^2\theta \,.\, \frac{p'\sec\theta}{a},$$

$$\therefore \frac{p''}{p} = -\frac{2}{r^3}\sec^3\phi p'^3 - \frac{1}{a^3}\sec^3\theta p'^3.$$

Let this expression be denoted by Qp'^3.

Now $\rho = p + p'' = p(1+Qp'^3) = p\left\{1+Q\left(\dfrac{ar}{2a\sec\phi + r\sec\theta}\right)^3\right\}$,

$$\therefore \rho = p\left\{1 + \left(\frac{-2a^3\sec^3\phi - r^3\sec^3\theta}{a^3r^3}\right)\left(\frac{ar}{2a\sec\phi + r\sec\theta}\right)^3\right\},$$

$$\therefore \rho = p\left\{1 - \frac{2a^3\sec^3\phi + r^3\sec^3\theta}{(2a\sec\phi + r\sec\theta)^3}\right\}.$$

Examples.

(1) Let $a = 0$,

S is at the centre and $\phi = 0$.

$$\therefore p = 0,\ p' = 0,\ \rho = 0.$$

The caustic reduces to a point at the centre.

(2) Let $$a = \frac{r}{3},$$

$$p = \frac{r}{3}\sin\theta = r\sin\phi,$$

$$p' = \frac{ar}{2a\sec\phi + r\sec\theta},$$

$$\rho = p\left\{1 - \frac{2a^3\sec^3\phi + r^3\sec^3\theta}{(2a\sec\phi + r\sec\theta)^3}\right\}.$$

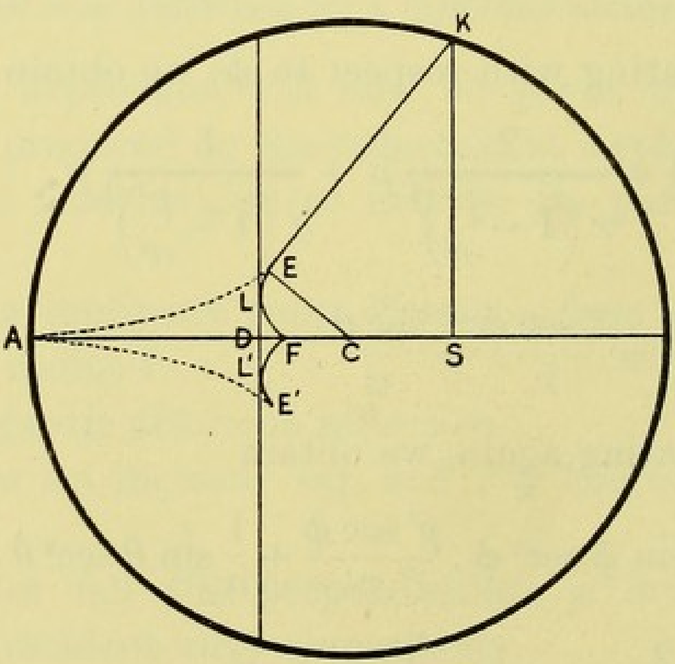

Fig. 44.

When $\theta = 0$,

$$p = 0,\ \phi = 0,\ p' = \frac{\frac{r^2}{3}}{\frac{2r + 3r}{3}} = \frac{r}{5},\ \rho = 0.$$

There is a cusp at a point on the diameter distant $\frac{r}{5}$ from C.

When $$\theta = \frac{\pi}{2},$$

$$p = \frac{r}{3},$$

$$\phi = \sin^{-1}\tfrac{1}{3} = 19^\circ\ 28'\cdot2731\ldots,$$

$$p' = 0.$$

The expression for ρ involves a fraction of the indeterminate form $\frac{\infty}{\infty}$ but it can be easily evaluated, for

$$\mathrm{L^t}_{\theta=\frac{\pi}{2}} \frac{2a^3 \sec^3\phi + r^3 \sec^3\theta}{(2a \sec\phi + r \sec\theta)^3} = \mathrm{L^t}_{\theta=\frac{\pi}{2}} \frac{2a^3 \cos^3\theta + r^3 \cos^3\phi}{(2a \cos\theta + r \cos\phi)^3}$$

$$= \frac{r^3 \cos^3\phi}{r^3 \cos^3\phi} = 1,$$

$$\therefore\ \rho = p\{1 - 1\} = 0.$$

There is a cusp at a point determined by

$$\theta = 90^\circ,\ \phi = 19^\circ\ 28'\cdot273\ldots,$$

$$\psi = 2\phi + \theta = 128^\circ\ 56'\cdot546\ldots,\ p = \frac{r}{3}.$$

There is another cusp at the corresponding point on the negative side of the diameter when $\theta = -\frac{\pi}{2}$.

When $\theta = \pi$, r is negative,

$$\phi = 0,\ \psi = \pi,$$

$$p = 0,\ p' = \frac{\frac{r^2}{3}}{\frac{2r - 3r}{3}} = -r,\ \rho = 0.$$

There is a cusp on the diameter distant r from C.

If $\psi = \frac{\pi}{2}$, θ can be found.

For $p = \frac{r}{3}\sin\theta = r\sin\phi = r\sin\frac{\psi-\theta}{2}$

$$= r\sqrt{\frac{1-\cos(90-\theta)}{2}} = r\sqrt{\frac{1-\sin\theta}{2}},$$

whence $$\tfrac{1}{3}\sin\theta = \sqrt{\frac{1-\sin\theta}{2}},$$

or $$\sin^2\theta = \frac{9}{2} - \frac{9}{2}\sin\theta,$$

$$\therefore \sin\theta = -\frac{9}{4} \pm \sqrt{\frac{9}{2}+\frac{81}{16}} = -\frac{9}{4} \pm \frac{\sqrt{153}}{4} = \cdot 84233 \left(\text{or} - \frac{21\cdot 36932}{4}\right),$$

$$\therefore p = \frac{r}{3}\cdot 84233\ldots.$$

This value of p and ψ will form a guiding tangent LDL' to part of the caustic curve.

The caustic surface is traced out by the curve when it is revolved round the axis SC.

(3) Let $a = \frac{r}{2}$ (Fig. 45),

$$p = \frac{r}{2}\sin\theta = r\sin\phi,$$

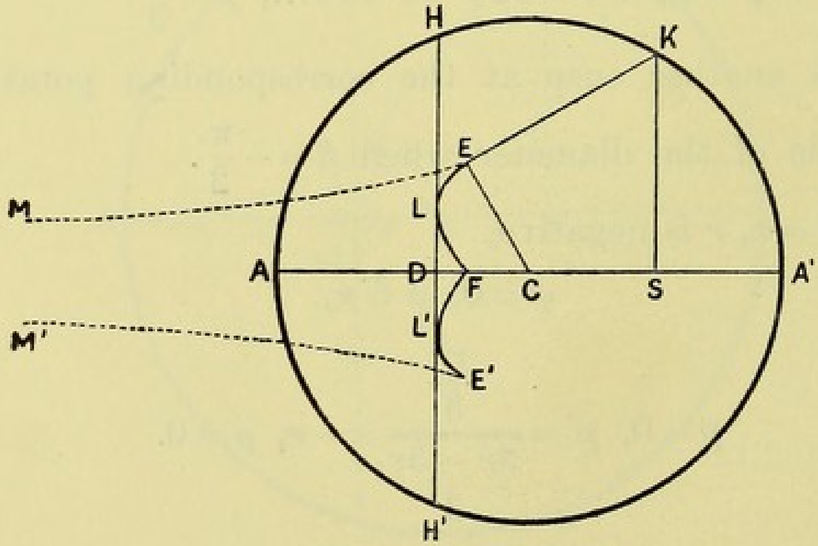

Fig. 45.

$$p' = \frac{r}{2(\sec\phi + \sec\theta)},$$

$$\rho = p\left\{1 - \frac{\frac{1}{4}\sec^3\phi + \sec^3\theta}{(\sec\phi + \sec\theta)^3}\right\}.$$

When $\qquad \theta = 0,$

$$p = 0,\ \phi = 0,\ p' = \frac{r}{4},\ \rho = 0.$$

There is a cusp at a point on the diameter distant $\frac{r}{4}$ from C.

When $\qquad \theta = \frac{\pi}{2},$

$$p = \frac{r}{2},$$

$$\phi = \sin^{-1}\frac{p}{r} = \sin^{-1}\tfrac{1}{2} = \frac{\pi}{6},$$

$$p' = \frac{r}{2\left(\sec\frac{\pi}{6} + \sec\frac{\pi}{2}\right)} = 0.$$

The expression for ρ involves again a fraction of the indeterminate form $\frac{\infty}{\infty}$.

As before,

$$\mathrm{L}^t_{\theta=\frac{\pi}{2}}\frac{\frac{1}{4}\sec^3\phi + \sec^3\theta}{(\sec\phi + \sec\theta)^3} = \mathrm{L}^t_{\theta=\frac{\pi}{2}}\frac{\frac{1}{4}\cos^3\theta + \cos^3\phi}{(\cos\theta + \cos\phi)^3} = \frac{\cos^3\phi}{\cos^3\phi} = 1,$$

$$\therefore\ \rho = p\{1 - 1\} = 0.$$

There is a cusp at the point determined by $\theta = \frac{\pi}{2},\ \phi = \frac{\pi}{6},$ $\psi = \frac{5\pi}{6},\ p = \frac{r}{2}.$

When $\qquad \theta = \pi,\ r$ is negative,

$$p = 0,$$

$$\phi = 0,\ \psi = \pi,$$

$$p' = \frac{r}{2(1-1)} = -\infty.$$

The expression for ρ involves the indeterminate form $0 \times \infty$ which on evaluation is seen to be equal to ∞.

The initial line is an asymptote to the caustic.

If $\psi = \frac{\pi}{2}$, θ can be found as before.

For $$p = \frac{r}{2}\sin\theta = r\sin\phi = r\sin\frac{\psi-\theta}{2}$$

$$= r\sqrt{\frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{2}} = r\sqrt{\frac{1-\sin\theta}{2}},$$

whence $$\tfrac{1}{2}\sin\theta = \sqrt{\frac{1-\sin\theta}{2}},$$

or $$\sin^2\theta = 2 - 2\sin\theta,$$

$$\therefore \sin\theta = -1 \pm \sqrt{2+1} = \cdot 73205\ldots \text{ (or } -2\cdot 732\ldots),$$

$$\therefore p = \frac{r}{2}\cdot 73205\ldots.$$

This value of p and ψ will form a guiding tangent LDL' to part of the caustic curve.

(4) Let $a > \frac{r}{2}$, $< r$. For example let $a = \frac{2r}{3}$ (Fig. 46),

$$p = \frac{2r}{3}\sin\theta = r\sin\phi,$$

$$p' = \frac{\frac{2r}{3}}{\frac{4\sec\phi}{3} + \sec\theta} = \frac{2r}{4\sec\phi + 3\sec\theta},$$

$$\rho = p\left\{1 - \frac{16\sec^3\phi + 27\sec^3\theta}{(4\sec\phi + 3\sec\theta)^3}\right\}.$$

When $$\theta = 0,$$

$$p = 0,\ \phi = 0,\ p' = \frac{2r}{7},\ \rho = 0.$$

There is a cusp at a point on the diameter situated $\frac{2r}{7}$ from C.

When $$\theta=\frac{\pi}{2},$$

$$p=\frac{2r}{3},$$

$$\phi=\sin^{-1}\tfrac{2}{3}=41^\circ\ 48'\cdot6185\ldots,$$

$$p'=0.$$

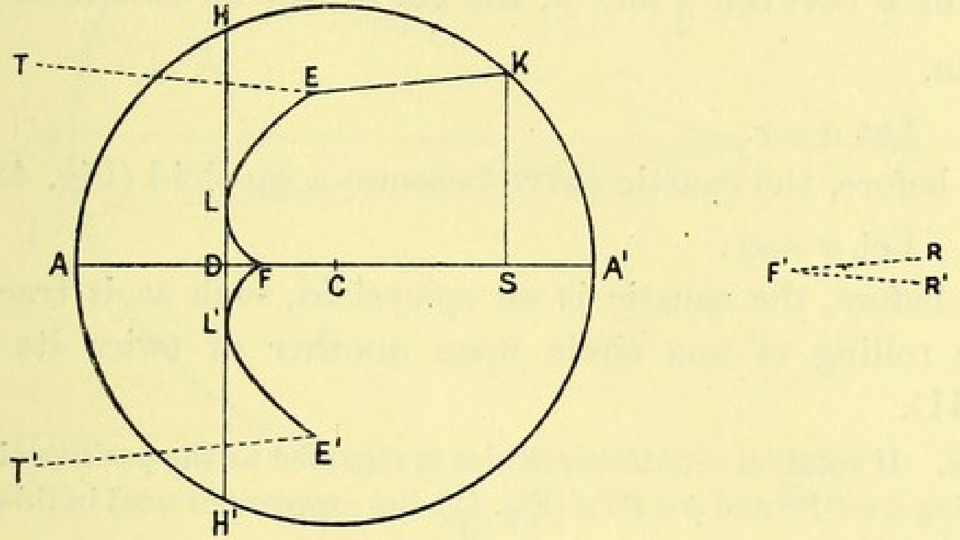

Fig. 46.

$$\mathrm{L^t}_{\theta=\frac{\pi}{2}}\frac{16\sec^3\phi+27\sec^3\theta}{(4\sec\phi+3\sec\theta)^3}=\mathrm{L^t}_{\theta=\frac{\pi}{2}}\frac{16\cos^3\theta+27\cos^3\phi}{(4\cos\theta+3\cos\phi)^3}$$

$$=\frac{27\cos^3\phi}{(3\cos\phi)^3}=1,$$

$$\therefore\ \rho=p\,(1-1)=0.$$

There is a cusp at a point determined by

$$\theta=90^\circ,\ \phi=41^\circ\ 48'\cdot6185,\ldots\ \psi=2\phi+\theta=173^\circ\ 37'\cdot237\ldots,\ p=\frac{2r}{3}.$$

If $$\psi=\frac{\pi}{2},$$

$$p=\frac{2r}{3}\sin\theta=r\sin\phi=r\sqrt{\frac{1-\sin\theta}{2}},$$

$$\sin^2\theta=\tfrac{9}{8}-\tfrac{9}{8}\sin\theta,$$

$$\therefore\ \sin\theta=-\tfrac{9}{16}\pm\sqrt{\tfrac{9}{8}+(\tfrac{9}{16})^2}=\cdot638085\ldots\ (\text{or }-1\cdot763085\ldots),$$

$$p=\frac{2r}{3}\cdot638085\ldots=\cdot42539\ldots r.$$

This value of p and ψ will form a guiding tangent LDL'.

When $\theta = \pi$, r is negative,

$$p = 0,\ \phi = 0,\ p' = 2r,\ \rho = 0.$$

There is a virtual cusp on the axis produced in the negative direction at a point situated $2r$ from c.

By finding other points on the figure corresponding to other values of θ between $\frac{\pi}{2}$ and π, the curve can be traced as in the diagram.

(5) Let $a = r$.

As before, the caustic curve becomes a cardioid (Fig. 42).

(6) Let $a = \infty$.

As before, the caustic is an epicycloid, such as is traced out by the rolling of one circle upon another of twice its radius (Fig. 41).

NOTE. If rotation counter-clockwise is regarded as the positive direction, on making $\phi = CPS$ and $\psi = PTA$ (Fig. 41), the expressions used in this section still hold good.

Aplanatic reflecting surfaces. It is an easy matter to discover the form of surface that shall reflect light originating from a given point S to a given focus F without aberration.

Real Focus. We have already seen from the "principle of same phase" (p. 102) that light originating from S must arrive at F by way of the mirror in the same phase, whatever point of the mirror reflects it. In other words, if S and F be the conjugate foci, the sum of the focal distances to every point of the mirror must be the same, so that the light from S may arrive at F in the same time whatever may be its point of reflection. As is well known an ellipse may be defined as a curve traced out by a point which moves in such a manner that the sum of its distances from two fixed points, called its foci, is always the same.

If then a concave mirror be of the form of an ellipsoid of revolution about its major axis SF, Fig. 47, it will reflect light from S, incident at any point of the mirror to the conjugate focus F without aberration.

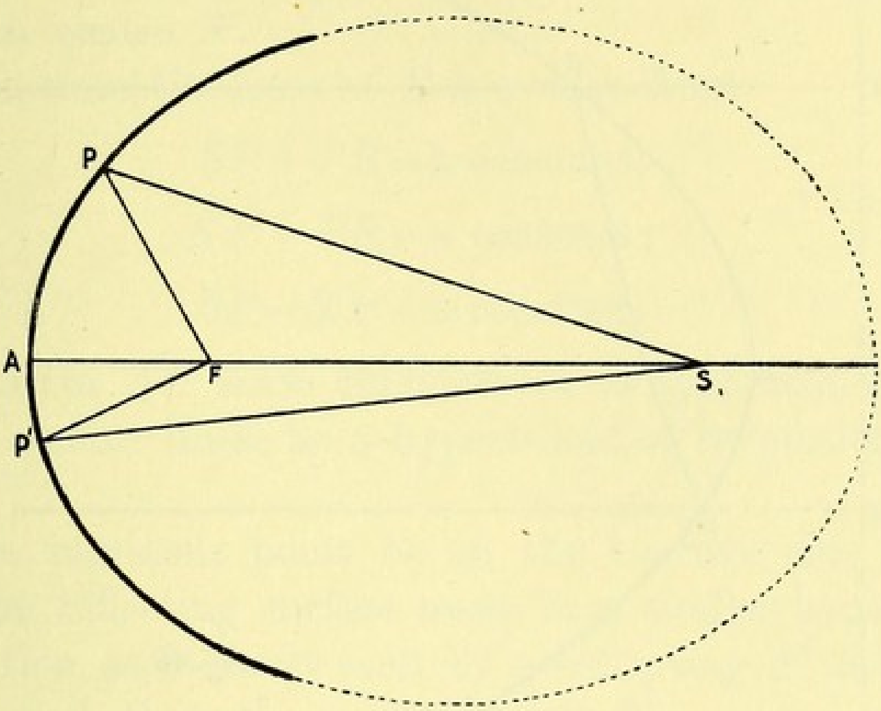

Fig. 47.

For $\quad SP + PF = SA + AF = SP' + P'F$ etc.

So the reflected light reaches F in the same phase whatever its point of incidence.

If the incident light present a plane wave-front, in other words, if the source of light be at an infinite distance, so that the incident rays are parallel, it will be readily seen that the surface of the mirror must be that of a paraboloid of revolution about its axis SA, Fig. 48.

Incident parallel rays of light will reach the point S in the same phase that they would reach the directrix HO before the interposition of the mirror for

$$PS = PH, \text{ and } P'S = P'H' \text{ etc.}$$

Therefore all the incident light is reflected to S without aberration.

Virtual Focus. If a mirror be required to form the virtual image of a point S at a given point S' without aberration, it will be found that the reflecting surface must present the form of a hyperboloid of revolution about SS'.

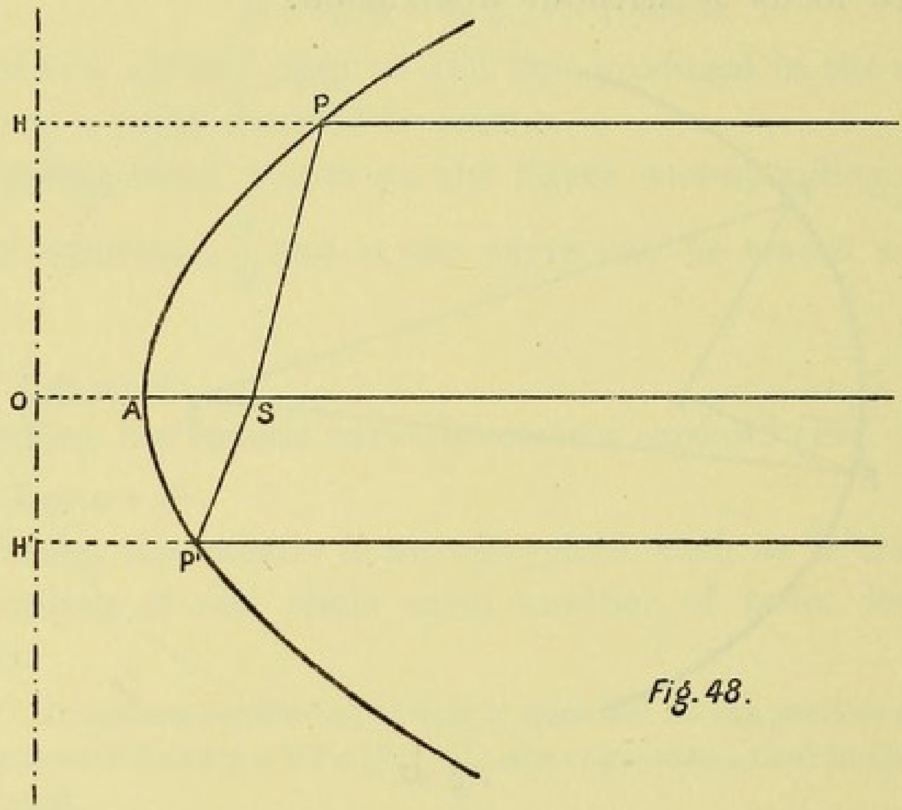

Fig. 48.

Let AP, Fig. 49, be the required reflecting surface, S and S' the given points. Let SP represent an incident ray, PR its corresponding reflected ray.

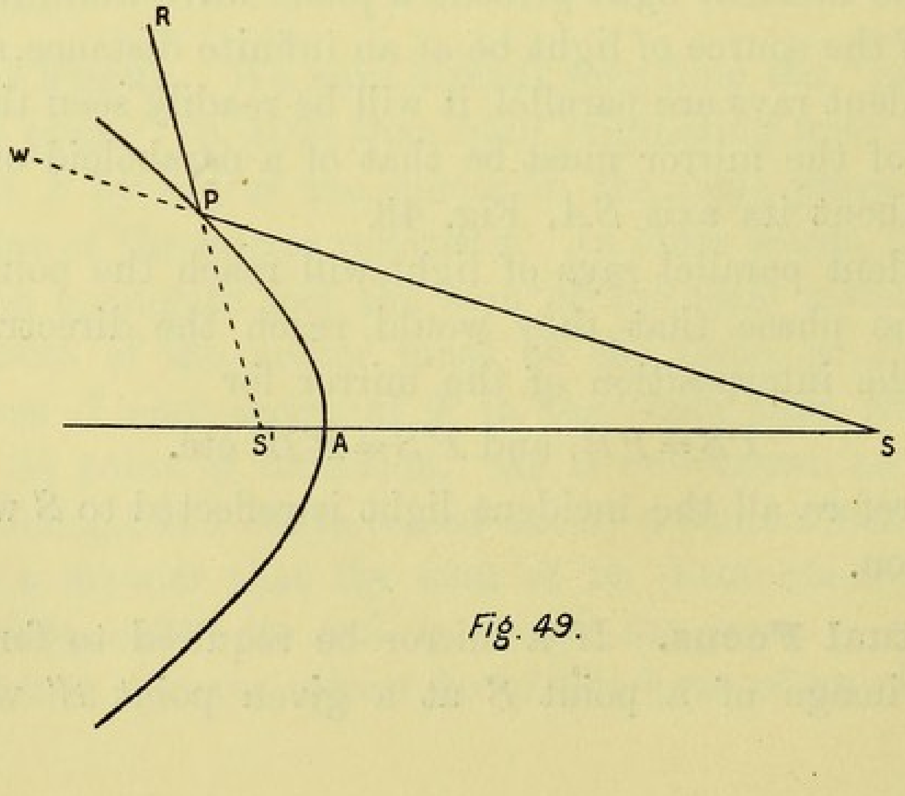

Fig. 49.

Then since the reflected light appears to come from S', it is clear that the locus of R must be a sphere whose centre is at S', for the reflected light must present a spherical wave-front with centre S'.

Again, since the locus of R is a wave-front

$$SP + PR = \text{a constant},$$

and

$$S'P + PR = \text{a constant};$$

$$\therefore\ SP - S'P = \text{a constant}.$$

The curve AP must therefore be a hyperbola, and the required surface must be a hyperboloid of revolution about SS'.

If the luminous point be on the concave side of the mirror, the reflecting surface must be a similar hyperboloid of revolution, as is easily seen by substituting S', S and W for S, S' and R in the preceding proof.

Neither elliptical nor hyperbolic mirrors are practically made, for in the first place there are very great mechanical difficulties in the way of grinding such surfaces, and secondly, even if made, each mirror would be only aplanatic for one specific distance of the object. An attempt is made by opticians to give a parabolic form to the concave mirrors which are used in reflecting astronomical telescopes. The wave-front of the incident light may be considered to be plane, and the function of the mirror is to impress a spherical form on the reflected waves.

This investigation will have served its purpose if it shews more clearly what aberration means, and that its occurrence is due to using a reflector of the wrong shape. A thin centric pencil is reflected by a spherical mirror without appreciable aberration, because the vertical portion of any conic section presents a curve that very nearly resembles the arc of a circle.

QUESTIONS.

(1) A pencil of parallel rays is incident at an angle of 60° on a small concave spherical mirror of diameter $1\frac{1}{2}$ ins., and of 6 ins. radius of curvature. Find the position of the focal lines, and the position and radius of the circle of least confusion.

(2) A pencil of parallel rays is incident at an angle of 45° on a small concave spherical mirror of diameter 1 in. Find the position of the focal lines and the position and radius of the circle of least confusion.

(3) In the above example suppose the incident pencil to be small so that the diameters of the effective reflecting surface is 1 in. in the primary plane, and $\frac{1}{\sqrt{2}}$ in. in the secondary plane. Find the position and the radius of the circle of least confusion in this case.

(4) A luminous point is situated on the principal axis of a concave spherical mirror at a distance of 12 feet from it. The diameter of the mirror subtends an angle of 120° at its centre of curvature, its radius of curvature is 1 foot. What is the longitudinal aberration? Use expression (1) p. 114.

(5) Parallel rays are directly incident upon a convex spherical mirror of which the angular aperture is 10° 20′, and the radius of curvature is -10 cm. Find approximately the longitudinal aberration, and the size and distance of the circle of least confusion from the principal focus. ($\sin 5° 10' = \cdot 09$.)

CHAPTER VII.

REFRACTION AT A PLANE SURFACE. PRISMS.

Refraction. So far we have been considering light travelling always in the same medium, and the alterations of its course ensuing from the interposition of certain reflecting surfaces in its route. We have now to investigate the alteration of course, or refraction, that it undergoes when it enters a different medium. We shall confine our attention to such media as are homogeneous and isotropic, and for the present we shall limit ourselves to the consideration of the behaviour of homogeneous light in such media. The term homogeneous light means light of the same kind, *i.e.* light of the same wave frequency or colour. We have already shewn how the velocity of light has been experimentally determined, and it will be remembered that light travels with different velocities in different media. Further, it has been shewn that its refraction on entering a new medium results simply from the alteration of the speed of its transmission.

The laws of the refraction of light in homogeneous and isotropic media given below were discovered experimentally by Willebrord Snell in 1621.

I. The refracted ray lies in the plane of incidence.

II. The sines of the angles of incidence and refraction are in a constant ratio for the same two media.

This constant ratio is really identical with the ratio of the velocity of light in the first medium to its velocity in the second; it is usually denoted by the symbol μ. We may therefore express the second law by the formula that we made use of in the Introductory chapter.

$$\frac{\sin i}{\sin r} = \frac{V_1}{V_2} = \mu \text{ a constant.}$$

Let us take as an example the refraction of light, or rather of a small element of a wave-front at a plane surface.

It is convenient to have some word by which to express the fact that light travels in a given medium with a relatively rapid or slow velocity. The terms *rare* and *dense* are universally used in Optics with this specific meaning; they do not however in books on Optics necessarily involve any other physical property. Adopting these terms let us consider a small element of a wave-front travelling in a given direction in a rare medium, and let us see how it changes its course on meeting a dense medium which is limited by a plane surface.

Let AB, Fig. 50, represent the bounding surface separating the two media, and let SI represent the incident ray, *i.e.* the direction in which the element is moving in the rare medium, then IR represents the refracted ray in the dense medium

$$\text{if } \frac{\sin NIS}{\sin NIR} \text{ or } \frac{\sin NIS}{\sin(\pi + N'IR)} \text{ or } \frac{\sin NIS}{-\sin N'IR} \text{ or } \frac{SN}{N'R} = \mu,$$

where μ denotes the constant ratio or the relative index of refraction for these two media. It will be noticed that as μ is the velocity ratio $\frac{V_1}{V_2}$, it must have a positive value, and SN and $N'R$ are both measured in the same direction. The sine of the angle of refraction is therefore sin NIR not sin $N'IR$. In Fig. 50 the angle NIR is measured clockwise.

If the first medium be air, and the second medium be water, the value of μ has been found by observation to be about $\frac{4}{3}$, so that when these two media are being considered, whatever the angle of incidence may be, the ratio $\frac{\sin i}{\sin r}$ is equal to $\frac{4}{3}$, and this fraction expresses the ratio of the speed of light in air to its speed in water.

Since the speed of light is less in dense media it is evident that the refracted rays are bent towards the normal when light passes from rare to dense media and vice versâ.

Let us take the case of light travelling from a dense medium to a rare medium. We may now regard RI as being the incident ray, it will become in the rare medium the refracted ray IS

$$\text{for } \frac{\sin N'IR}{\sin N'IS} \text{ or } \frac{\sin N'IR}{-\sin NIS} = \frac{RN'}{NS} = \frac{V_2}{V_1} = \mu' \text{ where } \mu' = \frac{1}{\mu}.$$

It is clear then that the course of light is reversible, that is to say, if light travels backwards in the direction of a refracted ray it will on emerging travel in the direction of the incident ray. This is obvious from the consideration that the alteration of its course depends entirely on the alteration in its speed of transmission in the new medium, and this speed depends upon the nature of the medium.

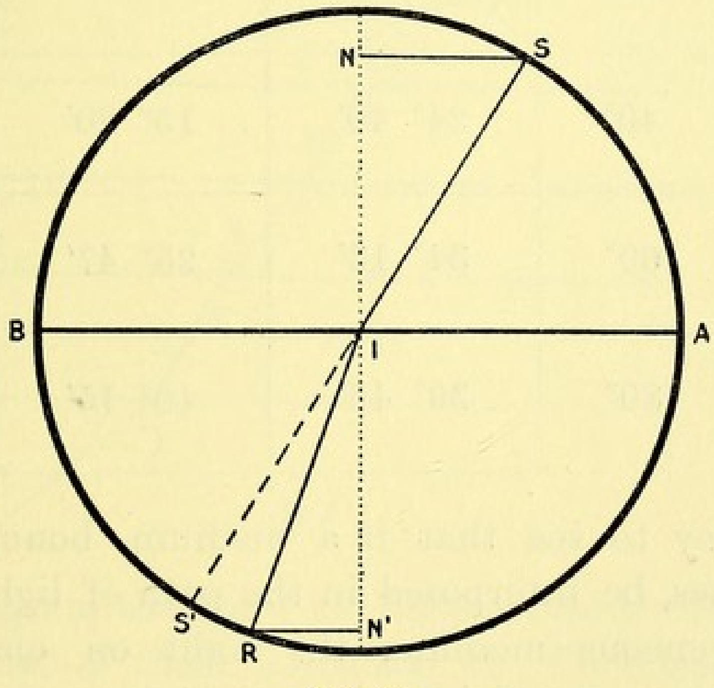

Fig. 50.

In Fig. 50 since SIS' represents the original path of the light and IR represents its path after refraction, the angle $S'IR$ is the deviation from its original course. Now since $\sin NIR = \sin(-N'IR) = \sin RIN'$, and $\sin NIS = \sin S'IN'$ the angles of refraction and incidence are very commonly regarded as $S'IN'$ and RIN'. The deviation $S'IR$ is consequently $i-r$. The angles of refraction that correspond to the given angles of incidence at the surface of glass ($\mu = 1{\cdot}54$) are arranged below in parallel columns. The third column gives the corresponding deviation.

It will be noticed that when the angle of incidence increases uniformly, the angle of refraction increases slower and slower, and consequently the deviation $(i-r)$ increases faster and faster. This is a law which holds universally for all media; a general proof will be found in a subsequent section (p. 201).

I	R	D
20°	12° 50′	7° 10′
40°	24° 40′	15° 20′
60°	34° 13′	25° 47′
80°	39° 45′	40° 15′

It is easy to see that if a medium, bounded by two parallel planes, be interposed in the path of light travelling in a homogeneous medium, the light on emerging will pursue a course parallel to its original course. We may extend the proposition to any number of media bounded by parallel planes, for if V_1, V_2, V_3,... represent the speed of light in the first, second, third,... medium respectively, and if ϕ_1, ϕ_2, ϕ_3,... denote the angles of incidence in these media,

$$\frac{\sin\phi_1}{\sin\phi_2}=\frac{V_1}{V_2},\quad \frac{\sin\phi_2}{\sin\phi_3}=\frac{V_2}{V_3} \text{ etc.,}$$

for since the media are bounded by parallel planes the angle of refraction at one surface is equal to the angle of incidence at the other surface. Thus if there are 4 media interposed, Fig. 51, and the angle of emergence into the original medium be denoted by θ

$$\sin\theta=\frac{V_1}{V_5}.\sin\phi_5, \text{ and } \sin\phi_5=\frac{V_5}{V_4}\sin\phi_4 \text{ etc.}$$

$$\therefore \sin\theta=\frac{V_1}{V_5}\frac{V_5}{V_4}\cdot\frac{V_4}{V_3}\cdot\frac{V_3}{V_2}\cdot\frac{V_2}{V_1}.\sin\phi_1$$

$\therefore \sin\theta=\sin\phi_1$ or in other words the final and initial rays are parallel to one another.

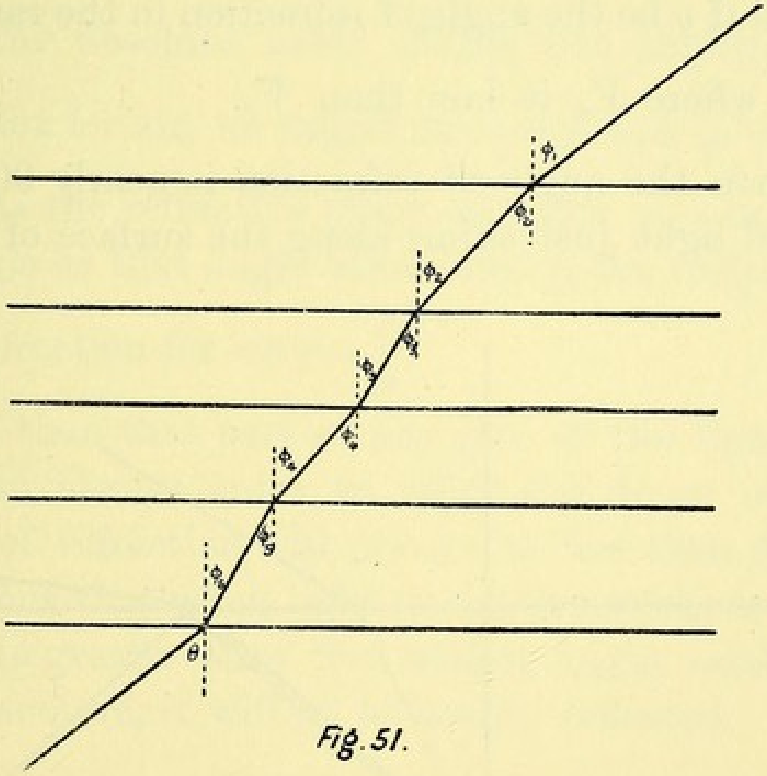

Fig. 51.

In the diagram the third medium is represented as more rare than either of those adjoining it. Consequently ϕ_4 is greater than ϕ_3 and ϕ_5.

Since the index of refraction is identical with the velocity ratio, we can easily find the relative index of refraction between any two media, if their absolute index is known. For example, the index of a certain kind of glass is $1\frac{1}{2}$, that of water is $1\frac{1}{3}$; what is the relative index of refraction from water to glass? If we denote by V_0, V_g, V_w the velocity of light in vacuo, in glass, and in water respectively,

$$\frac{V_0}{V_g} = \frac{3}{2} \text{ and } \frac{V_0}{V_w} = \frac{4}{3}.$$

The relative index of refraction from water to glass is

$$\frac{V_w}{V_g} \text{ and } \frac{V_w}{V_g} = \frac{V_0}{V_g} \cdot \frac{V_w}{V_0} \text{ or } \frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}.$$

Consideration of the second law of refraction reveals the curious fact that light in dense media incident at certain angles on their surfaces, is unable to get out.

For if x represent the angle of incidence in the dense medium and if y be the angle of refraction in the rare medium $\frac{\sin x}{\sin y} = \frac{V_d}{V_r}$ where V_d is less than V_r.

Now when the angle of refraction is nearly 90°, Fig. 52, the refracted light just skims along the surface of the dense

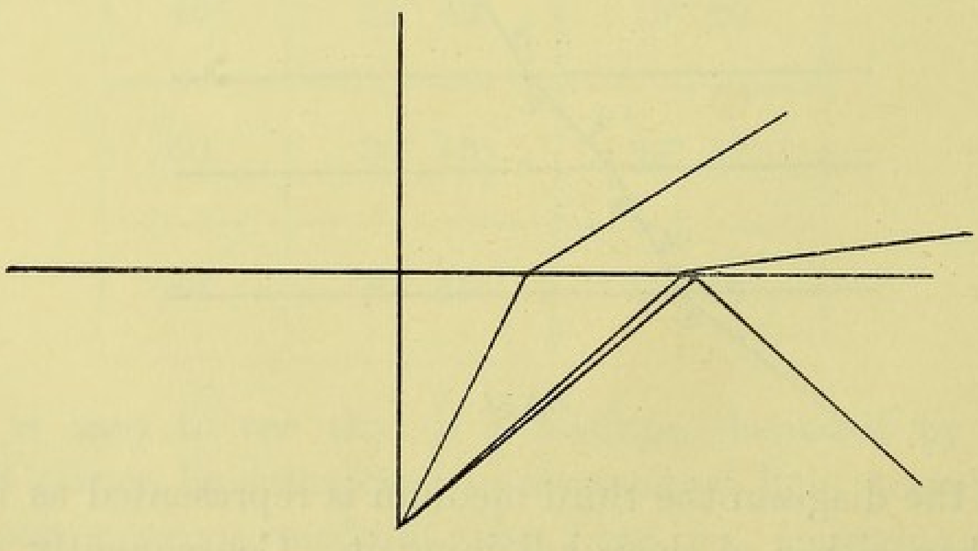

Fig. 52.

medium. If the angle of refraction be 90° or more than 90° the light will remain below the boundary that separates the two media, in fact it is unable to leave its original medium directly by transmission. Experiment shews that it then undergoes almost total internal reflection at the bounding surface.

The *critical angle* at which this phenomenon of total reflection occurs can be easily found. We have merely to give y its maximum value of 90° and the formula given above becomes $\frac{\sin x}{\sin 90} = \frac{V_d}{V_r}$, *i.e.* $\sin x = \frac{V_d}{V_r}$.

For example, the critical angle for water and air is about 48° 34′,

for $\frac{\sin x}{\sin 90} = \frac{V_d}{V_r} = \frac{3}{4}$ or more accurately $\frac{1}{1 \cdot 3336}$,

$$\therefore x = \sin^{-1} \frac{1}{1 \cdot 3336} \text{ or } 48° 34'.$$

Since the absolute index differs but slightly from the relative index for air, we might have replaced $\frac{V_d}{V_r}$ by $\frac{1}{\mu}$, where μ represents the refractive index of water, and expressed the critical angle as that angle whose sine is the reciprocal of the index of refraction for $\sin x = \frac{1}{\mu}$.

We see then that part at any rate of the light in a rare medium can always enter an adjoining dense medium, for the angle of refraction will always be less than the angle of incidence, but that when light in a dense medium is incident at any angle greater than the critical angle, none of it will leave the medium, it will all be totally reflected.

Images by Refraction at a Plane surface. The position of the image of a point in one refracting medium bounded by plane surfaces seen by an eye in another refracting medium that is in the normal to the boundary-plane.

Let P, Fig. 53, be a source of light in a dense medium which is separated from a rare medium by a plane surface

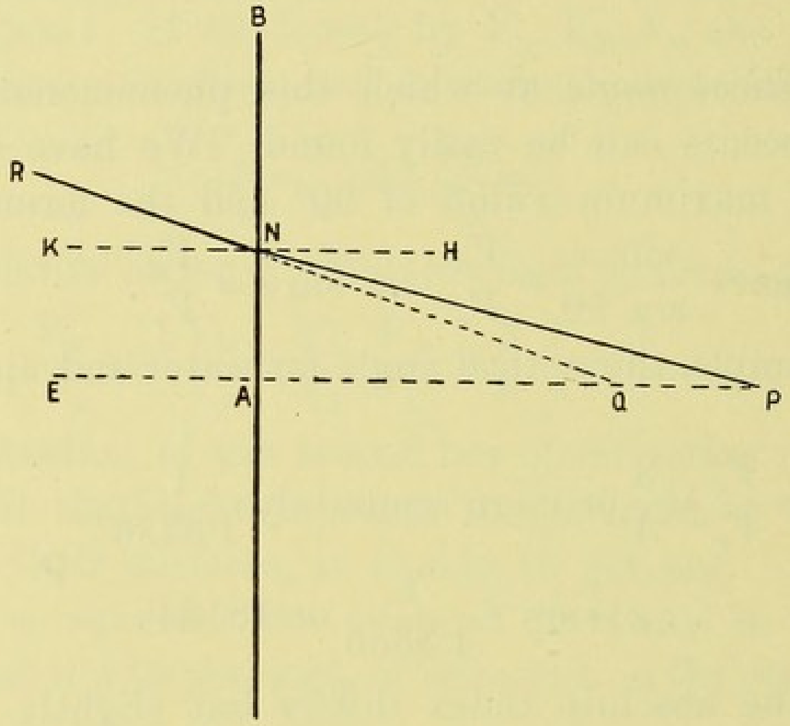

Fig. 53.

AB, part of the light which is diverging from P will take the path indicated by the ray PN; this will be refracted in such a way that its new course will be given by NR, where $\frac{\sin HNP}{\sin HNR} = \frac{V_d}{V_r}$. An observer then at R will receive light coming in the direction NR; we have now to determine from which point in this line the light will appear to him to have started, or in other words we have to determine the situation of the image of P under these circumstances. For this purpose we must find the point of intersection of the several rays that are directed towards the pupil of the observer's eye. We will take the simplest case, by considering that the observer's eye is at some point E in the line PA normal to the boundary AB. Consequently we shall only have to consider the narrow pencil of rays about the axis EAP, that enters the pupil of his eye.

Now NR and PAE are both in the same plane (Law I.) and the angles RNA and NAE are together greater than two right angles, consequently RN if produced backwards will meet the normal PA.

Let Q be this point of intersection.

Then $\angle APN = \angle HNP$ and $\angle AQN = \angle HNQ$,

$$\therefore \frac{\sin APN}{\sin AQN} \text{ or } \frac{\sin HNP}{\sin HNQ} \text{ or } \frac{\sin HNP}{-\sin KNR} = \frac{\sin HNP}{\sin HNR} = \frac{V_d}{V_r}.$$

Now if N be very close to A, the lines PN, QN will nearly coincide with PA, QA, and so the values of PA, QA may under these conditions be substituted for those of PN, QN. At the same time Q will assume a limiting position which we will proceed to determine,

$$\frac{\sin APN}{\sin AQN} = \frac{\frac{NA}{PN}}{\frac{NA}{QN}} = \frac{QN}{PN} = \frac{QA}{PA} \text{ ultimately,}$$

so that under the conditions named $\frac{QA}{PA} = \frac{V_d}{V_r}$.

If then we consider the very small pencil of rays that would reach an observer's eye at E on the normal from P to the bounding surface AB, they would diverge in the rare medium as if they came from Q, the distance QA being $\frac{V_d}{V_a} PA$. If for example a small object is placed at the bottom of a tumbler-full of water and if it is viewed from a point immediately above it, its apparent depth below the surface of the water will be $\frac{3}{4}$ its real depth, for the ratio $\frac{V_d}{V_r}$ with respect to water and air is $\frac{3}{4}$.

It appears then that a point P and its image Q lie on the normal drawn from the point to the bounding surface, provided that the eye of the observer be also on the normal, and they both lie on the same side of this surface. If P moves, Q moves also in the same direction and proportionally to the movement of P.

If the object be in the rare medium its virtual image, to an eye in the dense medium, will be further off than the object really is, for under these conditions the refracted ray is bent towards the normal.

It is important to remember that as in the case of reflection at spherical mirrors, this determination of the situation of the virtual image is only true, when very small pencils are considered; in this case indeed there is the further limitation that the incidence must be nearly normal to the surface.

The consideration of the refraction of oblique pencils must be deferred to a later section. It will be sufficient here to point out that an oblique pencil undergoes considerably greater refraction, so that if an object of appreciable size is placed so that its base is at P, its virtual image to an eye at E will be distorted and tilted, for its upper edge will only be seen by light the rays of which are oblique to the surface.

Refraction through a Plate. If an object be viewed through a medium bounded by parallel planes, it will appear distorted, and its position will apparently be altered, the extent of the displacement and distortion will depend on the thickness and density of the medium, as well as on the obliquity of the pencil by which the object is seen.

We will suppose that the observer is so situated that the narrow pencil of rays directed towards his eye from the luminous point P is nearly normal to the surfaces of the thick plate $ANMB$ that represents the dense medium.

The extreme ray PN on entering the glass will become the refracted ray NM which if prolonged backwards will intersect AP produced in Q' in such a way that

$$Q'A = \frac{V_r}{V_d} PA \text{ or } \mu PA.$$

Since the diverging pencil in the dense medium proceeds as if from Q' it will on emerging proceed as if from Q a point in PAB, such that $QB = \frac{V_d}{V_r} Q'B$ or $\frac{1}{\mu} Q'B$. It will be noticed that MR is parallel to PN.

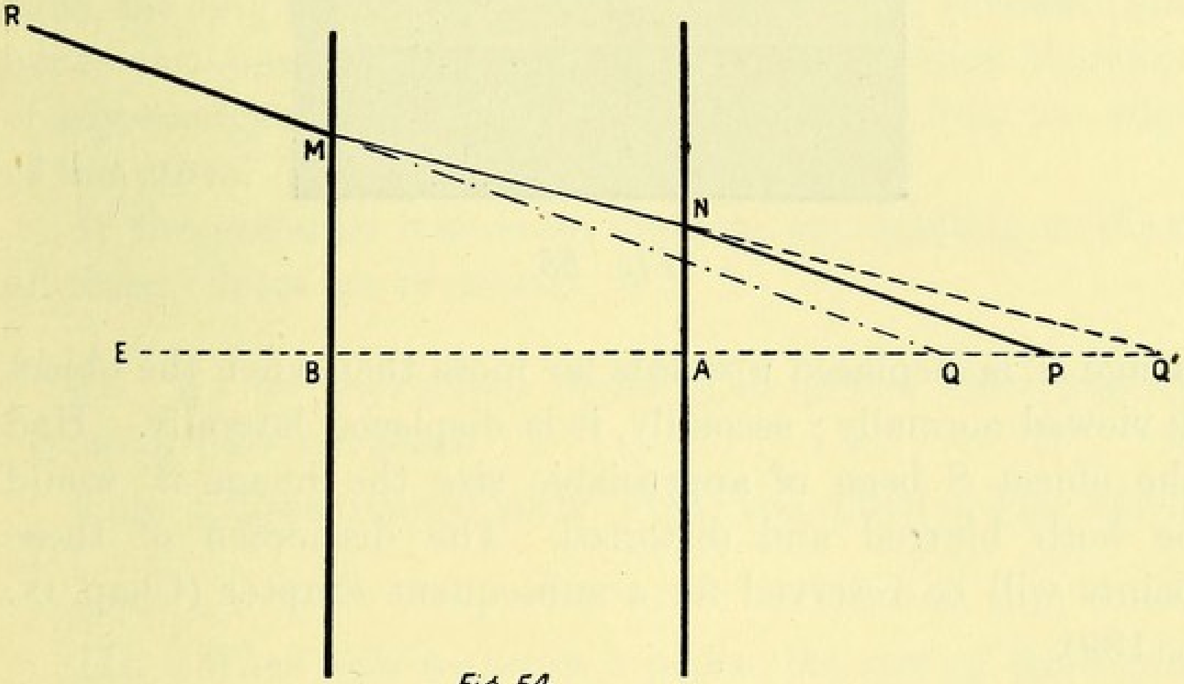

Fig. 54.

To determine the distance QA we have

$$QA = QB - AB \text{ or } \frac{1}{\mu} . Q'B - AB,$$

$$\therefore\ QA = \frac{1}{\mu}(Q'A + AB) - AB,$$

$$\therefore\ QA = \frac{1}{\mu} . \mu PA - AB\left(1 - \frac{1}{\mu}\right),$$

$$\therefore\ QA = PA - \frac{\mu - 1}{\mu} AB.$$

Thus a small object seen directly through a glass plate $1\frac{1}{2}$ ins. thick will appear to be half an inch nearer the observer than it really is if $\mu = 1{\cdot}5$. The displacement being equal to one-third of the thickness of the glass plate.

Fig. 55 represents a small object S seen obliquely through a glass plate. Two points will be noticed, firstly, the virtual

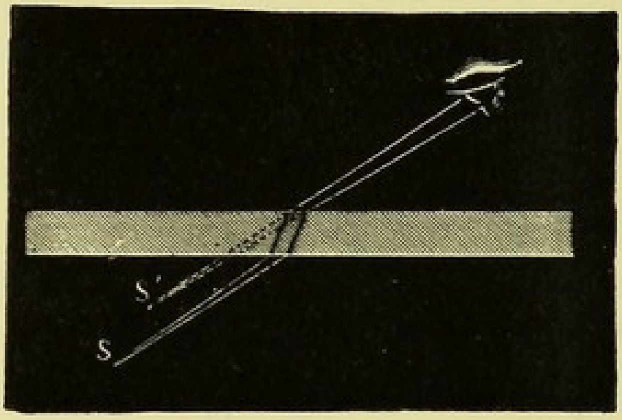

Fig. 55.

image S' is displaced upwards far more than when the object is viewed normally; secondly, it is displaced laterally. Had the object S been of appreciable size the image S' would be both blurred and distorted. The discussion of these points will be reserved for a subsequent chapter (Chap. IX. p. 189).

Prisms. Any refracting medium bounded by two plane surfaces which are inclined at an angle to one another is called a *prism*. The inclination of the faces to one another is called the refracting angle or apical angle of the prism. We proceed to demonstrate certain properties which are common to all prisms.

I. *When light passes through a prism, which is denser than the surrounding medium, it always undergoes a deviation in the direction opposed to the edge of the prism.*

When light traverses a medium bounded by parallel planes, each emergent ray is parallel to its corresponding incident ray (Fig. 54).

If the face *BM* be rotated through a small angle clockwise about *B* the plate will become a prism with its edge upwards, and the angle of incidence at this face will diminish until passing through the value 0 it becomes negative. Any ray therefore such as *NMR* will on emergence be deviated away from the edge.

If the rotation of the face *BM* be counter-clockwise, the prism formed will have its edge downwards. At the same time the angle of incidence at this face will increase and hence also the angle of refraction. The deviation therefore of any emergent ray will be upwards, *i.e.* away from the edge of the prism.

If the prism be less dense than the surrounding medium all these effects are reversed.

II. *As the refracting angle of a prism increases, the deviation also increases.*

This follows immediately from the proof given above of I.

III. *When light traverses a prism, the sum of the angles which any ray makes with the adjacent normals within the prism is a constant of the prism considered, and is equal to the refracting angle.*

Let *SIRT* denote a ray of light passing through the prism *ABC*, Fig. 56. Draw the normals at *I* and *R*. Let the angles of incidence and refraction at *I* be denoted by ϕ and ϕ', and let the angles of incidence and emergence at *R* be denoted by ψ' and ψ. The angles ϕ' and ψ' within the prism are to be reckoned positive when the ray which forms them lies above the intraprismatic portions of the normals.

Let the refracting angle, BAC, of the prism be denoted by A[1]. The angles ϕ' and ψ' are measured counter-clockwise.

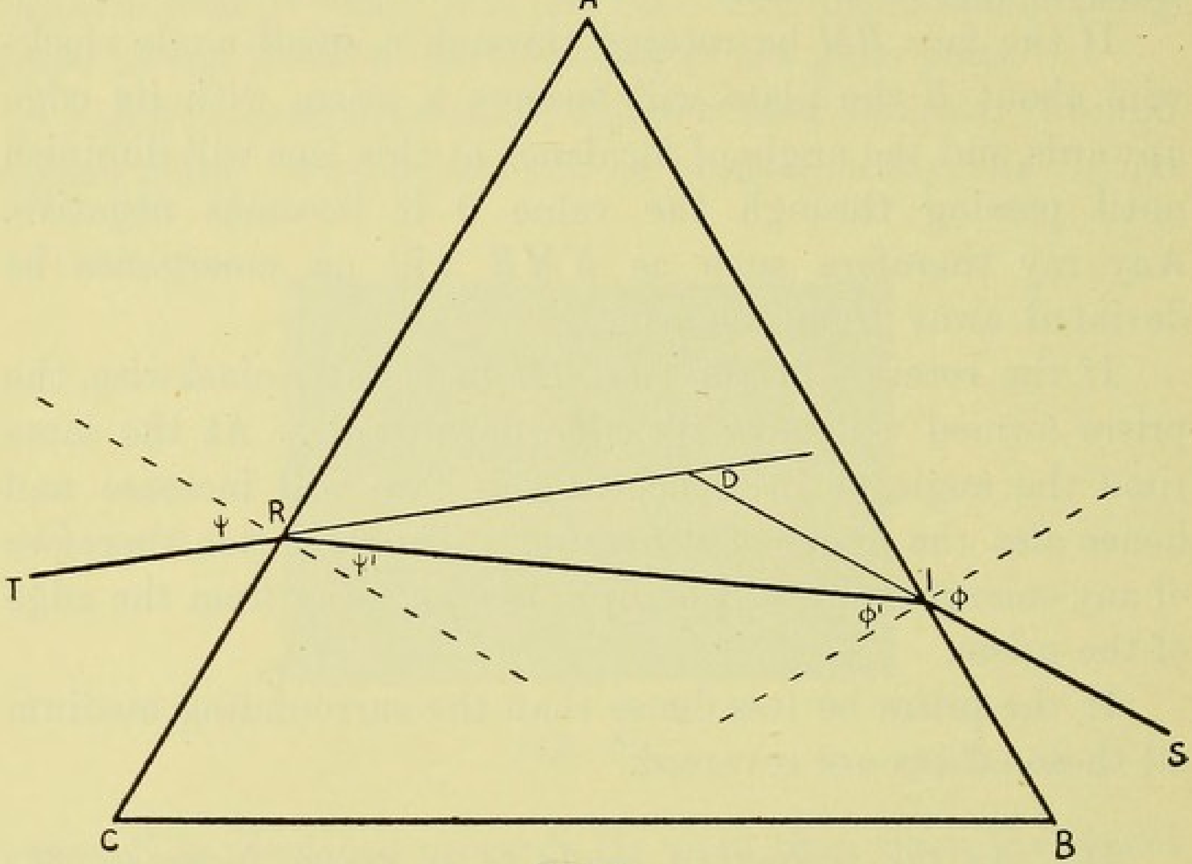

Fig. 56

Then in the triangle ARI the angle at I is equal to $90 - \phi'$ and the angle at R is equal to $90 - \psi'$.

$$\therefore A + 90 - \phi' + 90 - \psi' = 180,$$

$$\therefore A = \phi' + \psi'.$$

This result is universally true, for if the path of the ray within the prism is such that one of the angles of the triangle at I or at R is obtuse, the value of ϕ' or ψ' must be negative.

[1] If A is greater than twice the critical angle of the medium all light incident on one surface AB will reach the other face AC at an angle greater than the critical angle. It will consequently be totally reflected at this surface. A glass prism for example having a refracting angle of 82° will not transmit light. Any light which enters the prism can only get out after undergoing one or more internal reflections.

IV. *The total deviation is equal to the sum of the angles of incidence and emergence less the refracting angle of the prism.*

If the rays TR, SI be produced so as to intersect at D and we consider the triangle DRI its exterior angle D is equal to its two interior and opposite angles $\phi - \phi'$, and $\psi - \psi'$. And the angle D is the total deviation of the ray from its initial direction due to the prism

$$D = \phi - \phi' + \psi - \psi',$$

$$\therefore D = \phi + \psi - A.$$

The relation between ϕ and ϕ' is given by the equation

$$\sin\phi = \frac{V_r}{V_d}\sin\phi' \text{ or } \sin\phi = \mu\sin\phi';$$

while that between ψ' and ψ is $\sin\psi' = \frac{V_d}{V_r}\sin\psi$,

$$i.e. \sin\psi' = \frac{\sin\psi}{\mu} \text{ or } \sin\psi = \mu\sin\psi'.$$

V. *When a ray passes symmetrically through a prism the deviation is a minimum.*

A ray passes symmetrically through a prism when the angle of incidence is equal to the angle of emergence, *i.e.* when $\phi = \psi$.

If ϕ increases ϕ' increases also, at the same time ψ' diminishes and consequently ψ also.

But the deviation $\phi - \phi'$ increases faster than the deviation $\psi - \psi'$ diminishes (p. 142). Consequently the total deviation increases. If we consider the path of light reversed, it appears that when the angle of incidence is diminished the total deviation increases.

Hence this symmetrical position is the position of minimum deviation.

When the refracting angle of a prism is very small, and when the prism is placed in the position of minimum deviation, ϕ' and ϕ are both very small, hence $\sin\phi$ and $\sin\phi'$ may be replaced by ϕ and ϕ', and ϕ may be replaced by $\mu\phi'$.

Then D being minimum deviation, $\phi=\psi$ and $\phi'=\psi'$ $D=2\phi-A$ and $A=2\phi'$

$$\therefore D=2\mu\phi'-A=(\mu-1)A.$$

This is a most useful formula but it is frequently misapplied in ophthalmic literature from neglect of the two limiting conditions. We find that the deviation produced by a weak prism of glass is about half the apical angle. This is approximately true even if the prism is not exactly in the position of minimum deviation. For oblique pencils however the formula does not hold good. Thus if $A=10°$ and $\phi=40°$, $D=7°\,3'$, the refractive index of glass being 1·54; whereas the formula $\overline{\mu-1}\,A$ would give $5°\,24'$ as the value of D, involving an error of 30 per cent. If $\phi=20°$ the error introduced by the formula is about $10'$ or about 3 per cent. These examples will assist the reader in distinguishing the cases in which the use of the formula is legitimate from those in which its application is unjustifiable.

The relative index of refraction of a substance with respect to air may be determined in the following way. Take a small prism of the substance and measure its refracting angle A. Place it in the position of minimum deviation with respect to a beam of a certain kind of light, the rays of which are parallel. Each ray then undergoes the same deviation D. Measure this deviation.

Then because the deviation is a minimum

$$\phi=\psi \text{ and } \phi'=\psi',$$

$$\therefore D=2\phi-A \text{ and } A=2\phi',$$

$$\therefore \mu \text{ or } \frac{\sin\phi}{\sin\phi'}=\frac{\sin\frac{1}{2}(A+D)}{\sin\frac{1}{2}(A)}.$$

When the substance is a liquid its refractive index may be determined in precisely the same way if it be put into a hollow prism the faces of which are plane plates of glass.

The instrument by which these observations are made is called a spectrometer. It consists essentially of a horizontal graduated circle, Fig. 57, on which a collimator and a small telescope with cross wires are so mounted that they are

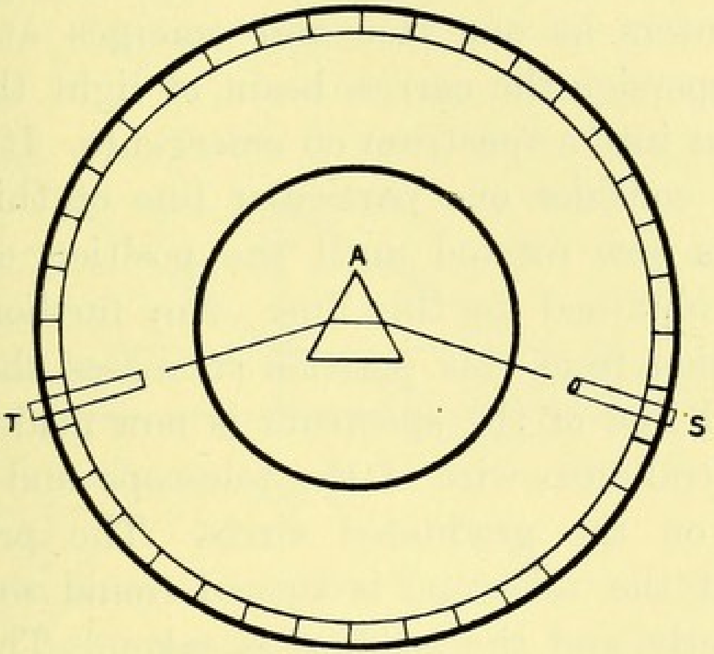

Fig. 57.

directed towards the centre of the circle. The collimator is an apparatus for obtaining a plane wave (of parallel rays); it is a tube with an achromatic lens at its proximal end and a narrow slit at its distal end at the focus of the lens. The source of illumination is placed close to the slit, the light which emerges from the collimator then forms a narrow beam of parallel rays. At the centre of the instrument there is a small support or table which can be rotated about a vertical axis; on this the prism is placed.

The first step is to measure the refracting angle (A) of the prism. The edge of the prism is directed towards the collimator. The telescope is then moved round the graduated circle until the reflected image of the slit from one face falls on its central cross wire. The angle on the circle is noted. The telescope is now turned into such a position that the reflected image from the other face of the prism occupies the centre of its field. Another reading is taken. The difference between the two readings is equal to twice the angle of the prism (p. 71). The angle of minimum deviation is now determined. The prism is so placed that the light from the collimator enters at one face and emerges at the other. Owing to dispersion the narrow beam of light that enters is broadened out into a spectrum on emergence. It is necessary therefore to consider one particular line of this spectrum. The prism is now rotated until the position of minimum deviation is obtained for this line. Any further turning in either direction from this position increases the deviation. The assigned line of the spectrum is now made to coincide with the central cross wire of the telescope and its position is read off on the graduated circle. The prism is now removed and the telescope is turned round so as to view the slit directly, and the reading is taken. The difference between the two readings gives the minimum deviation.

The method given above is unsuitable when the substance to be examined can only be procured in small quantities. In such cases the critical angle of the substance is determined by experiment and the value of μ is found from the formula $\sin x = \frac{1}{\mu}$. By this method the refractive index of a liquid may be determined even from a single drop. The drop is placed in the centre of the lower face of a right-angled prism immediately over a small marked hollow on a table. Part of the light entering the hypothenuse-face of the prism is reflected at that portion of the base to which the drop is attached and emerges at the other face to enter the eye of the observer, who is furnished with a telescope on a graduated scale. As he slowly lowers the telescope from that position in which he can see the mark through the drop he will find the spot at which the mark just disappears. The critical angle of the medium can then be determined by noting the angle (θ) of inclination of the telescope to the vertical.

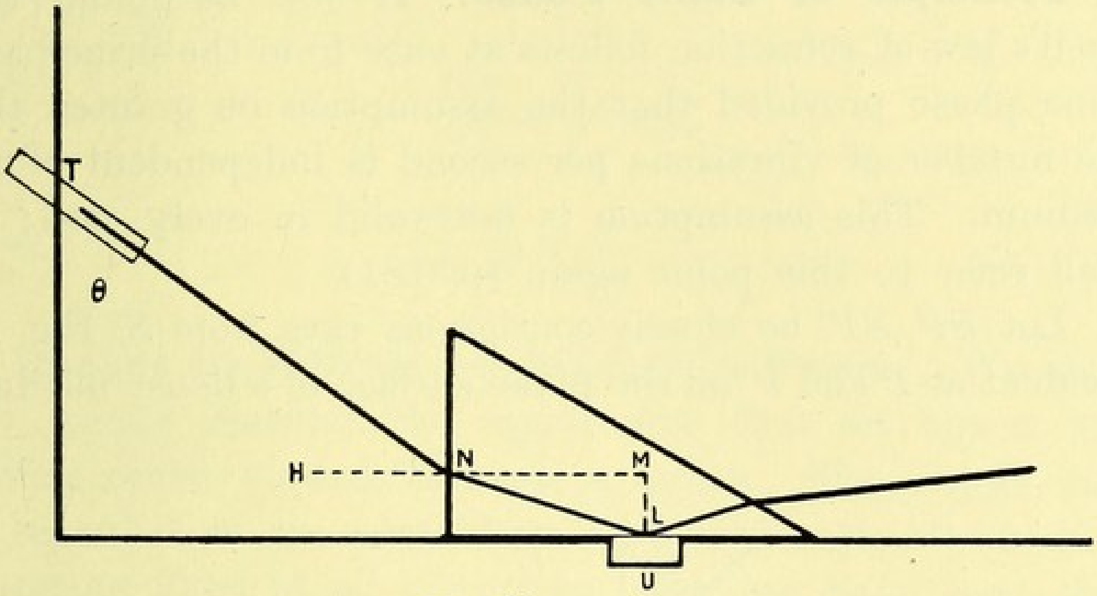

Fig. 58.

Since no light from the hollow U is received by the telescope the incident light must have been so refracted on entering the glass as to be totally reflected at the surface of the drop. The angle MLN is the critical angle α,

$$\angle\ MNL = 90 - \alpha, \text{ and } \angle\ TNH = 90 - \theta.$$

Then if the refractive index of the glass be denoted by μ or $\frac{V_a}{V_g}$,

$$\mu = \frac{\sin TNH}{\sin MNL} = \frac{\cos\theta}{\cos\alpha};$$

$$\therefore \cos\alpha = \frac{\cos\theta}{\mu}, \text{ and } \sin\alpha = \frac{\sqrt{\mu^2 - \cos^2\theta}}{\mu}.$$

And if the relative refractive index of the liquid and the glass be $\frac{V_x}{V_g}$,

$$\sin\alpha = \frac{V_g}{V_x}.$$

But μ' the refractive index of the liquid is equal to $\frac{V_a}{V_g} \cdot \frac{V_g}{V_x}$;

$$\therefore \mu' = \mu \sin \alpha \text{ or } \sqrt{\mu^2 - \cos^2 \theta}.$$

Principle of Same Phase. It will be found that Snell's law of refraction follows at once from the principle of same phase provided that the assumption be granted that the number of vibrations per second is independent of the medium. This assumption is not valid in every case; we shall refer to this point again (p. 181).

Let SP, SP' be closely contiguous rays from S, Fig. 59, incident at P and P' on the plane surface of a dense medium.

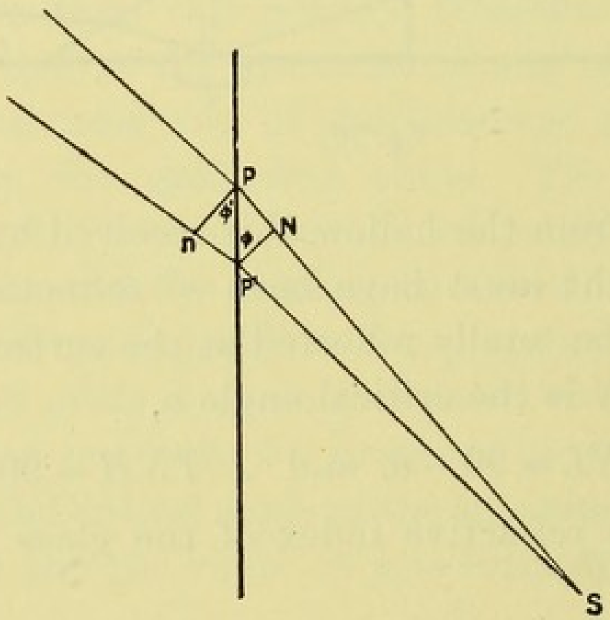

Fig. 59.

Let fall the perpendiculars $P'N$, Pn upon the incident and refracted rays respectively. Then since SP and SP' are contiguous we may regard SP' as equal to SN.

Therefore at P' and N there is the same phase at the same time.

The principle of same phase involves the condition of the points n and P in the refracted rays being in the same phase.

Therefore the time of traversing the distance NP $\left(\text{or } \frac{NP}{V_1}\right)$ is equal to the time of traversing the distance $P'n$ $\left(\text{or } \frac{P'n}{V_2}\right)$;

$$\therefore \frac{NP}{P'n} = \frac{V_1}{V_2} = \mu,$$

or
$$\frac{\sin\phi}{\sin\phi'} = \mu.$$

Images by Refraction through a Prism. We must now briefly consider the appearance that an object will present when viewed through a prism. The object may be regarded as an assemblage of points each of which is scattering light in all directions. But we have seen that the deviation produced by a prism is not proportional to the incidence. A difficulty therefore arises which we had not to face when dealing with reflection at a plane surface. For from each point of the object light will be incident on the prism at various angles and will consequently undergo various deviations. The corresponding emergent rays from the prism will form an astigmatic pencil, *i.e.* they will not intersect in a point if produced backwards. Now in each prism there is one angle of incidence which gives rise to the minimum deviation; whether the obliquity of the incident light is increased or diminished, the result is the same, the deviation is increased.

From what we have said above (p. 102) it follows that if we consider a very small cone of light of which the axial ray undergoes the minimum deviation, we shall not introduce any appreciable error by saying that each ray of the cone undergoes the same (minimum) deviation. The smaller the cone, *i.e.* the smaller its apical angle, the more nearly true will this approximation be.

If then we confine our attention to such a small pencil or cone of light we are justified in asserting that all its constituent rays after refraction will intersect in one point if produced backwards. In this way a point or focus corresponding to each point of the object may be found.

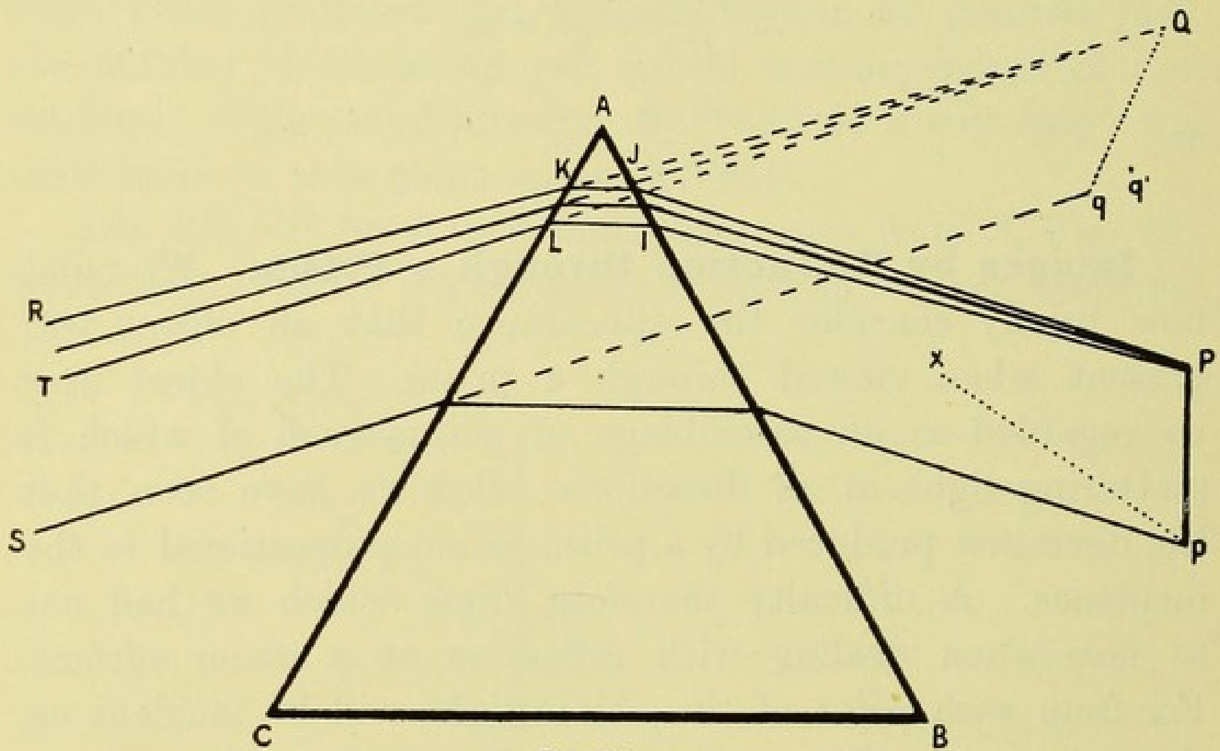

Fig. 60.

Let Pp represent an object placed before the prism ABC. Of all the light scattered in all directions from P, consider only the cone PIJ the axial ray of which undergoes minimum deviation. If then the angle IPJ be very small the extreme rays will also on emergence undergo the same deviation. Hence the emergent pencil must have the same angle of divergence that the incident pencil had. Produce the extreme rays $RKTL$ backwards to meet in Q. Then Q is the focus of P with respect to this small pencil. The point Q will lie on the axis of the emergent pencil produced. It will be situated considerably above P and nearer to the prism, by about one-third of the thickness of the glass through which it is viewed. Similarly a small divergent pencil from p undergoing minimum deviation will retain its original divergence, so that the intersection of the emergent rays produced backwards will be at q. This point q is then the virtual image of p to an eye suitably placed to receive this emergent pencil. It is however obvious that no observer could receive both sets of emergent pencils, so that it would be erroneous to conclude that Qq is the virtual image of Pp. If the pupil of the observer's eye be so placed that the narrow beam that enters it from P has a virtual focus at Q, the small pencil that reaches it from p must have traversed the prism by some other path than that of minimum deviation, and hence will not have a definite virtual focus at all; the image of p will consequently be blurred. It will be shewn in a later section that if the emergent rays of this aberrant pencil be produced backwards, they will intersect in two focal lines and not in a point; there will however be a spot between these focal lines where the cross section of this produced pencil will be very small. This is called the circle of least confusion and we may consider this spot as roughly representing the focus of the aberrant pencil.

It must be remembered that the cross section of the cone of light that enters an eye is exceedingly small, being limited by the size of the pupil. It is owing to this that the consideration of only small pencils gives fairly trustworthy results. A rough idea of the position of the circle of least confusion of an aberrant pencil may be formed by the following considerations. The axial ray of the aberrant pencil entering the eye must be incident on the prism at an angle either greater or less than that which corresponds to the minimum deviation. Referring to Fig. 60 let us suppose an eye at RT; the light which reaches this eye from p must have some such initial direction as pX. The angle of incidence of pX on AB is obviously greater than that which corresponds to the minimum deviation. Let us confine our attention to a small pencil of rays from p about an axis pX; it is clear that the lowest ray of the pencil will undergo less deviation than the uppermost ray of the pencil. Hence the point of intersection of these extreme rays will be further off than the point from which they originated, neglecting the influence of the thickness of the glass. We may presume then that the circle of least confusion where all the constituent rays of the pencil are closest together will be at a point q' somewhat further from the prism than q and above the level of q.

If however the eye be situated at S, the pencil that reaches it from p will be composed of rays that intersect at q. The pencil that reaches the eye from P will now be the aberrant pencil and will consequently undergo a greater deviation, the blurred image of P will be formed at a point Q' above the level of Q and slightly nearer the prism.

Some of these points can be very easily verified. If fine type be observed through a prism of 20° edge upwards, it will be found that in the neighbourhood of one position the type appears bright and fairly distinct, although displaced upwards by about half the apical angle of the prism. If the prism is rotated from this position in either direction the type appears to move upwards and to become blurred, as well as less bright. If the edge be turned towards the observer the height of the object appears diminished, for in this case the effective rays from the lower part of the object undergo a greater deviation than those from its upper part. If the rotation be in the opposite direction, the height of the object appears to be increased, for now the path of the rays from the lower part of the object is more nearly that of minimum deviation than that of those from its upper part.

The diminution of brightness in the image as the prism is rotated from the position of minimum deviation raises an interesting point, and one which is of fundamental importance in the explanation of the rainbow.

Let a widely divergent cone of light traverse a prism and let the axial ray undergo minimum deviation, the rays in its immediate neighbourhood undergo almost the same deviation, whereas the rays in the immediate neighbourhood of either of the extreme rays diverge widely. Consequently more light travels in or near the path of minimum deviation than by any other route. This phenomenon is often referred to as the condensation of rays that have undergone minimum deviation.

If the refracting angle of a prism be small, as is usually the case in the prisms used in ophthalmic practice, the confusion circles are very small, and the blurring of the virtual image is not noticeable. If the angle of the prism be 10° we may consider its deviation about 5° for all pencils whose aberration from the direction of minimum deviation does not exceed 20°.

An object viewed through the prism will appear bodily displaced upwards towards the edge of the prism, as though rotated through an angle of 5° about an axis passing through the prism.

Up to this point we have merely been considering the path of the light that traverses a prism in a plane at right angles to both its faces. The light however that travels in a direction that makes an angle with this plane does not traverse the prism with the same deviation. If a prism is placed edge upwards before the eye, and observation be directed towards a window, it will be noticed that the horizontal parts of the window-frame appear concave upwards, whereas the vertical sides still appear vertical and straight. The horizontal parts will also appear fringed with colour, this we shall for the moment neglect. Again, if the prism be rotated about a vertical axis the image will undergo a further distortion, one of its diagonals becoming lengthened while the other diagonal is shortened.

A popular explanation of this distortion is somewhat difficult to give. Let us confine our attention to a point at one end of the horizontal limb of the window-frame. The small cone of light from this point that eventually reaches the eye of the observer must have traversed the prism obliquely in the horizontal direction. The principle of same phase forces us to believe that the time taken by light to travel along each one of the constituent rays of the small pencil is the same, for under no other condition could an image of the point be formed. Now it is clear from what we have said above (p. 102) that this condition will be satisfied if we make the axial ray of this oblique pencil take the path which occupies minimum time. As the speed of light in glass is less than its speed in air, the longer its path in the glass the more time will the journey take, other conditions remaining the same. Now the light must preserve its horizontal obliquity unchanged if it is to reach the eye of the observer, there is therefore only one way of shortening its journey through the glass, namely, by traversing the prism higher up, *i.e.* nearer its edge. But this increases the length of its path through the air, so a limit is placed on the vertical obliquity of its path. There will thus be one route, which will take the least time to travel; this will be that in which the path through the glass is diminished at the expense of that through the air in a certain proportion.

When then the axial ray of the small cone that we are considering takes the route we have indicated all its constituent rays will represent journeys of the same length as regards time. This cone will consequently reach the eye as if it had started from a single point. This is the only cone which forms an image of the point and is consequently the only cone that we are considering. Since therefore this cone traverses the prism above the level of the cone from the middle point of the object, it will appear to the observer to be coming from a point considerably above the level of the middle point of the image. Both extremities then of a horizontal line viewed through a prism edge upwards will appear raised. The greater the obliquity of the incident pencil, the higher will be its course in the prism and the more elevated will the image be that it forms. If the edge of the prism be downwards the conditions will be reversed. In every case then when a line parallel to the edge of a prism is observed the image of the line will appear curved, the concavity of the curve being in the same direction as that of the edge of the prism.

If the prism, edge upwards, be rotated through an angle about a vertical axis, the cone of rays from one extremity of a horizontal line incident on the prism will become more oblique, while that from the other extremity will become less oblique, the image of the line will therefore be tilted as well as concave. If the object be a rectangular window, one side of the window will appear raised while the other side is depressed, consequently one diagonal will be increased while the other is diminished.

SUMMARY.

A square object presenting a plane surface when viewed through a prism, edge upwards, will give rise to an image with the following peculiarities.

The plane of the prism is the median vertical plane that bisects the apical angle.

1. When the plane of the prism is parallel to the plane of the object,

The image is raised above the level of the object, the sides are more raised than the mid-vertical line. The upper and lower edges are consequently concave upwards.

2. When the prism is rotated through a moderate angle about a horizontal axis parallel to its edge,

The image rises.

If the edge of the prism be turned towards the observer,

The height of the image is diminished.

If the edge of the prism be turned away from the observer,

The height of the image is increased.

3. When the prism is rotated about a vertical axis in such a way that its right side is turned from the observer,

The right margin of the image is raised above the left margin, its right superior and left inferior angles are consequently rendered more acute.

When the rotation is clockwise so that the left side of the prism is turned from the observer,

The left margin is raised above the right margin and the angles specified above become the obtuse angles.

4. When the prism is rotated about a horizontal axis at right angles to its edge, *e.g.* about the visual line of the observer,

The image of the object rotates also about the same axis. The image being always displaced towards the edge follows it in its rotation.

The last point demands a little further investigation, as it is closely related to a problem which frequently arises in practice.

Let us suppose that a thin pencil of light, the rays of which are parallel, traverses the centre of a prism in the path of minimum deviation. Let us now imagine a screen placed at a distance l from the centre of the prism Fig. 61, and so arranged that its surface is normal to the incident ray, which if produced would meet it at I, and to the plane of incidence and refraction. The deviation D undergone by the light

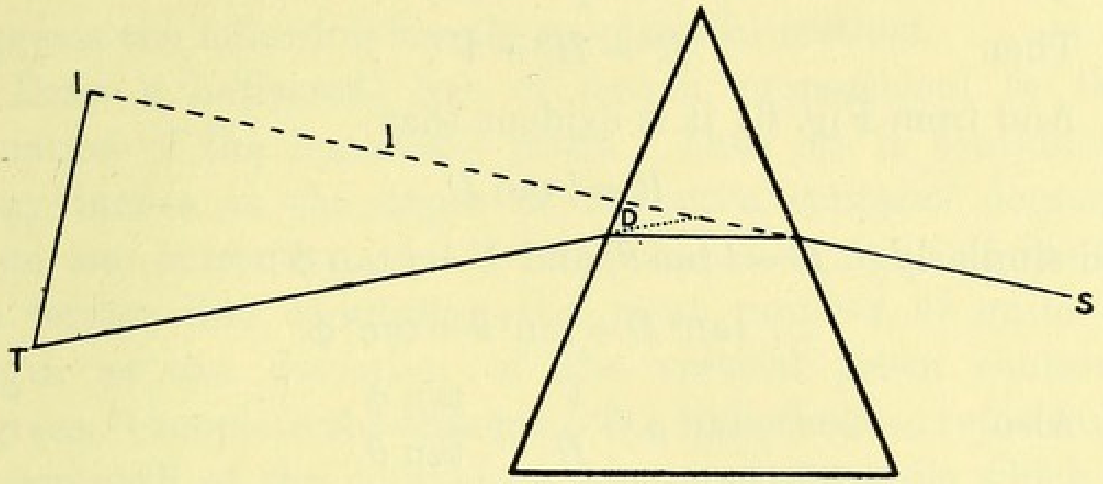

Fig. 61.

will be indicated on the screen by the distance IT that in this situation separates the refracted pencil from the incident pencil produced.

Suppose now that the prism were rotated about an axis SI the spot of light on the screen at T would trace out a circle of radius IT, the centre being at I. Fig. 62 represents

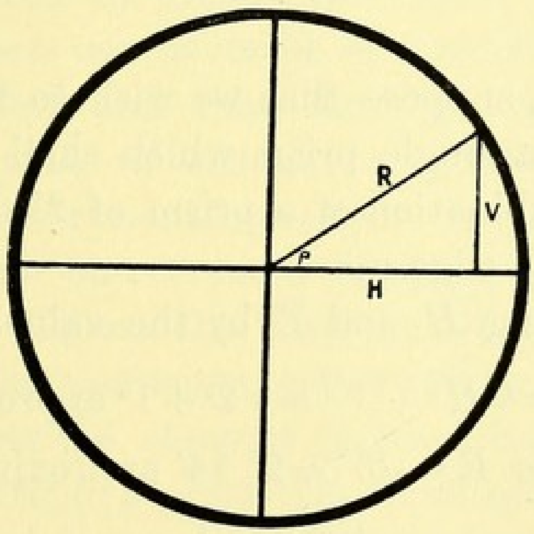

Fig. 62.

this circle where $R = IT$. If the prism be rotated through an angle ρ from the horizontal, the oblique deviation may be resolved into two components, one horizontal (θ) and one vertical (ϕ), and these would be represented on the screen by the lines R, H and V respectively.

Then $$R^2 = H^2 + V^2.$$

And from Fig. 61 it is evident that

$$R = l \tan D.$$

and similarly $$H = l \tan \theta, \text{ and } V = l \tan \phi;$$

$$\therefore \tan^2 D = \tan^2 \theta + \tan^2 \phi.$$

Also $$\tan \rho = \frac{V}{H} \text{ or } \frac{\tan \phi}{\tan \theta}.$$

If then D and ρ be given, ϕ and θ can be found, and vice versâ.

In ophthalmic practice such problems frequently arise, and their solution is rendered extremely easy from the fact that the prisms used clinically are so weak, that we may regard the tangents of their deviating angles as proportional to the angles themselves. Hence we may replace R, V and H by D, ϕ and θ in the equations $R = \sqrt{H^2 + V^2}$ and $\tan \rho = \frac{V}{H}$.

For example, suppose that we wish to find the strength and position of a single prism which shall be equivalent in effect to the combination of a prism of 2° dev. edge out and a prism of 1° dev. edge up.

Then replacing H and V by the values of θ and ϕ we have $$D \text{ or } R = \sqrt{H^2 + V^2} = \sqrt{2^2 + 1^2} \text{ approximately};$$

$$\therefore D \text{ or } R = \sqrt{5°} = 2°\ 14' \text{ approximately},$$

and $$\tan \rho = \tfrac{1}{2}.$$

This ratio is found on referring to a table of tangents to correspond to an angle of 26° 34′.

Hence a single prism of 2° 14′ if placed with its base apex line inclined 26° 34′ upwards from the horizontal will produce the desired effect.

If a set of mathematical tables is not at hand Maddox suggests the following simple geometrical method.

Draw a horizontal line of length proportional to the deviation of the horizontal prism. Thus let it contain as many inches as the angle of deviation contains degrees. From one extremity erect a vertical line at right angles to the former line containing the same number of units of length as the deviation of the vertical prism contains degrees. Complete the triangle. The hypothenuse represents the strength of the equivalent prism and the angle which it makes with the horizontal base line represents the angle at which this "resultant" prism should be set. An inch scale and a protractor or a circle divided into degrees is therefore all that is necessary to arrive at an approximate result. The error indeed will probably be considerably less than that of the optician who has the far more difficult task of making the prism to order.

After the consideration of the distortion and blurring of the images produced by prisms, the reader may be inclined to think that objects can never be seen satisfactorily through prisms of any kind, and that therefore they can not be practically used with advantage in ophthalmic cases. It would be almost as rational to object to glass windows on the ground that no object could be distinctly seen through a plate of glass. Unless the plate be thick and the pencil be very oblique, the confusion circles are so small that they only occupy one retinal element (retinal cone), and as long as they do not extend beyond this small area on the retina, the effect produced is precisely identical with the effect of a single point of light.

Similarly with very weak prisms the distortion and blurring of the image is so slight as to escape recognition by the eye under ordinary circumstances. When the deviation of a prism exceeds 2°, some distortion of the image may be just detected; this is especially noticeable with oblique pencils. Consequently it is only under exceptional circumstances that prisms of more than 2° dev. can be ordered for constant use with advantage, vision through them must be direct and not oblique, or the objects viewed will appear indistinct.

QUESTIONS.

(1) The refracting angle of a glass prism is 60°, and its refractive index is 1·54 for a certain kind of light. What is the minimum deviation for this kind of light on traversing the prism? Given that $\sin 50° 21' 14'' = {\cdot}77$.

(2) If the refracting angle of a prism be 60°, and the minimum deviation for the D line be 30°, what is the refractive index for the D line?

(3) If the refracting angle of a prism be 60, and the refractive index be $\sqrt{\frac{7}{3}}$ find the limit within which ϕ must lie in order that the light may be able to emerge at the second face.

(4) A prism of small refracting angle 2° with refractive index 1·5 is placed in water of refractive index $\frac{4}{3}$. Shew that its deviation is only a fourth of what it is in air.

(5) It is found that when a plate of glass 7·7 mm. thick is placed over a microscopic object, the microscope must be raised 2·7 mm. to bring the object into focus again. What is the refractive index of the glass?

(6) When viewing a distant object each eye of a patient is found to deviate outwards 1° 44′ (nearly $\sqrt{3}°$), while the right eye deviates above the level of the fixation-line of the left eye 1°. What prisms might be given which would relieve this defect?

CHAPTER VIII.

DISPERSION. ANOMALOUS DISPERSION. ACHROMATISM.

Dispersion. We have hitherto confined our attention to the refraction of homogeneous light, we must now consider what occurs if the incident light is not homogeneous.

Waves of widely different periods are being continually radiated from the sun through space. The waves however which are capable of setting up the specific stimulus in the retina that gives rise to the perception of light have but a limited range of period. The waves of longest period, when they reach the percipient structures, induce an impression of red light, those of shortest period induce an impression of violet light. The ultra-red waves are capable of stimulating other percipient structures but the sensation produced is then that of heat, not of light. Similarly the ultra-violet waves do not give rise to any sensation directly, though we have no reason to assume that they consequently may be neglected. Indeed the intimate structure of many substances, as for instance that of certain silver salts, is profoundly modified by these ultra-violet waves. It is probable indeed that they exercise an injurious influence on the delicate structure of the retinal cells if they are in excess.

The terms ultra-violet and ultra-red are not strictly applicable to waves at all. Colour is a subjective term which merely expresses the sensation produced by a certain kind of stimulus on a certain kind of cell. Before light waves reach a colour-seeing organ, they cannot be said to have colour. No one would apply the term nauseous to waves of the sea, though under certain circumstances they may induce nausea in certain individuals. If it be clearly understood that by the expression red waves or red light is meant a set of waves of that period that induces a sensation of red in most individuals, the term will be found a convenient abbreviation.

As we are dealing with light waves only, we need not now consider the properties of the ultra-red and ultra-violet waves, and our study is accordingly confined to waves whose frequency varies from about 391 billions per second to 759 billions per second.

Now it is found by observation that in free space and in air red waves (*i.e.* those of relatively long period and long wave-length) travel at the same rate as the shorter violet waves. If this were not the case, if for instance red waves travelled quicker than violet waves, a star reappearing after eclipse would appear first red and then gradually change tint until it became white when all the colours had had time to arrive. Now no change of tint has ever been observed, even in the case of variable stars, such as Algol, which are so distant that it takes several years for the light from them to reach us, and hence we may conclude that waves of any period between the limits that the eye can appreciate, travel at precisely the same rate through free space.

It has been found indeed that waves of electrical disturbance of a period a million times longer than any light wave travel through free space at precisely the same rate as light waves. The ether of free space is then regarded as homogeneous, or if it has any structural heterogeneity its parts must be so infinitesimally small, that it can deal with wave-lengths of a hundred-thousandth of an inch in precisely the same way as wave-lengths of a hundred miles.

We have however no right to assume that the ether in matter is also homogeneous. We have indeed some ground for thinking that in matter part of the ether is free, streaming freely through the pores of the matter, and part of the ether is bound, associated with the particles of matter in some way, so that it behaves differently. If the ether in matter is to be regarded as heterogeneous in this way we can no longer say that the velocity of transmission of an undulation depends solely on the nature of the medium, the wave-frequency or the wave-length of the undulation may prove to be an important factor in the expression for its velocity. We must resort to experiment to see what actually does happen.

If the white light of the sun be admitted through a narrow slit into a darkened room and a prism be held in the path of the entering sunbeam, the light will be deflected and form a broad coloured band on the opposite wall. The order of the colours is red, orange, yellow, green, blue, indigo, violet, of these red is refracted the least, and violet the most. Now we know that refraction depends upon the velocity of propagation; and we may infer that the white light of the sun is really composite, consisting of these several colours, but for some reason violet light is impeded more than red light when traversing glass, and hence the prism refracts violet more than red. It will be found that when the colours are brought together by another prism or by a lens, the colours disappear and a white patch is formed. The whole series of colours however is not necessary to produce white, it is found that they consist of pairs of complementary colours, each pair when in proper proportions, and received on the same part of the retina, producing the sensation of white.

Thus Red	is complementary to Bluish green,			
Orange	„	„	„	Sky blue,
Yellow	„	„	„	Violet blue,
Greenish yellow	„	„	„	Violet.

The sensation of white is always due to a mixture of at least two colours, but the eye alone is quite unable to distinguish a white produced by the mixture of all the spectral colours, from that produced by a mixture of only two or three of them. The difference is at once made manifest by examining the colour with a prism, which will decompose it into its constituent spectral colours. Certain interference experiments have proved most conclusively that in free ether one coloured light differs from another both in wave-length and in wave-period. The more refrangible colours, *i.e.* those nearer the violet end of the spectrum, have both wave-length and wave-period shorter than the less refrangible colours. And since the spectral colours are differently refracted by glass-prisms, the value of the refractive index of glass must vary according to the exact colour which is used as a standard, indeed the term index of refraction has no definite meaning unless the period, wave-length or some constant of the wave to which it refers is also given.

It will be necessary to digress somewhat to show what means we have for describing waves of a definite period and length independent of the colour-sense of the observer. On p. 155 a short description of the spectrometer is given; which is an instrument for obtaining what is called a pure spectrum. If its slit be directed towards a candle flame a succession of images of the slit is seen in every colour forming a continuous spectrum. If however the sun's light be examined, it will be found that colours corresponding to certain wave-periods are very faint, so that the spectrum appears to be crossed by certain dark lines. These lines are the images of the slit in those tints which are defective in sunlight, they appear dark against the bright background. With powerful instruments some 10,000 lines have been detected in the visible part of the solar spectrum, and by other means[1] the ultra-visible part of the spectrum has been shown to present nearly as many more. The most prominent of these lines have been named by the letters of the alphabet, thus *A*, *B* and *C* are in the red portion of the spectrum, *D* in the orange yellow, *E* in the green, *F* in the blue, *G* in the violet blue, and *H* in the extreme violet. These lines always correspond to a definite wave frequency and a definite wave-length, and, being easily recognizable, form convenient marks for determining refractive indices. Now it is found that all incandescent solids give continuous spectra, like that of a candle-flame which owes its luminosity chiefly to the incandescent carbon particles that it contains. However incandescent gases or vapours give rise to discontinuous spectra consisting of a finite number of bright lines. Thus incandescent hydrogen gives rise to three bright lines, two being in exactly the position of *C* and *F*, the other being near to *G*. Similarly incandescent sodium vapour gives rise to a double bright line in the position of *D* or rather $D_1 D_2$, for with powerful instruments this line is resolved into two adjacent lines. Incandescent vapours radiate light of specific period, and according to the theory of exchanges they must absorb undulations of that period which they radiate. The explanation of the dark lines of the solar spectrum is simple. The photosphere would give rise to a continuous spectrum but that it is surrounded by an atmosphere of incandescent vapours. If special means be

[1] The lines in the ultra-violet part of the spectrum have been mapped out by means of photography, while those in the infra-red part have been discovered by Langley with the bolometer.

taken to view this layer alone a discontinuous spectrum is seen of bright lines.

This layer is so much less bright than the photosphere that it absorbs far more undulations than it radiates, hence, when the glowing nucleus of the sun is seen through its atmosphere, undulations of certain periods are largely absorbed. The dark lines crossing the solar spectrum indicate where this selective absorption has occurred, and afford a means of recognizing many metals and gases which are present as glowing vapours in the atmosphere of the sun.

According to Captain Michelson's experiments in 1882, the velocity of light in free ether is 299,853 kilometres per second, and according to those made in the same year by Professor Newcomb, the velocity is 299,860 kilometres per second. In air its velocity is about 298,394·5 kilometres per second. Similarly in all gases the velocity of light is less than in free ether, but in most of them at the ordinary temperature and pressure no dispersion has been observed, in other words both long and short waves seem to be transmitted at the same rate. At the end of this chapter will be found a table of the wave-lengths, wave-frequencies and periods of the more prominent spectral lines in air. The index of refraction for all of these in air is practically the same, as air has no sensible dispersive power. When we come to deal with glass, we find that the index of refraction μ undergoes a slight progressive increase in value as it is measured for the A line or for the H_2 line. Further than this, different glasses disperse to different extents, the material that causes the greatest deviation does not necessarily possess the greatest dispersive power. The diamond with a refractive index of 2·5 has not such a high dispersive power as flint glass.

No completely satisfactory explanation of all the phenomena of dispersion has yet been given, but an attempt will here be made to shew in what direction the true explanation seems to be forthcoming.

We may note that it is found by experiment that the intensity of light exercises no influence on the amount of its deviation after traversing, for example, a glass prism. If however we supposed light to be due to an actual movement-vibration of the ether particles, we should expect some alteration of deviation to take place when the excursion of the particles was comparable to the intermolecular spaces of the glass. This, as far as it goes, supports the electric theory of light, *i.e.* that the ether particles do not actually move up and down or from side to side, but that they undergo periodical changes of electrical state. We shall continue to use the terms vibration and undulation, as they keep prominently before our minds the fundamental notion of periodicity.

Now since the vibrations of the ether in the glass are directly due to those adjacent to them in the air, there seems at first sight no reason why their period should be altered. Let us see how far we can explain dispersion on this hypothesis. We find by experiment that light, for instance that corresponding to the *A* line, travels quicker through air than through glass. And since $V = \frac{\lambda}{T}$, it follows that if the period T remains unchanged, the value of λ the wave-length must undergo a shortening in the glass corresponding to the retardation that this light undergoes in the glass. If then μ denote the relative refractive index between air and glass $\left(i.e. \text{ the ratio } \frac{V_a}{V_g}\right)$ for this *A* line, its wave-length in air must be μ times its wave-length in glass:—

$$\lambda_a = \mu\lambda_g \text{ and } \lambda_g = \frac{\lambda_a}{\mu}.$$

Now consider waves of a shorter period T', those for instance that correspond to the line H_2. Since these travel through air at the same rate as those of the period T it follows that their wave-length must be proportionately less, for

$$\frac{\lambda'}{T'} = V = \frac{\lambda}{T}.$$

In glass however the velocity of these short period waves is less than in air. On the assumption that their period T' remains constant, we must suppose that this diminution of velocity is due to a diminution of their wave-length.

If now we assume that the ether in the immediate neighbourhood of a matter particle behaves as if it were more dense, it is clear that the ether in a glass prism, for instance, assumes a heterogeneity depending on and resembling the molecular arrangement of the glass itself. This might impede those waves most which were of shortest wave-length, and so we might explain the fact of dispersion. Referring again to the analogy we made use of before, suppose a column of soldiers six or seven abreast were marching obliquely towards a wedge-shaped piece of rough ground. In the open country each soldier might be taking steps of different lengths and yet, if the frequency of the steps was rightly adjusted, all the soldiers would be travelling forwards at the same pace. On reaching the rough ground they all would have to take shorter steps, but the nature of the ground might be such as to affect most those who took the shortest steps. If the period remained unchanged, the velocity of the short waves would be especially diminished and so their refraction would be increased. If the obstruction were of a different nature, it might affect all the waves equally whatever their wave-length. If for instance our soldiers were provided with diving dresses, and were made to walk at the bottom of the sea, they would certainly each take shorter steps, but it is quite possible to conceive that the progression of each would be impeded to an equal extent. Similarly it is found by experiment that the dispersion produced by a substance does not depend on the mean deviation produced by it. Air for instance at the ordinary pressure has apparently no dispersive power, yet its refractive effects are readily noticeable, while fuchsin has an extraordinarily great dispersive action with quite a moderate refraction.

Anomalous Dispersion. The dispersion produced by fuchsin is very peculiar; the spectrum of light transmitted through a fuchsin prism shews the colours in the following order: violet, then follows an interval, then red, orange and yellow in their natural order. Green does not appear, for that colour is not transmitted by fuchsin. Such anomalous dispersion as it is termed is well displayed in most of the aniline dyes, and a slight form of it, irrational dispersion, is presented by almost all transparent bodies. It is found for instance that some substances disperse one part of the spectrum more than another part, in other words, the deviation of light is not proportional to its wave-length.

Theory of Dispersion. It is plain that the simple view we have been adopting does not explain the facts.

Kundt has shown that anomalous dispersion is best displayed in substances that have strongly marked absorption bands, and that in going up the spectrum from red towards violet, the deviation is abnormally increased below an absorption band, while above the band the deviation is abnormally diminished by the absorption.

The assumption is regarded as untenable that the undulation period remains constant, whatever the nature of the medium may be. Let us consider the case of fuchsin more minutely. When sunlight is transmitted through a rectangular glass cell containing a weak solution of fuchsin and examined with a spectroscope, the ordinary spectrum is seen with a strongly marked dark band cutting out the green and extending into the regions of the blue and yellow on either side. The surface colour of fuchsin is green, that is to say that fuchsin has the property of responding in a peculiar way to vibrations of the period of green light and reflecting them almost totally. Now we know that molecules of matter when in a gaseous state are vibrating, and sending out waves in all directions through the ether in which they are embedded. Some molecules are able to vibrate to many different periods, for instance the molecules of iron vapour vibrate to 450 different periods even within the limits of the visible spectrum, giving rise to 450 lines of light each of definite period and consequently of definite colour. The molecules of sodium vapour on the other hand are only known to vibrate to two different periods, $\cdot 197590 \times 10^{-14}$ sec. and $\cdot 197397 \times 10^{-14}$ sec. Waves of these periods produce a sensory impression of orange yellow light. When the molecules are packed closely together, as in the solid state, their vibrations are cramped, so that their specific periodicity is quite masked by the innumerable secondary vibrations that are set up. A white-hot bar of iron gives a continuous spectrum, shewing that its particles are vibrating to every period within the range of the visible spectrum. In the case of certain substances in solution it would seem that the particles may be permitted a certain kind of motion or vibration like a tuning-fork, so that they respond more readily to vibrations that they receive of their specific period than to any other. Their action is no doubt much cramped and the light which they reflect is not of definite period, but involves a more or less wide range of period. The green light reflected by fuchsin can be analysed by a prism into yellow, green and blue, and the absorption band seen by transmitted light is correspondingly broad.

Now presuming that in a fuchsin solution the fuchsin particles readily respond to vibrations of a period about $\cdot 176 \times 10^{-14}$ sec., the surface colour and the body colour of the dye are explained, and further consideration will go some way towards explaining its anomalous dispersion. Kundt found that when sunlight was sent through a fuchsin prism, the ether vibrations below the absorption band travelled slower through the fuchsin than those above it; red and yellow were deviated more than violet. If the vibration of the matter particles had altered the period of the ether vibrations, this might be accounted for.

We have a ready dynamical analogy. If to the bob of a pendulum P, executing horizontal vibrations, another pendulum p be attached, executing vibrations of a slightly shorter period, the effect of p will be to increase the period of P and vice versâ. It may then reasonably be maintained that the effect of the fuchsin particles, vibrating at a period of $\cdot 176 \times 10^{-14}$, will be to increase the period of the ether vibrations below the absorption band and to diminish the period of the ether vibrations above the band. And since $V = \frac{\lambda}{T}$, when the period T is increased the velocity is diminished and vice versâ, consequently the refraction of the light below the band is increased, whereas that above the band is diminished.

Now most substances, perhaps all, shew the properties of selective absorption, and to this fact is to be attributed the irrationality of their dispersion. Glass, or rather the ether in glass, which readily transmits vibrations of such a period as are sensible to the eye is quite opaque to vibrations that are two or three times quicker or slower.

If the dispersion produced by a prism were due solely to differences of wave-length, the prismatic spectrum would resemble in character the diffraction spectrum in which the deviation is simply proportional to wave-length. But all prismatic spectra, when compared with this standard, shew a relative contraction of the red end of the spectrum, and further most substances disperse different parts of the spectrum to specifically peculiar extents. This is what is called irrational dispersion, and it is probably best regarded as a form of anomalous dispersion due to absorption. According to this view we may conclude that prismatic dispersion is primarily due to the molecular structure of the prism, being such as to impede waves of short length more than those of longer length, and secondarily to the vibration of the molecular systems reacting on the ether undulations and tending to alter their period.

Dispersive Power. We shall now endeavour to find some proper expression for the measure of the dispersive power of a substance. We must first select some wave of definite period, the refraction of which through a prism of the substance we can regard as a measure of the deviation. We may conveniently take one of the fixed lines in the brightest part of the solar spectrum, *e.g.* the D line, and let μ represent the refractive index of the substance for yellow light of this wave frequency, 506×10^{12} per sec. Then if the refracting angle A of the prism be small and the prism be in the position of minimum deviation, the mean deviation is given by the expression $(\mu - 1) A$. Let μ_r denote the refractive index for the extreme red light of a definite wave frequency, *e.g.* 391×10^{12} per sec., and let μ_v similarly denote the refractive index for violet light of the wave frequency of 750×10^{12}.

Then $\mu_v A - A$ and $\mu_r A - A$ represent respectively the deviations of this violet light and of this red light produced by the prism. And the difference $(\mu_v - \mu_r) A$ represents the angular dispersion produced by the prism.

Now the dispersive power of a substance is independent of the refracting angle if it be small, and is measured by the ratio of its dispersion to its deviation. Therefore $\frac{(\mu_v - \mu_r) A}{(\mu - 1) A}$ or $\frac{\mu_v - \mu_r}{\mu - 1}$ measures the dispersive power.

Now the dispersive power of one kind of flint glass is ·053, while the dispersive power of a particular kind of crown glass is ·032. The prism of the crown glass ($\mu = 1·52$) whose refracting angle is 60° will in the position of minimum deviation produce a deviation of 38° 55′ and an angular dispersion of about $·032 \times 38° 55'$ or $1° 14'·72$. A flint glass prism of dispersive power ·053 will have the same dispersive effect when its minimum deviation is about 23° 30′ or ($\mu = 1·58$) when its refracting angle is 37° 26′. If therefore both prisms be placed in the position of minimum deviation, the edge of the one pointing in the same direction as the base of the other, the dispersion will be neutralized, whereas there will be a total deviation of 15° 25′ due to the preponderating effect of the crown glass prism. The emergent light will not however be totally free from colour, as the spectrum formed by a flint glass prism is relatively more extended in the blue part than the crown glass spectrum. In practice it is found that more perfect achromatism is obtained when two colours in the brightest part of the spectrum are united and the rest are allowed to take their chance. Theoretically only two definite colours can be fused by two prisms, three colours by three prisms, and so on, owing to the irrationality of dispersion; but if the above method is used the complementary colours will be to a great extent corrected, and the double combination is sufficiently achromatic for practical purposes. Professor Abbé and Dr Schott have now brought out many pairs of kinds of flint and crown glass, such that the dispersion in the various parts of the spectrum is for each pair as nearly as possible proportional. The achromatism that can be attained by this Jena glass with two prisms or with two lenses is far more perfect than was possible before with a double combination of the ordinary kinds of glass.

It is evident also that two prisms of different dispersive powers may be so arranged that the deviation produced by the one may be reversed by the other, so that after traversing both prisms the light is dispersed, but its mean direction is unchanged. If the refracting angles of the prisms are very small, we may take $(\mu - 1) A$ as the deviation. If the mean refractive indices of the two kinds of glass be 1·52 and 1·58, we have to make $\cdot 52\, A_1$ equal to $\cdot 58\, A_2$. Then a prism of crown glass with refracting angle $5^\circ\!\cdot\!8$ will have its deviation compensated by a prism of flint glass with refracting angle $5^\circ\!\cdot\!2$ placed in the reverse direction; while the emergent light will be dispersed through an angle of $3'\!\cdot\!8$, if the dispersive powers of the glasses be ·032 and ·053 respectively.

If the prisms be of greater angle, and if their adjacent faces be cemented together, the calculation becomes somewhat longer. Let the crown glass prism present a refracting angle of 60° in the position of minimum deviation; this will give a deviation of $38^\circ\, 55'$. If the flint glass prism be cemented to the crown glass prism it must have a refracting angle of $55^\circ\, 8'$ in order to neutralize the mean deviation produced by its neighbour, and the resultant angle of dispersion will be about $50'$. This is the principle of the direct-vision spectroscope.

Expression for Dispersion. When we wish to correct the chromatic dispersion of two specific colours, *e.g.* the orange and the blue, we cannot make use of the published tables of dispersive powers, as owing to the irrationality of dispersion in different substances we have no right to assume that the dispersion in this selected part of the spectrum is proportional to the total dispersion. We must therefore obtain another expression in terms of the refractive index, tables of which for various parts of the spectrum are published for many substances.

Let ϕ, ϕ' be the angles of incidence and refraction at the first surface of the prism, and let ψ', ψ be the angles of incidence and emergence at the second surface. Then these are connected by the equations

$$\sin\phi = \mu \sin\phi',$$
$$\sin\psi = \mu \sin\psi',$$
$$A = \phi' + \psi'.$$

To make the problem more general we will suppose that the incident light has been already dispersed before reaching the prism. Then let $\phi + d\phi$, $\phi' + d\phi'$, etc. denote the corresponding angles for any other light whose refractive index is $\mu + d\mu$.

By differentiation we obtain

$$\cos\phi \frac{d\phi}{d\mu} = \sin\phi' + \mu\cos\phi' \frac{d\phi'}{d\mu},$$
$$\cos\psi \frac{d\psi}{d\mu} = \sin\psi' + \mu\cos\psi' \frac{d\psi'}{d\mu},$$
$$0 = \frac{d\phi'}{d\psi'} + 1.$$

Then since $\frac{d\phi'}{d\psi'} = -1$, $\frac{d\phi'}{d\mu} = -\frac{d\psi'}{d\mu}$ and we can eliminate the last terms of the first two equations by multiplying the former by $\cos\psi'$ and the latter by $\cos\phi'$,

$$\cos\phi\cos\psi' \frac{d\phi}{d\mu} = \sin\phi'\cos\psi' + \mu\cos\phi'\cos\psi' \frac{d\phi'}{d\mu},$$
$$\cos\phi'\cos\psi \frac{d\psi}{d\mu} = \cos\phi'\sin\psi' + \mu\cos\phi'\cos\psi' \frac{d\psi'}{d\mu};$$

$$\therefore\ \cos\phi\cos\psi' \frac{d\phi}{d\mu} + \cos\phi'\cos\psi \frac{d\psi}{d\mu} = \sin(\phi' + \psi').$$

Or in differentials $\cos\phi\cos\psi' d\phi + \cos\phi'\cos\psi d\psi = d\mu \sin A$.

If the incident light is not dispersed the term involving $d\phi$ vanishes and the dispersion of the emergent light is

$$d\psi = \frac{d\mu \sin A}{\cos\phi' \cos\psi}.$$

When the incident light falls perpendicularly on the first surface $\phi' = 0$, $\psi' = A$ and $\sin\psi = \mu \sin A$,

$$\therefore\ d\psi = \frac{d\mu}{\mu} \tan\psi.$$

When the prism is in the position of minimum deviation for the standard wave-length, $\phi' = \psi' = \frac{1}{2}A$,

$$\therefore\ \sin A = 2 \sin\psi' \cos\phi' = \frac{2}{\mu} \sin\psi \cos\phi',$$

$$\therefore\ d\psi = \frac{2d\mu}{\mu} \tan\psi.$$

The position of minimum deviation is not therefore the position of minimum dispersion, which is that which makes the product $\cos\phi' \cos\psi$ a maximum. The maximum dispersion is obtained when this product is a minimum; this occurs when $\psi = 90°$. Hence the dispersion produced by a prism whose refracting angle is ever so great may be counteracted by the dispersion of another prism of the same material whose refracting angle is ever so small, provided it be placed in a suitable position.

Table of the wave-frequencies and wave-lengths of the chief spectral lines in air.

	A	B	C	D_1	D_2	
Wave-length	7621·31	6884·11	6563·07	5896·18	5890·22	$\times 10^{-10}$ metres
Wave-frequency	391·526	433·454	454·761	505·608	506·593	$\times 10^{12}$ per sec.
Period	·25541	·23070	·21989	·19759	·19739	$\times 10^{-14}$ sec.

	E	F	G	H_1	H_2	
Wave-length	5270	4861·51	4307·25	3968·1	3933	$\times 10^{-10}$ metres
Wave-frequency	566·213	613·789	692·774	751·983	758·694	$\times 10^{12}$ per sec.
Period	·17661	·16292	·14434	·13298	·13181	$\times 10^{-14}$ sec.

QUESTIONS.

(1) Shew that a longer spectrum will be obtained when a prism is put in the position of minimum deviation for violet rays, than when it is put in the position of minimum deviation for red rays.

(2) Shew that at a single refraction at a plane surface the dispersion is proportional to the tangent of the angle of refraction.

(3) The refractive indices of a medium for three particular kinds of light are 1·525, 1·533, 1·541. What are the dispersive powers between these kinds of light?

(4) The refractive indices of another medium for these same kinds of light are 1·628, 1·642 and 1·660. Give the dispersive powers, shewing that they are not proportional to those of the previous medium.

(5) What refracting angle should a prism of the first medium have in order to annul the chromatic dispersion of the two extreme kinds of light produced by a prism of 3° of the second medium?

CHAPTER IX.

OBLIQUE REFRACTION AT A PLANE SURFACE. FOCAL LINES. CONTOUR OF THE REFRACTED WAVE-FRONT. CAUSTICS.

Oblique Pencils. It has been already stated that when a pencil enters a refracting medium, that is bounded by a plane surface, obliquely, the constituent rays of the pencil on being produced backwards do not intersect in a single point.

Let O be a luminous point (Fig. 63) and APQ a plane refracting surface. Let OP represent the axial ray of an oblique pencil of rays, and let OQ represent one of the extreme rays of the pencil. Produce the refracted rays RP, $R'Q$ to meet in F_1, and let RP cut the normal to the surface from O in F_2. Now if we suppose the figure to rotate through a small angle about OA, PQ will trace out a small area on the refracting surface, and OPQ will trace out a solid cone incident upon it. Meanwhile the point F_1 will trace out a small arc, approximately a straight line. Also the line indicating the cross-section of the pencil at F_2 will trace out a slender 'figure of eight,' approximately a line. These lines at F_1 and F_2 are the primary and secondary focal lines respectively. The secondary focal line is in the primary plane, *i.e.* the plane that contains the axial ray of the pencil and the normal to the surface from the origin of the pencil. The primary focal line is in a plane at right angles to the primary plane. It is evident then that the refracted cone does not intersect in a point. The refracted pencil if produced backwards would become oval in section, then getting narrower would finally form a line at F_2. Beyond F_2 the lateral rays would have crossed, and the section would gradually widen and at the same time become thinner, until it would merely be represented by a line at F_1. At some spot between F_1 and F_2 the section would be circular: at this place the cross-section of the pencil is the smallest. It is called the circle of least confusion.

Position of the Focal Lines. The position of the primary and secondary foci may be determined in the following way.

Let $OP = u, \quad F_1P = v_1, \quad F_2P = v_2.$

Then $$\frac{\sin\phi}{\sin\phi'} = \frac{F_2P}{OP} = \frac{v_2}{u},$$

$$\therefore \mu u = v_2 \text{ or } \frac{\mu}{v_2} = \frac{1}{u}.$$

Again, $$\sin\phi = \mu\sin\phi'.$$

Differentiating we get

$$\cos\phi\, d\phi = \mu\cos\phi'\, d\phi'.$$

Now when the pencil is very small, $d\phi$ represents the angle POQ, and $d\phi'$ represents the angle PF_1Q.

And in the triangle POQ,

$$\frac{PQ}{OP} = \frac{\sin POQ}{\sin OQP} = \frac{\sin POQ}{\cos AOQ}.$$

Since the pencil is very small we may replace $\frac{\sin POQ}{\cos AOQ}$ by $\frac{\angle POQ}{\cos AOP}$.

Thus $$\frac{PQ}{u} = \frac{d\phi}{\cos\phi}.$$

Similarly

$$\frac{PQ}{v_1} = \frac{\sin PF_1Q}{\sin F_1QP} \approx \frac{\angle PF_1Q}{\sin F_2PA} = \frac{\angle PF_1Q}{\cos AF_2P} \text{ or } \frac{d\phi'}{\cos\phi'}.$$

The equation $\mu \cos\phi' d\phi' = \cos\phi\, d\phi$ may therefore be transformed into

$$\frac{\mu PQ \cos^2\phi'}{v_1} = \frac{PQ\cos^2\phi}{u},$$

or $$v_1 = \mu u \frac{\cos^2\phi'}{\cos^2\phi}.$$

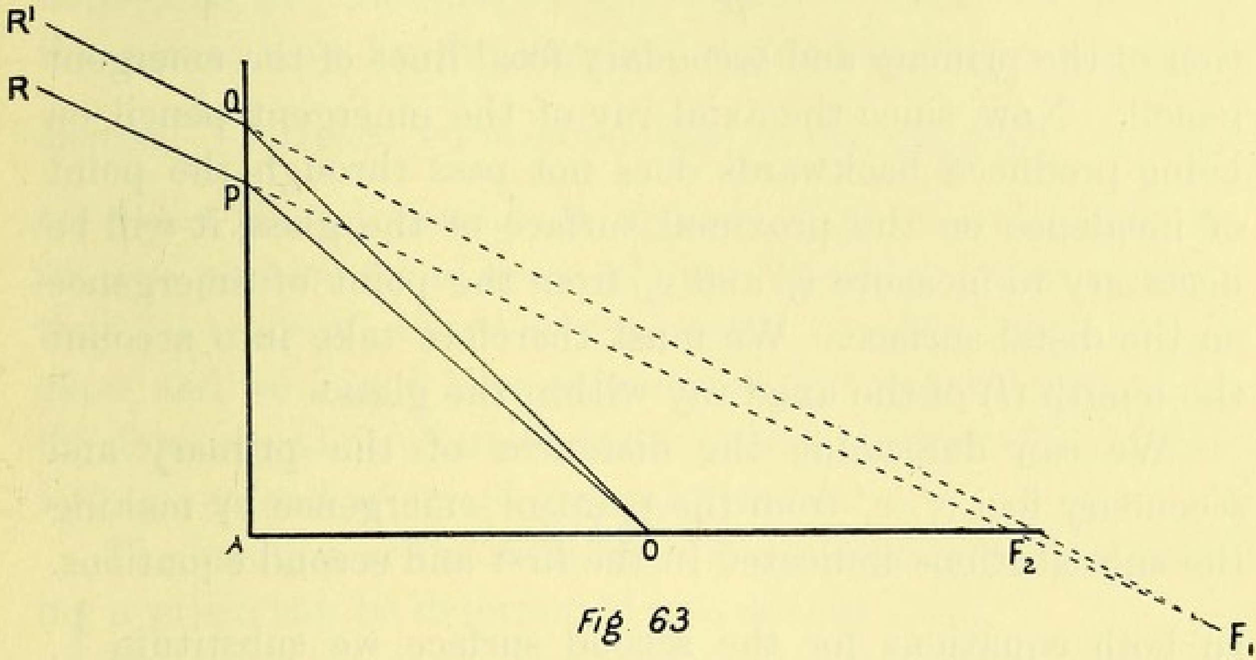

Fig. 63

The figure represents an oblique pencil of light passing from a rare medium to a dense medium. Had the light been travelling from a dense to a rare medium ($\mu < 1$) the point F_1 would have been situated above the axis OA.

These formulæ enable us to determine the position of the primary and secondary focal lines of an oblique pencil after any number of refractions, provided that each refracting medium is bounded by a plane surface.

Let us consider the form of an oblique pencil after traversing a glass plate of thickness w bounded by parallel plane surfaces[1].

[1] The reader is recommended to draw a diagram for himself.

Let the angle of incidence of the axial ray of the pencil be ϕ, and let the angle of refraction into the glass be ϕ'.

Then $$v_1 = \mu u \frac{\cos^2 \phi'}{\cos^2 \phi} \ldots\ldots(1), \text{ and } v_2 = \mu u \ldots\ldots\ldots(2).$$

The pencil in the glass will therefore proceed as if from these two focal lines; on reaching the distal surface of the glass the angle of incidence will be ϕ', and the angle of emergence or refraction into the air will be ϕ, since the faces of the plate are parallel. The index of refraction from the glass into the air will be $\frac{1}{\mu}$. It is required to find the position of the primary and secondary focal lines of the emergent pencil. Now, since the axial ray of the emergent pencil on being produced backwards does not pass through the point of incidence on the proximal surface of the glass, it will be necessary to measure v_1' and v_2' from the point of emergence on the distal surface. We must therefore take into account the length (l) of the axial ray within the glass.

We can determine the distances of the primary and secondary foci v_1', v_2' from the point of emergence by making the substitutions indicated in the first and second equations. In both equations for the second surface we substitute $\frac{1}{\mu}$, ϕ and ϕ' for μ, ϕ' and ϕ, and, while the expression $v_1 + l$ takes the place of u in the first equation, $v_2 + l$ takes its place in the second equation. Thus we get

$$v_1' = \frac{1}{\mu}(v_1 + l)\frac{\cos^2 \phi}{\cos^2 \phi'} \text{ or } u + l\frac{\cos^2 \phi}{\mu \cos^2 \phi'},$$

$$v_2' = \frac{1}{\mu}(v_2 + l) \text{ or } u + \frac{l}{\mu}.$$

Now if the angle of incidence or ϕ be 20°, the expression $\frac{\cos^2 \phi}{\cos^2 \phi'} = \cdot 92$ approximately, when the refraction takes place between air and glass.

If again $\phi = 80°$, $\frac{\cos^2 \phi}{\cos^2 \phi'} = \cdot 05$ approximately.

The distance $\frac{l}{\mu}\left(1 - \frac{\cos^2 \phi}{\cos^2 \phi'}\right)$ between the two foci may be taken as a measure of the indistinctness of the image.

If the pencil considered be small and direct, the two foci coincide at a point situated $u + \frac{l}{\mu}$ from the distal surface of the glass and the image is distinct. As the obliquity of the pencil increases, the separation of the foci becomes more noticeable, and the image appears blurred. However oblique the pencil may be the distance separating the foci cannot exceed $\frac{l}{\mu}$. Hence we may say that if the glass is very thin an object viewed through it appears distinct and nearly in its true position for $v_1' = u = v_2'$ approximately. It is evident that $l = w \sec \phi'$, where w represents the thickness of the glass, and we might have substituted this expression for l in the formulæ above, but they are easier to remember in the form given and are more generally useful.

The position of the focal lines of any small pencil traversing a prism can be determined in a similar way.

If u be the distance from the source of light to the point of incidence on the first surface,

$$v_1 = \mu u \frac{\cos^2 \phi'}{\cos^3 \phi}, \text{ and } v_2 = \mu u.$$

If l be the length of the axial ray of the pencil within the prism and ψ' be the angle of incidence at the distal surface and ψ the angle of emergence,

$$v_1' = \frac{1}{\mu}(v_1 + l)\frac{\cos^2 \psi}{\cos^2 \psi'} \quad \text{or} \quad u \frac{\cos^2 \phi'}{\cos^2 \phi} \cdot \frac{\cos^2 \psi}{\cos^2 \psi'} + \frac{l}{\mu}\frac{\cos^2 \psi}{\cos^2 \psi'},$$

and
$$v_2' = \frac{1}{\mu}(v_2 + l) \quad \text{or} \quad u + \frac{l}{\mu}.$$

Hence if the prism be in the position of minimum deviation $\phi = \psi$ and $\phi' = \psi'$, and we have

$$v_1' = u + \frac{l}{\mu}\frac{\cos^2\psi}{\cos^2\psi'},$$

$$v_2' = u + \frac{l}{\mu}.$$

The image is nearer the edge of the prism than the object, but its definition in the position of minimum deviation is as good as that produced by a plate of glass of corresponding thickness (*i.e.* l). This result is of fundamental importance in spectrum analysis, for if the prism of the spectroscope be placed in the position of minimum deviation, it is possible to obtain a definite image of the slit, and hence a pure spectrum. With the prism in any other position the coloured images of the slit will overlap and consequently the spectrum will be impure. When a collimating lens is used and is properly adjusted, it has the effect of virtually removing the slit to infinity, as the light coming from the slit after traversing the collimator presents a plane wave-front. In that case $u = \infty$ and the distance $v_2' - v_1'$ becomes practically negligible. Thus when a collimator is used and the prism or train of prisms is not in the position of minimum deviation, the effect will be to increase the breadth of the lines and to proportionally increase the length of that part of the spectrum.

If the prism is so thin that its thickness may be neglected, $v_1' = u = v_2'$ approximately, and we may say that both image and object lie at the same perpendicular distance from the principal plane of the prism.

From the consideration of the primary and secondary foci of thin oblique pencils we naturally pass to the combination of a thin direct pencil with pencils of varying degrees of obliquity, in other words to the locus of intersection of all the refracted rays from a wide pencil. The caustic curve formed by the refraction of the wide pencil AOP will be the locus of F_1 for all the small pencils of which AOP is composed (Fig. 63). The locus of F_2 is of course on the normal OA. We will first consider the contour or shape of the refracted wave-front.

Contour of the Refracted Wave-front[1]. Let S be a luminous point, LI the refracting surface, SI any incident ray. Draw SD perpendicular to the surface and produce it to S', making DS' equal to SD. Describe a circle about $SS'I$, and produce the refracted ray IR backwards to meet

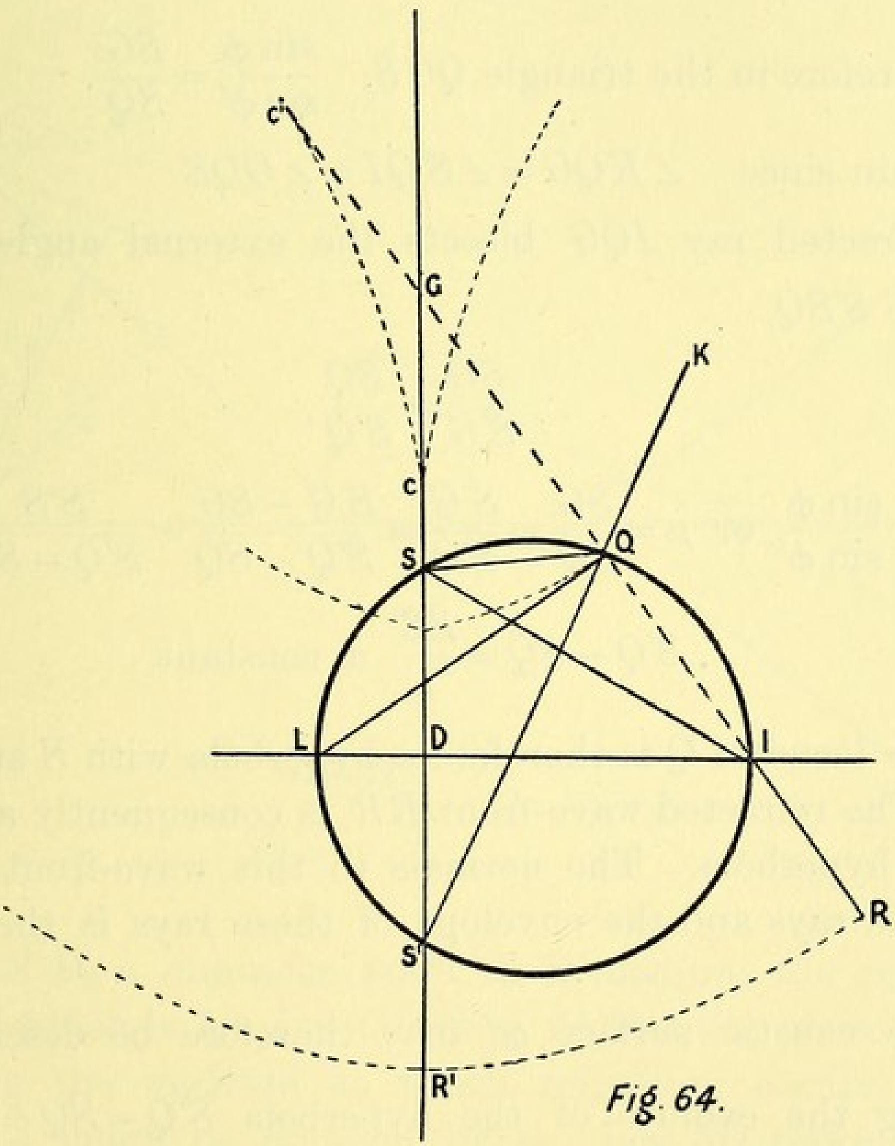

Fig. 64.

[1] The method adopted is taken from H. M. Preston's *Theory of Light.*

the circle at Q, and the normal $S'S$ in G. Through Q draw $S'QK$, and join QS, QL.

Then since LI bisects SS' at right angles, LI is a diameter of the circle and the angle LQI in the semicircle is a right angle,

$$\angle GQL = \angle LQI.$$

And since

$$SD = DS', \text{ arc } SL = \text{arc } LS' \text{ and } \angle SQL = \angle LQS',$$

$$\therefore \ \angle GQS = \angle S'QI = \angle S'SI = \phi,$$

the angle of incidence.

And $\angle QGS = \phi'$, the angle of refraction.

Therefore in the triangle QGS, $\dfrac{\sin\phi}{\sin\phi'} = \dfrac{SG}{SQ}$.

Again since $\angle KQG = \angle S'QI = \angle GQS$

the refracted ray IQG bisects the external angle of the triangle $S'SQ$,

$$\therefore \frac{SG}{S'G} = \frac{SQ}{S'Q},$$

$$\therefore \frac{\sin\phi}{\sin\phi'} \text{ or } \mu = \frac{SG}{SQ} = \frac{S'G}{S'Q} = \frac{S'G - SG}{S'Q - SQ} = \frac{S'S}{S'Q - SQ},$$

$$\therefore S'Q - SQ = \frac{SS'}{\mu} \text{ a constant.}$$

The locus of Q is therefore a hyperbola with S and S' for foci. The refracted wave-front RR' is consequently a parallel to this hyperbola. The normals to this wave-front are the refracted rays and the envelope of these rays is the caustic surface.

The caustic surface cc' may therefore be described by drawing the evolute of the hyperbola $S'Q - SQ = \dfrac{SS'}{\mu}$ and then causing the figure to revolve about the axis SS'.

Similarly it may be shewn that if the second medium be less refractive than the first,

$$SQ + S'Q = \frac{SS'}{\mu} \text{ a constant.}$$

The locus of Q is therefore an ellipse, and the caustic surface is the surface traced out by imagining the evolute of this ellipse to revolve round the axis SS'.

Caustics. The caustic curve produced by refraction at a plane surface can also be directly traced by employing a different method.

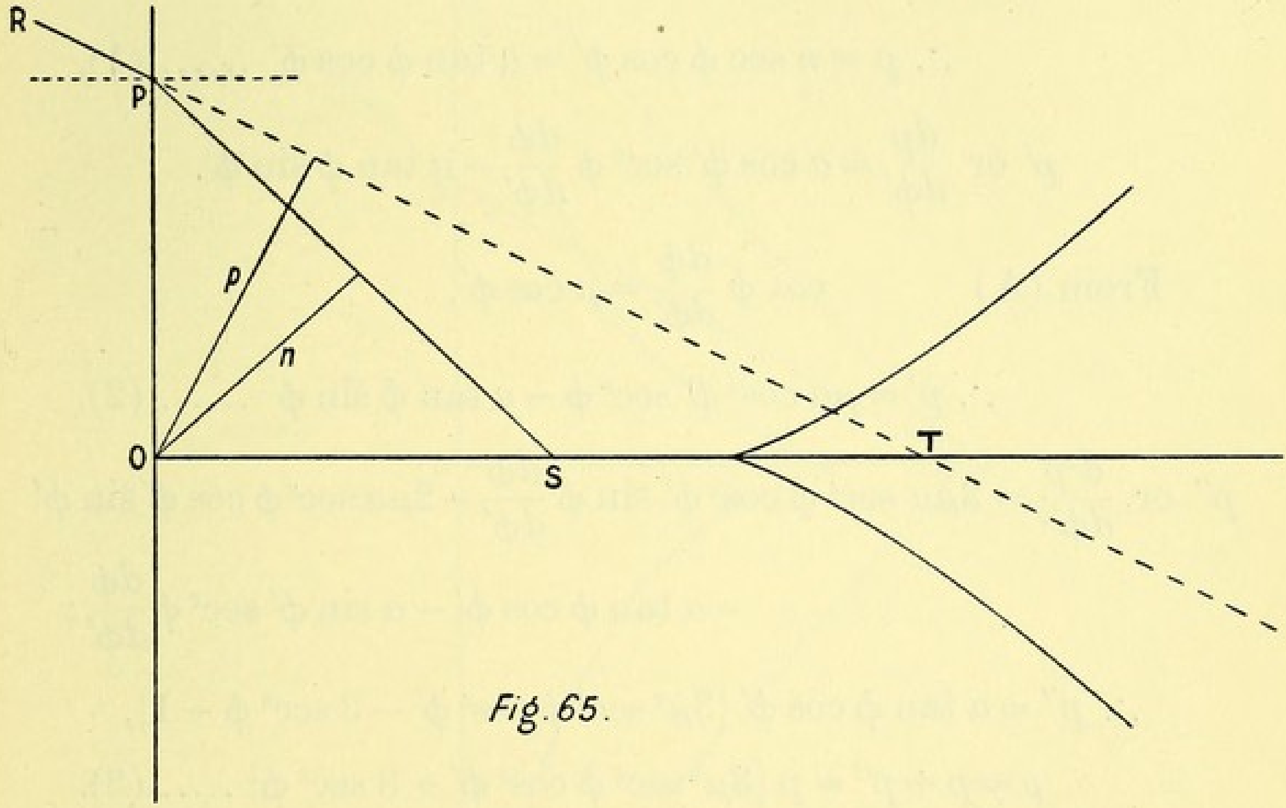

Fig. 65.

Let S be a luminous point in a medium the refractive index of which is 1. Let OP (Fig. 65) be the plane surface bounding the medium at which refraction occurs; let its refractive index be denoted by μ. Let SO represent the principal axis of the divergent pencil of rays issuing from S.

Then SO is normal to the plane surface OP. Let SO be denoted by a. Let SP represent an incident ray, PR the corresponding refracted ray. From O let fall the perpendiculars n and p upon the incident and refracted rays respectively.

Then the angle OTP or ψ which the refracted ray makes with the axis is equal to ϕ', the angle of refraction. Similarly the angle OSP is equal to the angle of incidence ϕ.

Now $$\sin\phi = \mu \sin\phi' \text{ .... (A)},$$

$$n = a\sin\phi,$$

$$OP = n \operatorname{cosec}\left(\frac{\pi}{2} - \phi\right) = p \operatorname{cosec}\left(\frac{\pi}{2} - \phi'\right) = n\sec\phi = p\sec\phi';$$

$$\therefore p = n\sec\phi\cos\phi' = a\tan\phi\cos\phi' \text{ ...... (1)},$$

$$p' \text{ or } \frac{dp}{d\phi'} = a\cos\phi'\sec^2\phi\frac{d\phi}{d\phi'} - a\tan\phi\sin\phi'.$$

From (A) $$\cos\phi\frac{d\phi}{d\phi'} = \mu\cos\phi',$$

$$\therefore p' = \mu a\cos^2\phi'\sec^3\phi - a\tan\phi\sin\phi' \text{ ...... (2)},$$

$$p'' \text{ or } \frac{d^2p}{d\phi'^2} = 3\mu a\sec^4\phi\cos^2\phi'\sin\phi\frac{d\phi}{d\phi'} - 2\mu a\sec^3\phi\cos\phi'\sin\phi'$$
$$- a\tan\phi\cos\phi' - a\sin\phi'\sec^2\phi\frac{d\phi}{d\phi'};$$

$$\therefore p'' = a\tan\phi\cos\phi'\{3\mu^2\sec^4\phi\cos^2\phi' - 3\sec^2\phi - 1\},$$

$$\rho = p + p'' = p\{3\mu^2\sec^4\phi\cos^2\phi' - 3\sec^2\phi\} \text{ ......(3)}.$$

Let $$\mu = 1\cdot5.$$

When $$\phi = 0,$$

$$\phi' = 0, \quad p = 0, \quad p' = \mu a, \quad \rho = 0.$$

There is a virtual cusp on the principal axis distant μa from O.

When $$\phi = \frac{\pi}{4},$$

$$\phi' = \sin^{-1}\frac{1}{\sqrt{2}\mu} \text{ say } 28^\circ\, 7'\cdot 53, \quad p = a\cos\phi' = \cdot 882\ldots a,$$

$$p' = 2\sqrt{2}\mu a\cos^2\phi' - a\sin\phi' = 2\cdot 8295\ldots a,$$

$$\rho = p\,\{12\mu^2\cos^2\phi' - 6\} = 15\cdot 0069\ldots p = 13\cdot 237\ldots a.$$

The adjoining figure (Fig. 66) represents the caustic formed by the refraction of a luminous point in a dense

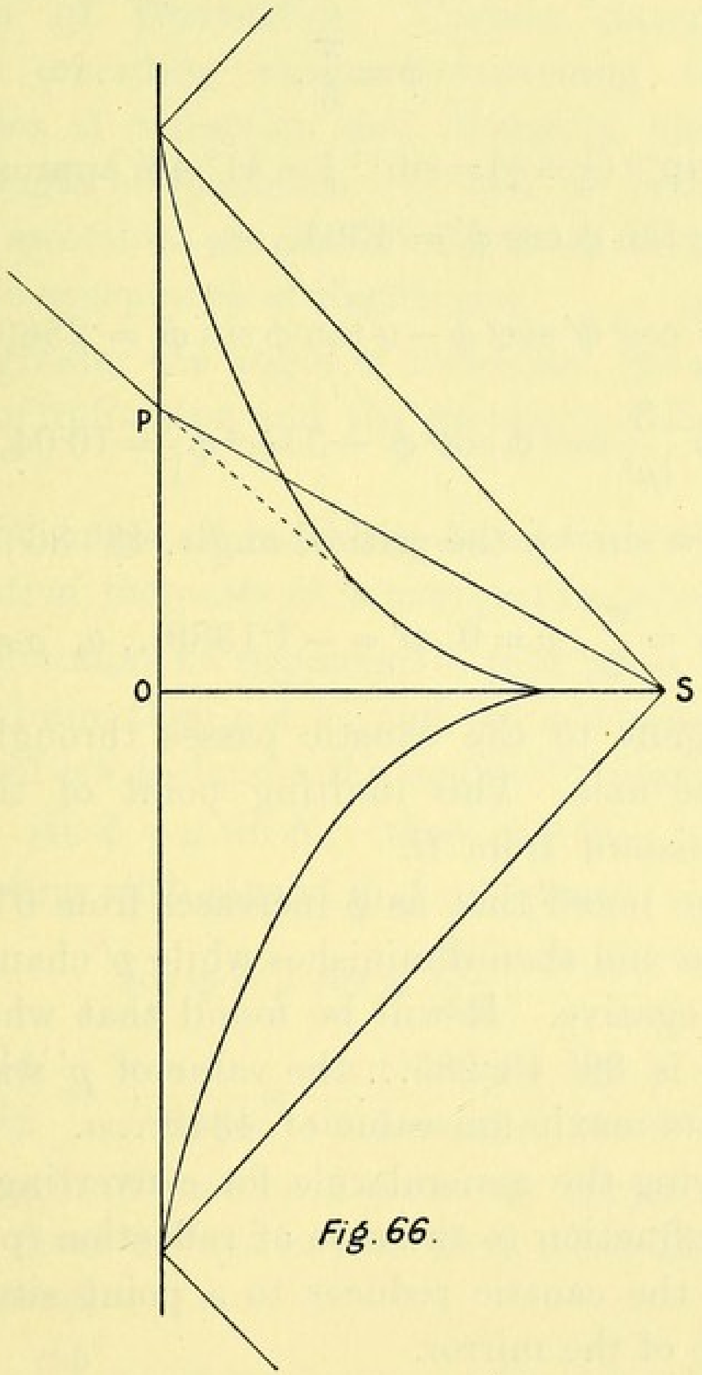

Fig. 66.

medium (*e.g.* water $\mu = \frac{4}{3}$). The line OP represents the bounding surface of the rare medium the refractive index of which is 1. The refractive index of the second medium is therefore $\frac{1}{\mu}$ of that of the first.

When $$\phi = 0,$$

$$\phi' = 0,\ \rho = 0,\ p' = \frac{a}{\mu},\ \rho = 0.$$

There is a cusp on the principal axis $\frac{3}{4}a$ from O.

When $$\phi = \frac{\pi}{6},$$

$$\phi' = \sin^{-1}(\tfrac{1}{2} \div \tfrac{3}{4}) = \sin^{-1}\tfrac{2}{3} = 41^\circ\ 48' \text{ approximately},$$

$$p = a \tan\phi \cos\phi' = \cdot 4304 \ldots a,$$

$$p' = \frac{a}{\mu}\cos^2\phi' \sec^3\phi - a\tan\phi\sin\phi' = \cdot 2569 \ldots a,$$

$$\rho = p\left\{\frac{3}{\mu^2}\sec^4\phi\cos^2\phi' - 3\sec^2\phi\right\} = 10{\cdot}04 \ldots a.$$

When $\phi = \sin^{-1}\frac{3}{4}$ the critical angle, $48^\circ\ 36'$,

$$\phi' = \frac{\pi}{2},\ p = 0,\ p' = -1{\cdot}1339 \ldots a,\ \rho = \infty.$$

The tangent to the caustic passes through O at right angles to the axis. This limiting point of the caustic is $1{\cdot}1339 \ldots a$ distant from O.

It will be noted that as ϕ increases from 0° to $48^\circ\ 36'$, p first increases and then diminishes while p' changes sign from positive to negative. It will be found that when the angle of incidence is $39^\circ\ 43'{\cdot}285\ldots$ the value of p' will be 0, while p will have its maximum value of $\cdot 4349 \ldots a$.

On applying the general rule for converting all formulæ relating to refraction to the case of reflection (p. 219), it will be seen that the caustic reduces to a point situated on the negative side of the mirror.

For on giving the values -1 to μ and $-\phi$ to ϕ',

(1) $p = a \tan\phi \cos(-\phi) = +a \sin\phi$,

(2) $p' = -a\cos^2(-\phi)\sec^3\phi - a\tan\phi\sin(-\phi)$

$$= -\frac{a}{\cos\phi}(1-\sin^2\phi) = -a\cos\phi,$$

(3) $\rho = p\{3\sec^4\phi\cos^2(-\phi) - 3\sec^2\phi\} = 0.$

A well-defined virtual image will be therefore formed of the point $-a$ from O whatever be the angle of incidence.

Variation of Deviation. Certain statements were made in the preceding chapter concerning the changes that the angles of refraction and deviation underwent on varying the angle of incidence. It will be found that the two following assertions are universally true; they are indeed involved in the acceptance of Snell's law.

(1) The greater the angle of incidence, the greater will be the angle of refraction and the greater will be the angle of deviation.

(2) As the angle of incidence increases uniformly, the angle of deviation increases at a continually increasing rate.

Let the refraction be supposed to take place from a rare to a dense medium, then $\mu < 1$; and let ϕ denote the angle of incidence and let ϕ' denote the angle of refraction,

$$\sin\phi = \mu\sin\phi'; \quad \text{then } \phi > \phi'.$$

Differentiating with regard to ϕ we obtain

$$\cos\phi = \mu\cos\phi'\frac{d\phi'}{d\phi},$$

$$\therefore \frac{d\phi'}{d\phi} = \frac{\cos\phi}{\mu\cos\phi'} = \frac{\sin\phi'\cos\phi}{\sin\phi\cos\phi'} = \frac{\tan\phi'}{\tan\phi}.$$

Now the angle of incidence ϕ cannot exceed 90°, consequently $\tan\phi'$ must always be less than $\tan\phi$ and bear the same sign as $\tan\phi$.

$\therefore \frac{\tan\phi'}{\tan\phi}$ or $\frac{d\phi'}{d\phi}$ must be always positive and less than 1.

The angle of deviation, we have observed, is given by the expression $\phi - \phi'$. And $\frac{d}{d\phi}(\phi - \phi') = 1 - \frac{d\phi'}{d\phi}$, which is always positive and less than 1.

Therefore the greater the angle of incidence, the greater will be the angle of refraction and the greater will be the angle of deviation. In this case, when $\mu > 1$ the increment in the angle of incidence is always greater than the corresponding increments in the angles of refraction and deviation, for both $\frac{d\phi'}{d\phi}$ and $\frac{d}{d\phi}(\phi - \phi')$ are less than 1.

However as

$$\frac{d^2\phi'}{d\phi^2} \text{ or } \frac{d}{d\phi}(\cot\phi\tan\phi') = \cot\phi\sec^2\phi'\frac{d\phi'}{d\phi} - \operatorname{cosec}^2\phi\tan\phi'$$

$$= \cot^2\phi\sec^2\phi'\tan\phi' - \operatorname{cosec}^2\phi\tan\phi'$$

$$= \operatorname{cosec}^2\phi\tan\phi'\left(\frac{\cos^2\phi}{\cos^2\phi'} - 1\right),$$

and $\frac{\cos^2\phi}{\cos^2\phi'}$ must be positive and less than 1,

$$\therefore \frac{d^2\phi'}{d\phi^2} \text{ is negative.}$$

Therefore as ϕ increases, the rate of increase of ϕ' decreases.

Again, $\frac{d^2}{d\phi^2}(\phi - \phi') = -\frac{d^2\phi'}{d\phi^2}$ and is therefore positive.

Therefore as ϕ increases the rate of increase of the deviation increases.

In other words, if the angle of incidence increases uniformly, the angle of deviation increases faster and faster.

The table given on p. 142 illustrates these statements in the special case where $\mu = 1{\cdot}54$.

If $\mu < 1$ it is evident that $\phi < \phi'$ and that the deviation $\phi - \phi'$ is negative, or measured in the reverse direction. It follows that in this case $\tan\phi < \tan\phi'$ and that $\frac{d\phi'}{d\phi} > 1$.

Consequently if ϕ increase by a certain amount, ϕ' increases by a still greater amount, and the negative quantity $\phi - \phi'$ increases also.

Again, in this case $\frac{d^2\phi'}{d\phi^2}$ is positive, and therefore $\frac{d\phi'}{d\phi}$ increases with ϕ, or in other words, as ϕ increases uniformly, ϕ' increases at a continually increasing rate, while the negative quantity $\phi - \phi'$ increases also faster and faster.

QUESTIONS.

(1) A small pencil of light, which is obliquely refracted through a plate of thickness w, is received by the eye. The angle of incidence being $\tan^{-1}\mu$, shew that the distance between the original point of light and the secondary focal line after emergence is $\frac{\mu^2 - 1}{\mu^2} w$.

(2) Shew why a straight stick that is partly in and partly out of water appears bent at the surface of the water, when viewed obliquely; further, why does the part that is in the water appear curved?

(3) A small white pebble at the bottom of a pool of water 10 ins. deep is seen by an observer standing on the margin. If the refractive index of the water is $\frac{4}{3}$ and the angle of incidence be 30°, shew that the distance between the focal lines is $\frac{35\sqrt{3}}{27}$ ins.

(4) In the last example the cross section of the astigmatic pencil that enters the eye has the following dimensions at the point of emergence: in the primary plane ·0047 ins., in the secondary plane ·0062 ins. Find the position and size of the circle of least confusion.

The secondary plane is that which contains the axis of the emergent pencil and the first focal line. Use expressions (1) and (2), p. 113.

CHAPTER X.

REFRACTION AT A SPHERICAL SURFACE. CONJUGATE FOCAL DISTANCES. SIZE OF IMAGE.

Concave Spherical Surface. When a widely divergent pencil of rays undergoes refraction at a spherical surface, its constituent refracted rays do not intersect in a single point, but they touch a certain caustic surface, somewhat similar to that which occurs when reflection takes place at a spherical surface. If however a point be taken on the principal axis of the refracting surface, and if we confine our attention to a very small centric pencil of rays incident on the surface, it will be found that after refraction the pencil will converge to a single point, or at any rate proceed as if it had arisen from a single point.

(1) Let the refracting medium ($\mu > 1$) be bounded by a concave spherical surface KA (Fig. 67), the centre of which is at C; and let P be a luminous point on the principal axis PAC. On the surface take a point K near to the vertex A so that the angle KPA is very small. The ray PK may then be regarded as an extreme ray of the small axial pencil incident on the spherical surface. From centre C draw the normal CK and let QKR represent the direction of the

refracted ray, which being produced backwards meets the principal axis in Q.

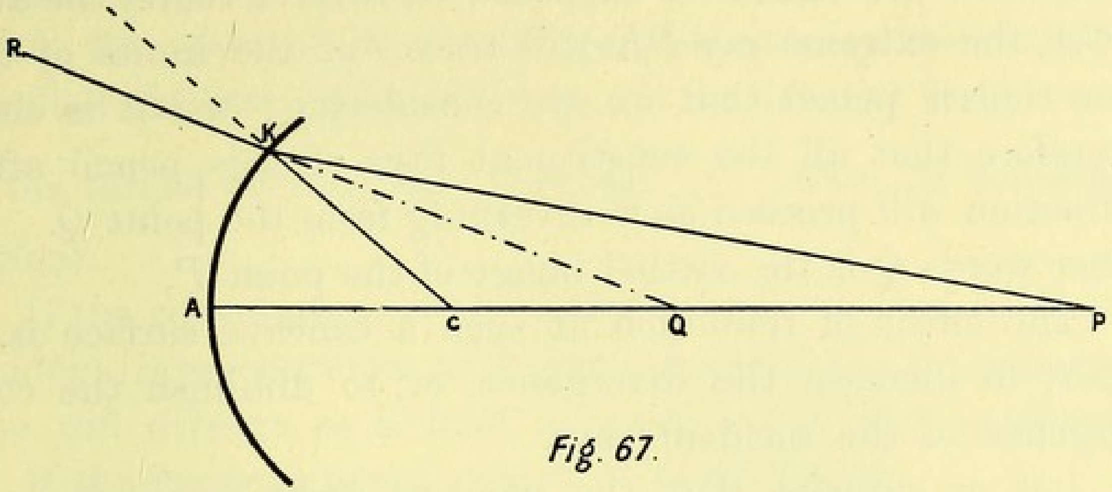

Fig. 67.

Then $\angle CKP$ is the angle of incidence ϕ, and $\angle CKQ$ is equal to the angle of refraction ϕ', and in the $\triangle PKC$,

$$\frac{PC}{CK} = \frac{\sin\phi}{\sin KPC},$$

and in the $\triangle CKQ$,

$$\frac{QC}{CK} = \frac{\sin\phi'}{\sin KQC}.$$

By dividing the first expression by the second, we get

$$\frac{PC}{QC} = \frac{\sin\phi}{\sin\phi'} \cdot \frac{\sin KQC}{\sin KPC}.$$

But $\sin KQC = \sin KQP$ and $\dfrac{\sin KQP}{\sin KPQ} = \dfrac{PK}{QK}$,

$$\therefore \frac{PC}{QC} = \mu \frac{PK}{QK}.$$

Now since K is very near to A, the distances PA and QA, or p and q, may be substituted for PK and QK,

$$\therefore \frac{p-r}{q-r} = \mu \frac{p}{q} \text{ approximately},$$

r denoting the radius CA or CK,

$$i.e.\ \mu pr - rq = pq(\mu - 1).$$

Dividing by pqr we obtain $\frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r}$.

If now the figure be supposed to revolve round the axis PCA, the extreme ray PK will trace out the limits of the thin centric pencil that we are considering; and it is clear therefore that all the constituent rays of this pencil after refraction will proceed as if diverging from the point Q. In other words Q is the virtual image of the point P.

The effect of refraction at such a concave surface is, if $p > r$, to increase the divergence, or to diminish the convergence of the incident rays.

Let us suppose that the incident light is convergent; after refraction its rays may be slightly convergent, parallel or even divergent, according to the degree of convergence of the incident pencil.

If the refracted rays are parallel (Fig. 68), $q = \infty$, so

$$\frac{\mu}{q} = 0,$$

and
$$-\frac{1}{p'} \text{ or } -\frac{1}{f'} = \frac{\mu - 1}{r},$$

$$\therefore \; f' = -\frac{r}{\mu - 1}.$$

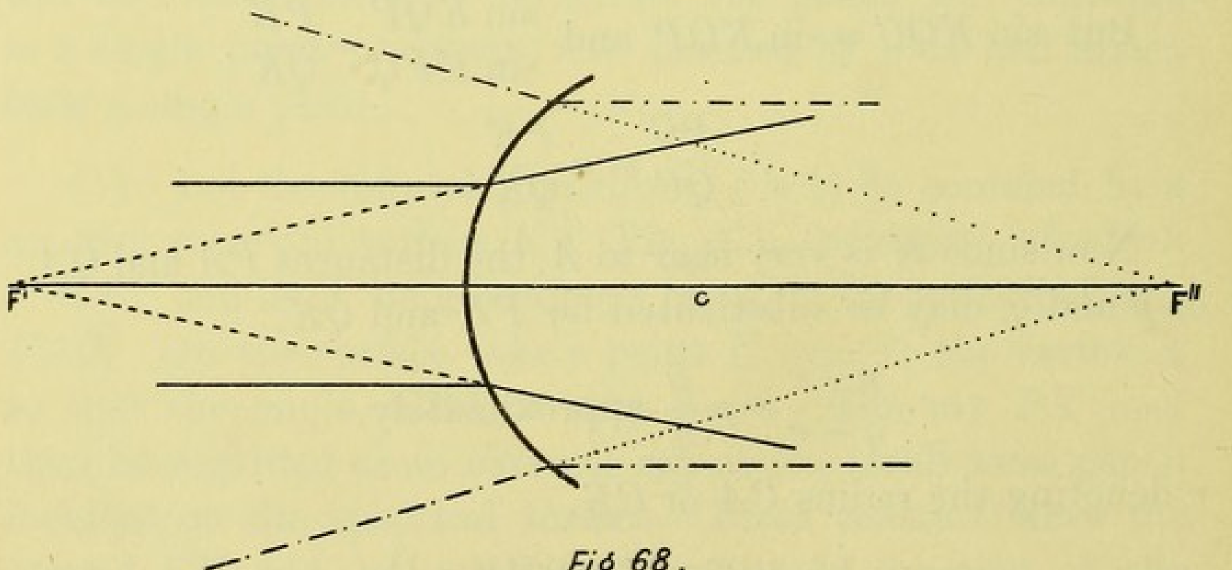

Fig. 68.

The negative sign shews that the distance f' is to be measured on that side of the refracting surface which is remote from the incident light. The point (F') towards which the incident rays converge under these circumstances is called the first principal focus. It is situated in this case in the second medium at a distance $\frac{r}{\mu - 1}$ from its bounding surface.

If the convergence of the incident light be less, *i.e.* if the incident rays converge to a point beyond F', the refracted rays will diverge as if from a point in the first medium.

If the incident pencil consist of parallel rays,

$$p = \infty \text{ and } \frac{1}{p} = 0,$$

and
$$\frac{\mu}{q} \text{ or } \frac{\mu}{f''} = \frac{\mu - 1}{r},$$

or
$$f'' = \frac{\mu r}{\mu - 1}.$$

If then the incident rays are parallel, they will after refraction diverge as if they had proceeded from a point F'' in the first medium situated $\frac{\mu}{\mu - 1} r$ from the vertex.

Now let the incident light diverge from a point P on the axis in front of the refracting surface. As P approaches C from infinity, its image Q moves from F'' up to C.

On reaching C the incident rays from P are normal to the surface, and therefore traverse the second medium without refraction. This is also evident from the formula

$$\frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r},$$

if
$$p = r; \; q = r.$$

As P moves onwards from C to A, Q also moves in the same direction at a gradually increasing rate, so that when P reaches A, Q is also coincident with A.

The distinction between the two focal distances should be carefully noted. It is evident that

$$f'' = -\mu f'$$

and that $$-(\mu - 1)f' = r,$$

that is, the numerical difference between the first and second principal focal distances is equal to the numerical value of the radius of curvature.

When the refracting surface ($\mu > 1$) is concave, the image of an actual object is always virtual and situated in the first medium. The formula therefore for the concave refracting surface may be conveniently regarded as the fundamental formula, as all the quantities p, q, and r as well as f'' lie on the same side of the refracting surface, and consequently have positive values. The formula may be applied to refraction at any spherical surface by paying due regard to the signs borne by these symbols when the conditions are changed. That is to say, when numerical values are substituted for the symbols they must be preceded by the negative sign, whenever they refer to distances of points in the second medium, distances in fact that lie on the side of the refracting surface remote from the object.

Convex Spherical Surface. If the surface be convex, r becomes negative. Two cases however now arise, for the image may be either virtual or real.

If q be positive, Q is virtual, for Q lies on the same side of the refracting surface as P.

If q be negative, Q is real, for Q now lies in the second medium.

This ready method of treating refraction at a convex spherical surface may however appear somewhat unsatisfactory to the reader, it may be advantageous therefore to discuss the subject in greater detail.

If then the refractive index is less than unity the preceding method of proof is applicable, for if RK be regarded as a ray incident on a convex spherical surface bounding a rare medium ($\mu < 1$), KP may be taken to represent its course in this medium after refraction.

If however the refractive index be greater than unity a slight modification in the proof will be necessary.

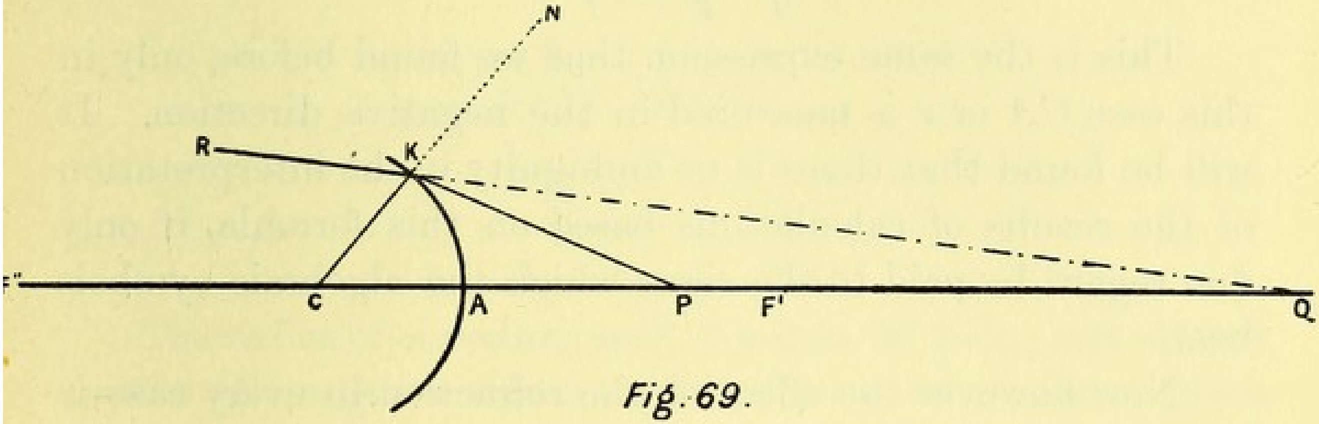

Fig. 69.

As before (Fig. 69) let P represent the luminous point and PK the extreme ray of the thin centric pencil considered, and let KR represent its course after refraction.

Then if RK produced meet the axis in Q, Q is the virtual image of P.

Now $\phi = \angle NKP$, $\therefore \sin\phi = \sin NKP = \sin CKP$,

and $\phi' = \angle NKQ$, $\therefore \sin\phi' = \sin NKQ = \sin CKQ$.

As before

$$\frac{PC}{CK} \div \frac{QC}{CK} = \frac{\sin\phi}{\sin KPC} \div \frac{\sin\phi'}{\sin KQC} = \mu\frac{\sin KQP}{\sin KPQ},$$

$$\therefore \frac{PC}{QC} = \mu\frac{PK}{QK},$$

or

$$\frac{PA + AC}{QA + AC} = \mu\frac{PK}{QK}.$$

As before let us denote PA, QA and CA by p, q and r.

Now as K approaches A, the distances PK and QK differ finally from PA and QA by a negligible quantity.

So in the limit we have

$$\frac{PA+AC}{QA+AC} \text{ or } \frac{PA-CA}{QA-CA}=\mu\frac{PA}{QA},$$

or
$$\frac{p-r}{q-r}=\mu\frac{p}{q},$$

i.e.
$$\frac{\mu}{q}-\frac{1}{p}=\frac{\mu-1}{r}.$$

This is the same expression that we found before, only in this case CA or r is measured in the negative direction. It will be found that there is no ambiguity in the interpretation of the results of calculations based on this formula, if only due regard be paid to the signs which the algebraic symbols bear.

Now however the effect of the refraction in every case is to render the incident rays more convergent. Consequently incident parallel rays converge to a point F'' in the second medium, whereas F' the point, from which those rays diverge, that after refraction become parallel, is situated in the first medium. We see then that f' is now positive and that f'' is negative.

The expression for the distances of the conjugate foci may also be put in another form which is sometimes convenient.

From
$$\frac{\mu}{q}-\frac{1}{p}+\frac{1}{f'}=0$$

we get
$$\frac{\mu f'}{q}-\frac{f'}{p}=-1,$$

or
$$\frac{f'}{p}+\frac{f''}{q}=1,$$

$$\therefore\ pq-pf''-qf'=0,$$

and on adding $f'f''$ to both sides of this equation we get

$$(p-f')(q-f'')=f'f''.$$

Ex. 1. What curvature must be given to the bounding surface of a refracting medium ($\mu=\frac{4}{3}$) for the virtual image of an object in the adjacent medium ($\mu=1$) at 4 ins. distance to be formed at a distance of 16 ins.?

As the image is virtual, it must be formed in the first medium, or in other words, q is positive.

And since $$\frac{\mu}{q}-\frac{1}{p}=\frac{\mu-1}{r},$$

$$\frac{4}{3\times 16}-\frac{1}{4}=\frac{1}{3r},$$

$$i.e.\ \frac{1}{3r}=-\frac{2}{12}.$$

$$\therefore\ r=-2 \text{ ins.}$$

The radius of curvature must therefore be 2 ins., and since it carries the negative sign it must be measured in the negative direction. The surface must therefore be convex.

Ex. 2. A refracting medium ($\mu=\frac{4}{3}$) is bounded by a convex spherical surface, its radius of curvature being 5 mm.: find the foci.

As the surface is convex, $r=-5$ mm.,

and since $$f'=-\frac{r}{\mu-1},$$

$$f'=-\frac{(-5)}{\frac{4}{3}-1} \text{ or } 15 \text{ mm.}$$

Also since $$f''=\frac{\mu r}{\mu-1},$$

$$f''=\frac{\frac{4}{3}(-5)}{\frac{4}{3}-1} \text{ or } -20 \text{ mm.}$$

The second principal focus (F'') therefore lies 20 mm. behind the refracting surface, while the first principal focus (F') lies 15 mm. in front of it.

It will be subsequently shewn that the complex optical system of the eye with its various refracting media may be roughly represented by a simple system consisting of a single refractive medium ($\mu = \frac{4}{3}$) bounded by a convex spherical surface, representing the cornea, with a radius of curvature of 5 mm. It would appear then that if the retina be 20 mm. behind the cornea, a fairly definite image of a distant object would be formed upon it. The first principal focus of this "reduced eye," as it is called, is situated 15 mm. in front of the cornea.

Formation of Images. The images formed by refraction at a single spherical surface may be either virtual or real. Virtual images are situated on the same side of the refracting surface as the object, that is they are in the first medium, whereas real images are situated in the second medium.

The geometrical construction for the image formed by refraction is very similar to that which we used for the image formed by reflection.

Let AB (Fig. 70) represent the object situated on the principal axis BCO which passes through C, the centre of the spherical surface, and let HOH' represent the principal plane, *i.e.* the plane touching the spherical surface at O where the axis cuts it. Let F' be the first principal focus, and let F'' be the second principal focus of the refracting system. From what has gone before it is clear that if the refracting surface be convex, and $\mu > 1$ (Fig. 71), F' lies on the object-side or in front of the surface and F'' lies behind it. If however the refracting surface be concave (Fig. 70), F' lies behind the surface, and F'' lies in front of it.

Two methods may be employed for determining the image of a point which is not on the principal axis.

For example, to find the image of the point A we may draw the ray AH parallel to the axis meeting the principal plane in H, then $F''H$ will be the corresponding refracted ray. Join AC and produce it if necessary to meet $F''H$

produced in a. Then a is image of the point A. (See Figs. 70, 71, 72.)

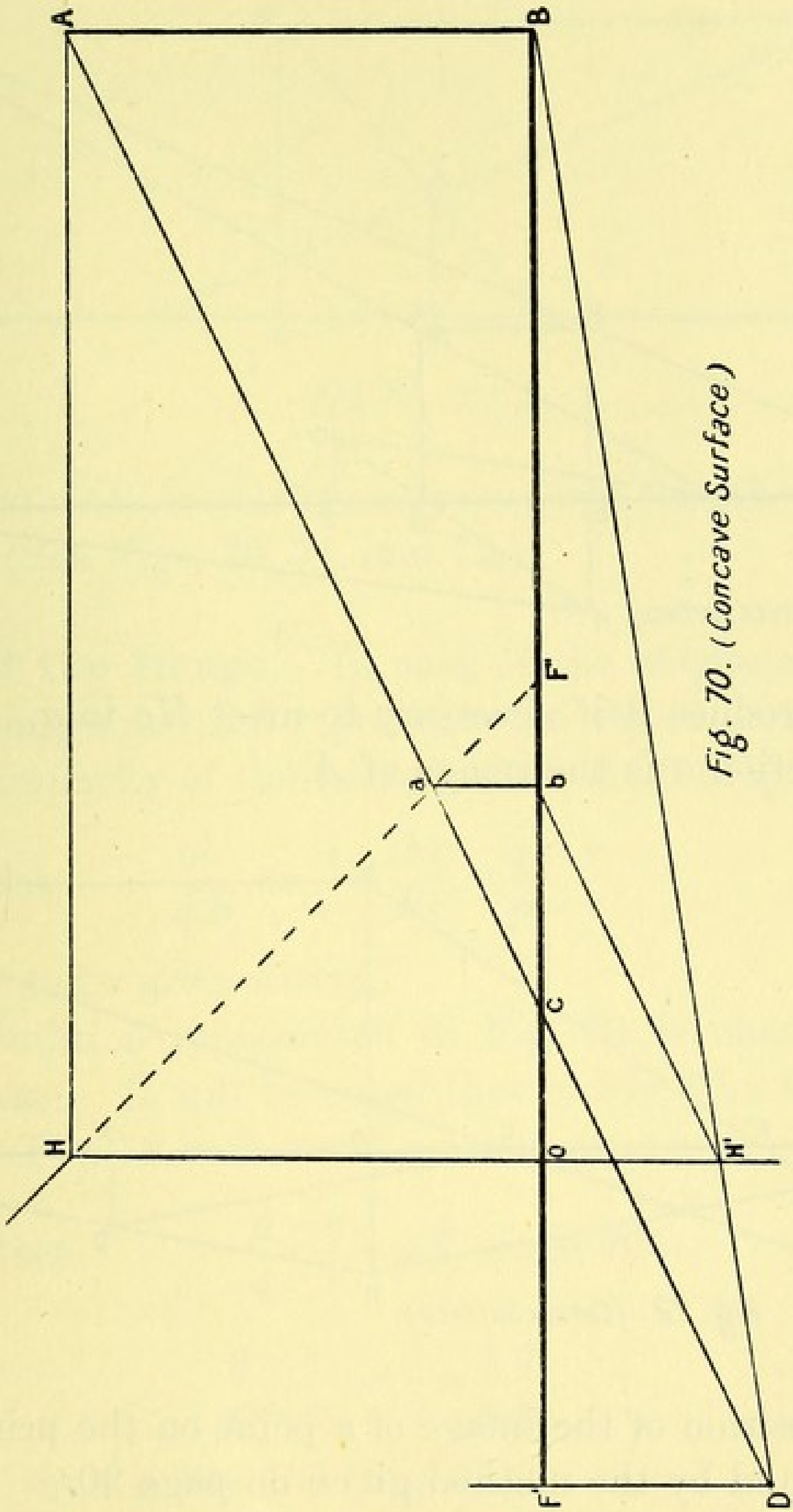

Fig. 70. (Concave Surface)

Or we may adopt the method employed in (Fig. 73).

Through A draw the ray $F'A$ passing through the first principal focus, and produce it to cut the principal plane at

H. Then the line *Ha* parallel to the axis represents the direction of the corresponding refracted ray. As before, join

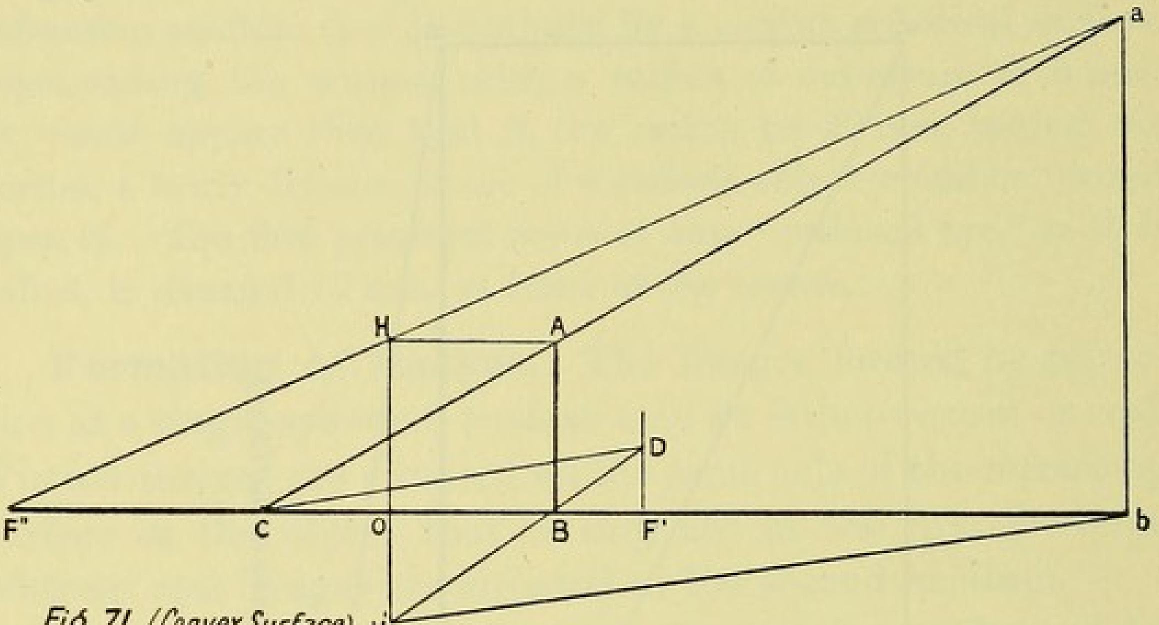

Fig. 71. (Convex Surface)

AC and produce it if necessary to meet *Ha* in *a*. The point of intersection *a* is the image of *A*.

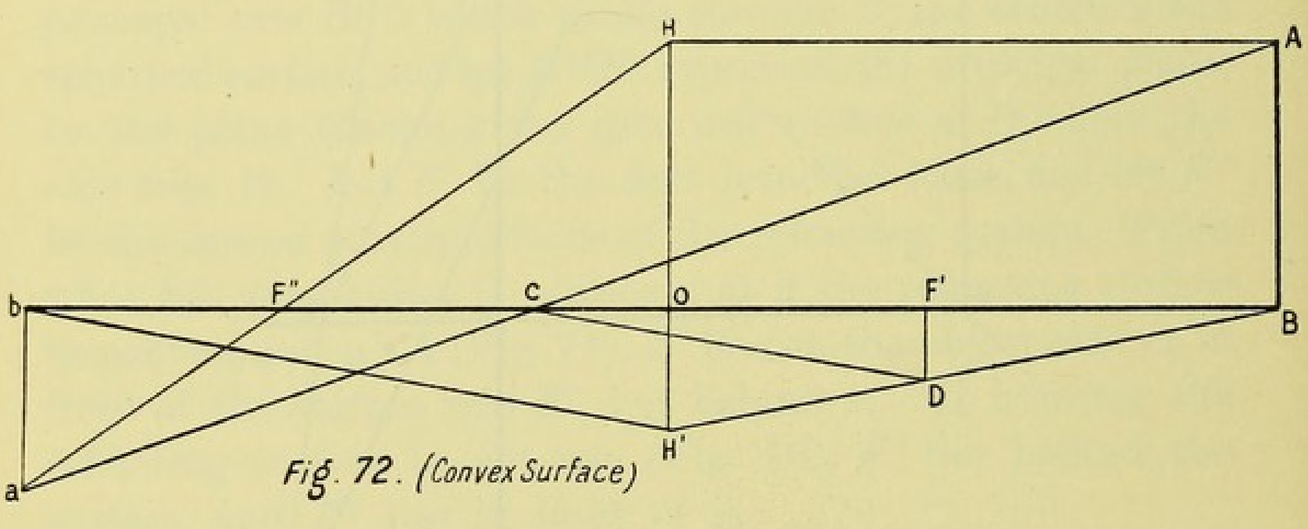

Fig. 72. (Convex Surface)

The position of the image of a point on the principal axis may be found by the method given on page 90.

If *B* is the point on the axis, take any ray *BD* cutting the first focal plane in *D*, and the principal plane in *H'*. Join *BC*, and through *H'* draw *H'b* parallel to *DC* cutting the principal axis in *b*. Then *b* is the image of *B*. For since D is a point in the first focal plane, the pencil of rays from D incident on the principal plane HH' will after refrac-

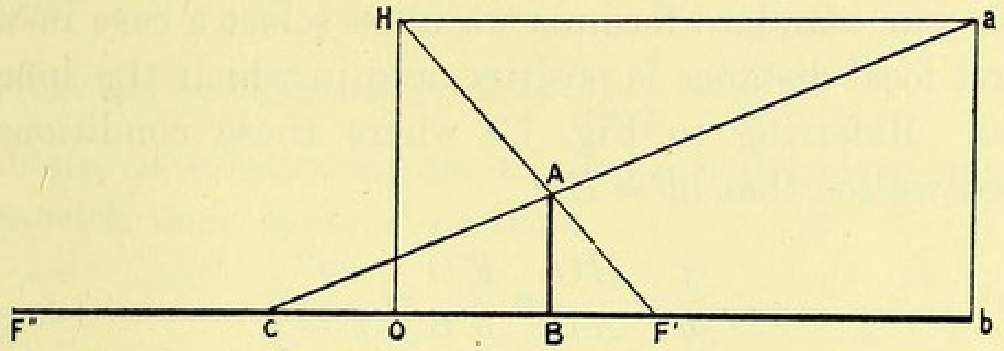

Fig. 73.

tion proceed with its constituent rays parallel to the secondary axis DC. (See Figs. 70, 71, and 72.)

Size of the Image. In each of the diagrams ab represents the image and AB represents the object, and it is clear from the similarity of the triangles acb, ACB (Fig. 70),

(1) that $$\frac{ab}{AB} \text{ or } \frac{i}{o} = \frac{bc}{BC} = \frac{q-r}{p-r},$$

where p, q and r are positive.

The condition represented in Fig. 70 is chosen as the standard case. It will be noted that in Fig. 71 r is negative, whereas in Fig. 72 both r and q are negative.

Also since $$\frac{p-r}{q-r} = \mu \frac{p}{q} \text{ (page 205)};$$

(2) $$\frac{q-r}{p-r} \text{ or } \frac{i}{o} = \frac{1}{\mu} \frac{q}{p}.$$

Again, since $AB = HO$ (Fig. 70),

(3) $$\frac{i}{o} = \frac{ab}{HO} = \frac{F''b}{F''O} = \frac{f''-q}{f''}.$$

In Fig. 71 f'' is negative but in Fig. 72 both f'' and q are negative. We may also express the relation between the dimensions of the image and object in terms of the distance of the object and of the first principal focus.

For our standard formula we must select a case in which the first focal distance is positive, and in which the image is virtual. Referring to Fig. 73 where these conditions are fulfilled we see that $ab = HO$.

$$(4) \qquad \therefore \frac{i}{o} = \frac{HO}{AB} = \frac{F'O}{F'B} = \frac{f'}{f'-p}.$$

No difficulty will be encountered in using these formulæ if due attention is paid to the geometrical meaning of the signs borne by the symbols.

Distances on the object-side, or in front of the refracting surface, are considered positive, and distances behind the refracting surface are considered negative. Similarly all distances above the principal axis are positive, while those below the axis are negative.

If then the ratios,

$$\frac{q-r}{p-r}, \quad \frac{1}{\mu}\frac{q}{p}, \quad \frac{f''-q}{f''} \text{ or } \frac{f'}{f'-p}$$

have positive values, the ratio $\frac{i}{o}$ has also a positive value, and consequently the line indicating the height of the image is measured in the same direction as that indicating the height of the object, and similarly with its other dimensions: in other words the image is erect.

If these ratios have negative values the image is inverted, for the height and the other dimensions of the image are measured in the reverse direction to the corresponding dimensions of the object.

It can be easily seen that all virtual images are erect, and that all real images are inverted.

Ex. 1. An object 4 mm. in height is placed 150 mm. from a convex refracting surface ($\mu = \frac{4}{3}$), the radius of curvature of which is 5 mm. Is the image real or virtual? What is its size and position?

From the fundamental formula $\frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r}$

we obtain on substituting for the symbols the given numerical values with their appropriate signs

$$\frac{4}{3q} - \frac{1}{150} = \frac{\frac{4}{3} - 1}{(-5)} \text{ or } -\frac{1}{15},$$

$$\therefore \; q = -\tfrac{200}{9} \text{ or } -22 \cdot \dot{2} \text{ mm.}$$

The negative sign shows that the image is formed in the second medium, or in other words that it is real. It is then situated $22 \cdot \dot{2}$ mm. behind the refracting surface.

We may determine its size by applying the formula (1) or (2).

Thus by (2) $\frac{i}{o} = \frac{1}{\mu}\frac{q}{p}$,

$$\frac{i}{4} = \frac{3}{4}\left(\frac{-22 \cdot \dot{2}}{150}\right) = - \cdot \dot{1}.$$

$$\therefore \; i = - \cdot \dot{4} \text{ mm.}$$

The image is then $\cdot \dot{4}$ mm. in height, and the negative sign shews that it is inverted.

If the data had been different, one of the other formulæ might have been employed.

Ex. 2. The first principal focus is situated 15 mm. in front of the bounding (spherical surface) of the refracting medium. What is the size of the image that it forms of an object 4 mm in height and 150 mm. distant?

Since $\frac{i}{o} = \frac{f'}{f' - p}$,

$$\frac{i}{4} = \frac{15}{15 - 150} = -\frac{1}{9} \text{ or } - \cdot \dot{1},$$

$$\therefore \; i = - \cdot \dot{4} \text{ mm.}$$

It has been already stated that as far as refraction is concerned the eye may be roughly represented by a single refracting medium ($\mu = \frac{4}{3}$), bounded by a spherical surface with a radius of curvature of 5 mm. An eye then (with relaxed accommodation) would form the image of an object 150 mm. off at a point $22\cdot\dot{2}$ mm. behind the cornea. If we take 20 mm. as representing the distance between the cornea and the retina of "the reduced emmetropic eye," we see that the image on the retina of this object would be blurred unless accommodation were called into play.

If however a certain eye is $2\cdot\dot{2}$ mm. longer than normal, its punctum remotum is situated at a point 150 mm. from the cornea, or in other words, the eye is myopic to the extent of $6\cdot\dot{6}$ dioptres.

It will indeed be shewn subsequently that each dioptre of axial ametropia corresponds approximately to an error of one-third of a millimetre in the antero-posterior diameter of the eye.

Ex. 3. What curvature must be given to the bounding surface of a refracting medium ($\mu = \frac{4}{3}$), in order that an object 150 mm. in front of it may give rise to a real image 20 mm. behind it? If the object is 4 mm. in height, what is the size of the image?

Since
$$\frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r},$$
$$\frac{4}{3}\left(\frac{1}{-20}\right) - \frac{1}{150} = \frac{\frac{4}{3} - 1}{r},$$
$$\therefore -\frac{1}{15} - \frac{1}{150} = \frac{1}{3r},$$
$$\therefore r = -\tfrac{50}{11} \text{ mm. or } -4\tfrac{6}{11} \text{ mm.}$$

The negative sign shews that the surface must be convex.

The dimensions of the image are given by the formula
$$\frac{i}{o} = \frac{1}{\mu}\cdot\frac{q}{p},$$
$$\frac{i}{4} = \frac{3}{4}\left(\frac{-20}{150}\right) = -\frac{1}{10},$$
$$\therefore i = -\cdot 4 \text{ mm.}$$

If then an eye of normal length be bounded by a cornea, the convexity of which corresponds to that of the case just considered, its far point will lie at a distance of 150 mm. The curvature myopia of this eye is equivalent in degree to the axial myopia of the eye in *Ex.* 2. The two conditions do not however resemble each other in every respect; for the retinal image of the same object is in the one case ·4 mm. and in the other case ·4̇ mm. in height. Consequently in axial myopia smaller objects should be recognized when placed at the far point, than in curvature-myopia of equivalent degree. It follows therefore that the results of testing the acuteness of vision by such methods are not directly comparable in the two cases. No reliable conclusions for instance can be drawn as to the sensibility of the retina in the two cases, without making due allowance for the difference in size of the respective retinal images.

Helmholtz' formula. We may also express the dimensions of the image and of the object in terms of the divergence from the axis of any ray before and after its refraction.

Let us denote by α the angle OBH' of divergence of the ray BH' from the principal axis, and by α' the angle ObH' which the corresponding ray $H'b$ makes with the axis (Figs. 70 and 72).

Then $$\tan\alpha = \frac{H'O}{BO} \quad \text{and} \quad \tan\alpha' = \frac{H'O}{bO},$$

and $$\frac{\tan\alpha}{\tan\alpha'} = \frac{bO}{BO} = \frac{q}{p}.$$

But $$\frac{i}{o} = \frac{1}{\mu}\frac{q}{p},$$

$$\therefore \frac{i}{o} = \frac{1}{\mu} \cdot \frac{\tan\alpha}{\tan\alpha'}.$$

Relation of the formulæ of refraction to those of reflection. In the case of refraction, $\mu = \frac{V_1}{V_2}$, the ratio of the velocity of light in the first medium to that in the second medium. In the case of reflection the reflected light is travelling at the same velocity in the same medium but in the reverse direction.

Hence in the case of reflection $\mu = -1$.

And since the angle of reflection is measured on the opposite side of the normal to that of incidence $\phi = -\phi'$.

On making these changes any formula relating to refraction at a single spherical surface can be changed into the corresponding formula for reflection at a spherical surface.

Thus instead of $\frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r}$, we write in the case of reflection

$$\frac{-1}{q} - \frac{1}{p} = \frac{-2}{r} \text{ or } \frac{1}{p} + \frac{1}{q} = \frac{2}{r}.$$

Instead of

$$\frac{i}{o} = \frac{1}{\mu}\frac{q}{p} = \frac{f'' - q}{f''} = \frac{f'}{f' - p} = \frac{1}{\mu}\frac{\tan\alpha}{\tan\alpha'},$$

we have in the case of reflection

$$\frac{i}{o} = \frac{-q}{p} = \frac{f - q}{f} = \frac{f}{f - p} = -\frac{\tan\alpha}{\tan\alpha'}.$$

Instead of $D = \phi - \phi'$, we have in the case of reflection

$$D = \phi + \phi = 2\phi.$$

QUESTIONS.

(1) When does a concave refracting surface with $\mu > 1$ have a converging effect?

(2) What curvature must be given to the bounding surface of a refracting medium for the formation of a real image 20 mm. behind it, of an object 300 mm. in front of it, (i) when $\mu = \frac{4}{3}$, (ii) when $\mu = \frac{2}{3}$? (iii) Compare the sizes of the images.

(3) The second principal focus is situated 20 mm. behind a curved refracting medium; a real image $\frac{1}{10}$ of the height of the object is formed. Where is the image formed, and if the index of refraction is $\frac{4}{3}$ where is the object?

(4) A sphere of glass $\mu = \frac{3}{2}$ has a speck within it, halfway between the centre and the distal side. Where will its image be formed as seen from either side?

(5) If the back of the sphere be silvered, where will be the image that is formed by one reflection and one refraction? Consider both cases (i) when the surface near the speck is silvered, (ii) when the distant surface is silvered.

CHAPTER XI.

THIN LENSES. CONJUGATE FOCAL DISTANCES. SIZE OF IMAGE.

A SPHERICAL lens is a portion of a refracting medium bounded by two spherical surfaces; the straight line that joins their centre is called the axis of the lens. The distance between the bounding surfaces, measured along the axis, is called the thickness of the lens.

A lens bounded by two concave surfaces is called a double concave or a biconcave lens, whereas a lens bounded by two convex surfaces is called a double convex or biconvex lens. A lens of which one face is convex and the other concave is called a meniscus. The terms plano-convex and plano-concave scarcely require explanation; the plane face may be regarded as a spherical surface of which the radius is infinite.

Thin Lenses. The relation of the conjugate focal distances of a thin spherical lens can be easily determined from the formulæ that we have already found.

Let us consider, for example, a meniscus of glass, the surrounding medium being air, Fig. 74, and let the curvature of the first surface be greater than that of the second surface.

Let P be a luminous point on the axis of the lens, and let P face the concave surface of the lens. Under these circumstances the conjugate focal distances and the radii of

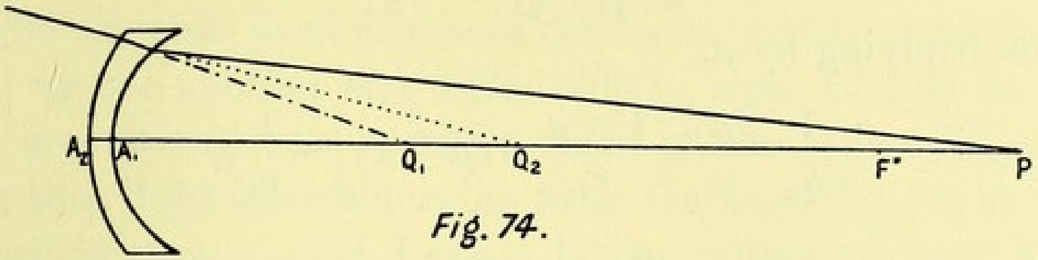

Fig. 74.

curvature of the two faces of the lens will all be on the same side of the lens as that of the incident light, all the measurements will consequently be positive.

Let Q_1 be the conjugate focus of P from refraction at the first concave surface of the lens, and let μ or $\frac{V_a}{V_g}$ be the relative refractive index between air and glass.

Then $$\frac{\mu}{q_1} - \frac{1}{p} = \frac{\mu - 1}{r_1} \quad \text{.......................}(a),$$

where $Q_1A_1 = q_1$ and $PA_1 = p$ and where r_1 denotes the radius of curvature of the first surface.

The light from P will traverse the substance of the lens as if it had originated from Q_1; on reaching the second surface it will again undergo refraction, so that on emerging from the lens it will proceed as though it had originated from a point Q_2.

Since the refraction now takes place from glass to air the index of refraction $\left(\frac{V_g}{V_a}\right)$ is now $\frac{1}{\mu}$, and if the lens is so thin that its thickness is negligible, we may regard Q_1A_2, the distance of the point from which the light appears to be proceeding, as equal to Q_1A_1 or q_1.

To determine the distance Q_2A_2 or q_2 we have therefore the formula,

$$\frac{1}{\mu}\cdot\frac{1}{q_2}-\frac{1}{q_1}=\frac{\frac{1}{\mu}-1}{r_2}.$$

Or multiplying by μ,

$$\frac{1}{q_2}-\frac{\mu}{q_1}=\frac{1-\mu}{r_2}=(\mu-1)\left(-\frac{1}{r_2}\right).$$

On adding (a), $$\frac{\mu}{q_1}-\frac{1}{p}=\frac{\mu-1}{r_1},$$

we obtain $$\frac{1}{q_2}-\frac{1}{p}=(\mu-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\ldots\ldots\ldots\ldots\ldots(b),$$

or changing the signs throughout

$$\frac{1}{p}-\frac{1}{q_2}+(\mu-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=0\ldots\ldots\ldots\ldots(b').$$

If p be infinite, in other words, if the incident rays be parallel, $\frac{1}{p}=0$ and q_2 becomes the second principal focal distance (f'').

$$\frac{1}{f''}=(\mu-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\ldots\ldots\ldots\ldots\ldots\ldots(c).$$

Thus parallel rays incident on the concave surface of such a lens as that represented in Fig. 75 will after traversing

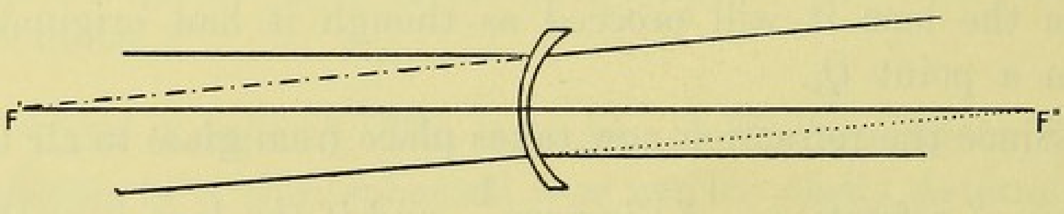

Fig. 75.

the lens diverge as though they were proceeding from the point F''.

If q_2 be infinite, that is if the emergent rays are parallel, p becomes the first principal focal distance f'.

$$\frac{1}{f'} = -(\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

The principal focal distances are consequently numerically equal to one another and differ only in sign, that is to say, they are situated on opposite sides of the lens. Thus in order that light after emerging from the lens, Fig. 75, should proceed in parallel rays, it must before refraction have been converging to a point F', situated on that side of the lens that is remote from the incident light, and at a distance equal to that of F'' from it.

On combining the equations (b') and (c) we obtain

$$\frac{1}{p} - \frac{1}{q_2} + \frac{1}{f''} = 0 \ldots\ldots \ldots\ldots\ldots\ldots (d).$$

We may suppress the subscripts if we remember that f denotes the second principal focal distance. These two formulæ (b') and (d) should be committed to memory, as they contain the whole theory of refraction of centric pencils through thin spherical lenses of any form. The only requisite, when numerical values are substituted for the symbols, is that due regard be paid to the signs that the symbols bear.

Ex. (1). The radius of curvature of the first surface of a meniscus is 6 ins., the radius of the second surface is 4 ins., where is the second principal focus of the lens? ($\mu = 1{\cdot}5$.)

$$\frac{1}{f''} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = (1\tfrac{1}{2} - 1)\left(\tfrac{1}{6} - \tfrac{1}{4}\right) = -\tfrac{1}{24},$$

$$\therefore f'' = -24 \text{ ins.}$$

Incident parallel rays will consequently converge to a point 24 inches from the lens on the side *opposite* to the incident light.

Such a lens is called a converging meniscus. It may be noted that in all converging lenses f' is positive and f'' is negative, whereas in all diverging lenses f' is negative and f'' is positive. Moreover all converging lenses are thickest in the middle, whereas all lenses which are thinnest in the middle are diverging in function, or what amounts to the same thing, their second principal focus (F'') is virtual, being situated on the object-side of the lens (p. 267).

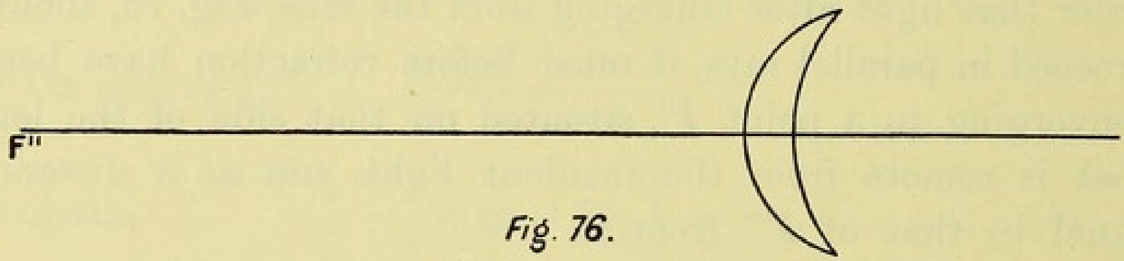

Fig. 76.

Ex. (2). A lens has one concave surface, its radius of curvature being 8 ins., what curvature must be given to the other surface, in order that a real image at 24 ins. distance may be formed of an object at 12 ins. distance?

Since the image is real, it must be formed on the side of the lens remote from the object, in other words q has a negative value.

Then since $$\frac{1}{p} - \frac{1}{q} + (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 0,$$

$$\frac{1}{12} - \frac{1}{(-24)} + (1\tfrac{1}{2} - 1)\left(\frac{1}{8} - \frac{1}{r_2}\right) = 0,$$

or $$\frac{1}{12} + \frac{1}{24} + \frac{1}{16} = \frac{1}{2r_2},$$

or $$\frac{4+2+3}{48} = \frac{1}{2r_2}.$$

$$\therefore\ r_2 = \tfrac{24}{9} \text{ or } 2\tfrac{2}{3} \text{ ins.}$$

Since the radius of the second surface is positive, it must be measured in the direction of the incident light. The lens is consequently a converging meniscus, the curvature of the second surface being three times greater than that of the first surface.

$$\left(\frac{1}{r_1} = \frac{1}{8} \text{ and } \frac{1}{r_2} = \frac{3}{8}\right).$$

Ex. (3). If the lens considered be bounded by two concave surfaces, r_2 bears a negative value, for the radius of the second surface is measured in a direction opposed to that of the incident light. If, for instance, the radii of the two faces of the lens are 8 ins. and -12 ins.,

$$\frac{1}{f''} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

$$i.e.\ \frac{1}{f''} = \frac{1}{2}\left(\frac{1}{8} - \frac{1}{(-12)}\right) = \frac{1}{2} \cdot \frac{3+2}{24} \text{ or } \frac{5}{48}.$$

$$\therefore\ f'' = \tfrac{48}{5} \text{ or } 9\tfrac{3}{5} \text{ ins.}$$

Since f'' is positive, F'' lies on the same side of the lens as the incident light. Incident parallel rays will proceed after refraction as if from a point $9\frac{3}{5}$ ins. from the positive side of the lens, that is to say, the parallel rays will diverge after refraction.

Ex. (4). An object is placed at 8 ins. distance from a biconcave lens whose focal distance is 4 ins.; where is the image?

As the lens is biconcave, the second principal focus (F'') is virtual, so f'' is positive.

Then since $$\frac{1}{p} - \frac{1}{q} + \frac{1}{f''} = 0,$$

$$\frac{1}{8} - \frac{1}{q} + \frac{1}{4} = 0.$$

$$\therefore\ \frac{1}{q} = \frac{3}{8} \text{ or } q = 2\tfrac{2}{3} \text{ ins.}$$

Since q is positive the image is virtual.

Ex. (5). A lens has one convex face the radius of its curvature being 9 ins., what must be the radius of curvature of the other surface, in order that F'' may be real and situated at a distance of 6 ins. from the lens?

Since F'' is real, it must be situated on the side opposite to the incident light, consequently f'' must have a negative value. As one surface is convex, let us place the lens so that its radius is measured in the positive direction, *i.e.* let 9 ins. be radius of curvature of the second surface.

$$\frac{1}{f''} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

$$\frac{1}{-6} = \frac{1}{2} \cdot \frac{1}{r_1} - \frac{1}{2} \cdot \frac{1}{9}.$$

$$\therefore \frac{1}{18} - \frac{1}{6} = \frac{1}{2r_1}.$$

$$\therefore \frac{1}{2r_1} = -\frac{2}{18} \text{ or } r_1 = -4\tfrac{1}{2} \text{ ins.}$$

The radius of the first surface must consequently be measured in the negative direction. The lens is therefore a biconvex, the curvature of one face being twice that of the other.

$$\left(\frac{1}{r_1} = \frac{2}{9} \text{ and } \frac{1}{r_2} = \frac{1}{9}\right).$$

It will be noticed that to the degree of approximation to which these formulæ apply, it makes no difference which surface faces the incident light. For had we put $r_1 = -9$,

$$\frac{1}{-6} = \frac{1}{18} - \frac{1}{2r_2} \text{ or } r_2 = 4\tfrac{1}{2} \text{ ins.},$$

i.e. the radius of the other surface would have the same numerical value as before, but it would have been measured in the positive direction.

Ex. (6). An object is placed 12 ins. in front of a biconvex lens, the focal length of which is 24 ins.; find the position of the image.

Since the lens is a converging lens, F'' lies on the side of the lens remote from the incident light, f'' consequently has a negative value.

$$\frac{1}{p} - \frac{1}{q} + \frac{1}{f''} = 0, \qquad \frac{1}{12} - \frac{1}{q} + \frac{1}{-24} = 0,$$

$$\text{or } \frac{1}{q} = \frac{1}{12} - \frac{1}{24} = \frac{1}{24}.$$

$$\therefore q = 24 \text{ ins.}$$

The image of the object is consequently virtual, for it is situated on the same side of the lens as the object, and its distance from the lens is 24 inches.

Ex. (7). If the object is placed 36 inches from the above lens, where is its image formed?

$$\frac{1}{36} - \frac{1}{q} + \frac{1}{-24} = 0,$$

$$\frac{1}{q} = \frac{1}{36} - \frac{1}{24} = \frac{2-3}{72} = \frac{-1}{72}.$$

$$\therefore \quad q = -72 \text{ ins.}$$

The image is therefore situated on the side of the lens opposite to the object, consequently the image is real.

Ex. (8). The focal distance of a symmetrical biconvex lens is 10 inches, what is the curvature of each surface, the refractive index of the glass being $1\frac{1}{2}$?

$$\frac{1}{f''} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Since the lens is biconvex, it is converging in function, so f'' is negative, and r_1 is also negative; and since it is symmetrical, r_1 is numerically equal to r_2.

$$\therefore \quad \frac{1}{-10} = \frac{1}{2}\left(-\frac{1}{r_2} - \frac{1}{r_2}\right) = -\frac{1}{r_2}.$$

That is to say, the radius of curvature of each surface is numerically equal to the focal distance.

A similar result is found for symmetrical biconcave lenses, and as the refractive index of the crown glass commonly used for lenses is about 1·54, the focal distances may in general be regarded as equivalent to their radii of curvature, when only rough approximate determinations are required.

Formation of images. As we are at present considering symmetrical lenses so thin that their thickness is negligible, we may regard the principal plane HOH' bisecting the lens symmetrically at right angles to the optic axis as the single surface at which refraction takes place.

All rays passing through the optical centre O of the lens may be considered to traverse the lens without refraction, for any deviation that they may undergo on encountering the first surface of the lens will be reversed on emerging from the second surface of the lens.

Let AB represent the object and let the image be denoted by ab (Fig. 77). For a point A not on the optic axis, draw AH parallel to the axis, then $HF''a$ cutting the axis

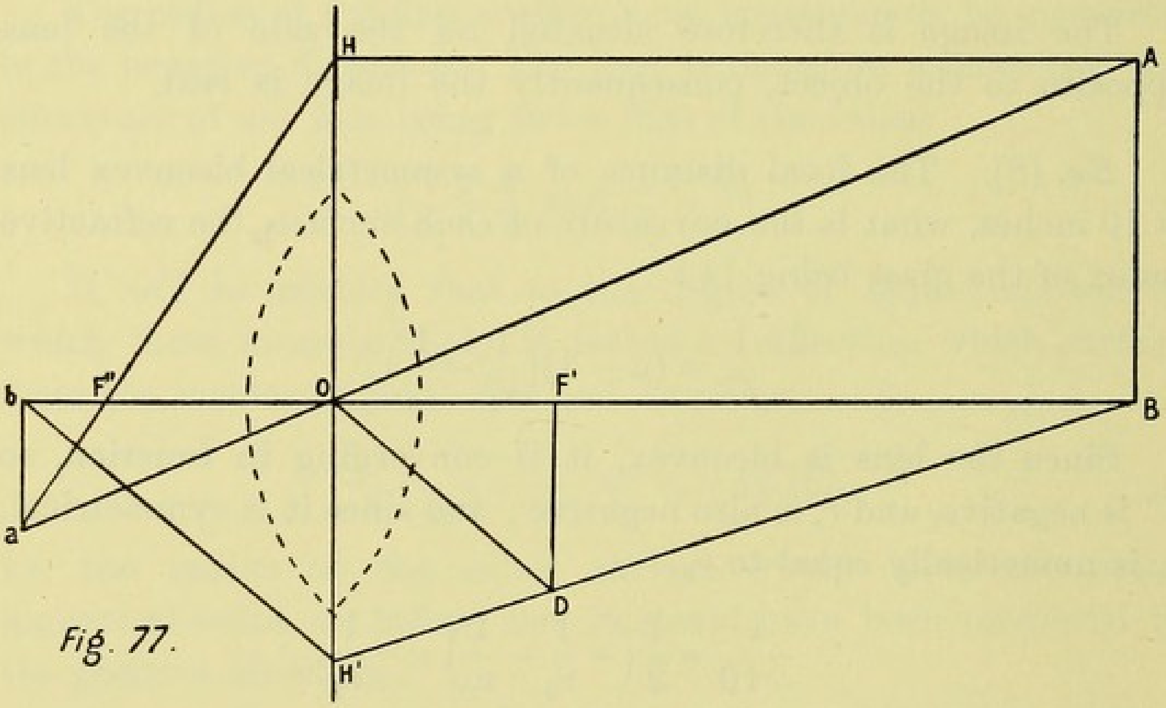

Fig. 77.

in F'' represents the corresponding refracted ray. The point a of intersection of this line with the ray AOa drawn through the optical centre of the lens determines the conjugate focus of A (Figs. 77, 79).

The following is an alternative method, Fig. 78. Through A draw the ray $AF'H'$ cutting the axis at the first principal focus F', then $H'a$ drawn parallel to the axis represents the corresponding refracted ray. The point a of intersection of this line with the ray AOa drawn through the optical centre of the lens determines the conjugate focus of A.

For a point B on the optic axis, Figs. 77, 79, take any ray BH' cutting the first focal plane in D. Join DO, and

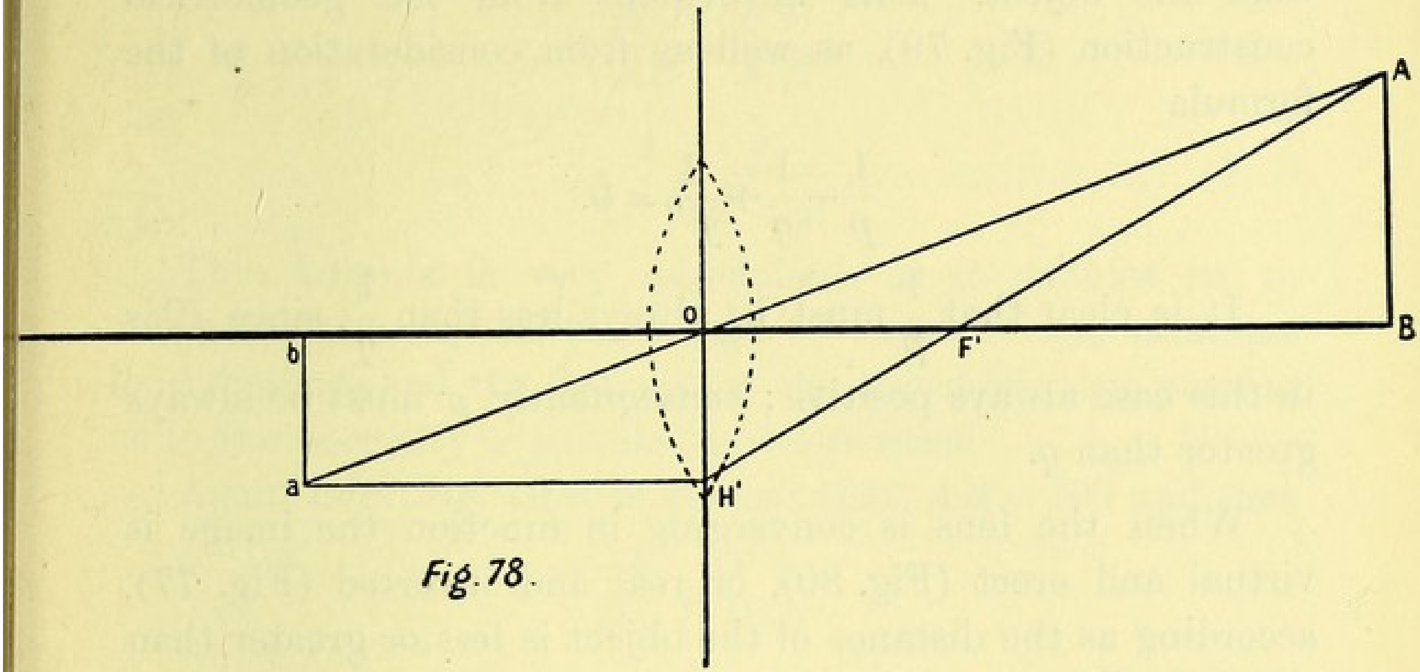

Fig. 78.

draw $H'b$ parallel to DO, cutting the axis in b. Then b is the conjugate focus of B, and ab is the image of the object AB.

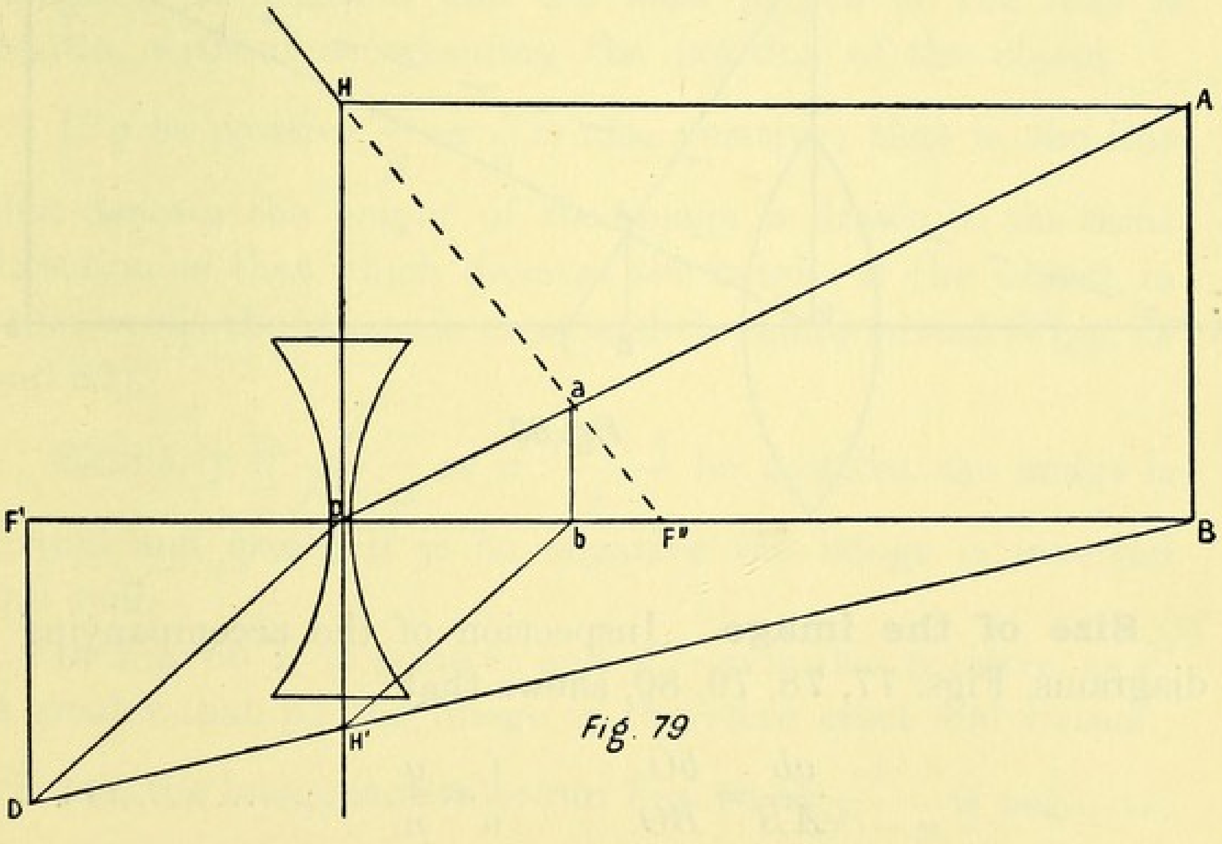

Fig. 79

When the lens is diverging in function the image is virtual and erect, and it is always situated nearer the lens than the object. This is evident from the geometrical construction (Fig. 79), as well as from consideration of the formula

$$\frac{1}{p}-\frac{1}{q}+\frac{1}{f''}=0.$$

It is clear that $\frac{1}{p}$ must be always less than $\frac{1}{q}$, since f'' is in this case always positive; consequently p must be always greater than q.

When the lens is converging in function the image is virtual and erect (Fig. 80), or real and inverted (Fig. 77), according as the distance of the object is less or greater than the focal distance of the lens.

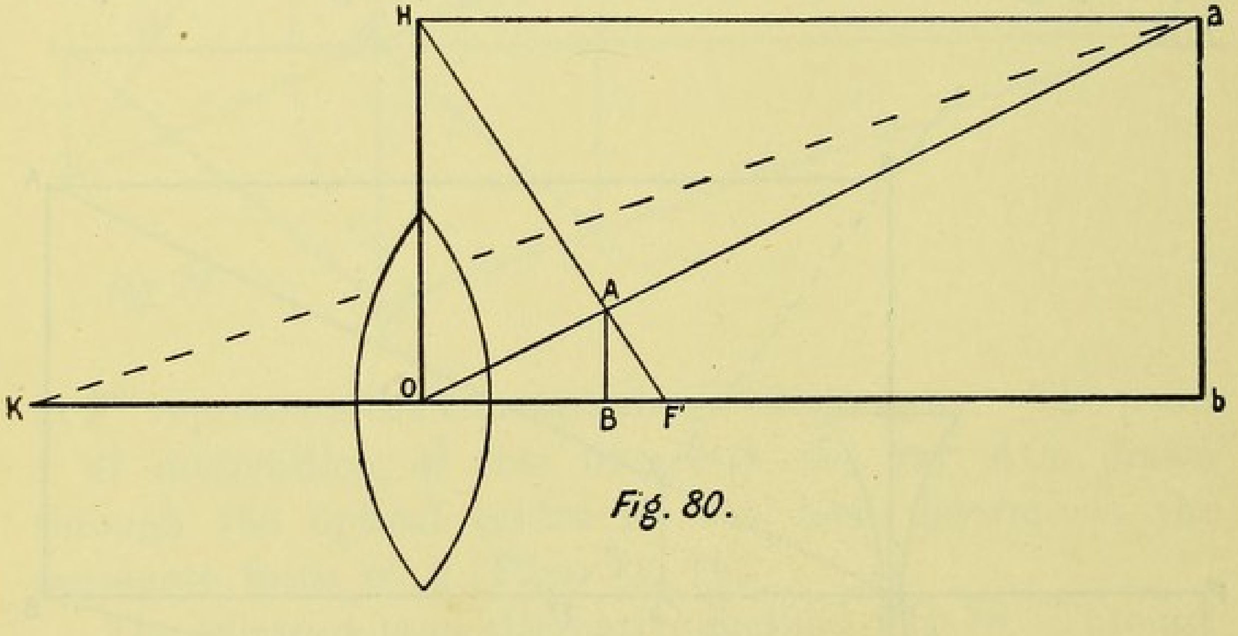

Fig. 80.

Size of the image. Inspection of the accompanying diagrams, Figs. 77, 78, 79, 80, shews that

$$\frac{ab}{AB}=\frac{bO}{BO} \quad \text{or} \quad \frac{i}{o}=\frac{q}{p}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Moreover from the construction employed in Fig. 80 it is clear that $ab = HO$, and that

$$\frac{HO}{AB} = \frac{F'O}{F'B}.$$

$$\therefore \quad \frac{i}{o} = \frac{f'}{f'-p} \text{......................(2).}$$

This formula is very convenient, as it enables us to determine the size of the image in terms of the principal focal distance and the distance of the object without putting us to the necessity of calculating its position.

Again from Fig. 79 it is evident that $AB = HO$ and that

$$\frac{ab}{HO} = \frac{F''b}{F''O}.$$

$$\therefore \quad \frac{i}{o} = \frac{f''-q}{f''} \text{......................(3).}$$

From this formula we can determine the size of the image, if its position and the focal length of the lens is known, without ascertaining the position of the object.

If q be positive, $\frac{q}{p}$ or $\frac{i}{o}$ is also positive; that is, the line that denotes the height of the image is drawn in the same direction as that which denotes the height of the object, in other words the image is erect and therefore virtual (Figs. 79 and 80).

Similarly if $\frac{f'}{f'-p}$ or if $\frac{f''-q}{f''}$ be positive, the image is virtual and erect; if it be negative the image is inverted and real.

In Fig. 80 f' is positive and $f'-p$ is also positive, as f' is greater than p; the image is therefore erect and virtual; whereas the image is inverted in Fig. 78 as $\frac{f'}{f'-p}$ is negative,

p being greater than f'. With diverging lenses (Fig. 79) f' is negative, the fraction $\frac{f'}{f'-p}$ has therefore a positive value, and the image is consequently always erect. If the lens be converging in function F'' lies on the opposite side of it as in Figs. 77, 78; f'' is therefore negative.

Ex. (1). A convex lens, the focal distance of which is 4 ins., forms the image of an object placed before it at a distance of 6 ins. behind the lens. What is the size of the image compared with that of the object?

Since the lens is convex f' is positive and f'' is negative, and since the image is formed behind the lens the value of q is also negative.

The formula $$\frac{i}{o} = \frac{f''-q}{f''},$$

$$\frac{i}{o} = \frac{(-4)-(-6)}{-4} = -\frac{1}{2}.$$

The image is therefore inverted, and its linear dimensions are half those of the object.

Had the situation of the object been given instead of that of the image, formula (2) might have been applied.

Ex. (2). An object 5 ins. in height is placed at 12 ins. distance from a convex lens, the focal length of which is 4 ins., what is the height of the image?

$$\frac{i}{o} = \frac{f'}{f'-p},$$

$$\frac{i}{o} = \frac{4}{4-12} = -\frac{1}{2}.$$

$$\therefore\ i = -2\tfrac{1}{2} \text{ ins.}$$

The image is inverted and $2\frac{1}{2}$ ins. in height.

Magnification. When the image is real it can be projected upon a screen, and the results of such investigations as to the size of the image can be actually verified with an ordinary foot-rule. But when the image is virtual it is not actually formed, but it only appears to be formed to the eye that sees it. The size therefore of a virtual image is only apparent[1]. Now the apparent size of an object or of an image, whether real or virtual, is determined by the tangent of the visual angle under which it is seen, as was mentioned in a previous chapter; and in the case of virtual images, where we only wish to know what size they appear to the eye, it is only necessary to consider the tangent of the angle subtended by the image at the anterior nodal point of the eye.

Let us consider first the apparent size of a real object *AB*, Fig. 81, situated on the optic axis *BK* where *K* denotes the position of the nodal point of the eye. Then the appa-

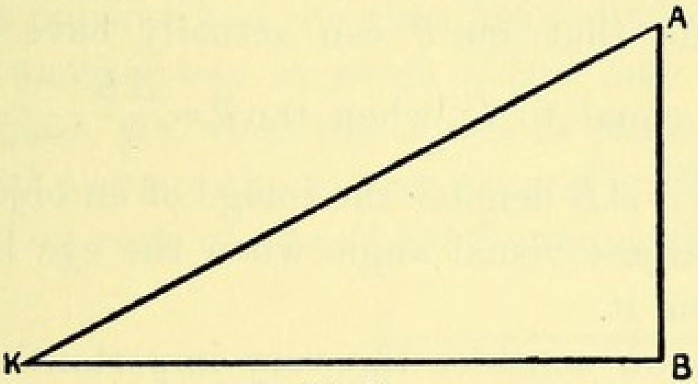

Fig. 81.

[1] The formulæ found above may be used in the case of virtual images, provided that their meaning is clearly understood. For instance $\frac{i}{o} = \frac{f'' - q}{f''}$ means that the light from the object after traversing the lens proceeds as if from an image situated at a distance q from the lens, the size of this imaginary or virtual image being $\frac{f'' - q}{f''}$ times that of the object. It will be found that it is only under one condition that the apparent size of the image to the eye is of this value.

rent size of AB is determined by $\tan\theta$ where θ denotes the angle subtended by AB at K.

$$\tan\theta = \frac{AB}{BK}.$$

It is evident that $\tan\theta$ could theoretically be indefinitely increased by diminishing BK indefinitely. In other words the apparent size of the object could be indefinitely increased by bringing the eye close to it. The nature of vision however forbids this, for the eye is incapable of distinct vision if the object perceived lies within a certain distance. This distance of the *punctum proximum*, as it is called, varies in different individuals, and increases with age so that it is impossible to assign to it any definite value which shall be applicable to all cases. It is usual to take 10 ins. as the distance at which most eyes can see objects distinctly without discomfort, and to estimate the magnifying power of an instrument on this assumption. If l denote the least distance for distinct vision of the individual eye considered, then the greatest value that $\tan\theta$ can actually have is given by making BK equal to l; when $\tan\theta = \frac{AB}{l}$.

Similarly if AB denotes the image of an object, it is seen under the largest visual angle when the eye is situated at distance l from it.

We will now consider the very important case where a convex lens is used as a magnifying glass (Fig. 80).

Let K denote the situation of the nodal point of the eye on the axis KOb, and let θ' represent the visual angle subtended at K by the virtual image ab.

Then $$\tan\theta' = \frac{ab}{bK} = \frac{i}{m+q},$$

where m represents the distance OK of the nodal point of the eye from the plane of the lens.

Now i increases with q, therefore if $m+q$ is to be equal to l the eye should be brought close up to the lens, making m as small as possible, and q nearly equal to l. As the nodal point of the eye is situated rather more than a quarter of an inch behind the cornea, the smallest value that can practically be assigned to m is about half an inch.

If $$m+q=l,$$

$$\tan\theta' = \frac{i}{l} = \frac{o}{l}\,\frac{f''-q}{f''}.$$

But when the object is placed in the most favourable position for viewing with the naked eye its greatest apparent size is represented by $\tan\theta$ or $\frac{o}{l}$.

Therefore the magnifying power of the glass so used is

$$\frac{\tan\theta'}{\tan\theta} = \frac{f''-q}{f''} = 1 - \frac{q}{f''}.$$

When the lens is convex f'' is negative; when the lens is concave f'' is positive, and the glass has a diminishing effect.

There is another way in which a lens may be used as a magnifying glass, viz. when the object is placed in the first principal focal plane of the convex glass (Fig. 82).

The incident cone of rays HAO diverging from the point

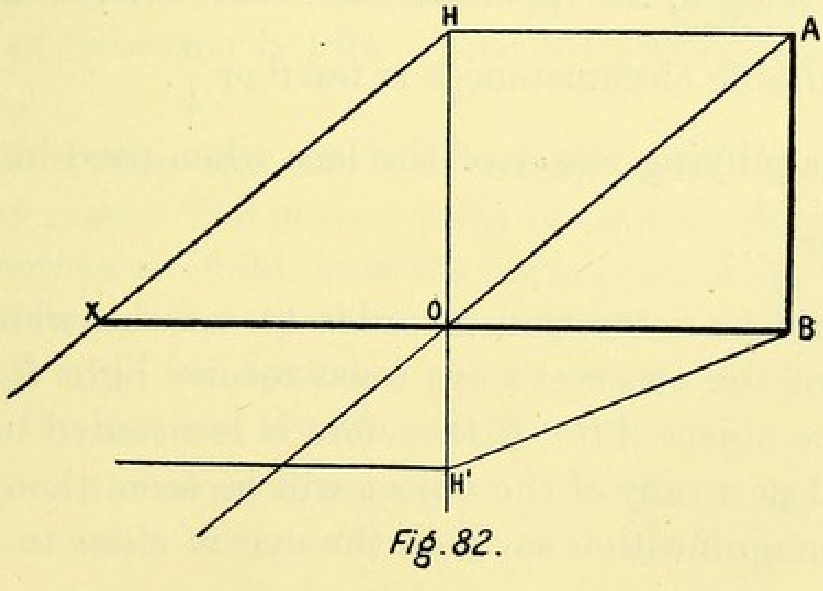

Fig. 82.

A of the object will after traversing the lens proceed as a pencil of parallel rays in the direction of the secondary axis AO.

Similarly all the incident rays diverging from B will after refraction proceed as a pencil of parallel rays in the direction of the principal axis BO. In this case the convex lens does not form an image, either real or virtual, so we must consider the image, if any, that is formed by the eye itself.

If an eye be placed on the axis BO behind the lens, an image of the point B will be formed at the second principal focus of the eye by the beam of parallel rays that it receives from B. Similarly an image of the point A will be formed at a certain point on the second principal focal plane of the eye by the obliquely incident beam that originally arose from A. In fact a complete image of the object AB is formed at the principal focal plane of the eye. It is clear that the size of the image in such a case is independent of the position of the eye, for the actual size of the retinal image depends upon the inclination of the pencil from A to the pencil from B, *i.e.* upon the angle BOA. The apparent size of the object when viewed through the lens is $\tan\theta'$, or $\tan BOA$, or $\frac{o}{f'}$, whatever may be the distance of the eye from the lens. The maximum visual angle under which the object may be seen distinctly being θ, its apparent size when viewed under the most favourable circumstances is $\tan\theta$ or $\frac{o}{l}$.

The magnifying power of the lens when used in this way is $\frac{\tan\theta'}{\tan\theta}$ or $\frac{l}{f'}$.

It should be noted that in order to see the whole of AB the pupil of the observer's eye must receive light from every point of the object AB. If therefore it is situated beyond X, the central part only of the object will be seen, though under the same magnification as when the eye is close to the lens.

The greatest linear extent of object visible through a lens in any position is called the field of view. It follows that the nearer the eye is to the lens the greater is the field of view.

On comparing the magnifying power of a convex lens used in these two different ways, we see that in the first case it is $1-\frac{q}{f''}$ or $1+\frac{q}{f'}$, and in the second case it is $\frac{l}{f'}$ or $\frac{m+q}{f'}$. Therefore if m is less than f' the first method gives the higher magnification and *vice versâ*.

QUESTIONS.

(1) The focal length of an equiconvex lens is 10 ins. If the index of refraction is 1·54, what is the radius of curvature of each surface?

(2) The radius of curvature of the first surface of a lens of equal power is -6 ins. What is the curvature of the second surface when $\mu = 1{\cdot}54$?

(3) The focal length of a convex lens is 6 ins.; an object is placed 36 ins. from it. What is the relative size of the image, and where is it formed?

(4) An object 5 ins. in height is placed 50 ins. from a planoconvex lens of which the radius of curvature is 7 ins., and the index of refraction is 1·54. Find the position and height of the image.

(5) A convex lens of focal length $\frac{1}{5}$ in. is used as a magnifying glass. The nearest point of distinct vision is $9\frac{3}{4}$ ins. from the cornea or 10 ins. from the nodal point of the eye. Find the magnifying power (i) when the lens is $\frac{1}{4}$ in. from the cornea, (ii) when it is $1\frac{1}{4}$ ins. from the cornea, (iii) when the object is $\frac{1}{5}$ in. from the lens and the eye is emmetropic.

(6) Give the magnifying powers of a convex glass of focal length 4 ins. under the same conditions.

CHAPTER XII.

REFRACTION OF A CENTRIC PENCIL THAT TRAVERSES A SYSTEM OF THIN CENTRED LENSES.

Equivalent Lenses. Let Fig. 83 represent a series of thin lenses of focal lengths $f_1, f_2, f_3, \ldots$[1] so placed that their centres of curvature all lie on the same straight line. Such a series is called a centred system.

Let us suppose that the lenses are in contact with each other and that their thicknesses are negligible.

Let P (Fig. 83) be a luminous point at distance p from the nearest lens. Then if q_1 denote the distance of the conjugate focus of P due to the refraction of this lens,

$$\frac{1}{p} - \frac{1}{q_1} + \frac{1}{f_1} = 0.$$

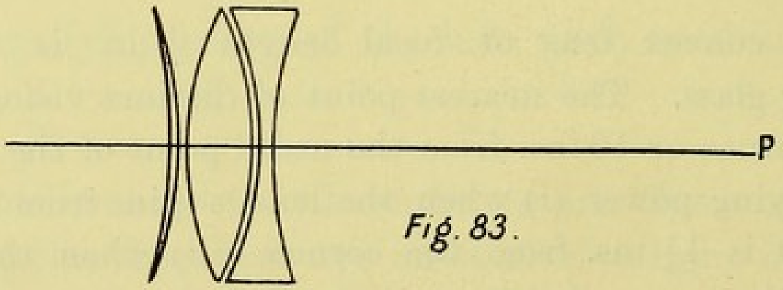

Fig. 83.

[1] In this chapter the term focal length and the symbols f_1, f_2, f_3 etc. refer always to the second principal focal distance.

The rays which reach the second lens will therefore proceed as though diverging from a point at this distance q_1. The refraction that they undergo at the second lens will so alter their course that they will appear to be diverging from a point at a distance q_2 from the second lens, such that

$$\frac{1}{q_1} - \frac{1}{q_2} + \frac{1}{f_2} = 0.$$

Similar equations may be formed for the remaining lenses of the system provided always that the thickness of each lens be neglected.

We thus obtain a series of equations,

$$\frac{1}{p} - \frac{1}{q_1} + \frac{1}{f_1} = 0,$$

$$\frac{1}{q_1} - \frac{1}{q_2} + \frac{1}{f_2} = 0,$$

$$\frac{1}{q_2} - \frac{1}{q_3} + \frac{1}{f_3} = 0,$$

....

By addition $$\frac{1}{p} - \frac{1}{q_3} + \left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) = 0.$$

If now a lens of focal length f' be taken such that

$$\frac{1}{p} - \frac{1}{q_3} + \frac{1}{f'} = 0,$$

it may be regarded as equivalent to the combination of lenses that we have been considering, and

$$\frac{1}{f'} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}.$$

In the case illustrated by the figure, f_1 and q_1 are positive, while f_3 and q_3 as well as f_2 and q_2 are negative.

This result is also obvious from the following considerations. The greater the refracting power of a lens the shorter is its focal length. The refracting power of a lens is represented by the deviation that it produces on a ray of incident light. If for instance, Fig. 84, the incident ray SH undergo the deviation HF'' after traversing the lens whose principal

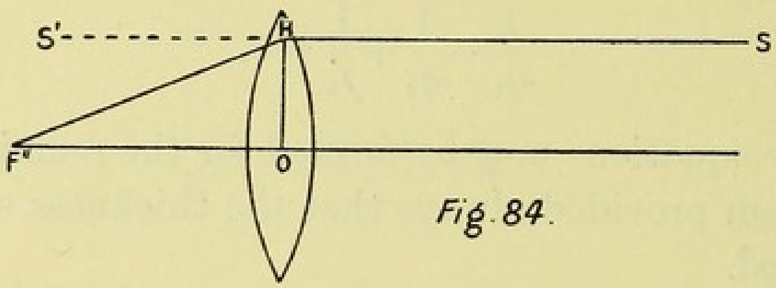

Fig. 84.

plane OH is at right angles to the incident ray, the refracting power of the lens is $\angle S'HF''$ or $\angle OF''H$. If only a rough approximation is required, we may replace this angle, when not too large, by its tangent, and say that the refracting power is measured by $\frac{HO}{OF''}$ or $\frac{1}{-f}$, if we agree to take H at unit distance from the principal axis. When a series of lenses is placed together, the refracting power of the combination is roughly represented by the algebraic sum of the refracting power of each member of the series.

In practice it is found very convenient to denote the strength of lenses not by their focal length but by their refracting power $\left(i.e. -\text{ the reciprocal of their focal length, } -\frac{1}{f}\right)$. The unit universally adopted is that of a lens of 1 metre focal length. This unit of refracting power is called a dioptre, and is denoted by the symbol D. Thus a $2D$ lens means a lens whose focal length is $\frac{1}{2}$ metre, a $10D$ lens means a lens whose focal length is $\frac{1}{10}$ metre, and so on. Further, it has been agreed to denote converging lenses by positive signs and diverging lenses by negative signs. Thus to find the refracting power of a series of lenses A, B, C in contact, where $A = +10D$ $(f'' = -100 \text{ mm.})$, $B = -2D$ $(f'' = 500 \text{ mm.})$,

$C = -4D$ ($f'' = 250$ mm.), we have only to add the respective dioptric powers

$$+10D - 2D - 4D = +4D.$$

The combination is therefore equivalent to a converging or convex lens, the focal length of which is $\frac{1}{4}$ metre or 250 millimetres.

So far we have been considering the lenses that compose the refracting system as being in contact with each other; if however they be separated from each other by measurable intervals these must be taken into account.

Let two concave lenses of focal length f_1 and f_2 respectively be situated at a distance a from each other on a common axis AB. Let SH_1 represent the axial ray of a thin pencil of incident light, the course of which is supposed to be parallel

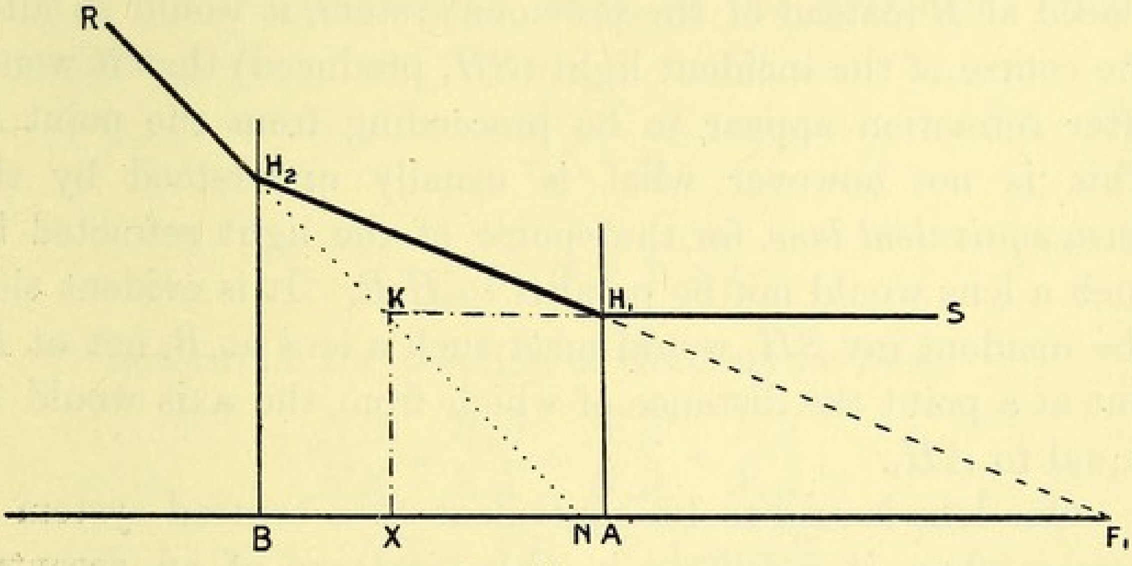

Fig. 85.

to the principal axis. Since the thickness of each lens is a negligible quantity, we may regard the refraction as taking place at the principal planes AH_1, BH_2. On meeting the first lens the incident ray SH_1 is refracted in the direction H_1H_2 as if it were diverging from the point F_1, the second principal focus of the first lens. On reaching the second lens a further refraction takes place, so that the ray now takes the direction H_2R as if it were proceeding from the point N.

And on substituting the corresponding values in the familiar formula

$$\frac{1}{p}-\frac{1}{q}+\frac{1}{f}=0,$$

we obtain

$$\frac{1}{F_1B}-\frac{1}{NB}+\frac{1}{f_2}=0,$$

$$\frac{1}{NB}=\frac{1}{F_1B}+\frac{1}{f_2} \text{ or } \frac{1}{F_1A+AB}+\frac{1}{f_2};$$

$$\therefore \frac{1}{NB}=\frac{1}{f_1+a}+\frac{1}{f_2} \text{ or } \frac{f_1+f_2+a}{f_2(f_1+a)}.$$

If then a lens of focal length NB or $\frac{f_2(f_1+a)}{f_2+f_1+a}$ were placed at B instead of the previous system, it would so alter the course of the incident light (SH_1 produced) that it would after refraction appear to be proceeding from the point N. This is not however what is usually understood by the term *equivalent lens,* for the course of the light refracted by such a lens would not be parallel to H_2R. It is evident that the incident ray SH_1 would meet such a lens at B, not at H_2 but at a point the distance of which from the axis would be equal to AH_1.

One lens is said to be equivalent to a centred system of lenses when it produces in the axial ray of an eccentric pencil, incident parallel to the axis, the same deviation as the system does[1].

The power and position of the equivalent lens can be

[1] It is important to remember that a lens the focus of which is $\frac{f_2(f_1+a)}{f_2+f_1+a}$ cannot, as is frequently stated, be regarded as equivalent to the combination of the two lenses. The virtual image formed by a lens of this power is of a different size to that formed by the combination. Several errors have found their way into certain books on ophthalmology from neglect of this point.

determined in the following way. Produce SH_1 to meet NH_2 in K, and from K draw KX perpendicular to the axis. It is clear that a lens at X of focal distance NX would be equivalent to the system considered, for the path of the light refracted by this lens would be identical with the final course of the light refracted by the system.

By similar triangles we have

$$\frac{NX}{NB} = \frac{KX}{H_2B} \text{ or } \frac{H_1A}{H_2B}.$$

Also $$\frac{H_1A}{H_2B} = \frac{F_1A}{F_1B} \text{ or } \frac{f_1}{f_1+a};$$

$$\therefore NX = \frac{f_1}{f_1+a} NB = \frac{f_1}{f_1+a} \cdot \frac{f_2(f_1+a)}{f_2+f_1+a}.$$

The focal distance NX or f_x of the equivalent lens is therefore

$$\frac{f_1 f_2}{f_2+f_1+a}$$

or $$\frac{1}{f_x} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{a}{f_1 f_2}.$$

To determine the position of the lens we have

$$\frac{NB}{NX} = \frac{H_2B}{KX} = \frac{H_2B}{H_1A} = \frac{F_1B}{F_1A};$$

$$\therefore \frac{NX+XB}{NX} = \frac{F_1A+AB}{F_1A};$$

$$\therefore XB = NX \cdot \frac{AB}{F_1A} \text{ or } a\frac{f_x}{f_1}.$$

The equivalent lens should be placed therefore at a distance XB or $a\dfrac{f_x}{f_1}$ in front of the second lens, or $a\left(1-\dfrac{f_x}{f_1}\right)$ behind the first lens of the system.

If the system consist of more than two lenses the power and position of the equivalent lens can be found by the same method.

For example, let there be three lenses A, B, C of focal lengths f_1, f_2, f_3 respectively, the interval between A and B being a_1 and that between B and C being a_2.

If a lens of focal length f_x be equivalent to the combination A and B when placed at a distance XB or x from B,

$$\frac{1}{f_x}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{a_1}{f_1 f_2}.$$

The system of three lenses may be replaced by two lenses of focal lengths f_x and f_3 separated from each other by the interval a_2+x. The focal length f_y of the lens equivalent to this combination is given by the equation

$$\frac{1}{f_y}=\frac{1}{f_x}+\frac{1}{f_3}+\frac{a_2+x}{f_x f_3}.$$

Substituting for x the value obtained above,

$$\frac{a_1 f_x}{f_1},$$

we get $$\frac{1}{f_y}=\frac{1}{f_x}+\frac{1}{f_3}+\frac{a_2}{f_3}\left(\frac{1}{f_x}\right)+\frac{a_1}{f_1}\left(\frac{1}{f_3}\right);$$

$$\therefore \frac{1}{f_y}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{a_1}{f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}\right)+\frac{a_2}{f_3}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)+\frac{a_1 a_2}{f_1 f_2 f_3}.$$

This equivalent lens must be placed at a point Y such that its distance YC (or y) from the third lens fulfils the condition

$$y=(a_2+x)\frac{f_y}{f_x}.$$

The distance AY from the first lens is

$$a_1+a_2-y, \text{ or } a_1+a_2-(a_2+x)\frac{f_y}{f_x}.$$

Precisely the same process may be applied for the determination of the power and position of the lens equivalent to any number of lenses. The expressions in such cases are somewhat cumbrous and their evaluation is laborious, but they present no actual difficulty in their application.

Practically we have rarely to deal with a combination of more than two lenses or two simple refracting systems.

From consideration of the expression

$$\frac{1}{f_x}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{a}{f_1 f_2}$$

it appears that the greater the interval that separates two concave glasses the stronger must be the refracting power of the equivalent lens.

Ex. (1). Let us consider the combination of a concave lens of focal distance $\frac{1}{3}$ m., with another similar lens of focal distance $\frac{1}{4}$ m., the interval between the first and the second lens being $\frac{1}{4}$ m.

Since
$$\frac{1}{f_x}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{a}{f_1 f_2},$$
$$\frac{1}{f_x}=3+4+\frac{12}{4}=0.$$

The equivalent lens has a positive focal distance of $\frac{1}{10}$ metre. We have however still to determine the position in which it should be placed in order that its action may be identical with that of the system considered.

$$x=a\frac{f_x}{f_1} \text{ or } \frac{1}{4}\cdot\frac{3}{10}=\cdot 075 \text{ m. or } 75 \text{ mm.}$$

The equivalent lens must therefore be a concave lens of focal distance 100 mm. and must be placed 75 mm. in front of the second lens or 250 − 75, *i.e.* 175 mm. behind the first lens.

If however the two lenses had been placed in juxtaposition the term involving the interval a vanishes and we have the power of the lens $\frac{1}{f_x}$ denoted by 3 + 4 or 7, so that in this case a lens of $\frac{1}{7}$ metre focal length might be substituted for the combination since its action will be precisely the same.

If any of the lenses that constitute a given system are converging in function, attention must be paid to the signs carried by the numerical values, when these are substituted for the symbols in the formula. For since the symbols f_1, f_2, f_3, etc. all refer to the second principal focal distances, these will have negative values when the lenses to which they refer are convex.

Ex. (2). Find the position and power of the lens, the action of which is equivalent to the system formed by one convex lens whose focal distance is 500 mm. combined with another convex lens whose focal distance is 250 mm., the interval between the two being $333 \cdot \dot{3}$ mm.

Let the incident light strike first the lens whose focal distance is 500 mm. or $\frac{1}{2}$ m. Then

$$f_1 = -\tfrac{1}{2}\text{m.}, \; f_2 = -\tfrac{1}{4}\text{m.}, \; a = \tfrac{1}{3}\text{m.}$$

From the formula

$$\frac{1}{f_x} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{a}{f_1 f_2}$$

we get

$$\frac{1}{f_x} = -2 - 4 + \tfrac{8}{3} = -\tfrac{10}{3}\text{ m.}$$

The focal distance f_x is then

$$\frac{-3}{10}\text{ m. or } -300\text{ m.};$$

the negative sign shews that the lens is convex.

If x be its distance from the previous position of the second lens,

$$x = a\frac{f_x}{f_1} = \frac{1}{3}\left(\frac{-3}{10}\right)(-2) = \tfrac{2}{10}\text{ m. or } 200\text{ mm.}$$

The lens is therefore to be placed 200 mm. on the positive side of the second member of the system or $133\cdot\dot{3}$ mm. on the negative side of the first member.

If however the system had been turned the other way, so that the incident light struck the lens of 250 mm. focal length first, we should have obtained a different result.

In this case

$$f_1 = -\tfrac{1}{4}\text{ m.},\ f_2 = -\tfrac{1}{2}\text{ m.},\ \ a = \tfrac{1}{3}\text{ m.}$$

As before,
$$f_x = -\tfrac{3}{10}\text{ m.},$$

but its distance from the second lens must now be increased.

$$x = a\frac{f_x}{f_1} = \frac{1}{3}\left(\frac{-3}{10}\right)(-4) = \tfrac{4}{10}\text{ m. or 400 mm.}$$

But 400 mm. is greater than the interval $333\cdot\dot{3}$ mm. between the two lenses. The equivalent lens must therefore be placed $66\cdot\dot{6}$ mm. in front of the previous position of the first lens of the system.

It will be found that if the interval between two convex lenses be greater than the sum of their focal distances ($a > f_1 + f_2$), the value of f_x becomes positive and the value of x becomes negative. In other words the combination is equivalent to a single *concave* lens placed on the distal side of the second lens. The reason will be apparent when it is remembered that such a combination would have the effect of twice inverting the image of a distant object. The final image will therefore be erect, as that formed by a concave lens.

Ex. (3). A screen is fixed at a distance of 7 ins. from a convex lens of focal distance 10 ins. What additional lens will be required, and where should it be placed in order that an image of the sun may be formed on the screen of the same size as that which would be formed by the original lens at a distance of 10 ins.?

Here $\qquad f_2 = -10, \qquad f_x = -10, \qquad XB = 3.$

Since

$$f_x = \frac{f_1 f_2}{f_2 + f_1 + a}, \quad -10 = \frac{f_1(-10)}{-10 + f_1 + a}; \quad \therefore a = +10.$$

Therefore a lens of focal length f_1 must be placed 10 ins. in front of the given lens, *i.e.* in its first focal plane.

And

$$XB = 3 = a\frac{f_x}{f_1} = 10\,\frac{-10}{f_1}; \quad \therefore f_1 = \frac{-100}{3} = -33\tfrac{1}{3}\text{ ins.}$$

The total deviation produced by the system of lenses at A and B is equal to the deviation D produced by the equivalent lens at X situated at its focal distance (10 ins.) from the screen.

In the figure (Fig. 86) N represents the position of the screen, F_2 the second principal focus of the lens at B. If the lens at B

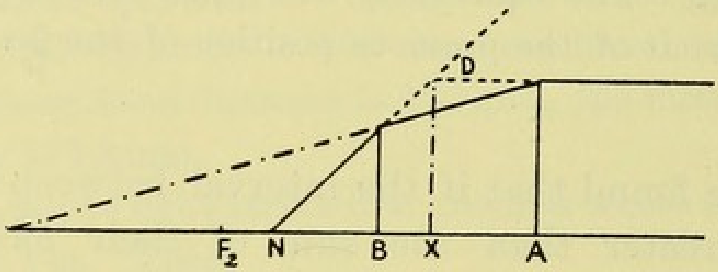

Fig. 86.

and the screen at N represent a badly focussed camera, the additional lens at A will cause a distinct image of a distant object to be formed on the screen of precisely the same size as would be formed by drawing out the lens to X. The determination of suitable correcting glasses for errors of refraction in the eye involves a problem of a precisely similar nature.

QUESTIONS.

(1) Two convex lenses, whose focal lengths are $3f$ and f, are placed at a distance apart equal to the difference of their focal lengths. (A Huygenian eyepiece is such a combination.) Find the focal length of the equivalent lens and its position.

(2) Two convex lenses, whose focal lengths are each equal to f, are placed at a distance apart equal to $\frac{2}{3}f$. (Ramsden's eyepiece is such a combination.) Find the focal length of the equivalent lens and its position.

(3) A real image 1 m. in diameter is formed by the object-glass of a compound microscope. When the eyepiece is so placed that the emergent rays are parallel, give the magnifying power of each of the above eyepieces.

(4) A concave lens of focal length 5 cm. is placed 3 cm. beyond a convex lens of focal length 5 cm. Find the focal length and the position of the equivalent lens.

CHAPTER XIII.

CARDINAL POINTS.

When after having traversed one medium light meets a second and then a third, and so on, each medium being bounded by a surface of different curvature and being of a different refractive index, the problem of calculating the position and size of the image formed by such a refractive system becomes much more complicated.

We have hitherto been considering lenses the thickness of which was negligible; if we were to consider the formation of an image by a lens of considerable thickness we might first find the position and size of the image formed by the first surface, and then regarding this image as the object for the second surface find the final image formed by it. If we had to calculate the refraction through several such media, as in the case of the eye, such a calculation would prove very tedious and wearisome. Fortunately the labour of such work has been considerably reduced by the investigations of Gauss[1] on the refraction of thin axial pencils. We proceed to give an account of some of the results of his mathematical investigations.

He has found that in every dioptric system, formed of any number of media bounded by centred spherical surfaces,

[1] Gauss, *Dioptrische Untersuchungen.*

there exist two pairs of cardinal points situated upon the axis; to these Listing has added another pair, the so-called nodal points.

These six cardinal points are the two principal foci, the two principal points, and the two nodal points.

The *first principal focus* (F') is the point on the principal axis where the incident rays intersect, or would intersect if produced, which emerge from the system parallel to the axis.

The *second principal focus* (F'') is the point of intersection of the emergent rays, whose direction when incident has been parallel to the principal axis.

The *principal points* have the following property: when an incident ray (produced if necessary) passes through the first principal point (H') the corresponding emergent ray (produced if necessary) passes through the second principal point (H''), but the incident and emergent rays are not necessarily parallel to each other. The principal points are each the image the one of the other.

The *nodal points* are two points on the principal axis such that every ray which before refraction is directed towards the first nodal point (K') (see Fig. 88), seems to come, after refraction through the system, from the second nodal point (K''), and takes a direction parallel to its direction on incidence.

These two parallel lines may be called the lines of direction; they play the same part in the complex system considered, that the straight line, drawn through the centre of curvature of the spherical surface bounding a single refractive medium, does in that simple system. The two nodal points are mutually the image of each other. The distance between the two nodal points is equal to the distance between the two principal points. The two planes drawn through F' and F'', and H' and H'' at right angles to the principal axis, in the case of axial pencils, are called the two focal planes and the two principal planes.

The rays that originate, or appear to originate, from a point on the *first focal plane*, are after refraction parallel to each other and to the lines of direction.

The incident rays which are parallel to each other, intersect after refraction in some point on the *second focal plane.* This point is where the corresponding line of direction cuts the second focal plane.

The *principal planes* have the following property. If through the point J_1 (Fig. 88) where the incident ray (produced if necessary) cuts the first principal plane J_1H', a line is drawn parallel to the principal axis meeting the second principal plane J_2H'' in J_2, J_2 will lie in the corresponding emergent ray produced if necessary. In other words, the directions of an incident ray and its corresponding emergent ray cut the two principal planes in two points situated on the same side and at the same distance from the principal axis.

The second principal plane is the optical image of the first and *vice versâ.* The principal planes are sometimes called planes of unit magnification, for if an area of definite shape and size be considered in the first principal plane, a virtual image of precisely the same shape and size will be formed in the second principal plane. They are the only two conjugate images which have the same size and are situated on the same side of the principal axis.

The first principal focal distance (f') is the interval $F'H'$ which separates the first principal focus (F') from the first principal point (H'). The second principal focal distance is the distance $F''H''$ between the second focal principal focus (F'') and the second principal point (H'').

We proceed to find the situation of the cardinal points of a lens the index of refraction of which is μ surrounded by a medium whose refractive index is 1. The thickness of the lens at the centre is t, and the radii of curvature of the two surfaces of the lens are denoted by r_1, r_2.

In order to obtain a general expression for the position of the cardinal points which may be applicable to every kind of lens, it will be convenient to take as our standard example a meniscus in which the principal points lie on the positive side of the lens. Lines measured in the direction of the incident light are considered positive, those in the reverse direction negative.

The Principal Points and the Principal Planes. (Fig. 87.) As the second principal plane is the optical image

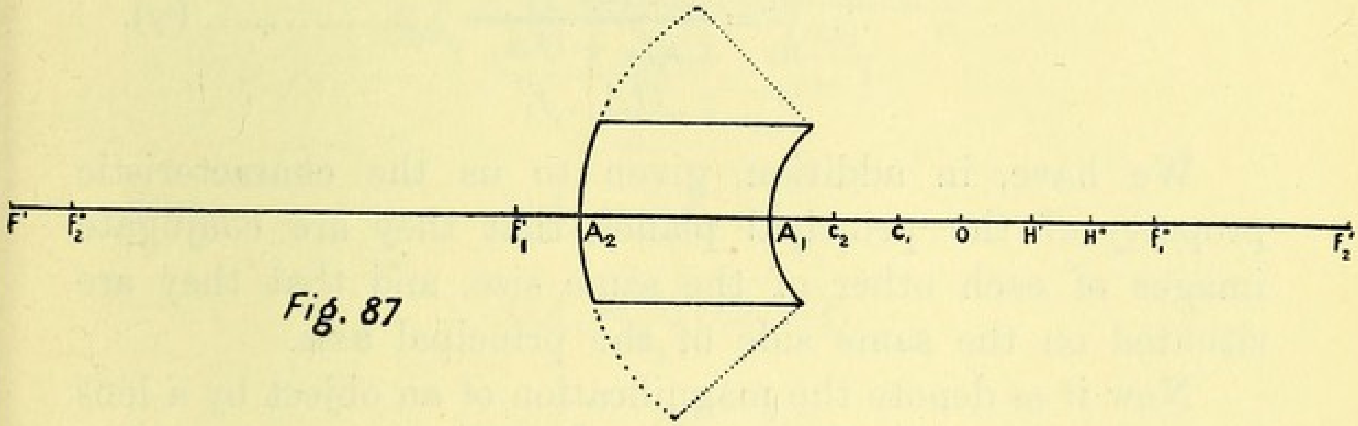

Fig. 87

of the first principal plane, we may regard the first as forming an image at O, distant OA_1 from the first surface due to the refraction on entering the lens; then the second principal plane will be the image formed on leaving the lens of the light that comes from this presumed image at O.

Let $h_1 = H'A_1$, the distance of the first principal point from the first surface of the lens; and let $h_2 = H''A_2$.

Then the following relation obtains between OA_1 and h_1,

$$\frac{\mu}{OA_1} - \frac{1}{h_1} = \frac{\mu - 1}{r_1} = -\frac{1}{f_1'} \quad \ldots\ldots\ldots\ldots\ldots (\alpha).$$

On leaving the lens an image is formed at h_2 as if from an object at O, distant OA_2 from the second (posterior) surface of the lens;

$$\therefore \frac{\frac{1}{\mu}}{h^2} - \frac{1}{OA_2} = \frac{\frac{1}{\mu} - 1}{r^2} = -\frac{1}{f_2'}.$$

On multiplying by μ we obtain

$$\frac{1}{h_2} = \frac{\mu}{OA_2} - \frac{\mu}{f_2'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

Also from (α)
$$\frac{1}{h_1} = \frac{\mu}{OA_1} + \frac{1}{f_1'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha');$$

$$\therefore \frac{h_1}{h_2} = \frac{\dfrac{\mu(f_2' - OA_2)}{OA_2 \cdot f_2'}}{\dfrac{\mu f_1' + OA_1}{OA_1 \cdot f_1'}} \quad \ldots\ldots\ldots\ldots\ldots (\gamma).$$

We have, in addition, given to us the characteristic property of the principal planes, that they are conjugate images of each other of the same size, and that they are situated on the same side of the principal axis.

Now if m denote the magnification of an object by a lens it may be regarded as consisting of two components, m_1 due to the refraction at the first surface, and m_2 at the second surface. Obviously $m = m_1 m_2$.

In this case $m_1 = \frac{i_1}{o_1} = \frac{OA_1}{\mu h_1}$,

$$m_2 = \frac{i_2}{o_2} = \frac{h_2}{\frac{1}{\mu} OA_2} = \frac{\mu h_2}{OA_2}.$$

And $m_1 m_2 = 1$;

$$\therefore \frac{OA_1}{\mu h_1} \cdot \frac{\mu h_2}{OA_2} = 1;$$

$$\therefore \frac{OA_1}{OA_2} = \frac{h_1}{h_2}.$$

Now from (γ) we have

$$\frac{h_1}{h_2} = \frac{OA_1 f_1' \mu (f_2' - OA_2)}{OA_2 f_2' (\mu f_1' + OA_1)}.$$

On dividing by $\frac{h_1}{h_2}$ or $\frac{OA_1}{OA_2}$ we get

$$1 = \frac{f_1' \mu (f_2' - OA_2)}{f_2' (\mu f_1' + OA_1)}.$$

And remembering that $\mu f_1' = -f_1''$ we obtain this relation

$$f_2' OA_1 - f_1'' f_2' = f_1'' OA_2 - f_1'' f_2',$$

$$\therefore \quad \frac{OA_1}{OA_2} = \frac{f_1''}{f_2'} = \frac{\dfrac{\mu r_1}{\mu - 1}}{\dfrac{-r_2}{\dfrac{1}{\mu} - 1}} = \frac{\dfrac{\mu r_1}{\mu - 1}}{\dfrac{\mu r_2}{\mu - 1}} = \frac{r_1}{r_2}.$$

But $$OA_2 = OA_1 + t,$$

$$\therefore \quad OA_1 = \frac{r_1}{r_2}(OA_1 + t),$$

or $$OA_1 \left(1 - \frac{r_1}{r_2}\right) = \frac{r_1 t}{r_2};$$

$$\therefore \quad OA_1 = \frac{r_1 t}{r_2 - r_1} = \frac{f_1'' t}{f_2' - f_1''},$$

and $$OA_2 = \frac{r_2 t}{r_2 - r_1} = \frac{f_2' t}{f_2' - f_1''} \text{*}.$$

Now from (α')

$$h_1 = \frac{OA_1 f_1'}{OA_1 + \mu f_1'} = \frac{\dfrac{f_1' f_1'' t}{f_2' - f_1''}}{\dfrac{f_1'' t}{f_2' - f_1''} - f_1''} = \frac{f_1' t}{t + f_1'' - f_2'}.$$

* It may be observed that O is in the position of the optical centre of the lens (p. 275).

Similarly from (β)

$$h_2 = \frac{OA_2 f_2'}{\mu(f_2' - OA_2)}$$

and

$$\frac{f_2'}{\mu} = \frac{1}{\mu}\left(\frac{-r_2}{\frac{1}{\mu} - 1}\right) = -f_2'',$$

$$\therefore\ h_2 = \frac{OA_2 \cdot f_2''}{OA_2 - f_2'} = \frac{\dfrac{f_2' f_2'' t}{f_2' - f_1''}}{\dfrac{f_2' t}{f_2' - f_1''} - f_2'} = \frac{f_2'' t}{t + f_1'' - f_2'}.$$

The distance $H''H'$ between the two principal points is now easily obtained for

$$H''H' = H''A_2 - A_1A_2 - H'A_1 = h'' - t - h',$$

$$\therefore\ H''H' = \frac{f_2''t - t(t + f_1'' - f_2') - f_1't}{t + f_1'' - f_2'}$$

$$= \frac{f_2''t - t(t - \mu f_1' + \mu f_2'') - f_1't}{t + f_1'' - f_2'}$$

$$= \frac{(\mu - 1)(f_1't - f_2''t) - t^2}{t + f_1'' - f_2'}.$$

The Principal Foci. In finding the situation of the principal foci we will first find their distances from the anterior (A_1) and posterior (A_2) faces of the lens. Knowing the position of the two principal points (H' and H'') we can then easily determine the principal focal distances ($F'H'$ and $F''H''$).

Let x, x' denote the distances of the object and its first image, with relation to the surface A_1, respectively; and let y, y' denote the distances of the final image and the first image with relation to the surface A_2, respectively.

Then from our fundamental formula we have

$$\frac{\mu}{x'} - \frac{1}{x} + \frac{1}{f_1'} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a),$$

$$\frac{1}{\mu y} - \frac{1}{y'} + \frac{1}{f_2'} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

and $$t = y' - x' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

But $$y' = \frac{\mu y f_2'}{f_2' + \mu y} \ldots\ldots\ldots\ldots \text{ from } (b),$$

and $$x' = \frac{\mu x f_1'}{f_1' - x} \ldots\ldots\ldots\ldots \text{ from } (a),$$

$$\therefore\ t = \frac{\mu y f_2'}{f_2' + \mu y} - \frac{\mu x f_1'}{f_1' - x}.$$

And since $-\mu f_1' = f_1''$ and $f_2' = -\mu f_2''$,

$$t = \frac{x f_1''}{f_1' - x} - \frac{y f_2'}{f_2'' - y};$$

$$\therefore\ t(f_1' - x)(f_2'' - y) = x f_1''(f_2'' - y) - y f_2'(f_1' - x)$$

or $$t f_1' f_2'' - x(t f_2'' + f_1'' f_2'') = y(t f_1' - f_2' f_1') - xy(t + f_1'' - f_2');$$

$$\therefore\ \frac{t f_1' f_2'' - x(t f_2'' + f_1'' f_2'')}{y} = t f_1' - f_2' f_1' - x(t + f_1'' - f_2').$$

If $y = \infty$, the emergent rays are parallel, and the corresponding value of x will give $F_1'A_1$ the situation of the first principal focus with regard to the first surface.

Therefore this value of x, *i.e.*

$$F'A_1 = \frac{t f_1' - f_1' f_2'}{t + f_1'' - f_2'}.$$

Again,

$$\frac{t f_1' f_2'' - y(t f_1' - f_2' f_1')}{x} = t f_2'' + f_1'' f_2'' - y(t + f_1'' - f_2').$$

If $x = \infty$, the incident rays are parallel, and the corresponding value of y will give $F''A_2$.

Therefore $$F''A_2 = \frac{tf_2'' + f_1''f_2''}{t + f_1'' - f_2'}.$$

The first focal point F' is situated on the negative side of A_1 in the figure (Fig. 87), whereas H' is on the positive side of A_1.

Therefore $F'H' = F'A_1 + A_1H' = F'A_1 - H'A_1$,

$$F'H' = \frac{tf_1' - f_1'f_2'}{t + f_1'' - f_2'} - \frac{f_1't}{t + f_1'' - f_2'} = \frac{-f_1'f_2'}{t + f_1'' - f_2'}.$$

And $$F''H'' = F''A_2 - H''A_2,$$

$$F''H'' = \frac{tf_2'' + f_1''f_2''}{t + f_1'' - f_2'} - \frac{f_2''t}{t + f_1'' - f_2'} = \frac{f_1''f_2''}{t + f_1'' - f_2'}.$$

It is easily seen that the principal focal distances $F'H'$, $F''H''$ are numerically equal to each other, but are measured in opposite directions from the corresponding principal planes.

For

$$f_1'f_2' = \left(\frac{-r_1}{\mu - 1}\right)\left(\frac{-r_2}{\frac{1}{\mu} - 1}\right) = \frac{\mu r_1}{\mu - 1} \cdot \frac{r_2}{1 - \mu} = \left(\frac{\mu r_1}{\mu - 1}\right)\left(\frac{\frac{1}{\mu} r_2}{\frac{1}{\mu} - 1}\right)$$
$$= f_1''f_2'',$$

$$\therefore\ F''H'' = -F'H'.$$

This numerical equality of the principal focal distances occurs in any combined system of refractive media, provided that the first medium and the last have the same refractive index.

The Nodal points. In every complex refracting system, in which the initial and final media have the same refractive index (*i.e.* when $F''H'' = H'F'$), the nodal points K', K'' coincide with the principal points H', H''. In order to give a geometrical construction for ascertaining their position generally, we must take a case in which the final medium has a different refractive index from the initial medium.

Fig. 88 represents the principal planes and the focal planes of a converging refractive system in which the final medium is denser than the initial medium. Such a system

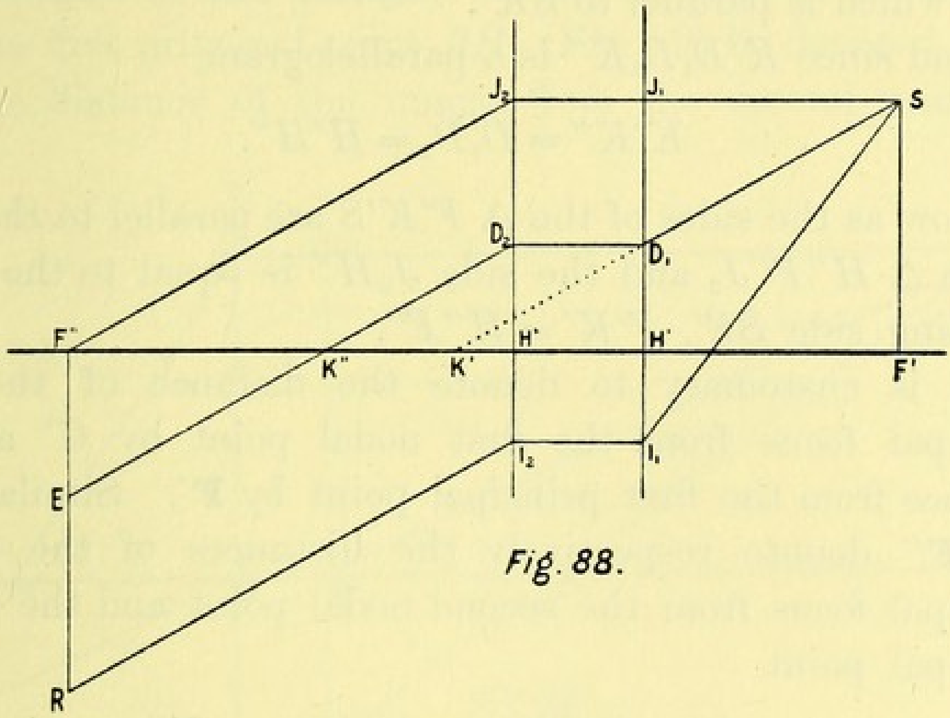

Fig. 88.

is found in the eye, the refraction of which we shall have to consider in detail in a subsequent chapter.

Let S be a luminous point in the first focal plane, and let SJ_1J_2 represent one of its rays, viz. that parallel to the principal axis. We know that it will be deviated on leaving the refracting system in the direction J_2F'' where J_2 is the point on the second principal plane, corresponding to J_1 on the first. Similarly any other incident ray SI_1 meeting the first principal plane in I_1 will emerge in the direction I_2R, parallel to J_2F'', where I_2 is the point on the second principal plane, corresponding to I_1 on the first.

In order to find the nodal points we have merely to draw a line SK' parallel to J_2F'', cutting the axis in K' and the first principal plane in D_1. Through D_1 draw D_1D_2 parallel to the axis, and through D_2 on the second principal plane draw D_2E parallel to J_2F'', meeting the axis in K'' and the second focal plane in E.

Then K', K'' are the two nodal points, for an incident ray SK' emerges, after traversing the system, in the direction $K''E$ which is parallel to SK'.

And since $K'D_1D_2K''$ is a parallelogram,

$$K'K'' = D_1D_2 = H'H''.$$

Now as the sides of the $\triangle\ F'K'S$ are parallel to the sides of the $\triangle\ H''F''J_2$ and the side J_2H'' is equal to the corresponding side SF', $F'K' = H''F''$.

It is customary to denote the distance of the first principal focus from the first nodal point by G' and its distance from the first principal point by $\mathbf{F}'$. Similarly G'' and $\mathbf{F}''$ denote respectively the distances of the second principal focus from the second nodal point and the second principal point,

$$\therefore\ G' = H''F'' = -F''H'' = -\mathbf{F}''.$$

Similarly as $F''E = J_2D_2 = J_1D_1$ it can be shewn that

$$\triangle\ K''F''E = \triangle\ SJ'D_1,$$

$$\therefore\ F''K'' = J_1S = H'F' = -F'H',$$

or

$$G'' = -\mathbf{F}'.$$

Now when $\mathbf{F}' = -\mathbf{F}''$, $-G'' = G'$ (Fig. 89),

$$\therefore\ F'H' = -F''H'' = F'K',$$

and similarly $F''H'' = F''K''$.

The nodal points consequently coincide with the principal points whenever $\mathbf{F}' = -\mathbf{F}''$.

The value of $\mathbf{F}'$ is considered positive if the first principal focus F' is on the positive side of H', that is, if it is on the same side of H' as the incident light.

Similarly G' is positive, if F' is on the positive side of K'.

The distances $\mathbf{F}''$, G'' are reckoned positive or negative according as the situation of the point F'' is on the positive or the negative side of the points H'', K'' respectively.

Distance of the Image. If the distance of the object from the first principal point BH' (Fig. 89) be denoted by p, and the distance of the image from the second principal

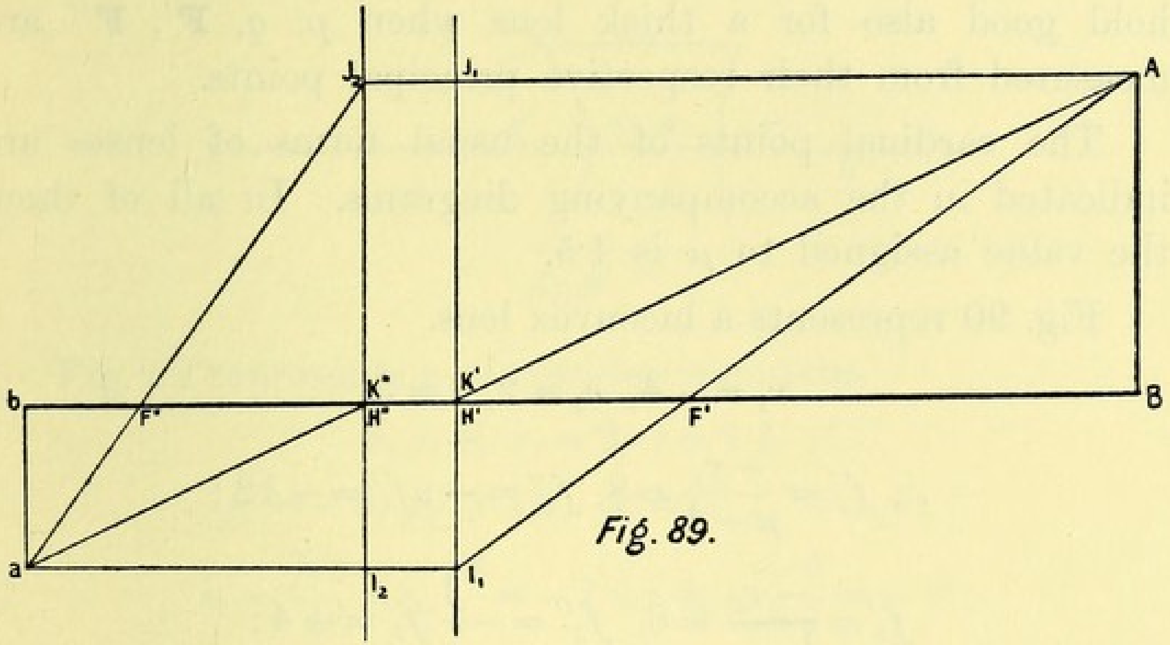

Fig. 89.

point bH'' be denoted by q, it will be found that the formula $\frac{F'}{p} + \frac{F''}{q} = 1$ still holds good when the proper values are given to the symbols.

For on consideration of the diagram (Fig. 89) it will be seen that

$$\frac{\mathbf{F}'}{p} = \frac{F'H'}{BH'} = \frac{F'H'}{AJ_1.} = \frac{I_1H'}{I_1J_1} = \frac{ab}{I_2J_2}$$

$$= \frac{ab}{-J_2I_2} = \frac{bF''}{-I_2a} = \frac{bF''}{bH''} = \frac{q - \mathbf{F}''}{q},$$

$$\therefore \frac{\mathbf{F}'}{p} + \frac{\mathbf{F}''}{q} = 1.$$

This is true universally.

And since $\mathbf{F}'' = -\mathbf{F}'$ in the case considered in Fig. 89,

$$\frac{1}{p} - \frac{1}{q} + \frac{1}{\mathbf{F}''} = 0,$$

whenever the final medium is of the same refractive index as the initial medium.

In a similar manner it may be shewn that the old formulæ for the size of the image formed by a thin lens, hold good also for a thick lens when p, q, $\mathbf{F}'$, $\mathbf{F}''$ are measured from their respective principal points.

The cardinal points of the usual forms of lenses are indicated in the accompanying diagrams. In all of them the value assigned to μ is 1·5.

Fig. 90 represents a biconvex lens.

$$r_1 = -4,\ r_2 = 2,\ t = 3.$$

$$\therefore f_1' = \frac{-r_1}{\mu - 1} = 8,\ f_1'' = -\mu f_1' = -12\,;$$

$$f_2' = \frac{-r_2}{\dfrac{1}{\mu} - 1} = 6,\ f_2'' = -\frac{1}{\mu} f_2' = -4\,;$$

$$h_1 = \frac{f_1' t}{t + f_1'' - f_2'} = \frac{f_1' t}{D} \text{ say, } = \frac{24}{-15} = -1{\cdot}6\,;$$

$$h_2 = \frac{f_2'' t}{D} = \frac{-12}{-15} = {\cdot}8\,;$$

$$F'H' = \frac{-f_1' f_2'}{D} = \frac{-48}{-15} = 3{\cdot}2\,;\ F''H'' = \frac{f_1'' f_2''}{D} = \frac{48}{-15} = -3{\cdot}2.$$

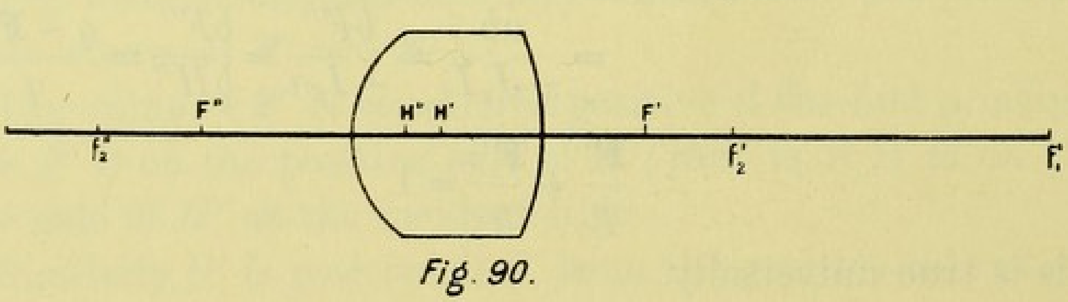

Fig. 90.

Fig. 91 represents a biconcave lens.

$$r_1 = 4,\ r_2 = -2,\ t = 2.$$

$$\therefore f_1' = -8,\ f_1'' = 12;\ f_2' = -6,\ f_2'' = 4;$$

$$h_1 = \frac{f_1' t}{D} = \frac{-16}{20} = -\cdot 8;\ h_2 = \frac{f_2'' t}{D} = \frac{8}{20} = \cdot 4$$

$$F'H' = \frac{-f_1' f_2'}{D} = \frac{-48}{20} = -2\cdot 4;\ F''H'' = 2\cdot 4.$$

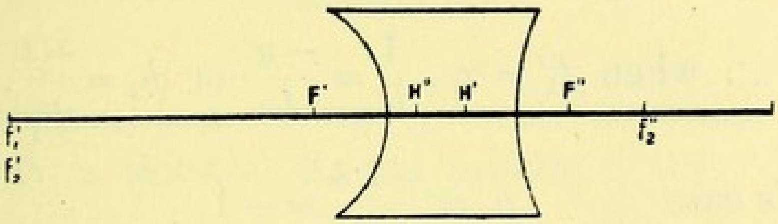

Fig. 91.

Fig. 92 represents a converging meniscus.

$$r_1 = 4,\ r_2 = 2,\ t = 1\cdot 5.$$

$$f_1' = -8,\ f_1'' = 12;\ f_2' = 6,\ f_2'' = -4;$$

$$h_1 = \frac{f_1' t}{D} = \frac{-12}{7\cdot 5} = -1\cdot 6;\ h_2 = \frac{-6}{7\cdot 5} = -\cdot 8;$$

$$F'H' = \frac{48}{7\cdot 5} = 6\cdot 4;\ F''H'' = -6\cdot 4.$$

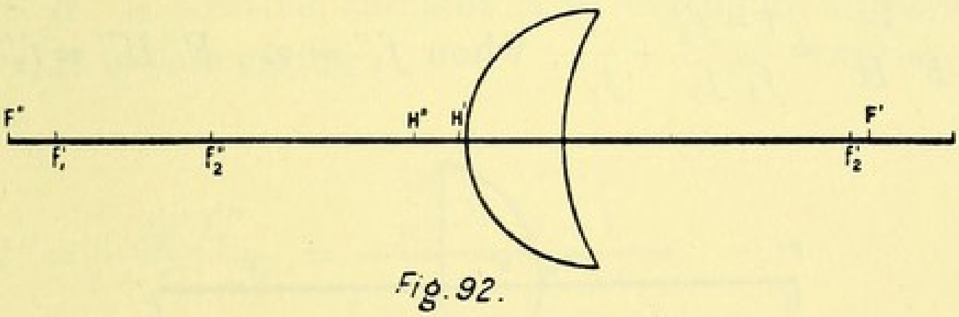

Fig. 92.

Fig. 93 represents a planoconvex lens.

$$r_1 = \pm\infty,\ r_2 = 2,\ t = 1\cdot 5.$$

$$\therefore f_1' = \mp\infty,\ f_1'' = \pm\infty;\ f_2' = 6,\ f_2'' = -4;$$

$$h_2 = \frac{f_2'' t}{t + f_1'' - f_2'} = 0.$$

The second principal point therefore is at the point of intersection of the convex surface with the optic axis.

The remaining cardinal points are of the indeterminate form $\frac{\infty}{\infty}$.

The expression for h_1 can be easily evaluated for

$$\frac{1}{h_1} = \frac{t + f_1'' - f_2'}{f_1' t} = \frac{t - f_2' - \mu f_1'}{f_1' t} = \frac{t - f_2'}{f_1' t} - \frac{\mu}{t},$$

$$\therefore \text{ when } f_1' = \infty, \ \frac{1}{h_1} = \frac{-\mu}{t} \text{ or } h_1 = \frac{-t}{\mu}.$$

In this case $\quad h_1 = \frac{-1{\cdot}5}{1{\cdot}5} = -1.$

Similarly since

$$\frac{1}{F'H'} = \frac{t + f_1'' - f_2'}{-f_1' f_2'} = \frac{t - f_2'}{-f_1' f_2'} + \frac{\mu}{f_2'},$$

when $\quad f_1' = \infty, \ \frac{1}{F'H'} = \frac{\mu}{f_2'},$

$$\therefore F'H' = \frac{f_2'}{\mu} = -f_2'', \text{ which in this case is 4.}$$

Also since

$$\frac{1}{F''H''} = \frac{t - f_2'}{f_1'' f_2''} + \frac{1}{f_2''}, \text{ when } f_1'' = \infty, \ F''H'' = f_2''.$$

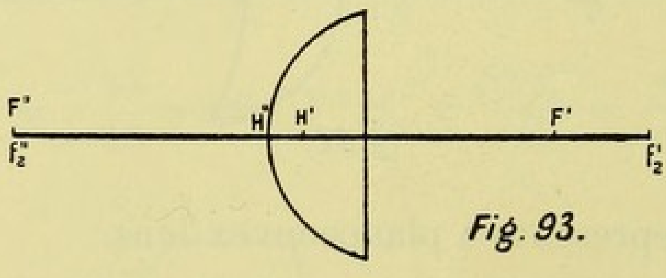

Fig. 93.

Fig. 94 represents a planoconcave lens.

$$r_1 = \pm \infty, \ r_2 = -2, \ t = 1{\cdot}5.$$

$$\therefore f_1' = \mp \infty, \ f_1'' = \pm \infty; \ f_2' = -6, \ f_2'' = 4.$$

As before

$$h''=0,\ h'=\frac{-t}{\mu}=-1\,;\ F'H'=-f_2''=-4,\ F''H''=f_2''=4.$$

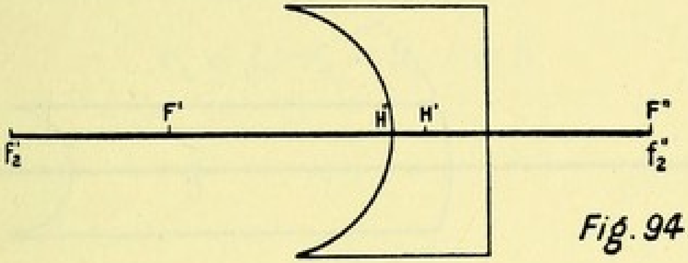

Fig. 94.

Fig. 87 represents a diverging meniscus which has this peculiarity: it is thickest in the middle.

$$r_1=2,\ r_2=4,\ t=3.$$

$$f_1'=-4,\ f_1''=6\,;\ f_2'=12,\ f_2''=-8\,;$$

$$h_1=\frac{f_1't}{D}=\frac{-12}{-3}=4\,;\ h_2=\frac{f_2''t}{D}=\frac{-24}{-3}=8\,;$$

$$F'H'=\frac{48}{-3}=-16\,;\ F''H''=\frac{-48}{-3}=16.$$

The following considerations will make the conditions on which this peculiarity depends apparent.

If $F''H''$ is positive, the lens is diverging in function: if negative, it is converging.

Now

$$F''H''=\frac{f_1''f_2''}{t+f_1''-f_2'}=\frac{\frac{\mu r_1}{\mu-1}\cdot\frac{-r_2}{\mu-1}}{t+\frac{\mu r_1}{\mu-1}-\frac{\mu r_2}{\mu-1}}$$

$$=\frac{-\mu r_1 r_2}{\overline{\mu-1}\,\{t\,\overline{\mu-1}-\mu(r_2-r_1)\}}.$$

Then in a meniscus, in which the values assigned to r_1, and r_2 are both positive.

If $r_2<r_1$, $F''H''$ must be negative.

That is, the lens must be converging in function, and be thickest in the middle (Fig. 92).

If $$r_2 > r_1.$$

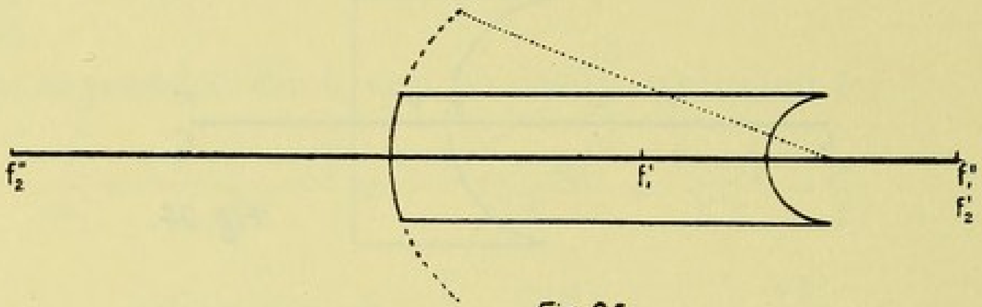

Fig. 95.

Three cases may now arise:

(1) If the centre of the first surface lies within the centre of the second surface, the lens is thinnest in the middle.

$$r_2 - r_1 > t.$$

(2) If the centre of the first surface coincides with the centre of the second surface, the lens is of uniform thickness.

$$r_2 - r_1 = t.$$

(3) If the centre of the first surface lies without the centre of the second surface, the lens is thickest in the middle.

$$r_2 - r_1 < t.$$

In this class three subdivisions may be made:

$$(3') \quad r_2 - r_1 > t\left(\frac{\mu - 1}{\mu}\right),$$

$$(3'') \quad r_2 - r_1 = t\left(\frac{\mu - 1}{\mu}\right),$$

$$(3''') \quad r_2 - r_1 < t\left(\frac{\mu - 1}{\mu}\right).$$

It is evident on consideration of the above formula that $F''H''$ is positive in cases (1), (2) and (3′), $F''H''$ is infinite in (3″), whereas $F''H''$ is negative in (3‴).

Fig. 95 illustrates case 3″.

$$r_1 = 1,\ r_2 = 3,\ t = 6.$$

QUESTIONS.

(1) A meniscus is made in which $r_2 - r_1 = t\dfrac{\mu - 1}{\mu}$. Show that incident parallel rays will emerge parallel, that, when the concave surface is turned towards a radiant point at finite distance, it acts as a converging lens, but that when turned the reverse way it acts as a diverging lens.

(2) Show that in the case of a sphere of any refracting medium the principal points are coincident with the centre of the sphere.

(3) A thick meniscus is made in which $r_2 - r_1 = t$. Show that this will act as a diverging lens as long as the radiant point lies to the right of the centre of curvature, but that it will act as a converging lens when the radiant point lies to the left of the centre of curvature.

(4) From Fig. 89 show that

$$\frac{i}{o} = \frac{q}{p} = \frac{\mathbf{F}'}{\mathbf{F}' - p} = \frac{\mathbf{F}'' - q}{\mathbf{F}''},$$

when $p = BH',\ q = bH'',\ \mathbf{F}' = F'H',\ \mathbf{F}'' = F''H''.$

CHAPTER XIV.

ECCENTRIC PENCILS. FOCAL LINES. CURVATURE OF THE IMAGE. CODDINGTON LENS.

In the preceding chapters we have confined our attention to the consideration of the refraction of thin centric pencils in media bounded by spherical surfaces. We have now to consider the more general problem when the incident pencils are not so limited.

Focal lines. We will first take the case of a thin pencil incident on an eccentric portion of a refracting medium bounded by a spherical surface.

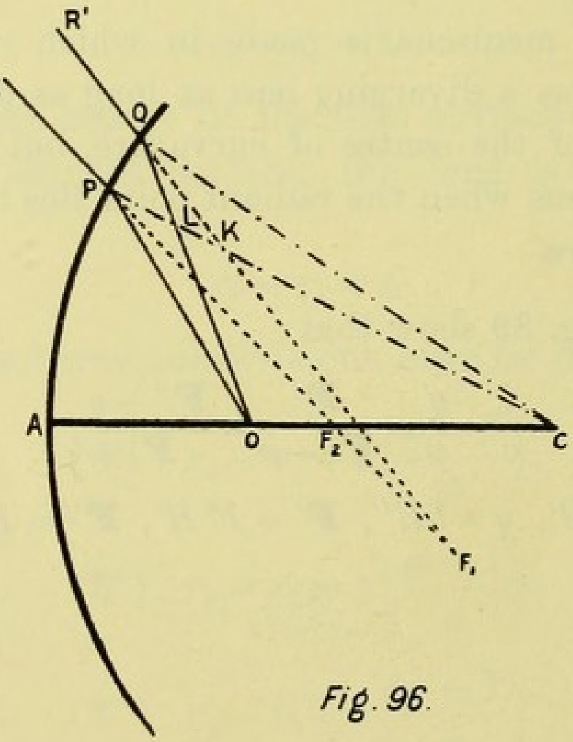

Fig. 96.

Let O be a luminous point, and let C be the centre of curvature of the spherical surface ($\mu > 1$). Join OC and produce it to meet the surface in A. Then OA is the principal axis. Let OP represent the axial ray of a thin eccentric pencil from O incident on the spherical surface, and let OQ represent an extreme ray of the thin pencil considered. Let PR, QR' represent the refracted rays; produce them to meet in F_1 and let RP produced cut the principal axis OA in F_2.

Now if we suppose the figure to rotate through a small angle about the axis COA, PQ will trace out a small segment of the spherical surface, OPQ a small solid cone incident upon it. Meanwhile the point F_1 will trace out a small arc, approximately a straight line, and the line at F_2, indicating the cross-section of the refracted pencil, will trace out a figure that also may be regarded as approximately a straight line. We proceed to determine the position of these two focal lines.

Let $\quad OP = u, \quad F_1P = v_1, \quad$ and $F_2P = v_2$.

Join PC and QC. Then the angles of incidence of OP and OQ are CPO and CQO respectively, whereas the angles of refraction are CPF_1 and CQF_1. Now when the incident pencil is very small, the angle POQ is very small, and the axial and extreme rays of the pencil may be considered to form equal angles of incidence (ϕ), and equal angles of refraction (ϕ'). Moreover, under these conditions the arc PQ may be replaced by the chord PQ, and the angle CQP may be regarded as a right angle.

Then in the triangle OPQ we have

$$\frac{PQ}{OP} = \frac{\sin POQ}{\sin OQP} = \frac{\sin POQ}{\sin (CQP - CQO)} = \frac{\sin POQ}{\cos CQO},$$

or $$\angle POQ = PQ \, . \, \frac{\cos \phi}{u}.$$

Similarly, in the triangle F_1PQ,

$$\frac{PQ}{F_1P} = \frac{\sin PF_1Q}{\sin F_1QP} = \frac{\sin PF_1Q}{\sin (CQP - CQF_1)} = \frac{\sin PF_1Q}{\cos CQF_1},$$

or
$$\angle PF_1Q = PQ \, . \frac{\cos \phi'}{v_1} .$$

And
$$\angle PCQ = \frac{PQ}{r} .$$

Now if we wish to find the limiting value of the ratio between the small quantities $CQO - CPO$ and $F_1QO - F_1PO$, we can no longer regard these differences as vanishing quantities.

Since the vertical angles of the triangles PLO, QLC are equal,

$$\angle POL + \angle LPO = \angle LCQ + \angle CQL,$$

or
$$\angle POQ + \phi = \angle PCQ + \phi + \triangle \phi.$$

$$\therefore \; \triangle \phi = \angle POQ - \angle PCQ.$$

Similarly, from the triangles PKF_1, QKC we have

$$\angle PF_1K + \angle KPF_1 = \angle KCQ + \angle CQK,$$

or
$$\angle PF_1Q + \phi' = \angle PCQ + \phi' + \triangle \phi'.$$

$$\therefore \; \triangle \phi' = \angle PF_1Q - \angle PCQ.$$

Then $\dfrac{\triangle \phi}{\triangle \phi'}$, or in the limit

$$\frac{d\phi}{d\phi'} = \frac{\angle POQ - \angle PCQ}{\angle PF_1Q - \angle PCQ} = \frac{PQ \left(\dfrac{\cos \phi}{u} - \dfrac{1}{r} \right)}{PQ \left(\dfrac{\cos \phi'}{v_1} - \dfrac{1}{r} \right)} .$$

But since
$$\sin \phi = \mu \sin \phi',$$

$$\cos \phi \, \frac{d\phi}{d\phi'} = \mu \cos \phi'.$$

$$\therefore \; \frac{\mu \cos \phi'}{\cos \phi} = \frac{\dfrac{\cos \phi}{u} - \dfrac{1}{r}}{\dfrac{\cos \phi'}{v_1} - \dfrac{1}{r}} .$$

or $$\frac{\mu \cos^2 \phi'}{v_1} - \frac{\cos^2 \phi}{u} = \frac{\mu \cos \phi' - \cos \phi}{r} \quad\text{............(1)}.$$

Again, since the area of the triangle CPO is equal to the sum of the areas of CPF_2, F_2PO,

$$\tfrac{1}{2} r (-u) \sin \phi = \tfrac{1}{2} r (-v_2) \sin \phi' + \tfrac{1}{2} v_2 (-u) \sin (\phi - \phi').$$

Dividing by $\dfrac{-uv_2 r \sin \phi'}{2}$, we have

$$\frac{\mu}{v_2} = \frac{1}{u} + \frac{\sin(\phi - \phi')}{r \sin \phi'},$$

$$\frac{\mu}{v_2} - \frac{1}{u} = \frac{\sin \phi \cos \phi' - \cos \phi \sin \phi'}{r \sin \phi'} = \frac{\mu \cos \phi' - \cos \phi}{r} \quad\text{...(2)}.$$

On combining (1) and (2) we obtain

$$\frac{\mu \cos^2 \phi'}{v_1} - \frac{\cos^2 \phi}{u} = \frac{\mu}{v_2} - \frac{1}{u} = \frac{\mu \cos \phi' - \cos \phi}{r} \quad\text{......(3)}.$$

A similar result is obtained, whatever may be the position of the source of light (O). An examination however of equation (3) shews that the relative magnitude of v_1 and v_2 depends upon the value of u; *i.e.* upon the position of O.

For equation (3) may be written

$$\frac{\mu}{v_1} - \frac{\mu \sin^2 \phi'}{v_1} - \frac{1}{u} + \frac{\sin^2 \phi}{u} = \frac{\mu}{v_2} - \frac{1}{u},$$

or $$\frac{\mu}{v_1} - \frac{\mu}{v_2} = \frac{\mu \sin^2 \phi'}{v_1} - \frac{\sin^2 \phi}{u} = \sin^2 \phi \left(\frac{1}{\mu v_1} - \frac{1}{u} \right);$$

$$\therefore\ v_2 - v_1 = \frac{v_2 \sin^2 \phi}{u \mu^2} (u - \mu v_1).$$

Hence $v_2 = v_1$ if $\phi = 0$; *i.e.* if the point O is at the centre of the concave refracting surface, when there is no refraction. In this case $u = r$, which excludes the case of the convex refracting surface, for u is positive and cannot be equal to r which is negative.

Also $v_2 \gtreqless v_1$ as $u \gtreqless \mu v_1$.

It is evident that v_2 cannot be equal to v_1 if the surface be convex. For, since u and μ are always positive, v_1 must also be positive (or virtual); and if $\mu > 1$, $v_1 > u$; and if $\mu < 1$, $v_1 < u$. Therefore if the refracting surface be convex u cannot be equal to μv_1.

If the refracting surface be concave $v_2 = v_1$, that is, there is no aberration, if $u = \mu v_1$. The oblique eccentric rays are in that case so refracted as to appear to come from the same point Q as the axial rays, where

$$\frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r}.$$

But then $\frac{u}{v}$ and consequently $\frac{p}{q} = \mu$.

$$\therefore \frac{\mu^2}{p} - \frac{1}{p} = \frac{\mu - 1}{r}.$$

Or $$(\mu + 1)\, r = p.$$

Whether the refracting surface be concave or convex, it will be evident that when $v_2 > v_1$, the cusp of the caustic points away from the surface, but that when $v_2 < v_1$, the cusp of the caustic points towards the surface. When the refracting surface is concave, there are two cases in which no caustic is formed, viz. when $p = r$ and when $p = r + \mu r$.

Hence it is evident that under certain conditions a spherical lens may be formed that shall be entirely free from aberration. The lens must be of the form of a meniscus with its concave surface facing the object P. If r_1 denote the radius of curvature of the first surface, and if PA_1 or $p = r + \mu r$, the result of the refraction at the first surface will be the formation of a definite virtual image at Q, such that $QA_1 = \frac{PA_1}{\mu}$ (Fig. 97). If now the second surface of the meniscus be such that $r_2 = QA_2$, the image at Q will be distinct and free from aberration.

Similarly an aplanatic converging meniscus for the point P may be obtained when $PA_1 = r_1$ and

$$PA_2 = r_2\left(\frac{1}{\mu} + 1\right),$$

or
$$r_2 = \frac{\mu PA_2}{\mu + 1}.$$

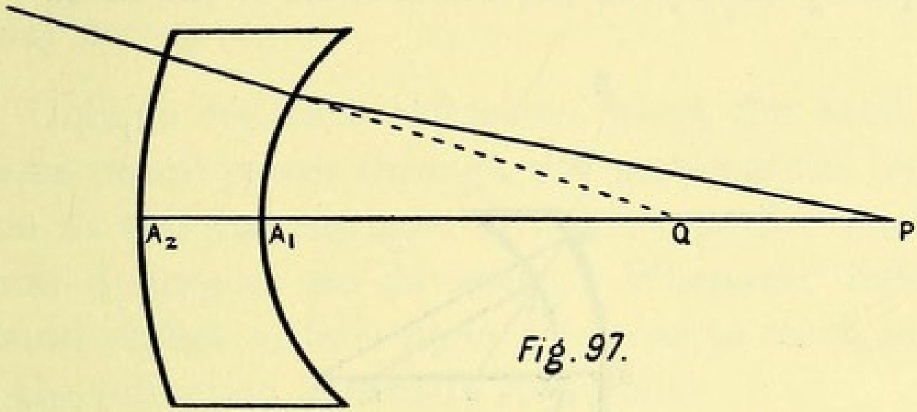

Fig. 97.

The distance of the image Q from the second surface is QA_2 and

$$\frac{QA_2}{PA_2} = \mu.$$

These are the only conditions under which an aplanatic lens can be formed of spherical surfaces.

Optical Centre. Before considering the refraction of small oblique pencils by thin lenses, it will be convenient to investigate the position and nature of a point which is known as *the optical centre* of the lens.

The optical centre is the point in which the line, joining the extremities of parallel radii of the two bounding surfaces, cuts the axis.

Let $BAOC_1C_2$ represent the axis of the lens (Fig. 98), and let C_1J_1 represent any radius (r_1) of the first surface, and let C_2J_2 be a radius (r_2) of the second surface parallel to C_1J_1. Join J_2J_1 and produce the line to meet the axis in O. Then O is the optical centre of the lens.

By similar triangles

$$\frac{C_1O}{C_2O}=\frac{C_1J_1}{C_2J_2} \text{ or } \frac{r_1}{r_2}.$$

The point O is therefore a fixed point on the axis whatever pair of parallel radii we employ, since its position on the axis depends merely on the dimensions of the radii.

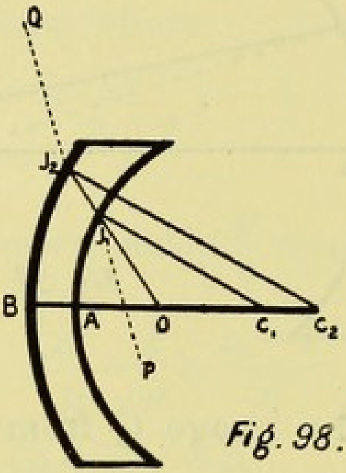

Fig. 98.

Let the thickness of the lens AB be denoted by t, then we have

$$\frac{C_1O}{C_2O} \text{ or } \frac{r_1-OA}{r_2-t-OA}=\frac{r_1}{r_2};$$

$$\therefore\ OA=\frac{r_1t}{r_2-r_1} \text{ and } t=\frac{r_2-r_1}{r_1}OA.$$

If the lens is biconvex r_1 is negative, if biconcave r_1 is positive but r_2 is negative, consequently under these conditions the optical centre lies within the lens.

The centre of a lens thus determined has the following important optical property. Any beam of light such as PJ_1, passing through a lens in such a manner that its direction while within the lens passes through the centre, will on emerging from the lens have a direction J_2Q parallel to its direction when incident on the lens; and conversely, every emergent ray that is parallel to its corresponding incident ray must have been directed towards the optical centre in its course through the lens.

This follows at once from the fact that the tangents at the two points where refraction in this case takes place, are parallel, and therefore the effect on this ray is the same as that due to refraction through a plate.

It is evident that two cases of oblique refraction through a lens may occur.

(1) Oblique centrical refraction, when the axial ray of the oblique pencil passes through the centre of the lens after refraction at the first surface. In this case the axial ray of the pencil undergoes no deviation. Whenever light from any natural object so falls upon a lens as to reach its whole surface, the refraction is usually centrical.

(2) Oblique eccentrical refraction, when the axis of the oblique pencil while within the lens does not pass through the centre of the lens. This usually occurs when the light considered falls only on a peripheral portion of the lens.

Oblique Centrical Refraction. Fig. 99 (1). Let P be the origin of light, PK' the axial ray of the pencil con-

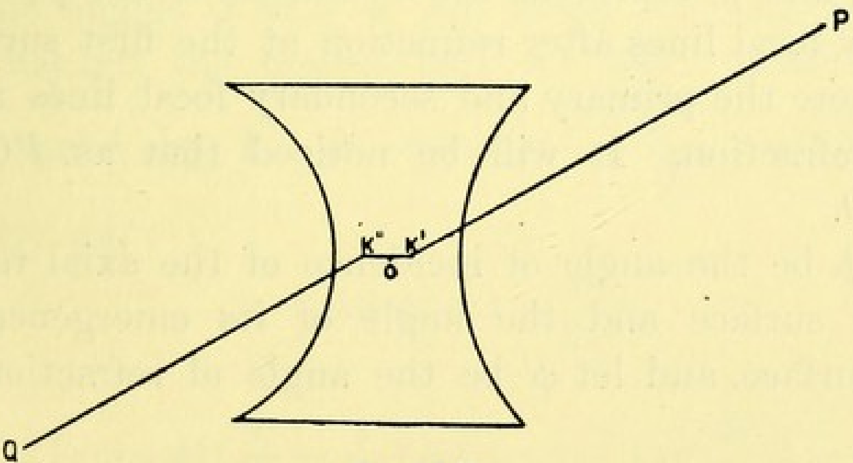

Fig. 99 (1)

sidered, and let $K'OK''$ be the direction of the axial ray within the lens, passing through O its optical centre. On emerging from the second surface of the lens this axial ray will proceed in a direction $K''Q$ parallel to its incident direction PK'.

If we regard the thickness of the lens as negligible we may replace the two nodal points K' and K'' by the optical centre O, and HO will represent the principal plane. Then in Fig. 99 (2) PO represents the axial ray of the incident

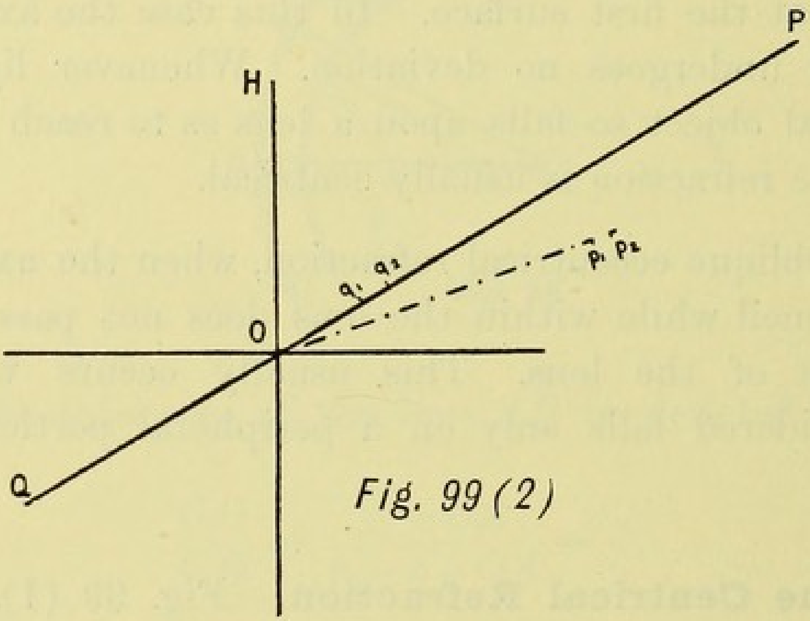

Fig. 99 (2)

pencil, p_1 and p_2 represent the situations of the primary and secondary focal lines after refraction at the first surface, and q_1, q_2 denote the primary and secondary focal lines after the second refraction. It will be noticed that as $PO > \mu p_1 O$, $p_2 O > p_1 O$.

Let ϕ be the angle of incidence of the axial ray PO at the first surface and the angle of its emergence at the second surface, and let ϕ' be the angle of refraction within the lens.

Let $\qquad PO = U, \qquad q_1 O = V_1, \qquad q_2 O = V_2.$

For the first refraction

$$\frac{\mu\cos^2\phi'}{p_1O}-\frac{\cos^2\phi}{U}=\frac{\mu\cos\phi'-\cos\phi}{R_1},$$

$$\frac{\mu}{p_2O}-\frac{1}{U}=\frac{\mu\cos\phi'-\cos\phi}{R_1}.$$

For the second refraction ϕ' is the angle of incidence, ϕ that of refraction, and $\frac{1}{\mu}$ is the refractive index. We have therefore the relations

$$\frac{\frac{1}{\mu}\cos^2\phi}{V_1}-\frac{\cos^2\phi'}{p_1O}=\frac{\frac{1}{\mu}\cos\phi-\cos\phi'}{R_2},$$

$$\frac{\frac{1}{\mu}}{V_2}-\frac{1}{p_2O}=\frac{\frac{1}{\mu}\cos\phi-\cos\phi'}{R_2}.$$

On multiplying these equations by μ and adding the first of the latter pair to the first of the former pair, we obtain

$$\frac{\cos^2\phi}{V_1}-\frac{\cos^2\phi}{U}=(\mu\cos\phi'-\cos\phi)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Similarly on adding the remaining equations together we have

$$\frac{1}{V_2}-\frac{1}{U}=(\mu\cos\phi'-\cos\phi)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

If the angles ϕ and ϕ' are small we may neglect all powers beyond the second, and we may replace $\cos\phi$ by $1-\frac{\phi^2}{\underline{|2}}$, $\cos^2\phi$ by $1-\phi^2$, $\cos\phi'=1-\frac{\phi'^2}{\underline{|2}}$, and, by Snell's law, ϕ by $\mu\phi'$;

$$\therefore\ \cos\phi'=1-\frac{\phi^2}{2\mu^2},$$

and

$$(\mu\cos\phi'-\cos\phi)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)=(\mu-1)\left(1+\frac{\phi^2}{2\mu}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Now let the second principal focal distance of the lens be denoted by f, then

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

and $$\cos^2\phi\left(\frac{1}{V_1} - \frac{1}{U}\right) = \frac{1}{f}\left(1 + \frac{\phi^2}{2\mu}\right) = \frac{1}{V_2} - \frac{1}{U}$$

approximately.

Or $$\frac{1}{V_1} - \frac{1}{U} = \frac{1}{f} \cdot \frac{1 + \frac{\phi^2}{2\mu}}{1 - \phi^2} = \frac{1}{f}\left\{1 + \phi^2\left(1 + \frac{1}{2\mu}\right)\right\}$$

approximately.

If x denote the distance of the circle of least confusion from the centre of the lens, and if we regard the cross-section (at O) of the astigmatic pencil, that enters the pupil of the observer's eye, as circular, we have

$$\frac{1}{V_1} + \frac{1}{V_2} = \frac{2}{x} \text{ (\textit{vid.} p. 113 (3))};$$

$$\therefore \frac{1}{x} - \frac{1}{U} = \frac{1}{f}\left\{1 + \phi^2\left(\frac{1}{2} + \frac{1}{2\mu}\right)\right\}.$$

If then a lens be so inclined that the plane of the glass makes an angle with the incident wave front, two focal lines are formed when the refraction is centric. Consider, for example, a plane wave incident on a biconvex lens at an angle of 20°.

$$\mu = 1{\cdot}54, \quad R_1 = -108 \text{ mm.}, \quad R_2 = 108 \text{ mm.}$$

When the angle of incidence is normal,

$$\frac{1}{f''} = \overline{\mu - 1}\left(\frac{1}{-108} - \frac{1}{108}\right) = -\frac{1}{100} \text{ mm.}$$

In other words the lens is $10D$ convex.

On inclining this lens 20° to the vertical or normal plane

$$\frac{\cos^2 20^\circ}{V_1} - \frac{\cos^2 20^\circ}{U} = (\mu \cos 12^\circ\, 49'\, 54'' - \cos 20^\circ)\left(\frac{-2}{108}\right)$$

and $$\frac{1}{V_2} - \frac{1}{U} = (\mu \cos 12^\circ \, 49' \, 54'' - \cos 20^\circ)\left(\frac{-2}{108}\right).$$

As the incident light is considered to be of plane waves U is infinite, and therefore the term containing $\frac{1}{U}$ vanishes.

Thus $$\frac{\cdot 883}{V_1} = -\frac{(1{\cdot}54 \times \cdot 975 - \cdot 9397)}{54} = -\frac{\cdot 5618}{54}$$

and $$\frac{1}{V_2} = -\frac{\cdot 5618}{54}.$$

Therefore $$V_1 = -\frac{54 \times \cdot 883}{\cdot 5618} = -84{\cdot}8 \text{ mm.}$$

and $$V_2 = -\frac{54}{\cdot 5618} = -96{\cdot}1 \text{ mm.}$$

The lens so inclined in fact refracts light in much the same way as a

$$+10{\cdot}4D \text{ sph.} \bigcirc + 1{\cdot}4D \text{ cyl. ax.} - 180.$$

This property of tilted lenses is sometimes of service to aphakic patients. After the lens has been removed by operation, the cornea is usually rendered less convex from above downwards than from side to side. As these patients usually require strong convex glasses, their astigmatism if slight may be corrected by inclining the convex spherical lenses instead of providing them with spherocylinders.

The cornea of a hypermetropic or of a myopic person is very often more curved from above downwards than from side to side. Hence astigmatic myopic patients often see best through biconcave lenses when they are tilted. Such a patient, if not provided with the appropriate spherocylinders, will wear his *pince-nez* inclined on his nose, and so will be able to correct his astigmatism if it be of low degree.

Oblique Eccentrical Refraction. Oblique eccentrical refraction takes place when for some reason the axial ray of the incident light does not traverse the optical centre of the lens. Sometimes an opaque obstacle is in the way. At other times the source of light is itself an image formed by reflection or refraction. The light emitted by such an image differs from that emitted by a real object in that it can only diverge from the image in the lines in which it had previously converged to form the image. Hence the pencil of light proceeding from any point of such an image is limited by its own mode of formation. Thus in the compound microscope eccentrical refraction occurs at the eyepiece by which the image formed by the objective is magnified.

The exact mathematical investigation of the form of a pencil after oblique eccentrical refraction through a lens is laborious and difficult, and moreover does not admit of any simple approximate expression. It will be sufficient for our present purpose to consider the principal consequences of oblique eccentrical refraction in the case of a microscope or telescope.

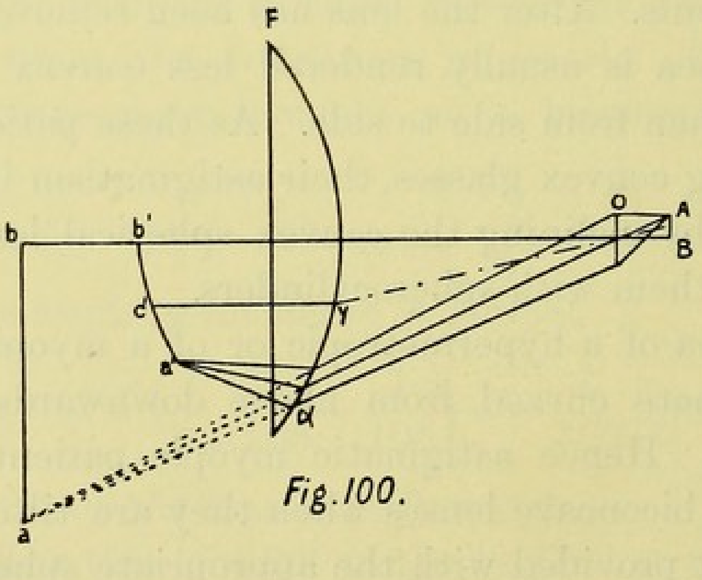

Fig. 100.

The compound microscope consists of a system of lenses called the objective, situated at *O*, the purpose of which is to form an inverted distinct plane image *ab* of the object *AB*. This image *ab* is subsequently magnified by the eyepiece. In practice it is found expedient to form the eyepiece of two plano-convex lenses separated by an interval; the lower lens is called the field-lens, for it increases the field of view of the instrument, the upper lens is called the eye-lens. In the Huygenian eyepiece, the most common form, the field-lens *F* is placed below the image formed by the objective. Consequently the image *ab* is not actually formed, but the converging pencils, proceeding towards the separate points of the image *ab*, are made to converge towards the separate points of the image $a'b'$. Now since the cone of light that corresponds to any point of the image meets only an exceedingly small portion of the field-lens, we may neglect the aberrations which occur within each of the incident cones. The point b' therefore will be distinct, for it is formed by a small direct centric pencil. Similarly we may regard each point of the image $a'b'$ as being fairly distinctly formed by a minute eccentric pencil, and we have to consider in what way the spherical aberration of the field-lens will affect their relative position. For this purpose we may consider the non-aplanatic field-lens as consisting of several annular zones, the refracting power of each zone increasing with its distance from the centre. The axial ray of the peripheral pencil $\alpha a'$ will consequently undergo a greater deviation than that of an intermediate pencil such as $\gamma c'$. The consequence of this will be that the image $a'b'$ will not only be smaller than *ab* but it will be distorted, for the surface elements of the image will be diminished in proportion to their distance from the axis.

Further, since the refracting power of the peripheral zone is greater than that of a more centrally situated zone, the focus a' of the peripheral pencil will be nearer the lens than the focus c' of the intermediate pencil. Hence the image will appear curved, its concavity being directed towards the incident light.

It is important to bear in mind that the distortion of the image does not depend upon its curvature, though it has been so explained in some of the older books on this subject. An image may be strongly curved yet may not appear distorted, and *vice versâ*.

If the image *ab* were represented as a network of squares (Fig. 101), the effect of the field-lens would be to form a distorted image (Fig. 102) owing to the peripheral parts of the image being more diminished than the central parts. If, however, a plano-convex lens were placed on the further side of *ab* so as to form a magnified virtual image of it, the distortion would be opposite in character as in Fig. 103, for

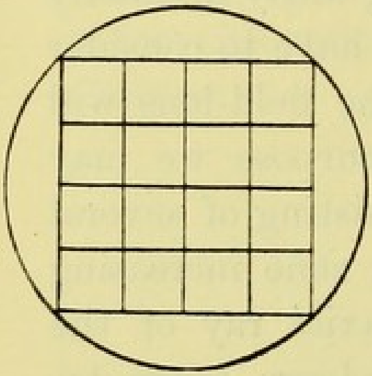

Fig. 101.

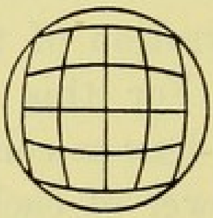

Fig. 102.

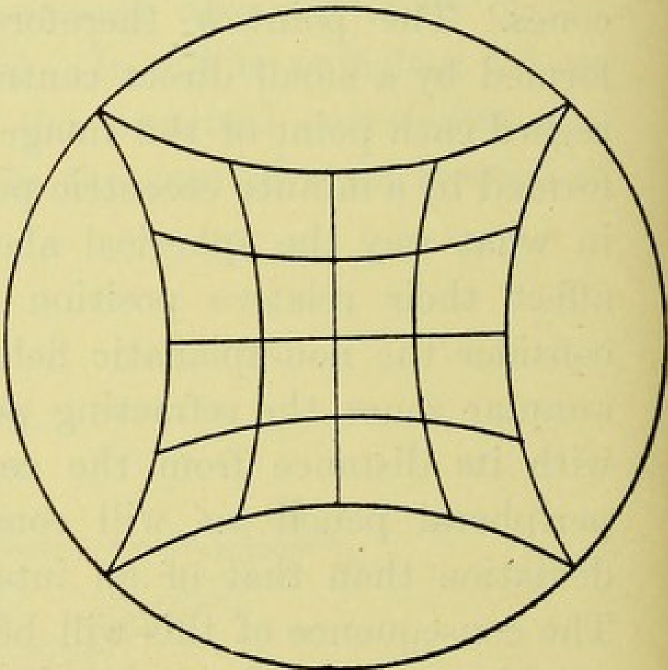

Fig. 103.

then the peripheral portions of the image would be more magnified than the more central portions. In like manner the image will be curved, its concave surface being directed upwards.

The construction of the Huygenian eyepiece depends upon the choice of such an eye-lens that the distortion and curvature of its virtual image shall be exactly equal and opposite to the distortion and curvature produced by the field-lens (Fig. 102). When then the image $a'b'$ is magnified by the eye-lens the resulting virtual image is undistorted and flat[1].

The Huygenian eyepiece consists of two plano-convex lenses separated by an interval a. The curved surface of each lens faces the incident light. If the focal distance of the first or field-lens be denoted by f_1, and that of the second or eye-lens be denoted by f_2,

$$f_1 = 3f_2 \text{ and } a = -2f_2.$$

Curvature of Images. The image of a plane object, whether real or virtual, formed by a spherical lens is not plane but appears curved.

The effects of spherical aberration may be practically eliminated by placing a diaphragm pierced with a small central aperture immediately in front of the lens. The refraction is then limited to the central part of the lens, and in such cases the curvature of the image is entirely due to the obliquity of the pencils that proceed from those points of the object that are more remote from the principal axis.

Let O represent the optical centre of a thin concave lens, the principal axis of which is OB. Let AB represent a section of the curved object and let ab be its image. The divergent pencil from B that falls on the lens whose axial ray is represented by BO is centric and direct, whereas the pencil from A will be centric and oblique. Hence the virtual image a of the point A will not be so distinct as

[1] It may be also noted that the chromatic aberration caused by the field-lens is obliterated in the final virtual image formed by this combination, owing to the inverse chromatic aberration induced by the eye-lens (p. 311).

the image of B, for the pencil from A will proceed after traversing the lens as though it were diverging from two focal lines in the neighbourhood of a. Let U be the distance from O of the point of the object under consideration, and let the angle of obliquity of the axial ray of the pencil considered be denoted by ϕ.

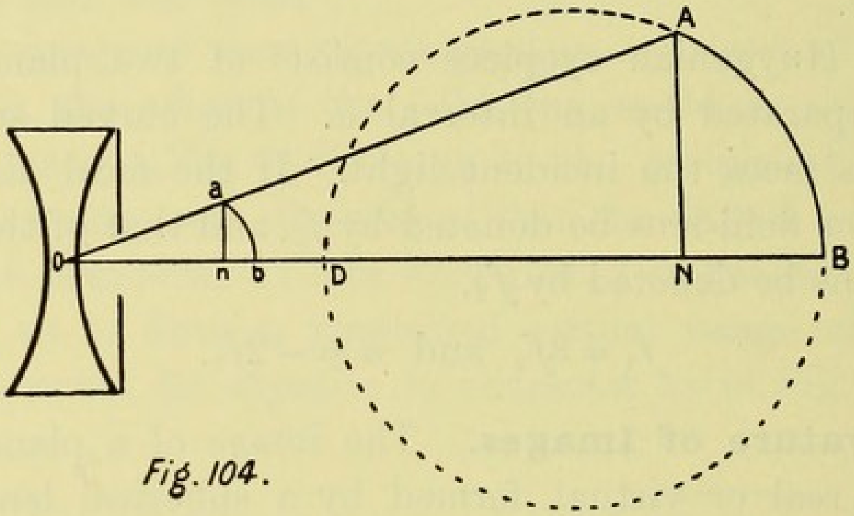

Fig. 104.

Then if ϕ be small

$$\frac{1}{V_1} - \frac{1}{U} = \frac{1}{f}\left\{1 + \phi^2\left(1 + \frac{1}{2\mu}\right)\right\} \text{ approximately,}$$

and

$$\frac{1}{V_2} - \frac{1}{U} = \frac{1}{f}\left(1 + \frac{\phi^2}{2\mu}\right).$$

Either formula may be replaced by the expression

$$\frac{1}{V} - \frac{1}{U} = \frac{1}{f}(1 + k\phi^2),$$

if it be remembered that when V refers to the primary focal line $k = 1 + \frac{1}{2\mu}$, and when V refers to the secondary focal line $k = \frac{1}{2\mu}$.

Draw the ordinates AN, an perpendicular to the axis. For the points B, b, the obliquity ϕ is 0;

$$\therefore \frac{1}{bO} - \frac{1}{BO} = \frac{1}{f},$$

and for the points A, a,

$$\frac{1}{aO}-\frac{1}{AO}=\frac{1}{f}(1+k\phi^2).$$

Multiplying the first of these equations by $\cos\phi$ or $1-\frac{\phi^2}{2}$,

we obtain $$\frac{nO}{aO\,.\,bO}-\frac{NO}{AO\,.\,BO}=\frac{1}{f}\left(1-\frac{\phi^2}{2}\right);$$

on subtracting this equation from

$$\frac{1}{aO}-\frac{1}{AO}=\frac{1}{f}(1+k\phi^2),$$

we get $$\frac{bn}{aO\,.\,bO}-\frac{BN}{AO\,.\,BO}=\frac{1}{f}(\tfrac{1}{2}+k)\,\phi^2.$$

But when ϕ is small, $AO\,.\,BO=(AO)^2$ approximately,

$$\therefore\ AO\,.\,BO\,.\,\phi^2=(AO)^2\,\phi^2=(AO\,.\,\sin\phi)^2=(AN)^2$$

approximately.

Similarly $aO\,.\,bO\phi^2=(an)^2$ approximately.

So we get $$\frac{bn}{(an)^2}-\frac{BN}{(AN)^2}=\frac{2k+1}{2f}.$$

Now if DB be the diameter $(2r)$ of the circle of curvature of the object corresponding to the part adjacent to B,

$$(AN)^2=BN\,.\,ND \text{ or } \frac{BN}{(AN)^2}=\frac{1}{ND},$$

but ND in the limit when A approaches B, is equal to BD or $-2r$;

$$\therefore\ -\frac{1}{2r}=\text{L}^{\text{t}}\,\frac{BN}{(AN)^2}.$$

Similarly if $\frac{1}{r'}$ represent the curvature of the image in the neighbourhood of b,

$$-\frac{1}{2r'} = \mathrm{L}^t \frac{bn}{(an)^2}.$$

So
$$-\frac{1}{2r'} + \frac{1}{2r} = \frac{2k+1}{2f},$$

or
$$\frac{1}{r'} - \frac{1}{r} = -\frac{2k+1}{f}.$$

This gives the relation between the curvatures of an object and its image.

If the object be plane, $\frac{1}{r} = 0$; and the radius of curvature of the image has a sign opposite to that of f, the second principal focus of the lens.

Thus if F'' is positive the radius of curvature of the image is measured in the direction opposed to that of the incident light; and *vice versâ*. In the figure, the curvature of the object is negative and that of the image is also negative. If the lens had been convex and the object had been plane, the image, whether real or virtual, would have been curved in the positive direction.

It should be remarked that the curvature of the image is independent of the distance of the object.

In assigning a value to k, we must determine whether the primary or secondary focal line is to be regarded as the focus; if the former $k = 1 + \frac{1}{2\mu}$, if the latter $k = \frac{1}{2\mu}$. If on the other hand the circle of least confusion is chosen as representing the focus, k has the value

$$\frac{1}{2} + \frac{1}{2\mu} \text{ (p. 280).}$$

The defects of an image formed by a lens when the light is homogeneous may be summed up in the following way:

I. When the lens is provided with a diaphragm or "stop" (as in the preceding case) and when the object is real.

The central part of the image is distinct, since it is formed by a direct and centric pencil.

The peripheral parts of the image are indistinct, since they are formed by oblique and centric pencils.

The whole image is curved owing to the obliquity of the incident pencils from the peripheral parts of the object.

II. When there is no stop, and the source of light is itself an image formed by some previous refraction or reflection (Figs. 100, 102, 103).

The individual parts of the image are approximately distinct if each individual cone of light meet only a very small portion of the lens.

The peripheral portions of the image are distorted from spherical aberration, *i.e.* from the greater refracting power of the peripheral zones of the lens.

The image is curved from spherical aberration.

III. When there is no stop and the source of light is a real object, the whole surface of the lens receives the incident cones of light.

The central part of the image is indistinct from spherical aberration.

The peripheral parts of the image are indistinct, both from spherical aberration and from the obliquity of incidence of the pencils that form them.

The image is curved chiefly from the varying degrees of obliquity of the incident pencils.

Refraction of a Sphere. The sphere may be considered as a kind of double convex lens, and there are certain advantages attending its use as a magnifying glass which we will proceed to investigate.

We have seen that refraction through a lens may be direct and centric, oblique and centric or eccentric, but it is clear that in the case of a sphere all centric pencils whether direct or oblique pass normally into the lens; further, the eccentric pencils have this peculiarity that the angle of incidence (ϕ) is equal to the angle of emergence (ψ), and the angle of refraction (ϕ') at the first surface is equal to the angle of incidence (ψ') at the second surface.

It occurred to Wollaston to cement two hemispherical lenses by their plane sides with a stop interposed so as to exclude all but centric pencils. The idea was followed up by Brewster and Coddington, who employed somewhat different methods for the same purpose. The common form of magnifying glass generally known as the "Coddington lens" is of the form represented in Fig. 105. The lens may be regarded as a sphere round which a deep equatorial groove has been ground, so that all the emergent light must have passed very nearly through the centre of the lens. All effective incident pencils may therefore be treated as though they were direct and centric pencils.

A simple expression for the relations of the conjugate distances may be readily obtained if we take the centre of the sphere as the point from which all distances are to be measured. Let x, x' denote the distances from the centre of the object and image due to the first refraction, and let y denote the distance of the final image due to the second refraction.

Then since $$\frac{p-r}{q-r}=\mu\frac{p}{q} \text{ (p. 210)},$$

$$\frac{x}{x'}=\mu\frac{x+r_1}{x'+r_1},$$

or $$xx'+xr_1=\mu xx'+\mu r_1 x'.$$

On dividing by $\mu x x' r_1$ we obtain

$$\frac{1}{\mu r_1} + \frac{1}{\mu x'} = \frac{1}{r_1} + \frac{1}{x}.$$

For the second refraction, by substituting $\frac{1}{\mu}$, x', y and r_2 for μ, x, x', and r_1 in the last expression, and on dividing throughout by μ, we obtain

$$\frac{1}{r_2} + \frac{1}{y} = \frac{1}{\mu r_2} + \frac{1}{\mu x'}.$$

On adding these expressions we get

$$\frac{1}{x} - \frac{1}{y} + \left(1 - \frac{1}{\mu}\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 0.$$

If x is infinite, *i.e.* if the incident rays are parallel,

$$\frac{1}{f''} = \left(1 - \frac{1}{\mu}\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \text{ and } \frac{1}{x} - \frac{1}{y} + \frac{1}{f''} = 0.$$

And since r_1 is numerically equal to r_2 but is negative in sign

$$\frac{1}{f''} = -\frac{2(\mu - 1)}{\mu r_2}.$$

If the sphere be glass ($\mu = 1{\cdot}5$) the second principal focus is at a distance of half a radius from the posterior surface of the sphere.

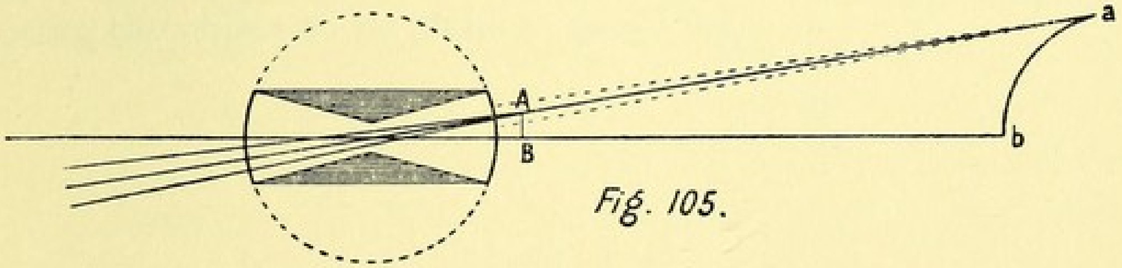

Fig. 105.

The "Coddington lens" is a very serviceable pocket magnifying glass and has this important advantage over an ordinary convex lens—the peripheral parts of the virtual image are as distinct as the central part, since the pencil from A is centric and direct as well as that from B. In practice it is found that the central aperture must not be greater than a fifth of the focal distance of the sphere or appreciable indistinctness will occur from spherical aberration. The image will be curved owing to the greater distance of the peripheral parts of the object from the centre of the sphere than that of the more central part of the object.

This form of lens however suffers from two serious defects; it has a very limited field of view, for only those emergent pencils are effective which can enter the pupil of the eye of the observer, and it has a very short working distance. It is advisable therefore for the eye of the observer to be brought as close as possible to the posterior surface of the lens.

The Stanhope lens is somewhat similar to the Coddington lens and may therefore be mentioned in this place. It is a short glass cylinder with its ends ground convex to an unequal degree of curvature to diminish the spherical aberration. The length of the cylinder is such that when the object is placed on the surface of lesser curvature, and the more convex surface is turned towards the eye, a distinct magnified image of the object is seen.

QUESTIONS.

(1) Parallel rays strike a thin lens at an angle of incidence of 30°. Give the ratio of the distances of the focal lines from the lens.

(2) An aphakic patient requires the following glasses for distance: $+ 12\,D$ sph. $\bigcirc + 1{\cdot}5\,D$ cyl. ax. $-$ 180°. What inclination should be given to biconvex spherical glasses to correct the astigmatism, and what should be their focal length if $\mu = 1{\cdot}54$?

(3) A spherical water bottle of radius 3 ins. is filled with water. Where is the focus for a small incident parallel pencil, neglecting the thickness of the glass and assuming $\frac{4}{3}$ as the refractive index of water?

(4) A Coddington lens of $\frac{1}{4}$ in. radius is placed so that its posterior surface is $\frac{1}{2}$ in. from the nodal point of an eye whose *punctum proximum* is 10 ins. from the nodal point. What is the magnifying power, and what is the curvature of the image, supposing the object to be plane? ($\mu = 1{\cdot}5$.)

CHAPTER XV.

LONGITUDINAL ABERRATION. APLANATIC LENSES. ACHROMATIC COMBINATIONS.

In this chapter we proceed to find an approximate expression for the aberration of a wide centric pencil when refracted at a single spherical surface and when refracted by a lens bounded by two spherical surfaces. The method adopted is taken from Professor Heath's *Geometrical Optics* with such alteration of the signs as to make the expressions consistent with the convention employed in this book—that the direction of the incident light is considered positive, and the reverse direction negative.

Let P be the source of light, Fig. 106, and let KAK' be the refracting surface. We have seen in the previous chapter that the thin eccentric pencil, whose axial ray is denoted by PK, will be so refracted that it appears to come from two focal lines at f_1 and f_2. Now if we consider the wide cone of incident light on the whole refracting surface, the locus of f_1 will be the caustic curve produced (shewn in dotted lines), while the longitudinal aberration is denoted by f_2Q, and the lateral aberration will be represented by $2QR$. The longitudinal aberration f_2Q is the more important, and is what is usually understood by the term aberration when used alone.

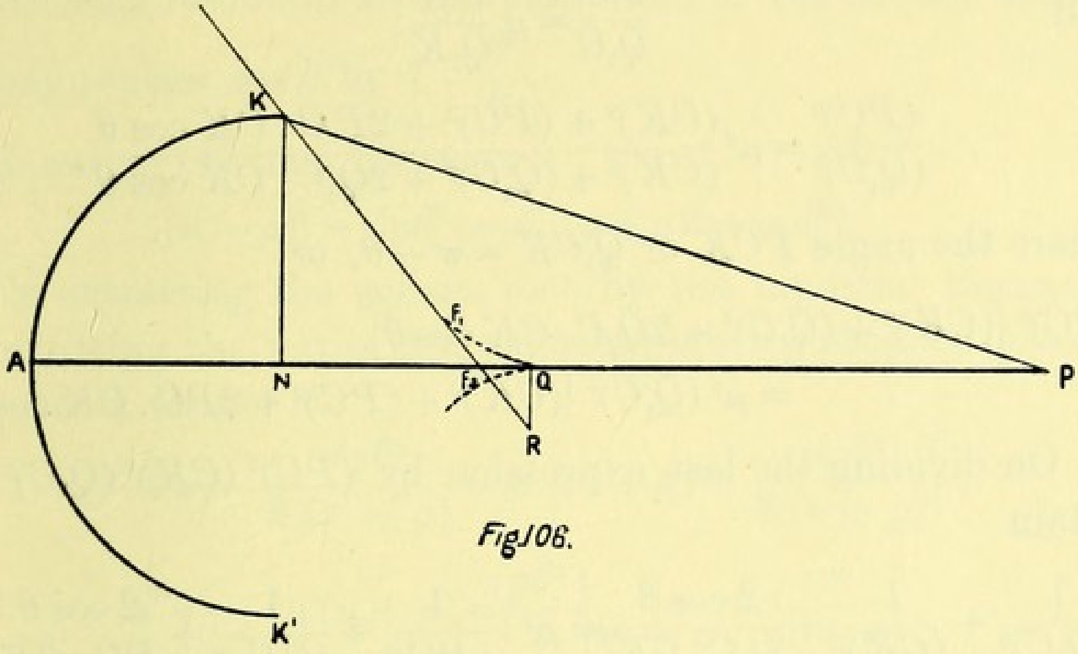

Fig 106.

Aberration at a single spherical surface. Let CA be the radius of curvature of the refracting surface KA (Fig. 107), let P be a point on the principal axis, and let Q_0 be the focus conjugate to P for the thin centric pencil from P. Let Q_1 denote the position of the second

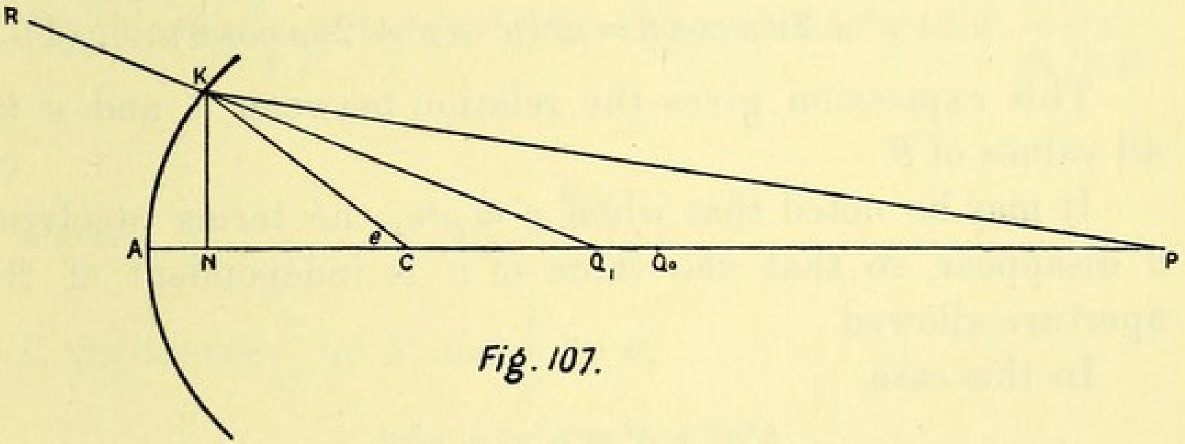

Fig. 107.

focal line formed by the refraction of the eccentric pencil PK. Let the angle KCA or θ denote half the angle of aperture of the surface considered.

Then we know (p. 205),

$$\frac{PC}{Q_0C} = \mu \frac{PA}{Q_0A} \text{ or } \frac{p-r}{q-r} = \mu \frac{p}{q},$$

and $$\frac{PC}{Q_1C} = \mu \frac{PK}{Q_1K};$$

$$\therefore \frac{(PC)^2}{(Q_1C)^2} = \mu^2 \frac{(CK)^2 + (PC)^2 + 2PC \, . \, CK \cos\theta}{(CK)^2 + (Q_1C)^2 + 2Q_1C \, . \, CK \cos\theta},$$

where the angle PCK or $Q_1CK = \pi - \theta$, or

$$(PC)^2 \{(CK)^2 + (Q_1C)^2 + 2Q_1C \, . \, CK \cos\theta\}$$
$$= \mu^2 (Q_1C)^2 \{(CK)^2 + (PC)^2 + 2PC \, . \, CK \cos\theta\}.$$

On dividing the last expression by $(PC)^2 (CK)^2 (Q_1C)^2$ we obtain

$$\frac{1}{(Q_1C)^2} + \frac{1}{(CK)^2} + \frac{2\cos\theta}{Q_1C \, . \, CK} = \mu^2 \left(\frac{1}{(PC)^2} + \frac{1}{(CK)^2} + \frac{2\cos\theta}{PC \, . \, CK} \right).$$

If we denote

$\frac{1}{PC}$ or $\frac{1}{p-r}$ by u, $\frac{1}{CK}$ or $\frac{1}{r}$ by ρ, $\frac{1}{Q_1C}$ or $\frac{1}{q'-r}$ by v',

and $\frac{1}{Q_0C}$ or $\frac{1}{q-r}$ by v_0, this expression becomes

$$v'^2 + \rho^2 + 2v'\rho\cos\theta = \mu^2 (u^2 + \rho^2 + 2u\rho\cos\theta) \ldots\ldots(1).$$

This expression gives the relation between v' and u for all values of θ.

It may be noted that when $v' = \mu^2 u$, the terms involving θ disappear, so that the value of v' is independent of the aperture allowed.

In this case,

$$\mu^4 u^2 + \rho^2 = \mu^2 u^2 + \mu^2 \rho^2,$$

or $$\mu^2 u^2 (\mu^2 - 1) = \rho^2 (\mu^2 - 1),$$

or $$\mu u = \pm \rho,$$

i.e. $\mu r = p - r$, as p must be positive. This is the same result that was given on p. 274.

Now when θ is a small quantity of the first order, and we merely wish to obtain an approximation of the second order, we may replace $\cos\theta$ by $1-\frac{\theta^2}{2}$.

Equation (1) then becomes

$$(v'+\rho)^2 - v'\rho\theta^2 = \mu^2\{(u+\rho)^2 - u\rho\theta^2\}.$$

On extracting the square root by the binomial theorem, and omitting the terms that contain higher powers of θ than the second, we get

$$(v'+\rho) - \frac{v'\rho\theta^2}{2(v'+\rho)} = \mu\left\{(u+\rho) - \frac{u\rho\theta^2}{2(u+\rho)}\right\},$$

or $$v'+\rho-\mu(u+\rho) = \frac{\rho\theta^2}{2}\left\{\frac{v'}{v'+\rho} - \frac{\mu u}{u+\rho}\right\}.$$

But when the pencil is thin and centric, $\theta = 0$, and

$$v_0+\rho = \mu(u+\rho);$$

$$\therefore\ v'-v_0 = \frac{\rho\theta^2}{2}\left\{\frac{v'}{v'+\rho} - \frac{\mu u}{u+\rho}\right\} \quad\text{.........(2).}$$

Since θ^2 is of the second order of small quantities, we may replace $\frac{v'}{v'+\rho}$ by its first approximate value $\frac{v_0}{v_0+\rho}$ or $\frac{r}{q}$.

Then $$v'-v_0 = \frac{\theta^2}{2}\left\{\frac{1}{q} - \frac{\mu}{p}\right\}.$$

If we denote $\frac{1}{q}$ by β, and $\frac{1}{p}$ by α,

$$v'-v_0 = \frac{\theta^2}{2}\{\beta - \mu\alpha\}.$$

Then $v'-v_0$ represents a small quantity of the second order; let it be represented by the differential dv. Similarly let the differential $d\beta$ represent a small quantity of the second order such that the fraction $\frac{d\beta}{dv}$ may be equal to the ratio, the result of differentiation which is similarly expressed.

Then $$dv = \frac{\theta^2}{2}\{\beta - \mu\alpha\}.$$

Now $$\frac{1}{\beta} = \frac{1}{v_0} + \frac{1}{\rho};$$

$$\therefore\ -\frac{1}{\beta^2}\frac{d\beta}{dv} = -\frac{1}{v_0^2} \text{ or } \frac{d\beta}{dv} = \frac{\beta^2}{v_0^2}.$$

Also from the same equation

$$\frac{1}{v_0} = \frac{\rho - \beta}{\rho\beta}.$$

Then $$d\beta = \frac{\beta^2}{v_0^2}\,dv = \frac{\theta^2}{2\rho^2}(\rho - \beta)^2(\beta - \mu\alpha).$$

Now we may replace $\frac{\theta^2}{\rho^2}$ by y^2, if y denote the ordinate KN or the semi-aperture considered, and within the bracket we may substitute $\frac{\mu\beta - \alpha}{\mu - 1}$ for ρ.

Then $$d\beta = \frac{y^2}{2(\mu - 1)^2}(\beta - \alpha)^2(\beta - \mu\alpha) \qquad \text{............(3).}$$

Since $$\beta = \frac{1}{q},\ dq = -q^2 d\beta,$$

$$\therefore\ dq = -\frac{y^2 q^2}{2(\mu - 1)^2}(\beta - \alpha)^2(\beta - \mu\alpha).$$

It will be noticed that the aberration Q_1Q_0 or dq is negative as in the figure unless β is less than $\mu\alpha$ or unless β is negative.

Aberration of a Lens. Let Q_1, Q_2 be the conjugate foci of P due to the refraction at the first and second surfaces of the lens in Fig. 108.

Let $$\alpha = \frac{1}{PA_1},\ \beta = \frac{1}{Q_1A_1},\ \beta' = \frac{1}{Q_1A_2},\ \alpha' = \frac{1}{Q_2A_2},$$

and let ρ_1, ρ_2 represent the curvatures of the first and second surfaces of the lens respectively.

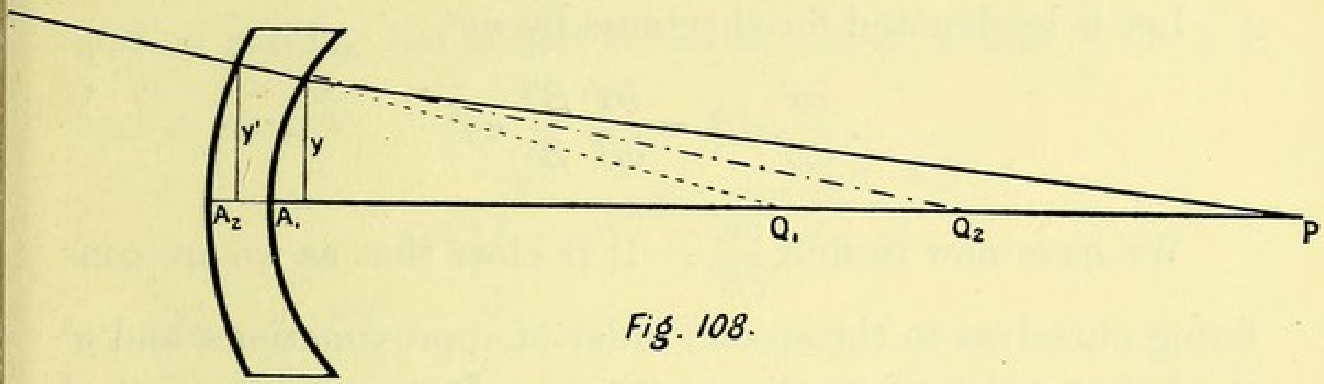

Fig. 108.

If the incident pencil be thin and centric,

$$(\alpha - \rho_1) = \mu(\beta - \rho_1) \text{ and } (\alpha' - \rho_2) = \mu(\beta' - \rho_2) \ldots(1),$$

since the index of refraction at the second surface is $\frac{1}{\mu}$.

When the aperture of the lens is not negligible, we must take into account the effective aperture at each surface of the lens. Let the semi-apertures y and y' be small quantities of the first order. Then if we consider the extreme pencil of the incident cone of light, the value of α' must depend on the values of β' and y', or $\alpha' = f(\beta', y')$.

$$\therefore d\alpha' = \frac{\partial \alpha'}{\partial \beta'} d\beta' + \frac{\partial \alpha'}{\partial y'} dy',$$

where $\frac{\partial \alpha'}{\partial \beta'} d\beta'$ means the partial differential of α' supposing that y' does not vary, and $\frac{\partial \alpha'}{\partial y'} dy'$ means the partial differential of α' supposing that β' does not vary.

Now it is evident that $\frac{1}{\beta'} - \frac{1}{\beta} = t$ where t denotes the axial thickness of the lens.

$$\therefore \frac{d\beta'}{d\beta} = \frac{\beta'^2}{\beta^2}.$$

But $d\beta$ denotes the variation of β due to a change of y the semi-aperture of the first surface, and the value of this was found in the previous investigation (3).

Let it be denoted for shortness by κy^2,

$$\therefore \frac{\partial\alpha'}{\partial\beta'}\,d\beta' = \frac{\partial\alpha'}{\partial\beta'}\,\frac{\beta'^2}{\beta^2}\,\kappa y^2.$$

We have now to find $\frac{\partial\alpha'}{\partial\beta'}$. It is clear that as we are confining ourselves to the second order of approximations, and y^2 is of that order of small quantities, a first approximation of the value of $\frac{\partial\alpha'}{\partial\beta'}$ is allowable; equation (1) will serve this purpose, whence we get $\frac{d\alpha'}{d\beta'} = \mu$.

Therefore $$\frac{\partial\alpha'}{\partial\beta'}\,d\beta' = \mu\kappa\,\frac{\beta'^2}{\beta^2}\,y^2.$$

Now $\frac{\partial\alpha'}{\partial y'}\,dy'$ means the partial differential of α' due to a change of y'; it is therefore comparable to $d\beta$ and may be denoted by $\kappa' y'^2$. A glance at the figure will shew that $\frac{y'}{y}$ is very nearly equal to $\frac{\frac{1}{\beta'}}{\frac{1}{\beta}}$, and that to the degree of approximation to which we are working $y'^2 = \frac{\beta^2}{\beta'^2}\,y^2$.

Therefore $$\frac{\partial\alpha'}{\partial y'}\,dy' = \kappa'\,\frac{\beta^2}{\beta'^2}\,y^2.$$

Then $d\alpha'$ or $\frac{\partial\alpha'}{\partial\beta'}\,d\beta' + \frac{\partial\alpha'}{\partial y'}\,dy' = y^2\left(\mu\kappa\,\frac{\beta'^2}{\beta^2} + \kappa'\,\frac{\beta^2}{\beta'^2}\right)$.

We have now to assign to κ and κ' their proper values.

From equation (3)

$$\kappa = \frac{1}{2(\mu-1)^2}(\beta-\alpha)^2(\beta-\mu\alpha),$$

and on making the proper substitutions β', α' and $\frac{1}{\mu}$ for α, β and μ, we get

$$\kappa' = \frac{\mu^2}{2(1-\mu)^2}(\alpha'-\beta')^2\left(\frac{\mu\alpha'-\beta'}{\mu}\right);$$

$$\therefore \quad d\alpha' = \frac{\mu y^2}{2(\mu-1)^2}\left\{\frac{\beta'^2}{\beta^2}(\beta-\alpha)^2(\beta-\mu\alpha) - \frac{\beta^2}{\beta'^2}(\beta'-\alpha')^2(\beta'-\mu\alpha')\right\} \quad \text{.........(2).}$$

From this expression the spherical aberration of a lens, whatever may be its thickness, may be determined to the second order of small quantities, if y the semi-aperture may be regarded as a member of the first order of small quantities.

When the thickness of the lens is negligible $\beta' = \beta$,

and $$d\alpha' = \frac{\mu y^2}{2(\mu-1)^2}\{(\beta-\alpha)^2(\beta-\mu\alpha) - (\beta-\alpha')^2(\beta-\mu\alpha')\}.$$

But from equation (1) to a first approximation

$$\beta-\alpha = \frac{\mu-1}{\mu}(\rho_1-\alpha) \text{ and } \beta-\mu\alpha = \frac{\mu-1}{\mu}\{\rho-(\mu+1)\alpha\},$$

and similar expressions are obtained for $\beta-\alpha'$ and $\beta-\mu\alpha'$.

We may substitute these expressions in the above equation and still to the second order of approximations,

$$d\alpha' = \frac{\mu-1}{2\mu^2}y^2\{(\rho_1-\alpha)^2(\rho_1-\overline{\mu+1}\alpha) - (\rho_2-\alpha')^2(\rho_2-\overline{\mu+1}\alpha')\} \quad \text{......(3).}$$

Let us now consider some of the common forms of lenses, and compare their aberration when the incident light presents a plane wave-front. The distance of the object is then infinite, so

$$\alpha = 0, \text{ and } \alpha' = \frac{1}{f''}.$$

First let us take a plano-spherical lens with its curved surface facing the light, then $\rho_2 = 0$,

$$d\alpha' = \frac{\mu - 1}{2\mu^2} y^2 \{\rho_1^3 + \alpha'^3 (\mu + 1)\};$$

and since

$$\alpha' = \frac{1}{f''} = \overline{\mu - 1}\rho_1,$$

$$d\alpha' = y^2 \rho_1^3 \frac{\mu - 1}{2\mu} (\mu^3 - 2\mu^2 + 2) \ldots\ldots\ldots\ldots \text{(A)}.$$

If the plano-spherical lens be turned the opposite way, so that its plane surface faces the incident light, $\rho_1 = 0$ and

$$\alpha' = -\overline{\mu - 1}\rho_2,$$

$$d\alpha' = -y^2 \rho_2^3 \frac{\mu^2}{2} (\mu - 1) \ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}.$$

If the lens be a double spherical lens, either biconvex or biconcave, which has surfaces of equal curvature, $\rho_1 = -\rho_2$,

$$d\alpha' = y^2 \rho_1^3 \frac{\mu - 1}{\mu} (4\mu^3 - 4\mu^2 - \mu + 2) \ldots\ldots\ldots \text{(C)}.$$

Now $\alpha' = \frac{1}{f}$ where f denotes the second principal focal distance,

$$\therefore \frac{d\alpha'}{df} = -\frac{1}{f^2} \text{ or } df = -f^2 d\alpha',$$

$\therefore$ in case (A) where $f = \frac{1}{\overline{\mu - 1}\rho_1}$,

$$df = -\frac{y^2 \rho_1}{2\mu\,\overline{\mu - 1}} (\mu^3 - 2\mu^2 + 2) = -\frac{y^2}{f} \frac{\mu^3 - 2\mu^2 + 2}{2\mu(\mu - 1)^2} \ldots \text{(A}'\text{)},$$

in case (B) where $f = \dfrac{-1}{\mu - 1\,\rho_2}$,

$$df = \frac{1}{2} y^2 \rho_2 \frac{\mu^2}{\mu - 1} = -\frac{1}{2}\frac{y^2}{f}\left(\frac{\mu}{\mu - 1}\right)^2 \text{.........(B}'\text{)},$$

in case (C) where $f = \dfrac{1}{\mu - 1\; 2\rho_1}$,

$$df = -\frac{y^2\rho_1}{4\mu\;\mu - 1}(4\mu^3 - 4\mu^2 - \mu + 2) = -\frac{y^2}{f}\frac{4\mu^3 - 4\mu^2 - \mu + 2}{8\mu\,(\mu - 1)^2}$$
$$\text{......(C}'\text{)}.$$

It will be noticed that the aberration is least in A′, greatest in B′ and intermediate in C′. Supposing the lenses to be made of a crown glass having an index of refraction 1·5 the aberration or df in A′ is $-\dfrac{7}{6}\dfrac{y^2}{f}$, in B′ it is $-\dfrac{9}{2}\dfrac{y^2}{f}$, while in C′ it is $-\dfrac{5}{3}\dfrac{y^2}{f}$. It must not be thought from this expression that the aberration depends upon the focal length; it depends essentially upon the curvature of the surfaces. Thus a biconvex of flint glass will shew considerably less spherical aberration than a biconvex of crown glass of the same focal length. It is found that the form of lens which will bring parallel rays of light to a focus with the minimum of aberration must be such that, if r_1 denote the radius of curvature of the first surface and r_2 that of the second surface,

$$\frac{r_1}{r_2} = \frac{2\mu^2 - \mu - 4}{2\mu^2 + \mu},$$

so that if $\mu = 1{\cdot}5$, $\dfrac{r_1}{r_2} = -\dfrac{1}{6}$; if $\mu = 1{\cdot}686$, the lens should have its posterior surface plane.

Lenses formed with this relation of their curvatures to minimize their spherical aberration are called "crossed lenses." It will be seen that in a crossed lens for an index of refraction 1·5, $\rho_1 = -6\rho_2$, and from the expression (3) it will be found that the aberration $df = -\frac{15}{14}\frac{y^2}{f}$, so that the crossed lens is little better than the plano-spherical lens for which the aberration $df = -\frac{7}{6}\frac{y^2}{f}$, when made of glass of the same refractive index.

Hence in most telescopes the object-glass consists of a planoconvex lens with its curved surface facing the incident parallel rays[1]. The lowest lens in the objective of a microscope consists of a hemispherical lens with its plane surface facing the object which is placed very nearly in the principal focus of the lens. The rays of light therefore which emerge from the convex surface are nearly parallel.

It is indeed impossible to construct a lens of spherical surfaces that shall be entirely free from spherical aberration, when the incident rays are parallel. When the incident pencil is divergent and direct, an aplanatic meniscus may be constructed on the principles described on p. 274. Such an

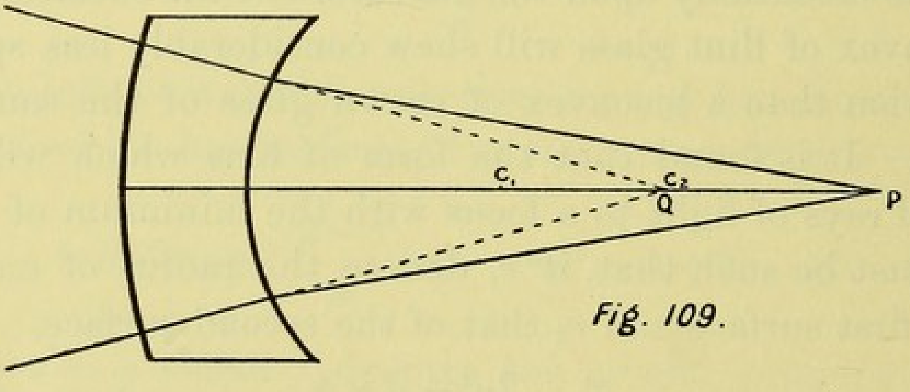

Fig. 109.

aplanatic meniscus is represented in Fig. 109, where Q is the virtual image of P formed without any spherical aberration.

Aplanatic Surfaces. We shall proceed to find what shape the surfaces of a lens should be in order to bring incident parallel rays to a focus without aberration. Hitherto

[1] For a similar reason, when sphero-cylindrical glasses are ordered for astigmatic patients for distance, the optician should so arrange them that the cylindrical side faces the eye, while the spherical side faces the incident light.

it has not been practicable to make lenses of this shape, for there are great mechanical difficulties in the way of grinding glass to any shapes other than plane, spherical or cylindrical. It should be remembered that white light consists of various coloured components, each of which travels with a different velocity through glass, or in other words μ has a different value for each of these components. Since the form of a lens of a given power depends on the value of μ, it is impossible to obtain a lens which shall give the minimum spherical aberration for more than one of the components of white light.

In order therefore to find the theoretical shape of a surface that shall refract incident spherical waves of light to a focal point without aberration we must consider the light to be homogeneous.

Let S (Fig. 110) be the monochromatic luminous point from which spherical waves are diverging in all directions

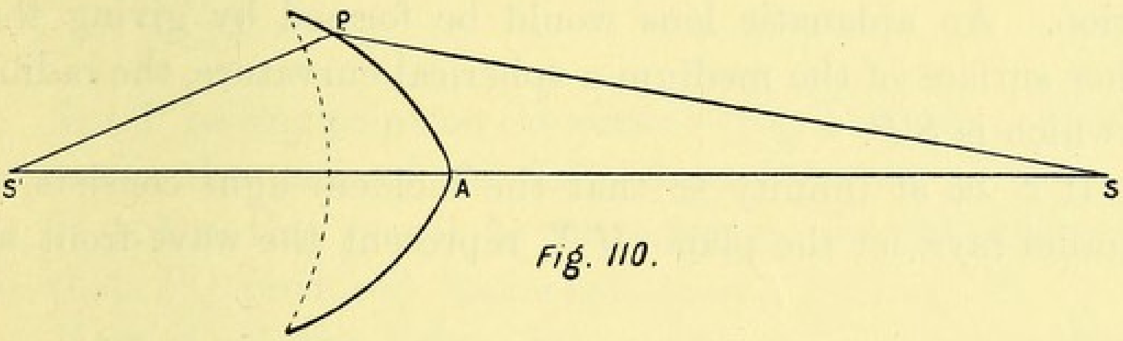

Fig. 110.

in the first medium; and let S' be the point in the second medium, which is bounded by the surface AP, towards which it is required that the incident waves should converge without aberration. A definite image of S will be formed at S' if all the incident light that enters the second medium reaches S' in the same phase. The surface AP must then be such that the light incident at any point of it P must take the same time to traverse the course $SP + PS'$.

Let the velocity of the light under consideration in the first medium be V and its velocity in the second medium be V'.

Then the time taken in traversing the path $SP + PS'$ must be constant,

$$\therefore \frac{SP}{V} + \frac{PS'}{V'} = \text{constant.}$$

Multiplying by V and remembering that $\frac{V}{V'}$ is the relative refractive index of the second medium,

$$SP + \mu PS' = \text{constant.}$$

The locus of P determined by this equation determines the form of the surface. But this is the equation to a Cartesian oval, its more usual form being $r \pm \mu r' = a$, where μ and a are constants. In the case given $r + \frac{3}{2}r' = 14\frac{1}{2}$. It is only the part of the curve in the neighbourhood of the vertex that resembles the arc of a circle, which explains why a spherical surface refracts a thin axial pencil without aberration. An aplanatic lens would be formed by giving the other surface of the medium a spherical curvature, the radius of which is $S'P$.

If S be at infinity so that the incident light consists of parallel rays, let the plane $H'X'$ represent the wave-front at

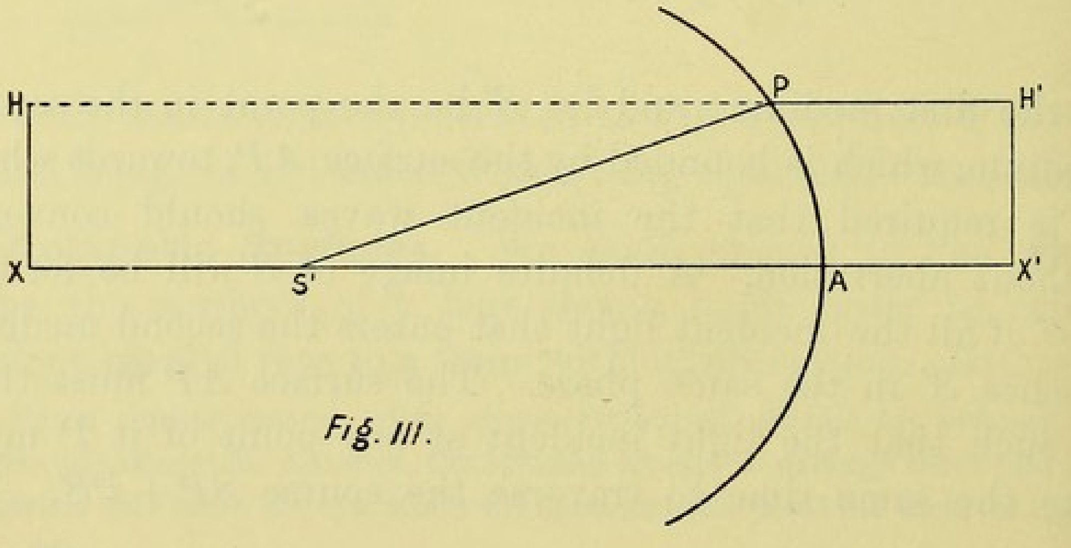

Fig. 111.

a finite distance from the surface (Fig. 111). Then if AP represent the surface of the medium, whose refractive index is μ, all the light incident on AP will come to a focus at S' if $H'P + \mu PS' = c$ a constant. Draw the plane HX parallel to $H'X'$, so that $H'P + PH = c$.

Then $$PH = \mu PS' \text{ or } \frac{S'P}{HP} = \frac{1}{\mu}.$$

The curve AP is therefore an ellipsoid of revolution if $\mu > 1$, and a hyperboloid of revolution if $\mu < 1$.

If it is required to form at S' a virtual image of the luminous point S a slight modification in the investigation of the form of the surface will be necessary.

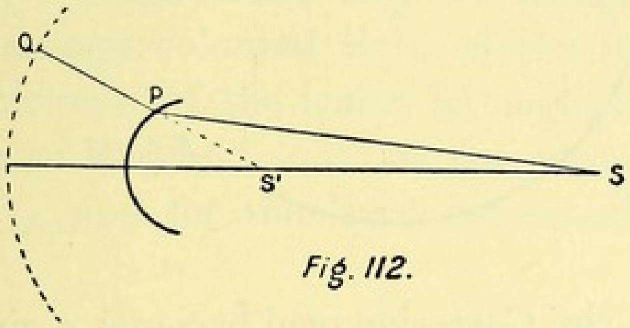

Fig. 112.

Let P be any point on the surface (Fig. 112), then SP will represent the path of the light from S which is incident at P. Let this pencil be refracted in the direction PQ so that PQ produced backwards passes through S'.

Now if all the light that enters the second medium appears to come from S', it is clear that this refracted light must present a spherical wave-front with centre at S'; the locus of Q is therefore part of a sphere;

$$\therefore \ S'P + PQ = c \text{ a constant};$$

$$\therefore \ \mu S'P + \mu PQ = \mu c.$$

But $$SP + \mu PQ = k \text{ a constant},$$

since the locus of Q is a wavefront;

$$\therefore \ SP - \mu S'P = k - \mu c \text{ a constant}.$$

The locus of P is then the second form of Cartesian oval usually denoted by the expression $r - \mu r' = a$.

If the constant $k = \mu c$, $SP - \mu S'P = 0$ and $\frac{SP}{S'P} = \mu$ (Fig. 113).

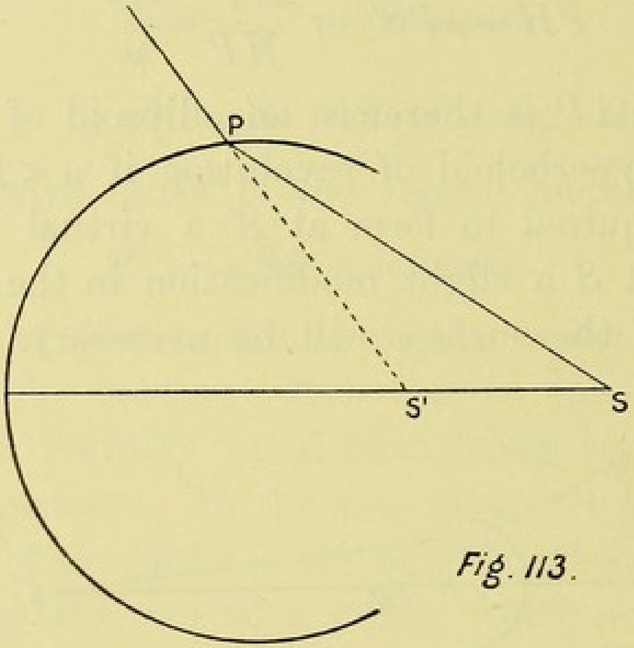

Fig. 113.

In this case the Cartesian oval becomes a circle, as indeed might be expected from the result obtained on p. 274 where we found that in refraction at a single spherical surface there was no aberration when $u = \mu v$.

If $k < \mu c$, the constant $k - \mu c$ is negative, and for one negative value the point S will coincide with the point S'. There will then be no refraction and the Cartesian oval again becomes a circle, for the equation reduces to $SP\ (1 - \mu) = -b$ a constant, and the radius of the circle is SP or $\frac{b}{\mu - 1}$.

Achromatic Combinations. When the incident light is heterogeneous, *i.e.* when it is composed of light of different wave-frequencies, the value of μ, the refractive index of the glass used, will have a different value for each wave-frequency. Hence the focal length will also have a different value for each wave-frequency, for $\frac{1}{f}=\overline{\mu-1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$. If then the incident light coming directly from the sun traverse a simple convex lens, a series of images of the sun of different colours from violet to red will be formed along the axis of the lens. Now owing to the different dispersive powers of different kinds of glass we have some means of partially getting rid of this defect by uniting two lenses of different kinds of glass. We cannot make a combination of two lenses perfectly achromatic owing to the irrationality of dispersion (p. 183). We shall shew in an elementary way how light of one wave-frequency, say that corresponding to the line F in the blue-green part of the spectrum, may be brought to the same focus as the orange coloured light of the line D.

Let the thickness of the lenses be negligible, and let the focal distance of the first lens for the blue-green light be denoted by f_{1F}, and for the light intermediate between F and D by f_1.

Then $$\frac{1}{f_{1F}}=\overline{\mu_F-1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

Similarly $$\frac{1}{f_{1D}}=\overline{\mu_D-1}\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

Therefore the difference between these is

$$\frac{1}{f_{1F}}-\frac{1}{f_{1D}}=(\mu_F-\mu_D)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)=\frac{\mu_F-\mu_D}{\mu-1}\,\frac{1}{f_1}=\frac{\varpi_1}{f_1},$$

where $\varpi_1=\frac{\mu_F-\mu_D}{\mu-1}$, the dispersive power of the lens considered.

Similarly $$\frac{1}{f_{2F}}-\frac{1}{f_{2D}}=\frac{\varpi_2}{f_2}.$$

Hence in order that the two images may coincide it is necessary that

$$\frac{\varpi_1}{f_1}+\frac{\varpi_2}{f_2}=0.$$

This is the condition that two thin lenses in contact may form an achromatic combination for F and D. It is obvious that f_1 and f_2 must carry opposite signs. Thus the object-glass of a telescope usually consists of a biconvex lens of crown glass cemented to a diverging meniscus of flint glass such that $\frac{f_1}{f_2} = -\frac{\varpi_1}{\varpi_2}$. In order that the two lenses may be cemented together it is necessary that the posterior curvature of the first lens should be equal to the anterior curvature of the second lens, *i.e.* $r_2 = r'_1$. As the achromatism of a combination of two lenses in contact can never be made perfect for sunlight, Herschel recommends the union of the kinds of light denoted by the D and F lines of the spectrum which are at the same time powerfully illuminating and very different in colour. He says, "The exact union of these will ensure the approximate union of all the rest, better on the whole than if we aimed at uniting the extremes of the spectrum, and a far greater concentration of light will be produced."

It is often necessary to correct a system of two lenses separated by an interval for chromatic aberration. The necessary conditions when each lens is made of the same kind of glass can be easily found. Let a denote the interval between the first lens and the second lens, then

$$\frac{1}{fx} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{a}{f_1 f_2},$$

or

$$\phi = \phi_1 + \phi_2 + a\,\phi_1\phi_2$$

where

$$\phi = \frac{1}{f} = \overline{\mu - 1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \overline{\mu - 1}\kappa.$$

$$\frac{d\phi}{d\mu} = \kappa_1 + \kappa_2 + a\,(\phi_2\kappa_1 + \phi_1\kappa_2).$$

This expression must be equal to 0 if the combination is to be achromatic;

$$\therefore \quad a = - \frac{\kappa_1 + \kappa_2}{\phi_2 \kappa_1 + \phi_1 \kappa_2},$$

or
$$a = - \frac{1}{2(\mu - 1)} \frac{\kappa_1 + \kappa_2}{\kappa_1 \kappa_2} = - \frac{1}{2(\mu - 1)} \left(\frac{1}{\kappa_2} + \frac{1}{\kappa_1} \right);$$

$$\therefore \quad a = - \frac{f_2 + f_1}{2}.$$

This condition is satisfied by Huygens' eyepiece which is formed of two plano-convex lenses, the focal length of the field-lens f_1 being three times the focal length of the eye-lens f_2, and the interval between the lenses being numerically equal to $2f_2$.

The condition is not satisfied by Ramsden's eyepiece in which $f_2 = f_1$ and $a = -\frac{2}{3}f$. Ramsden's eyepiece is therefore not achromatic.

QUESTIONS.

1. A refracting medium of index $\frac{4}{3}$ is bounded by a convex spherical surface with radius of curvature -5 mm. If it be provided with a diaphragm having a central aperture of diameter $1\frac{3}{5}$ mm., what will be the longitudinal aberration for incident parallel rays?

2. What is the radius of the circle of least confusion, and what is its distance from the principal focus? (See p. 117.)

3. Shew how an achromatic object-glass of 4 ft. focal length can be made for a telescope out of these two kinds of glass:

Flint $\mu_D = 1{\cdot}60$, $\mu_F = 1{\cdot}61$; Crown $\mu_D = 1{\cdot}52$, $\mu_F = 1{\cdot}526$.

Spherical aberration may be considered sufficiently corrected by making the combination a plano-convex.

CHAPTER XVI.

CONTOUR OF THE REFRACTED WAVE-FRONT. CAUSTICS.

Contour of the refracted Wave-front. We will first investigate the form of the wave-front of a system of spherical waves, that have undergone refraction at a single spherical surface. The method adopted is taken from Professor Heath's

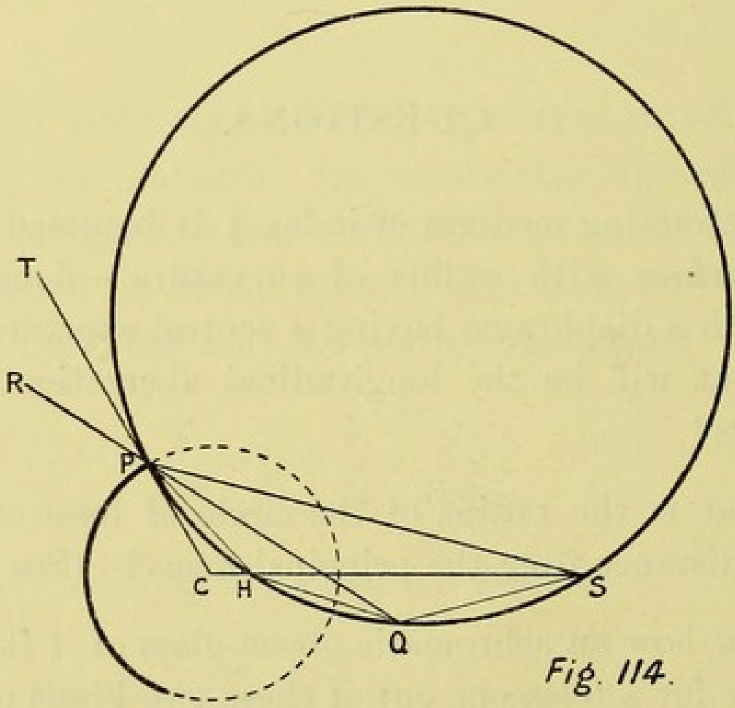

Fig. 114.

Geometrical Optics. Let C (Fig. 114) be the centre of the refracting surface, S the source of light, and SP an incident ray, PR the corresponding refracted ray. Through S describe a circle touching the radius CP in P; let this circle cut SC in H, and PR produced in Q. Join HP, HQ and SQ.

Then $$SC \,.\, CH = CP^2, \qquad \text{(Euc. III. 36.)}$$

and as SC and CP are constant, H is a fixed point, also the triangles PCH and SCP are similar, having their sides about their common angle proportional;

$$\therefore \frac{HP}{SP} = \frac{CP}{SC} \text{ which is a constant ratio.}$$

The angle of incidence $\phi = CPS =$ supplement of SPT $=$ supplement of SQP in the alternate segment.

(Euc. III. 32.)

And the angle of refraction $\phi' = CPQ = PSQ$ in the alternate segment.

And in the triangle PSQ,

$$\frac{SP}{PQ} = \frac{\sin SQP}{\sin PSQ} = \frac{\sin \phi}{\sin \phi'} = \mu \text{ a constant.}$$

Therefore also $\dfrac{HP}{PQ}$ is a constant ratio.

Now $$SP \,.\, QH + HP \,.\, SQ = PQ \,.\, SH;$$

(Euc. VI. D)

$$\therefore \frac{SP}{PQ} \cdot QH + \frac{HP}{PQ} \cdot SQ = SH.$$

If the variables QH and SQ be denoted by r and r', the ratios by μ and λ, and the constant SH by c, the equation becomes

$$\mu r + \lambda r' = c. \quad \text{..........................(1).}$$

The locus of Q is therefore a Cartesian oval, of which S and H are the foci.

On differentiating (1) with regard to s we get

$$\mu \frac{dr}{ds} + \lambda \frac{dr'}{ds} = 0. \quad \text{..................(2).}$$

But, as is well known, $\frac{dr}{ds}$ and $\frac{dr'}{ds}$ are the cosines of the angles which the radii vectores r and r' (QH and SQ) make with the tangent to the oval at the point Q.

If ω, ω' be the angles which the normal at Q makes with the radii vectores,

$$\frac{dr}{ds} = \cos(90 + \omega) = -\sin\omega, \text{ and } \frac{dr'}{ds} = \cos(\omega' - 90) = \sin\omega';$$

$$\therefore \frac{dr}{ds} : \frac{dr'}{ds} = -\sin\omega : \sin\omega'.$$

But from (2) $\quad \frac{dr}{ds} : \frac{dr'}{ds} = -\lambda : \mu;$

$$\therefore \sin\omega : \sin\omega' = \lambda : \mu.$$

Now it is seen from the figure that

$$\frac{HP}{SP} = \frac{\sin PSH}{\sin PHS} = \frac{\sin PQH}{\sin PQS},$$

and from (1) $\frac{HP}{SP} = \frac{\lambda}{\mu}$. Therefore $\frac{\sin PQH}{\sin PQS} = \frac{\sin\omega}{\sin\omega'}$.

In other words, the angle SQH between the radii vectores of the oval is divided by the line QPR into two parts, such that their sines are as $\lambda : \mu$, *i.e.* as $\sin\omega : \sin\omega'$.

The line QPR, the direction of the refracted ray at P, is therefore the normal to the Cartesian oval at Q.

Similarly every refracted ray in the new medium must be a normal to the Cartesian oval considered.

The wave-front in the new medium must therefore be parallel to the Cartesian oval.

The caustic formed by refraction at a spherical surface may be regarded as the envelope of the normals to a Cartesian oval.

In a similar way it may be shewn that if the refracting surface present a convex spherical surface to the incident light, and if $\mu > 1$, the locus of Q is a Cartesian oval of the form $\mu r - \lambda r' = c$.

The construction becomes impossible if the luminous point S be placed at an infinite distance, or if it be situated at C the centre of the refracting spherical surface. In the latter case there is no refraction, and the wave-front in the second medium maintains its spherical form.

There is one other case in which the form of the wave-front in the second medium is spherical, viz. when the constant c or SH vanishes, for then the ratio $\frac{r}{r'}$ is constant, and the oval reduces to a circle.

In all other cases, when spherical waves enter a refracting medium bounded by a spherical surface, the form of the refracted wave-front becomes a parallel to a Cartesian oval. Consecutive normals will not therefore intersect in a single point, but they will intersect along a certain caustic surface similar to that which obtained after reflection at a spherical surface.

When a widely divergent pencil of light, arising from a luminous point on the axis (as in Fig. 106), undergoes refraction at a spherical surface it may be considered to consist of an axial core and of a number of oblique pencils, which are incident upon eccentric portions of the refracting surface. It is only true of the axial core that light proceeds as if from a definite point Q, the position of which is given by the familiar formula $\frac{\mu}{q} - \frac{1}{p} = \frac{\mu - 1}{r}$.

Each of the oblique pencils after refraction loses its conical shape and assumes that of a sphenoid, converging to two focal lines at right angles to one another. The primary focal line touches the caustic surface, while the secondary focal line is the cross section in the primary plane of the refracted pencil where it crosses the axis.

It will be readily seen that the caustic surface may be regarded either as the locus of the primary focal lines, or as the envelope of the refracted rays, or as the evolute of the Cartesian oval.

Caustics. The caustic curve produced by refraction at a spherical surface for a luminous point can be traced directly by using the method that was adopted for finding the caustic produced by reflection.

Let S (Fig. 115) be the luminous point, distant a from C the centre of the spherical surface that bounds a refractive medium of index μ.

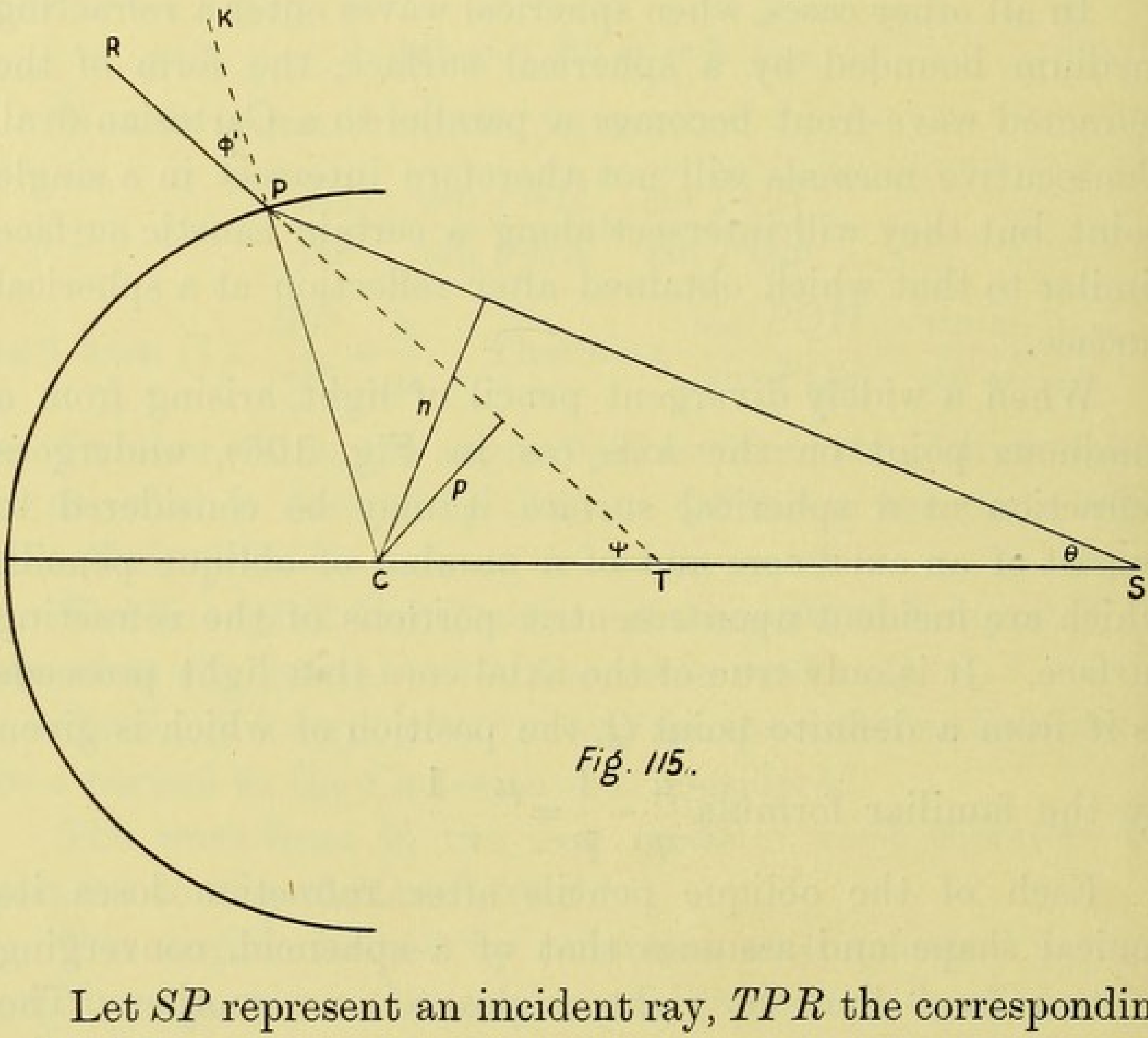

Fig. 115.

Let SP represent an incident ray, TPR the corresponding refracted ray.

Let PSC be denoted by θ, CPS by ϕ, KPR or CPT by ϕ', and PTC by ψ.

From C let fall the perpendiculars p and n on PT and SP, the refracted and incident rays.

I. We will first consider the case when the light is incident upon the concave surface of the medium.

Now $$\sin\phi = \mu \sin\phi' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

$$p = r\sin\phi' = \frac{r}{\mu}\sin\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.$$

$$n = a\sin\theta = r\sin\phi = \mu p \;\ldots\ldots\ldots\ldots\text{(C)}.$$

$$\psi = \theta + \phi - \phi' = \sin^{-1}\frac{\mu p}{a} + \sin^{-1}\frac{\mu p}{r} - \sin^{-1}\frac{p}{r} \ldots\text{(D)}.$$

$\therefore$ from (C), $$p = \frac{a}{\mu}\sin\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

From (D), $$1 = \frac{\mu}{a}\frac{1}{\left(1 - \frac{\mu^2 p^2}{a^2}\right)^{\frac{1}{2}}}\frac{dp}{d\psi} + \frac{\mu}{r}\frac{1}{\left(1 - \frac{\mu^2 p^2}{r^2}\right)^{\frac{1}{2}}}\frac{dp}{d\psi} - \frac{1}{r}\frac{1}{\left(1 - \frac{p^2}{r^2}\right)^{\frac{1}{2}}}\frac{dp}{d\psi}.$$

Let $\frac{dp}{d\psi}$ be denoted by p', and $\frac{d^2p}{d\psi^2}$ by p''.

Then $$\frac{1}{p'} = \frac{\mu}{a}\sec\theta + \frac{\mu}{r}\sec\phi - \frac{1}{r}\sec\phi';$$

$$\therefore\; -\frac{p''}{p'^2} = \frac{\mu}{a}\sin\theta\sec^2\theta\frac{\mu}{a}\sec\theta p' + \frac{\mu}{r}\sin\phi\sec^2\phi\frac{\mu}{r}\sec\phi p' - \frac{1}{r}\sin\phi'\sec^2\phi'\frac{1}{r}\sec\phi' p';$$

$$\therefore\; \frac{p''}{p} = -\frac{\mu^3}{a^3}p'^3\sec^3\theta - \frac{\mu^3}{r^3}p'^3\sec^3\phi + \frac{1}{r^3}p'^3\sec^3\phi'.$$

Now $$p' = \frac{ar}{\mu r\sec\theta + \mu a\sec\phi - a\sec\phi'} \;\ldots\ldots\; (2).$$

Let this expression be denoted by $\frac{1}{P}$, then $1 = Pp'$.

Similarly let the previous expression for $\frac{p''}{p}$ be denoted by Qp'^3.

Now

$$\rho = p + p'' = p\{1 + Qp'^3\} = p\{P^3p'^3 + Qp'^3\} = p\left\{\frac{P^3 + Q}{P^3}\right\},$$

$$\rho = p\left\{1 + \frac{Q}{P^3}\right\} = p\left\{1 + \frac{a^3 \sec^3\phi' - \mu^3r^3\sec^3\theta - \mu^3a^3\sec^3\phi}{(\mu r\sec\theta + \mu a\sec\phi - a\sec\phi')^3}\right\}$$
......(3).

These three equations are sufficient to enable the caustic curve to be traced whenever the light from a luminous point falls on the concave surface of the medium. In the following examples the value of μ will be supposed to be 1·5.

Ex. (1). Let $a = 0$.

$$p = 0, \quad p' = 0, \quad \rho = 0.$$

The caustic reduces to a point at the centre of curvature of the refracting surface, *i.e.* it is coincident with the situation of the luminous point.

Ex. (2). Let $a = \mu r$. (Fig. 116.)

$$p = r\sin\phi' = \frac{a}{\mu}\sin\theta = r\sin\theta;$$

$$\therefore \phi' = \theta \text{ and } \psi = \phi,$$

$$\rho = p + p'' = 0.$$

In the triangle SPC, $\frac{\sin\theta}{\sin\phi} = \frac{r}{a} = \frac{r}{\mu r}$.

In the triangle PTC,

$$\frac{\sin\phi'}{\sin\psi} = \frac{TC}{r} = \frac{\sin\theta}{\sin\phi} = \frac{1}{\mu};$$

$$\therefore TC = \frac{r}{\mu}.$$

The caustic reduces to a point on the diameter situated $\frac{r}{\mu}$ from C.

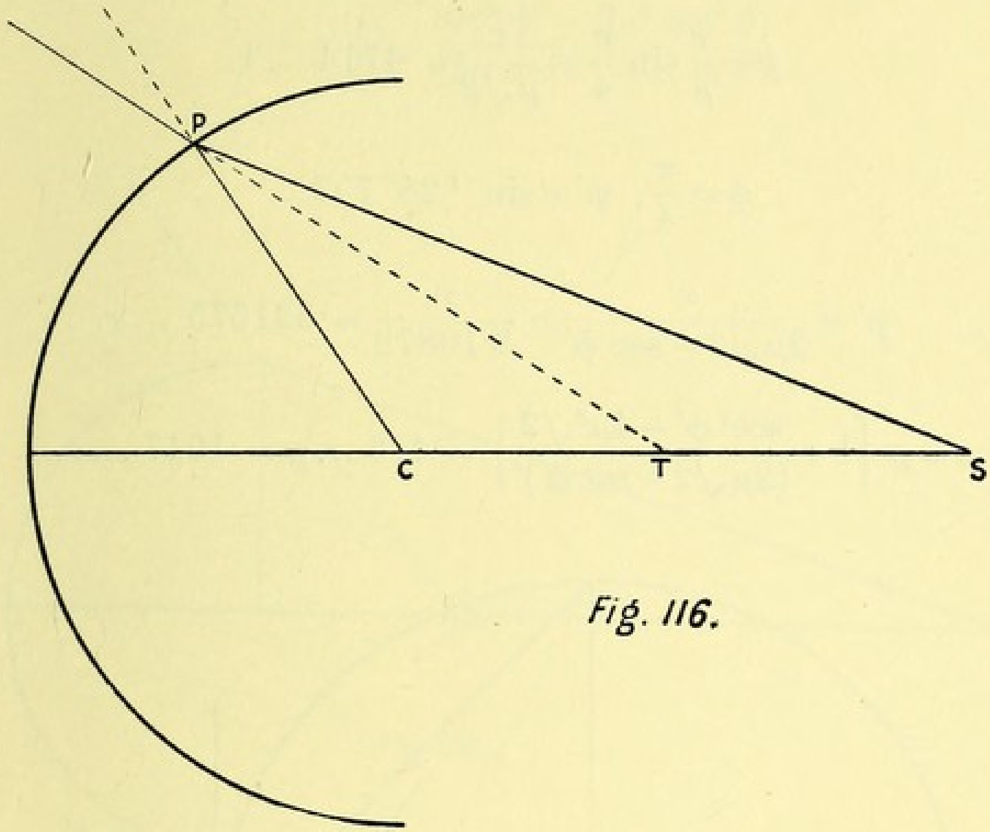

Fig. 116.

Ex. (3). Let $\qquad a = r.$ (Fig. 117.)

$$p = \frac{r}{\mu}\sin\phi = \frac{a}{\mu}\sin\theta = \frac{r}{\mu}\sin\theta;$$

$\therefore\ \phi = \theta$, and $\psi = 2\theta - \phi'$ when θ is not greater than $\frac{\pi}{2}$,

$$p' = \frac{r}{2\mu\sec\theta - \sec\phi'},$$

$$\rho = p\left\{1 + \frac{\sec^3\phi' - 2\mu^3\sec^3\theta}{(2\mu\sec\theta - \sec\phi')^3}\right\}.$$

When $$\theta = 0,$$

$$p = 0,\quad \phi = 0,\quad \phi' = 0,\quad \psi = 0,$$

$$p' = \frac{r}{2\mu - 1},$$

$$\rho = 0.$$

There is a cusp on the diameter situated $\dfrac{r}{2\mu - 1}$ from C.

When $$\theta = \frac{\pi}{4},$$

$$p = \frac{r}{\mu}\sin\frac{\pi}{4} = \frac{r}{\mu\sqrt{2}} = \cdot 4714 \dots r.$$

$$\phi = \frac{\pi}{4},\ \phi' = \sin^{-1} 28^\circ\, 7'\cdot 7 \dots$$

$$p' = \frac{r}{2\mu\sqrt{2} - \sec\phi'} = \frac{r}{3\cdot 10872} = \cdot 321675 \dots r.$$

$$\rho = p\left\{1 + \frac{\sec^3\phi' - 4\mu^3\sqrt{2}}{(2\mu\sqrt{2} - \sec\phi')^3}\right\} = \cdot 413 \dots p = \cdot 1947 \dots r.$$

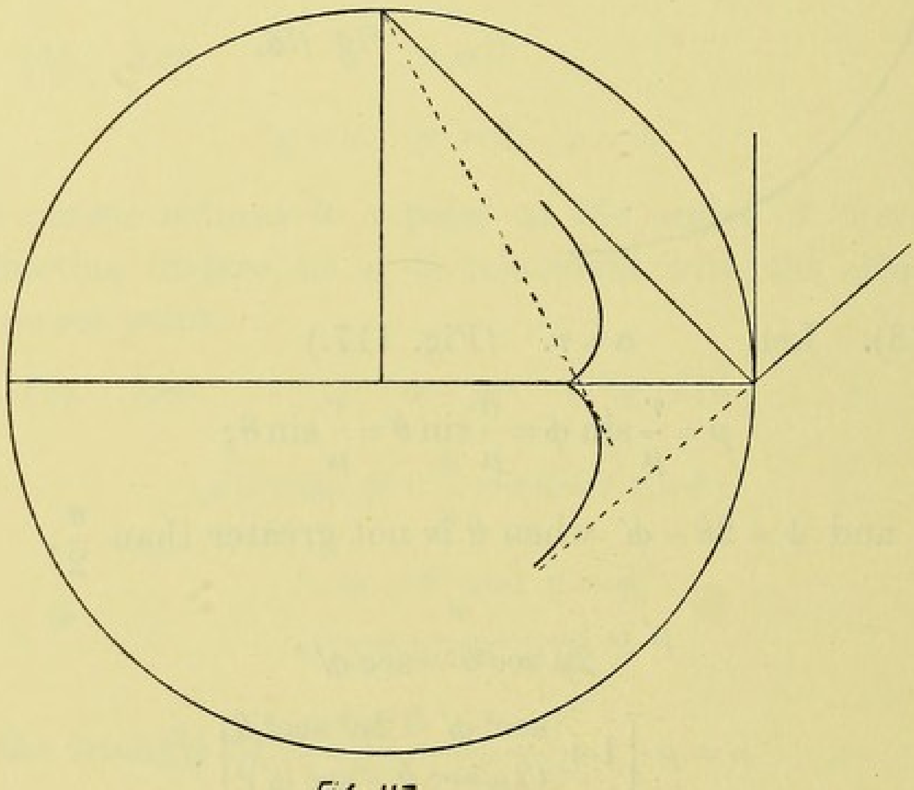

Fig. 117.

When $$\theta = \frac{\pi}{2},$$

$$p = \frac{r}{\mu}\sin\frac{\pi}{2} = \frac{r}{\mu} = \cdot\dot{6}\, r,$$

$$\phi = \frac{\pi}{2}, \quad \phi' = \sin^{-1}\frac{p}{r} = \sin^{-1}\frac{1}{\mu} = 41°\ 48'\cdot 6 \ldots$$

$$p' = 0,$$

$$\rho = p\left\{1 + \frac{\sec^3\phi' - 2\mu^3\sec^3\theta}{(2\mu\sec\theta - \sec\phi')^3}\right\}.$$

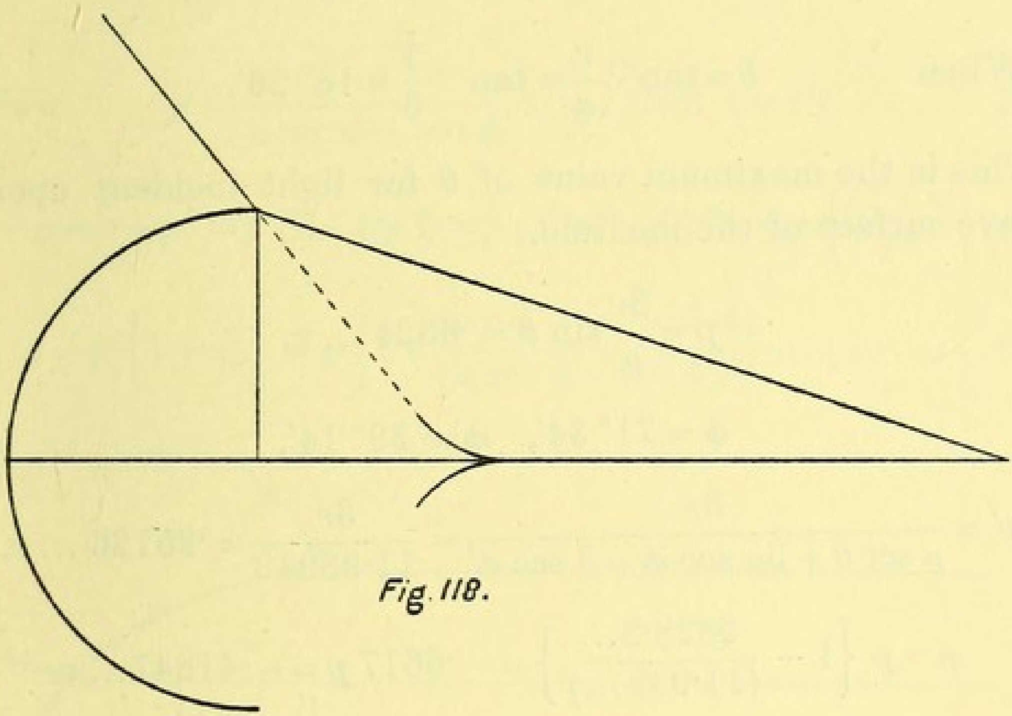

Fig. 118.

The fraction is of the indeterminate form $-\frac{\infty}{\infty}$ but it can be easily evaluated by putting it into the form $\frac{\cos^3\theta - 2\mu^3\cos^3\phi'}{(2\mu\cos\phi' - \cos\theta)^2}$ which is equal to $-\frac{1}{4}$ when $\theta = \frac{\pi}{2}$.

$$\therefore \rho = p\left\{1 - \frac{1}{4}\right\} = \frac{3}{4\mu}r = \cdot 5\, r.$$

Ex. (4). Let $a = 3r$. (Fig. 118.)

$$p = \frac{3r}{\mu}\sin\theta = \frac{r}{\mu}\sin\phi;$$

$$\therefore \phi = \sin^{-1} 3\sin\theta,$$

$$p' = \frac{3r}{\mu\sec\theta + 3\mu\sec\phi - 3\sec\phi'}.$$

When $$\theta = 0,$$

$$p = 0, \quad \phi = 0, \quad \phi' = 0, \quad \psi = 0, \quad \rho = 0,$$

$$p' = \frac{3r}{4\mu - 3} = r.$$

There is a cusp on the diameter $\frac{3r}{4\mu - 3}$ from C.

When $$\theta = \tan^{-1}\frac{r}{a} = \tan^{-1}\frac{1}{3} = 18^\circ\ 26'.$$

This is the maximum value of θ for light incident upon the concave surface of the medium.

$$p = \frac{3r}{\mu}\sin\theta = \cdot 6324 \ldots r,$$

$$\phi = 71^\circ\ 34', \quad \phi' = 39^\circ\ 14',$$

$$p' = \frac{3r}{\mu\sec\theta + 3\mu\sec\phi - 3\sec\phi'} = \frac{3r}{11\cdot 93949} = \cdot 25126 \ldots r,$$

$$\rho = p\left\{1 - \frac{2828\cdot 2\ldots}{(11\cdot 939\ldots)^3}\right\} = -\cdot 6617\,p = -\cdot 41847\ldots r.$$

Ex. (5). Let $a = \infty$. (Fig. 119.)

It is required to find the caustic produced by refraction at a concave spherical surface, when the incident light presents a plane wave-front.

Since the rays of incident light are parallel, $\theta = 0$ and $\phi = \psi + \phi'$. Hence we must take ϕ as the variable instead of θ.

$$p = r\sin\phi' = \frac{r}{\mu}\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\psi = \phi - \phi' = \sin^{-1}\frac{\mu p}{r} - \sin^{-1}\frac{p}{r};$$

$$\therefore 1 = \frac{1}{\sqrt{1 - \frac{\mu^2 p^2}{r^2}}}\frac{\mu}{r}p' - \frac{1}{\sqrt{1 - \frac{p^2}{r^2}}}\frac{1}{r}p';$$

$$\therefore \frac{1}{p'} = \frac{\mu}{r}\sec\phi - \frac{1}{r}\sec\phi' ;$$

$$\therefore \frac{-p''}{p'^2} = \frac{\mu}{r}\sin\phi\sec^2\phi\frac{\mu}{r}\sec\phi p' - \frac{1}{r}\sin\phi'\sec^2\phi'\frac{1}{r}\sec\phi' p' ;$$

$$\therefore \frac{p''}{p} = -\frac{\mu^3}{r^3}\sec^3\phi p'^3 + \frac{1}{r^3}\sec^3\phi' p'^3 = Qp'^3.$$

Now $$p' = \frac{r}{\mu\sec\phi - \sec\phi'} = \frac{1}{P}, \text{ then } 1 = Pp' \text{ .... (2)},$$

$$\rho = p + p'' = p\{1 + Qp'^3\} = p\{P^3p'^3 + Qp'^3\} = p\left\{\frac{P^3+Q}{P^3}\right\},$$

$$\rho = p\left\{1 + \frac{Q}{P^3}\right\} = p\left\{1 + \frac{\sec^3\phi' - \mu^3\sec^3\phi}{(\mu\sec\phi - \sec\phi')^3}\right\} \text{ .... (3)}.$$

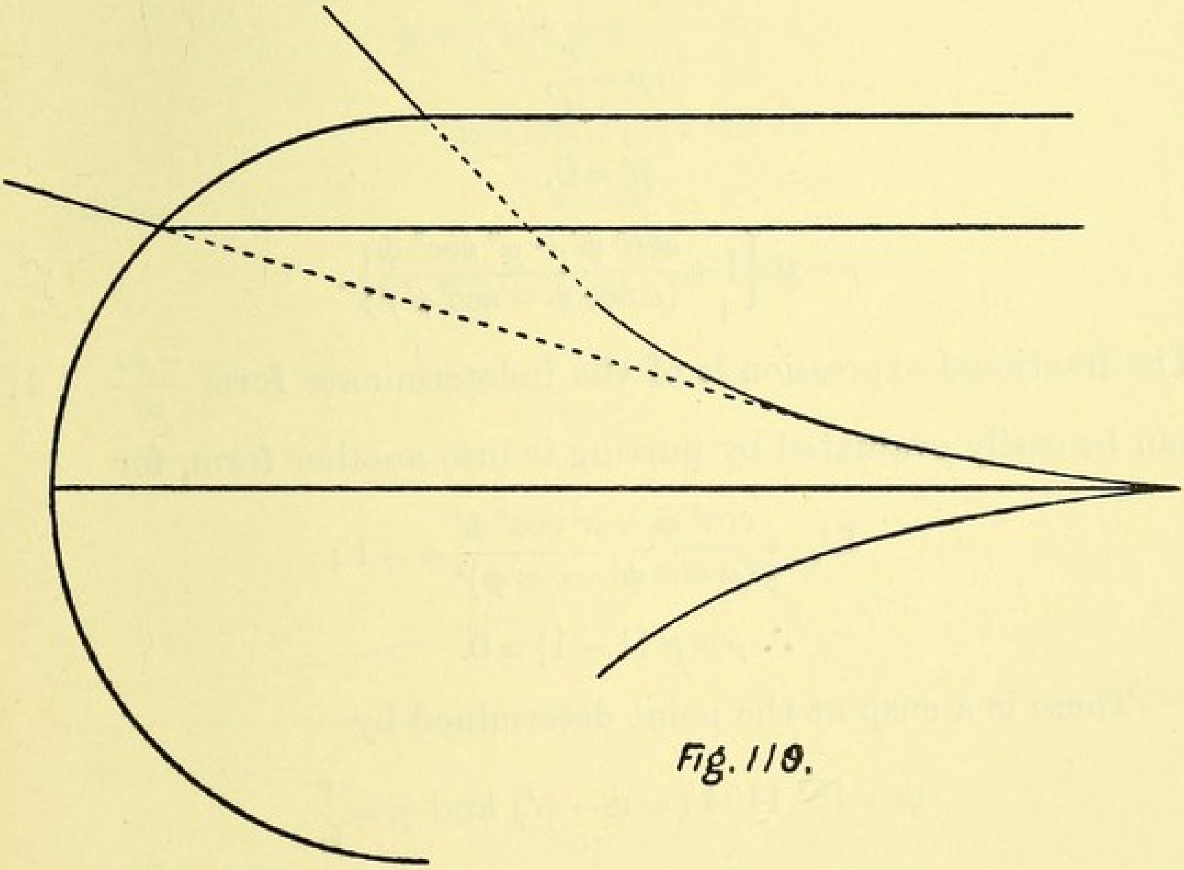

Fig. 110.

When $\phi = 0$,

$$p = 0,\ \phi' = 0,\ p' = \frac{r}{\mu - 1},\ \rho = 0.$$

There is a cusp on the principal axis or diameter distant $\frac{r}{\mu - 1}$ from C.

When
$$\phi = \frac{\pi}{4},$$

$$\phi' = 28^\circ\ 7'{\cdot}7\ldots$$

$$p = \frac{r}{\mu\sqrt{2}} = {\cdot}4714\ldots r.$$

$$p' = 1{\cdot}0127\ldots r,$$

$$\rho = p\left\{1 + \frac{-8{\cdot}0879\ldots}{({\cdot}98739\ldots)^3}\right\} = -7{\cdot}4016\ldots p = -3{\cdot}489\ldots r.$$

When $\phi = \frac{\pi}{2}$, its maximum value,

$$\phi' = 41^\circ\ 48'{\cdot}6\ldots,$$

$$p = \frac{r}{\mu},$$

$$p' = 0,$$

$$\rho = p\left\{1 + \frac{\sec^3\phi' - \mu^3\sec^3\phi}{(\mu\sec\phi - \sec\phi')^3}\right\}.$$

The fractional expression is of the indeterminate form $\frac{-\infty}{\infty}$. It can be easily evaluated by putting it into another form, for

$$\mathrm{L}^{\mathrm{t}}_{\theta=\frac{\pi}{2}} \frac{\cos^3\phi - \mu^3\cos^3\phi'}{(\mu\cos\phi' - \cos\phi)^3} = -1\,;$$

$$\therefore\ \rho = p\{1 - 1\} = 0.$$

There is a cusp at the point determined by

$$\psi = 48^\circ\ 11'{\cdot}4\ (= \phi - \phi') \text{ and } p = \frac{r}{\mu}.$$

II. When light is refracted by a medium bounded by a convex surface, the same formulae hold good provided that proper values are given to ϕ and ϕ'.

Thus in Fig. 115, if the surface were convex, *KPS* would be denoted by ϕ and *KPR* by ϕ', the angles being measured in a clockwise direction.

We have again, remembering that r is now negative,

$$p = \frac{r}{\mu}\sin\phi = \frac{a}{\mu}\sin\theta \quad\text{..................(1)},$$

$$p' = \frac{ar}{\mu r\sec\theta + \mu a\sec\phi - a\sec\phi'} \quad\text{.........(2)},$$

$$\rho = p\left\{1 + \frac{a^3\sec^3\phi' - \mu^3 r^3\sec^3\theta - \mu^3 a^3\sec^3\phi}{(\mu r\sec\theta + \mu a\sec\phi - a\sec\phi')^3}\right\} \quad\text{... (3)}.$$

Ex. (1). Let $a = -3r$. (Fig. 120.)

$$p = \frac{r}{\mu}\sin\phi = \frac{-3r}{\mu}\sin\theta;$$

$$\therefore \phi = \sin^{-1}(-3\sin\theta).$$

When $\theta = 0$,

$$p = 0, \ \phi = 0, \ \phi' = 0,$$

$$p' = \frac{a}{3 - 2\mu} = \infty.$$

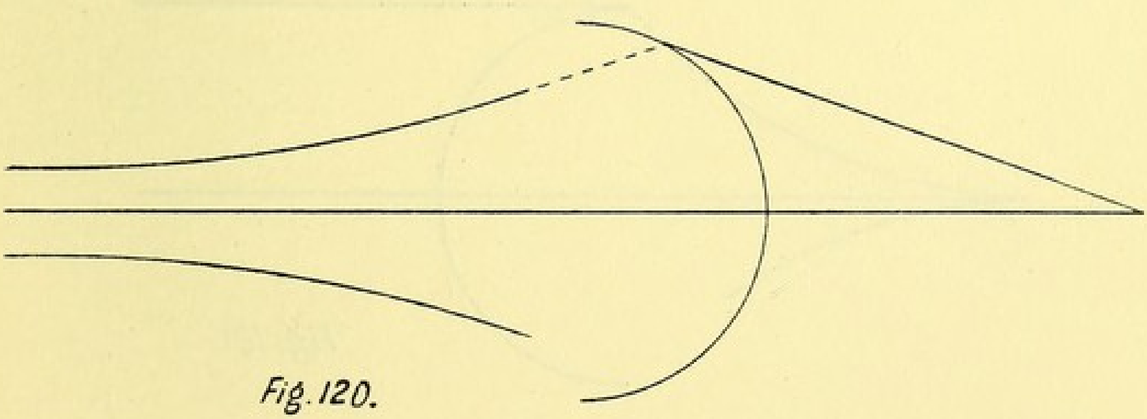

Fig. 120.

The small centric cone of rays diverging from S will traverse the second medium parallel to the principal axis.

When $\theta = \sin^{-1}\frac{1}{3} = 19^\circ\, 28'{\cdot}27\ldots.$

This is the maximum value of θ, for then

$$\phi = -90^\circ,\ \ \phi' = -41^\circ\, 48'{\cdot}6\ldots,$$

$$p = \frac{-r}{\mu},$$

$$p' = 0,$$

ρ will be found 0, for

$$\mathrm{L}^{\mathrm{t}}_{\phi=\frac{\pi}{2}} \frac{a^3 \sec^3 \phi' - \mu^3 r^3 \sec^3 \theta - \mu^3 a^3 \sec^3 \phi}{(\mu r \sec\theta + \mu a \sec\phi - a \sec\phi')^3} = -1.$$

There is a cusp at the point determined by

$$p = \frac{-r}{\mu},\ \psi = -28^\circ\, 43'{\cdot}13\ldots.$$

Ex. (2). Let $a = \infty$. (Fig. 121.)

As before $\theta = 0,\ \psi = \phi - \phi',$

$$p = \frac{r}{\mu}\sin\phi,$$

$$p' = \frac{r}{\mu\sec\phi - \sec\phi'},$$

$$\rho = p + p'' = p\left\{1 + \frac{\sec^3\phi' - \mu^3\sec^3\phi}{(\mu\sec\phi - \sec\phi')^3}\right\} = p\left\{\frac{-3\mu\sec\phi\sec\phi'}{(\mu\sec\phi - \sec\phi')^2}\right\}.$$

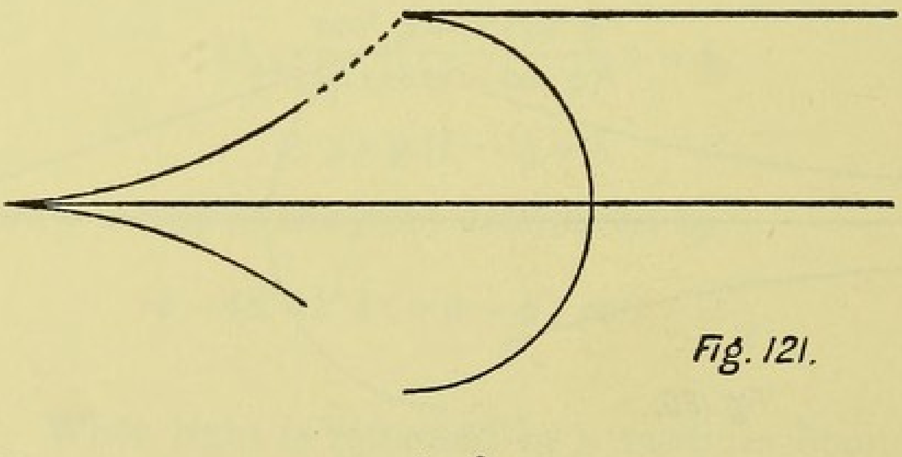

Fig. 121.

When $\phi = 0,$

$$\phi' = 0,\ p = 0,\ p' = \frac{r}{\mu - 1},\ \rho = 0.$$

There is a cusp on the principal axis at a point $\frac{r}{\mu - 1}$ from C, *i.e.* at a point $2r$ from C measured in the negative direction.

$$\phi = -\frac{\pi}{2},$$

$$\phi' = -41^\circ\, 48'{\cdot}6,\ p = \frac{-r}{\mu},\ p' = 0,\ \rho = -p\left\{\frac{3\mu \cos\phi \cos\phi'}{(\mu \cos\phi' - \cos\phi)^2}\right\} = 0.$$

There is a cusp at the point determined by

$$p = \frac{-r}{\mu},\ \phi' = \sin^{-1}\left(-\frac{1}{\mu}\right) = -41^\circ\, 48'{\cdot}6,\ \psi = -48^\circ\, 11'{\cdot}4.$$

CHAPTER XVII.

THE HUMAN EYE.

THE human eye is a nearly spherical ball, capable of turning in any direction in its socket. Its outermost coat is thick and horny, and is opaque except in its anterior portion. The opaque part is called the *sclerotic.* The transparent portion is more protuberant, and has the shape approximately of a very convex watch-glass. This is called the *cornea* (Fig. 123).

The eyeball has two other linings. Immediately within the sclerotic is a vascular coat called the *choroid.* This coat is attached to the inner surface of the sclerotic, but at the corneo-sclerotic junction it becomes free, and passes in a nearly vertical plane behind the cornea as the *iris.* The iris terminates in a circular aperture, called the *pupil.* The iris being opaque forms an annular diaphragm for the optical instrument.

Within the choroid is a thin membrane which is traversed by a ramified system of nerve fibres diverging from the optic nerve. This is the *retina*, the part of the eye that is sensitive to impressions of light. It is lined posteriorly by a layer of pigment cells which absorb the light. It is consequently in this region that the energy of light is transformed into energy of another kind; and the processes here started travel from the layer of rods and cones to the layer of nerve fibres on the inner surface of the retina, and thence pass as visual impulses along the optic nerve.

The space between the cornea and the iris is filled with a watery fluid called the *aqueous humour*.

Placed immediately behind the iris is the *crystalline lens*. This is of the form of a biconvex lens, the anterior surface of which is less curved than the posterior. It is composed of successive layers, whose refractive index increases towards the centre, its solid nucleus refracting light most strongly (Fig. 122).

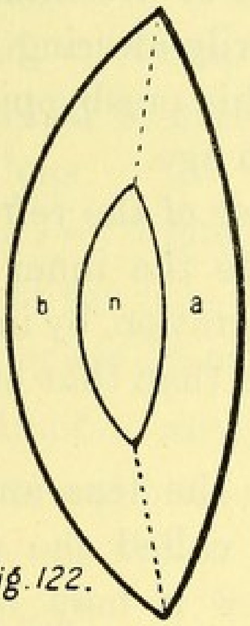

Fig. 122.

It will be easily seen that the action of the lens is more powerful than if it were composed of a homogeneous substance that had the same refractive index as the nucleus. For it may be regarded as the combination of a double convex lens *n* (the nucleus of a high degree of curvature) with two diverging menisci *a* and *b*. Consequently the higher the refractive index of the menisci, the less is the total refracting power of the lens. This explains the so-called acquired hypermetropia of old age. As Donders has shewn, at the age of 60 the refraction of the eye has diminished a third of a dioptre, at 68 it has diminished one dioptre, while at the age of 80 a person, who has been emmetropic until 53, becomes hypermetropic to the extent of 2·25 dioptres. With age the index of refraction of the peripheral layers of the lens increases; on oblique illumination therefore one sees an increased reflection from the surface of senile lenses, which may even be so intense as to simulate cataract (p. 62). An examination by transmitted light proves however in these cases that the lens is perfectly transparent. This acquired hypermetropia must be sharply distinguished from presbyopia which is a progressive diminution of the power of accommodation, *i.e.* a diminution of the power of voluntarily altering the curvature of the lens and hence its focus. This presbyopia is due to the increasing rigidity of the lens with age.

In youth the increase of the refracting power of the lens from the outer layers to the inner nucleus serves partly to correct the spherical aberration, by increasing the convergence of the central rays more than that of the extreme rays of the incident pencil.

The space between the lens and the retina is occupied by a semiviscous fluid called the vitreous humour. From an optical point of view it may be considered as a fairly homogeneous medium of a refractive index practically equal to that of the aqueous humour.

The Eye considered as an Optical Instrument. The mean values of the optical constants of the normal human eye have been elaborately worked out by Helmholtz from data obtained by direct observation of the eyes of living persons.

These values with some corrections by Stammeshaus are given by Landolt in the *Traité Complet d'Ophthalmologie*, and are as follows:

(*a*) The radii of curvature of the bounding surfaces have the following values:

(1) The anterior surface of the cornea

$$r_0 = -7{\cdot}829 \text{ mm.}$$

(2) The anterior surface of the lens

$$r_1 = -10{\cdot}000 \text{ mm.}$$

(3) The posterior surface of the lens

$$r_2 = 6{\cdot}000 \text{ mm.}$$

(*b*) The distances between the refracting surfaces are:

(1) From the anterior surface of the cornea to the anterior surface of the lens

$$A_0A_1 = 3{\cdot}6 \text{ mm.}$$

(2) From the anterior surface of the cornea to the posterior surface of the lens

$$A_0A_2 = 7{\cdot}2 \text{ mm.}$$

(3) The thickness of the lens therefore

$$A_1A_2 \text{ or } t = 3{\cdot}6 \text{ mm.}$$ [1]

(*c*) The indices of refraction are:

(1) For the aqueous and vitreous humours

$$\mu = 1{\cdot}3365.$$

(2) For the lens (total)

$$\mu' = 1{\cdot}4371.$$

As might be expected the surfaces presented by these different refractive media have not a geometrically true spherical form, nor indeed the form of any simple geometrical surface. The cornea for instance was regarded by Helmholtz as part of the surface of an ellipsoid of revolution. Sulzer[2] has however recently shewn that it has no axis of symmetry;

[1] This is the usually accepted value of the thickness of lens in the living eye. When the eye has been removed the thickness of the lens is much greater. *Glaucoma* p. 89, Mr Priestley Smith.

[2] *La Forme de la Cornée humaine.* Dr D. E. Sulzer.

that three-fourths of the total number of cases investigated by him were characterized by the following peculiarities:

1. The nasal parts of the cornea were more flattened than the temporal parts, and the superior parts than the inferior.

2. The visual line cut the cornea at a point within and either above or below the point of maximum curvature of the cornea.

The value for r_0 given above must be regarded as the radius of curvature of a small area of the cornea, the centre of which is the point of intersection of the visual line. This small area may be regarded as a small spherical cap terminating the unsymmetrical cornea.

Similarly the values given for r_1 and r_2 are the radii of curvature of minute areas of the anterior and posterior surfaces of the lens in the immediate neighbourhood of the points of intersection of the visual line.

The index of refraction of the cornea is not given, as practically it may be neglected. For the two surfaces of the cornea are very nearly parallel, and as the anterior surface is always moistened with a watery fluid, the conjunctival secretion, whose refractive index is the same as that of the aqueous humour, the cornea acts like a plate of refracting medium and produces no deviation in the incident rays.

What is called the total refractive index of the lens is determined in the following way. The focal length of the lens is found by experiment and, its shape being known, that value is assigned to the refractive index which the lens would have were it homogeneous. From what has been said it follows that this total refractive index is greater than that of the nucleus.

Before determining the position of the cardinal points of the eye according to the method previously given, there is one condition that ought to be satisfied. Is the optical system of the eye accurately centred? Unfortunately, as we have just observed, this is not the case; indeed the visual line cuts the cornea at a point usually situated on the inner side of the optic axis. However, this deviation being slight, an approximate result can be obtained by using Gauss's method.

We begin by determining the focal distances of the several refracting surfaces.

I. Surface of the cornea (or aqueous humour).

First focal distance

$$f_0' = \frac{-r_0}{\mu - 1} = \frac{7{\cdot}829}{{\cdot}3365} = 23{\cdot}2659732\ldots$$

Second focal distance

$$f_0'' = \frac{\mu r_0}{\mu - 1} = \frac{(1{\cdot}3365)(-7{\cdot}829)}{{\cdot}3365} = -31{\cdot}0949732\ldots$$

II. Anterior surface of the lens.

First focal distance

$$f_1' = \frac{-\mu r_1}{\mu' - \mu} = \frac{(-1{\cdot}3365)(-10)}{1{\cdot}4371 - 1{\cdot}3365} = 132{\cdot}8528827\ldots$$

Second focal distance

$$f_1'' = \frac{\mu' r_1}{\mu' - \mu} = \frac{(1{\cdot}4371)(-10)}{1{\cdot}4371 - 1{\cdot}3365} = -142{\cdot}8528827\ldots$$

III. Posterior surface of the lens.

First focal distance

$$f_2' = \frac{-\mu' r_2}{\mu - \mu'} = \frac{(-1{\cdot}4371)(6)}{1{\cdot}3365 - 1{\cdot}4371} = 85{\cdot}7117292\ldots$$

Second focal distance

$$f_2'' = \frac{\mu r_2}{\mu - \mu'} = \frac{(1{\cdot}3365)(6)}{1{\cdot}3365 - 1{\cdot}4371} = -79{\cdot}7117292\ldots$$

The principal point of the cornea is on its surface at the point of intersection with the optic axis.

The principal points (H_1, H_2) of the lens and the principal focal distances (ϕ_1, ϕ_2) can be easily found by the formulæ already given (pp. 257—260), for the surrounding mediums are of the same refractive index. The nodal points of the lens coincide with the principal points (Fig. 123).

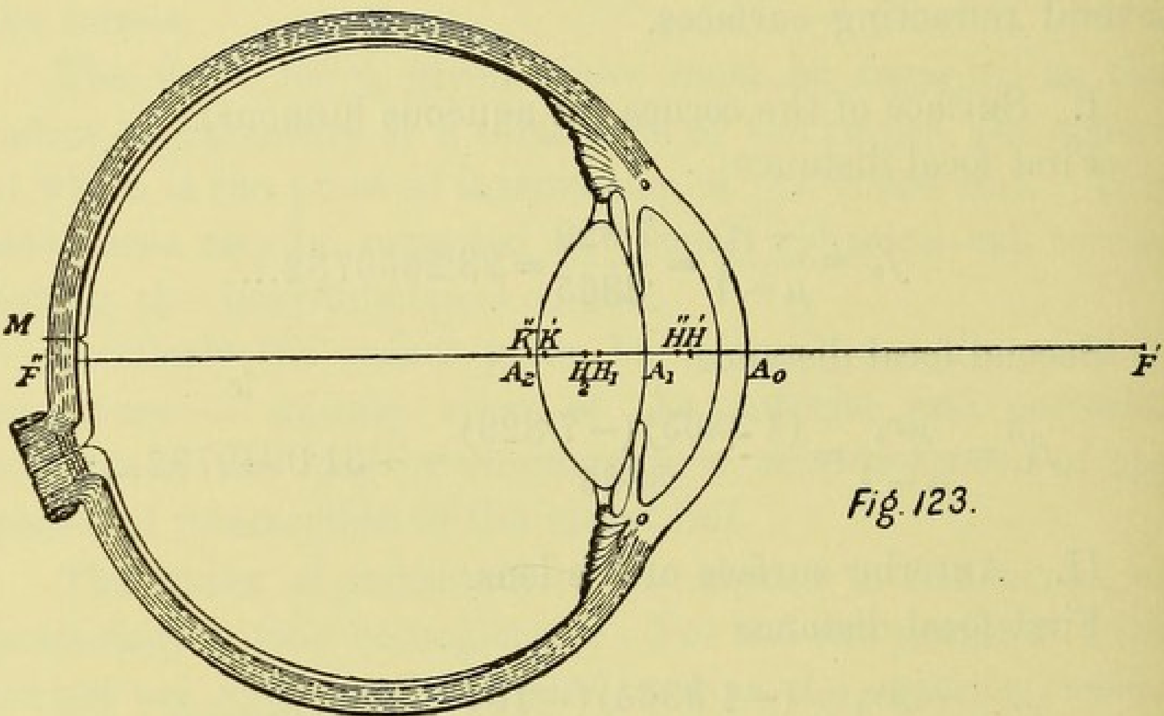

Fig. 123.

$$H_1A_1 = \frac{f_1' t}{t + f_1'' - f_2'} = \frac{(132 \cdot 85288)(3 \cdot 6)}{3 \cdot 6 - 142 \cdot 852882 - 85 \cdot 711729} = -2 \cdot 125980..,$$

$$H_2A_2 = \frac{f_2'' t}{t + f_1'' - f_2'} = \frac{(-79 \cdot 7117)(3 \cdot 6)}{-224 \cdot 96461} = 1 \cdot 275588...$$

The principal focal distances (ϕ', ϕ'') of the lens measured from H_1H_2 are

$$\phi' = \frac{-f_1' f_2'}{t + f_1'' - f_2'} = \frac{-(132 \cdot 85288)(85 \cdot 7117)}{-224 \cdot 96461} = 50 \cdot 617073..,$$

$$\phi'' = \frac{f_1'' f_2''}{t + f_1'' - f_2'} = \frac{(-142 \cdot 85288)(-79 \cdot 7117)}{-224 \cdot 96461} = -50 \cdot 617073...$$

We have now determined all the cardinal points of the two refracting systems of which the eye is composed. We proceed to find the cardinal points of the complex system presented by the eye.

Principal Points of the Eye. It is easy to see that the principal plane of the cornea plays the same part with respect to the principal planes of the crystalline lens, as the first surface of a lens plays with respect to its second surface.

To find the first principal point (H') of the system, we have only to substitute the appropriate symbols in the expression for the lens.

Instead of

$$\frac{f_1' t}{t + f_1'' - f_2'}, \text{ we write } \frac{f_0' e}{e + f_0'' - \phi'},$$

where

$$e = A_0 A_1 + A_1 H_1 = A_0 A_1 - H_1 A_1 = 3{\cdot}6 + 2{\cdot}12598 = 5{\cdot}72598;$$

$$\therefore\ H' A_0 = \frac{f_0' e}{e + f_0'' - \phi'} = \frac{(23{\cdot}26597)(5{\cdot}72598)}{5{\cdot}72598 - 31{\cdot}09497 - 50{\cdot}61707}$$
$$= -1{\cdot}753091\ldots$$

Similarly to find the distance of the second principal point (H'') of the system from the second principal point of the lens, we have

$$H'' H_2 = \frac{\phi'' e}{e + f_0'' - \phi'} = \frac{(-50{\cdot}61707)(5{\cdot}72598)}{-75{\cdot}98606..} = 3{\cdot}814749\ldots$$

Now

$$H'' A_0 = A_2 A_0 - (A_2 H_2 + H_2 H'') = -A_0 A_2 + (H_2 A_2 + H'' H_2)$$
$$= -7{\cdot}2 + (1{\cdot}275588.. + 3{\cdot}814749..) = -2{\cdot}109662\ldots$$

The second principal point is therefore 2·109662.. mm. behind the cornea, whereas the first principal point is 1·753091.. mm. behind the cornea.

The distance between the principal points,

$$H'H'' = A_0H'' - A_0H' = \cdot 356571 \text{ mm.}$$

Principal Focal Distances of the Eye. By similar reasoning we shall obtain the first principal focal distance ($F'H'$) of the eye by substituting the appropriate symbols in the expression for the lens.

Instead of

$$\frac{-f_1'f_2'}{t+f_1''-f_2'} \text{ we write } \frac{-f_0'\phi'}{e+f_0''-\phi'},$$

$$F'H' = \frac{-f_0'\phi'}{e+f_0''-\phi'} = \frac{-(23\cdot 26597)(50\cdot 61707)}{-75\cdot 98606..} = 15\cdot 498308.$$

Similarly

$$F''H'' = \frac{f_0''\phi''}{e+f_0''-\phi'} = \frac{(-31\cdot 09497)(-50\cdot 61707)}{-75\cdot 98606..}$$

$$= -20\cdot 713489.$$

The ratio $\frac{-F''H''}{F'H'}$ should be equal to the ratio of the index of refraction of the last medium to that of the first medium

$$\frac{-F''H''}{F'H'} = \frac{20\cdot 713489}{15\cdot 498308} = \frac{1\cdot 3365}{1}.$$

Now

$$F'A_0 = F'H' + H'A_0 = 15\cdot 498308 - 1\cdot 753091 = 13\cdot 745217,$$

and

$$F''A_0 = F''H'' + H''A_0 = -20\cdot 713489 - 2\cdot 109662$$

$$= -22\cdot 823151.$$

The Nodal Points of the Eye. We have already seen (p. 262) that $F'K' = H''F''$ and $F''K'' = H'F'$.

Then since

$$K'F' = F''H'', \text{ and } K'A_0 = K'F' - A_0F' = F''H'' + F'A_0,$$

$$K'A_0 = -20\cdot 713489 + 13\cdot 745217 = -6\cdot 968272.$$

And since

$$F''K'' = -F'H' \text{ and } K''A_0 = F''A_0 - F''K'' = F''A_0 + F'H',$$

$$K''A_0 = -22{\cdot}823151 + 15{\cdot}498308 = -7{\cdot}324843.$$

The distance between the nodal points

$$K'K'' = A_0K'' - A_0K' = 7{\cdot}324843 - 6{\cdot}968272$$

$$= {\cdot}356571 = H'H''.$$

The following table gives the distances of the cardinal points from the cornea, and from the principal points.

A_0 represents the anterior surface of the cornea.

H' and H'' represent the first and second principal points of the eye. The signs attached to the numerical values of the distances denote whether the points mentioned are on the positive or negative side of the cornea or the principal point from which they are measured.

It will be noticed that all the cardinal points, with the exception of the first principal focus, are situated on the negative side of the cornea, *i.e.* within the eye.

Cardinal points of the eye.

$H'A_0$	− 1·753091 mm.	$H'H''$	·356571 mm.
$H''A_0$	− 2·109662 mm.		
$K'A_0$	− 6·968272 mm.	$H'K'$ } $H''K''$ }	5·215181 mm.
$K''A_0$	− 7·324843 mm.		
		$K'K''$	·356571 mm.
$F'A_0$	13·745217 mm.	$F'H'$	15·498308 mm.
$F''A_0$	− 22·823151 mm.	$F''H''$	− 20·713489 mm.

It will be noticed that the two principal points and the two nodal points lie very close together, so that without introducing much error we may regard them as coinciding in one point H and in one point K, so that

$$HA_0 = -1{\cdot}931377,\ KA_0 = -7{\cdot}146557,\ HK = 5{\cdot}215180.$$

Then $F'H = 15{\cdot}676594$, $F''H = -20{\cdot}891775$.

On this supposition

$$\mu = \frac{-F''H}{F'H} = 1{\cdot}33267,$$

and r which must be equal to $-(\mu - 1)\mathbf{F}'$ must be $-5{\cdot}2151\ldots$ mm.

This is the principle on which the 'schematic eye of Listing' is founded. The figures given by Listing vary very slightly from those given above.

The 'schematic eye of Listing' corresponds to a single refracting medium bounded by a spherical surface, the vertex of which is at the principal point H.

We owe to Donders a further simplification of these figures. The 'reduced eye of Donders' is exactly equivalent to a single refracting medium ($\mu = \frac{4}{3}$) presenting a spherical surface whose radius of curvature is -5 mm. The principal point (H) is at the vertex of the spherical surface, and its centre of curvature coincides with the nodal point (K).

The first principal focus ($F'H'$) of this reduced eye is therefore 15 mm.

The second principal focus ($F''H''$) of this reduced eye is therefore -20 mm.

The simplicity of this reduced eye makes it exceedingly convenient to use in practical questions that relate to the refraction of the eye, as the calculations with regard to it can be executed in one's head; the results are sufficiently accurate for most practical purposes, as will be seen in the following examples.

The mode of using the cardinal points in any combined system. Let $H'H''$, $K'K''$, $F'F''$ be the first and second principal points, nodal points, and focal points of any combined system of refractive media. Let AB (Fig. 124) represent an object distant p from H', or g' from K', and let ab represent its image distant q from H'', or g'' from K''. In order to find the position of the image graphically we have merely to draw the line $AJ'J''$ parallel to the optic axis, cutting the first principal plane in J' and the second principal plane in J''. Then the line $J''F''$ will represent the direction of the corresponding emergent ray from the system.

If AK' represent an incident ray, $K''a$ will represent the corresponding emergent ray, where $K''a$ is parallel to AK'. For we know that every ray that on incidence is directed towards the first nodal point K', seems to come after refraction through the system from the second nodal point K'', and to take a course parallel to its original direction.

Or we may make use of the property of the first principal focus F'. Any incident ray as $AF'I'$ which passes through F', and meets the first principal plane in I', will emerge as $I''a$ parallel to the optic axis.

The point a is the point of intersection of all the emergent rays that originate from A. It is evident that the incident cone of light diverging from A (viz. $J'AI'$) becomes after refraction the cone of light ($J''aI''$) that converges towards a.

It is convenient to denote the distance $F'H'$ by $\mathbf{F}'$, and the distance $F''H''$ by $\mathbf{F}''$, and to regard them as positive or negative according as the points F', F'' are on the positive or negative side of H', H'' respectively.

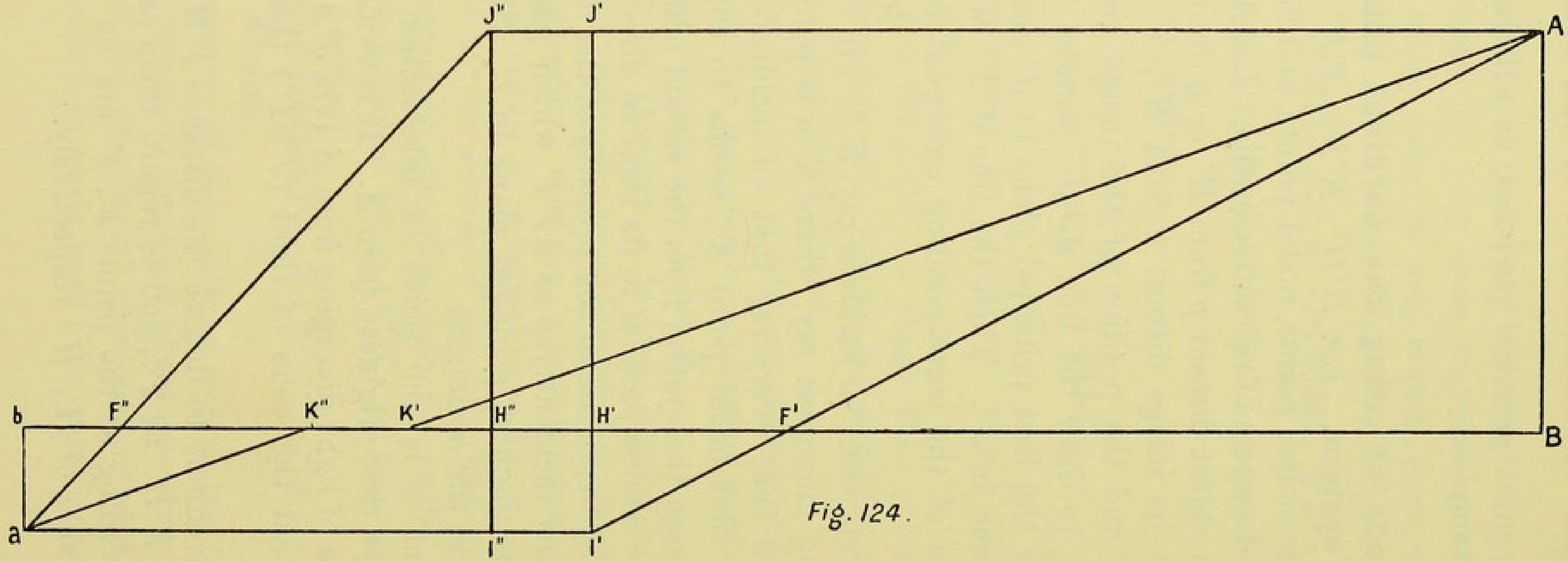

Fig. 124.

Similarly the distances $F'K'$ and $F''K''$ are denoted by the symbols G' and G'' which are regarded as positive or negative, according as the points F', F'' are on the positive or the negative side of K', K'' respectively.

Size of the image. Since the triangles ABK', abK'' are similar,

$$\frac{i}{o}=\frac{ab}{AB}=\frac{bK''}{BK'}=\frac{g''}{g'}.$$

In the figure g'' is negative while g' is positive, therefore the image i is negative or inverted.

Since $AB=J''H''$, and the triangles $J''H''F''$, abF'' are similar,

$$\frac{i}{o}=\frac{ab}{J''H''}=\frac{F''b}{F''H''}=\frac{H''b-H''F''}{F''H''}=\frac{F''H''-bH''}{F''H''}=\frac{\mathbf{F}''-q}{\mathbf{F}''}.$$

Again, since $ab=I'H'$, and the triangles $I'H'F'$, ABF' are similar,

$$\frac{i}{o}=\frac{I'H'}{AB}=\frac{F'H'}{F'B}=\frac{F'H'}{H'B-H'F'}=\frac{F'H'}{F'H'-BH'}=\frac{\mathbf{F}'}{\mathbf{F}'-p}.$$

These expressions are evidently analogous to the expressions for the similar ratios that we obtained when considering the size of an image formed by refraction at a single spherical surface, with the exception that $\frac{g''}{g'}$ is substituted for $\frac{q}{p}$.

Distance of the image. Since the triangle $aJ''I''$ is similar to the triangle $F''ab$,

$$\frac{\mathbf{F}'}{p}=\frac{F'H'}{BH'}=\frac{F'H'}{AJ'}=\frac{I'H'}{I'J'}=\frac{ab}{I''J''}=\frac{ab}{-J''I''},$$

and $$\frac{ab}{-J''I''}=\frac{bF''}{-H''b}=\frac{bH''-F''H''}{bH''}=\frac{q-\mathbf{F}''}{q}.$$

Then $$\frac{\mathbf{F}'}{p} = \frac{q - \mathbf{F}''}{q} \text{ or } \frac{\mathbf{F}'}{p} + \frac{\mathbf{F}''}{q} = 1,$$

or $$(p - \mathbf{F}')(q - \mathbf{F}'') = \mathbf{F}'\mathbf{F}''.$$

Ex. 1. If an eye be one millimetre longer than normal, where will its *punctum remotum* (p_r) be?

That is, if the antero-posterior diameter of an eyeball be 23·823151 mm. instead of 22·823151 mm. (the lens being normal and in its usual position 3·6 mm. behind the cornea), at what distance must an object be situated in order that a definite image of it shall be formed on the retina?

In this case $\mathbf{F}'$ and $\mathbf{F}''$ have their normal values, and q is negative, being equal to $\mathbf{F}'' - 1 = -20{\cdot}713489 - 1$.

$$\frac{\mathbf{F}'}{p} + \frac{\mathbf{F}''}{q} = 1, \therefore \frac{\mathbf{F}'}{p} = \frac{q - \mathbf{F}''}{q} = \frac{-1}{q},$$

$$\therefore p = -\mathbf{F}'q = -(15{\cdot}498308)(-21{\cdot}713489) = 336{\cdot}522\ldots \text{ mm.}$$

in front of the first principal point, or 334·769 mm. in front of the cornea.

Using Donders' 'reduced eye' and the second formula for convenience,

$$(p - \mathbf{F}')(q - \mathbf{F}'') = \mathbf{F}'\mathbf{F}'',$$

$$(p - 15)(-21 + 20) = (15)(-20);$$

$$\therefore p - 15 = 300 \text{ or } p = 315 \text{ mm.}$$

This is a fairly correct result for such a rough and ready method.

Ex. 2. In the above example if the object is 5 mm. in height what is the height of the image? How many retinal cones will it cover assuming that the cones are in contact with each other?

The transverse diameter of a foveal cone is ·002 mm.; in other parts of the retina the diameter of a cone is ·006 mm.

$$\frac{i}{o} = \frac{\mathbf{F}'' - q}{\mathbf{F}''},$$

$$\therefore \frac{i}{5} = \frac{1}{-20{\cdot}713489}, \therefore i = \frac{-5}{20{\cdot}713489} = -{\cdot}241388.. \text{ mm.}$$

The negative sign shews that the image is inverted.

The approximate result given by Donders' reduced eye is

$$\frac{i}{o} = \frac{\mathbf{F}'' - q}{\mathbf{F}''}, \quad \therefore \quad \frac{i}{5} = \frac{1}{-20}, \quad \therefore \quad i = \frac{-5}{20} = -\cdot 25 \text{ mm.}$$

The number of foveal cones it will cover is

$$\frac{\cdot 241388}{\cdot 002} = 120\cdot 69..$$

Ex. 3. In a normal eye what will be the minimum visual angle?

The extreme minimum will be the angle subtended by a foveal cone at the second nodal point.

If a denote the minimum visual angle and c denote the transverse diameter of a foveal cone,

$$\tan\frac{a}{2} = \frac{\frac{c}{2}}{\overline{K''F''}} = \frac{\frac{c}{2}}{\overline{FH'}} = \frac{\cdot 001}{15\cdot 498308} = \cdot 000064523,$$

$$\therefore \quad a = 26''\cdot 617 \text{ approximately.}$$

Errors of Refraction. The Refraction of the eye is the term used to denote its minimum power of altering the direction of incident rays of light, making parallel rays convergent, and divergent rays less divergent. The refraction of the eye can only be determined when the eye is at rest, *i.e.* without any effort of accommodation. The refraction of an eye is said to be *emmetropic* (ἔμμετρος, ὤψ) when incident parallel rays converge to an exact focus on the retina, while if the incident light is at all convergent, it comes to a focus in front of the retina. Incident divergent rays however may, by a special effort of accommodation, whether conscious or unconscious, be brought to a focus on the retina. Thus an emmetropic child of 8 or 9 is one to whom the stars appear as points of light, yet by a special effort of accommodation he may be able to focus light on his retina that is diverging from a point only $2\frac{3}{4}$ ins. from his eye.

If the curvatures and refractive indices of the media of the eye are normal the second principal focus of the eye must, as we have seen, be 22·823151 mm. from the cornea. The retina must therefore be situated at this distance from the cornea.

Ametropia denotes that condition of the eye in which the second principal focus of the eye is not situated on the retina. It may be subdivided into three varieties.

Hypermetropia when the retina is in front of the second principal focus. Incident rays must therefore be convergent in order that they may be adequately converged by the eye to come to a focus on the retina.

Myopia (μύω, ὤψ) when the retina is behind the second principal focus. The rays of light therefore which come to a focus on the retina must have been divergent on incidence. Hence the vulgar term for this defect is near-sightedness. The word myopia alludes to the habit many such patients have of half closing their eyelids when viewing objects at a distance. They obtain a clearer view of distant objects by looking at them through a narrow aperture as this diminishes the size of their confusion-circles.

Astigmatism (α, στίγμα) is the condition in which the refraction of the eye in different meridians is different. This is usually due to the curvature of the cornea being different in different meridians. It may be associated with either of the other forms of ametropia. The stars are not seen by astigmatics as points of light but as small ovals or some other distorted form.

In what is called regular astigmatism the refraction of the eye is such that an incident pencil converges to two focal lines which are at right angles to each other. The most common form of regular astigmatism is due to the curvature in the vertical meridian of the cornea being greater than that in the horizontal meridian.

In this case the first focal line is situated in the horizontal plane nearer to the cornea than the second focal line, which is in the vertical plane. To remedy such a defect cylindrical glasses are required, plane in the vertical direction but curved to an appropriate degree in the horizontal direction. Should the patient be also hypermetropic or myopic this defect can be easily remedied in addition, by giving the requisite spherical curvature to the opposite face of the cylindrical glass.

Irregular astigmatism does not admit of accurate correction by cylindrical lenses. It is usually due to what may be called a minute crumpling of the cornea resulting from the contraction of scars of previous corneal ulcers.

Hypermetropia or Myopia may depend on one or more of three physical conditions.

1. The most usual condition is a defect or excess in length of the antero-posterior diameter of the eyeball. If the eye has not developed to its full size, and the retina is in front of the second principal focus of the eye, hypermetropia results. If the eye has grown too big, or if in any way the retina is behind the second principal focus, the eye is myopic. Such abnormalities are called *axial hypermetropia* and *axial myopia* respectively.

2. Hypermetropia may also result from an undue flattening of one or more of the refracting surfaces. The cornea is the surface most usually at fault. An increased bulging of the cornea will similarly give rise to a curvature-myopia. These curvature defects are however relatively rare.

3. Ametropia may also result from a change in the index of refraction of the medium. Reference has been already made to "acquired hypermetropia" resulting from an increase of the refractive index of the cortical part of the lens. It may be added that sometimes in incipient cataract

a central myopia will be found. The first stage of the formation of senile cataract is usually sclerosis of the nucleus. If now spaces filled with serous fluid are formed in the cortical layers of the lens, the refractive index of the cortical part diminishes and consequently the total refracting power of the axial part of the lens is increased (*vid.* p. 329), whereas the refracting power of the peripheral parts of the lens is diminished. When the pupil is widely dilated the central refraction of the eye may be myopic, while the refraction of the peripheral zone near the pupillary margin may be hypermetropic.

Correction of ametropia. We must consider in greater detail the typical form (1) of ametropia, and the means that we have for its correction.

Suppose that an eye is l mm. too long; its retina is then situated l mm. behind the second principal focus of the eye. (Fig. 125.)

Let p denote the distance of the *punctum remotum* (P) from the first principal point (H') of the eye. Let q denote the distance of the retina from the second principal point (H''), and let the focal distances from their respective principal points be denoted by $\mathbf{F}'$ and $\mathbf{F}''$. Then $q - \mathbf{F}''$ is a negative quantity denoting the distance between the retina and the second principal focus of the eye,

$$\therefore q - \mathbf{F}'' = -l.$$

Light from P will on entering the eye converge to a focus on the retina. In order that distinct images of distant objects may be formed on the retina, a lens must be placed somewhere between P and the cornea such that incident parallel rays may after traversing the lens diverge as if they had come from P.

Now it will be shewn presently that there is a peculiar advantage in placing the correcting glass in the first principal focal plane of the eye, that is about half an inch in front of the cornea.

If the second principal focal distance of the glass be denoted by f'' when $f''=p-\mathbf{F}'$, the proper correction will be given. With such a glass incident parallel rays will be converged to a focus on the retina by the system formed of the correcting glass and the eye.

But we know (p. 342) that

$$(p-\mathbf{F}')(q-\mathbf{F}'')=\mathbf{F}'\mathbf{F}'',$$

$$\therefore q-\mathbf{F}''=\frac{\mathbf{F}'\mathbf{F}''}{p-\mathbf{F}'}$$

or $$-l=\frac{\mathbf{F}'\mathbf{F}''}{f''}=\frac{(15{\cdot}498308)(-20{\cdot}713489)}{f''}=\frac{-321{\cdot}024\ldots}{f''}.$$

This expression $l=\dfrac{321\ldots}{f''}$ gives the relation between the amount of axial ametropia and the glass required to correct it.

Ex. 1. If the eye be 3·21 mm. too long, what glass will correct it for distance?

$$f''=\frac{321}{3{\cdot}21}=100\,\text{mm}.$$

As the second focal distance is positive, the lens must be concave. The dioptric strength of the glass is given by the expression

$$\frac{1000}{-x}=f''.$$

In the case given $x=-10D$.

Ex. 2. If an eye be axially hypermetropic and require a glass of $+1D$ to correct its error of refraction, what will be its length?

Since $\dfrac{1000}{-x}=\dfrac{321}{l}$ and in this case $x=1$,

$$l=\frac{-321}{1000}=-{\cdot}321\,\text{mm}.$$

Since l is negative the length of the eye must be defective by $-\cdot 321$ mm.

or
$$q - \mathbf{F}'' = -l,$$

$$\therefore\ q = \mathbf{F}'' - l = -20\cdot 713 \ldots + \cdot 321 \ldots = -20\cdot 392 \ldots \text{ mm.}$$

The length of the eye is therefore

$$q + H''A_0 = -20\cdot 392 \ldots - 2\cdot 109 \ldots = -22\cdot 501 \ldots \text{ mm.}$$

The advantage that is gained by placing the correcting glass in the first principal focal plane of the eye is that when so placed the retinal image of a distant object formed by the combined system, is of precisely the same size as that which would be formed by an emmetropic eye. This renders all tests of visual acuity strictly comparable to the results of the same tests applied to an emmetropic individual.

Suppose an axial ametrope, suitably corrected with glasses, when at 6 m. distance from test types can only read as far as a normal sighted person can read at 9 m. distance. His visual acuity is then $\frac{6}{9}$ of normal. This defect cannot then be assigned to any incidental effect of the correcting glass, for the retinal image is of the same size as that of the normal emmetrope placed at the same distance from the test types. The cause of the defect must be sought for in an opacity of the media, a defect in the retina itself or in the brain and the nerve fibres uniting it to the retina.

Correcting Lens in First Focal Plane. We proceed to shew how this peculiar advantage arises when the correcting glass of an axial ametrope is placed in the first focal plane of his eye.

Consider an eye l mm. too long provided with a thin correcting glass in the plane at H, distant HH' or d from the first principal plane of the eye. Let f', f'' be the focal distances of the lens, and let h', h'' represent the principal points of the combined system of the eye and the lens. (Fig. 125.)

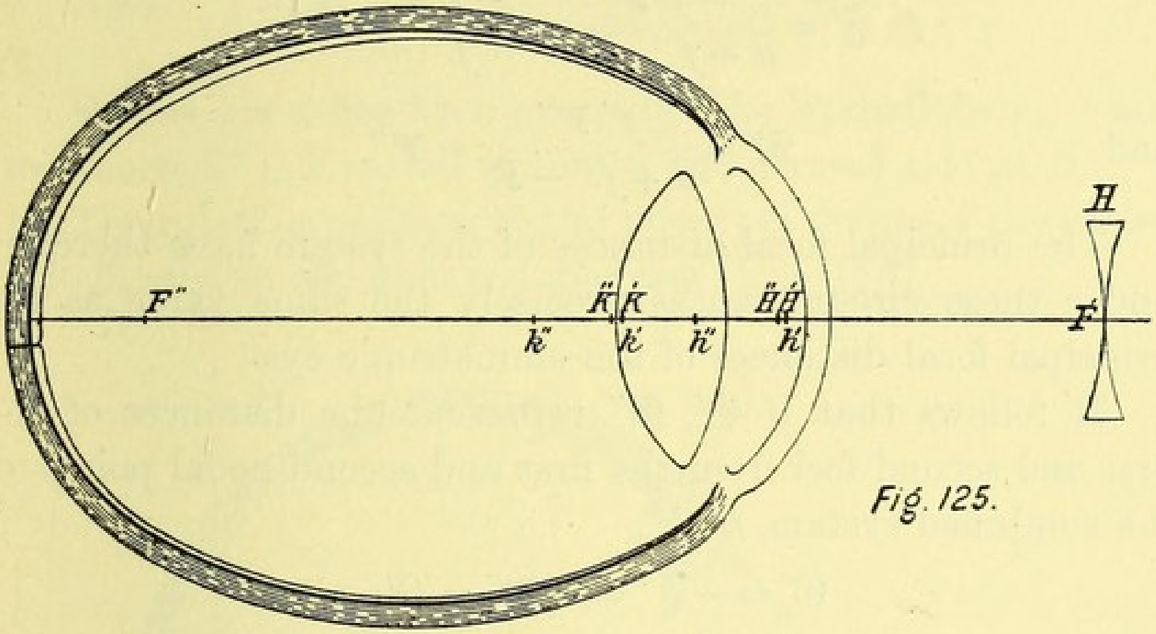

Fig. 125.

Then $$h'H = \frac{f'd}{d + f'' - \mathbf{F}'}.$$

If d is the first focal distance $\mathbf{F}'$,

$$h'H = \frac{f'\mathbf{F}'}{\mathbf{F}' + f'' - \mathbf{F}'} = \frac{f'\mathbf{F}'}{f''} = -\mathbf{F}';$$

but $$HH' = \mathbf{F}' \text{ or } H'H = -\mathbf{F}',$$

$\therefore$ the first principal point h' of the system coincides with the first principal point H' of the eye.

Also $$h''H'' = \frac{\mathbf{F}''d}{d + f'' - \mathbf{F}'},$$

and when $$d = \mathbf{F}',\ h''H'' = \frac{\mathbf{F}'\mathbf{F}''}{f''} = -l \text{ (p. 347)}.$$

The negative sign shews that the second principal point of the system (h'') is situated l mm. on the retinal side of the second principal point H'' of the eye. (Fig. 125.)

Now if $\mathfrak{F}'$ and $\mathfrak{F}''$ represent the distances of the first and second focal points of the system from the principal points h', h'',

$$\mathfrak{F}' = \frac{-f'\mathbf{F}'}{d + f'' - \mathbf{F}'} = \frac{-f'\mathbf{F}'}{f''} = \mathbf{F}',$$

and
$$\mathfrak{F}'' = \frac{f''\mathbf{F}''}{d + f'' - \mathbf{F}'} = \mathbf{F}''.$$

The principal focal distances of the system have therefore under these circumstances precisely the same value as the principal focal distances of the emmetropic eye.

It follows that if $\mathfrak{G}'$, $\mathfrak{G}''$ represent the distances of the first and second foci from the first and second nodal points of the combined system, k', k'',

$$\mathfrak{G}' = -\mathfrak{F}'' = -\mathbf{F}'' = G',$$
$$\mathfrak{G}'' = -\mathfrak{F}' = -\mathbf{F}' = G''.$$

The addition of a concave correcting glass in the first focal plane has no effect on the situation of the first nodal point, but the second nodal point is situated l mm. nearer the retina. The retinal image formed by the system is therefore of the same size as would be formed in an emmetropic eye in the same position[1].

If we consider the case of an axial hypermetrope whose eye is l mm. too short we shall find that on the proper addition of a convex correcting glass the second principal and nodal points are displaced towards the cornea l mm., so that their distance from the retina is that which obtains in emmetropia.

As before $h'H = \frac{f'd}{d + f'' - \mathbf{F}'} = -\mathbf{F}'$ when $d = \mathbf{F}'$.

[1] The mention of the displacement of the second nodal point has been given to make the discussion complete, but it is really unnecessary for we could infer the equality of the retinal image since

$$\frac{i}{o} = \frac{\mathbf{F}'}{\mathbf{F}' - p} = \frac{\mathfrak{F}'}{\mathfrak{F}' - p}.$$

Therefore h' coincides with H'.

Also $$h''H'' = \frac{\mathbf{F}''d}{d + f'' - \mathbf{F}'} = \frac{\mathbf{F}'\mathbf{F}''}{f''} = -l.$$

But as l is a negative quantity (the eyeball being l mm. too short) h'' is situated l mm. on the corneal side of H''.

The relative position of the principal points of the system is indicated in the annexed diagram (Fig. 126).

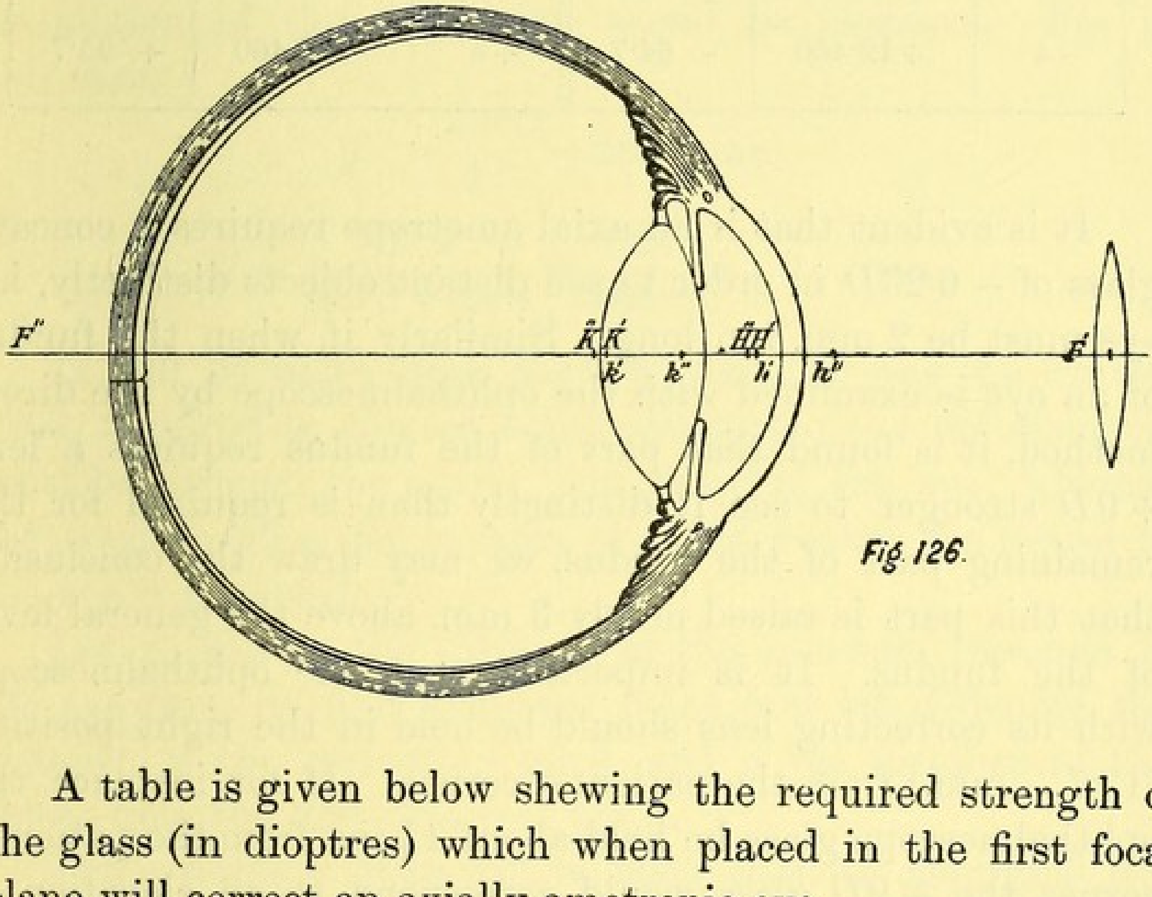

Fig. 126.

A table is given below shewing the required strength of the glass (in dioptres) which when placed in the first focal plane will correct an axially ametropic eye.

The first focal plane is situated 13·7452... mm. in front of the cornea.

The normal length of the eye is 22·8231... mm.

The excess and defect of this value is denoted by $+l$ and $-l$ respectively.

The distance of the *punctum remotum* (p_r) from the first principal point is given in a separate column. In hypermetropia the value of p_r is of course negative.

Hypermetropia.			Myopia.		
l (mm.)	D	p_r (mm.)	l (mm.)	D	p_r (mm.)
0	0	∞	0	0	∞
−1	+ 3·115	−305·5	+1	− 3·115	+336·5
−2	+ 6·230	−145·0	+2	− 6·230	+176·0
−3	+ 9·345	− 91·5	+3	− 9·345	+122·5
−4	+12·460	− 64·7	+4	−12.460	+ 95·7

It is evident that if an axial ametrope requires a concave glass of $-6{\cdot}23D$ in order to see distant objects distinctly, his eye must be 2 mm. too long. Similarly if, when the fundus of an eye is examined with the ophthalmoscope by the direct method, it is found that part of the fundus requires a lens $+9D$ stronger to see it distinctly than is required for the remaining part of the fundus, we may draw the conclusion that this part is raised nearly 3 mm. above the general level of the fundus. It is important that the ophthalmoscope with its correcting lens should be held in the right position (13·7... mm.) from the patient's cornea. If for instance the ophthalmoscope glass be held about 1½ ins. from the patient's cornea the $+9D$ glass would correspond to an elevation of about 3·7 mm. instead of 3 mm.

It is only true that $l = \frac{-\mathbf{F'F''}}{f''}$ if the glass of focal length f'' is placed in the first focal plane of the eye, *i.e.* at a distance of 13·7... mm. from the cornea. If, as in the above example, the glass is placed 1½ ins. or 38·1 mm. from the cornea, it is necessary to add the difference 24·4 mm. to the denominator of the expression.

Since $$\frac{1000}{-x} = f'', \; f'' = \frac{1000}{-9} = -111\cdot\dot{1},$$

$$l = \frac{-\mathbf{F'F''}}{-111\cdot\dot{1} + 24\cdot4} = \frac{321}{-86\cdot7} = -3\cdot7\ldots \text{ mm.}$$

If a concave glass of $-4\cdot5D$ had been required to see part of the fundus we may draw the conclusion that this part is depressed below the general level of the fundus by about $1\cdot\dot{4}$ mm. If however the ophthalmoscope be held at an improper distance from the cornea, say $1\frac{1}{2}$ ins., this estimation of the depression would be incorrect. For in this case,

$$f'' = \frac{1000}{4\cdot5} = 222\cdot\dot{2} \text{ mm.},$$

$$l = \frac{-\mathbf{F'F''}}{222\cdot\dot{2} + 24\cdot4} = \frac{321}{246\cdot6} = 1\cdot3\ldots \text{ mm.}$$

It will be easily seen that in a similar way the convex glass, required to correct a hypermetropic eye for distance, must be stronger the nearer it is placed to the eye; whereas the concave glass for myopia must be weaker the nearer to the eye it is placed.

Let p_r denote the distance of the *punctum remotum* of the eye from its first principal point, and let d denote the distance of the correcting glass, and p the distance of the object from the first principal point.

The correcting glass will be such that its conjugate foci are $p - d$ and $p_r - d$. Its second focal distance f'' will be given by the formula

$$\frac{1}{p-d} - \frac{1}{p_r - d} + \frac{1}{f''} = 0.$$

A. When the object for which the eye is adjusted is at a considerable distance (6 metres or more) $\frac{1}{p-d}$ may be regarded as equal to 0, and $$\frac{1}{f''} = \frac{1}{p_r - d};$$

or $$f'' = p_r - d.$$

Now in myopia p_r is positive.

Therefore if $p_r - d$ is positive, f'' is positive and is greater the smaller d is. In other words, the refracting power $\left(\frac{1}{f''}\right)$ of the concave lens must be diminished as d becomes less[1].

In hypermetropia p_r is negative.

Therefore $p_r - d$ or f'' is negative and diminishes with d.

The lens is therefore convex and its refracting power $\left(\frac{1}{f''}\right)$ must be increased as d is diminished.

B. When the object for which the eye is adjusted is fairly close to the eye (*e.g.* at reading distance or $\frac{1}{3}$ m.) the value of $\frac{1}{p-d}$ is no longer negligible.

We have then

$$\frac{1}{f''} = \frac{1}{p_r - d} - \frac{1}{p - d} = \frac{p - p_r}{(p_r - d)(p - d)}.$$

In myopia p_r is always positive, and if the degree of myopia be above $-3D$, concave spectacles must be worn in order to see distinctly an object at the distance of $\frac{1}{3}$ m.

When as is almost always the case $p_r - d$ is positive, it is clear that $\frac{1}{f''}$ becomes less as d is diminished, or that the strength of the correcting concave glass must be diminished the closer it is placed to the eye.

[1] If $p_r - d$ is negative, f'' is negative and the lens is convex. An inverted image of the object is consequently presented to the eye. Such a lens has been used in this way by a few extremely myopic patients as a hand-glass held at a distance of 15 cm. or 20 cm. from the eye.

A myopic patient for instance who requires a $-10D$ lens in the first focal plane to correct him for distance, will without exerting his accommodation be able to see distinctly an object at a distance of less than 1 metre by increasing the distance of the glass from his eye by 10 mm.

For a concave glass of $-10D$ ($f''=100$ mm.) at 23·745 mm. from the cornea has the same effect as a concave glass of about $-9D$ ($f''=112{\cdot}3$ mm.) at 13·745 mm. from the cornea.

The calculation is an easy one.

We have first to find the distance of the *punctum proximum* (p_r) from the principal point. We know that a lens of focal length 100 mm. corrects the eye for distance when placed in the first focal plane (*i.e.* 15·4983 mm. from the principal point or 13·745 mm. from the cornea).

$$f'' = p_r - \mathbf{F}',$$

$$p_r = f'' + \mathbf{F}' = 100 + 15{\cdot}4983.$$

Now when the glass is placed d mm. (or $\mathbf{F}' + 10$ mm.) from the principal point

$$\frac{1}{100} \text{ or } \frac{1}{f''} = \frac{p - p_r}{(p_r - d)(p - d)}$$

$$= \frac{p - 115{\cdot}4983}{(115{\cdot}4983 - 25{\cdot}4983)(p - 25{\cdot}4983)},$$

$$90p - 2294{\cdot}847 = 100p - 11549{\cdot}83,$$

$$\therefore\ 10p = 9254{\cdot}98\ldots \text{ mm.},$$

$$p = 925{\cdot}498\ldots \text{ mm.}$$

The distance consequently for which the $-10D$ glass now adjusts the eye is 925·498 mm. from its principal point, or 923·745 mm. from the cornea.

In hypermetropia p_r is always negative, and a variation in the distance of the glass from the eye will have a different effect according to the value of p_r or according to the degree of hypermetropia.

For $$\frac{1}{f''} = \frac{p - p_r}{(p_r - d)(p - d)} = \frac{p - p_r}{p_r p - d(p_r + p) + d^2}.$$

The numerator is always positive, and the denominator is always negative for d must be always less than p, for the correcting glass must be placed between the object viewed and the eye. Therefore f'' must be negative or the correcting glass must be convex.

If $p_r + p$ is a negative quantity or even if it is a positive quantity but smaller than d, the negative value of the denominator diminishes with an increase of d. If $p_r + p$ is a positive quantity and greater than d, the negative value of the denominator increases with an increase of d.

So we see that with a high degree of hypermetropia, when $p_r + p$ is positive and greater than d, the further the lens is removed from the eye, the weaker the lens should be. In other words, an increase of d will have the effect of virtually increasing the strength of the lens.

For example, suppose a hypermetropic (or aphakic) patient requires $+10D$ for seeing distant objects when the glass is 15·498... mm. from his first principal point,

$$p_r = -100 + 15{\cdot}498\ldots = -84{\cdot}502\ldots \text{ mm.}$$

Now if he have lost his power of accommodation, a convex lens of $+11D$ at the same distance from his eye will enable him to see an object 1015·498... mm. from his principal point. However the original glass ($+10D$) will serve for the same purpose, if placed about 1 inch from his eye, say 26·749 mm. from the principal point instead of 15·498 mm.; for

$$\frac{1}{f''} = \frac{p - p_r}{pp_r - d(p + p_r) + d^2},$$

or $$\frac{1}{f''} = \frac{1015{\cdot}498 + 84{\cdot}502}{-85811{\cdot}6 - 26{\cdot}749\,(930{\cdot}996) + (26{\cdot}749)^2} \text{ nearly}$$

$$= \frac{1100}{-110715 + 715} \text{ nearly } = \frac{1}{-100}.$$

Patients who have been rendered highly hypermetropic by the extraction of their cataractous lens, are of course deprived of their power of accommodation. They are however enabled in this way to virtually increase the strength of their correcting glass for distance when they wish to see distinctly objects at a distance of not less than about 1 metre.

Patients with a low degree of hypermetropia have not the same advantage. For if $p_r + p$ is negative or even if positive when less than d, an increase of the distance of the glass from the eye virtually diminishes the strength of the glass.

For example, suppose a hypermetropic patient, who has no power of accommodation requires $+5D$ for seeing at a distance of 250 mm. from his principal point—the glass being in his first focal plane; for what distance will his eye be adjusted if the glass be removed to 38 mm. ($1\frac{1}{2}$ ins.) from his principal point?

$$\frac{-5}{1000} \text{ or } \frac{-1}{200} = \frac{250 - p_r}{250p_r - 250\,(15{\cdot}4983) - p_r\,(15{\cdot}4983) + (15{\cdot}4983)^2},$$

$$\therefore\ p_r\,(200 - 250 + 15{\cdot}4983) = 50000 - 3874{\cdot}575 + 240{\cdot}197\ldots$$

$$p_r = \frac{46365{\cdot}622}{-34{\cdot}5017} = -1343{\cdot}98\ldots \text{ mm.}$$

Now if the $+5D$ lens is placed 38 mm. from the principal point we have

$$\frac{-1}{200} = \frac{p + 1343{\cdot}98}{-1343{\cdot}98p - 38p + (38)(1343{\cdot}98) + (38)^2},$$

$$\therefore\ p\,(1343{\cdot}98 + 38 - 200) = (200)(1343{\cdot}98) + 38\,(1343{\cdot}98) + (38)^2,$$

$$\therefore\ p = \frac{321311{\cdot}2}{1181{\cdot}98} = 271{\cdot}8\ldots \text{ mm.}$$

Removing the glass less than an inch from its previous position has in this case the effect of virtually diminishing its strength. An object 271·8 mm. from the principal point is now distinctly seen, whereas the eye was adjusted for a distance of 250 mm. when the glass was in its original position.

Presbyopic patients who are wearing convex glasses that are too weak for them are often noticed reading with their spectacles at the end of their nose. A popular but erroneous explanation of this habit is that by so doing they virtually increase the strength of the glass. As we have seen, unless the hypermetropia is extreme, increasing the distance of the glass from the eye virtually diminishes its strength. The true explanation is that by this action the second nodal point of the system is placed more anteriorly, hence the retinal images are larger, and the print though indistinct is easier to decipher.

Aphakia, (*α, φακός*), or *Absence of the crystalline lens.* This condition is produced after the extraction of the opaque lens in cataract operations and occasionally to relieve extreme degrees of myopia.

The eye is then reduced to the simplest possible refracting system, viz. one single medium ($\mu = 1{\cdot}3365$) bounded by a single spherical surface. We will consider the refractive condition of an eye that had been axially ametropic when the lens was present.

The principal point of the eye is situated on the cornea at the point of intersection of the optic axis. The second principal focus

$$= \frac{\mu r}{\mu - 1} = \frac{1{\cdot}3365\,(-7{\cdot}829)}{{\cdot}3365} = -31{\cdot}09497 \ldots \text{ mm.}$$

An eye then of this length when deprived of its crystalline lens will if normal in other respects have distinct distant vision. But such an eye is 8·2718 mm. longer than normal and we know that the strength of the glass required to correct its myopia when the lens was present is given by the formula (p. 347)

$$\frac{1000}{-x}=\frac{321}{8{\cdot}2718},\ \therefore\ x=-25{\cdot}76\ldots\ D.$$

If a patient then of this degree of myopia have his lens removed, he will require no lens for seeing distant objects, provided that the curvature of his cornea has undergone no change subsequent to the operation.

It is an easy matter to determine what correcting glass an emmetropic patient will require for distance if his lens is removed. His eye is 22·823151 mm. long; let it be denoted by q, then the position of his *punctum remotum* (p_r) is given by the expression

$$\frac{\mu}{q}-\frac{1}{p_r}=\frac{\mu-1}{r},$$

or

$$\frac{1}{p_r}=-\frac{\mu-1}{r}+\frac{\mu}{q}=-\frac{{\cdot}3365}{-7{\cdot}829}+\frac{1{\cdot}3365}{-22{\cdot}823151}$$

$$=\frac{1}{23{\cdot}26597}-\frac{1}{17{\cdot}0768},$$

$$\therefore\ p_r=-64{\cdot}1941\ldots\ \text{mm.}$$

We have therefore to provide the eye with a convex lens which will cause incident parallel rays to converge to a point 64·1941 mm. behind his cornea. We must first decide how far in front of his cornea it is to be placed. There is now no advantage in putting it in the first focal plane, as the retinal image must necessarily be larger than that of an emmetropic eye owing to the displacement forwards of the second nodal point. For convenience we may assume that the spectacles will be worn in the usual position, 13·7452 mm. in front of his cornea. The second focal distance (f'') will then be

$$f''=p_r-d=-64{\cdot}1941-13{\cdot}7452=-77{\cdot}9393\ \text{mm.}$$

The dioptric strength of the glass is given by the expression

$$x=\frac{1000}{-f''}=12{\cdot}83D \text{ nearly}^1.$$

In a similar way the glass required for any degree of axial ametropia may be found.

For instance, take an axial hypermetrope whose eye is 2 mm. too short ($l=-2$)

$$\frac{1}{p_r}=-\frac{\mu-1}{r}+\frac{\mu}{q-l}=\frac{1}{23{\cdot}26597}+\frac{1{\cdot}3365}{-22{\cdot}823151+2},$$

$$\frac{1}{p_r}=\frac{31{\cdot}094973-20{\cdot}823151}{-(20{\cdot}823151)(23{\cdot}26597)}=\frac{10{\cdot}271822}{-484{\cdot}471}=\frac{1}{-47{\cdot}165},$$

$$x=\frac{1000}{-(p_r-d)}=\frac{1000}{47{\cdot}165+13{\cdot}745}=+16{\cdot}4176\ldots D.$$

The following table gives the position of the *punctum proximum* (p_r) and the value of the correcting glass (D) required for distance by aphakic ametropes. The excess or defect in the length of the eye is denoted by l. The correcting glass which was required for distance before the lens was removed is denoted by D'. The column $D-D'$ gives the change of refraction that follows on removal of the lens. It will be noticed that this change of refraction increases with the length of the eye. The correcting glass, required in

[1] This is a higher value than is given in English books on the subject. Most English authorities give $+10D$ as the strength usually required after extraction of the lens from an emmetrope. Continental authorities have given a higher estimation of the change of refraction that ensues on removal of the lens. Thus Landolt gives $+11D$, Donders $\frac{1}{3{\cdot}5''}$ to $\frac{1}{3''}$ or $+11{\cdot}25D$ to more than $+13D$. It is essential that in judging from clinical experience one should not be misled by any preceding acquired hypermetropia (p. 330).

	l	p_r	D	D'	$D-D'$
Hyper-metropia	− 3 mm.	− 40·916	+ 18·294	+ 9·345	8·949
	− 2 mm.	− 47·165	+ 16·417	+ 6·230	10·187
	− 1 mm.	− 54·761	+ 14·597	+ 3·115	11·482
	0	− 64·194	+ 12·830	0	12·830
Myopia	+ 1 mm.	− 76·221	+ 11·115	− 3·115	14·230
	+ 2 mm.	− 92·084	+ 9·449	− 6·230	15·679
	+ 3 mm.	− 113·964	+ 7·831	− 9·345	17·176
	+ 4 mm.	− 146·089	+ 6·256	− 12·460	18·716
	+ 5 mm.	− 197·851	+ 4·729	− 15·575	20·304
	+ 6 mm.	− 295·181	+ 3·237	− 18·690	21·927
	+ 7 mm.	− 545·567	+ 1·788	− 21·805	23·593
	+ 8 mm.	− 2638·235	+ ·377	− 24·920	25·297
	+ 9 mm.	+ 1016·779	− ·997	− 28·035	27·038

aphakia, if 13·7... mm. in front of the cornea does not vary directly as the length of the eye. This would only occur if the correcting glass were placed in the first focal plane of the aphakic eye, *i.e.* 23·26597 mm. in front of the cornea; for then (as on p. 347)

$$-l' = q - \mathbf{F}'' = \frac{\mathbf{F}'\mathbf{F}''}{p - \mathbf{F}'}.$$

If then it is required to estimate correctly the amount of swelling of the optic disc in optic neuritis of an aphakic eye with a refraction-ophthalmoscope, it will be advisable to hold the instrument in such a way that the lens behind the eye-piece is about $\frac{11}{12}$ in. from the cornea. In that case $p - \mathbf{F}'$ is the second focal distance (f'') of the lens used.

$$\therefore\ -l' = \frac{\mathbf{F}'\mathbf{F}''}{f''} = \frac{(23{\cdot}26597)(-31{\cdot}09497)}{f''} = \frac{-723{\cdot}455\ldots}{f''},$$

or if X denote the dioptric strength of the glass used in this position

$$X = -\frac{1000l'}{723{\cdot}455} = -1{\cdot}38225\ldots\, l',$$

$$l' = -{\cdot}723455X.$$

Now $-l' = q - \mathbf{F}''$ or the difference between the actual length of the eye and its second focal distance.

In an aphakic eye of normal length

$$-l' = -22{\cdot}823151 + 31{\cdot}094973 = 8{\cdot}271822.$$

The eye is therefore 8·271822 mm. too short.

The correcting glass required 23·26597 mm. from the cornea is

$$-1{\cdot}38225l' \text{ or } +11{\cdot}4337\ldots\, D.$$

Whatever the length of the eye, the above expression will give the value correctly as determined by the dioptric strength of the correcting glass of the ophthalmoscope, provided that it be held in the right position and the curvature of the cornea is normal.

Suppose that the edges of the optic disc in an aphakic eye require a lens of $+3D$ more than that required to view the rest of the fundus,

$$l' = -{\cdot}723455X = -2{\cdot}17\ldots \text{ mm.}$$

The edges of the disc are therefore raised above the level of the fundus 2·17 mm. A similar observation of a normal eye (containing its crystalline lens) would indicate a swelling of barely 1 mm. if the ophthalmoscope was held $\frac{1}{2}$ in. from the cornea.

Size of the retinal image in aphakia. Since the size of the retinal image of a distant object varies directly as the distance of the second nodal point from the retina, and this distance is equal to $\mathfrak{F}'$ the first principal focal distance of the corrected eye, we must find the value of $\mathfrak{F}'$.

In an aphakic eye of normal curvature of cornea

$$F' = 23{\cdot}26597 \text{ mm.}$$

If the eye be of normal length it requires a lens

$$(f'' = -77{\cdot}9393)$$

at a distance of 13·7452 mm. from its cornea to correct it for distance

$$\mathfrak{F}' = \frac{-f'F'}{d+f''-F'} = \frac{-(77{\cdot}9393)(23{\cdot}26597)}{13{\cdot}7452 - 77{\cdot}9393 - 23{\cdot}26597},$$

$$\therefore \ \mathfrak{F}' = 20{\cdot}73327\ldots \text{ mm.}$$

In a normal emmetropic eye

$$\mathbf{F}' = 15{\cdot}498308,$$

$$\frac{i'}{i} = \frac{\mathfrak{F}'}{\mathbf{F}'} = \frac{20{\cdot}73327}{15{\cdot}498308} = 1{\cdot}33777.$$

It will be found that $\mathfrak{F}'$ undergoes a slight variation according to the degree of axial ametropia, but for practical purposes we may consider that a corrected aphakic eye forms retinal images about one third larger than it did before the extraction of its lens.

When testing an aphakic eye with test types, this fact should be remembered, for what would ordinarily represent a visual acuity of $\frac{6}{9}$, would represent in an aphakic eye a visual acuity of only $\frac{6}{12}$.

Means of estimating the refractive condition of the eye. A. *The direct examination of the eye with the ophthalmoscope.* There are three conditions which must be fulfilled to render this a satisfactory method. The accommodation both of the patient and of the observer must be relaxed, and the ophthalmoscope must be held about $\frac{1}{2}$ in. from the patient's cornea, so that the correcting glasses that are required may be placed in the first focal plane of the patient's eye. Under these circumstances if both patient and observer are emmetropic, a distinct view of the details of the fundus will be obtained without the use of any correcting glass. For since the patient is emmetropic, his retina lies in his second focal plane, and the light emerging from his retina will proceed to the eye of the observer in parallel rays. But the observer's eye is in such a condition that parallel rays come to a focus on his retina, hence a distinct image of the patient's fundus is formed upon it. The weakest convex glass in the eyepiece of the ophthalmoscope will blur the image, for the parallel rays which come from the patient's fundus will after traversing this lens be rendered convergent, and hence will come to a focus in front of the observer's retina. Similarly if the observer keep his accommodation relaxed, a weak concave glass will render the light from the fundus divergent and hence will tend to come to a focus behind the observer's retina. It is however exceedingly difficult for a young observer to keep his accommodation relaxed under these conditions. If he fail to do so, he will see a distinct image of the patient's fundus, owing to his neutralizing the effect of the concave glass by his own accommodation.

If it is found that on bringing successive convex glasses before the mirror of the ophthalmoscope, $+2D$ is the highest convex glass with which a distinct image is obtained, it may be inferred that $+2D$ is the glass the patient requires for distant vision. For $+2D$ is the lens required to render the emergent rays from the patient's retina parallel and therefore the same lens is required to bring incident parallel rays from a distant object to a focus on his retina. Similarly should $-5D$ be the weakest concave glass with which the details of the fundus are observed, it may be inferred that a glass of $-5D$ will correct the patient for distance.

Now suppose a case of compound hypermetropic astigmatism in which the distance of the cornea to the retina is about 1 mm. too short, and the curvature of the cornea is normal from above downwards but rather flattened from side to side. In such a case a $+3D$ glass will be the strongest glass with which the *horizontal* vessels are seen distinctly, while a stronger glass, say $+5D$ will be the strongest glass with which the *vertical* retinal vessels and the lateral edges of the disc are seen distinctly. The $+3D$ glass will render the vertical meridian of the patient's eye emmetropic and hence he will see horizontal lines distinctly, whereas a $+5D$ glass will enable him to see distinctly only vertical lines. His correcting glass will therefore be $+3D$ sph. $+2D$ cyl. axis vertical[1].

If the observer be ametropic he must subtract the amount of his own ametropia from the result obtained. Thus if the observer have $1D$ of hypermetropia and made the above observation the patient would require for distance

$$+2D \text{ sph.} + 2D \text{ cyl. axis vertical.}$$

Similarly if the observer have $-2D$ of myopia, the patient's correcting glasses will be $+5D$ sph. $+2D$ cyl. axis vertical.

For $$+3D-(-2D)=+5D.$$

This is much the readiest method of estimating a patient's refraction but it requires a good deal of practice to be at all proficient at it. Even experienced ophthalmologists cannot attain the same accuracy in their results, as can be reached by the next method described.

It is important to remember that though the optic disc is the easiest object to examine in this way, it is the very part to avoid in an estimate of a patient's refraction. For it must be remembered that the optic disc corresponds to the

[1] It should be noted that in a prescription for spectacles, the axis of the cylinder is always the plane axis, the "working axis" or axis of curvature being at right angles to the plane axis.

patient's blind spot, and moreover is often on a higher level than the macular region especially in cases of myopia. Now it is the macular region that embraces the area of most acute sight and consequently is the part that requires accurate correction. Unfortunately however there are no fine blood-vessels or other markings to serve as good test objects in the macular region. In addition to this unless a mydriatic has been previously employed, the pupil at once contracts when the light from the opthalmoscope mirror falls on the macula. There are almost always some fine horizontal blood-vessels passing from the optic disc outwards towards the macula, and these should be made use of by the observer as test-objects.

B. *Retinoscopy.* (Syn. Skiascopy, or Shadow Test).

This is the most accurate test for estimating the static refraction of an eye. It is usually carried out in the following way.

The patient is seated in a dark room with his back to a convenient form of artificial light—preferably an Argand burner. This light is placed some little distance behind the patient and just above the level of his head. The observer, provided with a perforated concave spherical mirror of 25 cm. focus, stations himself in front of the patient about 150 cm. in front of the light. He now reflects the light into the eye he wishes to examine, and looking through the central perforation or sight-hole of the mirror he will obtain the ordinary red fundus-reflex from the patient's eye. On slightly rotating the mirror from above downwards the illumination may appear to move across the pupil either in the same or in the reverse direction according to the refraction of the eye examined. A little consideration will show the reason of this.

If the light be represented by *A* (Fig. 127), 150 cm.

distant from the concave mirror of focal length 25 cm., a real inverted image of the light will be formed at a, 30 cm. from the mirror, for

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}, \therefore \frac{1}{q}=\frac{1}{25}-\frac{1}{150}=\frac{1}{30}.$$

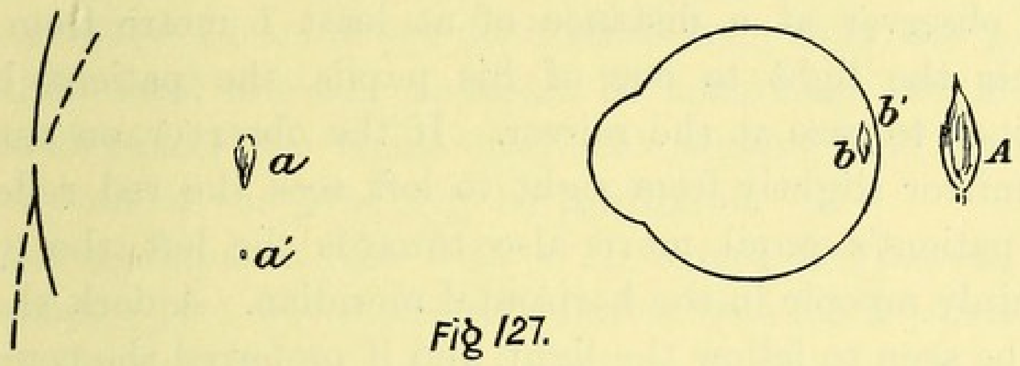

Fig 127.

The light from this image will illumine a small area of the patient's retina at b; in fact an inverted image of a will tend to be formed at b. Now on turning the mirror slightly downwards, the image at a will move downwards to a'. Consequently the illuminated area on the patient's retina at b will move upwards to b'. Whether the patient's eye be myopic or hypermetropic the illuminated area on his retina will move in the reverse direction to the mirror. If the patient's distance from the observer be less than that of his far point, the observer will see a magnified erect image of the patch of light through the patient's pupil. Suppose the patient be 1 metre from the observer; then if the patient be hypermetropic, or even myopic but to a less extent than $1D$, on turning the mirror, the observer will see the light move in the reverse direction across the patient's pupil. If the myopia is greater than $1D$ an inverted image of b will be formed at the patient's far point, which is situated somewhere between him and the observer. This inverted image will move in the reverse direction to b and therefore in the same direction as that in which the mirror is turned.

Hence in estimating a patient's refraction accurately by retinoscopy the following method must be adopted. Some mydriatic (*e.g.* homatropine) that will temporarily paralyse the muscle of accommodation must be first instilled into the patient's eyes. When his pupils are fully dilated he must be placed as above described with his back to the artificial light. The observer at a distance of at least 1 metre from him directs the light to one of his pupils, the patient being enjoined to gaze at the mirror. If the observer, on turning the mirror slightly from right to left, sees the red reflex in the patient's pupil move also towards the left, the eye is certainly myopic in the horizontal meridian. A dark shadow will be seen to follow the light, and if preferred the course of this dark shadow may be noted across the patient's pupil. Successive concave glasses are now placed in the trial frames before the patient's eyes until one is found with which the shadow just moves in the reverse direction. Suppose with $-2D$ the shadow moves in the same direction as the mirror, whereas with $-2{\cdot}25D$ the shadow moves against the mirror; we know that with $-2D$ the patient's far point is just situated on his side of the observer. If now the observer be a little more than 1 metre from the patient we may expect that $-2D$, $-1D$, or $-3D$ will correct his horizontal meridian for distance. If the same observations are made when the mirror is turned from above downwards, we may conclude that no astigmatism is present and that $-3D$ sph. will correct the eye for distance.

It is important that the patient should keep his eye fixed on the mirror, for then we are sure of correcting his macular region for distance. In myopia the macular region almost always bulges backwards behind the level of the rest of the fundus; hence a stronger concave glass will be necessary to correct the macular region than the adjoining parts of the retina. In hypermetropia there is usually no posterior staphyloma as it is called, so that if the patient relax his accommodation by directing his attention to some distant object, a fairly exact estimation of his refraction may be made even without the use of a mydriatic.

If astigmatism be present, the illuminated area on the retina at *b* will form an oval. In the usual form the cornea is more curved from above downwards than from side to side. The first focal line will therefore be horizontal, and if it be situated nearer the retina than the second focal line, the major axis of the oval will be horizontal.

Let us suppose that with $+2D$ before the patient's eye the shadow just moves with the mirror when it is tilted from above downwards or rotated about a horizontal axis. An inverted real image of the horizontal focal line is then formed between the observer and the patient, and the shadow will move at right angles to its direction. If attention be paid to the lateral margins of the image while the mirror is rotated about a vertical axis, the lateral shadows will be seen to move in the opposite direction to the mirror.

Let us suppose that $+4D$ has to be put before the patient's eye in order that these lateral shadows shall also move in the same direction as the mirror. Then the difference $(4D-2D)$ or $2D$ is the amount of astigmatism and the correcting glass will be $+1D$ sph. $+2D$ cyl. axis vertical.

It may be noted that from the appearance of the edge of the shadow and from the rate of its movement a rough estimate of the degree of ametropia may be made. Thus when nearly $-1D$ of myopia is present a fairly sharply defined image of the light will be formed bounded by a dark area with a fairly sharp edge. This will be seen by the observer as a real inverted image if the eye be myopic more than $-1D$, and as a virtual erect image if hypermetropic, or myopic less than $-1D$. Hence if the ametropia be not far removed from $-1D$ the edge of the shadow will appear fairly well defined. Also the greater the ametropia the nearer to the nodal point will the image of the illuminated patch on the retina be formed. If then the ametropia be of high degree, the shadow which is seen on slightly rotating the mirror will have ill defined margins and will move slowly. As the ametropia approaches correction by glasses placed in the trial-frame on the patient, the edge of the shadow will become more sharply defined, and its movements will be more rapid.

Some ophthalmologists prefer using a plane mirror in place of a concave one for retinoscopy. In that case since the image formed by a plane mirror is virtual and erect, it will move in the opposite direction to the mirror. The shadow therefore moves with the mirror in hypermetropia and against the mirror in myopia when a plane mirror is used.

If the ametropia be of high degree some details of the fundus may be seen, when the observer is at a distance of 1 metre or so from the patient, and reflects the light into his eye. It is clear that if the eye be highly myopic, an inverted image of the fundus will be formed between the observer and the patient. If the observer now moves his head from side to side, the retinal vessels or other details that he may see will appear to move in the reverse direction.

If the eye be hypermetropic, the observer may see an erect virtual image of the fundus behind the patient's eye. The retinal vessels will therefore now seem to move in the same direction as the observer's head.

The angles α, γ and κ, the visual line and the fixation-line. As we have said before (p. 333) the surfaces which bound the different media of the eye are not accurately centred; that is to say the several centres of curvature of the surfaces do not lie in one straight line. We have therefore to consider in some detail certain lines and angles which have received specific names in ophthalmic literature.

In Fig. 128 the line $AK'K''\phi''$ represents the optic axis, F represents the fovea, O represents the object viewed. What

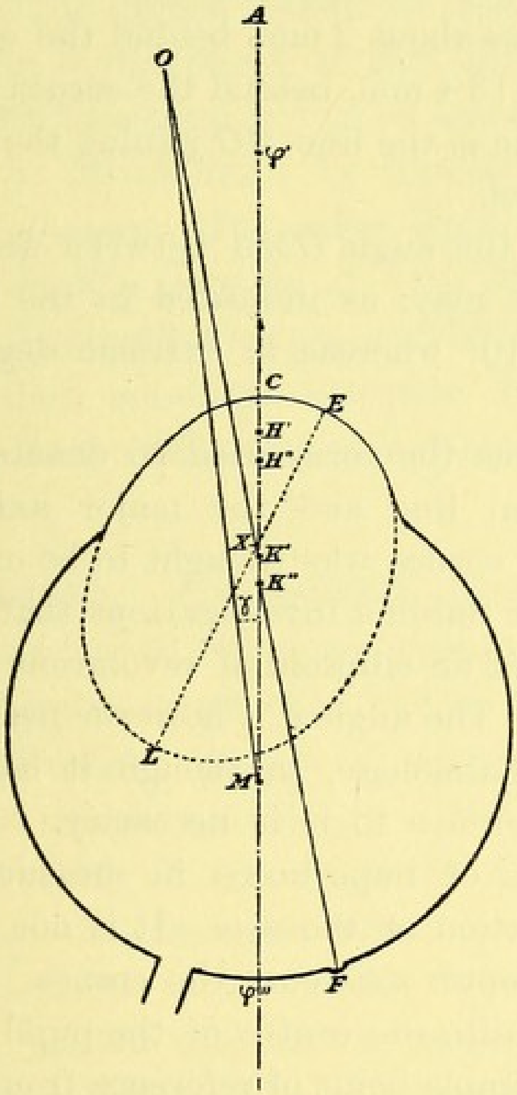

Fig. 128.

is called the visual line consists, strictly speaking, of two parallel lines, one OK' drawn from O to the first nodal point, the other $K''F$ drawn from the second nodal point K'' to the fovea. It will be noticed that the visual line cuts the cornea on the inner side of the optic axis in the diagram. This is always the case in emmetropia and hypermetropia. Thus if a hypermetrope view some distant object so that his visual lines are parallel, his optic axes will diverge. This divergence is less in myopia, while in extreme degrees of myopia the optic axes may even converge.

Now it is found that slight movements of the eyeball may be considered as movements of rotation about a fixed centre M, which is called the centre of motility. The position of this centre varies slightly in different eyes. Perhaps we may say that generally it lies about 2 mm. behind the centre of the optic axis, that is about 13·4 mm. behind the cornea[1].

The fixation-line is the line MO joining the centre of motility to the object viewed.

The angle γ is the angle OMA between the fixation-line and the optic axis. It may, as indicated in the diagram, attain a positive value of 10° whereas in extreme degrees of myopia it may be negative.

The angle α was the term used to denote the angle OXE between the visual line and the major axis of the corneal ellipsoid when the cornea was thought to be of that shape. We now know from Dr Sulzer's investigations that the shape of the cornea is not that of an ellipsoid of revolution and that it has no axis of symmetry. The angle α is however frequently referred to in books on ophthalmology, so though it has now no definite meaning some reference to it is necessary.

The angle κ is of importance in measuring the angle of strabismus or deviation of the eye. It is not easy to determine at what point the optic axis cuts the cornea, whereas the point of the cornea opposite the centre of the pupil is easily seen and therefore forms a simple point of reference from which to measure deviations of the eye. The angle κ is the angle that the fixation-line makes with the normal to the cornea that passes through the pupillary centre. As the pupil is usually rather to the inner side of the centre of the cornea, the angle κ is not equal to the angle γ, but being much easier to measure than γ, is far more

[1] Landolt gives 13·73 mm. behind the centre of the cornea, which would be 2·32 mm. behind the centre of the optic axis. Donders gives from 1·75 mm. in myopia to 2·17 mm. in hypermetropia as the mean distance between M and the centre of the optic axis.

convenient to make use of in the measurements of strabismus. For the way in which such measurements are made, books which deal with the subject must be consulted.

When an object is viewed at a distance for instance of 1 metre, it is essential for accurate binocular vision that a distinct image of the object should be formed on the macula of each eye. The distinctness of the image is attained by the exercise of the focussing power or accommodation of the eye, or, if that is insufficient, by means of the spectacles that may be necessary. In order that an image of the object may fall on the macula of each eye it is necessary that the fixation-lines of both eyes should meet at the object. Both eyes will then converge towards the object. It will be advantageous to consider accommodation and convergence separately.

Accommodation. Langenbeck, Donders, Cramer and Helmholtz all found that when attention is directed towards near objects, the anterior surface of the lens becomes more convex. The refracting power of the eye is in this way increased, so that the image of the object is formed on the retina. Until recently it was supposed that this change of shape in the lens was brought about in the following way. The longitudinal fibres of the ciliary muscle arise from the junction of the cornea and the sclerotic and are inserted into the choroid. When these fibres contract the choroid with the ciliary processes is drawn forward, and the suspensory ligament is relaxed. The tension of the suspensory ligament and the anterior capsule of the lens being diminished, the anterior surface of the lens from its elasticity bulges forwards, and so it becomes more convex.

Recently however Tscherning has shewn that the lens of the eye during accommodation has a shape that cannot be accounted for by this theory. While the central part of the lens is more convex, the peripheral part, behind the iris, is very much flattened. He has caused a similar alteration of shape in a lens that has been removed by increasing the tension of the suspensory ligament. According to Tscherning we must suppose that the contraction of the ciliary muscle increases the tension of the suspensory ligament. The effect of this will be that the peripheral parts of the lens are flattened, while the central part in front of the solid nucleus will become more curved. The cortex of the lens in young eyes must be considered as a plastic semifluid material that during accommodation is partially pushed aside by the bulging forwards of the more convex nucleus. The loss of accommodation in old age is due to the cortex of the lens becoming less plastic.

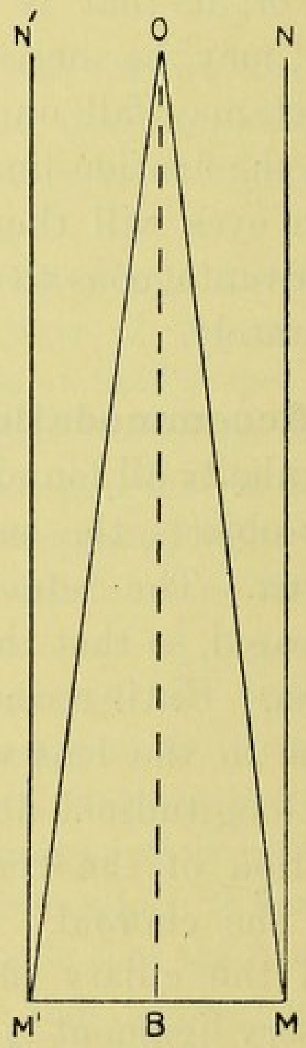

Fig. 129

Convergence. If the distance of the object, as compared to the distance between the eyes, may be considered infinite, the fixation-lines are parallel. In Fig. 129 MM' represent the centres of motility of the two eyes, O the object situated in the middle line at 1 metre's distance from either; MN, $M'N'$ represent the parallel fixation-lines when viewing some infinitely distant object in the middle line. Then NMO is the angle of convergence which each eye undergoes in viewing an object at O.

But $$NMO = BOM = \sin^{-1}\frac{BM}{OM}.$$

Now BM is half the distance between the centres of the eyes, which of course varies with different people. When OM is 1 metre the angle NMO is called one metre-angle and is denoted by 1 m.a. If the object be $\frac{1}{3}$ metre off, the angle of convergence is 3 metre-angles or 3 m.a. It is true that the angle of convergence is now rather more than three times the angle of convergence required for viewing an object at the distance of 1 metre, for angles are not simply proportional to their sines, but they are so nearly so within the limits that this notation is used, that the error hereby introduced is negligible.

In adults the average interocular distance MM' is 64 mm., so that the average value of BM is 32 mm.

$$\therefore \ 1 \text{ m.a. or } \sin^{-1}\frac{32}{1000} = 1^\circ\ 50'\ 1'',$$

and

$$3 \text{ m.a. or } \sin^{-1}\frac{3\,(32)}{1000} = 5^\circ\ 30'\ 32''.$$

Now if a case was presented in which the power of accommodation was normal although the power of (positive) convergence was completely lost, the patient would complain that when reading music at the distance of half a metre he saw double. Since his fixation lines remained parallel, the point for which he was focussing his eyes would form two distinct images on his retinæ, but they could not fall on the macular region of each eye respectively. What means have we to enable him to have macular images of his music in each eye while his fixation lines maintain their parallelism? Clearly an abducting prism of 2 m.a., the deviating angle of which is 3° 40′, must be placed before each eye. Such prisms with their edges directed outwards towards the temples worn as spectacles would so deviate the course of the incident light that it would enter the eyes in the same direction as the fixation-lines, and so an image of the object would be formed on the macula of each eye, and the object would therefore appear single to the patient.

Similarly in the case of a patient who could not entirely relax the convergence of his eyes, adducting prisms would be required for binocular vision at a distance. Suppose that when his convergence is relaxed, his fixation-lines meet at a point 1 metre off. He has then 1 m.a. of convergence, and to correct his diplopia for distance he would require adducting prisms of 1 m.a., or prisms of 1° 50′ d. with their edges inwards must be mounted before both eyes[1].

Suppose the patient had required concave glasses of $-4D$ for correcting his refraction for distance it would be found that a stronger prism than 1° 50′ d. would be required to correct the deviation of his eye.

We shall therefore have to determine what is the deviation induced by the combination of a lens and a prism.

Deviation of Prismospheres. In the figure (Fig. 130) AA' represents a lens with its optical centre at O. The part APQ may be regarded as a decentred lens with its geometrical centre at D distant DO from its optical centre or as a combination of the prism (marked in dotted lines) with a convex lens. Such a combination may be called a prismosphere. A narrow pencil incident in the direction SD will be deflected towards F, and if θ be the angle of deflection $\theta = OFD$.

[1] There are several different methods of numbering prisms, and some confusion has resulted therefrom. It will simplify matters if we consider only the effect of a prism, and number it either according to the angle of deviation it induces when in the position of minimum deviation, or according to the system of metre-angles. The apical angle of the required prism may be sufficiently exactly obtained by the use of the simple formula $d = (\mu - 1) A$ (p. 154).

Thus $\quad 1 \text{ m.a.} = 1° \, 50' \, 1'' \text{d.} = 3° \, 23' \, 44'' A$ if $\mu = 1{\cdot}54$.

In other words, a prism that produces a deviation of one metre-angle or of 1° 50′ 1″ would be formed by a prism of glass whose index of refraction was 1·54, the apical angle of which was 3° 23′ 44″.

If l denote the amount of decentration DO and if f denote the second focal distance of the lens, $\tan\theta = \frac{DO}{FO} = \frac{l}{f}$, or if the dioptric strength of the glass be denoted by x, $\tan\theta = \frac{-xl}{1000}$. When θ is small no great error is introduced by considering θ proportional to $\tan\theta$.

Then $$\theta = -xl\tan^{-1}\cdot 001 = -xl \times 3'\cdot 4376.$$

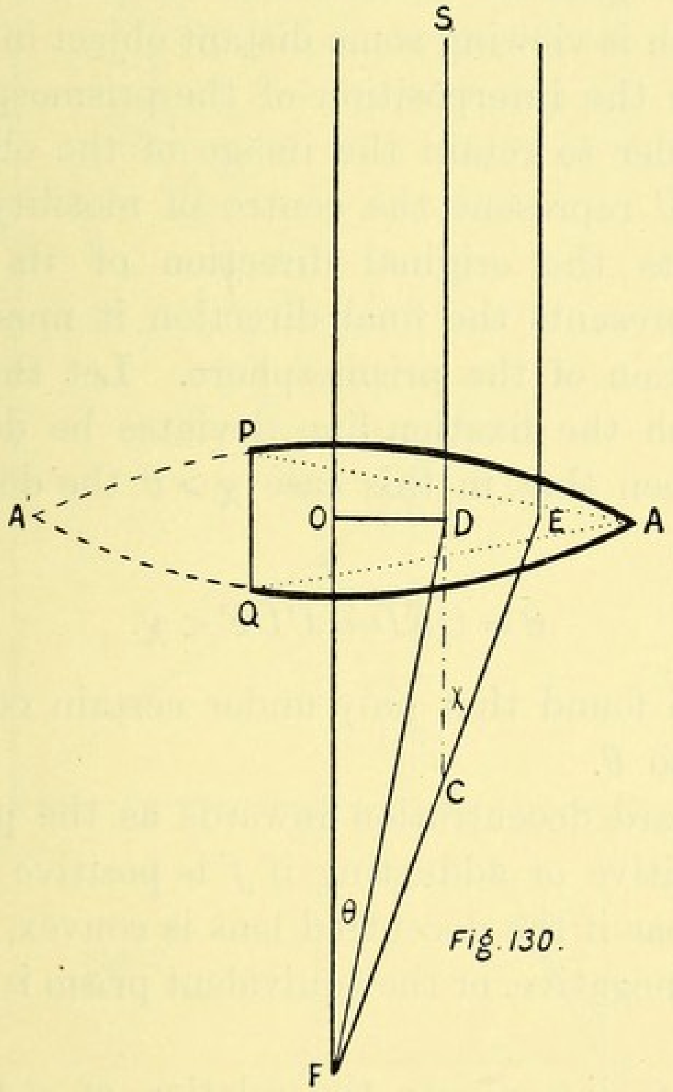

Fig. 130.

Thus if a $10D$ lens is decentred inwards 2·6 mm. it will have the same effect as the combination of a $+10D$ lens with an abducting prism the deviation of which is

$$-26 \times 3'\cdot 4376 = -1^\circ\ 29'\cdot 377.$$

If the lens were concave x would be negative, so θ would be positive, and the prism would be adducting in function.

This means that a lens decentred l mm. has the same effect on the course of light that passes through it, as a prismosphere the focal length of which is f, and the deviation θ or $\tan^{-1}\frac{l}{f}$.

An eye however which is placed behind the prismosphere no longer undergoes the deviation of the prism. An eye, for instance, which is viewing some distant object in the direction CD, will after the interposition of the prismosphere have to diverge in order to retain the image of the object upon its macula. If C represent the centre of motility of the eye, CD represents the original direction of its fixation-line while CE represents the final direction it must adopt after the interposition of the prismosphere. Let the angle DCE through which the fixation-line deviates be denoted by χ. It is easily seen that in this case $\chi > \theta$ the deviating angle of the prism.

For $$\theta = OFD = CDF < \chi.$$

It will be found that only under certain conditions will χ be equal to θ.

If we regard decentration inwards as the positive direction, θ is positive or adducting if f is positive as in concave lenses; whereas if the decentred lens is convex, f is negative and θ is also negative, or the equivalent prism is an abducting prism.

In order to investigate the relation of χ to θ we shall find it convenient to express θ in terms of the decentration of the lens considered, and then by a simple geometrical method we shall find a formula which will give the relation approximately.

Let us consider the case of a concave lens, of focal length

FO or *f*, its position represented by a principal plane *DM* which passes through its optical centre *O*. Let *C* represent

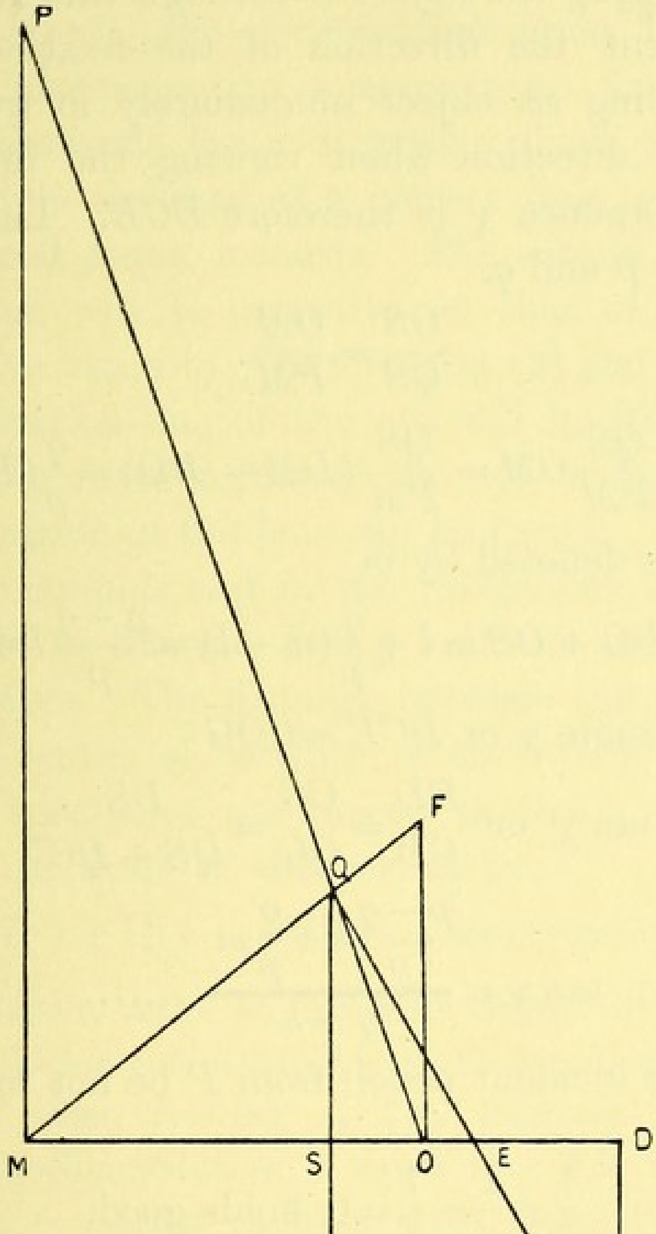

Fig 131.

the centre of motility of the eye situated *DC* (or *k* mm.) behind the lens. Then *D* represents the part of the lens opposite the centre of the eye, and the lens may be regarded as being displaced or decentred *DO* or (*l* mm.) inwards. Let *P* be an object situated in the middle line between the eyes, a virtual image of P will be formed at Q. The eye behind the lens will therefore direct itself towards the image at Q when viewing the object P through this decentred lens. If CD represent the direction of the fixation-line of the eye when viewing an object immediately in front of it, CE represents its direction when viewing the image Q. The angle of convergence χ is therefore DCE. Let PM and QS be denoted by p and q.

Now $$\frac{OS}{QS} = \frac{OM}{PM}.$$

$$\therefore \; OS = \frac{QS}{PM} OM = \frac{QS}{PM}(DM - DO) = \frac{q}{p}(DM - l).$$

Let DM be denoted by m

$$DS = DO + OS = l + \frac{q}{p}(m - l) = \frac{p-q}{p} l + \frac{q}{p} m.$$

Since the angle χ or $DCE = CQG$

$$\tan \chi \text{ or } \frac{ED}{CD} = \frac{CG}{QG} = \frac{DS}{QS + DC},$$

$$\therefore \; \tan \chi = \frac{\frac{p-q}{p} l + \frac{q}{p} m}{q + k} \text{ .... (1).}$$

Now if the incident pencil from P be not too oblique the formula

$$\frac{1}{p} - \frac{1}{q} + \frac{1}{f} = 0 \text{ holds good,}$$

$$\therefore \; \frac{1}{f} = \frac{p-q}{pq}, \quad \text{and} \quad q = \frac{fp}{f+p},$$

$$\therefore \; \tan \chi = \frac{\frac{q}{f} l + \frac{q}{p} m}{q + k} \text{ or } \frac{pl + fm}{fp + k(f+p)}, \text{ ... (2).}$$

If p be infinite, $q = f$, and it is seen from (2) that

$$\tan \chi = \frac{l}{f+k} \text{ .... (3).}$$

Whatever the value of p it is evident that the sign of $\tan\chi$ and therefore of χ must depend on the sign of l in the case of a concave lens.

In other words, the decentration must be positive or inwards for adaptation to a converging fixation-line, but negative or outwards for a diverging fixation-line.

Now consider the case of a convex lens with its optical centre displaced l mm. inwards. The second focal distance of such a lens will be negative so that on assigning the proper negative value to f in formulas (2) and (3) the direction of the fixation-line of the eye will be given; or if the direction of the fixation-line of the eye is fixed, the appropriate decentration of the lens can be found.

In these examples and in the tables following, the value assigned to m is 32 mm., being half the average distance between the eyes. The distance between the position of the lens and the centre of motility of the eye is denoted by k, which when spectacles are placed in their proper position (13·7... mm.) in front of the cornea is

$$13{\cdot}7\ldots + 13{\cdot}4\ldots = 27 \text{ mm. approximately.}$$

Ex. A patient with an extreme degree of hypermetropia requiring $+10D$ for correcting his sight for distance, complains of discomfort when reading at $\frac{1}{3}$ m. distance with the same glasses. His accommodation is equal to $+6D$, *i.e.* he can read test types at 6 metres distance with either eye alone when provided with $+10D$ or with any glass between that and $+4D$. Explain and state what should be done to relieve his discomfort. Clearly it is not due to the refractive condition of his eye; for reading at a distance of $\frac{1}{3}$ m. requires only $3D$ of accommodation when wearing the proper correction for distance. This patient has $6D$ of accommodation and symptoms of accommodative asthenopia only arise when more than $\frac{2}{3}$ of the total power of accommodation is continuously maintained. On examination it is found that his spectacles are normally centred for distance, *i.e.* that when his fixation-lines are parallel they intersect his spectacles at their optical centres.

The amount of convergence required when reading with these spectacles a book at $\frac{1}{3}$ metre from the centre of motility, is given by the formula

$$\tan\chi = \frac{pl + fm}{fp + k(p + f)}.$$

Here $l = 0$, $f = -100$, $p = 333\cdot\dot{3} - 27$ mm. $= 306$ mm. approximately,

$$\therefore \tan\chi = \frac{fm}{fp + k(p + f)} = \frac{(-100)(32)}{(-100)(306) + 27(306 - 100)} = \frac{3200}{25038},$$

$\therefore \chi = \tan^{-1}\cdot 1278\ldots = 7^\circ\, 17'\ldots$ which is about 4 m.a. of convergence.

The patient therefore when provided with these glasses for reading has to exert $3D$ of accommodation and about 4 m.a. of convergence. A normal person can only maintain 3 m.a. of convergence associated with $3D$ of accommodation, although he may by a special effort for a short time increase or relax his convergence beyond this point.

The patient is therefore presumably suffering from muscular asthenopia due to the improper position of the optical centres of his glasses for reading.

If the relation between his functions of accommodation and convergence is normal we may expect that relief will be given by so displacing his glasses that when reading, his ocular fixation-lines may pass through the optical centres of his glasses. The result will then be that the angle of convergence is 3 m.a. when exercising $3D$ of accommodation

$$\tan 3 \text{ m.a. or } \tan\chi = \frac{DO}{DC} = \frac{l}{k},$$

$$\therefore l = k \tan 5^\circ\, 30'\ldots = 27 \times \cdot 0963 = 2\cdot 6\ldots \text{ mm.}$$

We find that all discomfort is relieved when his glasses are displaced 2·6 mm. inwards, we consequently order him for reading spectacles with $+ 10D$ lenses decentred inwards 2·6 mm.

It is important to remember that whenever strong glasses, whether convex or concave, are required for reading or close work, they should be decentred inwards to this amount, or considerable discomfort may arise. Neglect of this precaution is a frequent source of trouble with aphakic patients who require $+13D$ or $+15D$ to read with.

Reading glasses should also be inclined downwards 15°, as the eyes are rotated downwards about this amount when using them. If this is not done oblique centrical refraction will occur, which, if the lenses are of high refractive power, will occasion much discomfort from the astigmatism induced. (p. 280.)

An examination of the formula (3) for distance shews that in the case of a convex lens decentration inwards must always be given for diverging fixation-lines, and decentration outwards for converging fixation-lines.

For $\tan \chi = \frac{l}{f+k}$ and with a convex lens f is always negative and in all the cases that can arise in practice f will be numerically greater than k, therefore the sign of χ must be opposite to the sign of l.

Similarly in formula (2)

$$\tan \chi = \frac{pl + fm}{fp + k(f+p)},$$

when a convex lens is under consideration the denominator $f(p+k)+pk$ is always negative. If then l be also negative χ will be positive; χ will also be positive even if l is positive provided that fm is numerically greater than pl.

If the object viewed is immediately in front of the eye $m=0$; if further the object is placed at the first principal focus of the convex glass, $f=-p$ and the expression for $\tan \chi$ becomes $\frac{l}{f}$ which is the expression for $\tan \theta$ where θ is the angle of deviation of the prism which is equivalent to the decentration of the lens. This is the only case in which the effect on convergence of a prismosphere is equal to that of the equivalent prism with plane sides.

It will be noticed that whether the lens be convex or concave, $\tan\chi=0$ when $pl=-fm$, and also that $\tan\chi=\tan 3$ m.a. when $\frac{l}{k}=\tan 3$ m.a. Tables II. and III. have been drawn up on these two data with the assumption that the tangents of the small angles considered are proportional to the angles themselves. Table I. has been drawn up from the formula $\tan\chi=\frac{l}{f+k}$. These tables though only approximately correct will be found accurate enough for all practical purposes, as they never involve an error as great as 1 per cent.

Examples illustrating the use of the Tables.

Table I. A hypermetrope of $+8D$ has an esophoria of 1 m.a. when his accommodation is relaxed; *i.e.* his eyes tend to converge to a point 1 metre from him. This error of convergence will be corrected by decentring the $+8D$ lens 3·1 mm. outwards, or what amounts to the same thing, by associating with it a prism of 1° 26′ d. (not 1° 50′) edge inwards.

Table II. A patient requiring $+12D$ glasses for reading who can only maintain convergence for a distance of $\frac{1}{2}$ metre (2 m.a.), must have his glasses decentred 4·6 mm. inwards (or associated with prisms of 3° 11′ d. edges outwards).

Table III. A myope requiring $-6D$ for reading, who can only maintain 2 m.a. of convergence, must have his glasses decentred 4 mm. outwards, or combined with prisms 1° 24′ edges outwards.

Again, the figures given in the tables may be used to give a rough estimate of the relative range of convergence. Suppose a myope using $-5D$ for reading at $\frac{1}{3}$ metre's distance can obtain binocular vision with 4° d. prisms held edge inwards before both eyes and also with abducting prisms of 12° d. before both eyes. His relative range of convergence for $\frac{1}{3}$ metre when provided with $-5D$ glasses is not however 16° d. but about $\frac{16°}{1° 13'}$ or about 13° 9′, *i.e.* a little over 7 m.a. which is $\sin^{-1} \frac{32 \times 7}{1000}$ or 12° 57′.

It is sometimes found that patients who require different correcting glasses for each eye, complain of a curious difficulty in using them which depends on this property of decentred lenses. For example, suppose a hypermetrope requires $+1D$ for his right eye and $+4D$ for his left eye. With these he may see satisfactorily when the visual line of each eye passes through the optical centre of the glass. On turning his eyes from side to side say 30° his visual lines will traverse his spectacles at points distant about 15·6 mm. from the optical centres. If he be viewing a distant object situated 30° to the left, in order to avoid diplopia his left eye must deviate outwards nearly 31° while his right eye must deviate inwards more than 34°. Discomfort may therefore arise from the 3° of convergence entailed, but the discomfort will be much more distressing on looking upwards or downwards. A difference of 1° in the elevation of the two eyes is intolerable to most persons. Double focus glasses have been suggested to obviate this difficulty, but they are seldom necessary for this purpose, as such patients soon acquire the habit of turning their heads rather than their eyes so that they always may look through the central parts of their glasses.

If double focus glasses are ordered for anisometropic presbyopes, so that the upper segment of the glass is adapted for distance, while the lower and smaller segment gives the correction for reading-distance, it is most important that the reading segment should be accurately centred. Since when reading the eyes converge 3 m.a., and are rotated downwards about 15°, the optical centres of the reading segments should be displaced 2·6 mm. inwards and 7·2 mm. downwards.

TABLE I.

			1 *D*	2 *D*	3 *D*	4 *D*	5 *D*	6 *D*	7 *D*	8 *D*	9 *D*	10 *D*	12 *D*	14 *D*	16 *D*	18 *D*	20 *D*
CONVEX. Divergence, decentration inwards. Convergence, " outwards.	4 m.a.	7° 21′ 14″	125·6	61·0	39·5	28·8	22·3	18·0	14·9	12·6	10·8	9·4	7·3	5·7	4·6	3·7	2·9
			7° 9′	6° 58′	6° 46′	6° 34′	6° 22′	6° 10′	5° 58′	5° 46′	5° 34′	5° 22′	4° 59′	4° 35′	4° 11′	3° 47′	3° 23′
	3 m.a.	5° 30′ 32″	93·8	45·6	29·5	21·5	16·7	13·5	11·2	9·4	8·1	7·0	5·4	4·3	3·4	2·7	2·2
			5° 22′	5° 13′	5° 4′	4° 55′	4° 46′	4° 37′	4° 28′	4° 19′	4° 10′	4° 1′	3° 43′	3° 25′	3° 8′	2° 50′	2° 32′
	2 m.a.	3° 40′ 9″	62·4	30·3	19·6	14·3	11·1	8·9	7·4	6·3	5·4	4·7	3·6	3·0	2·3	1·8	1·5
			3° 34′	3° 28′	3° 22′	3° 16′	3° 10′	3° 4′	2° 58′	2° 52′	2° 46′	2° 41′	2° 20′	2° 23′	2° 5′	1° 53′	1° 41′
	1 m.a.	1° 50′ 1″	31·1	15·1	9·8	7·1	5·5	4·5	3·7	3·1	2·7	2·3	1·8	1·4	1·1	·9	·7
			1° 48′	1° 44′	1° 41′	1° 38′	1° 35′	1° 32′	1° 29′	1° 26′	1° 23′	1° 21′	1° 14′	1° 9′	1° 3′	56′	50′
		1°	16·98	8·25	5·34	3·89	3·02	2·44	2·02	1·71	1·47	1·27	·98	·77	·62	·49	·40
			58′	57′	55′	53′	52′	50′	49′	47′	46′	44′	40′	37′	34′	31′	27′
		0															
CONCAVE. Convergence, decentration inwards. Divergence, " outwards.		1°	17·93	9·20	6·28	4·84	3·96	3·38	2·96	2·65	2·41	2·22	1·93	1·72	1·56	1·44	1·34
			1° 1′	1° 3′	1° 5′	1° 6′	1° 8′	1° 10′	1° 11′	1° 13′	1° 15′	1° 16′	1° 19′	1° 23′	1° 26′	1° 29′	1° 33′
	1 m.a.	1° 50′ 1″	32·9	16·9	11·5	8·9	7·3	6·2	5·4	4·9	4·4	4·1	3·5	3·2	2·9	2·6	2·4
			1° 53′	1° 56′	1° 59′	2° 2′	2° 5′	2° 8′	2° 11′	2° 14′	2° 17′	2° 20′	2° 26′	2° 32′	2° 38′	2° 44′	2° 50′
	2 m.a.	3° 40′ 9″	65·9	33·8	23·1	17·8	14·6	12·4	10·9	9·8	8·9	8·1	7·0	6·3	5·7	5·3	4·9
			3° 46′	3° 52′	3° 58′	4° 4′	4° 10′	4° 16′	4° 22′	4° 28′	4° 34′	4° 40′	4° 52′	5° 3′	5° 15′	5° 27′	5° 39′
	3 m.a.	5° 30′ 32″	99·1	50·8	34·7	26·7	21·9	18·7	16·4	14·7	13·3	12·3	10·6	9·5	8·6	8·0	7·4
			5° 40′	5° 48′	5° 57′	6° 6′	6° 15′	6° 24′	6° 32′	6° 41′	6° 50′	6° 59′	7° 17′	7° 33′	7° 49′	8° 10′	8° 28′
	4 m.a.	7° 21′ 14″	132·6	68·0	46·5	35·8	29·3	25·0	21·9	19·6	17·8	16·4	14·3	12·7	11·6	10·7	10·0
			7° 33′	7° 45′	7° 56′	8° 8′	8° 20′	8° 32′	8° 45′	8° 56′	9° 7′	9° 19′	9° 42′	10° 6′	10° 29′	10° 52′	11° 15′

The object of observation is presumed to be at a distance of more than 6 metres from the patient.

The figures in larger type indicate the amount of decentration in millimetres.

The figures in smaller type represent the deviating power of the prisms whose action is equivalent to that of the decentration of the lenses.

TABLE II (Convex).

			1 D	2 D	3 D	4 D	5 D	6 D	7 D	8 D	9 D	10 D	12 D	14 D	16 D	18 D	20 D
Diverging	4 m.a.	7° 21′ 14″	240·4 13° 31′	118·4 13° 19′	77·8 13° 8′	57·4 12° 56′	45·3 12° 46′	37·1 12° 34′	31·3 12° 23′	27·0 12° 11′	23·6 11° 59′	20·9 11° 48′	16·8 11° 25′	13·9 11° 2′	11·7 10° 38′	10·0 10° 16′	8·7 9° 53′
Diverging	3 m.a.	5° 30′ 32″	206·4 11° 40′	101·9 11° 31′	67·0 11° 22′	49·6 11° 13′	39·2 11° 5′	32·2 10° 56′	27·2 10° 48′	23·5 10° 39′	20·6 10° 31′	18·3 10° 22′	14·8 10° 4′	12·3 9° 47	10·4 9° 29′	9·0 9° 12′	7·8 8° 54′
Diverging	2 m.a.	3° 40′ 9″	172·4 9° 47′	85·3 9° 41′	56·3 9° 35′	41·8 9° 30′	33·1 9° 24′	27·3 9° 18′	23·1 9° 12′	20·0 9° 6′	17·6 9° 1′	15·7 8° 55′	12·8 8° 43′	10·7 8° 31′	9·1 8° 19′	7·9 8° 8′	6·9 7° 56′
Diverging	1 m.a.	1° 50′ 1″	138·5 7° 53′	68·8 7° 50′	45·6 7° 48′	34·0 7° 45′	27·0 7° 41′	22·4 7° 38′	19·0 7° 35′	16·5 7° 32′	14·6 7° 29′	13·1 7° 27′	10·7 7° 21′	9·1 7° 15′	7·8 7° 9′	6·9 7° 3′	6·1 6° 57′
		0	104·5 5° 58′	52·2 5° 58′	34·8 5° 58′	26·1 5° 58′	20·9 5° 58′	17·4 5° 58′	14·9 5° 58′	13·1 5° 58′	11·6 5° 58′	10·4 5° 58′	8·7 5° 58′	7·5 5° 58′	6·5 5° 58′	5·8 5° 58′	5·2 5° 58′
Converging	1 m.a.	1° 50′ 1″	70·5 4° 2′	35·7 4° 5′	24·1 4° 8′	18·3 4° 12′	14·8 4° 14′	12·5 4° 17′	10·8 4° 20′	9·6 4° 23′	8·6 4° 26′	7·8 4° 29′	6·7 4° 35′	5·8 4° 41′	5·2 4° 46′	4·7 4° 53′	4·3 4° 58′
Converging	2 m.a.	3° 40′ 9″	36·6 2° 6′	19·2 2° 12′	13·3 2° 17′	10·4 2° 23′	8·8 2° 31′	7·5 2° 35′	6·7 2° 41′	6·1 2° 48′	5·6 2° 54′	5·2 2° 59′	4·6 3° 11′	4·2 3° 23′	3·9 3° 36′	3·7 3° 47′	3·5 3° 59′
Converging	3 m.a.	5° 30′ 32″	2·6 9′	2·6 18′	2·6 27′	2·6 36′	2·6 45′	2·6 54′	2·6 1° 3′	2·6 1° 12′	2·6 1° 21′	2·6 1° 30′	2·6 1° 48′	2·6 2° 6′	2·6 2° 24′	2·6 2° 42′	2·6 2° 59′
Converging	4 m.a.	7° 21′ 14″	−31·3 (1° 48′)	−13·9 (1° 36′)	−8·1 (1° 24′)	−5·2 (1° 12′)	−3·5 (1° 0′)	−2·3 (48′)	−1·5 (36′)	−·87 (24′)	−·38 (12′)	·001 3″	·58 24′	1·0 48′	1·3 1° 12′	1·5 1° 36′	1·7 2°
Difference for 1°			18·5 1° 4′	9·0 1° 2′	5·8 1°	4·3 59′	3·3 57′	2·7 55′	2·2 53′	1·9 52′	1·6 50′	1·4 49′	1·1 46′	·9 42′	·7 39′	·6 36′	·5 33′

The object of observation is presumed to be in the middle line between the eyes and $\frac{1}{3}$ metre distant from the vertical plane passing through their centres of motility.

The figures in larger type give the amount of decentration in millimetres.

− sign indicates decentration outwards, + sign decentration inwards.

The figures in smaller type represent the deviating power of the prisms whose action is equivalent to that of the decentration of the lenses. The prisms are abducting in function and should be placed edges outwards, unless they are enclosed in brackets, when they are adducting in function and should be placed edges inwards.

TABLE III (CONCAVE).

		1 D	2 D	3 D	4 D	5 D	6 D	7 D	8 D	9 D	10 D	12 D	14 D	16 D	18 D	20 D
Diverging	4 m.a.	−247·3	−125·4	−84·8	−64·4	−52·2	−44·1	−38·3	−33·9	−30·6	−27·9	−23·8	−20·9	−18·7	−17·0	−15·7
		13° 53′	14° 5′	14° 16′	14° 27′	14° 38′	14° 50′	15° 1′	15° 12′	15° 23′	15° 34′	15° 56′	16° 19′	16° 41′	17° 3′	17° 24′
	3 m.a.	−211·6	−107·1	−72·3	−54·9	−44·4	−37·4	−32·5	−28·7	−25·8	−23·5	−20·0	−17·5	−15·7	−14·2	−13·0
		11° 57′	12° 6′	12° 14′	12° 22′	12° 31′	12° 40′	12° 48′	12° 57′	13° 5′	13° 14′	13° 31′	13° 48′	14° 5′	14° 22′	14° 38′
	2 m.a.	−175·9	−88·8	−59·8	−45·3	−36·6	−30·8	−26·6	−23·5	−21·1	−19·2	−16·2	−14·1	−12·6	−11·4	−10·4
		9° 59′	10° 4′	10° 10′	10° 16′	10° 22′	10° 28′	10° 33′	10° 39′	10° 45′	10° 51′	11° 2′	11° 14′	11° 25′	11° 37′	11° 48′
	1 m.a.	−140·2	−70·5	−47·3	−35·7	−28·7	−24·1	−20·8	−18·3	−16·3	−14·8	−12·5	−10·8	−9·6	−8·6	−7·8
		7° 59′	8° 2′	8° 5′	8° 8′	8° 10′	8° 13′	8° 16′	8° 20′	8° 22′	8° 25′	8° 31′	8° 37′	8° 43′	8° 49′	8° 55′
	0	−104·5	−52·2	−34·8	−26·1	−20·9	−17·4	−14·9	−13·1	−11·6	−10·4	−8·7	−7·5	−6·5	−5·8	−5·2
		5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′	5° 58′
Converging	1 m.a.	−68·8	−33·9	−22·3	−16·5	−13·1	−10·7	−9·1	−7·8	−6·9	−6·1	−4·9	−4·1	−3·4	−3·0	−2·6
		3° 56′	3° 53′	3° 50′	3° 47′	3° 44′	3° 41′	3° 39′	3° 35′	3° 32′	3° 29′	3° 23′	3° 17′	3° 11′	3° 5′	2° 59′
	2 m.a.	−33·1	−15·7	−9·9	−7·0	−5·2	−4·0	−3·2	−2·6	−2·1	−1·7	−1·2	−·7	−·43	−·19	0
		1° 54′	1° 48′	1° 42′	1° 36′	1° 30′	1° 24′	1° 18′	1° 12′	1° 6′	1°	48′	36′	24′	12′	
	3 m.a.	2·6	2·6	2·6	2·6	2·6	2·6	2·6	2·6	2·6	2·6	2·6	2·6	2·6	2·6	2·6
		(9′)	(18′)	(27′)	(36′)	(45′)	(54′)	(1° 3′)	(1° 12′)	(1° 21′)	(1° 30′)	(1° 48′)	(2° 6′)	(2° 24′)	(2° 42′)	(2° 59′)
	4 m.a.	38·3	20·9	15·1	12·2	10·4	9·3	8·5	7·8	7·3	7·0	6·4	6·0	5·6	5·4	5·2
		(2° 12′)	(2° 24)	(2° 36′)	(2° 47′)	(2° 59′)	(3° 11′)	(3° 23′)	(3° 34′)	(3° 47′)	(3° 59′)	(4° 23′)	(4° 47′)	(5° 11′)	(5° 34′)	(5° 58′)
Difference for 1°		19·4	9·9	6·8	5·2	4·3	3·6	3·2	2·8	2·6	2·4	2·0	1·8	1·6	1·5	1·4
		1° 7′	1° 9′	1° 10′	1° 12′	1° 13′	1° 15′	1° 17′	1° 18′	1° 20′	1° 22′	1° 25′	1° 28′	1° 31′	1° 35′	1° 38′

The object of observation is presumed to be in the middle line between the eyes and $\frac{1}{3}$ metre distant from the vertical plane passing through their centres of motility.

The figures in larger type give the amount of decentration in millimetres.

− sign indicates decentration outwards, + sign decentration inwards.

The figures in smaller type represent the deviating power of the prisms whose action is equivalent to that of the decentration of the lenses. The prisms are abducting in function and should be placed edges outwards, unless they are enclosed in brackets, when they are adducting in function and should be placed edges inwards.

Circle of Least Confusion. A few words may be added about the circle of confusion. We have seen (p. 113) that the size of the circle of confusion depends on the cross-section of the reflected or refracted pencil in the neighbourhood of the bounding surface. But the size and shape of this cross-section depends on the size and shape of the receiving surface. Practically we have only to consider the cases when a real image is formed at the back of an eye or at the back of a photographic camera. If the position of the focal lines is known, as well as the distance of the camera from the bounding surface and the size of the stop used for the lens, the size of the circle of confusion can be calculated.

Let us consider the size and position of the circle of least confusion in the case represented in Figs. 33 and 35 for an eye at K, distant KP or l from the point of incidence of the axial ray of the eccentric pencil considered. Now if R_1 be the radius of the pupil in the primary plane it is easily seen that

$$\frac{QP}{F_1P} = \frac{R_1}{F_1K} = \frac{R_1}{F_1P - KP},$$

or $\dfrac{a}{v_1} = \dfrac{R_1}{v_1 - l}$ where QP or a is half the thickness of the reflected pencil at P.

Similarly if R_2 be the radius of the pupil in the secondary plane (Fig. 35)

$$\frac{PR_1}{F_2P} = \frac{R_2}{F_2K} = \frac{R_2}{F_2P - KP},$$

or

$$\frac{b}{v_2} = \frac{R_2}{v_2 - l}.$$

Now in all ordinary cases the pupil is circular so that $R_1 = R_2$.

But DP or $x = \dfrac{v_1 v_2 (a+b)}{a v_2 + b v_1}$ from (1) p. 113.

$$\therefore \quad x = \frac{v_1 v_2 \left(\dfrac{R v_1}{v_1 - l} + \dfrac{R v_2}{v_2 - l} \right)}{\dfrac{R v_1 v_2}{v_1 - l} + \dfrac{R v_1 v_2}{v_2 - l}} = \frac{2 v_1 v_2 - l (v_1 + v_2)}{v_1 + v_2 - 2l}.$$

Also if r be the radius of the circle of least confusion

$$r = \frac{ab (v_2 - v_1)}{a v_2 + b v_1} = \frac{R (v_2 - v_1)}{v_1 + v_2 - 2l}.$$

The same formulæ will be found to hold good in the case of refraction. For instance, if in Fig. 63 PR denote the axial ray of the astigmatic pencil, entering the pupil of the eye at R, and if a represent half the thickness of the refracted pencil at P, since $RP = l$ and is negative,

$$\frac{a}{v_1} = \frac{R}{F_1 P + PR} = \frac{R}{F_1 P - RP} = \frac{R}{v_1 - l},$$

and

$$\frac{b}{v_2} = \frac{R}{v_2 - l}.$$

Therefore as before,

$$x = \frac{2 v_1 v_2 - l (v_1 + v_2)}{v_1 + v_2 - 2l} \text{ and } r = \frac{R (v_2 - v_1)}{v_1 + v_2 - 2l}.$$

In health the value of R, the radius of the pupil, varies from about 1·2 mm. to 2·8 mm. The pupil contracts on exposure to bright light, and during accommodation for near objects. Knowing the size and position of the circle of least confusion, we can easily find the size of its image formed on the retina. This has an important bearing on the manufacture of optical instruments. In a fine instrument the circle of least confusion corresponding to each point of the object will form an image on the retina, that is not greater than the sectional area of a retinal cone.

QUESTIONS.

(1) Two hypermetropes require $+5D$ for seeing distinctly test types at a distance of 6 m. The hypermetropia of A is due to a defect of curvature, that of B is due to a defect of length.

Compare the sizes of their retinal images.

(2) A fragment of glass is imbedded perpendicularly in the vitreous. The media are clear enough for the following observation to be made with the ophthalmoscope. The fundus is distinctly seen with $+1D$, except the part behind the glass which requires $+1{\cdot}5D$.

Estimate the thickness of the glass if its index of refraction is 1·54.

(3) An axial myope requires $-20D$ to correct him for distance. After the removal of his lens what correcting glass will he require for distance, assuming that the curvature of his cornea has undergone no alteration after the operation.

(4) It has been declared that four of Jupiter's satellites have been seen with the unaided eye. Is this optically possible?

Given that Jupiter's radius = 44100 miles, distance of the so-called first satellite from Jupiter's centre = 262000 miles, mean distance of Jupiter from the sun = 483 million miles, which may be taken as the mean distance of Jupiter from the earth. Diameter of a foveal cone ·002 mm.

(5) An observer's eye is $11\sqrt{2}$ cm. above some water in a basin, and 33 cm. from it in the direction in which he sees the point of a needle that is lying at the bottom. The water is 14·7 cm. deep. The radius of the pupil is 1·6 mm. Find the position and radius of the circle of least confusion, and state whether the needle point will appear blurred. Diameter of a foveal cone ·002 mm.

MISCELLANEOUS QUESTIONS.

(1) The equation $y = a \sin \frac{2\pi}{\lambda}(vt - x)$ represents a wave disturbance in which v is the velocity of propagation, λ the wave-length, y the displacement of a particle from its position of rest at the time t, and x the distance from the origin of the same particle. The amplitude of the vibration is denoted by a, it is the greatest displacement of any particle from its position of rest.

Prove that such a disturbance is periodic both in space and time. (p. 3.)

(2) A myope who can only see distinctly at a distance of 24 ins. sees an image of his eye in a concave mirror of focal length 5 ins. What is his distance from the mirror, and what is the size of the image?

(3) Where must a hypermetrope whose far point is -24 ins. be situated in order to see an image of his eye in the same mirror without exerting his accommodation? What is the size of the image?

(4) A convex lens of focal length f is placed at the centre of curvature of a concave mirror of radius r; so that the axes of the two coincide. If light diverging from a point at the distance p from the lens, after refraction through the lens, reflection at the mirror, and a second refraction through the lens, emerges as a pencil of parallel rays, prove that

$$\frac{1}{p} + \frac{2}{r} + \frac{2}{f''} = 0.$$

(5) The focal length of a biconcave lens, whose refractive index is $\frac{3}{2}$, is 5 ins. Prove that the distance from the lens of the image of a distant object formed (i) by reflection at the first surface is $2\frac{1}{2}$ ins., (ii) by reflection at the second surface and refraction at the first surface is $-1\frac{1}{4}$ ins.

(6) Trace the caustic formed by the refraction of a plane wave-front of light by a $+10D$ planospherical lens, of which the thickness is 6·4 mm. and $\mu = 1{\cdot}54$; (i) when the plane surface faces the incident light, (ii) when the curved surface faces the incident light.

(7) Trace the caustic formed by the refraction of a plane wave-front of light by a $+10D$ biconvex ($\mu = 1{\cdot}54$, $t = 25{\cdot}6$ mm.).

(8) The *punctum proximum* is 100 cm. from the anterior nodal point in a hypermetropic eye and 5 cm. from the same point in a myopic eye. Give the magnifying power of a $+10D$ lens placed 1·5 cm. in front of the nodal point in each case.

(9) If in the last example the *punctum remotum* of the hypermetrope is 25 cm. behind the nodal point, and the *punctum remotum* of the myope is 7 cm. in front of the nodal point, give the power of accommodation of the two eyes in terms of a lens situated at the nodal point.

(10) Shew that the action of an opera-glass may be compared to that of holding a strong convex lens at a little distance from a very hypermetropic eye.

(11) Trace the caustic formed by the refraction of a plane wave-front of light at a spherical surface whose radius of curvature is -5. ($\mu = \frac{4}{3}$.)

(12) A hollow spherical shell of glass ($\mu = \frac{3}{2}$) is filled with water ($\mu = \frac{4}{3}$). Shew that a pencil of parallel rays after passing through the whole will converge at a distance from the surface of the glass equal to $(r + t)\dfrac{t + 3r}{t - 3r}$ where t is the thickness of the glass shell, and r is the radius of the sphere of water.

(13) A transparent sphere, radius a, is silvered at the back, and there is a speck within it, halfway between the centre and the silvered side. Shew that the distance between the images formed (1) by one refraction, (2) by one reflection and one refraction is $\dfrac{2\mu a}{(3-\mu)(\mu-1)}$.

(14) Shew that if a star is seen eccentrically its retinal image will cover more than one cone. For instance, suppose the radius of the pupil is 1·6 mm., and the angle of incidence is 10°, find the radius of the circle of least confusion on the retina. The diameter of a retinal cone (outside the macula) is ·006 mm.

ANSWERS.

CHAPTER I.

(1) The angular velocity is $2n\pi$. In t seconds an angle a or $2n\pi t$ is described. $\therefore$ the pencil of light describes an angle of $2a$ or $4n\pi t$.

But since this angle is small, its circular measure is approximately

$$\frac{SS'}{RS} = \frac{\cdot 13085}{10}. \quad \therefore t = \frac{\cdot 013085}{4n\pi}.$$

Now

$$v = \frac{2RM}{t} = \frac{8n\pi RM}{\cdot 013085} = \frac{8 \times 256 \times 610 \times \pi}{\cdot 013085} = 299{,}941{,}010 \text{ metres per second.}$$

(2) 1·000294.

(3) Velocity in water, 139825·25 miles per second, in the crown glass 124289·2 miles per second.

(4) ·00002315 ins.

(5) The statements are correct; the reasoning is unsound. Light cannot be said to have colour, until it reaches the percipient organ; and the wave-length of any light on reaching the retina depends on its velocity in that medium (*i.e.* on the index of refraction of the percipient structures) no matter what has been its previous course.

(6) Draw pQ parallel to CT, meeting TE in Q.

Then $$\frac{pQ}{EK} = \frac{pQ}{pE} \cdot \frac{pE}{EK} = \frac{CT}{CE} \cdot \frac{CE}{ED} = \frac{CT}{ED} = \frac{V_2}{V_1}.$$

Therefore the hemispherical disturbance in the second medium originating from p will be of radius pQ in the time that the disturbance in the first medium has taken in passing from K to E.

Since $$\angle EQp = \angle ETC,$$
ET, which is a tangent at T to the circle of radius CT, must also be a tangent at Q to the circle of radius pQ.

(7) See p. 147.

CHAPTER II.

(1) 45 feet. (2) 70 feet.

(3) The quantity of light L falling on the surface S illuminates it with intensity I; the same quantity of light illuminates another surface S' with intensity I', and L is constant. $\therefore$ $SI = S'I'$.

$$\therefore \quad \frac{I'}{I} = \frac{S}{S'} = \frac{\pi r^2}{\pi r'^2} = \frac{d^2}{d'^2}.$$

If $$d = I, \; I' = \frac{I}{d'^2}.$$

(4) The electric light is 900 times greater than the gas-light.

(5) (i) The shadow cast by the gas-flame will be approximately 8 times more intense than that cast by the electric light.

(ii) 7200 : 1 approximately.

(6) 239526 miles approximately.

CHAPTER IV.

(1) 1 ft., 5 ft., 7 ft.

(2) Three images are seen. The third image is due to two reflections and undergoes the last reflection at the mirror remote from the object.

(3) Seven images when $a = 45°$. One image when $a = 135°$. Two images would be seen if the angle subtended at C, between the observer and the mirror remote from the object, were less than 60°.

(4) The last reflection occurs at the mirror remote from the object.

CHAPTER V.

(1) A real inverted image 2 cm. in height is formed 30 cm. from the mirror.

(2) 24 ins., -8 ins., -56 ins.

(3) 10 cm. (4) -6 ins.

(5) The candle is $6\frac{6}{11}$ ins. from the mirror, the image is 11 times the height of the candle.

(6) 9·06... mm.

CHAPTER VI.

(1) $v_1 = 1\frac{1}{2}$ ins., $v_2 = 6$ ins., $x = 3$ ins., $R = \frac{3}{8}$ ins.

(2) $v_1 = \frac{r\sqrt{2}}{4}$, $v_2 = \frac{r\sqrt{2}}{2}$, $x = \frac{r}{2}$, $R = \frac{2-\sqrt{2}}{4}$.

(3) $x = \frac{r\sqrt{2}}{3}$, $R = \frac{\sqrt{2}}{12}$.

(4) $a = -5\frac{6}{23}$ ins.

(5) $a = {\cdot}02025$ cm., $XI = {\cdot}01519$ cm. nearly, $R = {\cdot}00091\ldots$ cm.

CHAPTER VII.

(1) 40° 42′ 28″.

(2) $\sqrt{2}$.

(3) $\phi < 30°$.

(5) 1·54.

(6) Left prism 1° 44′ d. edge out. Right prism 2° d. edge out and up, its base-apex line inclined at an angle of 30° to the horizontal.

CHAPTER VIII.

(3) (i) ·01512287. (ii) ·014897579.... Total ·03001876....

(4) (i) ·022047244.... (ii) ·027649769.... Total ·04968944....

(5) 6°.

CHAPTER IX.

(4) $x = \frac{327\sqrt{3}}{75{\cdot}27} = 7{\cdot}5246\ldots$ ins., $r = {\cdot}0004065$ ins. nearly.

CHAPTER X.

(1) When $p < r$.

(2) (i) $r = -4\frac{16}{21}$ mm., (ii) $r = 9\frac{1}{11}$ mm., (iii) $\frac{i}{i'} = \frac{1}{2}$.

(3) $q = -22$ mm., $p = 165$ mm.

(4) (i) $q = 2r$; *i.e.* the image is at the distal surface, (ii) $q = \frac{2r}{5}$.

(5) (i) $q = -2r$. The image is at a diameter's distance on the side of the sphere nearest the observer.

(ii) $q = \frac{10r}{7}$.

CHAPTER XI.

(1) $r_1 = -10{\cdot}8$ ins., $r_2 = 10{\cdot}8$ ins. (2) $r_2 = 54$ ins.

(3) $\frac{i}{o} = -\frac{1}{5}$, $q = -7{\cdot}2$ ins.

(4) $q = -17{\cdot}5$ ins., $i = -1{\cdot}75$ ins.

(5) (i) $48\frac{1}{2}$, (ii) $43\frac{1}{2}$, (iii) 50.

(6) (i) $3\frac{3}{8}$, (ii) $3\frac{1}{8}$, (iii) $2\frac{1}{2}$.

CHAPTER XII.

(1) $\frac{3f}{2}$, $-f$ in front of the stronger posterior lens.

(2) $\frac{3f}{4}$, $-\frac{f}{2}$ in front of the posterior lens.

(3) H., $\frac{2l}{3f'}$; R., $\frac{4l}{3f'}$.

(4) $f_x = -8\frac{1}{3}$ cm.; 5 cm. in front of the concave lens.

CHAPTER XIII.

(1) When $p > \frac{r_1(\mu r_1 - r_2)}{\mu(r_2 + r_1)}$.

CHAPTER XIV.

(1) $\frac{V_1}{V_2} = \frac{3}{4}$.

(2) The lenses should be inclined 19° 28′ 16″ from the vertical plane the focal length $-86{\cdot}5\ldots$ mm., *i.e.* about $+11{\cdot}5\ldots$ *D.*

(3) -6 ins. from the centre. (4) $25\frac{2}{3}$, $\frac{1}{r'} = 7\frac{1}{9}$.

CHAPTER XV.

(1) ·144 mm. (2) ·00144 mm., ·108 mm.

(3) Biconvex of crown $r_1 = -19{\cdot}2$ ins., $r_2 = 12{\cdot}8$ ins.
Planoconcave of flint $r_1 = 12{\cdot}8$ ins., $r_2 = \infty$.

CHAPTER XVII.

(1) Using Donders' reduced eye, $\frac{A}{B}=\frac{40}{37}$.

(2) 1·21468... mm. (3) +2·6... D.

(4) If Jupiter's four furthest satellites were sufficiently bright there is no reason why they should not be seen with the naked eye. The angle subtended at the nodal point by a foveal cone is 26·617″... whereas the angle subtended by the distance between the first satellite and Jupiter's edge is 3′ 6·1″.

(5) $x=9·7966...$ cm. $r=·16447...$ mm.

Radius of retinal image ·00618 mm. nearly.

The point of the needle will therefore appear blurred.

ANSWERS TO MISCELLANEOUS QUESTIONS.

(1) Let $x'=x\pm\lambda$.

Then $y'=a\sin\frac{2\pi(vt-x\mp\lambda)}{\lambda}=a\sin\left\{\frac{2\pi}{\lambda}(vt-x)\mp 2\pi\right\}=y$.

Therefore the disturbance is periodic in space at intervals of λ.

Let $t_1=t\pm\frac{\lambda}{v}$.

Then

$$y_1=a\sin\frac{2\pi}{\lambda}\left\{v\left(t\pm\frac{\lambda}{v}\right)-x\right\}=a\sin\left\{\frac{2\pi}{\lambda}(vt-x)\pm 2\pi\right\}=y.$$

Therefore the disturbance is periodic in time at intervals of $\frac{\lambda}{v}$ or T.

(2) $p=30$ ins. or 4 ins.

If $p=30$ ins., an inverted image one-fifth the height of the eye is seen.

If $p=4$ ins., an erect image five times the height of the eye is seen.

(3) $p=6$ ins.

An inverted image five times the height of the eye is seen.

(8) Magnification of the lens for the hypermetropic eye is 10·85, for the myopic eye is 1·35.

(9) Accommodation of the hypermetropic eye is $+5D$, of the myopic eye is $+5\frac{5}{7}D$.

(14) $r=·0169...$ mm.

INDICES OF REFRACTION.

Solids.

Lead Chromate	*D*	2·5 to 2·97
Diamond	*D*	2·47 to 2·75
Lead Nitrate	*D*	1·866
Ruby	*D*	1·779
Emerald	*D*	1·585
Rock Salt	*D*	1·54418
Sugar	*A*	1·535
Canada Balsam	*A*	1·528
Tallow, Wax	*D*	1·492
Alum	*D*	1·45601
Tabasheer	*D*	1·1115

Uniaxal Crystals	Positive	Quartz	*D*	Ord.	1·544
				Extraord.	1·553
		Ice	*A*	Ord.	1·3060
				Extraord.	1·3073
	Negative	Iceland Spar	*D*	Extraord.	1·48639
				Ord.	1·65844
		Tourmaline	*D*	Extraord.	1·6193
				Ord.	1·6366
		Sodium Nitrate	*D*	Extraord.	1·3369
				Ord.	1·5854
Biaxal Crystals		Topaz	*D*	Max.	1·62109
				Med.	1·61375
				Min.	1·61161
		Mica	*D*	Max.	1·5997
				Med.	1·5941
				Min.	1·5609
		Nitre and Potassium Nitrate	*D*	Max.	1·5064
				Med.	1·5056
				Min.	1·3346

Seven varieties of optical glass made by Messrs Chance.

	D	*F*		*D*	*F*
Soft Crown	1·514580	1·520994	Dense Flint	1·622411	1·634748
Hard Crown	1·517116	1·523145	Extra Dense Flint	1·650374	1·664246
Extra Light Flint	1·541022	1·549125	Double Extra Dense Flint	1·710224	1·727257
Light Flint	1·574013	1·583881			

Liquids.

Carbon Bisulphide	*A*	1·61136	Alcohol	*A*	1·3596
	D	1·63034		*D*	1·3633
	H	1·70277		*H*	1·3745
Chloroform	*A*	1·4440	Water	*A*	1·32889
	D	1·4492		*D*	1·33298
	G	1·4611		*H*	1·34343

Gases at 0° and 760 mm. for light of sodium-flame.

Air	1·0002923	Nitrous Oxide	1·0005159
Hydrogen	1·0001387	Cyanogen	1·0008216
Carbonic Acid	1·0004544		

The above table has been mainly taken from Everett's *C. G. S. System of Units.*

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