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AN ELEMENTARY TREATISE

ON

GEOMETRICAL OPTICS.

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ON

GEOMETRICAL OPTICS

BY

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PREFACE TO THE FIRST EDITION.

THE present work is essentially an abridgment of my larger treatise on Geometrical Optics, and is primarily intended for the use of students who require an exposition of the principles of Optics and their application in the use and construction of optical instruments, without very extended and complicated mathematical analysis. It is elementary, inasmuch as it uses no mathematics beyond trigonometry.

In the task of selection and arrangement, I have departed somewhat from the traditions of previous writers of elementary text-books on Optics. My object has been to include only those parts of the theory which could be investigated completely and satisfactorily by elementary methods, and to treat those parts as fully as possible. Thus, while giving an account of the method of correcting optical instruments for their most important defect, that due to chromatic dispersion, I have omitted entirely the theories of aberration and of thin pencils, believing that they are not suited to elementary treatment and that they should be postponed until they can be investigated by more advanced and comprehensive methods. On the other hand the theory of lenses, as developed by Gauss, has been worked out completely, and the description and theory of the ordinary optical instruments are given in much greater detail than has been usual in elementary treatises. The theory of vision through lenses is based upon Cotes' theorem, after the manner of the older

English writers on Optics, Cotes and Smith. An elegant geometrical construction for the deviation of a ray at a refraction, due to Prof. P. G. Tait, furnishes an elementary theory of the rainbow. Numerous easy exercises are scattered through the text, and several typical examples are fully worked out, while the more difficult are collected at the end of the chapters. The articles marked with an asterisk may be omitted at a first reading.

Suggestions which may improve and extend the usefulness of the book and notifications of errors will be very thankfully received by the author.

R. S. HEATH.

MASON COLLEGE, BIRMINGHAM,
August, 1888.

PREFACE TO THE SECOND EDITION.

The present edition exhibits nearly the same changes as were made in the second edition of the larger work. The theory of lenses is developed throughout on a uniform plan with a uniform notation. The later portions now include the theories of diaphragms, angular aperture, field of view and magnifying power treated after the manner of the modern German writers on Optics.

My thanks are due to my colleague, Mr Lawrence Crawford, M.A., Fellow of King's College, Cambridge, for his valuable aid in reading through the proofs and his many criticisms and suggestions.

R. S. HEATH.

MASON COLLEGE, BIRMINGHAM,
March 1st, 1897.

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CHAPTER I.

The Nature and General Properties of Light.

1. Light may be defined as the external conditions which, acting through the instrumentality of the eyes, produce the sensation called sight.

When a lighted lamp is brought into a dark room, we become sensible by sight of the existence not only of the lamp itself, but of the walls of the room and the other objects contained in it, which before were invisible to us. Bodies, such as the flame of a lighted lamp and the sun, whose presence is necessary to enable us to see anything, are called *self-luminous* bodies. Bodies which in themselves are not luminous, become luminous in the presence of other luminous bodies and are then visible to us. The sensation of sight is caused by certain vibrations, which pass from the things seen into the eye; it is these vibrations which we shall call *light.*

2. We know by experience that we can *see through* certain bodies, such as air, glass and water, but not through others, such as wood, stone and iron; in other words light can be transmitted through the former, but not through the latter. Bodies through which light can be freely transmitted are called *transparent,* the others are called *opaque.* There is an intermediate class of bodies, such as oiled paper, thin porcelain, thin gold-leaf, which will allow a little light to pass through them in an irregular manner, but through which we cannot see an object distinctly. Such bodies are called *translucent.* They

H. O.

will not be considered further in this book, because the light transmitted through them does not obey any simple geometrical law.

Any space through which light can pass, whether occupied by matter or not, is called a *medium.*

3. When the medium is homogeneous, light travels through it in straight lines. This is a law which is being constantly verified by experience. In fact, to test roughly whether a given line is straight or not, it is not unusual to *look along it;* if the line is straight the different points along it appear to be superimposed, but if the line is not straight the points cannot be made to appear superimposed.

Light consists of separable and independent parts; if part of the light proceeding from a luminous body be intercepted by an opaque obstacle this will not in any way affect the other portion which is allowed to pass. When we look at any bright body, it is only a very small portion of the light emanating from the body which can strike the pupils of our eyes; if we shift our position our eyes will intercept a different small portion of the light. It is thus often convenient to consider the portion of light which travels along some particular line in space apart from the rest; such a portion of light is called a *ray*, and it will be supposed to have the form of an indefinitely slender cone, whose axis is the line under consideration. A collection of rays which during their course never deviate far from some fixed central ray is called a *pencil* of rays, and the central ray is called the *axis* of the pencil. If the rays of a pencil meet in a point, that point is called the *focus* of the pencil.

As we shall have continually to mention the eye in the course of our work, it may be well here to refer very briefly to the theory of the eye. Every point of a luminous body (by which we mean every indefinitely small area on the surface of the body) is giving out light in all directions; a pencil of rays limited by the aperture of the eye is by the aid of the refracting surfaces of the eye brought to a focus on the retina, forming an image there; and it is by means of this image that the point is seen. Corresponding to all the bright points of the luminous body there will be images on the retina, and these images enable us to form a mental picture of an extended surface.

4*. In general, light from two independent sources may travel along the same path without interference. From this experimental fact we infer that light is capable of quantitative measurement. For the present we shall suppose that the light with which we are dealing is all of the same kind and homogeneous, and that its quantity or intensity is measured in terms of some fixed standard.

If lines be drawn from a point so as to generate a conical surface of any form, the solid angle of the cone is measured by the area intercepted within it on a sphere of unit radius whose centre is at the vertex of the cone.

If a bright point be emitting light, and the quantity of light emitted within a cone of solid angle ω be Q, then the mean intensity of emission within the cone is Q/ω. When the cone is indefinitely slender, the mean emission within it is called the *intensity of emission* in the direction of the axis of the cone.

A bright body emits light in all directions, but the intensity of emission is different for different directions. The law of emission is given by a well-known experiment. Luminous bodies appear of the same brightness whatever be the inclination of the bright surface to the line of sight. Thus if a cylinder of silver be heated till it becomes luminous and taken into a dark room, it cannot be distinguished from a perfectly flat bar; and similarly, a luminous sphere (like the sun as seen through a mist) appears like a flat disc. The same experiment is true of the intensity of the heat rays radiated from a hot body; in this form it is intimately associated with the Theory of Exchanges.

This experiment shows that *the intensity of emission of light from any element of a bright surface in any direction* *is proportional to the cosine of the inclination of the direction of emission to the normal* to the element of the surface.*

For suppose that a bright body is viewed through a tube of small aperture; when the tube is directed so that the element of the bright surface seen is normal to the line of sight, let the area of the element be ω. Then when the tube is directed so that the normal to the element of the bright surface seen through the tube makes an angle θ with the line of sight, the area of the element will be $\omega \sec \theta$. Let $f(\theta)$ be the intensity of emission per unit area in a direction making an angle θ with the normal to the element; then the whole amount of light transmitted to the eye when the element is inclined to the line of sight at an angle θ is $\omega \sec \theta \, . f(\theta)$.

But this, by experiment, is independent of θ, and therefore $$f(\theta) \propto \cos \theta.$$

Let β be the area of an element of the bright surface and let $\mu\beta$ denote the intensity of the light emitted in the direction of the normal to the element. Then μ may be called the *intrinsic brightness* of the element.

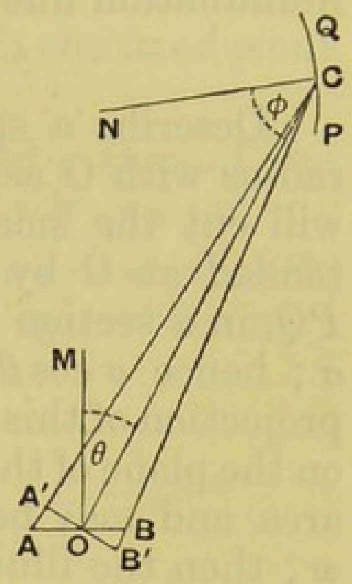
5*. If Q be the quantity of light which falls on an area α of illuminated surface, then Q/α is called the mean intensity of illumination of the area. When the area of the element of bright surface is indefinitely small, the mean intensity of illumination is called the *intensity of illumination* within the element.

We shall now find the illumination of a small area α due to an element of any bright surface β. Let C be the centre of the element of the luminous surface PCQ and O the centre of the illuminated area AOB, and let $OC = r$. Let θ be the inclination of OC to the normal at O, and ϕ that of OC to the normal at C.

Then if α subtend a solid angle ω at C, the quantity of light it receives will be $\mu\beta \cos \phi \, . \, \omega$, where μ is the intrinsic brightness of the element.

* The normal at any point of a surface is the straight line through that point perpendicular to the tangent plane to the surface.

We must next find the value of ω. With centre C and radius OC describe a spherical surface cutting the cone, and let $A'OB'$ be the part of the spherical surface enclosed within the cone. Then the solid angle is given by the equation



$$\omega = A'OB'/r^2.$$

The small areas AOB and $A'OB'$ may be regarded as plane; the inclination of their planes is θ. Also since $A'OB'$ cuts the generators of the cone at right angles, $A'OB'$ is the projection of the area AOB, and therefore

$$(\text{area } A'OB') = (\text{area } AOB) \cos\theta = \alpha \cos\theta.$$

Thus $$\omega = \frac{\alpha \cos\theta}{r^2};$$

and therefore the quantity of light received by α from the element β is

$$\mu\alpha\beta \frac{\cos\theta \cos\phi}{r^2}.$$

This is symmetrical with regard to the two elements, and would therefore represent the quantity of light received by β from the element α, were it of intrinsic brightness μ.

Let σ be the solid angle subtended at O by the bright element β, then it may be shown as before that

$$\sigma = \frac{\beta \cos\phi}{r^2};$$

then the illumination of α due to the element β is

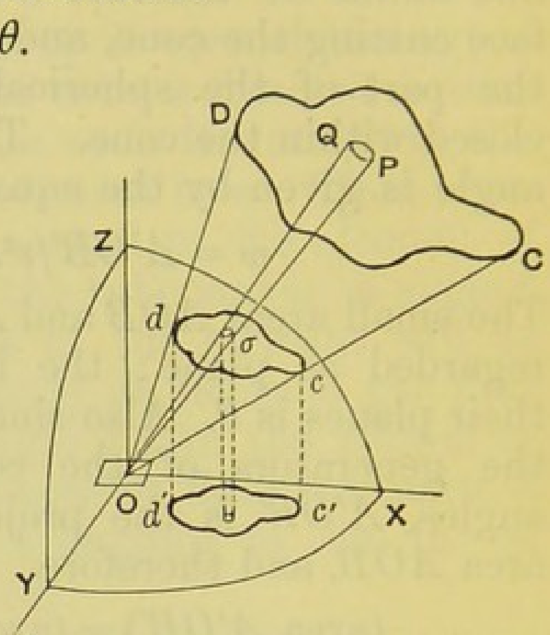
$$\mu\sigma \cos\theta.$$

6*. The illumination of a small area α due to any finite surface of uniform brightness may now be found.

Take any small element of the bright surface PQ and as before let σ be the solid angle subtended by it at the centre O of the illuminated area. Let θ be the angle between the line OC and the normal at O. Then the illumination due to the bright element is

$$\mu\sigma \cos\theta.$$

Describe a sphere of unit radius with O as centre; this will cut the small cone subtended at O by the element PQ, in a section whose area is σ; hence $\sigma\cos\theta$ will be the projection of this small section on the plane of the illuminated area and may be denoted by ϖ; then the illumination due to the small element is $\mu\varpi$.



By addition we arrive at the following method of determining the illumination of a small area at O due to any finite bright surface.

From O draw radii to all points of the boundary of the bright surface as seen from O, forming a conical surface. Let the part of the surface of the sphere of unit radius whose centre is O, intercepted within the cone, be projected on the plane of the area illuminated. If ϖ be the area of the projection, the illumination of the area will be given by the equation

$$I = \mu\varpi.$$

Ex. To find the illumination due to a spherical luminary.

Let a be the semi-vertical angle of the cone whose vertex is at the centre of the area and which envelopes the bright sphere. The curve in which this cone cuts the sphere of unit radius is a circle whose radius is $\sin a$. Hence if θ be the zenith distance of the luminary, the illumination on a small horizontal area is

$$I = \mu\pi \sin^2 a \cos\theta.$$

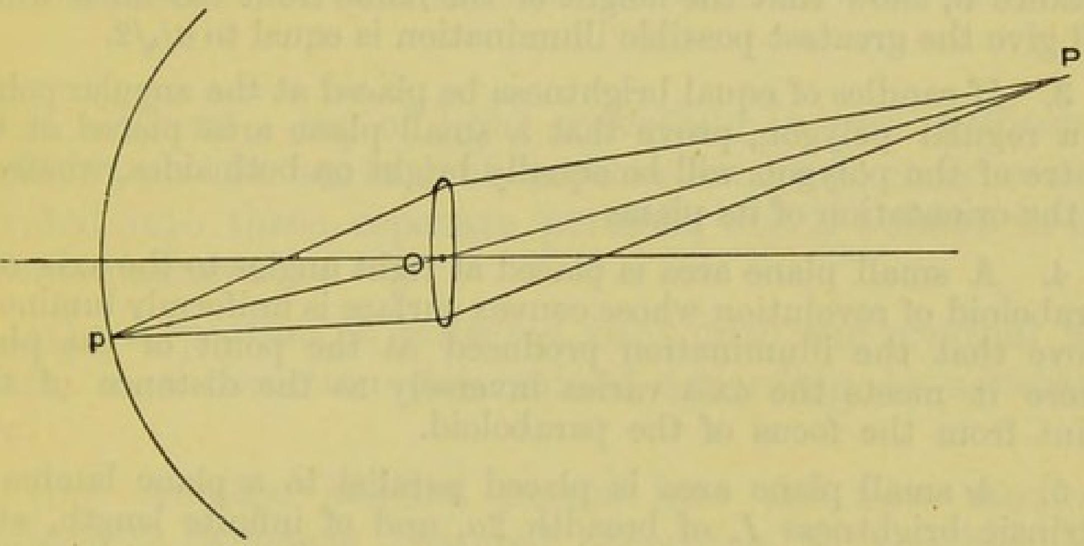
7*. *Objects appear equally bright at all distances.*

The apparent brightness of an object may be measured by the whole quantity of light entering the eye from the object divided by the area of the picture of the object on the retina of the eye.

Let P be any point of the object, p the corresponding point of the picture on the retina; then it will be shown afterwards that the line Pp passes through the fixed point O, the optical centre of the eye.

Let S be the area of a small object and s that of the picture and let $OP = R$, $Op = r$. Then $S : R^2 = s : r^2$.

Now the quantity of light entering the eye is $\mu S\omega/R^2$, where ω is the area of the aperture of the eye. This may



be written $\mu s\omega/r^2$; hence, dividing by s, we get the intrinsic brightness of the image, which is equal to $\mu\omega/r^2$.

We shall assume for the present that as the eye adjusts itself to different distances, r does not change.

Thus *the apparent brightness is independent of the distance of the object.*

The area of the aperture of the eye changes according to the brightness of the light. But if we suppose the aperture to remain the same, as the object is removed, no change in brightness has taken place, so that the aperture does not need further adjustment.

When the object is very distant the area of the picture on the eye gets to be very small indeed, so that the nerves of the retina cannot distinguish it from a point. In this case the brightness must be measured simply by the quantity of light; and therefore, by the same investigation, *the brightness varies inversely as R^2.*

EXAMPLES*.

1. The light from two sources is allowed to fall upon the same screen. One light is at a distance a and the light falls directly from it on the screen. From the other, which is at a distance $3a$, the light falls at an obliquity of 60°. The illuminations of the screen from the two sources being equal, show that one source is 18 times as bright as the other.

2. A small white surface being placed horizontally on a table, and illuminated by a lamp or candle placed at a given horizontal distance a, show that the height of the flame from the table which will give the greatest possible illumination is equal to $a/\sqrt{2}$.

3. If candles of equal brightness be placed at the angular points of a regular polygon, prove that a small plane area placed at the centre of the polygon will be equally bright on both sides, whatever be the orientation of its plane.

4. A small plane area is placed at right angles to the axis of a paraboloid of revolution whose convex surface is uniformly luminous. Prove that the illumination produced at the point of the plane where it meets the axis varies inversely as the distance of this point from the focus of the paraboloid.

5. A small plane area is placed parallel to a plane lamina of intrinsic brightness I, of breadth $2a$, and of infinite length, at a distance c from the centre of the lamina in a line perpendicular to the lamina. Prove that the illumination at the centre of the plane area is $\pi aI/\sqrt{(a^2+c^2)}$.

6. The sides a of triangle are the bases of three infinite rectangles of the same brightness, whose planes are perpendicular to the plane of the triangle; show that all points within the triangle are equally illuminated. Find the position of a point in the plane of the triangle, such that the illuminations at that point received from the three rectangles may be equal.

7. A luminous point is placed on the axis of a truncated conical shell; prove that the whole illumination of the surface of the shell varies as

$$\frac{c'}{(c'^2+a'^2)^{\frac{1}{2}}} \pm \frac{c}{(c^2+a^2)^{\frac{1}{2}}},$$

where a, a' are the radii of the circular ends of the shell and c, c' the distances of the luminous point from their planes.

8. A right cone of vertical angle 2θ is described about a given self-luminous sphere, and at the points of the sphere in which the axis of the cone cuts it, tangent planes are drawn; prove that the mean illumination of that part of the cone which is enclosed between these two planes varies as $\cos\frac{1}{2}\theta\cos^2\theta$.

CHAPTER II.

REFLEXION AND REFRACTION OF RAYS OF LIGHT.

8. WHEN a ray of light travelling in one medium is incident on the surface of another medium, it is usually divided into three separate portions which behave in different manners.

(i) A portion is reflected back into the original medium, in a direction determined according to a certain law.

(ii) Another portion passes into the new medium, having its direction changed according to another law; this portion is said to be *refracted* into the new medium.

(iii) A third portion is said to be *scattered* by the surface bounding the two media; the bounding surface becomes illuminated and itself acts like a source of light sending rays in all directions.

When a ray of light is incident on a solid opaque body, the second portion does not exist, and all the light is either reflected or scattered. The quantity of light reflected depends upon the nature of the surface; the smoother and more highly polished the surface is, the more light is reflected. The scattering of light is probably due to the unevenness of the surface; the incident light is reflected by minute portions of the surface which act as mirrors distributed irregularly in all directions. It is by the scattering of light that non-luminous bodies become visible when in the presence of a bright body.

Thus if the rays of light from the sun strike a window, part of the light passes into the glass and out again into the room; for an observer inside the room can see the sun distinctly. Another part of the sun's light is reflected back into the air; for a person stationed outside can see a picture of the sun in the window as in a looking-glass. A third portion serves to render visible the specks and marks on the window-pane; this light is said to be scattered.

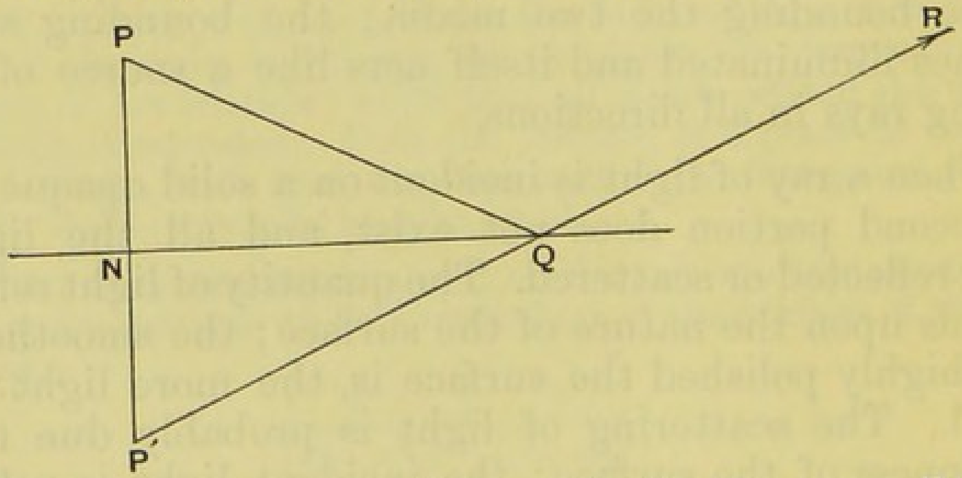
9. The plane containing the incident ray and the normal* to the surface separating the two media, is called *the plane of incidence,* and the acute angle between the incident ray and the normal is called *the angle of incidence,* and the acute angle between the reflected ray and the normal, *the angle of reflexion.*

When the direction of a ray of light is changed by reflexions or refractions, the angle through which the original ray produced must be turned in order to bring it into the position of the final ray, is called the *deviation* of the ray.

The law according to which a ray of light is reflected at a surface may be thus stated.

The angles of incidence and reflexion always lie in the same plane and are equal to each other.

10. If the ray be incident on a plane surface the reflected ray may be found by a simple geometrical con-



struction. If P be any point on the incident ray PQ, and if from P a perpendicular PN be drawn to the reflecting plane and be produced to P' so that $P'N$ is equal to PN, then in the triangles PNQ, $P'NQ$, the two sides PN, NQ are equal to the two sides $P'N$, NQ, each to each, and the

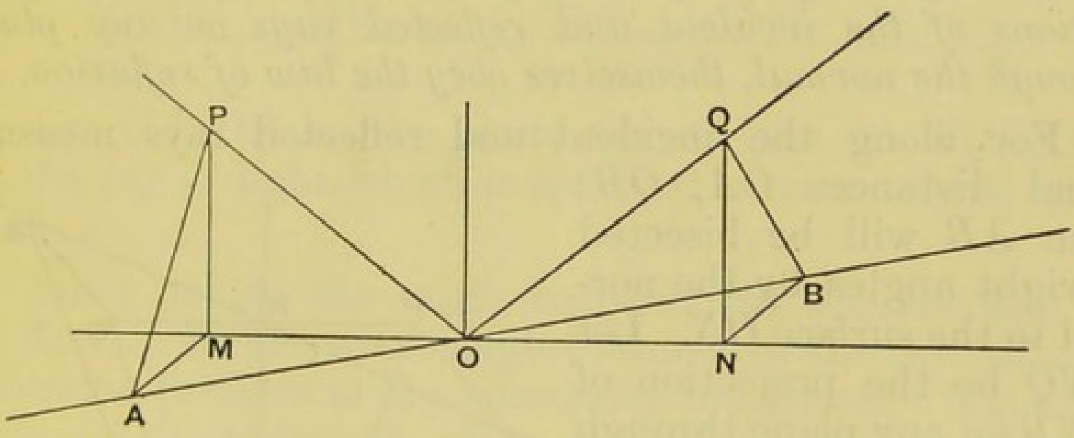
* See note to p. 4.

angles PNQ, $P'NQ$ are both right angles, therefore the triangles are equal in all respects, and the angle PQN is equal to the angle $P'QN$. Hence $P'Q$ produced and PQ make equal angles with the plane NQ and therefore with the normal to it. Thus $P'Q$ produced will be the reflected ray.

If the surface be not plane, we may substitute the tangent plane to the surface at Q, for the plane of the mirror in the previous construction.

11. *When a ray is reflected at a plane surface the incident ray and the reflected ray make equal acute angles with any line in or parallel to the reflecting plane.*

For let PO, OQ be the incident and reflected rays, and let the plane of incidence meet the reflecting plane in the



line MON. Also let AOB be a line drawn through O parallel to the given line. On the lines OP, OQ measure equal lengths OP, OQ, and through P, Q draw planes perpendicular to the line MON meeting this line in the points M, N and the line AOB in the points A, B, respectively. Then in the triangles POM, QON, the angles at M and N are right angles and the angles POM, QON are equal, by the law of reflexion, and OP is equal to OQ; therefore the triangles are equal in all respects, so that $OM = ON$, $PM = QN$. Also in the triangles AOM, BON, the angles at M and N are right angles and the angle AOM is equal to the angle BON; therefore $AM = BN$ and $AO = OB$. Also in the triangles PMA, QNB the two sides PM, MA are equal to the two sides QN, NB each to each, and the angles at

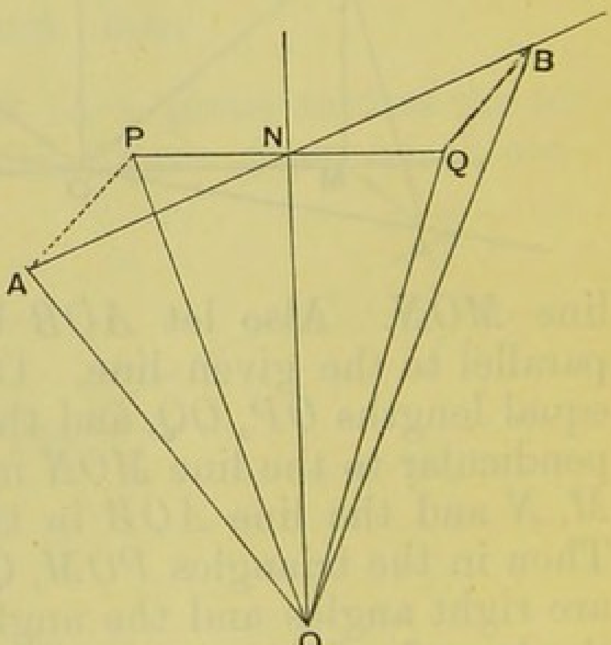
M and N are right angles; therefore $AP = BQ$. Finally in the triangles AOP, BOQ, the three sides of the one are equal to the three sides of the other, each to each; therefore the angles AOP, BOQ are equal. This proves the proposition.

Conversely, if two lines PO, OQ lie in a plane normal to the reflecting plane and make equal acute angles with any given line in the plane, they may be taken to represent an incident and the reflected ray, respectively. The proof is similar to the preceding.

It follows from the preceding proposition that *if a ray of light be reflected in any manner successively at two plane surfaces, the initial and final rays are equally inclined to the line of intersection of the plane surfaces.*

12. *If a ray of light be reflected at a surface, the projections of the incident and reflected rays on any plane through the normal, themselves obey the law of reflexion.*

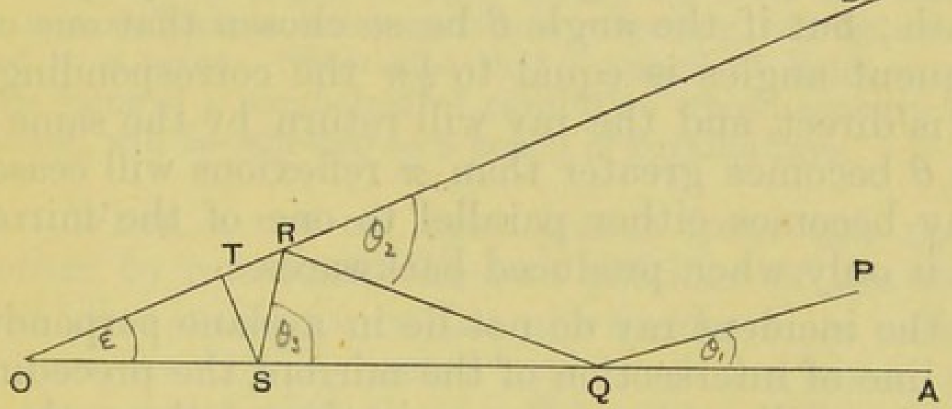
For along the incident and reflected rays measure equal distances OA, OB; then AB will be bisected at right angles by the normal to the surface ON. Let PNQ be the projection of ANB on any plane through the normal, so that OP, OQ are the projections of the incident and reflected rays. Then in the triangles APN, BQN, the angles at P and Q are right angles and the angles ANP, BNQ are equal to each other, and $AN = NB$. Therefore $PN = NQ$. Hence the triangles PNO, QNO are equal in all respects, and therefore OP, OQ are equally inclined to the normal ON.



Further, it is easily seen that the triangles AOP, BOQ are equal in all respects, and therefore the angle AOP is equal to the angle BOQ. In other words, *the incident and* *reflected rays are equally inclined to any plane through the normal.*

13. Let a ray of light be reflected successively at two plane mirrors, to find the direction of the ray after any number of reflexions.

We shall first consider the case in which the reflexions take place in a plane perpendicular to both mirrors.



Let OA, OB be the plane mirrors and let $PQRST$... be the ray of light which is reflected successively at Q, R, S, T...

Let ϵ denote the angle between the mirrors, and let $\theta_1, \theta_2, \theta_3$... be the acute angles formed by the ray with the reflecting surfaces at the successive incidences. Thus the angles at Q are each θ_1, those at R, θ_2, and so on; so that from the triangle QOR we find $\theta_2 = \theta_1 + \epsilon$, and similarly $\theta_3 = \theta_2 + \epsilon$, &c. These equations may be written

$$\theta_2 - \theta_1 = \epsilon,$$
$$\theta_3 - \theta_2 = \epsilon,$$
$$\cdots\cdots\cdots\cdots$$
$$\theta_{n+1} - \theta_n = \epsilon;$$

and therefore, by addition

$$\theta_{n+1} - \theta_1 = n\epsilon.$$

When n is even, the angles θ_{n+1} and θ_1, are measured from the same mirror, and therefore $\theta_{n+1} - \theta_1$ is the angle between the initial and final rays; therefore *the total deviation is equal to n times the inclination of the mirrors.*

The deviation is the same whatever the angle of incidence, so that *any two rays are inclined at the same angle to each other after reflexion as before incidence.*

When the ray is reflected twice, once at each mirror, the deviation of the ray is twice the angle between the mirrors. This is the principle of Hadley's Sextant.

At each reflexion the value of θ is increased by ϵ. When θ becomes greater than $\frac{1}{2}\pi$ the ray will begin to come back from the angle outwards, generally by a different path; but if the angle θ be so chosen that one of the subsequent angles is equal to $\frac{1}{2}\pi$ the corresponding incidence is direct, and the ray will return by the same path. When θ becomes greater than π reflexions will cease; for the ray becomes either parallel to one of the mirrors or meets it only when produced backwards.

If the incident ray do not lie in a plane perpendicular to the line of intersection of the mirrors, the preceding investigation will apply to the projection of the path of the ray on such a plane. If, further, we remember that the inclination of the ray to this plane changes at each reflexion just as if the ray were reflected at it, the direction of the emergent ray is completely determined. After any even number of reflexions the ray makes with the principal plane the same angle as at first, and after an odd number of reflexions, an equal angle on the other side of the plane.

EXAMPLES.

1. If a ray of light after reflexion at each of the sides of a triangle in succession retrace its path, show that it must proceed along the lines joining the feet of the perpendiculars drawn from the angular points to the opposite sides.

2. What must be the inclination of two mirrors in order that a ray incident parallel to one of them may, after two reflexions, be parallel to the other? Ans. 60°.

3. Two rays emanate from a point in the circumference of a reflecting circle in the plane of the circle; supposing that their n^{th} points of incidence coincide, prove that the angle between their original directions is any one of a series of $(n-1)$ angles in arithmetical progression.

4. Two plane mirrors inclined at an angle θ intersect in O; P is a point between the mirrors and PQR a ray emanating from P reflected at the mirrors in succession so as to return to P; show that OP bisects the angle QPR and that the length of the path is $2OP \sin \theta$.

5. If the angle of a hollow cone, polished internally, be any sub-multiple of two right angles, a cylindrical pencil of rays incident parallel to the axis will, after a certain number of reflexions, be a cylindrical pencil parallel to the axis, and of the same diameter as the incident pencil.

6. Show that a pencil of light emanating from the focus of a prolate spheroid whose inner surface is reflecting, will be accurately reflected to a point. Show also that a pencil of light emanating from the focus of a paraboloid of revolution whose concave surface is reflecting, will be reflected as a pencil of parallel rays.

14. When a ray of light passes from one medium to another by refraction, the two portions of the ray before and after incidence on the new medium are called *the incident ray and the refracted ray*; and the acute angles which they make with the normal to the surface of separation at the point of incidence, are called the *angles of incidence and refraction*, respectively.

The angles of incidence and refraction lie always in the same plane, and their sines are to one another in an invariable ratio.

This is the fundamental law of refraction. The constant ratio depends on the nature of the two media and the kind of light transmitted; it is called the *refractive index* from the first medium to the second.

If a ray of light pass from a *vacuum* into a given medium, the constant ratio is the *absolute refractive index* of that medium.

If ϕ be the angle of incidence and ϕ' the angle of refraction, as a ray passes from one medium into another, the law of refraction is expressed by the equation

$$\frac{\sin \phi}{\sin \phi'} = \mu,$$

where μ is the refractive index from one medium to the other.

15. It is an experimental law, that the path of a ray of light is reversible; in other words, if a ray travel backwards through the second medium along the direction of the refracted ray, it will after refraction into the first medium retrace the path of the incident ray.

If we denote the two media by A, B and the refractive index from A into B by μ_{ab}, and the refractive index from B into A by μ_{ba}, this experiment shows that

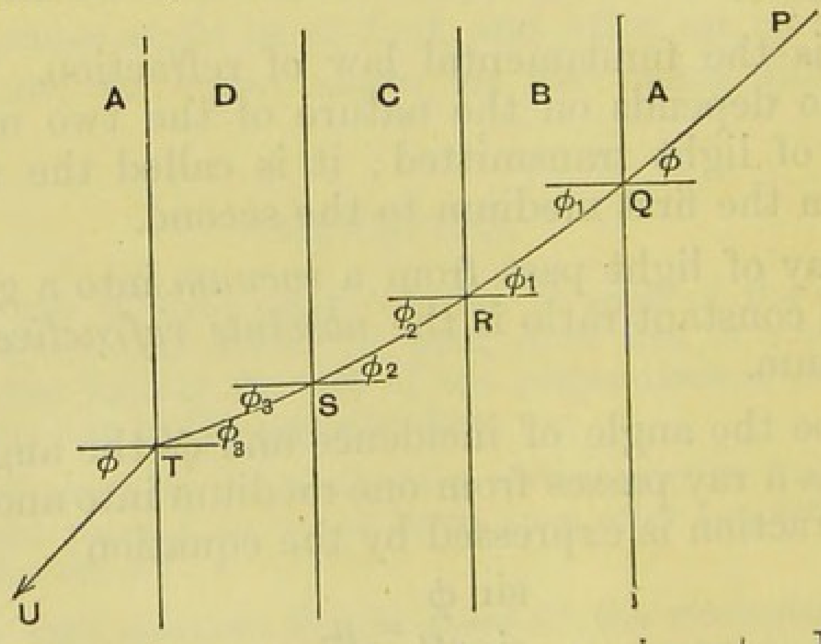
$$\frac{\sin \phi}{\sin \phi'} = \mu_{ab}, \quad \frac{\sin \phi'}{\sin \phi} = \mu_{ba},$$

with the previous notation; and therefore, eliminating ϕ and ϕ',

$$\mu_{ab} \cdot \mu_{ba} = 1.$$

16. Also, it is found by experiment that if a ray of light pass through any number of media bounded by parallel planes, into a medium of the same nature as that in which it was originally travelling, the initial and final directions of the ray are parallel to each other.

Let A be the original medium, B, C...... the other media. Let ϕ be the angle of incidence on B, ϕ_1 the corresponding angle of refraction. Then ϕ_1 will be the angle of incidence on C, and so on. The final angle of refraction



into A is shown to be ϕ by the experiment. Using the same notation as before to express the law of refraction at the successive surfaces, we arrive at the relations

$$\frac{\sin\phi}{\sin\phi_1} = \mu_{ab},$$

$$\frac{\sin\phi_1}{\sin\phi_2} = \mu_{bc},$$

$$\dots\dots\dots\dots\dots$$

$$\frac{\sin\phi_n}{\sin\phi} = \mu_{ka}, \text{ say.}$$

By multiplication,

$$\mu_{ab} \cdot \mu_{bc} \cdot \mu_{cd} \dots\dots\dots \mu_{ka} = 1.$$

If there are only three media, this relation becomes

$$\mu_{ab} \cdot \mu_{bc} \cdot \mu_{ca} = 1,$$

or

$$\mu_{ac} = \mu_{ab} \cdot \mu_{bc}.$$

For example, let us take the three media, air, glass and water. The values of the refractive indices from air to glass, and air to water are, respectively, $\mu_{ag} = \frac{3}{2}$, $\mu_{aw} = \frac{4}{3}$.

The preceding formula enables us to find the refractive index from glass to water.

For

$$\mu_{gw} = \mu_{ga} \cdot \mu_{aw}$$

$$= \tfrac{2}{3} \cdot \tfrac{4}{3} = \tfrac{8}{9},$$

that is, the refractive index from glass to water is $\frac{8}{9}$.

Also, let μ, μ' be the absolute refractive indices of the media A and B. Then if we denote the vacuum by the suffix v, $\mu_{ab} = \mu_{av} \cdot \mu_{vb}$. But μ_{av} is the reciprocal of μ_{va} or the reciprocal of μ, and therefore

$$\mu_{ab} = \frac{\mu'}{\mu};$$

that is, *the relative refractive index between any two media may be found by dividing the absolute refractive index of the second by that of the first.*

The law of refraction can now be more symmetrically expressed in terms of the absolute refractive indices of the two media, μ and μ'; using the previous notation, the relation between the angles of incidence and refraction becomes

$$\mu \sin \phi = \mu' \sin \phi'.$$

17. Suppose that μ' is greater than μ; that is, suppose B to be a more highly refracting medium than A. Then if ϕ be given, the equation to determine ϕ' is,

$$\sin \phi' = \frac{\mu}{\mu'} \sin \phi.$$

This value for $\sin \phi'$ is always less than unity whatever the value of ϕ, so that a value of ϕ' can always be found for any value of ϕ. Thus when a ray of light travelling in any medium is incident on a more highly refracting medium, the law of refraction always gives a direction for the refracted ray.

But when a given ray is passing from the medium B into the medium A which is less refractive, the angle ϕ' is given, and the equation to determine ϕ is

$$\sin \phi = \frac{\mu'}{\mu} \sin \phi'.$$

If $\sin \phi'$ is greater than μ/μ' the corresponding value for $\sin \phi$ becomes greater than unity; so that the law of refraction fails to give a real direction for the refracted ray.

The angle $\sin^{-1} (\mu/\mu')$, or, the greatest angle at which a ray of light proceeding in the more highly refractive medium can be incident on the other so as to be refracted into it, is called the *critical angle* between those media. The value of the critical angle from water to air is about $48° \, 27' \, 40''$, and from crown-glass to air, $40° \, 30'$.

When a ray of light is incident on a medium less refractive than the medium in which it is moving, at an angle greater than the critical angle, the whole of the light is found by experiment to be reflected; the refracted part does not exist. This is known as *total internal reflexion*. Since no part of the light is lost by refraction, the reflected light is much more brilliant than when it is reflected by an ordinary mirror.

Total internal reflexion may be exhibited by holding a glass of water above the level of the eye; the under surface of the water will appear very bright from the light internally reflected at it, and any object in the water will be seen by reflexion at the under surface more brilliantly than when it is reflected in a mirror. A glass prism may easily be held so that the eye may receive light through it after internal reflexion at one of its faces. That face will appear very bright. An arrangement of prisms fixed in this manner is often used to light under-ground rooms.

The subject of this article may also be illustrated by describing the appearance presented to an eye placed under the surface of still water. All external objects would appear compressed within a conical space whose axis is vertical and semi-vertical angle 48° 27′ 40″, the objects near the horizon being much distorted. Beyond this conical space objects within the water would be seen reflected by the surface of the water.

If we suppose that $\mu' = -\mu$, then $\phi = -\phi'$, and the refraction becomes a reflexion. All the subsequent theorems relating to refraction will give corresponding theorems for reflexion by making the same substitution $\mu' = -\mu$.

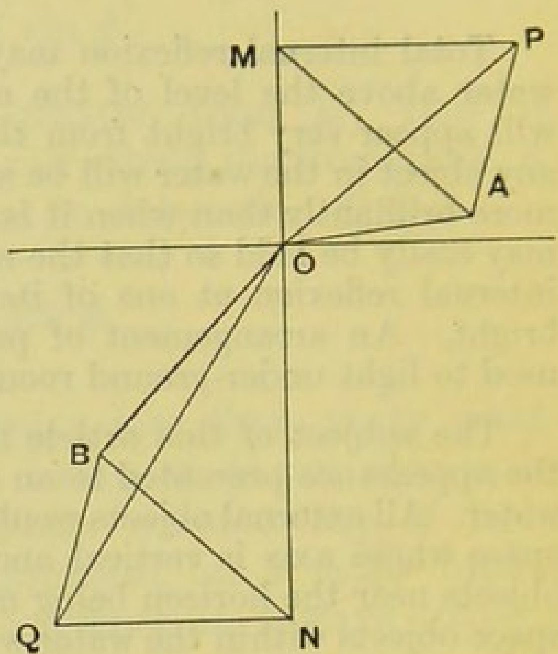
18. There are two other useful theorems relating to the incident and refracted rays.

The angles which the incident and refracted rays make with any plane through the normal to the refracting surface, obey the law of refraction.

Also *the projections of the incident and refracted rays on any plane through the normal are connected by a law of refraction, with a refractive index depending on the inclinations of the rays to the plane.*

For let AO, OB be the incident and refracted portions of any ray, and let the number of units of length in AO, OB be taken equal to μ and μ', the refracting indices of the two media, respectively. Then if AM, BN be drawn from A and B perpendicular to the normal to the refracting surface, AM, BN will be equal and parallel, since they are equal to $\mu \sin \phi$, $\mu' \sin \phi'$, respectively.

Let PO, OQ be the projections of AO, OB on any plane through the normal, P and Q being the projections of the points A, B respectively. Then the triangles APM, BQN are equal in all respects, by Euclid i. 26.



Let η, η' be the acute angles which the incident and refracted rays make with the plane; ϕ, ϕ' the acute angles which the projections of these rays on the plane make with the normal. Then

$$AP = \mu \sin \eta,\ BQ = \mu' \sin \eta',$$

and therefore, since AP is equal to BQ,

$$\mu \sin \eta = \mu' \sin \eta'.$$

This proves the first theorem.

Also $OP = \mu \cos \eta$, $OQ = \mu' \cos \eta'$; and therefore, since PM is equal to QN,

$$\mu \cos \eta \sin \phi = \mu' \cos \eta' \sin \phi',$$

which proves the second theorem.

19. *In any refraction, the greater the angle of incidence, the greater will be the angle of deviation.*

For if ϕ, ϕ' be the angles of incidence and refraction,

$$\sin \phi = \mu \sin \phi',$$

and therefore $$\frac{\sin \phi - \sin \phi'}{\sin \phi + \sin \phi'} = \frac{\mu - 1}{\mu + 1};$$

that is, $$\frac{\tan \frac{1}{2}(\phi - \phi')}{\tan \frac{1}{2}(\phi + \phi')} = \frac{\mu - 1}{\mu + 1},$$

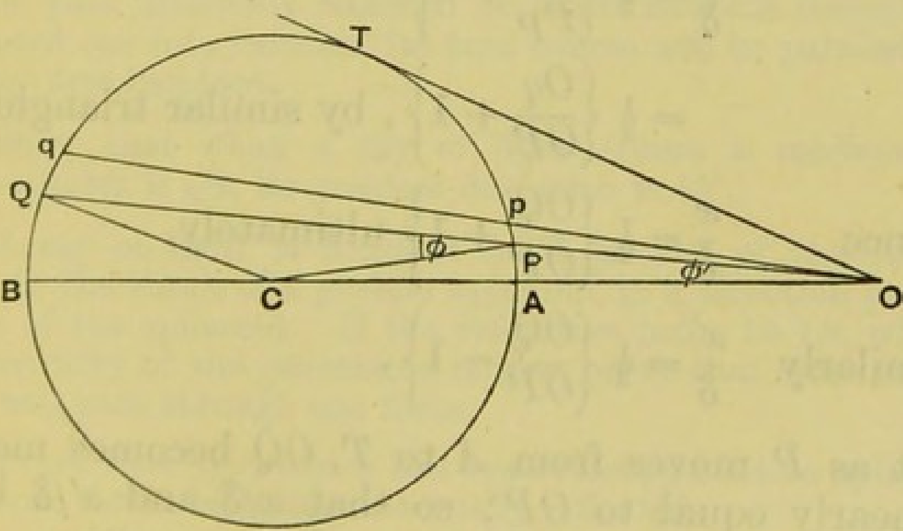
or finally, $$\tan \tfrac{1}{2}(\phi - \phi') = \frac{\mu - 1}{\mu + 1} \tan \tfrac{1}{2}(\phi + \phi').$$

But the deviation is equal to $\phi - \phi'$. If ϕ increase, and therefore also ϕ', $\tan \frac{1}{2}(\phi + \phi')$ will increase, since $\frac{1}{2}(\phi + \phi')$ is less than $\frac{1}{2}\pi$, and therefore the deviation will increase.

When the ray is passing into a rarer medium, we have only to suppose the ray reversed; then since the angle of refraction increases as the angle of incidence increases, the proof comes under the case just considered.

20. We shall now give a geometrical proof of this theorem, due to Professor P. G. Tait.

Let C be the centre of any circle of radius r; take an external point O, such that $OC = \mu r$, and draw any line OPQ through O to meet the circle in P, Q, and join CP, CQ. Then if we denote the angle CPQ by ϕ, and COP by ϕ',



$$\sin \phi : \sin \phi' = CO : CP = \mu : 1,$$

or
$$\sin \phi = \mu \sin \phi'.$$

The angles ϕ and ϕ' are therefore related like angles of incidence and refraction of a ray of light. The deviation $\phi - \phi'$ will be the angle PCO, or D say. By varying the direction of the line OPQ from the position OAB, to the position OT in which it touches the circle, the angle ϕ will be made to increase from O to $\frac{1}{2}\pi$, and during this change D is increasing also. This proves that the deviation increases with the angle of incidence.

The angle of refraction increases from zero to the value COT; this angle represents the critical angle.

But further, *as the angle of incidence or that of refraction increases uniformly, the deviation increases faster and faster.*

For let Opq be another chord of the circle close to OPQ. Then the change in D is the angle subtended at the centre by the arc Pp. And since the angle PCQ is $\pi - 2\phi$, the increase in ϕ is represented by the arc $\frac{1}{2}(Qq + Pp)$; and therefore, by subtraction, the increase in ϕ' is represented by an arc $\frac{1}{2}(Qq - Pp)$.

If we suppose ϕ, ϕ' and D to have become $\phi + x$, $\phi' + x'$ and $D + \delta$ respectively, we have

$$\frac{x}{\delta} = \frac{1}{2}\left\{\frac{Qq}{Pp} + 1\right\}$$

$$= \frac{1}{2}\left\{\frac{Oq}{OP} + 1\right\}, \text{ by similar triangles.}$$

Hence $$\frac{x}{\delta} = \frac{1}{2}\left\{\frac{OQ}{OP} + 1\right\} \text{ ultimately,}$$

and similarly $$\frac{x'}{\delta} = \frac{1}{2}\left\{\frac{OQ}{OP} - 1\right\}.$$

But as P moves from A to T, OQ becomes more and more nearly equal to OP; so that x/δ and x'/δ become smaller and smaller, which proves the proposition.

EXAMPLES.

1. The angle of incidence being 60°, and the index of refraction $\sqrt{3}$; find the angle of refraction. *Ans.* 30°.

2. The absolute refractive indices of two media being $\sqrt{5} - 1$ and 2, respectively, find the angle of refraction of a ray travelling in the first medium and incident on the second at an angle of 30°. *Ans.* 18°.

3. A ray of light is incident on a refracting surface at an angle whose tangent is equal to the refractive index. Prove that the angle of refraction is the complement of that of incidence.

4. The height of a cylindrical cup is 4 inches and the diameter of its base 3 inches. A person looks over its rim so that the lowest point of the opposite side visible to him is $2\frac{1}{4}$ inches below the top. The cup is filled with water; looking in the same direction he can just see the point of the base farthest from him. Find the refractive index of water. *Ans.* $\frac{4}{3}$.

5. A ray is incident on a refracting sphere whose refractive index is $\frac{3}{2}$, at an angle whose sine is $\frac{3}{8}\sqrt{6}$. Show that if the ray be refracted into the sphere, that portion of it which emerges after having been twice internally reflected will be in the same direction as the original ray.

6. A ray of light is incident on a refracting sphere, whose refractive index is $\sqrt{3}$. It is refracted into the sphere and when it is incident on the inner surface of the sphere, part is reflected internally, and part is refracted out into vacuum. Show that if the original angle of incidence be 60°, these two parts will be at right angles.

If the part internally reflected be again incident internally and be refracted out into vacuum, its final course will be parallel to that of the ray first incident.

7. Show that when a ray of light enters a medium whose refractive index is $\sqrt{2}$, its greatest deviation is 45°.

8. A ray of light is incident on a portion of the refracting medium in the shape of a prolate spheroid, in a direction parallel to the axis of the spheroid. If the refractive index be $1/e$, where e is the eccentricity of the generating ellipse, prove that after refraction the ray will pass through one focus.

9. Prove that light which has been refracted into a sphere from vacuum can never be totally internally reflected.

10. If light be incident on the curved surface of a hemisphere of a refracting medium in a direction parallel to its axis, show that there will be no total internal reflexion at the plane surface, unless the refractive index be greater than $\sqrt{2}$.

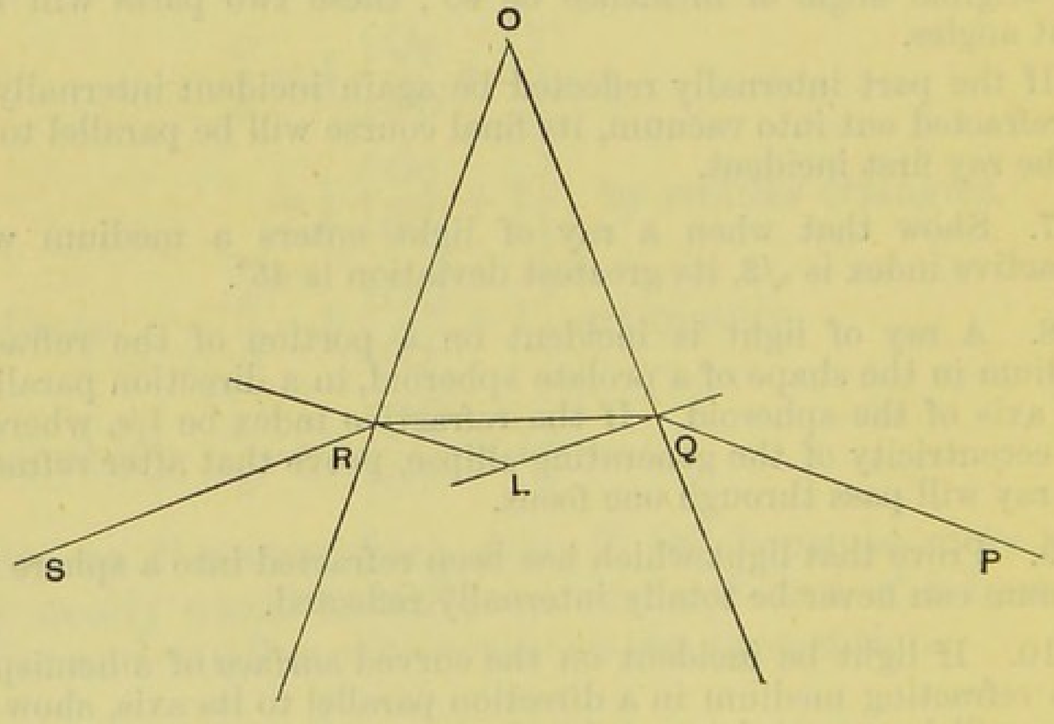
11. If ϕ and ϕ' be the angles of incidence and refraction of a ray of light, show that $\sin(\phi-\phi')/\sin(\phi+\phi')$ increases as ϕ increases.

21. Any medium bounded by two plane faces meeting in an edge, is called a *prism*. The inclination of the faces to each other is called the *refracting angle* of the prism.

At present we shall only consider the path of rays of light which pass through the prism in a plane perpendicular to both its faces, and therefore perpendicular to the edge of the prism; we shall call such a plane a *principal section* of the prism.

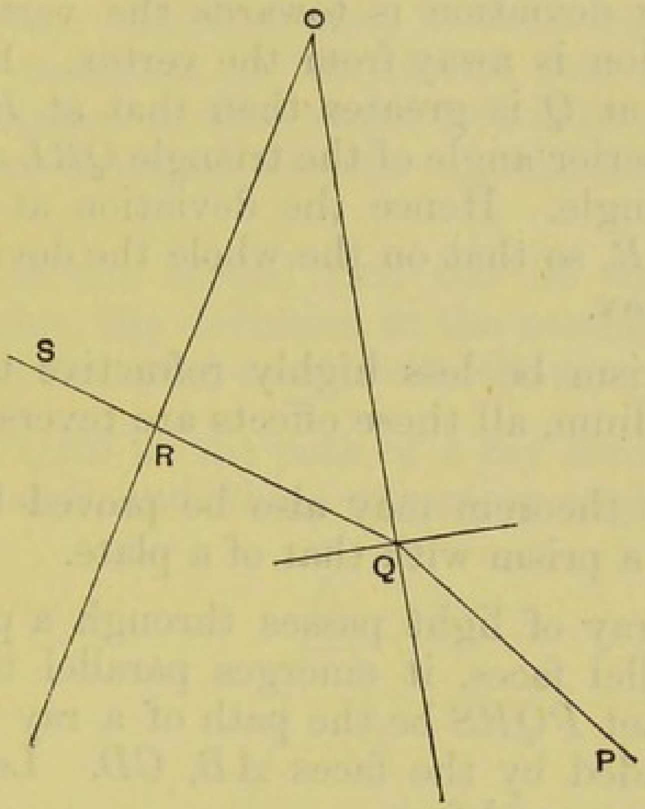
When a ray of light passes through a prism which is more highly refractive than the surrounding medium, the deviation is, in all cases, from the refracting angle towards the thicker part of the prism.

Let $PQRS$ be the course of a ray of light through a prism in a principal section QOR. Draw the normals at Q and R meeting in L. There are three cases to be considered, according as the triangle OQR is acute angled, or contains a right angle or an obtuse angle.

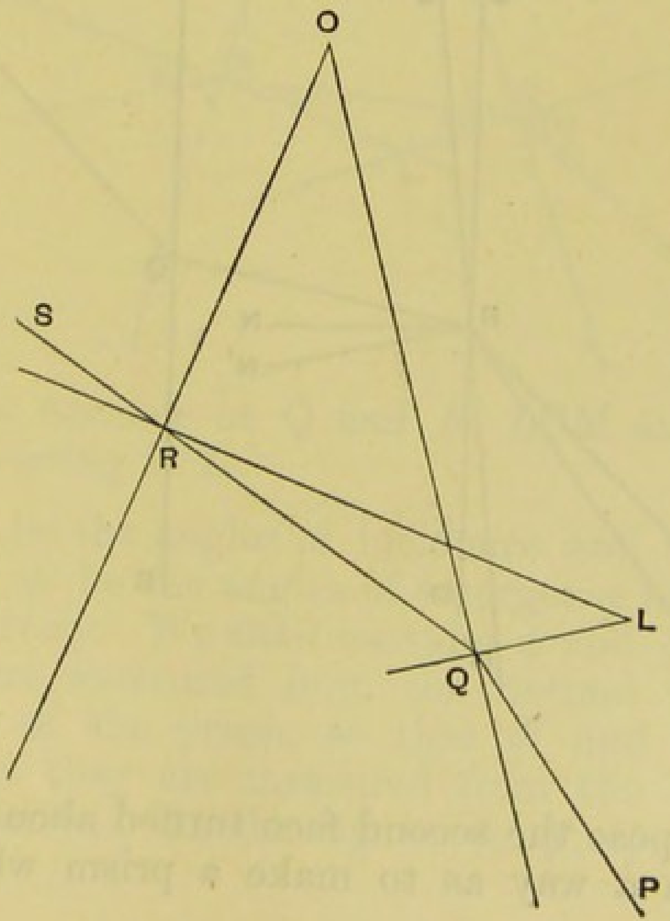


In the first case the rays PQ and RS lie on the sides of the normals away from the vertex, and therefore the deviations both at ingress and egress will be away from the edge of the prism.

In the second case let one of the angles of the triangle OQR, namely the angle ORQ, be a right angle; at the point of incidence R there will be no deviation and at the other point of incidence the deviation is away from the vertex.



In the last case, one of the angles, ORQ, is obtuse, the

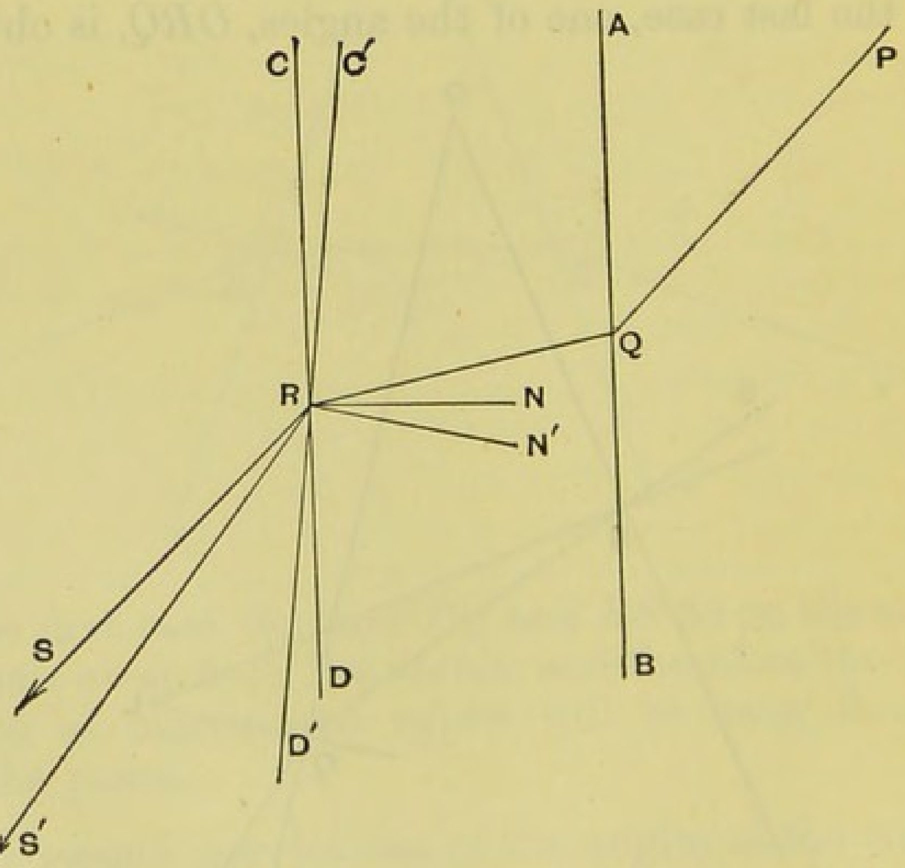


other angle, *OQR*, being acute. Then the ray *SR* lies on the side of the normal towards the vertex, so that the corresponding deviation is towards the vertex, while at *Q* the deviation is away from the vertex. But the angle of refraction at *Q* is greater than that at *R*, the former being the exterior angle of the triangle *QRL* and the latter an interior angle. Hence the deviation at *Q* is greater than that at *R*, so that on the whole the deviation is away from the vertex.

If the prism be less highly refractive than the surrounding medium, all these effects are reversed.

22. This theorem may also be proved by comparing the action of a prism with that of a plate.

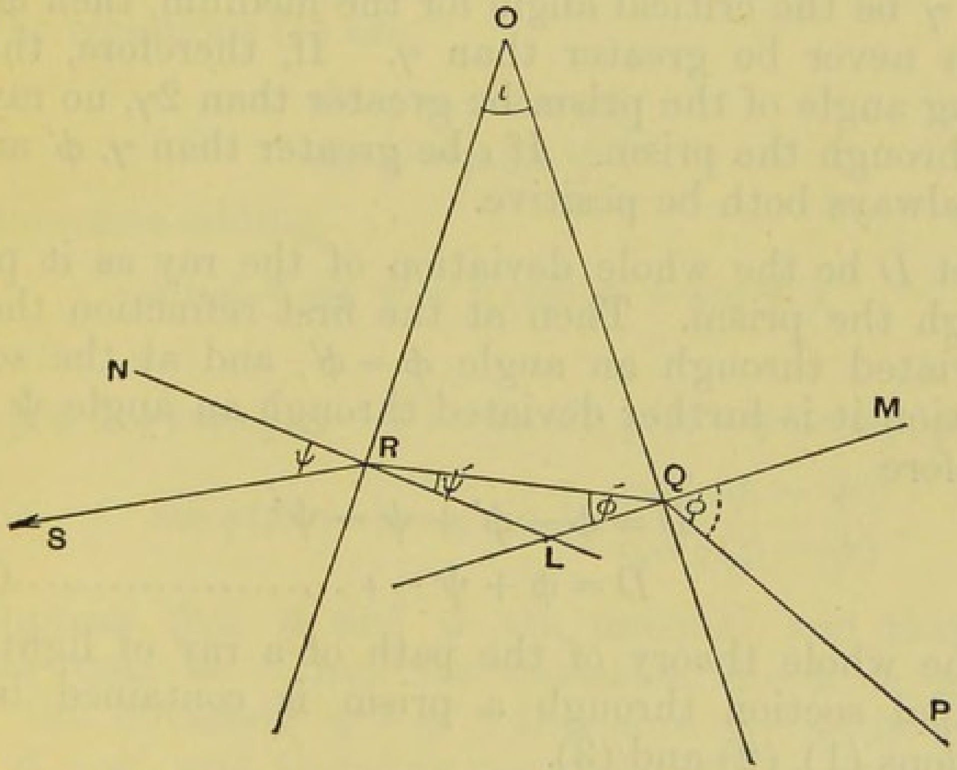
When a ray of light passes through a plate bounded by two parallel faces, it emerges parallel to its original direction. Let *PQRS* be the path of a ray through such a plate bounded by the faces *AB*, *CD*. Let *RN* be the normal at the second face.



Now suppose the second face turned about *R* towards *AB*, in such a way as to make a prism whose edge is perpendicular to the plane of the ray. Let RN' be the new position of the normal to the second face and RS' the emergent ray. Then, as may be seen from the figure, the angle of incidence at the second face is increased; hence the deviation at the second face is increased. The ray is therefore deviated towards the thicker part of the prism.

Similarly, if the second face CD be turned in the opposite direction, the deviation at the second face will be diminished and the same result will follow.

23. Let $PQRS$ be the path of a ray through a prism whose edge is at O, and whose refracting angle is ι.



Draw the normals at Q and R, LQM and LRN respectively, meeting in L.

Let ϕ, ϕ' be the angles of incidence and refraction at Q, and let ψ, ψ' be the angles of emergence and incidence at R, respectively. We shall consider ϕ and ψ as positive when they are measured from the normal towards the thicker part of the prism, so that ϕ' and ψ' will be positive when they are measured from the normals towards the vertex. In the figure ϕ, ϕ', ψ, ψ' are all positive.

By the law of refraction we have

$$\left.\begin{aligned}\sin\phi &= \mu\sin\phi' \\ \sin\psi &= \mu\sin\psi'\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, the angles at the base of the triangle OQR are respectively $\frac{1}{2}\pi - \phi'$, and $\frac{1}{2}\pi - \psi'$, hence

$$\iota + \tfrac{1}{2}\pi - \phi' + \tfrac{1}{2}\pi - \psi' = \pi,$$

or
$$\phi' + \psi' = \iota \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This result is also true when the triangle OQR is obtuse; in this case one of the angles ϕ' or ψ' would be negative.

If γ be the critical angle for the medium, then ϕ' and ψ' can never be greater than γ. If, therefore, the refracting angle of the prism be greater than 2γ, no ray can pass through the prism. If ι be greater than γ, ϕ' and ψ' must always both be positive.

Let D be the whole deviation of the ray as it passes through the prism. Then at the first refraction the ray is deviated through an angle $\phi - \phi'$, and at the second refraction it is further deviated through an angle $\psi - \psi'$. Therefore

$$D = \phi - \phi' + \psi - \psi',$$

or
$$D = \phi + \psi - \iota \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The whole theory of the path of a ray of light in a principal section through a prism is contained in the equations (1), (2) and (3).

24. *The deviation is a minimum when the ray of light passes symmetrically through the prism.*

Let ϕ_0 be the value of ϕ for this symmetrical path, and let ϕ gradually increase from ϕ_0. Then ϕ' and ψ' increase and decrease, respectively, by equal increments by virtue of equation (2); hence, since ϕ' becomes greater than ψ', the deviation at the first face increases faster than that at the second face diminishes, so that on the whole the total deviation increases. The same result is easily seen to be true even after ψ' becomes negative (if it does become negative before ϕ reaches $\frac{1}{2}\pi$). Hence as ϕ increases from ϕ_0 the deviation continually increases.

If ϕ diminishes from ϕ_0, then ψ increases from ψ_0, and we have only to consider the reversed ray to see that the same result follows.

Hence, *when the ray of light passes symmetrically through the prism, the deviation is a unique minimum.*

The theorem may also be proved by means of the formulæ of the preceding article.

The equations (1) are

$$\left.\begin{aligned}\sin\phi &= \mu\sin\phi' \\ \sin\psi &= \mu\sin\psi'\end{aligned}\right\},$$

and therefore adding,

$$\sin\phi + \sin\psi = \mu(\sin\phi' + \sin\psi'),$$

or

$$2\sin\tfrac{1}{2}(\phi+\psi)\cos\tfrac{1}{2}(\phi-\psi) = 2\mu\sin\tfrac{1}{2}(\phi'+\psi')\cos\tfrac{1}{2}(\phi'-\psi'),$$

that is $$\sin\tfrac{1}{2}(D+\iota) = \mu\sin\tfrac{1}{2}\iota\,\frac{\cos\frac{1}{2}(\phi'-\psi')}{\cos\frac{1}{2}(\phi-\psi)}.$$

Suppose that ϕ and ψ are unequal, and that ϕ is greater than ψ. Then the deviation $\phi-\phi'$ is greater than the deviation $\psi-\psi'$; therefore $\phi-\psi$ is greater than $\phi'-\psi'$, and therefore $\cos\frac{1}{2}(\phi'-\psi')$ is greater than $\cos\frac{1}{2}(\phi-\psi)$.

Similarly $\cos\frac{1}{2}(\phi'-\psi')$ is greater than $\cos\frac{1}{2}(\phi-\psi)$ if ψ be greater than ϕ. Hence in all cases in which ϕ and ψ are unequal $\sin\frac{1}{2}(D+\iota)$ is greater than $\mu\sin\frac{1}{2}\iota$.

But when $\phi=\psi$, $\sin\frac{1}{2}(D+\iota) = \mu\sin\frac{1}{2}\iota$.

Hence when $\phi=\psi$, D is a unique minimum.

EXAMPLES.

1. Rays are incident at a given point of a prism so as to be refracted in a plane perpendicular to its edge. If ι be the angle of the prism and a the critical angle, show that the ray will pass through the prism if the angle of incidence be such that

$$\sin\phi \sin a > \sin(\iota - a).$$

2. If the angle of a prism be 60°, and the refractive index $\sqrt{\frac{7}{3}}$, find the limits between which ϕ must lie in order that the ray may be able to emerge at the second face.

Ans. 30° and 60°.

3. If the angle of a prism be 60°, and the refractive index $\sqrt{2}$, show that the minimum deviation is 30°.

4. The minimum deviation for a prism is 90°. Show that the least value possible for the refractive index is $\sqrt{2}$.

5. Prove that in prisms of the same material, as the refracting angle increases, the minimum deviation also increases.

6. The refractive indices of three rays with respect to a given prism are μ_1, μ_2, μ_3; show that if D_1, D_2, D_3 their minimum deviations through it are in Arithmetical Progression, then

$$\frac{\sin\frac{1}{2}D_2}{\mu_2} = \frac{\sin\frac{1}{2}D_1 + \sin\frac{1}{2}D_3}{\mu_1 + \mu_3}.$$

7. Two prisms of the same vertical angle but of different refractive indices are placed in contact with their edges parallel and their angles turned opposite ways; prove that the deviation due to the system of a ray which is incident perpendicularly on the first surface of the system increases with the angle of the prisms.

8. If the minimum deviation for rays incident on a prism be a, the refractive index cannot be less than $\sec\frac{1}{2}a$, and the angle of the prism cannot be greater than $\pi - a$.

25. When the refracting angle of the prism is small, then the deviation will be small. By equations (2) and (3),

$$\psi' = \iota - \phi',$$

$$\psi = \iota + D - \phi.$$

Hence, $\sin(\iota + D - \phi) = \mu \sin(\iota - \phi')$.

But, since ι and D are small, their sines may be very nearly represented by their circular measures, and their cosines do not differ sensibly from unity, and we get

$$(\iota + D)\cos\phi - \sin\phi = \mu\iota\cos\phi' - \mu\sin\phi';$$

therefore $$D\cos\phi = \iota\,\{\mu\cos\phi' - \cos\phi\},$$

or $$D = \iota\left\{\frac{\mu\cos\phi'}{\cos\phi} - 1\right\}.$$

If the ray passes nearly perpendicularly through the prism ϕ and ϕ' will both be small, and if we exclude the third order of small quantities, the value of the deviation becomes

$$D = (\mu - 1)\,\iota,$$

which, to this approximation, is *independent of the angle of incidence.*

26. We shall next suppose that the ray does not lie in a principal plane of the prism.

Let the same notation as before be applied to the projections of the path of the light on a principal plane. Also let η, η' be the inclinations of the incident and refracted rays to the principal plane at the first refraction, ξ, ξ' the inclinations of the refracted and incident rays to the same plane, at the second refraction, respectively. Then by § 20

$$\left.\begin{aligned}\sin\eta &= \mu\sin\eta'\\ \sin\xi &= \mu\sin\xi'\end{aligned}\right\}.$$

Also, ξ' and η' denote the inclination of the same ray to the same plane, and therefore $\xi' = \eta'$ and $\xi = \eta$.

This proves that *the incident and emergent rays are equally inclined to the principal plane, or to the refracting edge of the prism.*

Further, there are the equations of refraction

$$\left.\begin{aligned}\sin\phi\cos\eta &= \mu\sin\phi'\cos\eta'\\ \sin\psi\cos\eta &= \mu\sin\psi'\cos\eta'\end{aligned}\right\},$$

and $$\phi' + \psi' = \iota.$$

These equations contain the whole theory of the refraction of a ray through a prism.

MISCELLANEOUS EXAMPLES ON CHAPTER II.

1. A hemispherical bowl whose inner surface is polished, is just filled with water; and a ray is incident on the surface of the water in a vertical plane, which passes through the centre of the sphere. After two internal reflexions it emerges, when it makes the same angle with the normal to the surface of the water as at incidence. Show that the point of incidence must lie on a ring concentric with the rim of the vessel and having its bounding radii respectively $\frac{1}{2}\sqrt{2}$ and $\sqrt{\frac{2}{7}}$ of the radius of the vessel, the index of refraction for water being taken to be $\frac{4}{3}$.

2. The concave side of an equiangular spiral being polished, prove that a ray of light once a tangent to the spiral will always be a tangent to the spiral, however often it may be reflected at the curve.

3. Rays proceeding from the vertex of a parabola are reflected, each one at the diameter where it meets the curve. Prove that the reflected rays all touch a parabola of eight times the dimensions of the given parabola.

4. A ray of light is reflected a number of times between two plane mirrors, not in a principal plane; prove that every segment of the ray reflected from one mirror intersects every segment reflected from the other mirror.

5. A prism, refractive index μ' and refracting angle 60°, is enclosed between two others of refractive indices μ and angle 60°, their edges being turned the opposite way to that of the first. Show that if a ray passes through without deviation, its course must be symmetrical, and that $3\mu^2=\mu'^2+\mu'+1$.

6. Two right-angled prisms each of refracting index λ, and a prism whose angle is 60° and refracting index μ, are placed so that each of the former touches one face of the latter, and the angle of the middle prism is turned in a direction opposite to that of the angles of the other two. A ray passes through the system in such a direction that its deviation by the middle prism is a minimum, and it emerges parallel to its incident direction; prove that

$$4\lambda^2=\mu^2+3.$$

7. A direct-vision spectroscope is composed of three prisms, two of which are exactly alike and are placed each with a face in contact with the faces of the third and their vertices turned towards its blunt end. Find equations for the angles of the prisms and their refractive indices in order that a ray refracted through the three prisms may be able to emerge parallel to its direction of incidence.

If the refractive indices of the two similar prisms and the third be $\sqrt{6}$ and $\sqrt{3}+1$, respectively, and the angle of the third prism be 120°, show that the angle of the two like prisms is $\tan^{-1}(6+3\sqrt{3})$.

8. If n equal and uniform prisms be placed on their ends with their edges outwards, symmetrically about a point on the table, find the angle of each prism in order that a ray refracted through each of them in a principal plane may describe a regular polygon. Show that the distance of the point of incidence of such a ray on each prism from the edge of the prism, bears to the distance of each edge from the common centre the ratio of

$$\sqrt{\mu^2 - 2\mu \cos\frac{\pi}{n} + 1} \ : \ \mu + 1.$$

9. A ray is refracted at one face of a triangular prism in the principal plane, and after being reflected at each of the other faces emerges through the first face; show that the whole deviation is greater or less than two right angles, according as the vertical angle of the prism is less or greater than a right angle. Show also that if the angle of emergence of the ray be equal to the angle of incidence, the deviation will be a minimum when the vertical angle is less, and a maximum when it is greater, than a right angle.

10. A ray enters a prism of quadrilateral section in a principal plane and after reflexion at three sides in order emerges from the one at which it entered, making the angle of emergence equal to that of incidence but on the opposite side of the normal. Show that the section of the prism by the principal plane can be inscribed in a circle.

11. Sunlight falls on a small isosceles prism standing on a horizontal table and emerges after reflexion at the base, the edge of the prism being inclined at any angle to the sun's rays. Show that the result is the same as if the sunlight had been simply reflected at the table.

12. There are two confocal reflecting ellipses; a ray proceeds from a point P of either of them in a direction passing through one of the foci and is continually reflected between the curves. If, after $2n-1$ reflexions, it returns to the point P, the length of the path is equal to n times the difference of the major axes.

13. A cylindrical pencil of light is incident on a refracting prolate spheroid in a direction parallel to the axis, the excentricity of the spheroid being e, and the refractive index μ; find the positions of the rays which emerge parallel to the axis, supposing $\mu > 1/e^2$, and show that none of the emergent rays will be parallel to the axis if $\mu < 1/e^2$.

14. Three plane mirrors are placed with their planes at right angles to one another. If a ray be reflected by all of them successively, its direction will be parallel to its direction at incidence.

15. A ray is reflected at three plane mirrors successively, so as to be parallel to its original directions after the reflexions, and the three directions which it takes are mutually at right angles to each other. Prove that the mirrors are mutually inclined at angles of 60°.

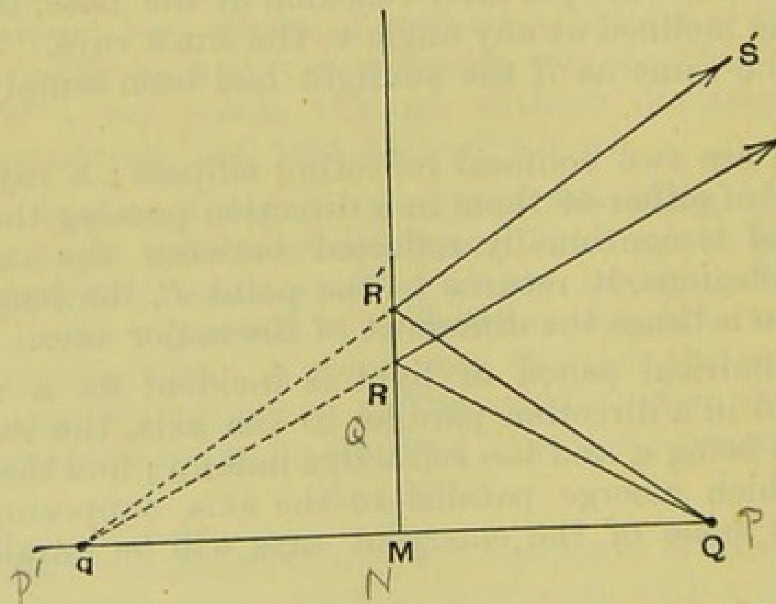
CHAPTER III.

Reflexion and Refraction of Direct Pencils.

27. Hitherto we have considered the reflexion and refraction of single rays only; we shall now consider the modifications produced in pencils of rays, by reflexion and refraction.

A pencil of rays is incident on a plane reflecting surface; to find the form of the pencil after reflexion.

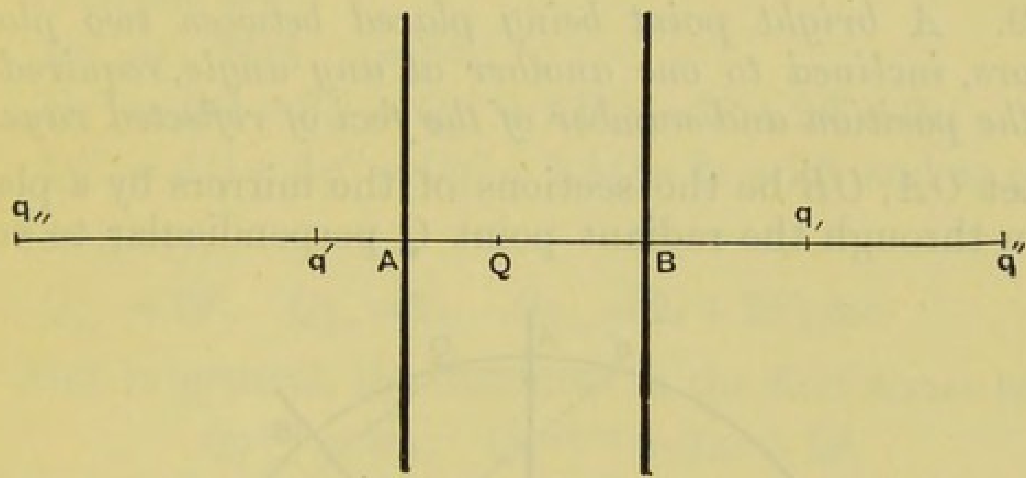
Let QR be any ray proceeding from a fixed point Q, and RS its course after reflexion at the mirror.



Draw QM perpendicular to the mirror and produce RS backwards to meet it in q; this can always be done, for the lines QM, QR and RS lie in one plane. Then by the law of reflexion it follows that the angle qRM is equal to the angle QRM, and therefore the triangles QRM, qRM are equal in all respects, so that $qM = QM$.

The position of q is independent of the particular ray chosen, and the pencil after reflexion diverges from q. Thus *the foci of the incident and reflected pencils lie on the same perpendicular to the mirror, at equal distances from it, on opposite sides.*

28. *A bright point being placed between two parallel plane mirrors, required to find the foci of the reflected rays.*



Let Q be the radiant point; through Q draw a line AQB perpendicular to both surfaces and produce it indefinitely both ways.

Then, taking Aq' equal to AQ, q' will be the focus of the rays from Q after reflexion at the first mirror. These reflected rays, diverging from q', will become incident on the second mirror. If, therefore, we take Bq'' equal to Bq', q'' will be the focus of the rays after a second reflexion, and so on. Again, the rays diverging from Q and incident on the second mirror will have a focus $q_{,}$, where $Bq_{,} = BQ$; the rays diverging from this focus and incident on the first mirror will have a focus $q_{,,}$, where $Aq_{,,} = Aq_{,}$, and so on. Thus there are an infinite number of foci all arranged on the line AB, and becoming more and more distant after each reflexion.

The distances Qq', Qq''... may be easily calculated. For, making $QA = a$, $QB = b$, and $AB = a + b = c$, we find

$$Qq' = 2AQ = 2a,$$
$$Qq'' = BQ + Bq' = Qq' + 2BQ = 2a + 2b = 2c,$$
$$Qq''' = AQ + Aq'' = Qq'' + 2AQ = 2c + 2a,$$
$$Qq'''' = BQ + Bq''' = Qq''' + 2BQ = 2a + 2b + 2c = 4c,$$

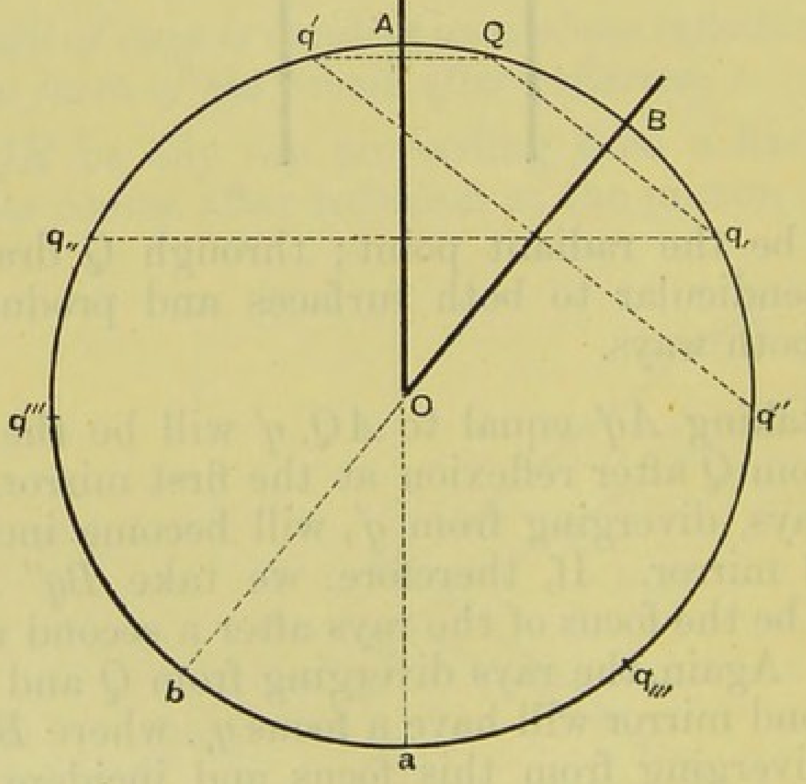
and so on.

In like manner we find

$$Qq_{,} = 2b,\ Qq_{,,} = 2c,\ Qq_{,,,} = 2c + 2b,\ Qq_{,,,,} = 4c, \text{ \&c.}$$

29. *A bright point being placed between two plane mirrors, inclined to one another at any angle, required to find the position and number of the foci of reflected rays.*

Let OA, OB be the sections of the mirrors by a plane drawn through the radiant point Q perpendicular to both



mirrors; and let a perpendicular be drawn from Q to the mirror OA and produced to q', so that Qq' may be bisected by the mirror; then q' will be the focus of the rays after one reflexion at OA. Again, letting fall a perpendicular from q' on OB and producing it to q'', at an equal distance on the other side, q'' will be the focus of the rays after a second reflexion, and so on. In like manner we find another series of foci $q_{,}, q_{,,}, q_{,,,}\ldots$ if we first take the rays which are incident on the mirror OB.

Now, by the construction, it easily follows that $Oq' = OQ$, and in the same manner we find that $Oq'' = Oq' = OQ$. Therefore all the foci lie on the circumference of a circle whose centre is O, and radius OQ.

To determine the positions of the foci, let the arc $QA = \theta$, $QB = \theta'$ and $AB = \theta + \theta' = \iota$.

Then the arc

$$Qq' = 2QA = 2\theta,$$
$$Qq'' = BQ + Bq' = Qq' + 2BQ = 2\theta + 2\theta' = 2\iota,$$
$$Qq''' = AQ + Aq'' = Qq'' + 2AQ = 2\iota + 2\theta, \text{ and so on.}$$

Similarly,

$$Qq_{,} = 2\theta', \quad Qq_{,,} = 2\iota, \quad Qq_{,,,} = 2\iota + 2\theta', \text{ \&c.}$$

And, in general, the distances in the first series are

$$Qq^{(2n)} = 2n\iota, \quad Qq^{(2n+1)} = 2n\iota + 2\theta,$$

and in the second series,

$$Qq_{(2n)} = 2n\iota, \quad Qq_{(2n+1)} = 2n\iota + 2\theta'.$$

The number of images is limited; for when any one of the images falls on the arc ab, between the mirrors produced, it lies behind both mirrors, and therefore no further reflexion takes place. If the image $q^{(2n)}$ be the first to fall on the arc ab, then, since this is one of the images which lie behind the second mirror, we must have the arc $Qq^{(2n)} > QBa$; that is, $2n\iota > \pi - \theta$, or

$$2n > \frac{\pi - \theta}{\iota}.$$

If the first image which falls on the arc ab be one of those behind the first mirror, say $Qq^{(2n+1)}$, we must have

$$Qq^{(2n+1)} > QAb;$$

that is, $2n\iota + 2\theta > \pi - \theta'$, or $2n\iota + \theta + \theta' > \pi - \theta$,

or finally, $$2n + 1 > \frac{\pi - \theta}{\iota}.$$

This is the same result as before, $2n$ being the number of images in the first case, $2n+1$ in the second. Therefore the whole number of images in the first series is the integer next greater than $(\pi-\theta)/\iota$; and, in like manner, the number of images in the second series may be shown to be the integer next greater than $(\pi-\theta')/\iota$.

If ι be a submultiple of two right angles, π/ι will be a whole number, and the number of images in each series will be π/ι, since θ/ι and θ'/ι are proper fractions; so that the total number of images will be $2\pi/\iota$. But in this case it happens that two of the images of the different series coincide. For if π/ι be an even integer, say $2n$, then

$$Qq^{(2n)} + Qq_{(2n)} = 2n\iota + 2n\iota = 2\pi,$$

and therefore the images $q^{(2n)}$, $q_{(2n)}$ coincide. And if π/ι be an odd integer, say $2n+1$,

$$Qq^{(2n+1)} + Qq_{(2n+1)} = 4n\iota + 2(\theta+\theta') = (4n+2)\iota = 2\pi,$$

and the images $q^{(2n+1)}$, $q_{(2n+1)}$ coincide.

If therefore we include the radiant point in the number, *the total number of foci is* $2\pi/\iota$.

This theory contains the principle of the *kaleidoscope.*

30. A kaleidoscope is made in the form of a cylinder, with slips of mirror inside, arranged so that they form two faces of the equilateral prism inscribed in the cylinder. In the centre of one end of the cylinder is an aperture for the eye, and at the other end are bits of coloured glass. The reflecting surfaces of the mirrors are inwards and they give six images symmetrically arranged of any bit of glass lying in the space between them. The kaleidoscope often contains three strips of mirror forming an equilateral prism inscribed in the cylinder, the reflecting surfaces being turned inwards. There will then be a symmetrical pattern arranged about each edge of the prism.

Ex. Two mirrors are inclined at an angle of 50°, show that there will be 7 or 8 images of a bright point between them according as its angular distance from the nearer mirror is or is not less than 20°.

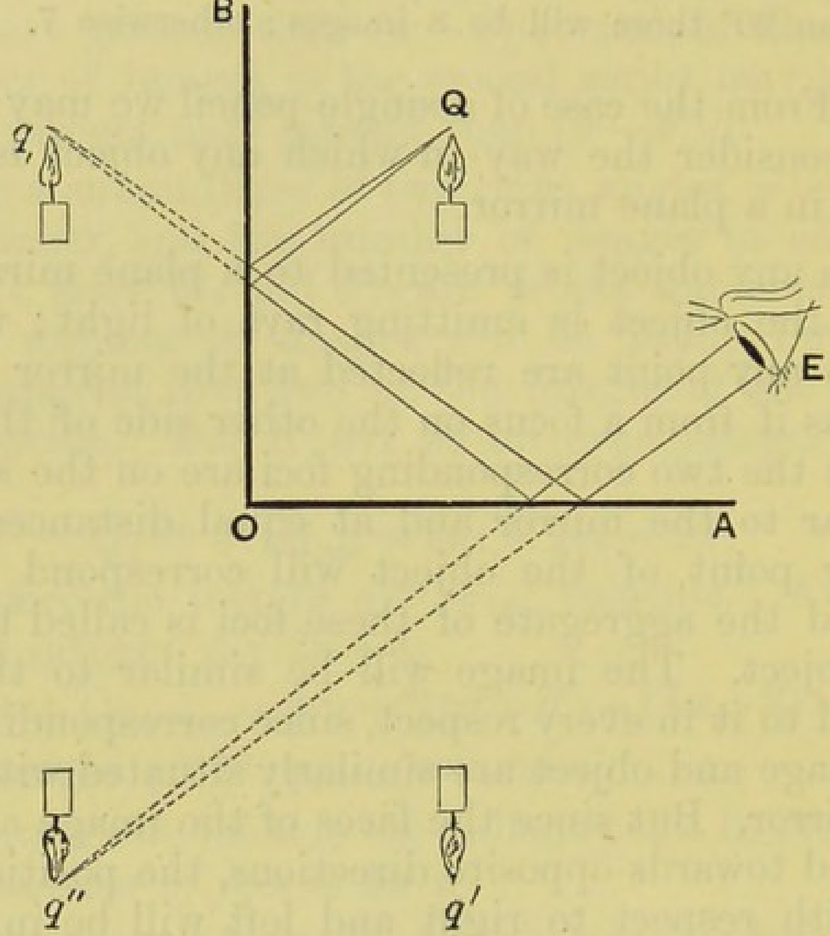
Let x, x' be the number of degrees in the angular distances of the bright point from the mirrors, respectively. Then as x assumes all values from 0 to 50°, the fraction $(180-x)/50$ takes all values from $3\frac{3}{5}$ to $2\frac{3}{5}$, and when $x=30°$ the fraction is equal to 3. Now let x be the angular distance from the nearer mirror. Then the integer next greater than $(180-x)/50$ is always 4, and the integer next greater than $(180-x')/50$ is 4 or 3 according as x' is less or greater than 30°, that is according as x is greater or less than 20°. Hence if the angular distance of the point from the nearer mirror be greater than 20° there will be 8 images; otherwise 7.

31. From the case of a single pencil we may now proceed to consider the way in which any object is seen by reflexion in a plane mirror.

When any object is presented to a plane mirror, every point of the object is emitting rays of light; when the rays from any point are reflected at the mirror they will proceed as if from a focus on the other side of the mirror, such that the two corresponding foci are on the same perpendicular to the mirror and at equal distances from it. To every point of the object will correspond one such focus, and the aggregate of these foci is called the *image* of the object. The image will be similar to the object and equal to it in every respect, since corresponding points of the image and object are similarly situated with respect to the mirror. But since the faces of the image and object are turned towards opposite directions, the position of the object with respect to right and left will be inverted in the image. If the eye be placed so as to receive reflected rays, they will produce the same impression as if they were radiating from a real object behind the mirror in the position occupied by the image. We may trace the rays by which the eye sees any point of the object, by drawing a pencil of lines bounded by the pupil of the eye, towards the corresponding point of the image as far as the mirror, and then joining the points of the section of the small pencil by the mirror, to the point of the object.

In illustration of the preceding article, the accompanying figure shows the manner in which an eye receives rays after reflexion at each of two mirrors OA, OB at right angles to each other. By § 29 it appears that there will be three images of any object Q, which with the object will form a symmetrical arrangement of images. To trace the rays by which the eye sees the image q'', we draw the pencil of rays from q'' bounded by the pupil of the eye; only the part of the rays between the eye and the first mirror are drawn continuously, the rest dotted. We next draw the pencil from

q, bounded by the section of the former pencil made by the mirror OA; the rays of this pencil are only drawn continuously between



the two mirrors, and the rest dotted. Finally, we draw the pencil from Q to the points of the section of the last pencil made by the mirror OB. Rays issuing from Q are reflected first at the mirror OB, then again at the mirror OA, and finally enter the eye in the same direction as if they proceed from an object situated at q''.

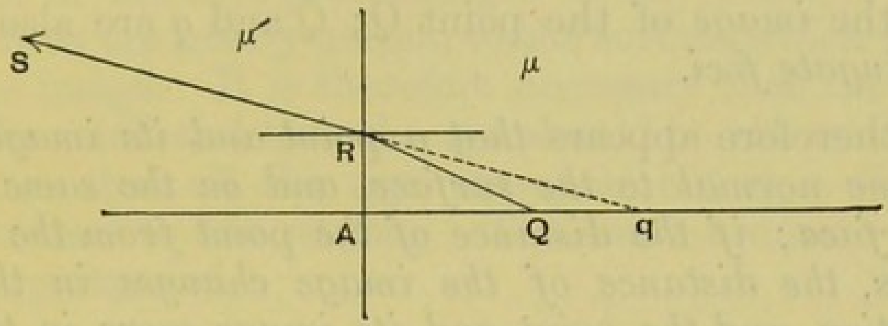
32. The pencils we shall now consider will be very slightly divergent, or in other words, the solid angles of the pencils will be very small.

When the axis of the pencil coincides with the normal to the surface on which it is incident, the incidence is said to be *direct;* in other cases the incidence is *oblique.*

In general, the rays of the pencil after refraction or reflexion do not accurately pass through a point; but there are many useful cases, where the incidence is direct, in which the rays very approximately meet in a point. We shall now consider a few of these cases.

A small pencil of light is incident directly on a plane refracting surface; to find the form of the pencil after refraction.

Let the pencil diverge from a point Q, the axis of the pencil, QA, being normal to the plane refracting surface.



Let QRS be the path of any ray of light, and let RS produced backwards meet the axis in q. Then the angle AQR is equal to the angle of incidence of the ray, and the angle AqR, equal to the angle of refraction. But if μ, μ' be the refractive indices of the two media, the law of refraction, expressed in the usual notation, is

$$\mu \sin \phi = \mu' \sin \phi'.$$

This may be written,

$$\frac{\mu AR}{RQ} = \frac{\mu' AR}{Rq},$$

or

$$\frac{\mu}{RQ} = \frac{\mu'}{Rq}.$$

When we consider different rays of the pencil the position of the point R will vary, and therefore the position of q will vary. But if we suppose the pencil very small, then AR will be very small and RQ will be very nearly equal to AQ, and Rq to Aq, and therefore

$$\frac{AQ}{\mu} = \frac{Aq}{\mu'},$$

or as it is more usually expressed

$$\frac{u}{\mu} = \frac{u'}{\mu'},$$

where u and u' denote the lengths AQ and Aq, respectively. To this approximation, the point q is fixed; that is, its position does not depend upon the particular ray chosen; so that all the refracted rays produced backwards cut the axis AQ in the same point q. The point q is therefore the focus of the refracted pencil. It is sometimes called the *image* of the point Q; Q and q are also said to be *conjugate foci.*

It therefore appears *that a point and its image lie on the same normal to the surface, and on the same side of the surface; if the distance of the point from the surface changes, the distance of the image changes in the same proportion, and the point and its image move in the same direction.*

33. From the case of a single pencil we may deduce the manner and position in which an eye sees an object situated in a medium whose refractive index is different from air, as for instance an object under water. Every point of the object under water is emitting rays of light; when the rays from any point emerge in air, they will proceed from the focus conjugate to the given point. Assuming the refractive index from air to water to be $\frac{4}{3}$, the focus conjugate to a given point will lie on the same normal to the surface at $\frac{3}{4}$ of the depth. To every point of the object there will be such a corresponding focus, and the aggregate of these foci is called the image of the object. To an eye in air, the emergent rays will produce the same impression as if they proceeded from a real object occupying the position of the image. The rays by which the eye sees any point may be traced by drawing lines bounded by the pupil of the eye towards the corresponding point of the image, as far as the refracting surface, and then joining the points of the section of the small pencil by this surface, to the point of the object.

The forms of the images corresponding to different forms of object may be deduced by geometry from the preceding construction. Thus it is clear that the image of a plane, will be another plane, the two planes meeting the refracting surface in the same line at different inclinations; to a sphere, will correspond an ellipsoid of revolution whose axis of revolution is normal to the surface, and so on.

This representation of the image is only approximately true; for of the rays proceeding from any point, it is only those which are nearly normal to the surface which emerge from the image. It is therefore necessary that the object should be small and that the eye should be almost directly over it, so that all the rays may pass out in a direction nearly normal to the surface. The more accurate theory must be postponed.

EXAMPLES.

1. Three plane mirrors are all perpendicular to a given plane. Show that if a bright point be placed anywhere on the circumference of the circle which is described round the triangle formed by the intersection of the mirrors with the given plane, the three images of the point formed by one reflexion at each mirror, respectively, will all lie in a straight line.

2. The locus of the image of a luminous point reflected in a plane mirror is a circle. Prove that the mirror always touches a conic section or passes through a fixed point.

3. Show that when an eye is placed to view any image formed by successive reflexions at two mirrors, the apparent distance of the image from the eye is equal to the distance actually travelled by the light in arriving to the eye from the original point of light.

4. A luminous point is placed at the centre of an equilateral triangle whose side is a; show that the distance from the luminous point to the image formed by $2n$ reflexions at the sides of the triangle in succession is na, and to that formed by $2n+1$ reflexions is $a\sqrt{(n^2+n+\frac{1}{3})}$.

5. A speck within a solid cube of glass is viewed directly through each face in succession; prove that the six images will form the angular points of an octahedron, not generally regular but having all its diagonals equal.

6. A luminous point moves about between two plane mirrors which are inclined at an angle of 27°. Prove that at any moment the number of images of the point is 13 or 14 according as the angular distance of the point from the nearer mirror is less or not less than 9°.

7. Two circular plane reflectors, the radii of which are a, b, are placed so that the line joining their centres is perpendicular to the plane of each, and a bright point is placed midway between the centres; prove that the greatest number of images visible to an eye looking over the edge of the larger reflector, is expressed by the greatest integer in $(a+b)/2(a-b)$.

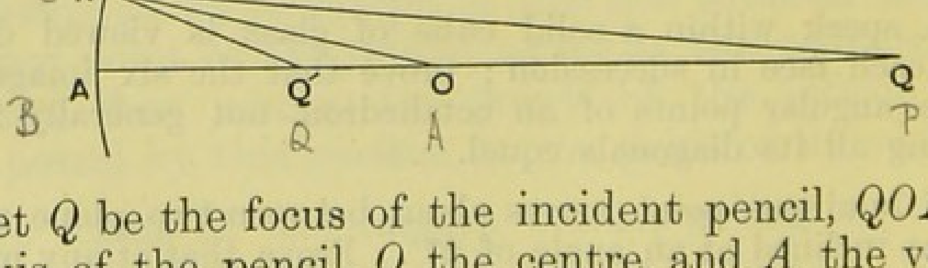
8. When the angle between a stick under water and its image is a maximum, the stick makes an angle whose tangent is $\sqrt{\mu}$ with the water, and the sum of the angles which the stick and its image make with the water is $\frac{1}{2}\pi$.

34. We shall next consider the case of a small pencil directly reflected or refracted at a spherical surface.

The most general case will not be a plane problem, but it may easily be derived from a plane problem. We shall first consider a pencil issuing from a bright point on the axis of the spherical surface; the whole system of rays will then be symmetrical about the axis, and we need only consider rays lying in one plane through the axis, and afterwards suppose the plane system to be revolved about the axis. It will be shown that after reflexion or refraction the pencil will approximately diverge from a focus also situated on the axis. If the bright point be not on the axis of the spherical surface, we join it to the centre of the surface; the joining line is normal to the surface and therefore may be considered as itself an axis, and the problem is the same as before.

35. *Direct reflexion of a small pencil at a spherical surface.*

We first treat this as a plane problem.



Let Q be the focus of the incident pencil, QOA being the axis of the pencil, O the centre, and A the vertex of the spherical reflecting surface. Let QR be any incident ray, which after reflexion cuts the axis in Q'.

Let $AQ = x$, $AQ' = x'$, and let the radius of the circle be r.

Then by the law of reflexion, the normal RO bisects the angle between RQ and RQ', and therefore, by Euclid VI. 3,

$$RQ : RQ' = QO : OQ'.$$

Now if the pencil be small, AR will be small, and RQ will not differ much from AQ nor RQ' from AQ', and therefore $x : x' = x - r : r - x'$,

that is $x(r - x') = x'(x - r)$;

which may be written,

$$\frac{1}{x} + \frac{1}{x'} = \frac{2}{r} \quad \text{.......................(1).}$$

To this degree of approximation we may say that all rays passing through Q will after reflexion pass through Q', and *vice versa*. The points Q, Q' will be called conjugate foci; either of them may be taken to be the image of the other.

36. The formula we have proved connecting the distances of a pair of conjugate points from the surface includes all cases that may arise. If, for instance, the reflecting surface is convex, so that AO is measured in the opposite direction, we must change the sign of r in the formula. All distances measured to the right of A are considered positive, those measured to the left, negative.

If the incident rays are parallel to the axis, so that x is infinite and positive, or infinite and negative, the corresponding value of x' in each case is

$$x' = \tfrac{1}{2}r = f, \text{ say.}$$

Hence if F be the middle point of AO, F is the focus for parallel rays proceeding in either direction. It is called the *principal focus* of the mirror.

The formula (1) may be written

$$xx' - xf - x'f = 0,$$

or

$$(x - f)(x' - f) = f^2.$$

Let u, u' be the distances of a pair of conjugate points measured from the principal focus in the same direction, so that

$$\left.\begin{aligned} u &= x - f \\ u' &= x' - f \end{aligned}\right\};$$

then $$uu' = f^2.$$

From this it appears that the whole theory of reflexion at a spherical surface, whether the surface be convex towards the incident light or concave, and whether the incident pencil be convergent or divergent, may be very briefly stated by the aid of the principal focus. For let F be the principal focus, which is the point midway between the vertex of the mirror and its centre; then *a pair of conjugate foci always lie on the same side of F, and at distances, u and u' from it, such that*

$$uu' = f^2, \text{ where } f = \tfrac{1}{2}r.$$

If now the system be revolved about the axis QOA we shall have considered all the rays issuing from the point Q.

37. Next let the axis $QOQ'A$ be turned about O in all planes through a small angle, the points on it being fixed. Since the line is still normal to the surface, the points Q, Q' will still be conjugate foci. All the fixed points on the line will describe small elements of spheres whose common centre is at O. To the approximation to which we are limiting ourselves, these small elements may be taken to be plane, all these plane elements being at right angles to AO. Then, corresponding to a small plane object at Q, we shall have a plane image at Q'; the image and object will be similar and similarly situated, the lines joining corresponding points always passing through the centre of the mirror. The principal focus F will also describe a small element of a plane. This plane is called the *principal focal* plane. All pencils of parallel rays inclined at small angles to the axis of the mirror will have for foci points on the principal focal plane.

38. Let η, η' represent the linear dimensions of the object and its image, then since corresponding points lie on the same line through the centre of the sphere,

$$\frac{\eta}{\eta'} = -\frac{QO}{OQ'}$$

$$= -\frac{AQ}{AQ'}.$$

That is, $$\frac{\eta}{\eta'} = -\frac{x}{x'},$$

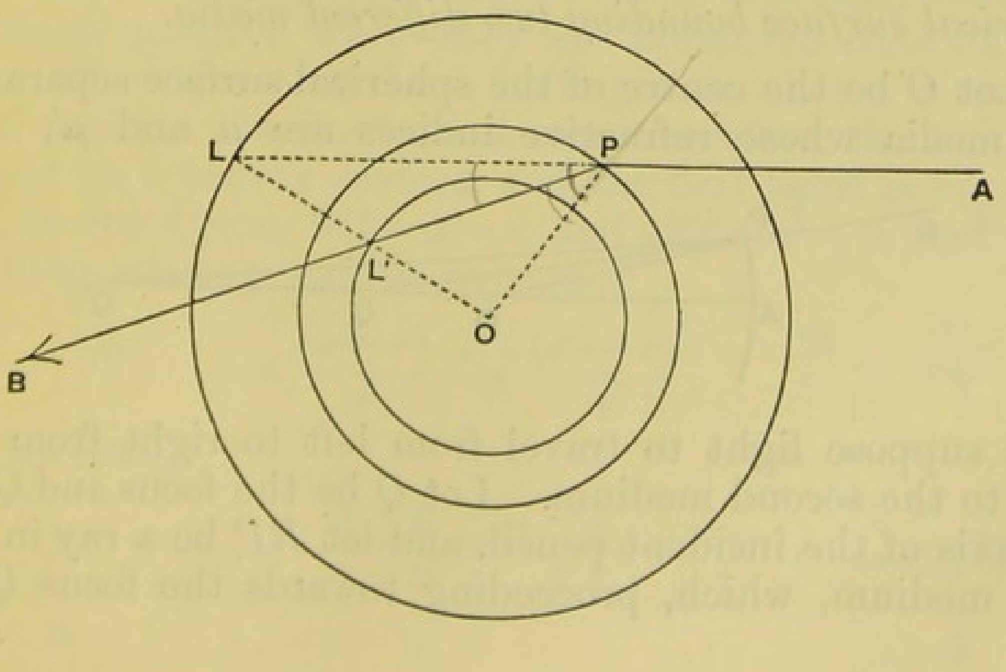
or $$\frac{\eta}{x} + \frac{\eta'}{x'} = 0.$$

It will be noticed that *if the conjugate foci lie on the side of the principal focus on which the centre of the surface lies, the image will be inverted, otherwise erect.*

Further details and constructions for conjugate foci will be given in the case of the direct refraction of a small pencil at a spherical surface; these will all be applicable to the present case, with the usual modifications.

39. *To find the path of a ray after refraction at any spherical surface.*

Let AP be any ray incident on a sphere of radius r, and centre O. To find the path of the refracted ray.



Let the original medium in which the ray moves have a refractive index μ, and the sphere an index μ'. With centre O describe two other spherical surfaces whose radii are respectively $r\mu'/\mu$ and $r\mu/\mu'$. Produce the incident ray to meet the first sphere in L; join OL cutting the second sphere in L', then PL' is the refracted ray.

For, by construction $OL \,.\, OL' = OP^2$; hence OP is the tangent at P to the circle circumscribing the triangle LPL', and therefore the angle OPL' is equal to the angle OLP, in the alternate segment. But from the triangle OLP

$$\sin LPO : \sin OLP = \mu' : \mu,$$

that is, $$\sin LPO : \sin L'PO = \mu' : \mu,$$

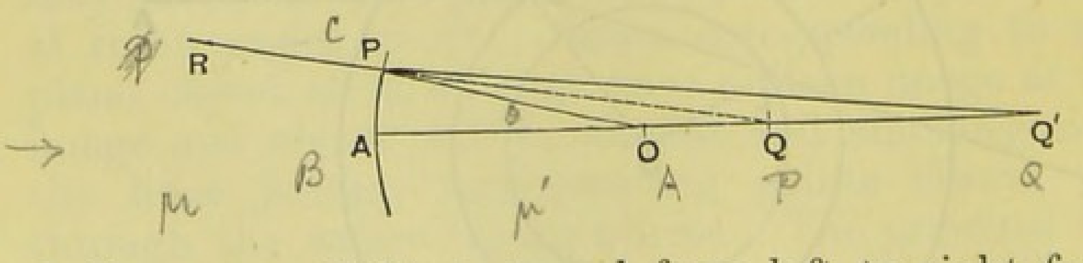
or $$\mu \sin LPO = \mu' \sin L'PO.$$

Hence PL' is the refracted ray.

From this construction it appears that all rays passing through the point L are refracted so as accurately to pass through L'. Thus two points on any radius of the sphere whose distances from the centre are $r\mu'/\mu$ and $r\mu/\mu'$ respectively, form a pair of conjugate points, *for all angles of incidence.* This property is of importance in the construction of strong objectives for microscopes, as was first pointed out by Amici.

40. *Direct refraction of a small pencil through a spherical surface bounding two different media.*

Let O be the centre of the spherical surface separating two media whose refractive indices are μ and μ'. We



shall suppose light to travel from left to right from the first to the second medium. Let Q be the focus and QOA the axis of the incident pencil, and let RP be a ray in the first medium, which, proceeding towards the focus Q, is refracted into the second medium along the line PQ'. Draw the normal to the surface at P and let ϕ, ϕ' be the angles of incidence and refraction, respectively; then

$$\mu \sin \phi = \mu' \sin \phi'.$$

Let the angle AOP be denoted by θ; then from the triangles OPQ and OPQ', we obtain the relations

$$\frac{\sin \phi}{\sin \theta} = \frac{OQ}{QP},$$

and

$$\frac{\sin \phi'}{\sin \theta} = \frac{OQ'}{Q'P};$$

and therefore

$$\mu \frac{OQ}{QP} = \mu' \frac{OQ'}{Q'P}.$$

We shall suppose that the inclinations to the axis, of the rays we are considering, are small; then we can write approximately QA for QP, and $Q'A$ for $Q'P$, so that all rays in the plane through the axis, diverging from Q, will after refraction pass through the point Q', determined by the relation

$$\mu \frac{OQ}{QA} = \mu' \frac{OQ'}{Q'A} \quad \text{.....................(1).}$$

The relation between the points Q and Q' is reciprocal; they are called *conjugate foci.*

Let $AQ = x$, $AQ' = x'$, and $AO = r$, x, x' and r being considered positive when they are measured from left to right. The figure has been arranged so that x, x' and r are all positive; in the typical case, therefore, the light is incident on a convex surface, and the first focus is virtual. Then, in the figure, $OQ = x - r$, and $OQ' = x' - r$, and the preceding relation may be written

$$\mu \frac{(x - r)}{x} = \mu' \frac{(x' - r)}{x'},$$

and therefore,

$$\frac{\mu}{x} - \frac{\mu'}{x'} = \frac{\mu - \mu'}{r} \quad \text{.....................(2).}$$

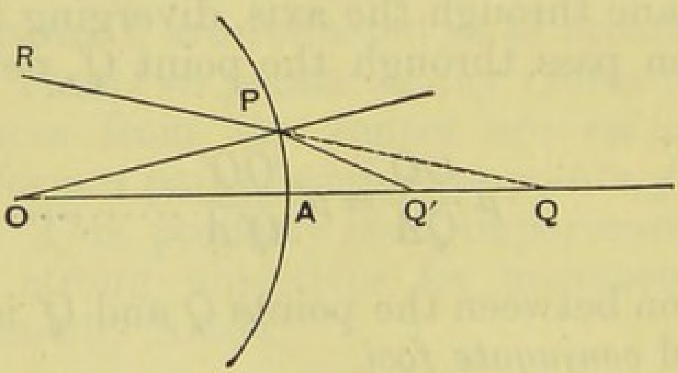
From this equation it appears that x and x' will increase together or decrease together, so that *a pair of conjugate foci always move in the same direction.*

41. We might have taken the centre O for origin. Thus let $OQ = p$, $OQ' = p'$, $OA = r$, all these distances being positive when measured to the right. The figure is drawn so as to make p, p' and r positive. In the typical case, therefore, light is incident on a concave surface, and the first focus is virtual. Then, writing the relation (1) in the form

$$\mu' \frac{QA}{OQ} = \mu \frac{Q'A}{OQ'},$$

it becomes
$$\mu' \frac{p - r}{p} = \mu \frac{p' - r}{p'}$$

or
$$\frac{\mu'}{p} - \frac{\mu}{p'} = \frac{\mu' - \mu}{r}.$$



42. It is often more convenient to choose a different convention with regard to sign. Suppose, as usual, that light is travelling from left to right, and let $QA = x$,

$Q'A = x'$, x being considered positive when Q lies in front of A, and x' being positive when Q' lies behind A. Then changing the sign of x the equation (2) becomes

$$\frac{\mu}{x} + \frac{\mu'}{x'} = \frac{\mu' - \mu}{r}.$$

The *focal points* are defined to be the points conjugate to the points at infinity in the two media, and play an important part in the theory of other conjugate points.

First, let us suppose that x' is infinite, so that the rays are parallel after refraction into the second medium; the conjugate point F is then determined by the equation

$$x = \frac{\mu r}{\mu' - \mu} = f, \text{ say.}$$

Similarly, if the rays are parallel to the axis in the first medium, they will, in the second medium, converge to a point F', such that

$$x' = \frac{\mu' r}{\mu' - \mu} = f', \text{ say.}$$

These points are the *focal points* of the surface, and f, f' are called its *principal focal lengths.*

The relation between x and x' may now be written in the form

$$\frac{f}{x} + \frac{f'}{x'} = 1 \quad\quad\quad (3).$$

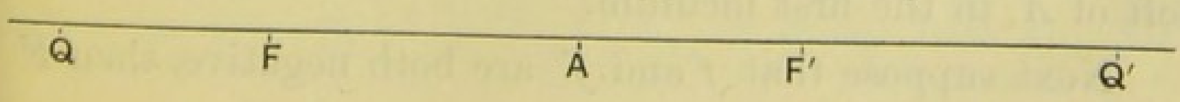
The course of a ray of light is always reversible, so that this formula includes the case in which rays are incident on a concave spherical surface.

The focal lengths are always both positive, or both negative, and therefore the product ff' is always positive. They are both positive if the medium in which lies the centre of the bounding surface is more highly refractive than the other, and both negative if this medium is less refractive than the other. In all cases one of the principal foci lies in each medium, and $f : f' = \mu : \mu'$.

43. The relation between the abscissæ of a pair of conjugate foci, when referred to the focal points as origins, takes a very simple and convenient form.

For let A be the vertex of the spherical surface separating the two media, F, F' the focal points of the surface, Q the focus of the rays in the first medium, Q' the focus of the rays in the second medium.

Let $QF = u$, $Q'F' = u'$, u being considered positive when



Q is in front of F, and u' being considered positive when Q' is behind F'. Then the relation between x, x' may be written in the form

$$(x-f)(x'-f')=ff';$$

that is, $$uu'=ff' \quad \text{..............(4).}$$

44. The relative positions of conjugate foci may be easily traced by means of the equation (4).

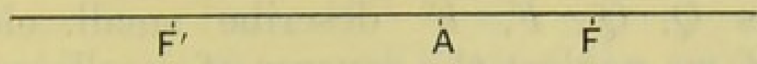
Since f and f' are of the same sign, it follows that u, u' will have the same sign. Two cases will have to be considered. First, suppose that the medium which is bounded by the convex surface, that is, the medium in which lies the centre of the spherical surface, is the more highly refractive, so that f and f' are both positive.

When Q is at infinity on the left, Q' will be at F'. Now Q and Q' always move in the same direction, so that as Q moves from infinity, Q' recedes beyond F', until Q reaches F, and then Q' is at infinity on the right. As Q passes F, u becomes negative and is at first very small, so that u' is negative and at first very large; thus Q' emerges from infinity on the left and follows Q as it moves along. When Q is at A, Q' coincides with it. When Q passes A, Q' follows it, but moves more slowly than Q, and when Q reaches infinity on the right, Q' coincides with F'.

So long as Q lies in the first medium, the rays in the first medium form a pencil diverging from Q; but when Q passes beyond A into the second medium, the rays in the first medium form a pencil which if produced would converge to Q, but they are intercepted by the refracting surface and never actually pass through Q. In this case, Q is a virtual focus. All these remarks apply also to the point Q'; it is a real focus only when it lies in the second medium, and a virtual focus when it lies to the left of A, in the first medium.

Next suppose that f and f' are both negative, then F

lies to the right of A, in the second medium; and F' lies to the left of A, in the first medium.

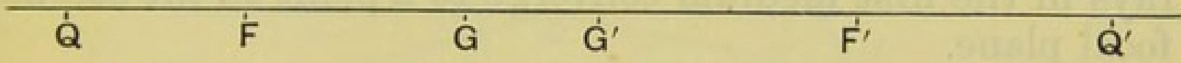


If we suppose Q to begin at infinity on the right, and to move backwards through all positions to infinity on the left, the motion of Q' may be described in nearly the same words as before. Q will be a real focus only when it lies to the left of A, and in other cases it will be a virtual focus; similarly Q' will be a real focus when it lies to the right of A, and in other cases a virtual focus.

45. A formula similar to (3), may be found for the positions of a pair of conjugate foci, if we take as origins, *any* fixed pair of conjugate foci. Let G, G' be any given pair of conjugate foci, whose distances from the focal points, *measured inwards*, are respectively g, g', so that

$$gg' = ff'.$$

Let the distances of Q, Q' from G, G' be respectively



v, v', the signs being fixed by the same convention as before. Then, with the previous notation,

$$\left.\begin{aligned} u &= v - g \\ u' &= v' - g' \end{aligned}\right\},$$

and the relation between the abscissæ of conjugate foci is

$$(v-g)(v'-g') = ff' = gg';$$

which may be written in the form

$$\frac{g}{v} + \frac{g'}{v'} = 1.$$

46. We shall next consider points not on the axis of the surface.

Let the line OA turn about O through a small angle, carrying the points Q, Q', F, F', A along with it. The new line OA will still be normal to the surface, and Q, Q'

will still be conjugate foci. If OA be turned through all positions making a small angle with its initial direction, the points Q, Q', F, F' describe small elements of spheres; if we neglect the squares of small quantities as before, these may be regarded as planes, cutting the axis at right angles. The planes at Q, Q' perpendicular to the axis may be called conjugate planes. Rays diverging from a point on one of the planes, after refraction at the spherical surface will converge to a point on the other plane.

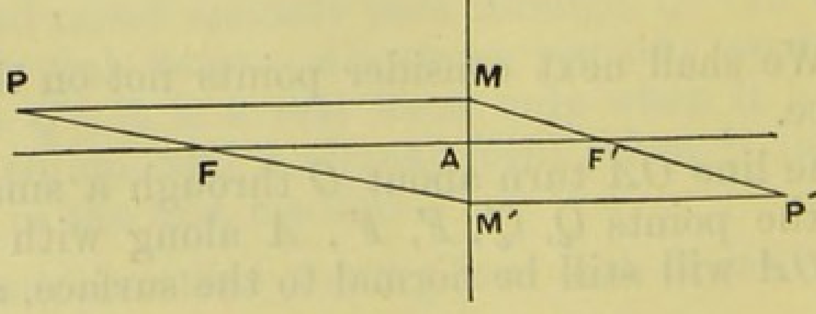
Corresponding to a small object in a plane perpendicular to the axis there will be a similar image in the conjugate plane, such that the lines joining the corresponding points of the object and its image all pass through the centre of the refracting surface.

The planes at F, F' perpendicular to the axis are called *focal planes.* Rays diverging from any point on the first focal plane will be parallel after refraction into the second medium, and conversely, any system of parallel rays in the first medium converge to a point on the second focal plane.

We have already regarded the spherical surface as approximately plane near A, and shall continue to do so. We may call it the *principal plane* of the surface.

47. We can now give simple geometrical constructions for determining the focus conjugate to a given point, and for drawing the emergent ray when the incident ray is given.

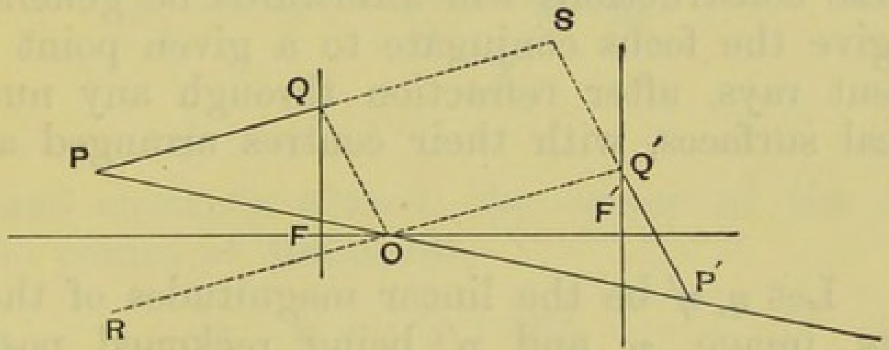
Let P be a point whose conjugate focus is required. If we can trace any two rays through P they will meet in



the required point. For one of the rays choose PM, parallel to the axis of the surface, meeting the principal plane at A in M. Then MF' is the corresponding emergent ray. Also, let the ray PF meet the principal plane in M'; this ray will emerge parallel to the axis, so that if we draw $M'P'$ parallel to the axis it will meet MF' in the required point.

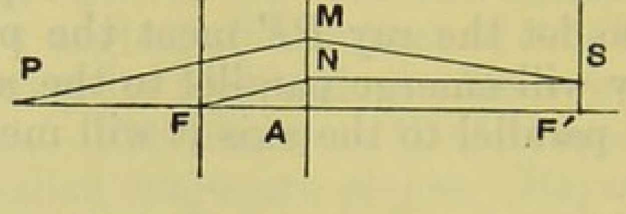
For either of these two rays we might have substituted the ray PO, which passes into the second medium without deviation.

48. Let P be a point, PQ any ray through it meeting the first focal plane in Q; the conjugate focus of P, and the emergent ray may also be constructed in the following way.



Draw the ray PO; it will pass through into the second medium without deviation, and the conjugate focus will lie on this line. Again let ROQ' be drawn through O, parallel to the ray PQ, meeting the second focal plane in Q'. Join OQ, and from Q' draw $Q'P'$ parallel to OQ. This is the emergent ray corresponding to the incident ray PQ, and will meet PO produced in the required point. For the rays RO and PQ are initially parallel, and therefore will meet on the second focal plane; and therefore the ray PQ after refraction will pass through Q'. Also PQ and QO are two rays proceeding from Q, a point on a focal plane, and therefore they will emerge parallel to each other after refraction. But the ray QO passes through into the second medium without deviation; therefore the emergent ray $Q'P'$ is parallel to QO.

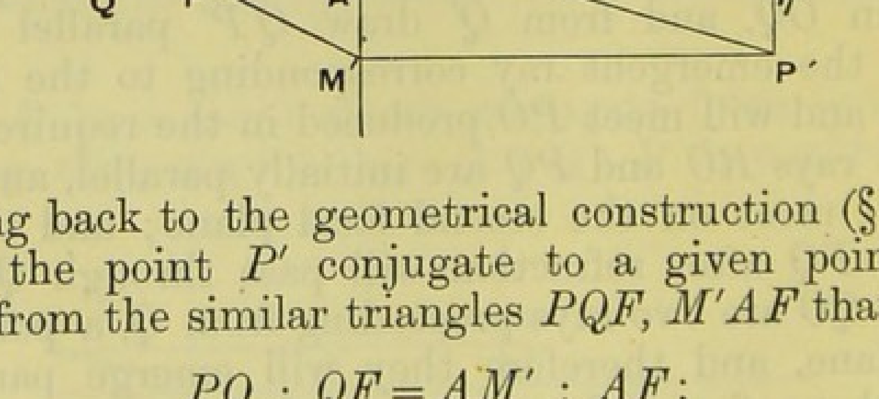
49. The emergent ray may be also constructed as follows:—



Let the incident ray meet the principal plane in M. Draw FN parallel to the incident ray to meet the principal plane in N; the ray FN will emerge along NS parallel to the axis, meeting the focal plane at F' in the point S. Then since PM and FN are parallel initially, they will converge to a point on the second focal plane, and therefore MS is the emergent ray.

These constructions will afterwards be generalised so as to give the focus conjugate to a given point and the emergent rays, after refraction through any number of spherical surfaces, with their centres arranged along an axis.

50. Let η, η' be the linear magnitudes of the object and its image, η and η' being reckoned positive or negative according as they are above or below the axis.



Referring back to the geometrical construction (§ 47) for finding the point P' conjugate to a given point P, it follows from the similar triangles PQF, $M'AF$ that

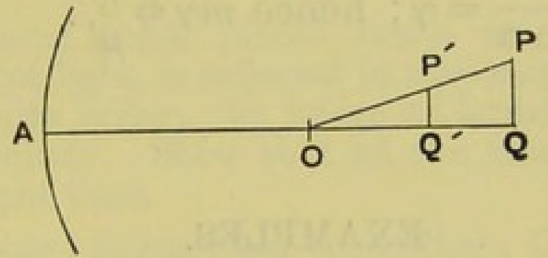
$$PQ : QF = AM' : AF;$$

that is, with the same notation as before, $\frac{\eta}{u} = -\frac{\eta'}{f}$,

$$\left.\begin{aligned} &\text{or} && \frac{\eta}{\eta'} = -\frac{u}{f} \\ &\text{and similarly,} && \frac{\eta'}{\eta} = -\frac{u'}{f'} \end{aligned}\right\}.$$

51. Helmholtz has shown how to find an expression for the ratio of the linear magnitudes of the object and its image in terms of the divergence of the rays before and after refraction, which is independent of the distance and focal lengths of the refracting surface.

Let PQ, $P'Q'$ be the object and image, which are



similar and similarly placed, the centre of the sphere being the centre of similitude.

Thus $$\eta : \eta' = OQ : OQ'.$$

But it follows immediately from the law of refraction, as was shown in § 40, that

$$\mu \frac{OQ}{QA} = \mu' \frac{OQ'}{Q'A};$$

and therefore, if we denote QA by x, and $Q'A$ by x',

$$\frac{\mu\eta}{x} = \frac{\mu'\eta'}{x'};$$

this is a very useful formula.

But if α be the angle of divergence of any ray through Q, and α' the angle of divergence of the corresponding ray through Q',

$$\tan\alpha : \tan\alpha' = \frac{1}{x} : \frac{1}{x'};$$

and therefore,

$$\mu\eta \tan\alpha = \mu'\eta' \tan\alpha',$$

which is Helmholtz's formula.

The ratio of the linear dimensions of the image to those of the object is called the *lateral magnification*, or simply the *magnification*, and will usually be denoted by m.

Similarly, the ratio of the angle of divergence of the final pencil to that of the original pencil is called the *angular magnification*, and will usually be denoted by γ. Thus $\eta'/\eta = m$, and since for the small angles α, α' the tangent does not sensibly differ from the circular measure, we may write $\dfrac{\tan\alpha'}{\tan\alpha} = \gamma$; hence $m\gamma = \dfrac{\mu}{\mu'}$.

EXAMPLES.

1. A luminous point is placed within a reflecting circle; prove that its distance from the centre is a harmonic mean between the distances, from the centre, of the geometrical foci after reflexion at the opposite portions of the surface.

2. Prove that the mercury in a cylindrical thermometer tube appears completely to fill the external surface of the tube, if the bore be $\frac{2}{3}$ths of the external diameter.

3. A pencil of rays is refracted directly through a hemisphere of glass, show that the position of the geometrical focus will be unaltered when the hemisphere is reversed, if

$$r^2 + (\mu + 1)\,pr - \mu\,(\mu - 1)\,p^2 = 0,$$

where p denotes the distance of the origin of light from the first refracting surface in each case, and r the radius of the hemisphere.

4. A ray is refracted through a sphere of radius r, its shortest distance from the centre of the sphere being r/n: show that if n be large the total deviation of the ray will be $2\,(\mu - 1)/n$.

5. A concave glass mirror bounded by two concentric spheres is silvered at the back; show that the displacement of an image, due to the thickness t of the glass, is approximately

$$2\frac{x^2}{r^2}\frac{\mu - 1}{\mu}t$$

towards the centre of curvature, where x is the distance of the image from that point, and r is the radius of curvature of the mirror.

6. Two small arcs of a circle at the extremities of a diameter are polished and a luminous point is placed in the diameter at a distance a from the centre. Show that the distance v of the focus from the centre after m reflexions is given by the equation

$$\frac{1}{v}=(-1)^m\left(\frac{1}{u}\pm\frac{2m}{r}\right),$$

the upper or lower sign being taken according as the first reflexion takes place at the nearer or further arc.

MISCELLANEOUS EXAMPLES ON CHAPTER III.

1. A luminous point, placed inside an equilateral triangle whose sides reflect light, is reflected in succession at the three sides taken in a definite order; the image so formed is again reflected, and so on indefinitely. Show that all the images so formed lie on one of two straight lines.

2. Two concave mirrors face each other; O, O' are their centres, and the distance AA' between the mirrors is greater than the sum of the radii. Prove that if Q, q be conjugate foci for each mirror, Qq will be the diameter of a circle which cuts orthogonally the two circles on AO, $A'O'$ as diameters.

3. If a pencil be reflected between two concave mirrors, radii ρ, σ, facing each other on the same axis at a distance a apart, show that there are two positions of the geometrical focus such that after any even number of reflexions the geometrical focus coincides with its first position, unless either both ρ and σ are greater than a, or both ρ and σ are less than a, and $\rho+\sigma>a$.

4. A pencil issuing from a point is incident upon a convex spherical refracting surface of index μ; show that the distance of the point from its conjugate focus will be a minimum, when the distance of the point from the surface is to the radius of the surface as $1 : 1+\sqrt{\mu}$.

5. A ray of light, traversing a homogeneous medium is incident upon a spherical cavity within it; supposing the limit of the magnitude of the deviation of the ray, produced by its passage through the cavity to be θ, show that the index of refraction of the medium is equal to $\sec\frac{1}{2}\theta$.

6. Rays converging to a point Q fall on a spherical surface whose centre is C; if, after one refraction, more than three rays in any plane through QC pass through the same point Q' on the axis QC, then will all the rays pass through the same point Q'.

7. Parallel rays fall on a sphere, and emerge after one internal reflexion; show that rays which are reflected at the same point of the surface are parallel after emergence; show also that, when the refractive index is greater than 2, no two rays will be reflected at the same point.

8. Find the geometrical focus after direct refraction through a hollow spherical shell bounded by two concentric spherical surfaces and filled with fluid of refractive index different from that of the shell.

9. Two spherical surfaces A, B have the same centre O; P is the geometrical focus of rays from a luminous point Q after reflexion first at the surface A and then at the surface B, and R is the geometrical focus after reflexion first at B and then at A; show that OP, OQ, OR are in harmonic progression.

10. A hemisphere of glass has its spherical surface silvered; light is incident from a luminous point Q, in the axis of figure produced, on the plane surface, show that if q is the geometrical focus of the emergent pencil, A the centre of the hemisphere, O its vertex and μ the refractive index for glass,

$$\frac{1}{Aq} - \frac{1}{AQ} = \frac{2\mu}{OA}.$$

11. A ball of glass contains a concentric spherical cavity; show that, provided the radius of the cavity do not exceed the radius of the ball divided by the index of refraction μ of the glass, it will appear to an eye at any distance from the ball to be μ times greater than it really is.

12. A sphere of a refracting substance whose index is $\sqrt{3}$ has a concentric spherical nucleus which is a reflector, whose radius is such that a ray which just enters the sphere grazes the surface of the nucleus. Prove that, if a ray, which is incident at an angle 60°, return to the point of incidence after internal reflexions, the path within the medium will be $\frac{4}{3}$ of what it would have been if there had been no nucleus.

13. Explain why, in looking down the axis of a smooth gun-barrel with an eye close to one end, a series of dark rings, images of the other end of the barrel, are seen on the surface, at distances from the eye equal to $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$... of the length of the barrel.

14. Two equal concave mirrors of radius r are placed exactly opposite one another at a distance a, supposed greater than $2r$, apart. Rays emanating from a point on the line joining their centres are reflected alternately at the mirrors. Show that after an infinite number of reflexions the conjugate foci are distant $\frac{1}{2}\sqrt{(a^2 - 2ar)}$ from the middle point of the line joining the centres of the mirrors.

15. A transparent sphere is silvered at the back; prove that the distance between the images of a speck within it formed (1) by one direct refraction, (2) by one direct reflexion and one direct refraction is

$$2\mu ac(a-c) \div (a+c-\mu c)(\mu c+a-3c),$$

where a is the radius of the sphere, and c the distance of the speck from the centre towards the silvered side.

16. A pencil diverges from a point P and passes directly through a transparent sphere whose centre is O. If Q_0 be the focus when it is not reflected inside the sphere, Q_n the focus when the pencil has been reflected $2n$ times inside the sphere, show that OQ_0, OQ_1, $OQ_2 \ldots OQ_n$ form a series in harmonical progression, and that

$$\frac{1}{OQ_{n+1}} - \frac{1}{OQ_n} = \frac{4}{\mu r}.$$

CHAPTER IV.

Theory of Refraction through Lenses.

52. A lens is a portion of a refracting medium bounded by two surfaces of revolution which have a common axis, called the axis of the lens. In general, the surfaces of revolution are spherical or plane. If these surfaces do not meet, the lens is supposed to be bounded by a cylinder having the same axis, in addition to the surfaces of revolution.

The distance between the bounding surfaces, measured along the axis, is called the *thickness* of the lens. The thickness will generally be small in comparison with the radii of curvature of the bounding surfaces.

Lenses are classified according to their forms. A lens bounded by two convex surfaces is called a double-convex lens.

A lens bounded by two concave surfaces is called a double-concave lens.

A lens of which one face is convex and the other concave is called convexo-concave or concavo-convex, according as the light first falls on the convex or concave surface, respectively.

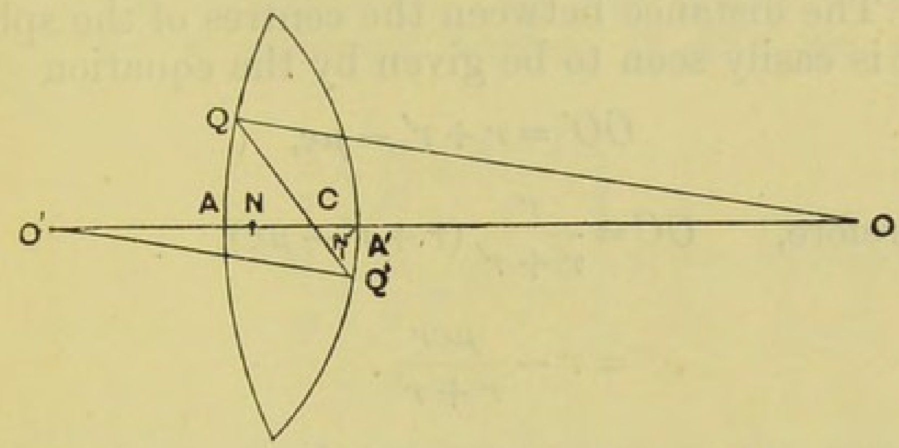
The terms plano-convex, convexo-plane, plano-concave and concavo-plane need no further explanation.

53. We shall now consider the refraction of light through a single double-convex lens, the radii of whose faces are r, r', the light, as usual, travelling from left to

right. The following abbreviations will be found to be convenient: let

$$\frac{r}{\mu - 1} = f, \quad \frac{r'}{\mu - 1} = f',$$

and let the thickness of the lens be μc, μ being the refractive index of the substance of which the lens is made, when that of air is taken to be unity.



There exist two points on the axis of the lens, which are most useful in the determination of the positions of conjugate foci, and corresponding incident and emergent rays. They are a pair of conjugate foci, such that any incident ray passing through one of them, will emerge in a parallel direction through the other. These points are called the *nodal* points, and also from another property which will be pointed out later, the *principal* points of the lens.

We proceed to find the position and properties of the *nodal* points. Draw any two parallel radii OQ, $O'Q'$ of the spherical surfaces, and join QQ' meeting the axis in C. Then from the similar triangles OCQ, $O'CQ'$,

$$OC : O'C = r : r',$$

and therefore C is a fixed point. Any ray of light which in its path through the substance of the lens passes through C will emerge parallel to its original direction, because the tangent planes at Q, Q' are parallel to each other, and the lens will act on such a ray like a plate with parallel faces. If therefore we take N, N' the conjugate foci of C with respect to the two surfaces, a ray of light proceeding in a direction towards N will after the first refraction pass through C, and therefore after the second refraction will pass through N' and will emerge parallel to the original direction; in other words N, N' are the nodal points. The point C is called the *centre* of the lens.

The position of the nodal points can now be determined. The distance between the centres of the spherical surfaces is easily seen to be given by the equation

$$OO' = r + r' - \mu c,$$

and therefore, $$OC = \frac{r}{r+r'}(r + r' - \mu c)$$

$$= r - \frac{\mu c r}{r + r'}.$$

Thus $$AC = \frac{\mu c r}{r + r'} = \frac{\mu c f}{f + f'},$$

and similarly $$A'C = \frac{\mu c r'}{r + r'} = \frac{\mu c f'}{f + f'}.$$

Let h be the distance of N from A, h' the distance of N' from A', both distances being measured from the surface of the lens, inwards. Then, by § 40, since N and C are conjugate foci,

$$\frac{1}{h} - \frac{\mu}{AC} = \frac{1-\mu}{r},$$

that is, $$\frac{1}{h} = \frac{f+f'}{cf} - \frac{1}{f}.$$

From this we deduce the value of h, namely,

$$h = \frac{cf}{f + f' - c}.$$

Similarly, $$h' = \frac{cf'}{f + f' - c}.$$

54. There will be two images of a given object, formed by refraction at the two surfaces in succession, and we shall use a symmetrical notation for their positions along the axis.

Let x, x' denote the distances of the object and its first image, in front of, and behind the surface A, respectively; and let y, y' denote the distances of the final image and the first image behind, and in front, of the second surface, respectively. By the theory of a single refraction at a spherical surface (§ 42), we get the equations

$$\left.\begin{aligned} \frac{1}{x}+\frac{\mu}{x'}&=\frac{\mu-1}{r} \\ \frac{1}{y}+\frac{\mu}{y'}&=\frac{\mu-1}{r'} \\ \text{and} \qquad x'+y'&=\mu c \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If planes be drawn perpendicular to the axis of the system at the nodal points, these planes will be *planes of unit magnification;* that is, any object lying in the first plane, will have an image in the second plane, equal in all respects to the object. This theorem may also be enunciated in a slightly different manner; the line joining the points where the incident and emergent rays meet the first and second planes, respectively, is parallel to the axis of the system. The two planes are called the *principal planes*, and the points where they meet the axis (in this case coinciding with the nodal points), the *principal points.*

To prove this theorem, let η, $\eta_{,}$, η' denote the linear magnitudes of the object and its images, respectively. Then, by § 51,

$$\left.\begin{aligned} \frac{\eta}{x}+\frac{\mu\eta_{,}}{x'}&=0 \\ \frac{\eta'}{y}+\frac{\mu\eta_{,}}{y'}&=0 \end{aligned}\right\},$$

so that

$$\frac{\eta}{\eta'}=\frac{xy'}{yx'}.$$

But by § 53 $AC : A'C = r : r'$, and therefore at the nodal points $x'/y' = r/r'$; it easily follows from the equations (1), that each of these ratios is equal to x/y. Hence $\eta = \eta'$.

55. If we eliminate x', y' from the equations (1), we get

$$c = \frac{1}{\frac{1}{f} - \frac{1}{x}} + \frac{1}{\frac{1}{f'} - \frac{1}{y}};$$

that is
$$c = \frac{fx}{x-f} + \frac{f'y}{y-f'}.$$

By reduction, this equation becomes

$$xy(f+f'-c) - fy(f'-c) - f'x(f-c) = cff' \ldots (2).$$

By means of this equation the positions of the *focal points* may be found; these are points such that rays diverging from them are made parallel by refraction through the lens; in other words, they are the points conjugate to the points at infinity, in both directions.

If in equation (2) we divide by y and then make y indefinitely large, we get the first focal point, $x = g$, where

$$g = \frac{f(f'-c)}{f+f'-c}.$$

Similarly, the other focal point will be given by the equation $y = g'$, where

$$g' = \frac{f'(f-c)}{f+f'-c}.$$

The distance between the first focal point and the first principal point is found to be equal to that between the second principal point and the second focal point, and this distance is called the *focal length* of the lens; it is considered to be positive if the first focal point is to the left of the first principal point. If we denote this focal length by ϕ, we have

$$\phi = g + h = g' + h',$$

which gives $$\phi = \frac{ff'}{f+f'-c},$$

or $$\frac{1}{\phi} = \frac{1}{f} + \frac{1}{f'} - \frac{c}{ff'}.$$

The reciprocal of the focal length is called the *power* of the lens.

Introducing these values g, g', ϕ into the equation (2), it becomes, on dividing by $f+f'-c$,

$$xy - gy - g'x = c\phi,$$

or $$(x-g)(y-g') = gg' + c\phi$$
$$= \phi\left\{\frac{(f'-c)(f-c)}{f+f'-c} + c\right\};$$

and therefore by reduction,

$$(x-g)(y-g') = \phi^2.$$

Let the distances of a pair of conjugate points measured respectively in front of and behind the focal points, be denoted by u, v; the values of u, v are then connected by the simple formula

$$uv = \phi^2.$$

56. Let the distances of a pair of conjugate points measured from the principal points in accordance with the same convention of sign as before be denoted by ξ, ξ'; then

$$u = \xi - \phi, \quad v = \xi' - \phi',$$

and therefore

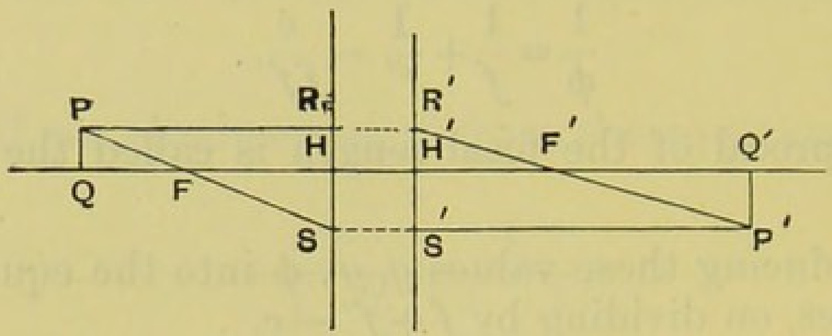
$$(\xi - \phi)(\xi' - \phi) = \phi^2,$$

which gives the relation

$$\frac{1}{\xi} + \frac{1}{\xi'} = \frac{1}{\phi}.$$

57. The position of the point P' conjugate to a given point P may now be determined by a geometrical construction. Let F, F' be the focal points, H, H' the principal points. If we can trace two rays emerging from

P after refraction by the lens, these will meet in the required point P'. For one of these rays choose the ray



PR parallel to the axis, meeting the first principal plane in R; then the corresponding emergent ray will pass through R', where RR' is drawn parallel to the axis to meet the second principal plane in R'. But PR and QH are two parallel incident rays, and therefore after refraction they will meet in the focal point F'; hence $R'F'$ is the emergent ray. For the second ray choose the ray PF, meeting the principal plane in S; then the emergent ray will be parallel to the axis, through the point S', the projection of S on the second principal plane. This determines the position of P'.

58. Let η, η' represent the linear magnitudes of the object PQ and its image $P'Q'$ as constructed by this process, reckoned positive if above the axis, negative if below. Then, by similar triangles,

$$PQ : QF = SH : HF.$$

But $PQ = \eta$, $QF = u$, $SH = P'Q' = -\eta'$, and $HF = \phi$; so that the relation becomes

$$\frac{\eta}{\eta'} = -\frac{u}{\phi};$$

similarly

$$\frac{\eta'}{\eta} = -\frac{v}{\phi}.$$

59. Two special cases may be noticed.

First, suppose that the thickness of the lens is very small compared with the radii of its faces; such a lens will be called a *thin* lens. In this case the points A, A' and C

coincide, and the nodal points also coincide with these points. The equations then become

$$\left.\begin{aligned}\frac{1}{x}+\frac{\mu}{x'}&=\frac{\mu-1}{r}\\ \frac{1}{y}+\frac{\mu}{y'}&=\frac{\mu-1}{r'}\end{aligned}\right\},$$

and $$x'+y'=0.$$

The quantities x', y' will disappear on addition, and we get

$$\frac{1}{x}+\frac{1}{y}=(\mu-1)\left\{\frac{1}{r}+\frac{1}{r'}\right\}=\frac{1}{\phi}.$$

As before, we have two focal points, each at a distance ϕ from the lens. If the distances of a pair of conjugate points measured from these focal points be u, v, so that

$$\left.\begin{aligned}u&=x-\phi\\ v&=y-\phi\end{aligned}\right\},$$

then $$uv=\phi^2.$$

60. Next, suppose that the lens consists of a perfect sphere. In this case, we shall measure all distances from the centre of the sphere.

Let x, x' be the distances of the object and its first image, in front of, and behind, the centre, respectively, and y, y' the distances of final and first image behind, and in front of, the centre. Then from § 41, changing the sign of p and r we get in the new notation

$$\frac{\mu}{x}+\frac{1}{x'}=\frac{\mu-1}{r}.$$

Similarly, $$\frac{\mu}{y}+\frac{1}{y'}=\frac{\mu-1}{r};$$

also $$x'+y'=0.$$

Hence $$\frac{1}{x}+\frac{1}{y}=\frac{2(\mu-1)}{\mu r}=\frac{1}{\phi}, \text{ say.}$$

Let $OF = OF' = \phi$, so that F, F' are the focal points. Then if P, P' be a pair of conjugate points, and $PF = u$, $P'F' = v$, the same relation holds between u, v as before, namely,

$$uv = \phi^2.$$

61. We shall next find the relations between the abscissæ and magnitudes of an object and its image after refraction at any number of spherical surfaces arranged symmetrically along an axis. This will include as a particular case the refraction by any number of lenses of any thicknesses arranged at intervals along the axis.

We shall suppose that there are n refracting surfaces, and that the absolute refracting indices of the several media are μ, μ_1, μ_2......μ_n and that light travels from left to right from the first medium to the last. Let

$$r, r_1, r_2 \ldots\ldots r_{n-1}$$

be the n radii of the surfaces, considered positive if the line joining the vertex of any surface to its centre is in the positive direction, as in § 40; and, for brevity, suppose that

$$\frac{\mu-\mu_1}{r}=k_0, \quad \frac{\mu_1-\mu_2}{r_1}=k_1 \ldots\ldots \quad \frac{\mu_{n-1}-\mu_n}{r_{n-1}}=k_{n-1}.$$

Also, let the thicknesses of the media, measured along the axis, be $\mu_1 t_1$, $\mu_2 t_2$......$\mu_{n-1} t_{n-1}$.

Finally, let the distance of the object from the first surface be denoted by μv, the distance of the first image also measured from the first surface by $\mu_1 v_1$, the distance of the second image measured from the second surface by $\mu_2 v_2$, and so on, and the distance of the last image measured from the last surface, by $\mu_n v_n$; all these distances being measured from left to right. We shall find the relations between these quantities, beginning at the end and reckoning backwards.

The distances of the last two images reckoned from the last surface are easily seen to be, respectively, $\mu_n v_n$, and $\mu_{n-1}(v_{n-1}-t_{n-1})$; and since these are conjugate focal distances with respect to the last surface, we have by § 40, formula (2)

$$\frac{\mu_n}{\mu_n v_n}-\frac{\mu_{n-1}}{\mu_{n-1}(v_{n-1}-t_{n-1})}=\frac{\mu_n-\mu_{n-1}}{r_{n-1}},$$

or

$$\frac{1}{v_n}-\frac{1}{v_{n-1}-t_{n-1}}=-k_{n-1}.$$

This equation may be written in the form

$$v_{n-1}=t_{n-1}+\cfrac{1}{k_{n-1}+\cfrac{1}{v_n}}.$$

In exactly the same manner it may be proved that

$$v_{n-2}=t_{n-2}+\cfrac{1}{k_{n-2}+\cfrac{1}{v_{n-1}}},$$

and therefore

$$v_{n-2}=t_{n-2}+\frac{1}{k_{n-2}+}\,\frac{1}{t_{n-1}+}\,\frac{1}{k_{n-1}+}\,\frac{1}{v_n}.$$

Continuing this process backwards, we arrive at the equation

$$v_1=t_1+\frac{1}{k_1+}\,\frac{1}{t_2+}\,\frac{1}{k_2+}\,\ldots\ldots+\frac{1}{k_{n-1}+}\,\frac{1}{v_n}.$$

Also the distances μv, $\mu_1 v_1$, being conjugate focal distances with reference to the first surface, are (cf. § 40) connected by the relation

$$\frac{\mu}{\mu v}-\frac{\mu_1}{\mu_1 v_1}=\frac{\mu-\mu_1}{r},$$

or

$$\frac{1}{v}=k_0+\frac{1}{v_1},$$

and therefore, finally

$$\frac{1}{v} = k_0 + \frac{1}{t_1 +}\frac{1}{k_1 +}\frac{1}{t_2 +}\frac{1}{k_2 +} \ldots\ldots + \frac{1}{k_{n-1} +}\frac{1}{v_n}.$$

Let g/h, k/l, be the last two convergents of the continued fraction

$$k_0 + \frac{1}{t_1 +}\frac{1}{k_1 +} \ldots\ldots + \frac{1}{k_{n-1}},$$

so that, by the properties of such fractions, $gl - hk = 1$; then the value of v will be given by the equation

$$\frac{1}{v} = \frac{v_n k + g}{v_n l + h}.$$

It will be convenient to represent the distances of the object and its final image from the first and final surfaces, respectively, by ξ, ξ'; then $\xi = \mu v$, $\xi' = \mu' v_n$, where μ' is written instead of μ_n for the refractive index of the final medium. The relation between ξ and ξ' is

$$\frac{\mu}{\xi} = \frac{\xi' k + \mu' g}{\xi' l + \mu' h},$$

or

$$k\xi\xi' + \mu' g\xi - \mu l\xi' - \mu\mu' h = 0.$$

62. The *focal planes* of the system are the planes conjugate to the planes at infinity.

To find the first focal plane, we must make ξ' infinite, then the rays will be parallel in the final medium. The corresponding value of ξ is

$$\xi = \frac{\mu l}{k} = \gamma_1, \text{ say.}$$

Similarly, if we make ξ infinite, so that the rays are parallel in the first medium, the value of ξ' becomes

$$\xi' = -\frac{\mu' g}{k} = \gamma_2, \text{ say.}$$

The relation between ξ, ξ' may now be written

$$\xi\xi' - \gamma_2\xi - \gamma_1\xi' = \frac{\mu\mu' h}{k},$$

or $$(\xi - \gamma_1)(\xi' - \gamma_2) = \frac{\mu\mu' h}{k} - \frac{\mu\mu' lg}{k^2}$$

$$= -\frac{\mu\mu'}{k^2}\{gl - hk\},$$

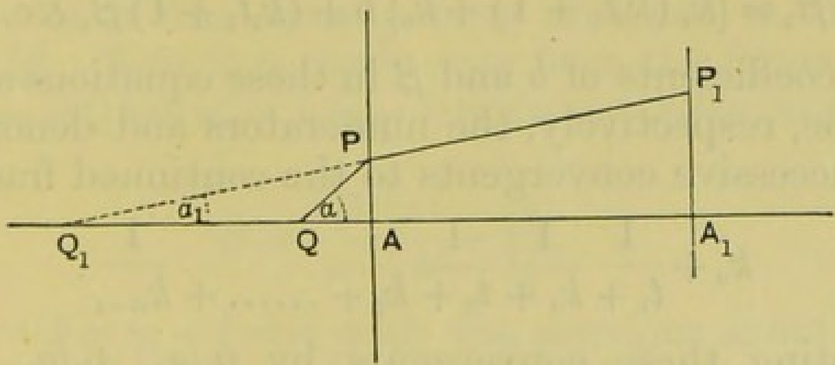
that is $$(\xi - \gamma_1)(\xi' - \gamma_2) = -\frac{\mu\mu'}{k^2}.$$

Let u, u' denote the distances of the conjugate planes from the focal planes, the same convention of sign being observed as in § 55; then $u = \gamma_1 - \xi$, $u' = \xi' - \gamma_2$. Also let $f = -\mu/k$, $f' = -\mu'/k$. Then the relation between the abscissæ of conjugate points takes the form

$$uu' = ff'.$$

63. Let $\alpha, \alpha_1, \alpha_2 \ldots\ldots$ be the successive inclinations to the axis of a ray in the same plane as the axis as it moves onwards through the different media; and let $b, b_1, b_2 \ldots\ldots$ be the distances from the axis at which it meets the successive spherical surfaces. Also let

$$\mu \tan \alpha = \beta,\ \mu_1 \tan \alpha_1 = \beta_1 \ldots.$$



In the figure, suppose that QAA_1 represents the axis of the system, QP the incident ray, Q_1PP_1 the course of the ray after one refraction, produced backwards to meet the axis in Q_1. Then $AQ = b \cot \alpha = b\mu/\beta$. This relation may be expressed in the form $\mu/AQ = \beta/b$. In exactly the same manner it may be shown that $\mu_1/AQ_1 = \beta_1/b$.

But by § 40, formula (2), since Q, Q_1 are conjugate foci at refraction at the spherical surface and AQ, AQ_1 are both negative in the figure, we have

$$-\frac{\mu}{AQ}+\frac{\mu_1}{AQ_1}=\frac{\mu-\mu_1}{r};$$

and therefore $$-\frac{\beta}{b}+\frac{\beta_1}{b}=k_0,$$

or $$\beta_1=\beta+k_0 b.$$

Also, referring back to the figure, it is easy to see that

$$b_1=b+\mu_1 t_1 \tan\alpha_1;$$

that is $$b_1=b+t_1\beta_1.$$

In exactly the same manner it may be proved that

$$\left.\begin{aligned}\beta_2&=\beta_1+k_1 b_1\\ b_2&=b_1+t_2\beta_2\end{aligned}\right\},$$

and so on.

64. By these equations all the quantities β_1, b_1, β_2, b_2...... may be expressed in terms of b and β; their values become

$$\beta_1=k_0 b+\beta,$$
$$b_1=(k_0 t_1+1)\,b+t_1\beta,$$
$$\beta_2=\{k_1(k_0 t_1+1)+k_0\}\,b+(k_1 t_1+1)\,\beta, \text{ \&c.}$$

The coefficients of b and β in these equations are easily seen to be, respectively, the numerators and denominators of the successive convergents to the continued fraction

$$k_0+\frac{1}{t_1+}\,\frac{1}{k_1+}\,\frac{1}{t_2+}\,\frac{1}{k_2+}\cdots\cdots+\frac{1}{k_{n-1}}.$$

Denoting these convergents by p_1/q_1, p_2/q_2...... the equations may be written in the forms,

$$\beta_1=p_1 b+q_1\beta,$$
$$b_1=p_2 b+q_2\beta,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$b_{n-1}=p_{2n-2}b+q_{2n-2}\beta,$$
$$\beta_n=p_{2n-1}b+q_{2n-1}\beta,$$

there being n spherical surfaces.

We shall denote the last two convergents by g/h, k/l, respectively, remarking that the quantities g, h, k, l are connected by the relation $gl - hk = 1$, by the theory of continued fractions. Also, instead of the final values b_{n-1}, β_n, μ_n we shall write b', β', μ'; then the last two equations of the series become

$$\left.\begin{array}{l} b' = gb + h\beta \\ \beta' = kb + l\beta \end{array}\right\}.$$

If we solve these equations, and express b, β in terms of b', β', we find by virtue of the relation $gl - hk = 1$,

$$\left.\begin{array}{l} b = lb' - h\beta' \\ \beta = -kb' + g\beta' \end{array}\right\}.$$

65. We shall next find the relation between the linear dimensions of a point and its final image.

Let η, η_1, $\eta_2 \ldots$ denote the linear magnitudes of the object and its successive images; then by Helmholtz' theorem

$$\mu\eta \tan\alpha = \mu_1\eta_1 \tan\alpha_1 = \mu_2\eta_2 \tan\alpha_2 \ldots;$$

that is,

$$\eta\beta = \eta_1\beta_1 = \ldots = \eta'\beta',$$

where η' denotes the linear magnitude of the final image. The value of β' has already been obtained in the form $\beta' = kb + l\beta$. Now it is easily seen from the figure of § 63 that $AP = AQ \tan\alpha$, or $b = -\xi \tan\alpha = -\xi\beta/\mu$, and therefore

$$\beta' = \frac{k\beta}{\mu}\left\{\frac{\mu l}{k} - \xi\right\}.$$

But $\mu l/k - \xi = \gamma_1 - \xi = u$, with the previous notation, and $f = -\mu/k$; with these abbreviations, the preceding equation becomes

$$\frac{\beta'}{\beta} = -\frac{u}{f}.$$

The relation between the linear magnitudes of the object and image is therefore

$$\frac{\eta}{\eta'} = -\frac{u}{f},$$

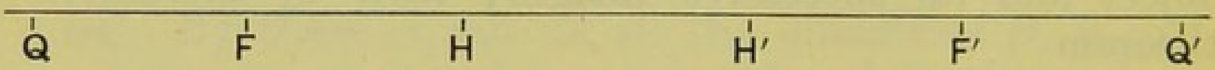
and from this we deduce

$$\frac{\eta'}{\eta} = -\frac{u'}{f'}.$$

Either of these equations determines the magnification of the system.

66. If we take $u = -f$ and therefore $u' = -f'$, these equations give $\eta = \eta'$; this shows that the planes $u = -f$, $u' = -f'$ are planes of unit magnification; in other words, any ray passing through the system meets these planes in two points such that the line joining them is parallel to the axis. They are called the *principal planes*, and the points where they meet the axis, the *principal points* of the system.

Let H, H' be the principal points, Q, Q' any pair of conjugate foci. Let $QH = x$, $Q'H' = x'$, the distances being



measured according to the same convention of sign as before. Then the equation $uu' = ff'$ is equivalent to

$$(x - f)(x' - f') = ff',$$

from which we deduce the equation

$$\frac{f}{x} + \frac{f'}{x'} = 1.$$

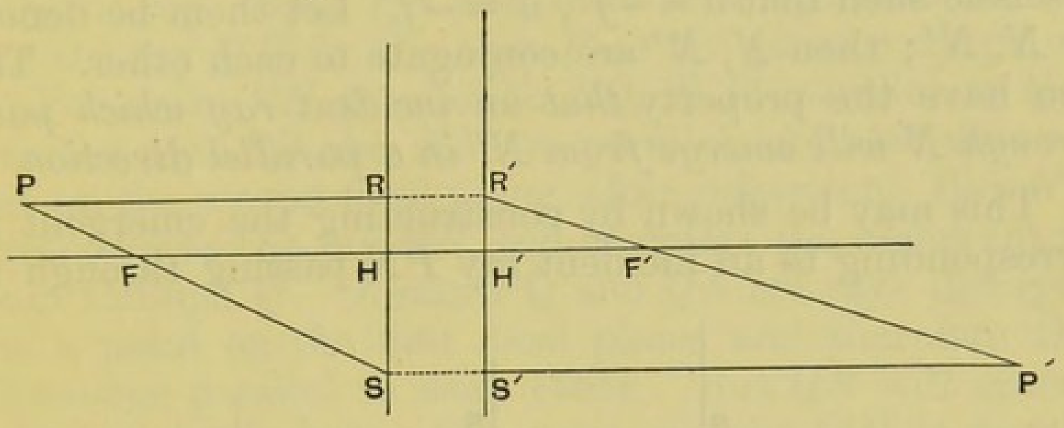
The lengths f, f' are called the *principal focal lengths* of the system.

67. We can now give simple geometrical constructions for the focus conjugate to a given point and for an emergent ray when the incident ray is given.

Let F, F' be the principal foci, H, H' the principal points of the system.

Let P be a given point, it is required to find its conjugate focus. If we can trace the course of any two rays from P, we shall be able to find P'. Take PF as one ray; let PF meet the principal plane HS in S. Draw SS'

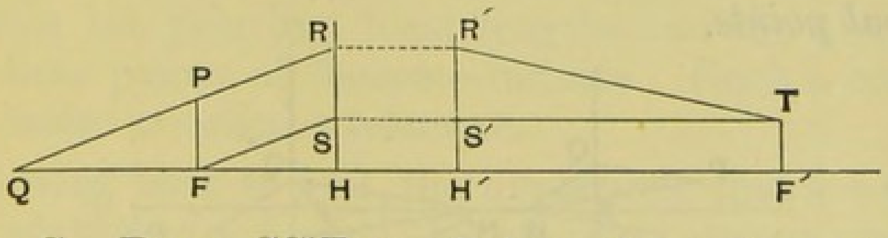
parallel to the axis to meet the other principal plane in S'; then the emergent ray will pass through S'. Also the



rays FH and FS, since they diverge from a point on a focal plane, will emerge parallel to each other; if therefore we draw $S'P'$ parallel to the axis, $S'P'$ will be the emergent ray corresponding to PF, and will pass through the required point. For the other ray, take PR, parallel to the axis, meeting the first principal plane in R. Draw RR' parallel to the axis to meet the other principal plane in R'. Then $R'F'$ is the corresponding emergent ray; produce $R'F'$ to meet $S'P'$ in P', then P' is the point required.

The emergent ray may be constructed as follows:

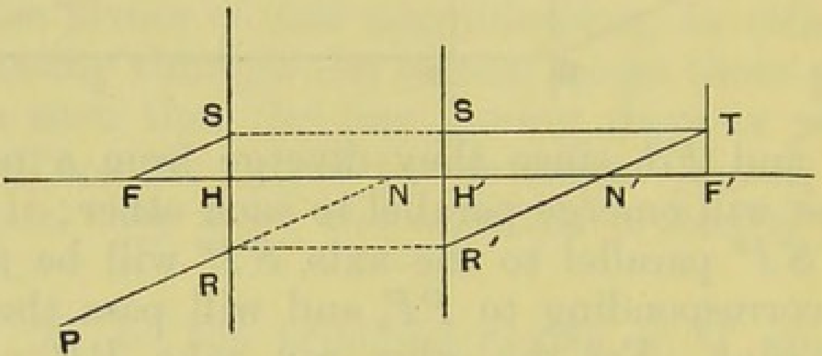
Let QPR be the incident ray, meeting the first focal plane in P, and the first principal plane in R. Draw RR' parallel to the axis to meet the second principal plane in R'; the emergent ray will pass through R'. Again, draw a parallel incident ray from F, meeting the first principal



plane in S. Draw $SS'T$ parallel to the axis meeting the second principal plane and the second focal plane in S', T respectively; $S'T$ is the emergent ray corresponding to FS. But PR and FS are parallel, and therefore after refraction they will converge to a point on the focal plane at F'. Hence $R'T$ is the emergent ray required.

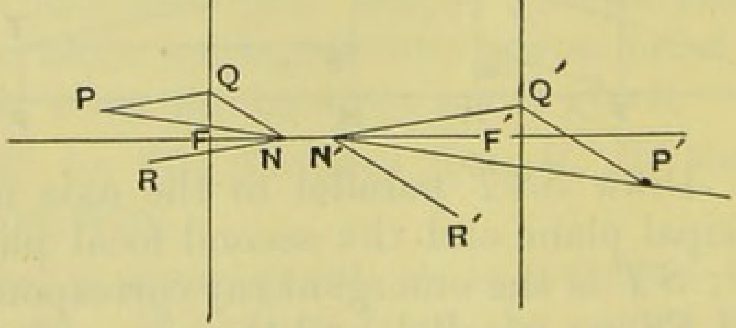
68. The best construction is effected by means of two points, called *nodal points.* These points have their abscissæ such that $u = -f'$, $u' = -f$. Let them be denoted by N, N'; then N, N' are conjugate to each other. They also have the property *that an incident ray which passes through N will emerge from N' in a parallel direction.*

This may be shown by constructing the emergent ray corresponding to an incident ray PN passing through the



point N. Let the points R', T be constructed as in § 67, then the emergent ray will be the line joining R' and T. But if N' be the second nodal point $F'N' = FH$ and therefore the triangles $TN'F'$, SFH are equal in all respects. Again, $H'N' = HN$, and therefore the triangles $R'N'H'$, RNH are equal in all respects. And therefore since FS, PR are parallel, the lines $N'T$, $N'R'$ are in the same straight line. This shows that the emergent ray corresponding to PN passes through N', and is parallel to the incident ray.

If the initial and final media are the same we have $f = f'$, and therefore *the nodal points coincide with the principal points.*



Let PQ be any incident ray through P. Let N, N' be the nodal points. Let PQ meet the first focal plane in Q.

Draw $N'Q'$ parallel to PQ, meeting the second focal plane in Q'; and draw $Q'P'$ parallel to QN. Then $P'Q'$ is the emergent ray. Join PN and draw $N'P'$ parallel to it to meet the ray $Q'P'$ in P'; then P' is the point conjugate to P. For draw RN parallel to PQ, $N'R'$ parallel to $Q'P'$. Then the rays PQ and RN are parallel, and therefore will meet on the second focal plane, after refraction. But $N'Q'$ corresponds to the ray RN, and therefore the emergent ray passes through Q'. Again PQ and QN are rays diverging from a point on the first focal plane, and therefore they will emerge parallel to each other. But QN will emerge parallel to itself; hence the emergent ray $Q'P'$ is parallel to QN. Finally, the ray PN will emerge from N' in a parallel direction, and therefore P' is conjugate to P.

69. In all cases of refraction through lenses used in air, the initial and final media are the same, and therefore $\mu=\mu'$, $f=f'=-\mu/k$, and the relation between the abscissæ of conjugate points becomes

$$uu'=f^2.$$

The nodal points also coincide with the principal points, and all the constructions depend in a simple manner on the positions of four planes and the points where they meet the axis, namely, the two focal planes and the two focal points, and the two principal planes and the two principal points.

70. The foregoing representation of the effect of a system of lenses by means of cardinal points and principal focal lengths fails altogether in one important case; for if k vanishes, the principal focal lengths and the abscissæ of the cardinal points all become infinite. Such a combination of lenses is called a *telescope*.

Referring back to § 61, the hypothesis that $k=0$, gives us the relation $gl=1$; and the relation between conjugate abscissæ becomes

$$\mu'g\xi-\mu l\xi'-\mu\mu'h=0.$$

If c, c' represent the abscissæ of any other pair of conjugate points

$$\mu'gc-\mu lc'-\mu\mu'h=0;$$

and therefore by subtraction,

$$\mu' g (\xi - c) = \mu l (\xi' - c').$$

Hence *the abscissæ of conjugate points measured in the same direction from any fixed pair of conjugate points are in a constant ratio.*

It is, in general, possible to find a *self-conjugate* point which may be called the *centre* of the telescope. For ξ, ξ' are measured in the same direction from the first and last refracting surfaces, respectively; and therefore if t denote the distance between these surfaces, ξ and ξ' will denote the same point, if

$$\xi - \xi' = t.$$

We then have

$$\left.\begin{array}{r}\mu' g\xi - \mu l\xi' = \mu\mu' h \\ \xi - \xi' = t\end{array}\right\};$$

and therefore

$$\xi = \frac{\mu(\mu' h - lt)}{\mu' g - \mu l},$$

$$\xi' = \frac{\mu'(\mu h - gt)}{\mu' g - \mu l}.$$

If now x, x' denote the abscissæ of conjugate points referred to the centre of the telescope as origin and measured in the same direction,

$$\frac{x'}{x} = \frac{\mu'}{\mu} \cdot \frac{g}{l}.$$

This ratio is called the *elongation* of the telescope.

There is no centre in the particular case in which

$$\mu' g = \mu l.$$

But in this case the relation between the abscissæ of conjugate points becomes

$$\mu' g (\xi - \xi') = \mu\mu' h,$$

or

$$\xi - \xi' = \frac{\mu h}{g}.$$

That is, the distance between an object and its image is constant.

The effect of the instrument is then to alter the position of an object by a certain distance measured along the axis, as in the case of refraction through a plate of glass bounded by parallel planes.

Again, by § 64, we find

$$\beta' = l\beta,$$

that is,
$$\mu' \tan \alpha' = l \,.\, \mu \tan \alpha,$$

or
$$\frac{\tan \alpha'}{\tan \alpha} = \frac{\mu}{\mu'} \,.\, l.$$

In other words, *the angular magnification of the instrument is the same for all rays.*

Rays which are parallel in the first medium will also be parallel when they emerge into the final medium.

Lastly, if η, η' denote the linear magnitudes of the object and its final image, we have in § 65, by Helmholtz' theorem

$$\eta\beta = \eta'\beta'.$$

Hence
$$\frac{\eta'}{\eta} = \frac{1}{l} = g;$$

and therefore the *magnification of the instrument is the same for all objects.*

If the elongation, angular and lateral magnification be denoted by e, γ and m, respectively, we have

$$\left.\begin{aligned} m\gamma &= \frac{\mu}{\mu'} \\ e &= \frac{\mu'}{\mu} \,.\, m^2 \end{aligned}\right\}.$$

Also
$$e = \frac{m}{\gamma}.$$

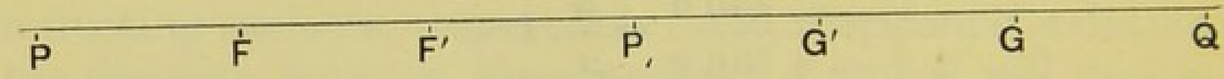
If the initial and final media are of the same kind

$$\left.\begin{aligned} \gamma &= \frac{1}{m} \\ e &= m^2 \end{aligned}\right\}.$$

A telescope is completely determined when its *centre* (or any pair of conjugate points) and the magnification and elongation are given.

71. We shall next consider the combination of two known lens-systems.

Let F, F' be the focal points, and f, f' the principal focal lengths of the first system; G', G the focal points, and



g', g the principal focal lengths of the second system. Let P be any object, $P_,$ its image after refraction through the first system, Q the image of $P_,$ after refraction through the second system. Let u, u' be the distances of P, $P_,$ from the focal points F, F' respectively, the distances being measured in the ordinary way, and also let v', v be the distances of $P_,$, Q from the focal points G', G respectively, and let $F'G' = c$, then

$$u' + v' = c.$$

Also

$$\left.\begin{aligned} uu' &= ff' \\ vv' &= gg' \end{aligned}\right\}.$$

Eliminating u', v', we get the equation

$$\frac{ff'}{u} + \frac{gg'}{v} = c.$$

The last equation may be written

$$\left(u - \frac{ff'}{c}\right)\left(v - \frac{gg'}{c}\right) = \frac{ff'gg'}{c^2}.$$

Hence the focal points of the combination are determined by the abscissæ

$$\left.\begin{aligned} u &= \frac{ff'}{c} \\ v &= \frac{gg'}{c} \end{aligned}\right\}.$$

Let η, $\eta_{\prime}$, η' denote the linear magnitudes of the images at P, $P_{\prime}$, Q, respectively; then

$$\left.\begin{aligned}\frac{\eta}{\eta_{\prime}} &= -\frac{u}{f}\\ \frac{\eta'}{\eta_{\prime}} &= -\frac{v}{g}\end{aligned}\right\}.$$

The principal points are points of unit magnification, and therefore to find them we must make $\eta = \eta'$. From this we deduce the equation

$$\frac{u}{f} = \frac{v}{g}.$$

Hence, also
$$\frac{u'}{f'} = \frac{v'}{g'} = \frac{c}{f'+g'}.$$

Therefore the principal points are determined by the equations

$$\left.\begin{aligned}u &= \frac{ff'}{u'} = \frac{f(f'+g')}{c}\\ v &= \frac{gg'}{v'} = \frac{g(f'+g')}{c}\end{aligned}\right\}.$$

If ϕ, ϕ' be the principal focal lengths, we have

$$\phi = \frac{ff'}{c} - \frac{f(f'+g')}{c},$$

that is,
$$\left.\begin{aligned}\phi &= -\frac{fg'}{c}\\ \phi' &= -\frac{f'g}{c}\end{aligned}\right\}.$$
and similarly

The positions of the cardinal points of the system and the principal focal lengths have now been found, and therefore the solution is complete.

72. The combination becomes telescopic if F' the second focal point of the first system coincides with G' the first focal point of the second system. An incident system

of parallel rays will then converge to a point on the common focal plane, and therefore will emerge as a system of parallel rays. In the previous investigation c will vanish and ϕ, ϕ' will be infinite, as will also the abscissæ of the cardinal points. The ratio of the focal lengths will however remain finite, viz.,

$$\frac{\phi}{\phi'} = \frac{f}{f'} \div \frac{g}{g'}.$$

The relation between the abscissæ of conjugate points becomes

$$\frac{ff'}{u} + \frac{gg'}{v} = 0.$$

The centre of the instrument is found by combining this equation with the equation

$$u + v = -b,$$

where b is the distance between the first focal point of the first system and the second focal point of the other system. Therefore the centre of the telescope is determined by either of the equations

$$\left.\begin{aligned} u &= -\frac{bff'}{ff' - gg'}, \\ v &= \frac{bgg'}{ff' - gg'} \end{aligned}\right\}.$$

If x, x' be the distances of a pair of conjugate points measured from the centre *in the same direction*, then since

$$\frac{v}{u} = -\frac{gg'}{ff'},$$

we have

$$\frac{x'}{x} = \frac{gg'}{ff'},$$

and therefore

$$e = \frac{gg'}{ff'}.$$

Again,
$$\frac{\eta'}{\eta} = \frac{v}{u} \cdot \frac{f}{g}$$
$$= -\frac{g'}{f'};$$

therefore
$$m = -\frac{g'}{f'}.$$

73. If one of the systems is a telescopic system the method of procedure is somewhat different.

Let F, F' be the focal points, f, f' the focal lengths of the first system; also let O be the centre, m the lateral magnification, e the elongation of the telescopic system. Then it is clear that F is also the first focal point of the combined system. For parallel rays which emerge from the telescopic system must have been parallel when they have passed through the first system; hence they must have proceeded from a point on the focal plane at F. If OF'' be taken equal to $e \, . \, OF'$, F'' will be the second focal point of the combined system; for rays initially parallel to the axis will converge to F', and therefore after emergence through the telescope will meet in F''.

To find the principal points of the combination we consider the magnitudes of the images formed by the two systems. If η, $\eta_{,}$, η' be the initial, intermediate and final magnitudes of the images, we have

$$\frac{\eta_{,}}{\eta} = -\frac{f}{u},$$

$$\frac{\eta''}{\eta_{,}} = m;$$

$$\therefore \frac{\eta''}{\eta} = -\frac{mf}{u}.$$

The planes of unit magnification are therefore found by taking $u = -mf$.

Measure FH backwards, equal to mf, and let $H_,$ be the image of H in the first system, H' the image of $H_,$ in the telescope.

Then $$H_,F' = \frac{ff'}{u} = \frac{f'}{m}.$$

Also $$OH' = e \,.\, OH_,.$$

But $$OF'' = e \,.\, OF';$$

$$\therefore\ H'F'' = e \,.\, H_,F'$$

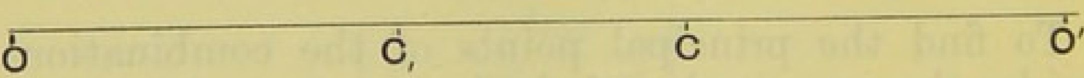
$$= \frac{ef'}{m}.$$

The principal focal lengths of the combination are therefore

$$\left.\begin{array}{l} F = fm \\ F' = \dfrac{f'e}{m} \end{array}\right\}.$$

The focal points are F, F'', and the principal points H, H'; the solution is therefore complete.

74. Lastly, let us suppose that both systems are telescopic.



Let O, O' be the centres of the two telescopes and let $OO' = b$. Also let e', m', γ' be the elongation, lateral magnification and angular magnifications, respectively, of the first system, and e'', m'', γ'' similar quantities for the second system.

The centre of the combined system may now be found. Let C be the centre, $C_,$ its conjugate point in the first telescope; then $C_,$ and C must be conjugate in the second telescope. Hence if $OC = x$, $O'C = b - x$ we must have

$$\left.\begin{array}{l} OC_, = e' \,.\, OC \\ O'C = e'' \,.\, O'C_, \end{array}\right\},$$

or
$$\left.\begin{aligned} OC_{,} &= e'x \\ O'C_{,} &= \frac{b-x}{e''} \end{aligned}\right\}.$$

But $$OC_{,} + O'C_{,} = b,$$

therefore
$$e'x + \frac{b-x}{e''} = b$$

or
$$x = \frac{b(e''-1)}{e'e''-1}.$$

Next consider the lateral magnification of the combined system. If η, $\eta_{,}$, η' be the lateral magnitudes of an object, its first and final images, we have

$$\frac{\eta_{,}}{\eta} = m',$$

$$\frac{\eta'}{\eta_{,}} = m''.$$

Therefore
$$\frac{\eta'}{\eta} = m'm''.$$

The magnification of the combined instrument is

$$m = m'm''.$$

Exactly the same argument may be used to prove that the angular magnification is

$$\gamma = \gamma'\gamma''.$$

Finally, the elongation is found from the equation

$$e = \frac{m}{\gamma}$$

$$= \frac{m'm''}{\gamma'\gamma''}.$$

Hence
$$e = e'e''.$$

Theory of Equivalent Lenses.

75. A lens is said to be *equivalent* to any number of lenses arranged at intervals along an axis when, if placed in a proper position, it will produce the same deviation in rays inclined at small angles to the axis of the system, as would be produced by the system of lenses.

We shall first suppose the incident rays to be parallel to the axis of the system, so that the position of the equivalent lens is immaterial.

The deviation produced by a thin lens may be found by supposing the lens to act like a thin prism formed by the tangent planes to the spherical surfaces at the points of incidence and emergence of the ray. The deviation will therefore be independent of the angle of incidence, for all small angles of incidence. To find the deviation, we suppose the incident ray to be parallel to the axis, and then the emergent ray will proceed to the principal focus of the lens. If y be the distance from the axis at which the ray strikes the lens, and f the focal length of the lens, the deviation is clearly $\partial = -y/f$, the lens being supposed collective. This expression will therefore represent the deviation caused by the lens in *any* incident ray.

Now suppose that there are n thin lenses whose focal lengths are, respectively, $f_1, f_2 \dots f_n$, arranged at intervals $a_1, a_2, \dots a_{n-1}$, along an axis. For brevity, let $k = -1/f$, for all suffixes. Let any ray originally parallel to the axis strike the lenses in succession at distances $y_1, y_2 \dots y_n$ from the axis, and let $\partial_1, \partial_2, \dots \partial_n$ be the total deviations of the ray, after passing through the several lenses. Then, using the value of the deviation just given, and expressing the distances $y_2, y_3 \dots$ in terms of the deviations, we obtain the equations

$$\partial_1 = k_1 y_1,$$
$$y_2 = a_1\partial_1 + y_1,$$
$$\partial_2 = k_2 y_2 + \partial_1,$$
$$y_3 = a_2\partial_2 + y_2,$$
$$\dots\dots\dots\dots\dots$$
$$\partial_n = k_n y_n + \partial_{n-1}.$$

From these equations it is easy to see that ∂_n is the numerator of the last convergent of the continued fraction

$$\frac{y_1}{1+}\,\frac{1}{k_1+}\,\frac{1}{a_1+}\,\frac{1}{k_2+}\dots\dots+\frac{1}{k_n}.$$

If F_n be the focal length of the equivalent lens, $\partial_n = -y_1/F_n = y_1 K_n$, say. Then K_n is equal to the numerator of the last convergent of the continued fraction

$$\frac{1}{1+}\,\frac{1}{k_1+}\,\frac{1}{a_1+}\,\frac{1}{k_2+}\dots\dots+\frac{1}{k_n}.$$

The values of the first few numerators are

$$1,\ k_1,\ a_1k_1+1,\ a_1k_1k_2+k_2+k_1,\ a_1a_2k_1k_2+a_2(k_1+k_2)+a_1k_1+1,$$
$$a_1a_2k_1k_2k_3+a_2k_3(k_1+k_2)+a_1k_1(k_2+k_3)+k_1+k_2+k_3,$$

from which we deduce the values

$$\frac{1}{F_2}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{a_1}{f_1f_2},$$

$$\frac{1}{F_3}=\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}-\frac{a_1}{f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}\right)-\frac{a_2}{f_3}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)+\frac{a_1a_2}{f_1f_2f_3}.$$

These results might also have been obtained directly from the equations.

76. If the incident pencil be of any form, the position of the equivalent lens is not immaterial, and must be found.

Let the incident ray make an angle ∂ with the axis; then using the same notation as before, all the equations remain the same except the first, which is

$$\partial_1 = k_1y_1 + \partial = y_1\left(k_1+\frac{\partial}{y_1}\right)$$

and therefore the final value of ∂_n will be the same as before, with $k_1 + \partial/y_1$ written for k_1. If the negative reciprocal of the focal length of the equivalent lens be denoted by K, since K involves k_1 only in the first degree, the new value of K will be

$$K' = K + \frac{\partial}{y_1} K_1,$$

where K_1 is the coefficient of k_1 in the expression for K; so that

$$\partial_n = K y_1 + \partial K_1.$$

Let the distance of the equivalent lens behind the first lens of the system be x; then the incident ray will meet the lens at a distance from the axis equal to $y_1 + x\partial$, and therefore the inclination of the ray to the axis after refraction through it will be

$$\begin{aligned} \partial' &= K(y_1 + x\partial) + \partial \\ &= K y_1 + \partial(1 + Kx). \end{aligned}$$

Equating this value to the inclination ∂_n, we get

$$1 + Kx = K_1,$$

so that $$x = \frac{1}{K}(K_1 - 1).$$

This determines the position of the lens so that it may be equivalent to the given system of lenses.

When there are two lenses, we have shown the value of K to be

$$K = a_1 k_1 k_2 + k_2 + k_1,$$

and therefore

$$K_1 = a_1 k_2 + 1.$$

Thus in this case

$$x = \frac{a_1 k_2}{K} = \frac{a_1 F_2}{f_2},$$

or finally $$x = \frac{a_1 f_1}{f_1 + f_2 - a_1}.$$

EXAMPLES.

1. Four convex lenses whose focal lengths are a, b, b, a, are placed at intervals $a+b$, $2b(a+b)/(a-b)$, $a+b$, on the same axis; show that any emergent ray is in the same straight line with the corresponding incident ray.

Let rays diverge from an object and pass through the lenses. Let u, v be the distances of the object and its first image, in front of and behind the first lens, respectively, and let u', v', u'', v'', u''', v''' denote similar quantities for the other lenses in succession.

Then $$\frac{1}{u}+\frac{1}{v}=\frac{1}{a},$$

and therefore $$v=\frac{au}{u-a}.$$

Next, $$u'=a+b-v=\frac{bu-a^2-ab}{u-a}.$$

At the next lens we have

$$\frac{1}{u'}+\frac{1}{v'}=\frac{1}{b},$$

from which we get $$v'=\frac{b(a^2+ab-bu)}{a^2}.$$

Also $$u''=\frac{2b(a+b)}{a-b}-v'$$

$$=\frac{b[a(a+b)^2+bu(a-b)]}{a^2(a-b)}.$$

At the next lens we have

$$\frac{1}{u''}+\frac{1}{v''}=\frac{1}{b},$$

and therefore $$v''=\frac{a(a+b)^2+bu(a-b)}{3a^2+ab+u(a-b)}.$$

Also $$u'''=a+b-v''$$

$$=\frac{2a^2(a+b)+au(a-b)}{3a^2+ab+u(a-b)}.$$

Lastly, by refraction at the remaining lens

$$\frac{1}{u'''}+\frac{1}{v'''}=\frac{1}{a},$$

and therefore

$$v''' = \frac{2a(a+b)+u(a-b)}{a(b-a)},$$

so that

$$v''' = -\left[a+b+\frac{2b(a+b)}{a-b}+a+b+u\right].$$

This equation shows that the position of the final image coincides with the position of the object.

We shall next show that the final image has the same magnitude as the object.

Let x, x', x'', x''', x'''' be the linear magnitudes of the object and the images in succession. Then

$$\frac{x'}{x} = -\frac{v}{u},\quad \frac{x''}{x'} = -\frac{v'}{u'}, \text{ \&c.},$$

and therefore

$$\frac{x''''}{x} = \frac{vv'v''v'''}{uu'u''u'''}$$

$$= \frac{a}{u-a}\times\left[-\frac{b(u-a)}{a^2}\right]\times\frac{a^2(a-b)}{b[3a^2+ab+u(a-b)]}\times\frac{3a^2+ab+u(a-b)}{a(b-a)};$$

that is

$$x'''' = x.$$

It follows by Helmholtz' theorem, or by elementary geometry, that the divergence of the emergent pencil is the same as that of the incident pencil, and therefore any ray passing through the lenses emerges in the same straight line as before incidence.

2. If an eye be supposed to consist of a sphere of fluid (radius r, refractive index $\frac{11}{10}$), in which is placed at a distance $\frac{2}{3}r$ from the centre a convex lens whose axis coincides with the diameter and whose focal length and refractive index in air are, respectively, $\frac{1}{3}r$ and $\frac{6}{5}$; show that the distance from the centre of the sphere for clear vision is $\frac{213}{503}r$.

3. From a cubic inch of glass, the inscribed sphere is removed, a film of glass remaining at the points of contact. The cavity is filled with water. A bright point is placed on the axis at a distance of one inch from one face of the cube. Prove that the conjugate focus is at the point of the cube nearest to the luminous point.

4. Prove that the magnifying power of a thin double-convex lens, the radius of each surface being ρ, when the space between the lens and an object at distance a is filled with fluid of index μ', is given by

$$\frac{1}{m} = 1-\frac{a}{\rho}\,\frac{2\mu-\mu'-1}{\mu'}.$$

5. If f be the focal length of a double-convex lens, show that the smallest distance between an object and its image is $4f$.

6. Two thin lenses of equal numerical focal length f are placed on the same axis at a distance a apart, the one nearest the origin of light being concave and the other convex; show that the least distance between an object and its final image is $a+4f^2/a$.

7. Two lenses of equal focal length f are placed so as to be on a common axis at a distance $2e$ from one another, and midway between them is placed a glass sphere of radius r and index μ. A thin pencil diverges from a point distant c behind the first lens, and after refraction through it, through the sphere and the second lens, converges to a point at a distance c behind the second lens; prove that

$$\frac{\mu r}{\mu-1}=\frac{ec+cf+fe}{c+f}.$$

8. A pencil of rays is directly refracted through a series of thin lenses separated by finite intervals $a_1, a_2 \dots a_{n-1}$, the axes being coincident. Show that if the focal lengths of the lenses (considered as concaves in the typical case) be $1/k_1, 1/k_2 \dots 1/k_n$, the abscissæ of a pair of conjugate points reckoned from the first and final lens, respectively, are connected by the equation

$$\frac{1}{v}=\frac{1}{k_n+}\,\frac{1}{a_{n-1}+}\,\frac{1}{k_{n-1}+}\cdots\frac{1}{k_1+}\,\frac{1}{u}.$$

9. If m, m', m'' be the magnifying powers of a combination of any number of lenses on the same axis for objects at distances u, u', u'' from the first lens, show that

$$\frac{u'-u''}{m}+\frac{u''-u}{m'}+\frac{u-u'}{m''}=0.$$

10. If x be the distance between two objects and x' the distance between the corresponding images due to any system of lenses, and if m be the magnification of the first image and n that of the second, show that

$$\frac{x'}{x}=\frac{\mu'}{\mu}mn,$$

where μ and μ' are the refracting indices of the initial and final media.

11. Three lenses A, B, C (of which A and C are double-concave and B is double-convex) are mounted on an axis in the order named, so that the foci of A and C coincide at the centre of B. An object beyond C is viewed through the system by an eye behind A; show that the distance through which it would have to be displaced in order that, when viewed directly, it may have the same apparent

magnitude as when viewed through the system is independent of the position of both the eye and the object, if the focal lengths of the lenses are connected by the relation

$$\frac{1}{f}+\frac{2}{f'}+\frac{1}{f''}=0.$$

12. A thin lens has one face silvered so as to form a mirror. If Q be the image of a point P, formed by the mirror (by two refractions and one reflexion), show that Q will be the same as if the lens were replaced by a spherical mirror whose radius R is given by the equation

$$\frac{1}{R}=\frac{\mu}{s}-\frac{\mu-1}{r},$$

r and s being the radii of the surfaces of the lens.

13. Show that the image of an arc of a conic whose focus is at one principal point of a thick lens, is an arc of a conic whose focus is at the other.

14. A double-convex lens is formed by two equal paraboloidal surfaces cut off by planes through the focus perpendicular to the axis. Prove that for rays passing in the neighbourhood of the axis, the focal length measured from the posterior surface of the lens is $2a/(\mu^2-1)$, and the distance between a bright point and its image is a minimum when it is $2a(\mu+1)/(\mu-1)$, $4a$ being the latus rectum of either of the generating parabolas, and μ the refractive index of the glass.

15. A system of $2n$ thin convex lenses of equal numerical focal length, f, are placed with their axes in the same straight line, and their centres at a distance $4f$ apart, except the two middle ones, which are at a distance $8f$ apart. Show that the focal length of a lens which must be placed midway between the two middle ones in order that the image of a bright point at a distance $4f$ in front of the first lens may be formed at an equal distance behind the last lens is

$$\frac{2(n+1)}{2n+1}f.$$

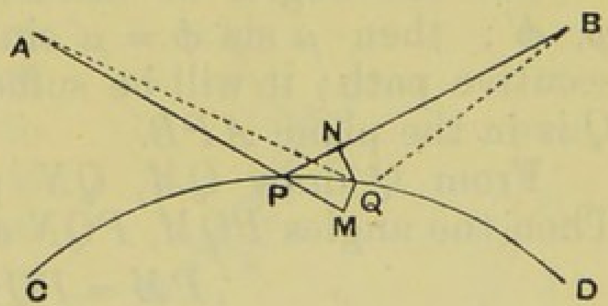
CHAPTER V.

General Theorems. Caustics.

77. If a ray of light pass from a point A to another point B, through any number of media, undergoing any number of reflexions and refractions, then the actual laws of reflexion and refraction are such as to make $\Sigma(\mu\rho)$ a minimum, where ρ represents the length of the path of the ray situated in the medium whose refractive index is μ. Conversely if we assume the path of light to be such as to make $\Sigma\mu\rho$ a minimum, we are led to the actual laws of reflexion and refraction. The expression $\Sigma\mu\rho$ is frequently called the *reduced path*.

We shall first prove this general theorem for a single reflexion and a single refraction, and afterwards extend it to any number of reflexions and refractions.

Let APB be the path of a ray of light which travels in a homogeneous medium from a point A to a point B, undergoing one reflexion at a surface CD; then the total path between A and B is a minimum, that is, $AP + PB$ is less along the actual path than along any consecutive path as AQB.



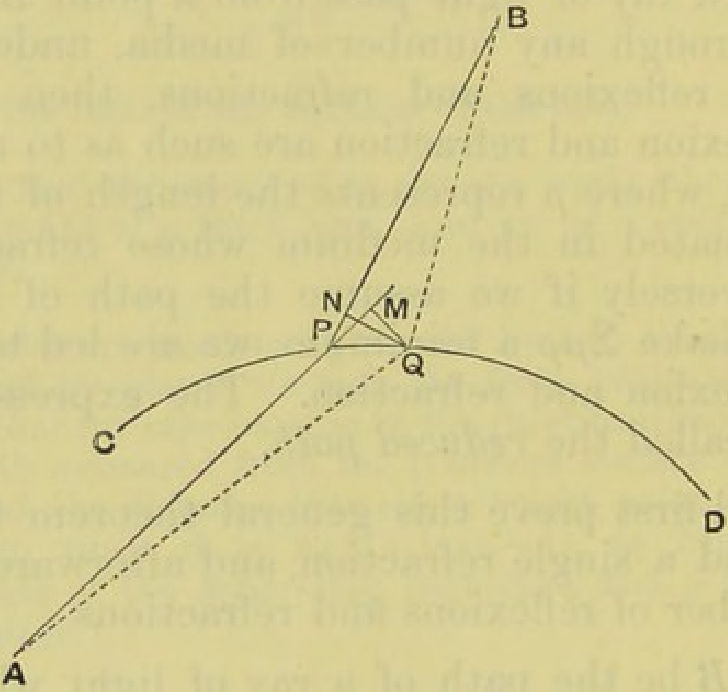
For a variation of P perpendicular to the plane APB, this proposition is clearly true. Let AQB be a consecutive path in the plane APB. From Q draw QM, QN perpendicular to AP and PB. Then by the law of reflexion the lines PM, PN are equally inclined to PQ. Hence the triangles MPQ, NPQ are equal in all respects, and PM is equal to PN.

Now AQ is $> AM$, and therefore $AQ - AP$ is $> PM$.
Also BQ is $> BN$, and therefore $BP - BQ$ is $< PN$.

Hence $AQ - AP$ is $> BP - BQ$.

Thus, $AQ + QB$ is $> (AP + PB)$, which proves that the total path is a minimum

A similar theorem holds if we take the path from A to B, supposing the ray to suffer a refraction at a surface CD. Let μ, μ' be the refractive indices of the two media, then $\mu AP + \mu' PB$ is a minimum for the actual path.



Let the angles of incidence and refraction at P be ϕ, ϕ'; then $\mu \sin \phi = \mu' \sin \phi'$. Let AQB be a consecutive path; it will be sufficient to take the case when Q is in the plane APB.

From Q draw QM, QN perpendicular to AP, PB. Then the angles PQM, PQN are the angles ϕ, ϕ'. Also

$$\left.\begin{aligned} PM &= PQ \sin \phi \\ PN &= PQ \sin \phi' \end{aligned}\right\};$$

therefore $\mu PM = \mu' PN$.

Now AQ is $> AM$, and therefore

$$\mu AQ - \mu AP \text{ is } > \mu PM.$$

Also BQ is $> BN$, and therefore

$$\mu' BP - \mu' BQ \text{ is } < \mu' PN.$$

Hence $\mu AQ - \mu AP$ is $> \mu' BP - \mu' BQ$;

that is $\mu AQ + \mu' BQ$ is $> \mu AP + \mu' BP$.

This shows that for the actual path, $\mu AP + \mu' PB$ is a minimum.

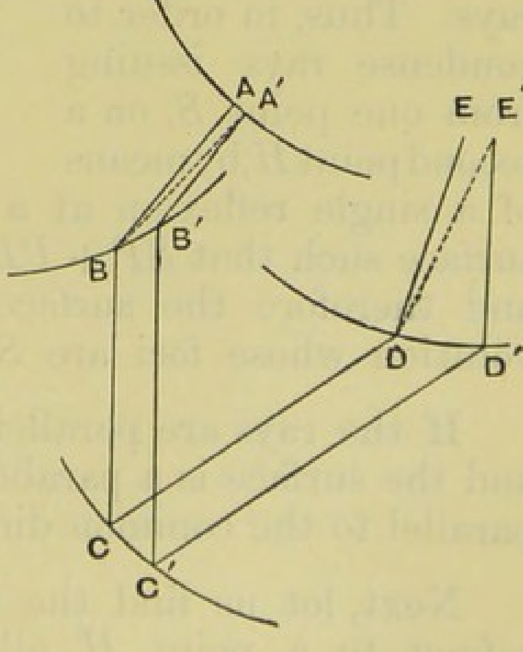
The previous theorem is a particular case of this; we have only to put $\mu' = -\mu$ to deduce it from the more general theorem.

Next, suppose that the ray of light in its passage from A to B undergoes any number of refractions or reflexions. Let ρ be the length of the path in any medium whose refractive index is μ. Then it has been shown that $\Sigma\mu\rho$ is a minimum for separate variations of the points of incidence between consecutive media; and therefore by the principle of superposition of small variations, it will be a minimum when simultaneous variations are admitted. The actual path, therefore, makes $\Sigma\mu\rho$ a minimum between any two points.

78. Another important proposition, enunciated by Malus, easily follows from the preceding.

Any system of rays originally normal to a surface, will always retain the property of being normal to a surface after any number of reflexions or refractions.

Let $ABCDE$, $A'B'C'D'E'$... be a series of rays normal to a surface at A, which undergo any number of refractions and reflexions. Measure off along these rays distances to E, E'......, such that $\Sigma\mu\rho$ is the same along each ray; then we shall show that the rays are finally normal to a surface EE'. Join $A'B$ and $E'D$. Then $\Sigma\mu\rho$ along $ABCDE$ is the same as along $A'B'C'D'E'$. But by what has been shown above for any ray and its consecutive it follows that $\Sigma\mu\rho$ along $A'BCDE'$ is the same as along $A'B'C'D'E'$, and

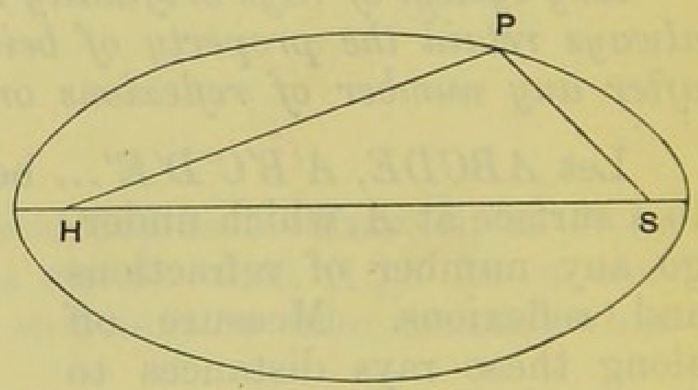


therefore the same as along $ABCDE$. Take away the common parts; then if μ, μ' belong to the initial and final media, there remains the equation, $\mu A'B + \mu' DE' = \mu AB + \mu' DE$. But, since AB is normal to the surface AA', $A'B = AB$ ultimately, and therefore, $DE' = DE$; that is, EE' is perpendicular to DE. The same may be proved for every point E' near E, and thus the surface EE' near E is perpendicular to the ray DE, and by similar reasoning to every other ray of the system.

79. A system of rays which can be cut at right angles by a surface, we shall call an *orthotomic* system.

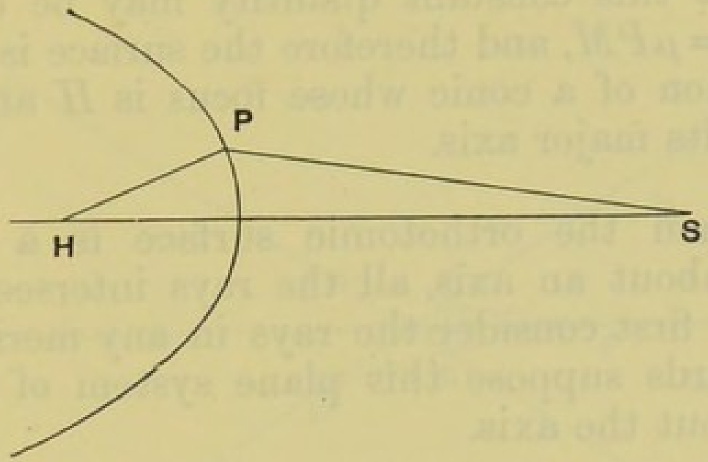
A system of rays diverging from a point, or such that by any combination of mirrors or refracting surfaces they can be made to meet in a point, is clearly orthotomic; for a sphere whose centre is the point through which all the rays pass, will cut them all at right angles.

If a system of rays diverging from a point converge to another point after any number of reflexions and refractions, the values of $\Sigma\mu\rho$ taken from one point to the other will be the same for all rays. Thus, in order to condense rays issuing from one point S, on a second point H, by means of a single reflexion at a curved surface, we choose our surface such that $SP + PH$ may be the same for all paths, and therefore the surface must be an ellipsoid of revolution whose foci are S and H.



If the rays are parallel, the point S will be at infinity, and the surface is a paraboloid of revolution whose axis is parallel to the common direction of the rays.

Next, let us find the form of the surface which will refract to a point H all the rays proceeding from a point S. Let μ, μ' be the refractive indices of the



media; then if P be any point of the surface, the surface must be such that

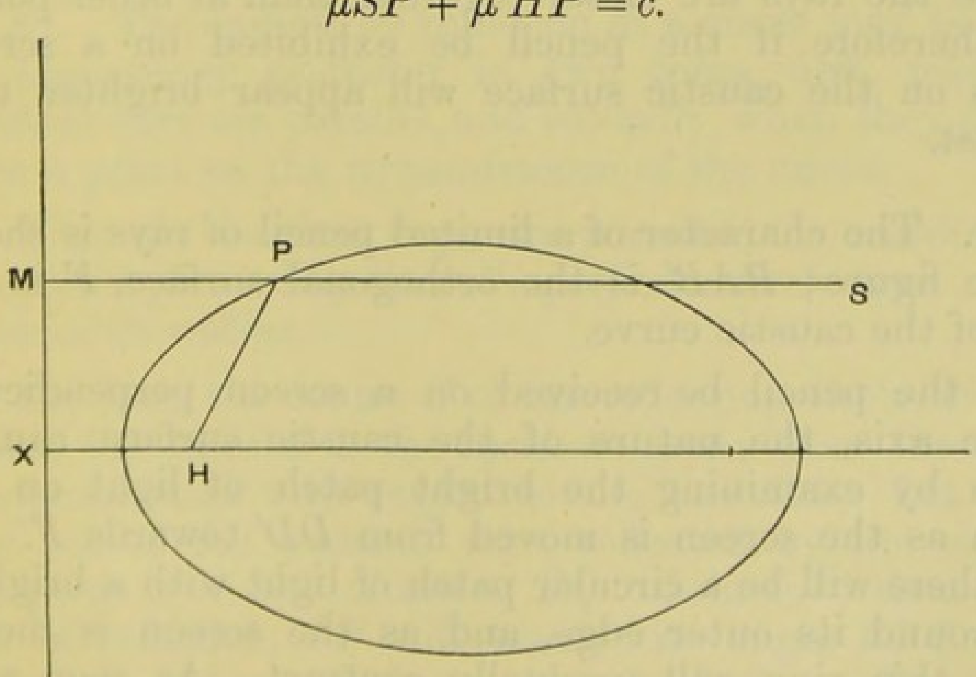
$$\mu SP + \mu' HP = c,$$

where c is a constant.

Hence the surface is formed by the revolution of a Cartesian oval of which S and H are foci. The theory of the Cartesian oval may be found in Williamson's *Differential Calculus*, Chapter 20.

As a particular case suppose the rays parallel, so that S is at infinity. Draw a plane MX perpendicular to the rays, and let any ray be produced to meet this plane, in M. Then

$$\mu SP + \mu' HP = c.$$



But $\mu SP + \mu PM$ is also constant. Choose the plane MX so that this constant quantity may be equal to c; then $\mu' HP = \mu PM$, and therefore the surface is formed by the revolution of a conic whose focus is H and directrix MX, about its major axis.

80. When the orthotomic surface is a surface of revolution about an axis, all the rays intersect the axis, and we may first consider the rays in any meridian plane and afterwards suppose this plane system of rays to be revolved about the axis.

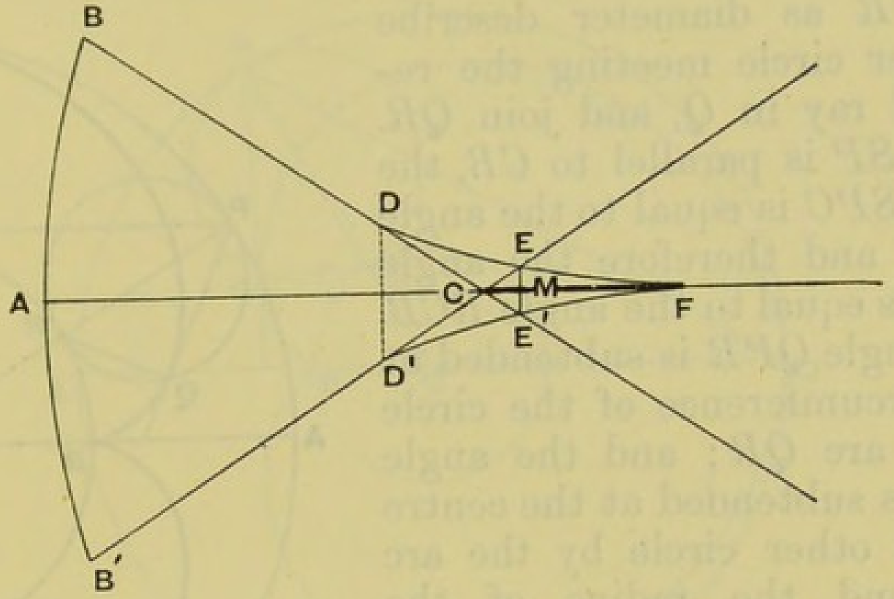
The rays in any meridian plane are a series of normals to a curve. Consecutive rays will intersect each other in points lying on a curve which is called the evolute of the given curve, and each ray will touch this evolute. The evolute which is touched by all the rays is called a *caustic curve;* and the surface formed by its revolution about the axis a *caustic surface.* A caustic curve of such a symmetrical system as we are considering always has a cusp on the axis.

When the system is revolved about the axis consecutive rays along the circle traced out by a point will meet on the axis, and therefore the axis may be considered as a second caustic surface. At points on a caustic surface the rays are closer together than at other points, and therefore if the pencil be exhibited on a screen, points on the caustic surface will appear brighter than the rest.

81. The character of a limited pencil of rays is shown in the figure; BAB' is the orthogonal surface, F is the cusp of the caustic curve.

If the pencil be received on a screen perpendicular to the axis, the nature of the caustic surface can be shown by examining the bright patch of light on the screen as the screen is moved from DD' towards F. At DD', there will be a circular patch of light with a brighter ring round its outer edge, and as the screen is moved along, this ring will gradually contract. As soon as C

is reached, the other part of the caustic surface is shown,



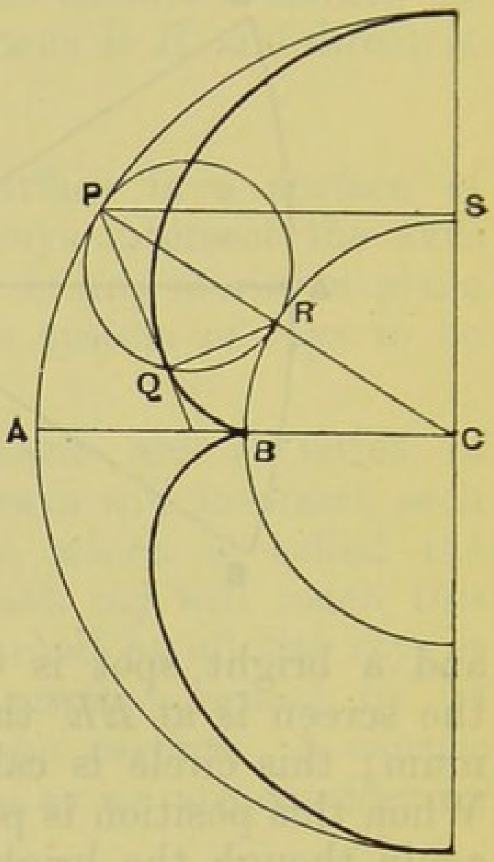
and a bright spot is developed in the centre. When the screen is at EE' the circle of light reaches its minimum; this circle is called the *least circle of aberration.* When this position is passed, the outer boundary expands again though the bright ring still contracts. Beyond F, no part of the screen is specially illuminated.

If any ray BCE meet the axis in C, then FC is called the *longitudinal aberration* of the ray.

82. The caustic by reflexion at a circle may be found by elementary geometry in two cases, first, when the incident rays are parallel, and secondly, when they diverge from a point on the circumference of the circle.

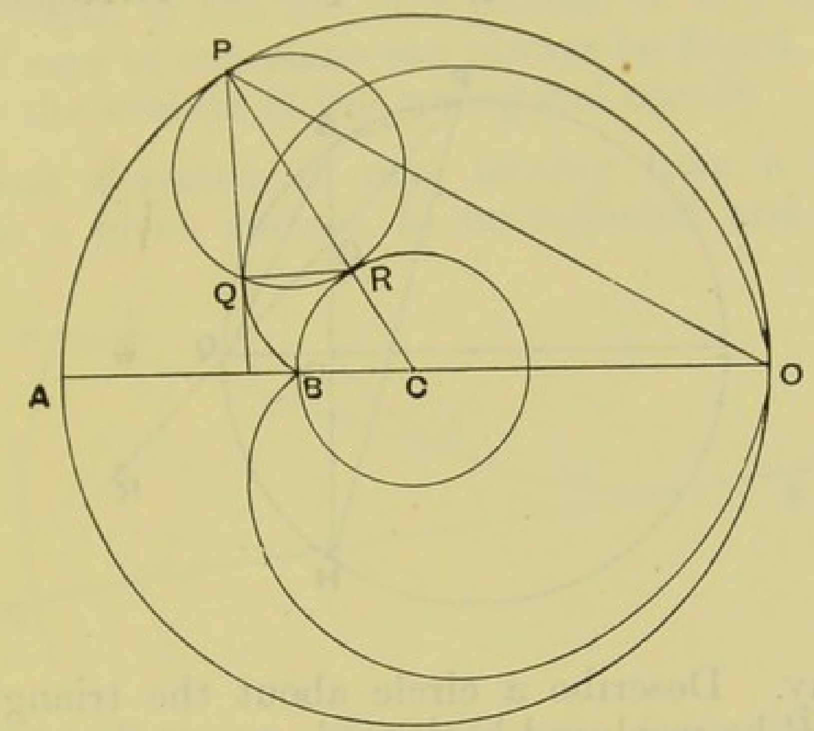
When the incident rays are parallel, the caustic is an epicycloid formed by the rolling of one circle upon another of twice its radius.

For from the centre C of the reflecting circle, draw the radius CA parallel to the incident rays; then the caustic is symmetrical with regard to the line CA. Let SP be any one of the incident rays, reflected by the circle at the point P in the direction PQ. Join CP; then by the law of reflexion, CP will bisect the angle SPQ. With centre C and a radius equal to half the radius of the given circle, describe the circle BR bisecting the radii CA, CP in B, R, respectively. On PR as diameter describe another circle meeting the reflected ray in Q, and join QR. Since SP is parallel to CB, the angle SPC is equal to the angle PCB; and therefore the angle QPR is equal to the angle RCB. The angle QPR is subtended at the circumference of the circle by an arc QR; and the angle RCB is subtended at the centre of the other circle by the arc RB, and the radius of the second circle is double the radius of the first, and therefore the arc QR is equal to the arc RB; and if the circle PQR were to roll along the circle RB, the point Q would finally coincide with B. Now as Q begins to move, the point of contact R is for an instant fixed, so that the motion of Q is perpendicular to QR; and therefore the reflected ray PQ touches the curve described by Q. This is true whatever the position of the point P. The locus of Q is an epicycloid, and this is the caustic curve required.



83. *If the incident rays diverge from a point in the circumference of the reflecting circle, the caustic curve is a cardioid*, or, in other words, the caustic may be described as an epicycloid in which the rolling circle is equal to the fixed circle.

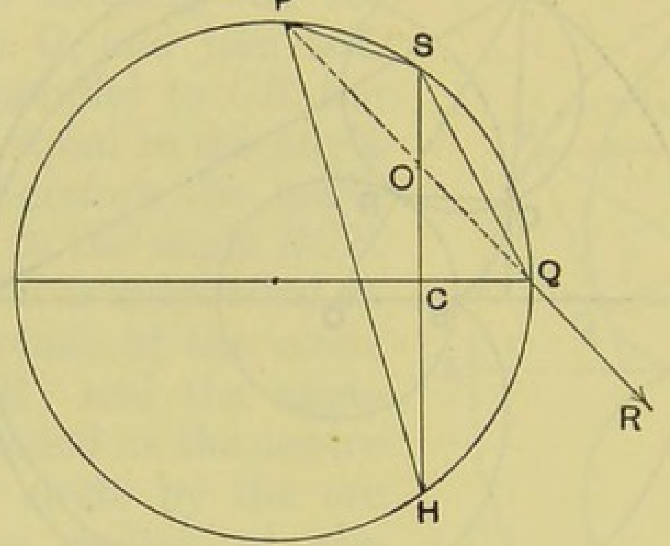
Let O be the origin of the incident rays, OCA the diameter of the reflecting circle; then the caustic curve will be symmetrical about the line OCA. Let OP be any incident ray which is reflected at P by the circle in the direction PQ. Join CP; then by the law of reflexion, CP will bisect the angle OPQ. With centre C and radius equal to one-third of the radius of the given circle,



describe a circle meeting CA and CP in B and R, respectively, and on PR as diameter describe another circle cutting the reflected ray in Q; join QR. The radii of the two smaller circles will be equal to each other. Now, since the triangle CPO is isosceles, the external angle PCB is double of the angle CPO, and therefore double of the angle QPR. Hence the arcs RB, QR subtend equal angles at the centres of their respective circles, and therefore these arcs are equal. If the circle PQR were to roll along the circle RB, the point Q would finally come to B. As the circle PQR begins to roll, the point of contact R is for a moment stationary, and therefore Q begins to move perpendicular to QR along PQ. From this it follows that the reflected ray touches the curve described by the point Q. This is true whatever the position of the point P. The locus of Q is a cardioid, and this is the caustic required.

84. *To find the caustic by refraction at a straight line, for rays issuing from a point.*

Let S be the bright point; draw SC perpendicular to the line, and produce it to H, so that $CH = CS$. Let SQ be any ray incident at Q, and QR the corresponding re-



fracted ray. Describe a circle about the triangle SHQ, and let QR be produced backwards to cut the circle in P; then PQ bisects the angle SPH. Let ϕ be the angle of incidence and ϕ' the angle of refraction at Q; then the angle $POS = \phi'$, and $\phi = \angle HSQ = \angle HPQ = \angle SPO$.

Hence $$SO : SP = \sin\phi : \sin\phi',$$

and therefore $$\mu SO = \mu' SP.$$

But since the angle P is bisected,

$$HO : HP = SO : SP,$$

and therefore $$\mu HO = \mu' HP.$$

By addition, $\mu SH = \mu' (SP + HP)$.

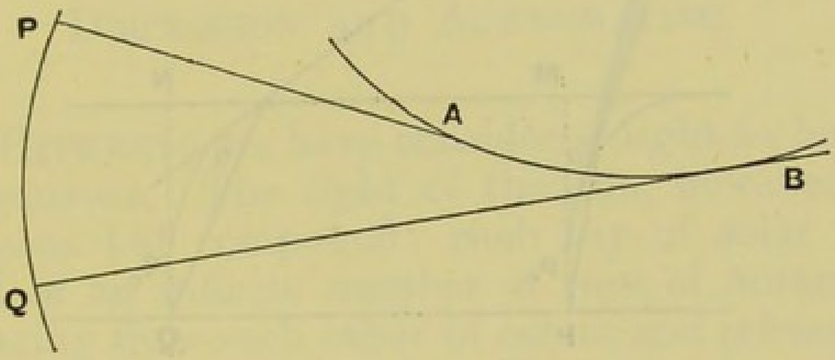
Thus the locus of P is an ellipse whose foci are S and H and whose eccentricity is μ'/μ; and PQ is normal to the ellipse, and therefore the ellipse is an orthotomic curve. The evolute of this ellipse is the caustic required.

If the second medium is more highly refractive than the first, it may be shown in the same way that the caustic is the evolute of a hyperbola whose foci are S and H.

85. *To find the length of the arc of a caustic.*

The length of the arc of a caustic of any orthotomic system of rays in one plane can always be found. For the caustic is the evolute of the orthogonal curves.

Suppose a system of rays issuing from a point, or normal to a given surface, to be reflected and refracted



any number of times. For each ray, form the function $\Sigma\mu\rho$, and let $V = \Sigma\mu\rho$. Let the final medium be of refractive index μ, and let $V = V_0$ be the value of the reduced path for an orthogonal curve in this medium, say the curve PQ. Let AB be any arc of the caustic, and let PA, QB be the rays touching at A, B. Then the arc $AB = QB - PA$, by the properties of evolutes.

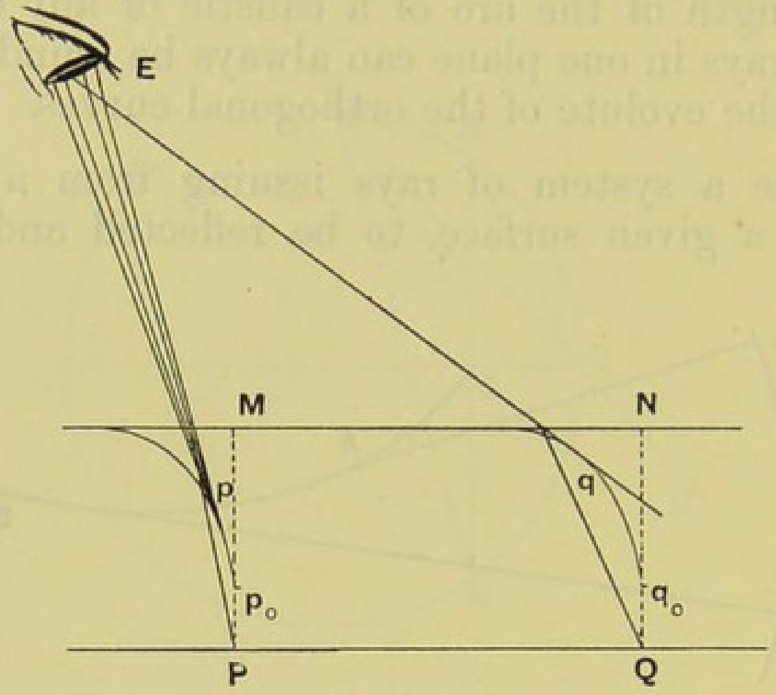
Also
$$V_A = V_0 + \mu PA,$$
$$V_B = V_0 + \mu QB;$$

and therefore by subtraction,

$$V_B - V_A = \mu\,(\text{arc } AB).$$

86. We can now, by means of caustics, indicate more accurately the manner and position in which an object under water is seen by an eye outside.

Suppose for instance that the water had a horizontal level bottom not very deep. Let P be a point on the bottom, let us trace the pencil of rays by which an eye sees the point P. Draw the normal PM and consider rays in the plane EPM. Construct the caustic in this plane which is touched by refracted rays originally diverging from P. We must draw the two extreme tangents to



this caustic which will meet the eye, and then these lines will bound the part of the pencil which traverses the air; if we join the points where these tangents meet the surface to P, the joining lines will bound the pencil as it passes through the water. The two tangents to the caustic meet at the point of contact of either of them, very nearly. Thus to an eye outside the point P appears to be at p.

CHAPTER VI.

DISPERSION AND ACHROMATISM.

87. HITHERTO we have considered light to be simple or homogeneous. The light of the sun, however, is not homogeneous but compound; each ray of solar light is composed of an infinite number of rays of homogeneous light differing from each other in colour and refrangibility. This fact was first established by Newton.

In Newton's first experiments his room was darkened and a beam of the sun's light admitted through a small circular hole in the shutter of one of the windows. This beam of light made a small circular spot of white light on the opposite wall. He then placed a triangular prism of glass near the hole, with its edge downwards and perpendicular to the beam of sunlight, so that the rays passed through the prism close to its edge. The patch of light on the wall was no longer circular and white, but elongated and coloured with vivid and intense colours. The sides of the coloured image or *spectrum* were both straight and perpendicular to the edge of the prism, and the ends appeared semicircular. The breadth of the spectrum was the same as that of the circular white spot, while its length was about five times greater.

This elongation of the image can only be explained by supposing that the rays of the beam of sunlight are refrangible in different degrees. The rays from the sun are not quite parallel, for some might proceed from the upper and others from the lower limb of the sun's disc. But when the prism is placed in its position of minimum deviation, a small difference of incidence will produce no appreciable difference of deviation; consequently the inclination of the emergent rays will be the same as those of the incident rays; and therefore if the beam of light were homogeneous it would cause a circular spot of white light of the same dimensions as before, but in a displaced position.

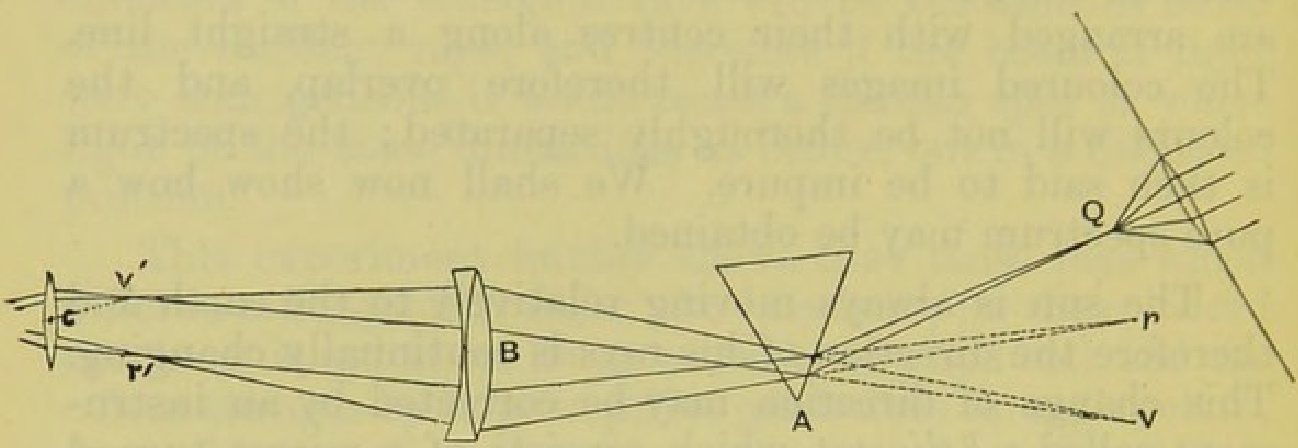
This experiment further shows that those rays which differ in refrangibility differ also in colour; for the coloured spectrum is red at its lower or least refracted end, and the colour changes by imperceptible gradations through yellow, green, blue, until at the upper or most refracted end it is violet. Newton distinguished seven principal colours; these arranged in order of their refrangibility are red, orange, yellow, green, blue, indigo, violet. Of these the orange and yellow are the most luminous, the red and green next in order, and the indigo and violet weakest.

88. After trying several ways of explaining those phenomena Newton was finally led to the following *experimentum crucis*, which is described almost in Newton's own words. He took two boards, and placed one of them close behind the prism at the window, so that the light might pass through a small hole, made in it for the purpose, and fall on the other board, which was placed at about twelve feet distance, a small hole having first been made in it also for some of that incident light to pass through. Then he placed another prism behind this second board, so that the light passing through the two boards might pass through that also, and be again refracted before it reached the wall. This done, he took the first prism and turned it slowly to and fro about its axis, so as to make the several parts of the image cast on the second board successively pass through the hole in it, and observed to what places on the wall they were refracted by the second prism. He saw that the light tending towards the violet end of the spectrum was considerably more refracted than the light tending towards the red end. Hence he concluded that sunlight is not homogeneous, but consists of rays of different colours, some of which are more refrangible than others.

89. In this form of the experiment the different coloured images of the sun are of considerable size, and are arranged with their centres along a straight line. The coloured images will therefore overlap, and the colours will not be thoroughly separated; the spectrum is then said to be impure. We shall now show how a pure spectrum may be obtained.

The sun is always moving relatively to the earth and therefore the direction of his rays is continually changing. This change of direction may be corrected by an instrument called a *heliostat*, which consists of a mirror turned by clockwork in such a way that the light is always reflected in the same direction. The reflected rays of the sun are allowed to fall on a convex lens of short focal length, so as to make a very small image of the sun at the focus of the lens; this image may easily be made so small that it may be regarded as a point. A small pencil may be selected from the rays passing through this point by making them fall on a very narrow slit between two carefully worked plates of metal. If a cylindrical lens with its generating lines parallel to the slit be used, the rays may be concentrated on the slit throughout its whole length, and a very bright thin pencil can be obtained. The pencil of light is allowed to fall on a prism near the refracting edge, this edge being parallel to the slit. The prism must be placed in the position of minimum deviation for rays of mean refrangibility, and then it will be nearly in a position of minimum deviation for all rays.

Let Q be the small focus or the section of the slit through which the rays pass. Then after refraction at the prism the red rays will diverge from a point r, and the violet rays from a point v, where $Av = Ar = AQ$. If the colours be received on a screen, they will overlap, and though by moving the screen farther away from the edge of the prism, the colours become more and more separated, yet they become fainter at the same time. The pencil is therefore made to pass through an achromatic lens (the construction of which will be hereafter described), whose centre is B, after which the red rays will converge to a focus r', and the violet rays to a focus



v', where rBr', vBv' are straight lines. The colours are now perfectly separated, but the spectrum $v'r'$ is very small, so that it needs to be magnified before it can be accurately measured. The spectrum is therefore viewed through another lens or eye-piece (also corrected for chromatic dispersion). The two lenses constitute an ordinary astronomical telescope. If therefore the rays from the prism be received on a telescope, by focusing the telescope we shall be able to see a pure spectrum.

If we wish to exhibit the spectrum on a screen, the lens must be removed. In this case it is better to put between Q and A a lens whose focus is at Q. Then the rays after passing this lens are parallel and the points v and r are at an infinite distance; and by moving the screen further from A we separate the colours more and more without weakening their intensity.

Newton himself described fully how a pure spectrum might be obtained by the use of a lens and a narrow slit in front of the prism.

90. If a pure solar spectrum be examined carefully, it is found that it is not a continuous coloured band, but that there are at certain intervals abrupt deficiencies of light, forming dark lines across the spectrum. These lines are always seen irregularly disposed along the spectrum whatever refracting substance may be used. When the refracting substance is varied, the positions of the lines change, but they and the coloured rays always appear in

the same order, so that any line can be recognised. As these lines are sharp and definite and are always present, they can be used as marks for determining refractive indices; the refractive indices of the rays to which they correspond can be determined for any substance with an accuracy equal to that of astronomical measurements. The positions of these lines, to the number of seven hundred, have been carefully measured and mapped out by Fraunhofer and others, and the refractive indices of the corresponding rays accurately determined for a very large number of substances. By using prisms of the same substance but of different refracting angles, Fraunhofer verified the law of refraction for the rays corresponding to any one of the fixed lines, with extreme accuracy. These dark lines are not characteristic of light in general, but only of solar light; for if the slit be illuminated by a gas-flame, a perfectly continuous spectrum is observed.

The brightness of the solar spectrum is by no means uniform; it is brightest in the yellow and the neighbouring colours, orange and light green, and falls off gradually on both sides. It may be observed here, though this scarcely belongs to the province of optics, that the solar rays as separated into a spectrum differ from each other also in heating and chemical effects. The heating effect increases as we pass from the violet to the red rays, and still continues to increase for a certain distance beyond the visible spectrum, at the red end. Similarly, if the action of the different rays on a sheet of sensitive paper be observed, the action is very feeble in the red, strong in the blue and violet, and is sensible to a great distance beyond the violet end of the spectrum.

91. There are three different kinds of spectra depending upon the nature of the source of the light employed.

i. The solar spectrum is a continuous spectrum, except that it is interrupted by a definite system of dark lines. The spectra of fixed stars also contain dark lines, different for different stars.

ii. The spectra afforded by incandescent solids and liquids are continuous, containing light of all refrangibilities, from the extreme red to a higher limit depending on the temperature.

iii. Flames not containing solid particles in suspension, but emitting the light of incandescent gases, give discontinuous spectra, consisting of a definite number of bright lines.

92. Modern experiments have proved that the missing rays in the solar and similar spectra have been removed by absorption. For according to the theory of exchanges it is known that every substance which emits certain kinds of rays to the exclusion of others, absorbs the same kind as it emits; and when the temperatures are the same in the two cases, the amount emitted and the amount absorbed are equal. When an incandescent vapour emitting only rays of certain definite refrangibilities is interposed between the observer and a very bright source of light giving a continuous spectrum, the gas absorbs from the incident light just those rays which itself emits, the light emitted by the gas being substituted for the light it absorbs. It depends on the relative brightness of the two sources whether these particular rays be in excess or defect. If the two sources be at all comparable in brightness the rays will be greatly in excess, and will appear as bright lines across the spectrum; for these rays constitute the whole light of the one, but only a very small fraction of the light from the other source. But if the brilliancy of the gas be diminished, while that of the source of the continuous spectrum be increased sufficiently, the rays emitted by the gas become less intense than those which have been absorbed, and so by contrast the corresponding lines of the spectrum appear dark. The dark lines in the solar spectrum would therefore be accounted for by supposing that the principal portion of the sun's light comes from an inner mass which gives a continuous spectrum, and that a stratum external to this contains vapours which absorb particular rays and thus produce dark lines.

For further details connected with the subject of Spectrum Analysis we refer to works dealing specially with that subject.

93. When a ray of light from the sun falls on a prism of glass, we have seen that it is separated into rays of different colours; this fact is called *dispersion*. We shall now seek a proper measure of the *dispersive power* of a substance.

We must first select some ray of the spectrum as a standard ray; we might with advantage choose the ray corresponding to some well-defined dark line occurring about the middle of the spectrum. Let μ be the refractive index of the standard ray.

The measure of the dispersive power of a substance must be independent of the refracting angle of the prism which is used in the experiment. Take a prism of small refracting angle ι, and let D be the deviation for the standard ray; then

$$D=(\mu-1)\,\iota,$$

when the light passes through the prism in a direction nearly perpendicular to its faces. If μ', D' correspond to any other ray of the spectrum, we shall have

$$D'=(\mu'-1)\,\iota;$$

and therefore by subtraction,

$$D'-D=\iota\,(\mu'-\mu).$$

To eliminate ι, we divide this by the former result, so that we get

$$\frac{D'-D}{D}=\frac{\mu'-\mu}{\mu-1}.$$

This is taken as the measure of the dispersive power of the substance for the ray whose refractive index is μ', and is often denoted by ϖ; thus

$$\varpi=\frac{\mu'-\mu}{\mu-1}.$$

94. We may next choose a standard substance. Herschel proposed that water at its temperature of maximum density should be used as a standard, so that any ray might be identified by its refractive index referred to water, or, as we might say, by its position on the water scale.

The dispersive power of any other substance can be expressed as a function of that of the standard substance. It is found by theory combined with Fraunhofer's experiments that the ratio of the dispersive powers is nearly constant; this constant ratio may be called the dispersive power of the substance in terms of the standard, for all rays. This ratio is not, however, quite constant, and this fact is called the *irrationality of dispersion*. If two prisms be constructed, one of the standard substance and the other of the substance under consideration, then if the spectrum given by each be examined, the fixed lines and coloured rays will occur in the same order in each, but since the dispersions of corresponding rays by the two substances are not proportional, the spectra will not be geometrically similar. If the prisms be arranged side by side so as to give spectra of equal lengths and so that the extreme rays in each may correspond in position, the intermediate rays will not exactly correspond in position.

95. If a ray of light be made to pass through two prisms in succession, it is always possible to adjust their refracting angles, so that the dispersion produced by the first may be counteracted approximately by the second, and consequently that the emergent ray may be without colour.

This Newton conceived to be impossible, without at the same time making the deviations of the two prisms counteract one another, so that the whole deviation of the pencil would disappear. This made him despair of improving refracting telescopes, and led him to turn his attention to the application of mirrors to these instruments.

Newton's mistake was first discovered by a gentleman named Chester Moor Hall, who made the first achromatic telescope. This discovery, however, was allowed to fall into oblivion, until the experiment was again tried by Dollond, an optician in London, who found that the dispersion could be corrected without destroying the deviation, and therefore that Newton's conclusion was not correct.

We have seen, however, that different coloured rays are not dispersed in the same proportion by different substances; or in other words, that the spectra formed by prisms of different substances are not geometrically similar. Hence, if the prisms be arranged so as to unite two rays (for example, the extreme red and the extreme violet rays) in the emergent beam, there will be still a small dispersion of the other rays. Thus the beam instead of emerging quite colourless, will form a second but much smaller spectrum; this is called the *secondary spectrum.*

Also, it will be found that by using three prisms of three different materials, three rays of the emergent beam (for example, the red, green and violet) may be united; but still, owing to the irrationality of dispersion, the other rays will not be quite united, and there will be another still smaller spectrum called a tertiary spectrum; and so on indefinitely. In theory, therefore, it is impossible to attain perfect achromatism, without the use of a very large number of different media; yet in practice these successive spectra rapidly grow fainter and become insensible; so much so, that it is seldom deemed necessary to combine more than two rays. The two rays selected will not be the extreme red and violet rays, because these are comparatively faint; it is better to combine the two rays whose brightness and difference of colour are greatest, such as a ray from the yellow-orange and one from the green-blue.

The first successful attempt to get rid of the secondary spectra was made by Blair; an account of his work was published in the *Phil. Trans. Edin.*, 1791. He found that in the spectrum of hydrochloric acid the more refrangible part of the spectrum, green to violet, was much more contracted, and the less refrangible part of the spectrum more dilated, than in most metallic solutions; and by mixing the chlorides of antimony and of mercury in suitable proportions with hydrochloric acid, or with salammoniac, he obtained a fluid which, while having a different absolute dispersion from crown-glass, gave a spectrum geometrically similar to that of crown-glass. When a combination of two lenses or two prisms was constructed out of this fluid medium and crown-glass, in such a way that in the emergent beam of light two differently coloured rays should be united, the emergent beam was absolutely without colour. Blair's object-glasses were considered as of singular merit at the time, but through certain inconveniences attending lenses made of fluid media they never came into use.

What Blair effected with fluid lenses, Professor Abbé of Jena has now achieved by his discoveries of new kinds of glass. In 1881, Professor Abbé, assisted by Dr Schott, commenced the work of examining the optical properties of all glasses, that is, of all known substances which undergo vitreous fusion and solidify in non-crystalline transparent masses. The work was continued till the end of 1883, and directed towards the solution of two practical problems. The first of these was the production of pairs of kinds of flint and crown-glass, such that the dispersion in the various regions of the spectrum should be, for each pair, as nearly as possible proportional. The second problem was the production of a greater multiplicity in the gradations of optical glass, in respect of the two chief optical constants, the index of refraction and the mean dispersion. The first problem has been satisfactorily solved, with the result that achromatic lenses of a much more perfect kind than have ever before been attainable are now being manufactured; and the second has also been successfully carried out, and a whole series of new glasses of graduated properties are at the service of the optician.

Moreover the same experiments have resulted in the production of glasses which having the same refractive index, offer considerable latitude in dispersive power, and also of glasses having the same dispersive power, which vary considerably in refractive index; whereas formerly an increase of refractive index was always accompanied by an increase of dispersive power.

Achromatism of lenses.

96. By the proper combination of lenses the dispersion of differently coloured lights may approximately be destroyed; for the dispersion produced by one lens may be approximately counteracted by that produced by a second lens, so that the emergent rays may be without colour.

We shall confine our attention to the approximate theory of lenses, in which the thickness of the lens is neglected and the principal points considered as coinciding in one point called the centre of the lens. For the accurate theory of lenses becomes in the general case much complicated by the fact that the principal points of the lenses, from which all distances are usually measured, themselves vary in position according to the refractive index of the particular ray we are considering.

In all cases we shall let μ be the refractive index of the standard ray, and μ' the refractive index of any other ray. The focal lengths of the lenses will be supposed to be expressed in terms of the refractive index of the standard ray.

It will be useful to find the change in the focal length of a lens, as the ray changes from the standard ray, to any other. The value of the focal length of a double convex lens, the radii of whose bounding surfaces are r, s, respectively, is given by the equation,

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r} + \frac{1}{s}\right),$$

where μ is the refractive index of the substance for the standard ray. Giving a small variation to μ, so that it becomes μ', this equation gives

$$\frac{1}{f'} = (\mu' - 1)\left(\frac{1}{r} + \frac{1}{s}\right).$$

By subtraction, we get

$$\frac{1}{f'} - \frac{1}{f} = (\mu' - \mu)\left(\frac{1}{r} + \frac{1}{s}\right)$$

$$= \frac{\mu' - \mu}{\mu - 1}\frac{1}{f};$$

and therefore, if we denote the dispersive power of the medium by ϖ, the variation of the focal length is determined by the equation

$$\frac{1}{f'} = \frac{1 + \varpi}{f}.$$

97. When an image is formed by a lens or system of lenses which is not achromatic, the light being not homogeneous, it will be affected by dispersion in the lenses in two particulars; first, the different coloured images will be distributed in different positions along the axis of the system, and secondly, the coloured images will have different magnitudes. In certain cases both these defects can be removed, in other cases only one of them can be removed, and to choose which correction shall be made, it will be necessary to consider the use to which the system is to be applied, so as to remove the defect which is of the most consequence.

For the object-glass of a telescope two lenses are used, and are placed close together so as to act as one lens. Then a point and its image always lie on the same line through the centre of the lens, so that if the lenses be corrected so that the differently coloured images all lie in the same plane perpendicular to the axis, they will all have the same magnitude. It will therefore be necessary only to make the first correction, and then the other will be satisfied.

These object-glasses are usually made of a double convex lens of crown-glass outside, combined with a double concave lens of flint-glass, which has a higher dispersive power than crown-glass. It is easy to see in a general way how the correction may be effected. By the convex lens the coloured images will be formed at different distances along the axis, the violet image being the nearest to the lens, and the red image the most remote from it. The effect of the concave lens on these images will be to throw them farther away from the lens, and the effect on the violet image will be stronger than that on the red image. By a proper adjustment of the lenses, the final violet image may be made to coincide with the final red image, or any two other colours may be united in the final image. If the lenses were of the same kind of glass, in order that the dispersion produced by the one should be neutralized by that produced by the other, the lenses would have to be such that the deviation produced by the two lenses would also destroy each other, and therefore the combination would not produce an image at all. But it has been seen that for different kinds of glass the dispersion is not proportional to the deviation, but that flint-glass has a higher dispersive power than crown-glass, so that it is possible to destroy the dispersion without destroying the deviation.

98. We shall now investigate the condition that a combination of two lenses made of different kinds of glass, placed close together, may be achromatic for two given colours.

We shall suppose that one of the colours is the standard colour, and that the focal lengths of the two lenses are f, f', respectively. There will be two images; the first being the image of the object formed by the first lens, and the second being the image of this first image formed by the second lens. Let x, x' be the distances of the object and the first image in front of, and behind, the centre of the first lens, y', y the distances of the first and second images in front of, and behind, the centre of the second lens, respectively. Then

$$\frac{1}{x}+\frac{1}{x'}=\frac{1}{f},$$

$$\frac{1}{y}+\frac{1}{y'}=\frac{1}{f'}.$$

If we neglect the thicknesses and the distance between the lenses, $y'=-x'$, and therefore

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{f}+\frac{1}{f'}.$$

The condition that the system should be achromatic is that y should be the same for the two colours; and therefore, since x is independent of the colour,

$$\frac{1}{x}+\frac{1}{y}=\frac{1+\varpi}{f}+\frac{1+\varpi'}{f'},$$

and therefore, by subtraction,

$$\frac{\varpi}{f}+\frac{\varpi'}{f'}=0.$$

This is the condition of achromatism for the combination.

This condition is independent of x and y, so that *the combination will be achromatic for objects at all distances.* It is immaterial in what order the lenses are placed.

In the construction of microscopic object-glasses, achromatic couples of this kind are very generally used, each consisting of a plano-concave lens of flint cemented to a double convex of crown, the plane face being exposed to the incident light.

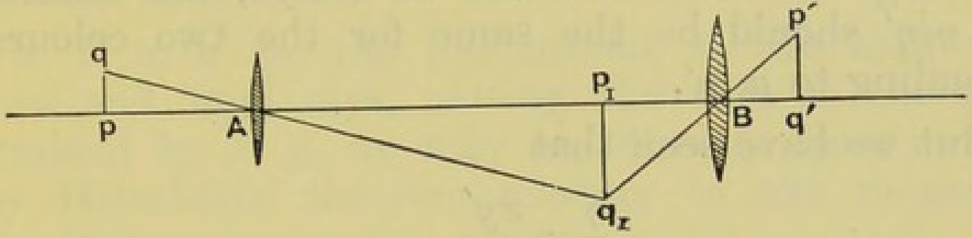
99. If three thin lenses, formed of media of different dispersive powers, be combined into a single lens, the system may be made achromatic to a higher degree of approximation; the coloured images formed by three different kinds of light may be united. More generally, if n lenses form a combination, whose thickness may be

neglected, the system will unite the images formed by rays whose refractive indices are μ and μ', provided that

$$\Sigma\left(\frac{\varpi}{f}\right)=0.$$

This may be proved in the same way as before. The equation of condition can be satisfied for $n-1$ systems of values of $\mu'-\mu$, and therefore the images corresponding to n lines of the spectrum may be united.

100. When the two lenses forming a combination are separated by an interval, it is impossible simultaneously to effect the two corrections for dispersion.



For let x, x' be the distances of the object and its first image in front of, and behind, the first lens, y', y the distances of the first and final image in front of, and behind, the second lens, respectively, and let η, η_1, η' be the linear magnitudes of the object and its images. Then the following ratios must hold:

$$\left.\begin{aligned}\frac{\eta}{\eta_1}&=-\frac{x}{x'}\\ \frac{\eta_1}{\eta'}&=-\frac{y'}{y}\end{aligned}\right\},$$

and therefore
$$\frac{\eta}{\eta'}=\frac{xy'}{x'y}.$$

If the coloured images corresponding to refractive indices μ, μ' be formed at the same distance and also have the same magnitude, we must have x and y fixed and also the ratio $xy' : x'y$. Hence the ratio $y' : x'$ is fixed. But $x'+y'=a$, where a denotes the distance between the lenses; so that it is necessary that x', y' should both be the same for the two colours. In other words, each lens must be achromatic of itself. This cannot be effected unless each lens of the combination be itself an achromatised couple of lenses in contact.

101. It is often necessary, however, to correct a system of two lenses separated by an interval, for errors due to dispersion, as far as possible; so that we must choose which of the two corrections should be effected, and which left.

It is then usual to make the coloured images have the same magnitude; for the eye is a better judge of the magnitude of an object than of its distance.

Using the same notation as before, the condition is that η/η' should be the same for the two colours corresponding to μ, μ'.

But we have seen that

$$\frac{\eta}{\eta'} = \frac{xy'}{yx'}$$

$$= \frac{x}{x'}\left\{\frac{y'}{f'} - 1\right\},$$

by virtue of the equation $\frac{1}{y} + \frac{1}{y'} = \frac{1}{f'}$.

Also $x' + y' = a$; and therefore

$$\frac{\eta}{\eta'} = \frac{x}{x'}\left(\frac{a - x'}{f'} - 1\right)$$

$$= \frac{x}{x'}\left(\frac{a}{f'} - 1\right) - \frac{x}{f'}$$

$$= \left(\frac{x}{f} - 1\right)\left(\frac{a}{f'} - 1\right) - \frac{x}{f'},$$

or finally,

$$\frac{\eta}{\eta'} = 1 - \frac{x}{f} - \frac{x+a}{f'} + \frac{ax}{ff'}.$$

For the ray of refractive index μ', this becomes

$$\frac{\eta}{\eta'} = 1 - \frac{x(1+\varpi)}{f} - \frac{(x+a)(1+\varpi')}{f'} + \frac{ax(1+\varpi)(1+\varpi')}{ff'}.$$

Equating these expressions, and neglecting the product $\varpi\varpi'$, we get

$$\frac{x\varpi}{f} + \frac{(x+a)\,\varpi'}{f'} = \frac{ax\,(\varpi + \varpi')}{ff'}.$$

This is therefore the condition for the partial achromatism of the two lenses. In general, it is not independent of the position of the object.

102. If we consider the inclinations of rays to the axis of the instrument, instead of the magnifying power, it will be seen that we have ensured that two differently coloured rays diverging from the object will emerge parallel to each other.

For if α, α' be the inclinations to the axis of the original and final rays, cutting the axis at the points determined by x, y, we may see directly from a figure, or by Helmholtz' theorem relating to the magnifying power, that

$$\frac{\eta}{\eta'} = \frac{\tan\alpha'}{\tan\alpha} = \frac{xy'}{x'y};$$

so that if the condition previously found be satisfied, then α' is the same for the two colours; and the final rays emerge parallel to each other.

103. The most useful application of this condition is to the achromatism of eye-pieces. The rays strike the eye-piece excentrically diverging from the image formed by the object-glass. The images formed by the lenses of the eye-pieces are formed exactly as if the rays diverged from a real object, except that the rays from any point of the image do not fill the whole of the lens.

It will be shown later that the axes of pencils passing through the instrument pass through the centre of the object-glass. The centre of the object-glass is usually very distant as compared to the focal lengths of the lenses of the eye-piece. If we make x very large in the previous equation of condition, it becomes

$$\frac{\varpi}{f} + \frac{\varpi'}{f'} = \frac{a\,(\varpi + \varpi')}{ff'},$$

or
$$a = \frac{\varpi f' + \varpi' f}{\varpi + \varpi'}.$$

This condition may be derived in a shorter manner for this particular case by making the focal length of the equivalent lens the same for two colours.

There is a special advantage in making the lenses of the same kind of glass, because then if we make two coloured images coincide, all the coloured images will be united. The condition for achromatism then becomes

$$a = \frac{f + f'}{2};$$

or in words, *the distance between the lenses must be half the sum of their focal lengths.*

EXAMPLES.

1. Shew that at a single refraction at a plane surface the dispersion is proportional to the tangent of the angle of refraction.

2. The refractive index of a medium for the two rays at the red and violet ends of the spectrum being 1·63 and 1·66 respectively, calculate the dispersive power. *Ans.* $\frac{2}{43}$.

3. Calculate the dispersive power of a medium for which the refractive indices for the same two rays are 1·53 and 1·54 respectively, and find the ratio between the focal lengths of two lenses formed of the media in this and the last example, that the combination may be achromatic when the lenses are placed in contact. *Ans.* $\frac{2}{107}$, 43 : 107.

4. Prove that if f be the focal length of a lens, ϖ its dispersive power, v the distance from the centre of the lens of the point to which a pencil of standard rays is made to converge, the distance between the foci of the red and violet rays for the same incident ray is approximately $\varpi v^2/f$.

5. The dispersive power of a medium is ·036. The focal length of a lens formed of it being 3 feet for standard rays, find the distance between the extreme images of the sun formed by the lens. *Ans.* 0·108 feet.

6. If μ, ν be the indices of refraction for the red and violet rays, respectively, for crown-glass, and μ', ν' be the indices for the same rays for flint-glass; and if two thin lenses be constructed, one double convex of crown-glass with each surface of radius r, and one double concave of flint-glass with its surfaces of radii r and s, and they be placed in contact so that the light is incident on the surface of radius s; then the combination will be achromatic if

$$r+s : 2s = \mu - \nu : \mu' - \nu'.$$

7. A small pencil of parallel rays of white light, after transmission in a principal plane through a prism, is received on a screen whose plane is perpendicular to the direction of the pencil; prove that the length of the spectrum will be proportional to

$$(\mu_v - \mu_r) \sin \iota \div \cos^2 D \cos (D + \iota - \phi) \cos \phi';$$

where ι is the refracting angle, ϕ, ϕ' the angles of incidence and refraction at the first surface, and D the deviation of the mean ray.

8. If an achromatic eye-piece for an astronomical telescope be composed of two convex lenses of different materials, prove that the distance between them must be intermediate between f' and $lf/(l-f)$, where f is the absolute focal length of the field-glass, f' that of the eye-glass, and l the length of the telescope from object-glass to field-glass.

9. Prove that a system of three thin convex lenses made of the same material, placed so that the distance between the first and second is a, and that between the second and third is b, is achromatic for a pencil coming from a point on the axis whose distance from the first lens is

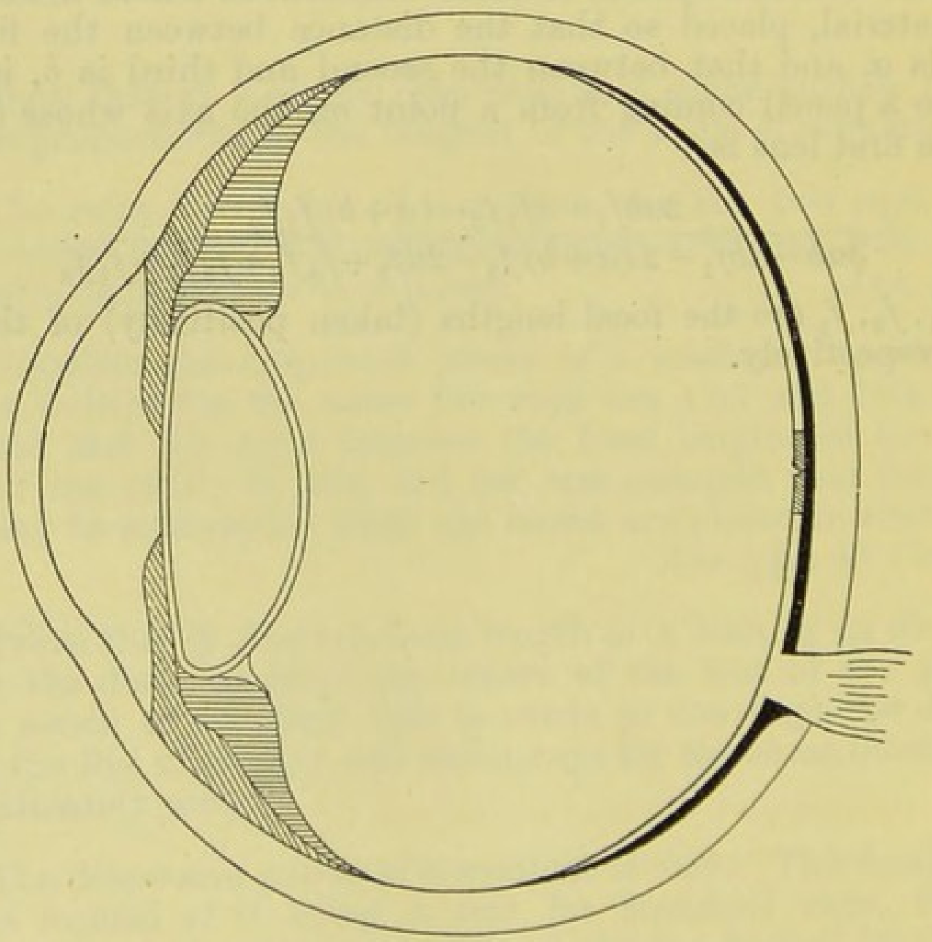
$$\frac{2abf_1 - af_1f_3 - (a+b)f_1f_2}{3ab - 2bf_1 - 2(a+b)f_2 - 2af_3 + f_2f_3 + f_3f_1 + f_1f_2}$$

where f_1, f_2, f_3 are the focal lengths (taken positively) of the three lenses, respectively.

CHAPTER VII.

THE EYE, AND VISION THROUGH LENSES.

104. THE eye is an optical instrument consisting essentially of a series of refracting media bounded by curved surfaces, and a delicate network of small nerve-fibres forming part of the optic nerve; a pencil of light incident upon the eye is refracted at the curved surfaces and brought to a focus on the network of nerve-fibres, and the impression is carried to the brain along the optic nerve.



The human eye is nearly spherical in shape, except in front, where it bulges out a little more than elsewhere. It is invested in a thick tough coat which, except in the small protuberant front part, is opaque and white and is called the *sclerotic*. This is partly exposed in the living eye, and is in common language termed the white of the eye. The more protuberant part of the ball is covered with a thick, strong, transparent membrane called the *cornea*.

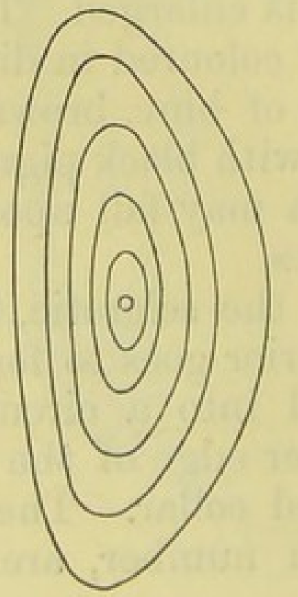
105. The eyeball has two other linings; immediately within the sclerotic is a thin membrane called the *choroid*, and within that there is another thin lining called the *retina*.

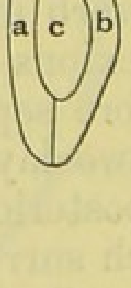
The interior of the choroid coat is covered with black pigment, which gives it a velvety appearance; the function of this is to absorb rays of light which have passed through the retina and prevent them from being thrown back on the retina, so as to interfere with the distinctness of the images there formed. The anterior portion of the choroid, separating from the sclerotic, is thickened and forms the *iris*, which is a contractile curtain perforated in the centre by an aperture called the *pupil*. The outer edge of the iris is fixed, but the inner edge may be contracted by a strong muscular band running round it, and thus the size of the pupil may be changed. The use of the iris is to regulate the quantity of light allowed to fall on the sensitive part of the eye. In strong lights the pupil contracts automatically and in feeble lights it is enlarged. The anterior surface of the iris is differently coloured in different persons, varying through all shades of blue, brown, and grey. The posterior surface is covered with black pigment, which serves to absorb any light which may fall upon it, due to internal reflexions or other causes.

Just before separating from the sclerotic, the choroid splits into two layers; the anterior goes to form the iris, while the posterior is gathered into a circular plaited curtain which surrounds the outer edge of the lens (to be presently described) like a plaited collar. These plaits or folds, seventy to seventy-two in number, are called the *ciliary processes.* Beneath this dark plaited collar, and therefore in contact with the sclerotic, is a muscular collar, with radiating fibres, called the *ciliary muscle.*

The retina is a delicate semi-transparent membrane resulting from the spreading out of the optic nerve, and is composed of the terminal fibres of this nerve and nerve cells; it covers the whole of the interior of the ball as far as the ciliary collar. Exactly in the centre of the retina is a round yellowish elevated spot, about $\frac{1}{20}$th of an inch in diameter, having a minute indentation, called the *fovea centralis,* at its summit. This is the point of distinct vision and the fovea centralis is the most sensitive part of the retina. About $\frac{1}{10}$th of an inch on the inner side of the yellow spot is the point at which the optic nerve spreads out its fibres to form the retina; this is the only spot on the retina which is not sensitive to light rays, and is known as the blind spot.

106. Within the eye, a little behind the iris, is suspended a soft transparent body, called the *crystalline lens,* of the form of a double convex lens, whose anterior surface is less curved than the posterior. The crystalline lens is contained in a thin transparent capsule, and is kept ~~in its place by the ciliary processes.~~ It is composed of successive layers, whose refractive indices increase towards the centre, its solid nucleus, which is of very small radius of curvature, refracting light most powerfully.





It is easy to see that the action of the lens is more powerful than if it were composed of homogeneous substance having the same refractive index as the nucleus. For it may be regarded as the combination of a double convex lens *c*, with two other concave lenses *a* and *b*. These concave lenses will neutralise the effect of the lens *c* to a certain extent; but not so much as if their refractive indices were as high as that of *c*. The focal length of the lens may be found by experiment, and its shape being known, its so-called total refractive index may be found; that is, the refracting index which the lens would possess were it homogeneous. From what has been previously said, it follows that this total refractive index is greater than that of the nucleus.

The increase of refracting power from the outer portions to the inner portions of the lens serves partly to correct the aberration, by increasing the convergence of the central rays more than that of the extreme rays of the pencil.

107. The space between the cornea and the crystalline lens is filled with a transparent fluid resembling water, and thence termed the *aqueous humour*. The space between the crystalline lens and the retina is filled with another transparent fluid, somewhat more viscous than the former, and called the *vitreous humour*. These two humours, like the crystalline lens, are contained in transparent membranous capsules of great delicacy.

In their refractive indices the aqueous and vitreous humours differ very little from water, while the total refractive index of the crystalline lens is a little greater than that of water.

108. To determine the manner in which a pencil of light incident on the eye is refracted by it, we must know the refractive indices of the different media of which the eye is composed, and the forms and positions of the bounding surfaces.

The anterior surface of the cornea is very nearly that of a segment of an ellipsoid of revolution, the axis of revolution being the major axis. The form of the posterior surface is not very accurately known. But the two surfaces of the cornea are very nearly parallel, and as the anterior surface is always moistened with water, whose refractive index is the same as that of the aqueous humour, the cornea acts like a plate of refracting medium, and produces no deviation in an incident ray. The cornea itself may therefore be entirely neglected, and we may for optical purposes suppose the aqueous humour extended to the anterior surface of the cornea.

The anterior surface of the crystalline lens is part of the surface of an oblate spheroid, and the posterior is supposed to be part of the surface of a paraboloid of revolution.

109. There are therefore three surfaces at which refraction takes place, the first surface of the cornea and the two surfaces of the crystalline lens. The centres of curvature of these surfaces are very nearly in a straight line, called the optic axis. For rays whose deviations from the axis are not large, the surfaces may be supposed to coincide with the spheres of curvature at their respective vertices. Gauss' theory of refraction at any number of spherical surfaces whose centres lie along an axis is therefore applicable to this case, and the positions of the focal points, the principal points, and the nodal points may be found by calculation, as soon as the radii of curvature, the positions of the refracting surfaces and the indices of refraction of the media are known. Listing has given the following numbers as representing very closely the constants of an average eye; in reckoning refractive indices, the refracting index of the air is taken to be unity.

(*a*) The radii of curvature of the bounding surfaces have the following values:

1. The anterior surface of the cornea...... 8 mm.
2. The anterior surface of the lens.........10 mm.
3. The posterior surface of the lens ...... 6 mm.

(*b*) The distances between the refracting surfaces are:

From 1 to 2 .... 4 mm.
From 2 to 3 (thickness of the lens).......... 4 mm.
From 3 to the retina.................................13 mm.

(*c*) The indices of refraction are:

1. For the aqueous humour......................... $\frac{103}{77}$,
2. For the lens (total) .... $\frac{16}{11}$,
3. For the vitreous humour .... $\frac{103}{77}$.

From these data he calculates the positions of the cardinal points according to Gauss' theory, and finds that the two principal points lie very close together, as do also the two nodal points, so that without introducing much error, we may regard them as coinciding in each case. The single principal point lies 2·3448 mm. behind the cornea, and the nodal point ·4764 mm. in front of the second surface of the lens. Such an eye is exactly equivalent to a single refracting spherical surface, whose vertex is at the principal point and centre at the nodal point, the refractive index being $\frac{103}{77}$ as before. A point and its image on the retina will lie on a line passing through the nodal point; and therefore if we wish to find in what direction lies a point whose image is in a given position on the retina, we have only to join the image to the nodal point and produce the line outwards.

110. When the eye is passive, it is clear that only the points which lie in a single surface will have images falling exactly on the retina. The form of this surface and its position may be determined from the optical constants of the eye. Any object lying on this surface will have an image on the retina similar to the original figure, but inverted, the lines joining corresponding points of the object and image all passing through the nodal point. But if a point does not lie on this surface, its image will

be not on the retina, but in front of or behind it. In both cases the retina cuts the pencil of refracted rays not in a single point, but in a circle of diffused light. Hence it follows that an immoveable eye can only see distinctly objects lying in one surface, and if we consider only rays of light making small angles with the axis of the eye, this surface may be considered plane. All objects or portions of objects not lying in this plane give indistinct images, in which circles of diffusion correspond to luminous points of the object.

Experience teaches us, however, that an eye is capable of seeing distinctly at almost any distance; there must therefore exist an arrangement for altering the eye, and adapting it for seeing at different distances at will. The changes which occur as the result of this arrangement are included under the term *accommodation.* It is not known with absolute certainty for what distance an eye is adjusted when it is not actively accommodated, but it is almost universally supposed that a normal eye when passive is adjusted for objects at an infinite distance, so that the second focal point of the eye at rest is on the retina. It follows from this that accommodation only occurs in one direction, the eye being actively accommodated for near objects.

111. It has been found by experiment that accommodation is effected by change of form in the refracting surfaces of the eye. When the eye is accommodated for near objects, the anterior surface of the crystalline lens becomes more strongly curved, and approaches nearer to the cornea; this is especially the case with the part not covered by the iris, which arches forwards through the pupil.

112. It has been seen that when the eye is at rest in any position and accommodated for an object, there is one point, the *fovea centralis*, where the vision is distinct, but that the vision is distinct only for a very small area about this spot. But the eye is usually in very rapid motion, and in an incredibly short space of time brings the various

points of an object into distinct view. We are thus enabled to form a clear conception of a considerably extended object or surface. This is aided also by the duration of the impression produced by a light. It has been found by experiment that this duration depends on the character of the light. For strong lights Helmholtz gives $\frac{1}{24}$th of a second, and for weak lights $\frac{1}{10}$th of a second, as the duration of the impression. Lissajou and others assign about $\frac{1}{30}$th of a second as the lowest limit of the duration. If a spot on the retina be stimulated by a regular periodic light, whose period is sufficiently short, there will arise a continuous impression, which in intensity is equal to what would be produced were the whole incident light of any period uniformly distributed over the whole period.

113. The retinæ of both our eyes receive impressions when we look at any external object and in certain positions of our eyes we see two images, arising from the two retinæ, while in other positions we see only one image. To each point of one retina there is a *corresponding point* on the other; and when the images of an external point formed by the two eyes fall on corresponding points of the two retinæ, the point is seen single, but in other cases it is seen double. The points on the retina of an eye may be referred to two meridians formed on the retina by two planes through the axis of the eye. When the eye is directed forwards in a horizontal position, the points on the horizon have images lying on a meridian, which we may call the *retinal horizon*. Similarly certain lines appear vertical to an eye; the retinal image of these vertical lines is a meridian, which we may call the *apparently vertical meridian*. By experiment, Helmholtz concludes that the retinal horizon is actually horizontal for both eyes, but that the apparently vertical meridians are not quite perpendicular to the retinal horizon; they diverge outwards at their upper extremity. The inclination of each of these meridians to the real vertical is the same, and they include between them an angle varying from 2° 22′

to 2° 33′. Helmholtz also finds that in normal eyes, the points of distinct vision, as well as the retinal horizons and apparent verticals in the two eyes *correspond;* and further that corresponding points are equally distant from each retinal horizon and from each apparently vertical meridian.

114. Our most accurate estimate of the distances of visible objects depends upon our having two eyes. As we fix our gaze successively upon points at different distances we have to change the convergence of the axes of the two eyes, and from the degree of convergence of these axes when we look at any point we form an estimate of the distance of the point. Distances can however be estimated by a single eye, by observing the relative changes of position of objects, when the observer's position is changed.

Our idea of solidity also depends upon vision with two eyes. The views presented to the two eyes are slightly different, because the eyes have slightly different positions; and it is by the blending of the two impressions received upon the two retinæ that we receive the idea of solidity. This can be well shown by aid of the stereoscope. This instrument was invented by Wheatstone for the purpose of combining two different photographic pictures, one of which is presented to each eye. These pictures are not exactly alike, but are taken by a camera with two lenses placed a small distance apart, so that they represent two different views such as might be presented to two eyes observing the scene. By means of mirrors or prisms the pictures are seen superimposed, and the impression produced on the mind by these superimposed views is exactly the same as if we were looking at the real scene, each object appearing in relief as it would in nature. For a perfect stereoscopic representation, the points at an infinite distance must fall on corresponding points of the two retinæ when the axes of the eyes are parallel. If the pictures are brought nearer to each other in the same plane than in the positions thus determined the impression produced is exactly that of a relief picture.

Spectacles and Reading Glasses.

115. The distinctness of objects as seen by the naked eye depends on the accurate convergence of the rays of different pencils to points on the retina. We have seen that the eye is furnished with a mechanism for adapting itself for seeing distinctly objects at different distances. A normal eye when not actively accommodated is adapted for rays coming from a distant object, or for parallel rays; and it must be accommodated for seeing objects which are near, the range of distinct vision extending from five or six inches to infinity. Eyes for which the greatest distance of distinct vision is finite are called *short-sighted*, or *myopic;* these eyes can only bring divergent pencils to a focus on the retina. On the other hand, eyes which can bring to a focus on the retina not only parallel rays but convergent pencils are called *long-sighted* or *hypermetropic*. The defects in these eyes depend on the length of the axes of the eyes; in a short-sighted eye, the axis is too long, and in a long-sighted eye it is too short. In both short-sighted and long-sighted eyes the accommodating mechanism may be quite perfect. When this is the case, the defects may be entirely remedied and the eyes made normal by the use of spectacles.

116. Let the range of distinct vision by the naked eye extend from points distant a, b from the eye. In a normal eye, b will be infinite; in a short-sighted eye b will be finite and positive, and in a long-sighted eye b will be finite and negative. Suppose the eye to view an object through a lens of focal length f, placed close to the eye, f being positive for a collective lens, and negative for a dispersive lens. Then if x, x' be the distances from the eye (or from the lens) of an object and its image, respectively, measured in the same direction, outwards,

$$\frac{1}{x}-\frac{1}{x'}=\frac{1}{f}.$$

The rays striking the eye will appear to diverge from the image; and therefore the rays may be brought to a focus provided x' lies between the limits a, b. If we substitute for x' the values a, b in succession, the corresponding values of x will be the limits of the range of distinct vision through the spectacles.

When the accommodating mechanism is perfect, we have only to choose f, so that the farther limit of distinct vision is at an infinite distance. We must therefore make x infinite when $x' = b$, and thus we find the focal length of the spectacle glass, namely, $f = -b$. The nearer limit of the range of distinct vision becomes

$$x = \frac{ab}{b-a};$$

and therefore the range of distinct vision through the spectacles will extend from $ab/(b-a)$ to infinity.

In a short-sighted person b is finite and positive, and therefore f is negative; he must therefore use dispersive lenses, generally double concave lenses, whose focal length is equal to the greatest distance of distinct vision by the naked eye. Thus if the range of distinct vision extends from 3 to 6 inches from the eye, the use of a concave lens whose focal length is 6 inches, will cause the range of distinct vision to extend from 6 inches to infinity.

On the other hand, in a long-sighted eye b is negative, and therefore f is positive. For example, if the range of distinct vision extend from 12 inches outwards through infinity to -12 inches, the spectacles chosen must be collective lenses of 12 inches focal length; substituting these values in the general formula, we find that the range is then from 6 inches to infinity.

Practically, these glasses may be chosen by making the person look at a distant object; then the weakest concave glasses which will enable a short-sighted person to see this object distinctly, and the strongest convex glasses which will enable a long-sighted person to see it distinctly, are the glasses suitable to the eyes.

The limiting points of the range of distinct vision may be measured by making the person look through suitably chosen convex lenses, so that the points in question are brought within 12 inches from the eye, and then their distances can be measured on a divided scale. They are generally not the same for both eyes, so that the two eyes require different glasses.

Short-sighted persons who have to do delicate work, have sometimes to bring things close to the eyes; in this case they should use rather weaker concave glasses, than those prescribed above. For the same purpose achromatised prismatic glasses, which are thicker towards the sides next the nose, and thinner towards the sides next the temples are used, because the objects can then be seen with less convergence of the axes of the eyes.

117. As the age of a person advances, the eye gradually loses its power of accommodation; it is supposed that the outer layers of the crystalline lens lose their elasticity, so that the lens becomes less capable of changing its form and curvature. This defect is known as *presbyopia*. It is entirely different from the defect described above, called long-sightedness; though aged persons are sometimes said to be long-sighted. The structure of an eye does not alter with age, so that a person with normal eyes can still see distant objects when he becomes old; but the range of accommodation of the eye is then less than before, so that it cannot bring to a focus on the retina pencils of rays issuing from points very near to it; in other words, the nearer limit of distinct vision has receded from the eye. Presbyopic eyes therefore need convex glasses to enable them to see near objects, as in reading or writing; but they must be laid aside to look across a room or at a distant view. Usually the glasses are chosen so as to bring the nearer limit of distinct vision to 10 or 12 inches from the eye. For very aged persons, whose sight has lost its keenness, it is sometimes advisable to use spectacles which will bring this nearer limit to within 8 or even 7 inches from the eye, so that objects may be seen under a greater angle.

From what has been said, it is evident that presbyopia may exist along with the other defects previously mentioned. Both long-sighted eyes and short-sighted eyes can be made normal by the use of spectacles, as we have seen. When presbyopia sets in, these eyes will need two pairs of spectacles, one for walking and another for reading and writing.

118. A convex lens of considerable aperture and magnifying power is often used as a reading glass, or for viewing the details of small objects. Such a glass may be used by both short and long-sighted people. For suppose that the glass is placed, so that the object is in the principal focus of the glass, then the rays emerging from the lens are parallel. If the glass be now moved a little nearer to the object, the emergent rays will diverge, and can be brought to a focus on the retina by a short-sighted eye; if on the other hand the glass be moved a little farther away from the object, the emergent rays will converge and will be adapted for distinct vision by a long-sighted eye.

119. *On vision through any number of thin lenses.*

Before entering upon the modern theory of vision through any optical instrument it will be interesting to give an account of the method of treating the problem of vision through thin lenses used by the earlier English writers, Cotes and Smith, taken from Smith's *Opticks*, Cambridge, 1738, book ii. chapter v. The method is founded upon a theorem of Cotes, giving the apparent distance of an object seen by an eye in any position through any number of lenses.

By *apparent distance* is meant the distance at which the object would have to be placed so as to appear by direct vision of the same apparent magnitude as through the lenses. Thus, if η be the linear magnitude of the object, and α' the visual angle under which it is seen through the lenses, the apparent distance ξ will be determined by the relation

$$\frac{\eta}{\xi} = \tan \alpha'.$$

If α be the visual angle under which the object is seen by the naked eye, and x be the distance of the object from the eye, we have also

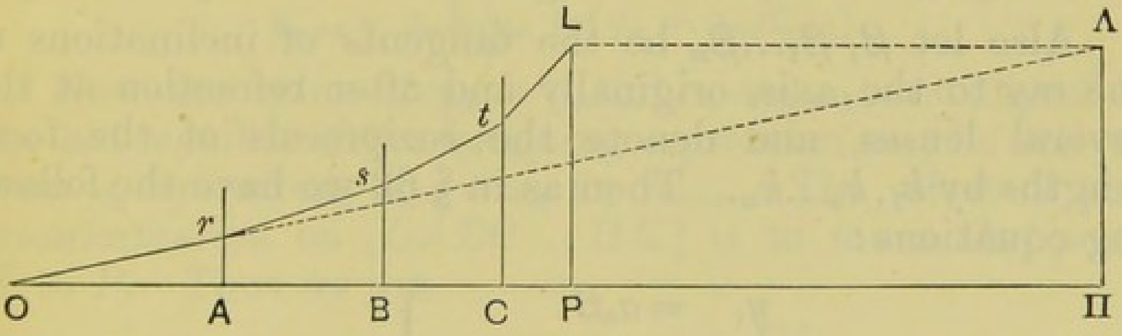
$$\frac{\eta}{x} = \tan \alpha.$$

The ratio $\tan \alpha' : \tan \alpha$ is defined as the *angular magnifying* power of the lens system.

Hence it follows at once that the *angular magnifying power is equal to the ratio of the true distance of an object to its apparent distance when seen through the lenses.*

120. *Cotes' Theorem.*

Let an object PL be viewed by an eye at O through any number of lenses placed at A, B, $C \dots K$, whose focal



lengths are a, b, $c \dots k$, and whose common axis is the line $OABC \dots KP$. In the standard case the lenses are supposed to be concaves. Then the distance OP may be considered as divided by each lens into two parts, as OA, AP; OB, $BP \dots$; and by each pair of lenses into three parts, such as OA, AB, $BP \dots$; and by each combination of three lenses into four parts, as OA, AB, BC, $CP \dots$; and so on, as far as the number of lenses permits. All the several products of such corresponding parts are to be divided by the focal length, or product of focal lengths of the lenses which are placed at the point or points of division. Then the sum of OP and these various quotients will be the apparent distance of the object.

If $O\Pi$ be the apparent distance, the statement for three lenses is

$$O\Pi = OP + \frac{OA\,.\,AP}{a} + \frac{OB\,.\,BP}{b} + \frac{OC\,.\,CP}{c}$$
$$+ \frac{OA\,.\,AB\,.\,BP}{ab} + \frac{OB\,.\,BC\,.\,CP}{bc} + \frac{OA\,.\,AC\,.\,CP}{ac}$$
$$+ \frac{OA\,.\,AB\,.\,BC\,.\,CP}{abc},$$

and the law of formation is general. We shall represent the apparent distance for any number of lenses, by the symbol $[OABC\ldots KP]$.

Denote the distances OA, $AB \ldots KP$ by a_0, $a_1 \ldots a_n$, supposing that there are n lenses, and let any ray $Orst\ldots L$ be supposed to flow from O to the object and cut the lenses at points whose distances from the axis are $y_1, y_2 \ldots y_n$, and let y_{n+1} be the linear magnitude of the object.

Also let β, $\beta_1 \ldots \beta_n$ be the tangents of inclinations of the ray to the axis, originally and after refraction at the several lenses, and denote the reciprocals of the focal lengths by $k_1, k_2 \ldots k_n$. Then as in § 63 we have the following equations:

$$\left.\begin{aligned}
y_1 &= a_0\beta \\
\beta_1 &= k_1 y_1 + \beta \\
y_2 &= a_1\beta_1 + y_1 \\
\beta_2 &= k_2 y_2 + \beta_1 \\
y_3 &= a_2\beta_2 + y_2 \\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
\beta_n &= k_n y_n + \beta_{n-1} \\
y_{n+1} &= a_n\beta_n + y_n
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1).$$

Also
$$y_2 \;\;= \beta\,[OAB]$$
$$y_3 \;\;= \beta\,[OABC]$$
$$\dots\dots\dots\dots\dots\dots$$
$$y_{n+1} = \beta\,[OABC\dots KP].$$

Solving the equations (1) we find

$$y_2 = \beta\,\{a_0 + a_1 + k_1 a_0 a_1\},$$
$$y_3 = \beta\,\{a_0 + a_1 + a_2 + k_1 a_0 (a_1 + a_2) + k_2 (a_0 + a_1)\, a_2 + k_1 k_2 a_0 a_1 a_2\},$$

and so on. These equations show that

$$[OAB] \;= OB + \frac{OA\,.\,AB}{a},$$

$$[OABC] = OC + \frac{OA\,.\,AC}{a} + \frac{OB\,.\,BC}{b} + \frac{OA\,.\,AB\,.\,BC}{ab}.$$

The general law of formation may be proved by induction.

Eliminating β_n between the last two of equations (1), we find

$$y_{n+1} = a_n k_n y_n + (y_n + a_n \beta_{n-1}).$$

Now the value of y_n is $a_{n-1}\beta_{n-1} + y_{n-1}$, and therefore the last member of the foregoing expression may be derived from y_n by writing $a_{n-1} + a_n$ for a_{n-1}. The effect of this transformation on $[OABC\dots HK]$ is to move the point K to P. Thus we get

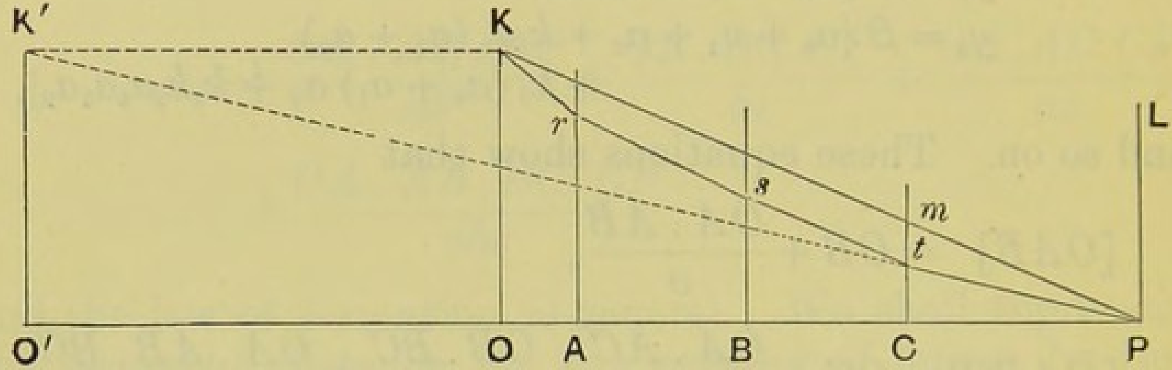
$$[OAB\dots HKP] = [OAB\dots HK]\frac{KP}{k} + [OAB\dots HP].$$

This equation proves that if the law of formation be true for $(n-1)$ lenses, it will be true for n lenses.

121. The form of the expression for the apparent distance shows at once that

$$[OAB\dots HKP] = [PKH\dots BAO].$$

Thus while the lenses are fixed, if the eye and object be supposed to change places, the apparent distance, magnitude and situation of the object will be the same as before. If therefore O, P be the optic centres of the eyes of two persons looking at each other through any set of lenses arranged along an axis between them, the pupils will be seen as circles of equal apparent magnitude and distance by the two observers, supposing their pupils to be of equal size in reality.



122. When an object PL is seen through any number of lenses, the breadth of the principal pencil where it falls on the eye at O, is to its breadth at the object-glass (that is, the lens nearest to the object) as the apparent distance of the object to its real distance from the object-glass.

For let a ray $PtsrK$ flowing from P cut the glasses in succession in tsr ... and finally meet the plane of the pupil in K. Then if PO' be the apparent distance and the rectangle $OKK'O'$ be completed PK' will be the direction of OK as seen from P, so that PtK' is a straight line.

Thus $$O'K' : Ct = O'P : CP,$$

or $$OK : Ct = O\Pi : CP.$$

If the object be very remote, as is always the case in the use of telescopes, CP and OP may be taken to be the same, and therefore

$$Ct : OK = OP : O\Pi = \gamma,$$

where γ is the angular magnifying power.

The image of the surface of the object-glass as seen through the instrument is called the *eye-ring*. Every ray which passes through the instrument will emerge within the eye-ring, at the image, namely, of the point at which the ray strikes the object-glass.

If the instrument be directed to an illuminated surface, or to the sky, each point of the eye-ring receives light from all points of space whose rays can traverse the instrument, that is, from all points of space which can be seen by help of the instrument. If the eye be placed so that its centre is at, or close to, the centre of the eye-ring, it will therefore embrace the entire field of the instrument. The centre of the eye-ring is therefore the best position for the eye, and is called the *eye-point.*

All the incident pencils have the object-glass as their common base, and therefore all the emergent pencils have the eye-ring as a common base. It follows from the preceding proposition that *the angular magnifying power of the instrument is equal to the ratio of the radius of the object-glass, to that of its image as seen through the telescope.*

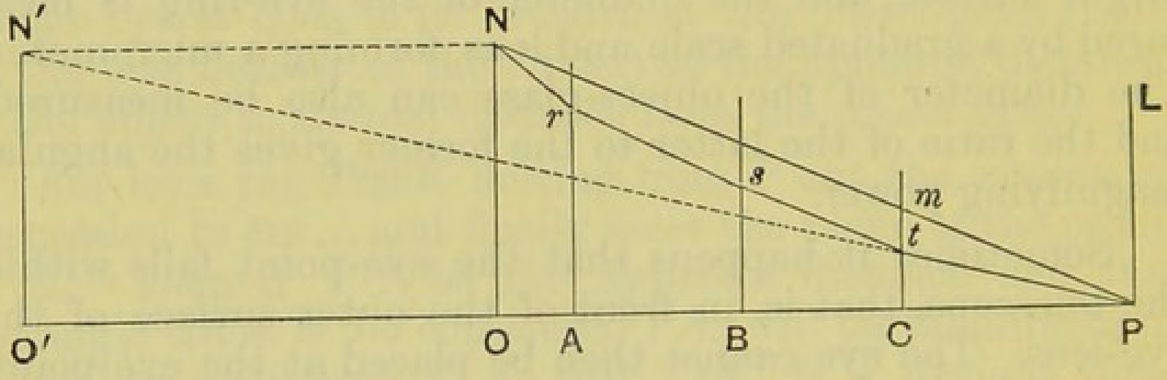
This gives a practical way of measuring the magnifying power of a telescope. The telescope is pointed to a bright surface, and the diameter of the eye-ring is measured by a graduated scale and lens, forming a micrometer. The diameter of the object-glass can also be measured, and the ratio of the latter to the former gives the angular magnifying power.

Sometimes it happens that the eye-point falls within the telescope, that is, in front of the outer surface of the eye-lens. The eye cannot then be placed at the eye-point, but is placed as close to it as possible; it is therefore placed close to the eye-glass. In estimating the field of view the radius of the pupil must be used instead of the radius of the eye-lens. Also the object-glass ought not to be considered as the common base of the incident pencils which go to form the picture on the retina, because parts of the full pencils are stopped. The image of the pupil of the eye as formed by the instrument, will then be the common base of all the incident pencils; this image has been called the *entrance-pupil.* The axes of all the incident pencils which go to form the picture pass through the centre of the entrance-pupil, and the field of view is limited by the cone of rays issuing from this point and filling the object-glass.

123. *Brightness of images.*

The magnitude of the pupil is subject to variation by varying degrees of light; let ON be its semi-diameter when the object PL is viewed by the naked eye from the distance OP. If the breadth of the principal pencil at O be not less than ON, the area of the pupil will be totally illuminated by the pencil that flows from P. Let $PtsrN$ be a ray of that pencil cutting the object-glass in t; and supposing the glasses to be removed, let an unrefracted ray PmN cut the line Ct in m.

Then the quantity of refracted rays which fall upon NO is to the quantity of unrefracted rays as the angle CPt to the angle CPm; or, in the ratio of the apparent magnitude of ON to the true. And therefore by turning the figure round about the axis, the quantity of refracted rays which fill the pupil is to the quantity of unrefracted rays



which would fill it as the apparent magnitude of any surface at O seen from P, to the true; or as the apparent magnitude of any surface at P seen from O, to the true; and consequently as the apparent magnitude of the least surface or physical point P, to the true; that is as the picture of the point P formed on the retina by those refracted rays, to its picture formed by the unrefracted rays. These pictures of the point P are therefore equally bright and cause the appearance of P to be equally bright in both cases.

Next let the pupil be larger than the greatest area at O illuminated by the pencil from P; and supposing a smaller pupil equal to this area, we have shown that the pictures of P upon the retina made by refracted and unrefracted rays would be equally bright. Consequently the picture will be less bright than when the larger pupil is filled by unrefracted rays in the proportion of the smaller pupil to the larger.

Thus it is proved that an object seen through lenses may appear as bright as to the naked eye, but never brighter, even though all the incident light be transmitted through the lenses.

EXAMPLES.

1. A person who can see distinctly at a distance of three feet, finds that with a pair of plano-convex spectacles he can see distinctly at a distance of one foot. Find the radius of the curved surface, the refractive index of glass being $\frac{3}{2}$. *Ans.* 9 inches.

2. A wafer is viewed through a convex lens of 8 inches focal length, placed half-way between it and the eye; find the diameter of the lens when the whole is seen, the diameter of the wafer being half an inch, and its distance from the eye 8 inches. *Ans.* $\frac{1}{3}$ inch.

3. Three-convex lenses of focal lengths f_1, f_2, f_3 are separated by intervals a, b; find the magnifying power of the combination, and prove that it is independent of the position of the object if

$$(f_1-a)(f_3-b)+f_2(f_1+f_3-a-b)=0.$$

4. The light after passing through an optical instrument symmetrical about an axis is reflected by a plane mirror perpendicular to its axis so as to pass through it again in the reverse direction; show that the compound instrument so formed is equivalent in every respect, if spherical aberration be neglected, to a simple spherical mirror, with its vertex in the position conjugate to the plane mirror and its centre of curvature at the corresponding principal focus.

5. If in any optical instrument formed of lenses and mirrors on the same axis, γ is the angular magnifying power when the instrument is adjusted for an eye which sees clearly with the incident light parallel, and if the eye-glass (focal length f) is moved till the instrument is in adjustment for an eye whose distance of distinct vision is δ, show that the magnification is increased by $\gamma f/\delta$.

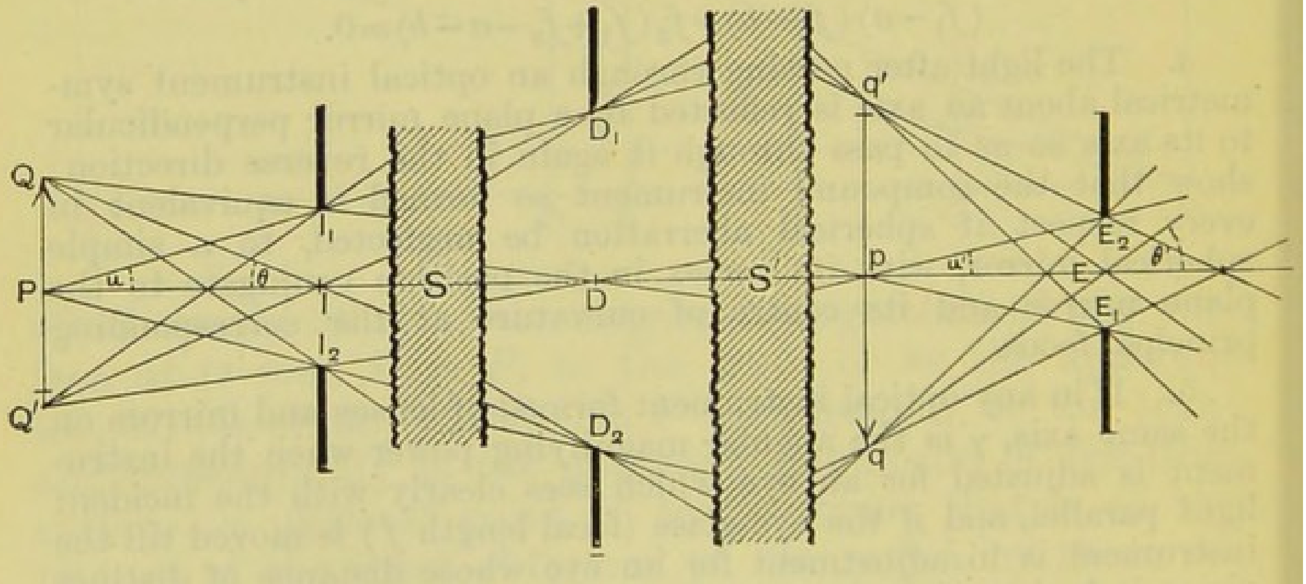
6. A stereoscope is constructed of two glass prisms ($\mu=\frac{3}{2}$) with their edges coincident, and placed so that the faces of each are equally inclined to the plane on which the two pictures are placed, and at a distance of 6 in. The eyes of an observer are $2\frac{1}{2}$ in. apart; find their distance from the prism when the axes of the pencils from the middle points of the two pictures have minimum deviation and cross at the point half-way between them, the points being 4 in. apart. Show that the angles of the prisms must be nearly $\tan^{-1}\frac{3}{4}$.

CHAPTER VIII.

General Theory of Optical Instruments.

General properties of Optical Instruments.

124. The space in the interior of an optical instrument which is available for the transmission of rays is always limited, either by the finite opening of the lenses, or by the opening of the pupil of an observer's eye, or by artificial diaphragms. These natural or artificial diaphragms are always circular and have their centres on the axis of the instrument.



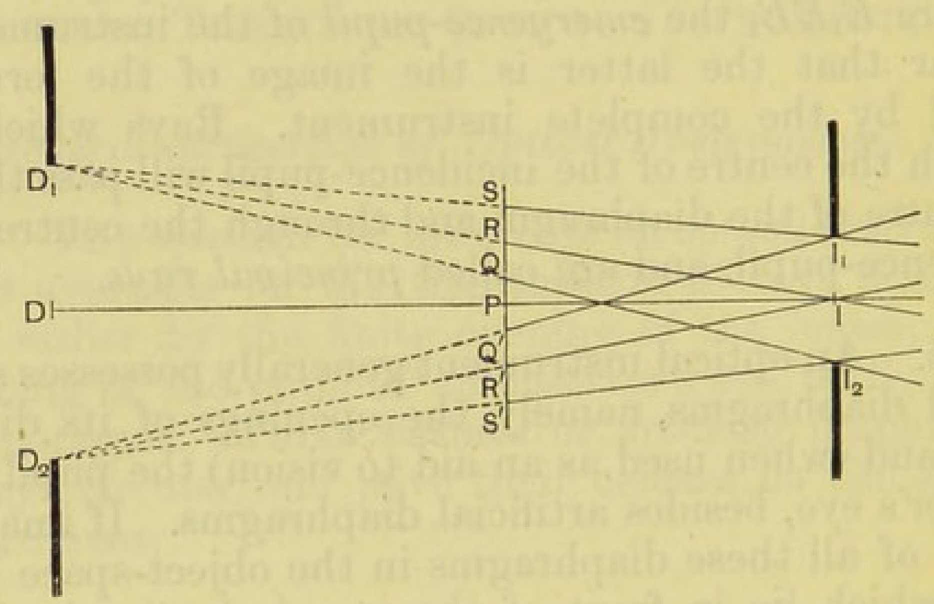
Suppose that D_1DD_2 represents a diaphragm placed anywhere in an optical instrument, and denote the systems of lenses which lie in front of the diaphragm (that is on the side nearest the object) by S, and that which lies behind it by S'. Let the image of the diaphragm formed by the lens-system S be I_1II_2, and that formed by the system S', E_1EE_2. Then every ray which in the instrument passes within the aperture in the diaphragm D_1DD_2, must have entered within the aperture I_1II_2 and must emerge within the aperture E_1EE_2. The aperture I_1II_2 is thus the common base of all pencils incident on the instrument which ultimately emerge from it, and E_1EE_2 is the common base of all the emergent pencils. From analogy to the pupil of the eye, the aperture I_1II_2 has been called by Professor Abbé, the *incidence-pupil*, and the aperture E_1EE_2 the *emergence-pupil* of the instrument. It is clear that the latter is the image of the former as formed by the complete instrument. Rays which pass through the centre of the incidence-pupil will pass through the centre of the diaphragm and through the centre of the emergence-pupil, and are called *principal rays*.

125. An optical instrument generally possesses several natural diaphragms, namely, the openings of its different lenses and (when used as an aid to vision) the pupil of the observer's eye, besides artificial diaphragms. If images be formed of all these diaphragms in the object-space by the lenses which lie in front of them, and also in the image-space by the lenses which lie behind them, the question arises, which of all these images in the object-region is the true incidence-pupil, and which image in the image-region is the true emergence-pupil? In the object-region, that diaphragm image which subtends the smallest angle at the central point of the object is the true incidence-pupil. The angle ($2u$) subtended by this incidence-pupil at the central point of the object is called the *angular aperture* of the instrument. The image of the incidence-pupil, formed in the image-region, is then the emergence-pupil.

126. The *angular field of view* is the angle between the incident portions of the extreme principal rays which pass through the instrument. That diaphragm image in the object-region which subtends the smallest angle at the centre of the incidence-pupil, will therefore determine the angular field of view, and may be called the *field of view diaphragm*. When the incidence-pupil and the field of

view diaphragm are settled, the effective apertures of all the other diaphragms are determined. If the apertures of the diaphragms are properly graded, the diaphragm images in the object-region should all fit into the full cone of incident light.

The angular field of view is not the greatest field of view if partial pencils are taken into account.



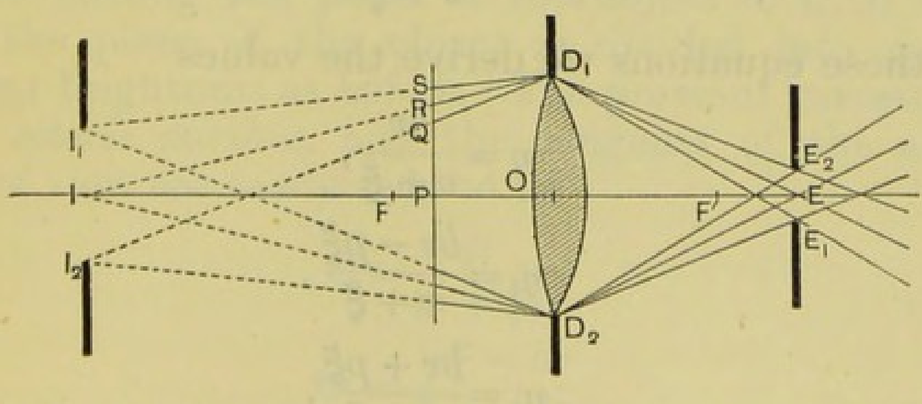
For let I_1II_2 be the incidence-pupil, D_1DD_2 the field of view diaphragm, and let the edges of the latter be joined to the edges and centre of the incidence-pupil. Let the three cones thus formed cut the plane of the object in three circles whose radii are PQ, PR, PS. The full pencil from any point of the object fills the cone whose vertex is the point and whose base is the incidence-pupil, and thus it is clear that points in the ring QR will give images formed by partial pencils each greater than half the full pencils, and points in the outer ring RS will yield images formed by less than half the full pencils, the other parts of the pencils being stopped by the diaphragm D_1DD_2. No point of the object outside the circle S will be able to transmit rays through the instrument at all. Thus the angular field of view formed by principal rays is the angle subtended at I by the circle PR, but the extreme field of view is the angle subtended at I by the circle PS, and the field of view for an image formed by full pencils only is the angle subtended at I by the circle whose radius is PQ.

Field of view through a Single Lens.

127. The two natural diaphragms which limit the pencils by which an object is seen through a single lens are the aperture of the lens and the pupil of the observer's eye. In accordance with § 126 we must form the images of these diaphragms by such lenses as lie in front of them. We thus get two diaphragm images on the side of the system in which the object lies, viz. the lens-aperture, and the image of the eye-pupil formed by the lens. Of these apertures, that which subtends the smallest angle at the centre of the object must be taken as the incidence-pupil, and this aperture is to be regarded as the common base of all full pencils proceeding from different points of the object. Two cases will have to be considered, first, that in which the image of the eye-pupil is the incidence-pupil, and secondly, that in which the lens-aperture is the incidence-pupil.

128. First, then, suppose that the image of the eye-pupil subtends at the centre of the object an angle smaller than that subtended by the lens-aperture.

Let D_1OD_2 be the lens-aperture, E_1EE_2 the eye-pupil and I_1II_2 the image of the eye-pupil formed by the lens; let PQ be the plane of the object, and suppose that the angle subtended at P by the aperture I_1II_2 is less than the angle subtended by the lens-aperture D_1OD_2. Then



I_1II_2 is the incidence pupil, and the field of view is limited by the lens-aperture. Join D_1 to the centre and edges of the incidence-pupil, cutting the plane of the object in Q,

R and S; then the portion of the object within a circle whose centre is P and radius PQ is seen by full pencils, those points of the object lying on the ring between the circles whose radii are PQ, PR are seen by portions of pencils in each case greater than half the full pencil, while those points on the ring between the circles whose radii are PR, PS are seen by portions of pencils in each case less than half the full pencil.

The parts of the object within the radius PQ will thus appear brightest, and the rings outside this circle will have less and less brilliancy, until beyond the circle whose radius is PS the light will be cut off altogether. The angular fields of view corresponding to these rings are the angles the rings subtend at I, the centre of the incidence-pupil. Their values can easily be found in terms of the apertures and distances.

Let PQ, PR, PS be denoted by η_1, η, η_2; also let the radius of the incidence-pupil be p, and that of the lens, b. Denote the distances IP, PO by v and ξ.

Then by similar triangles,

$$\frac{\eta}{v} = \frac{b}{v+\xi}.$$

Again $$\frac{\eta_1 + p}{v} = \frac{b+p}{v+\xi},$$

and finally $$\frac{\eta_2 - p}{v} = \frac{b-p}{v+\xi}.$$

From these equations we derive the values

$$\eta = \frac{bv}{v+\xi},$$

$$\eta_1 = \frac{bv - p\xi}{v+\xi},$$

$$\eta_2 = \frac{bv + p\xi}{v+\xi}.$$

Hence if the tangents of the angular fields of view corresponding to these regions be denoted by Θ, Θ_1, Θ_2, we get

$$\Theta = \frac{b}{(v+\xi)},$$

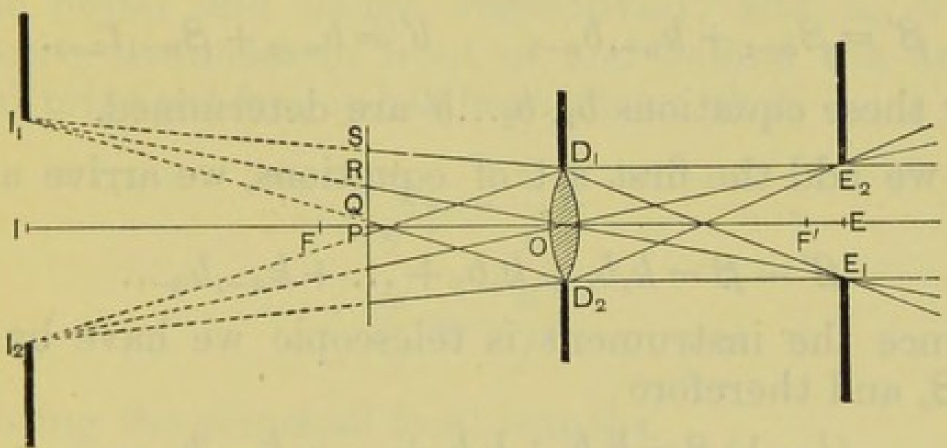
$$\Theta_1 = \frac{bv - p\xi}{v(v+\xi)},$$

$$\Theta_2 = \frac{bv + p\xi}{v(v+\xi)}.$$

The angular field of view, as determined by the principal rays, has the first of these values. It is clear that

$$2\Theta = \Theta_1 + \Theta_2.$$

129. Secondly, let the lens-aperture subtend a smaller angle at P than the aperture I_1II_2, then D_1OD_2 is the true incidence-pupil, and the field of view will be limited by the aperture I_1II_2.



We must now join I_1 to the edges and centre of D_1OD_2 by lines cutting the plane of the object in Q, R and S. Then the plane of the object is divided into rings of different brightness as before. The previous investigation holds, *ceteris paribus*, and the tangents of the angular fields of view are easily seen to be

$$\Theta = \frac{p}{(v+\xi)},$$

$$\Theta_1 = \frac{p\xi - bv}{\xi(v+\xi)},$$

$$\Theta_2 = \frac{p\xi + bv}{\xi(v+\xi)}.$$

130. The field of view for any telescope as determined by principal rays may easily be found in terms of the notation previously used (§ 63) as soon as the incidence-pupil and the emergence-pupil are known. The principal rays pass through the centre of the incidence-pupil and will meet the various refracting surfaces at different distances from the axis, determining the effective apertures of each. Let β correspond to the inclination of the extreme principal rays, then taking the incidence-pupil as the first surface, b vanishes and the effective apertures will be determined by the equations.

$$\beta_1 = \beta, \qquad b_1 = \beta_1 t_1,$$
$$\beta_2 = \beta_1 + k_1 b_1, \qquad b_2 = b_1 + \beta_2 t_2,$$
$$\beta_3 = \beta_2 + k_2 b_2, \qquad b_3 = b_2 + \beta_3 t_3,$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$
$$\beta' = \beta_{n-1} + k_{n-1} b_{n-1} \qquad b' = b_{n-2} + \beta_{n-1} t_{n-1}.$$

By these equations b_1, $b_2 \ldots b'$ are determined.

If we add the first set of equations, we arrive at the result,

$$\beta' - \beta = k_1 b_1 + k_2 b_2 + \ldots + k_{n-1} b_{n-1}.$$

But since the instrument is telescopic we have by § 70, $\beta' = l\beta$, and therefore

$$(l-1)\,\beta = k_1 b_1 + k_2 b_2 + \ldots + k_{n-1} b_{n-1}.$$

This shows that *the field of view is continually increased by adding more convex lenses;* for corresponding to both surfaces of a convex lens k is positive.

If any lens have its aperture diminished, the values of all the other apertures and of the field of view are diminished in the same proportion. It is useless to make the aperture of any of the lenses greater than its effective aperture.

On magnifying power.

131. Before defining the magnifying power of an optical instrument the use to which the instrument is to

be put should be considered. Optical instruments are generally of one or two kinds:—(1) those used for the purpose of projecting an image on a screen, such as lenses for photographic cameras or for magic lanterns, (2) those used in conjunction with the eye, as an aid to vision.

In instruments used to project images on a screen, the magnifying power is defined to be the ratio of the linear dimensions of the image to those of the object. To distinguish this definition from that which follows for instruments used as an aid to vision, the magnifying power in the present case may be called the *objective magnifying power*.

The value of the objective magnifying power for any optical instrument (except those using wide-angled pencils) has already been found. If η, η' be the linear dimensions of the object and image, respectively, and u, u' their respective distances in front of, and behind the first and second principal foci, it has been shown that

$$\left.\begin{aligned} \frac{\eta}{\eta'} &= -\frac{u}{f} \\ \frac{\eta'}{\eta} &= -\frac{u'}{f'} \end{aligned}\right\},$$

f, f' being the principal focal lengths.

132. We shall next consider the case of a *telescope*. The characteristic property of a telescope is, as we have already seen, that a series of incident parallel rays proceeding from a distant object shall emerge parallel to each other. The proper measure of the magnifying power in this case is obtained by comparing the linear dimensions of the retinal images, formed in the one case when the object is viewed directly by the naked eye, and in the other when the object is viewed through the telescope. In the case of a telescope both of these images will be small and their linear dimensions will be proportional to the tangents of the inclinations of the rays to the axis of the instrument. If α be the inclination of the incident rays,

α' that of the emergent rays to the axis of the instrument, the proper measure of the magnifying power will be

$$\gamma = \frac{\tan \alpha'}{\tan \alpha}.$$

It has already been shown that for a telescope this ratio is constant for all rays, and that in the notation of § 70, supposing that the initial and final media are both air,

$$\gamma = l.$$

It is also shown that if a telescope be used to see an object not at an infinite distance, the objective magnifying power is

$$m = \frac{1}{l}$$

for all objects.

This last property gives a simple method of measuring the magnifying power of a telescope. The telescope is pointed to a bright surface, and the image of the object-glass as seen through the instrument is measured by a graduated scale and lens forming a micrometer. Then *the ratio of the diameter of the object-glass to the diameter of its image is the angular magnifying power of the telescope.*

133. Lastly, consider the optical instruments, not telescopic, which are used as aids to vision; this will include the simple lens and the compound microscope.

To obtain a correct measure of the magnifying power of an instrument, we must compare the magnitudes of the retinal images, first when the eye is used in combination with the instrument, and secondly when the eye is used alone. But before this comparison can be definite, we must say where the object and the image formed by the lens-system must be placed, in order that the retinal images formed may be fit for the determination of the magnifying power. To make this comparison correct, the eye, and the combination of the eye and instrument, must be compared as much as possible under analogous circumstances; this may be realised by comparing them while working as favourably as possible, that is, when they give the largest possible images on the retina. For the eye alone, the object must therefore be placed at the nearest point for distinct vision. But the smallest distance for distinct vision is very different for different persons; whereas the magnifying power ought to give an idea of the amplification of the instrument for the eye in general. It has therefore been agreed to place the object at a distance conventionally fixed, a distance not too great for the retinal images to be near their greatest dimensions, and which is large enough for the great majority of eyes to remain accommodated for it during a long time. The distance chosen is 10 inches, and is generally called the "distance of distinct vision." The phrase is not a happy one, for at every distance at which an eye can accommodate itself, it sees equally distinctly. The distance chosen for the position of the image formed by the lens-system is the same; for then the retinal images will be proportional to the linear magnitudes of the object and image themselves.

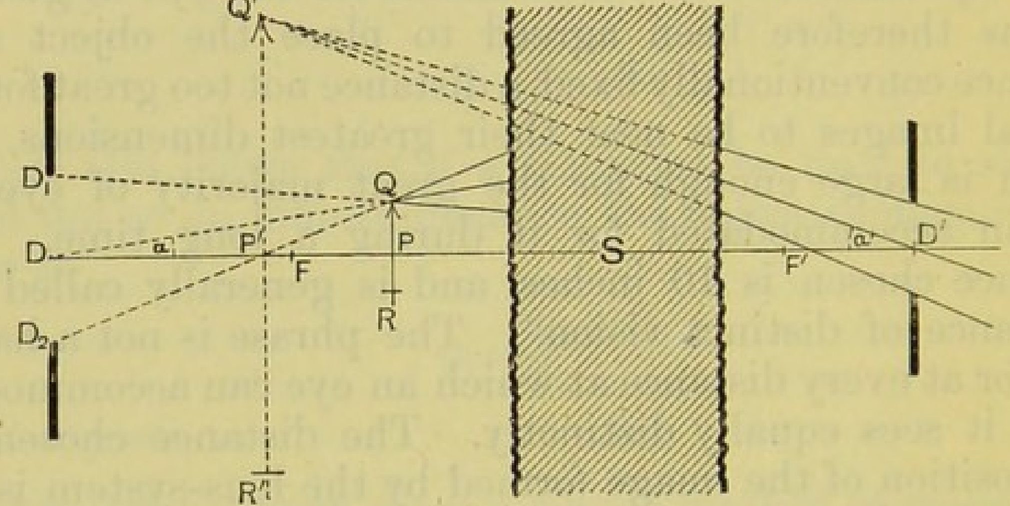
Instead of the "distance of distinct vision" it would be better to use the phrase "distance of projection of the image." For short-sighted people, if the object be placed at this distance, the image would not be distinct; in these cases we must take the centres of the circles of indistinctness instead of the sharp image points which would be formed on the retina of a normal-sighted person. For brevity we shall use λ, for this distance of projection of the image.

134. Let F, F' be the first and second focal points of the complete lens-system, and, as usual, let the light travel from left to right. Let u, u' be the distances of the object and the final image, in front of, and behind the focal points, respectively. In the figure u, u' are negative. Also let x' be the distance of the pupil of the eye behind the posterior focal point, x the distance of its image in front of the first focal point.

Then
$$\left.\begin{aligned} uu' &= ff' \\ xx' &= ff' \end{aligned}\right\}.$$

Also let α, α' be the inclinations to the axis of one incident ray and its corresponding emergent ray, and η, η' the linear dimensions of the object and final image. Then if the final image is formed at a distance λ from the eye,

$$\tan \alpha' = \frac{\eta'}{\lambda}.$$



Hence the magnifying power becomes the ratio of

$$\frac{\eta'}{\lambda} : \frac{\eta}{\lambda};$$

that is
$$m = \frac{\eta'}{\eta}.$$

Now by the theory of optical instruments

$$\frac{\eta'}{\eta} = -\frac{u'}{f'}.$$

But since the image is formed at a distance λ from the eye, we have

$$x' - u' = \lambda,$$

or
$$u' = x' - \lambda.$$

Hence

$$m = \frac{\lambda - x'}{f'}.$$

In a microscope the eye-point practically coincides with the posterior principal focus of the instrument, so that x' is very small.

Hence for a microscope

$$m = \frac{\lambda}{f'}.$$

With a hand-magnifier, however, the eye may change its position to some extent, and the accurate formula is

$$m = \frac{\lambda - x'}{f'}.$$

If the eye be placed close to the lens, we have $x' = -f'$ nearly, and consequently

$$m = \frac{\lambda}{f'} + 1.$$

135. It may be worth noticing that if the instrument be arranged for a normal-sighted eye, so that the emergent rays are parallel, the ratio $\tan \alpha' : \frac{\eta}{\lambda}$ is independent of the position of the eye altogether.

For
$$\tan \alpha' = \frac{\eta'}{x' - u'},$$

and
$$\frac{\eta'}{\eta} = -\frac{u'}{f'}.$$

Hence
$$\frac{\lambda \tan \alpha'}{\eta} = -\frac{\lambda u'}{f'(x' - u')}$$
$$= \frac{\lambda}{f'} \cdot \frac{1}{1 - \frac{x'}{u'}}.$$

But in the case before us, u' is infinite.

Thus
$$\frac{\lambda \tan \alpha'}{\eta} = \frac{\lambda}{f'}.$$

Professor Abbé proposes to make the measure of the magnifying power for microscopes independent of the conventional distance λ; he would measure the magnifying power according to the definition

$$m = \frac{\tan \alpha'}{\eta}.$$

According to this definition, the numerical measure of the magnifying power of a microscope will be

$$m = \frac{1}{f'}.$$

If we make $\eta = 1$, we get $\tan \alpha' = m$, and therefore the numerical measure of m gives the visual angle under which an object whose linear amplitude is *unity* is seen through the instrument.

CHAPTER IX.

Optical Instruments.

136. We have already treated the theory of vision through a single lens and its application to spectacles and reading-glasses. The next optical instrument in the order of simplicity is the simple microscope.

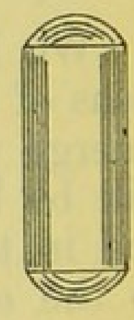
We have seen that when an object is placed at the focus of a convex lens, the rays of the several pencils will emerge parallel to each other, and therefore each pencil will be brought to a focus on the retina without effort; and in this position the angle under which it will appear to the eye is the angle it would subtend at a distance equal to the focal length of the lens. Consequently the image will be distinct and magnified. A lens of high power thus used is called a simple microscope. It has been shown that in this case the magnifying power is independent of the position of the eye, and equal to λ/f.

The lens is usually made plano-convex, and is used with its plane side towards the eye. When the eye is placed near the lens, a field of view of about $\frac{1}{5}$ of the focal length may be obtained, practically flat, and tolerably free from distortion. The chromatic defects are however very considerable.

Single lenses answer very well so long as the focal length is not smaller than one inch; but when higher powers are required, combinations of more than one lens are preferable.

A form of simple magnifier, which possesses certain advantages over a double convex lens, is that commonly known as a "Coddington lens." The lens is spherical, but the rays are made to pass nearly through the centre of the lens. The first idea of it is due to Wollaston, who proposed to unite two hemispherical lenses by their plane sides, with a stop interposed, the central aperture of which should be equal to one-fifth of the focal length. The same end was shown by Brewster to be attained more satisfactorily by grinding a deep groove round the equatorial part of a spherical lens, and filling it with something opaque. The great advantage of this lens is that the oblique pencils as well as the central pencils, pass normally into the lens, so that they are but little subject to defects of aberration.

The Stanhope lens consists of a cylinder of glass with its ends ground into spherical convex surfaces of unequal curvature; the length of the cylinder is so arranged that when the more convex end is turned towards the eye, objects placed *on* the other end shall be in the focus of the lens. This furnishes an easy way of mounting light objects for examination.



A modified form of the Stanhope lens, in which the further surface is plane, has been used extensively in France for the enlargement of minute pictures photographed on the plane surface; it is called a "Stanhoscope."

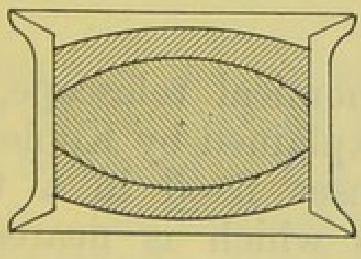
137. Wollaston was the first to use a combination of two lenses instead of a single lens; this combination is still known as *Wollaston's Doublet*. It was suggested by an inverted Huyghens' eye-piece, to be described presently. It consists of two plano-convex lenses whose focal lengths are in the proportion of 1 : 3, the plane surfaces being turned towards the object, and the lens of shorter focal length being placed next the object. The distance between the lenses can be adjusted to suit different eyes, but is usually $\frac{3}{2}$ of the shorter focal length.

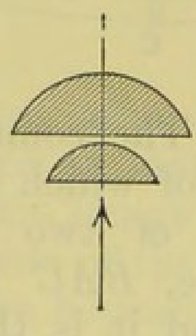
Pritchard, who made doublets which magnified 200 to 300 diameters, performing excellently, made the distance between the lenses equal to the difference of their focal lengths, while the latter could vary in ratio from 1 : 3 to 1 : 6.

A better doublet was invented by Chevalier, who placed two plano-convex lenses of equal focal lengths but of different diameters, very close together, the larger being the nearer to the object; between them he fixed a diaphragm. In this way he obtained more light and secured a greater distance between the lens and the object.

138. Triplets have been constructed on the same principles. The combination with sufficient care of three plano-convex lenses gives even better results than doublets. They can be made comparatively free from aberration both spherical and chromatic.

Among compound lenses made by cementing together lenses of different kinds of glass Steinheil's Aplanatic Lens deserves special mention. This consists of a crown-glass double convex lens enclosed between two equal meniscus lenses of flint-glass, as shown in the first figure. The inner surfaces have about double the curvature of the outer surfaces. This lens gives a beautiful, distinct and flat image, possesses a wide field of view and permits a



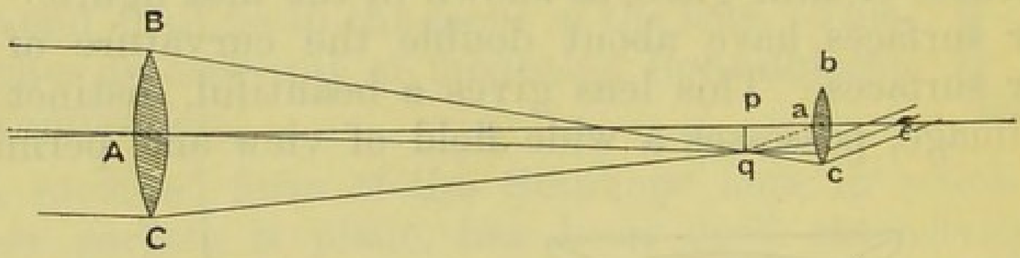


considerable distance of object. The second figure represents a modern form of doublet made by Zeiss of Jena, drawn four-times the natural size. It has a focal length of 2 mm. and magnifying power about 70 and field of view 1·2 mm.

All the simple microscopes are however so much inferior to the modern compound microscope that they are only used for rough observations or for dissecting.

139. The refracting telescopes and the compound microscope, in their simplest forms, consist of two lenses. The lens placed nearer to the object receives rays directly from the object and forms a real inverted image of the object; this lens is called the *object-glass*, or the *objective.* The inverted image is viewed by the eye through the other lens, which is called the *eye-glass* or *eye-piece;* this eye-glass alters the divergence of the small pencils which form the first image, so that they can be brought to a focus on the retina without effort, and increases the visual angle under which the image is seen. In general, an eye is accommodated for rays emerging parallel to each other; the eye-glass is therefore placed so that the first image is in the principal focus of this lens. In microscopes, however, where the magnifying power is very important, the instrument is arranged so that the final image is at the conventional distance λ from the eye.

The Astronomical Telescope.



140. The common Astronomical telescope, the construction of which was first explained by Kepler, consists primarily of two convex lenses fixed in a tube. In the figure, *BAC* is the lens which is nearest to the object, and it is therefore called the object-glass. This lens forms an inverted image *pq*, of the object, corresponding points of image and object lying on the same line through *A*, the centre of the object-glass. *Bq*, *Aq*, *Cq* are three rays diverging from any one point of the object which, after refraction by the object-glass, are made to meet in *q*, the corresponding point of the image. These rays after crossing at *q*, fall upon the convex lens *bac*, called the *eye-glass*, and after refraction they are in general

made to emerge parallel to each other. This will be effected by adjusting the position of the eye-glass, so that the image pq shall lie in its principal focus.

Let f, f' be the focal lengths of the object-glass and eye-glass, respectively. Then the angle qAp is the angle which the object subtends at the centre of the object-glass, and this will not differ sensibly from that subtended at the eye. By the naked eye, therefore, the object is seen under an angle whose tangent is $-\eta/f$, where η is the linear dimensions of the image. Also, the image pq will be seen through the lens at an angle whose tangent is η/f', wherever the eye be placed, supposing pq to be in the principal focus of the eye-glass. The magnifying power is therefore

$$\gamma = -\frac{f}{f'}.$$

141. In order to take in the whole extent of this field the eye must be placed at the point in which the axes of the extreme pencils, diverging from the centre of the object-glass, meet the axis of the telescope on their final emergence. The place of the eye is therefore the focus conjugate to the centre of the object-glass as seen through the eye-glass.

The image of the objective aperture formed by the instrument will be beyond the eye-piece, so that the eye can always be placed at this image. This image is called the eye-ring. The position and magnitude of the eye-ring can easily be found. If x be the distance beyond the eye-piece, we have

$$\frac{1}{x} + \frac{1}{f+f'} = \frac{1}{f'},$$

so that

$$x = \frac{f'}{f}(f+f').$$

Let b, b' be the semi-apertures of the objective and the eye-lens respectively, then the semi-diameter of the eye-ring will be given by the usual equation

$$\frac{e}{b} = -\frac{x}{f+f'}$$

$$= -\frac{f'}{f},$$

or $$e = \frac{b}{m}.$$

When the magnifying power of the instrument is large, the diameter of the eye-ring is less than that of the pupil of the eye, and therefore the eye can take in all the rays which emerge within the eye-ring. Thus the eye-ring is the emergence-pupil of the instrument and the objective aperture the incidence-pupil.

In the construction of the instrument, the tube is prolonged to the required distance and is there furnished with an eye-stop, and in looking through the instrument the eye is placed close to the end of the tube.

142. To find the field of view we must form the image of the eye-lens due to the object-glass, as was explained in the general theory of the field of view; let D_1DD_2 be this image, then if D_1DD_2 have a greater diameter than the object-glass (as will usually be the case), D_1DD_2 is the field-of-view diaphragm. If ξ be the distance of D_1DD_2 from the object-glass,

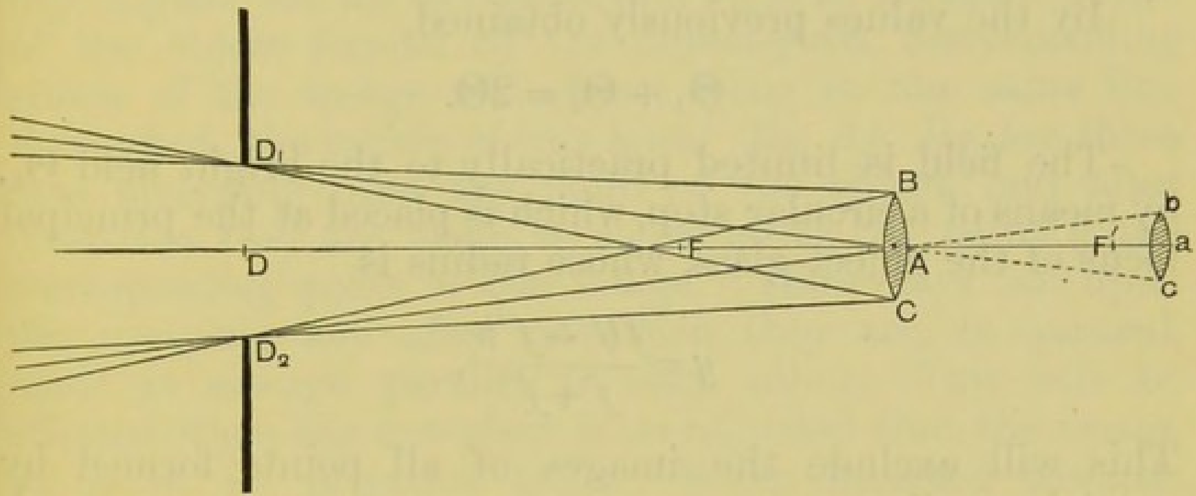
$$\frac{1}{\xi} + \frac{1}{f+f'} = \frac{1}{f},$$

and therefore $$\xi = \frac{f(f+f')}{f'}.$$

Also the semi-aperture of the field-of-view diaphragm is given by the equation

$$d = \frac{b'\xi}{f+f'} = \frac{b'f}{f'}.$$

If now the edges of the field-of-view diaphragm be joined to the edges and centre of the object-glass, and the lines produced backward to infinity to cut the object-plane, the bounding rays of the three regions are parallel to D_1B, D_1A and D_1C. Pencils of parallel rays filling the object-



glass whose inclinations to the axis do not exceed that of D_1B pass through the instrument as full pencils; but if the inclination is equal to that of D_1A, it is clear that half the pencil will be stopped by the field-of-view diaphragm, and if the inclination increases up to that of the line D_1C more and more of the pencil is stopped, until at last only the extreme edge of the pencil is admitted. Denoting the tangents of the inclinations of the bounding rays to the axis by Θ_1, Θ, Θ_2, respectively, we have

$$\Theta_1 = \frac{\dfrac{b'f}{f'} - b}{\dfrac{f(f+f')}{f'}} = \frac{b'f - bf'}{f(f+f')},$$

and
$$\Theta = \frac{\dfrac{b'f}{f'}}{\dfrac{f(f+f')}{f'}} = \frac{b'}{f+f'},$$

and finally
$$\Theta_2 = \frac{b'f + bf'}{f(f+f')}.$$

143. If $b'/b = f'/f$, that is, if the apertures of the lenses are proportional to their focal lengths, Θ_1 vanishes; in this case the brightness of the field decreases from the

centre to the circumference. If b'/b be less than f'/f, the value of Θ_1 becomes negative, and no part will be illuminated by full pencils.

By the values previously obtained,

$$\Theta_1 + \Theta_2 = 2\Theta.$$

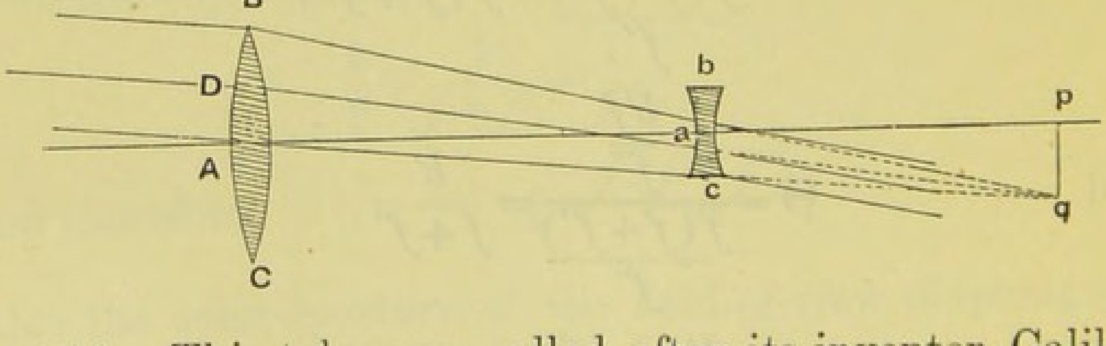
The field is limited practically to the bright field Θ_1, by means of a circular stop, which is placed at the principal focus of the object-glass, whose radius is

$$y = \frac{fb' - f'b}{f + f'}.$$

This will exclude the images of all points formed by partial pencils.

In an Astronomical telescope there is usually fixed a network of fine wires, vertical and horizontal, the plane of the wires being the focal plane of the object-glass. The image of the object given by the object-glass will then lie in the plane of the wires, and the image and the wires are viewed together through the eye-lens. By the aid of these wires the position of the image of any point can be accurately measured.

Galileo's Telescope.



144. This telescope, called after its inventor, Galileo, was the first whose construction was explained on theoretical principles. It differs from the astronomical telescope chiefly in the form of its eye-glass, which is a double concave lens, and is placed between the object-glass and its principal focus. A pencil of light diverging from the

object is brought to a focus by the object-glass; but before the rays reach this focus, some part of the pencil is caught by the eye-glass. In the annexed figure, BAC is the object-glass, bac the eye-glass, and pq is an inverted image of the object formed by the object-glass, corresponding points of the image and object lying on the same line through A, the centre of this lens. Bq, Aq, Dq, are three rays diverging from any point of the object, and after refraction they are made to converge to the point q, the corresponding point of the image. These rays fall upon the eye-glass and after refraction they are, in general, made to emerge parallel to each other. This will be effected when the eye-glass is so adjusted that the image pq is in its principal focus. When directed towards distant objects, pq is also in the principal focus of the object-glass, so that the distance between the lenses is then equal to the difference between the focal lengths of the two glasses.

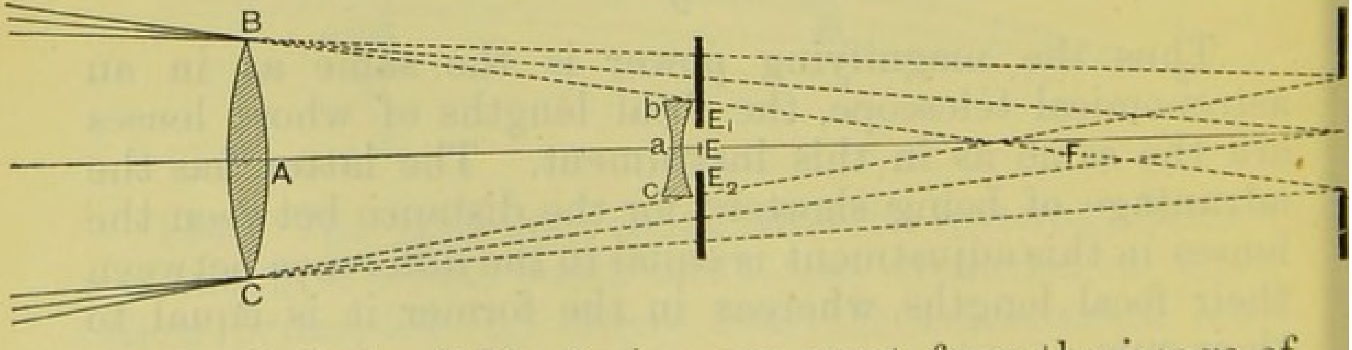
Let η be the linear magnitude of the image pq, and f, f' the numerical focal lengths of the object-glass and the eye-glass, respectively. Then the angle under which the object is seen by an eye placed at A is equal to the angle qAp, and this will not differ sensibly from the angle under which it will be seen by the eye in its proper position. The tangent of the angle is $-\eta/f$. Also the image pq will be seen through the lens under an angle whose tangent is $-\eta/f'$. The magnifying power is therefore

$$\gamma = \frac{f}{f'}.$$

Thus the magnifying power is the same as in an astronomical telescope, the focal lengths of whose lenses are the same as in this instrument. The latter has the advantage of being shorter; for the distance between the lenses in this adjustment is equal to the difference between their focal lengths, whereas in the former it is equal to their sum.

A more important advantage which this instrument possesses is that through it objects are seen erect and not inverted, as in the Astronomical telescope. This is readily seen by following the course of the axes of extreme pencils as they diverge from the centre of the object-glass. When they meet the eye-glass they are made to diverge still more by it; and therefore the pencil flowing from the uppermost part of the object will proceed to the lower part of the retina, and *vice versa;* and therefore the object is seen in the same position as by the naked eye. On this account the instrument is convenient for viewing terrestrial objects. The ordinary opera-glass consists of a pair of Galileo's telescopes placed with their axes parallel, and arranged so that the distance between the lenses can be altered so as to adopt the telescopes for seeing objects at different distances.

145. The field of view in this instrument is very limited. For the axes of the pencils flowing from the several parts of the object, diverging from the centre of the object-glass, will diverge still more after refraction by the concave eye-glass, and therefore, for the most part, they will fall outside the pupil of the eye and be lost. The image of the objective opening lies within the instrument, so that the eye cannot be placed at it. In order that the eye may receive as many rays as possible, it must be placed as near as possible to the image of the objective opening; the eye is therefore placed close to the eye-lens. The effective aperture of the eye-lens is therefore reduced to that of the pupil of the eye. Let E_1EE_2 be the eye-pupil.



To find the field of view, we must form the image of the eye-pupil by the lenses which are in front of it. Let I_1II_2 denote this image. Then since the eye is close to the eye-lens, that lens will not affect the image, so that I_1II_2 is the image of the pupil formed by the object-glass.

Let ξ be the distance of this image from the object-glass; then in the figure AI represents ξ. Then

$$\frac{1}{f-f'} - \frac{1}{\xi} = \frac{1}{f};$$

and therefore
$$\xi = \frac{f(f-f')}{f'}.$$

Also the semi-diameter of the image is given by the equation

$$p = \varpi \frac{\xi}{f-f'} = \frac{\varpi f}{f'},$$

where ϖ is the radius of the eye-pupil. Usually this semi-diameter is smaller than that of the object-glass, so that I_1II_2 is the true incidence-pupil, and the object-glass is the field-of-view diaphragm. If the edges of the object-glass be joined to the edges and centre of the incidence-pupil, and the rays so formed be produced outwards to infinity to cut the object-plane, the bounding rays of the three regions of the field of view are parallel to BI_1, BI, BI_2. If we denote the tangents of the inclinations of these rays to the axis by Θ_1, Θ, Θ_2 we have

$$\Theta_1 = \frac{b - \dfrac{\varpi f}{f'}}{\dfrac{f(f-f')}{f'}} = \frac{bf' - \varpi f}{f(f-f')},$$

$$\Theta = \frac{b}{\dfrac{f(f-f')}{f'}} = \frac{bf'}{f(f-f')},$$

and
$$\Theta_2 = \frac{bf' + \varpi f}{f(f-f')}.$$

It might happen that the object-glass were of smaller aperture than the image of the pupil. In that case the object-glass BAC would be the incidence-pupil and I_1II_2

the field-of-view diaphragm. The three values of the field of view could then be found exactly as before to be

$$\Theta_1 = \frac{\varpi f - bf'}{f(f-f')},$$

$$\Theta = \frac{\varpi}{f-f'},$$

$$\Theta_2 = \frac{\varpi f + bf'}{f(f-f')}.$$

In this telescope the image formed by the object-glass is virtual, and therefore no stop or network of fine wires for measuring, can be used.

146. *Ex.* To find the magnifying power and the field of view in Galileo's telescope, when it is arranged for seeing distant objects, the final image being at a distance λ from the eye.

The distance of the eye-lens from the principal focus of the object-glass will no longer be f'. Let the distance of the principal focus in front of the eye-lens be denoted by u, and let v be the distance of the final image beyond the eye-lens, then

$$v = -\lambda.$$

Also
$$\frac{1}{u} + \frac{1}{v} = -\frac{1}{f'},$$

thus
$$u = -\frac{\lambda f'}{\lambda - f'}.$$

Let α be the angle subtended by the object at the centre of the object-glass, and x the linear magnitude of the first image. Then

$$\tan \alpha = \frac{x}{f}.$$

Also if α' be the angle of vision, then

$$\tan \alpha' = -\frac{x'}{\lambda},$$

where x' is the magnitude of the final image.

$$\therefore \ \gamma = -\frac{f}{x}\frac{x'}{\lambda}.$$

But
$$\frac{x'}{x} = -\frac{v}{u},$$

hence
$$\gamma = \frac{f(\lambda - f')}{\lambda f'}.$$

In working out the field of view we have only to substitute $\frac{\lambda f'}{\lambda - f'}$, instead of f' in the formulæ previously obtained.

Thus the field of view as determined by axes of principal pencils is

$$\Theta = \frac{\lambda b f'}{f[\lambda(f-f') - ff']},$$

supposing that the object-glass is the field of view diaphragm.

Object-glasses.

147. We shall next apply the preceding theoretical considerations to the construction of good object-glasses.

One advantage of a telescope over the naked eye, in viewing a distant object, is the quantity of light which the instrument admits. The eye admits a small cone of rays issuing from each point of the object, just sufficient to fill the pupil; whereas a telescope admits a cone large enough to fill the whole object-glass. Thus a telescope enables us to see stars which are too faint to be perceived by the naked eye. The larger the aperture of the object-glass, the more light will be admitted. The first requisite of an object-glass is therefore a wide aperture.

We have seen that the brightness of an image is equal to that of the object; so that when the light from the image completely fills the pupil, just as light from the object does, they will appear of equal brightness. But when the magnifying power of the instrument is large, the emergent pencil never fills the pupil. When the telescope is directed towards a bright surface the emergent pencil fills the eye-ring. Let r be the radius of the eye-ring, and p the radius of the pupil; then r is usually smaller than p, and the apparent brightness will be less than the brightness of the object in the proportion of the areas of the eye-ring to that of the pupil. The brightness is therefore given by the equation

$$I = I_0 \left(\frac{r}{p}\right)^2.$$

But if γ be the magnifying power, $\gamma = b/r$, where b is the semi-aperture of the object-glass. Hence

$$I = I_0 \left(\frac{b}{\gamma p}\right)^2.$$

Thus the brightness depends on the magnifying power and on the aperture of the object-glass; and if the magnifying power be large, the aperture of the object-glass must be large too, otherwise the brightness of the image will be impaired.

Ex. In making with an astronomical telescope an observation for which it is essential that the brightness of the image on the retina should be at least a hundredth part of that of the object, show that if the diameter of the object-glass be 25 inches and that of the pupil $\frac{1}{4}$ inch, the greatest magnifying power that can be used is 1000. What is the highest magnifying power that can be used without any diminution of brightness?

In Galileo's telescope the eye is placed close to the eye-lens, and the pupil is filled when points are seen by full pencils, and therefore the brightness of the image is very nearly equal to that of the object, and it does not depend on the aperture of the object-glass. But in this instrument the field of view depends on the aperture of the object-glass. This aperture, however, cannot be made very large, because the refraction through the lens is excentrical, and if the aperture be large, the extreme pencils will be refracted at such a distance from the axis as to make the chromatic aberration considerable.

148. Object-glasses are usually made of two lenses, a convex lens of crown glass being combined with a concave lens of flint glass. The pencils of light are incident centrically on the first lens, and if there were an interval between the lenses, the incidence on the second lens would be excentrical; this would be disadvantageous, and the two lenses are placed close together.

We have therefore four quantities at our disposal, namely, the radii of curvature of the four surfaces of the two lenses.

The focal lengths of the two lenses are immediately determined by two essential conditions. These are, that the combination must have a given focal length, and must be achromatic. Let f, f' be the focal lengths of the lenses, and F the focal length of the combination. Then

$$\frac{1}{F}=\frac{1}{f}+\frac{1}{f'}.$$

Also the condition for achromatism is

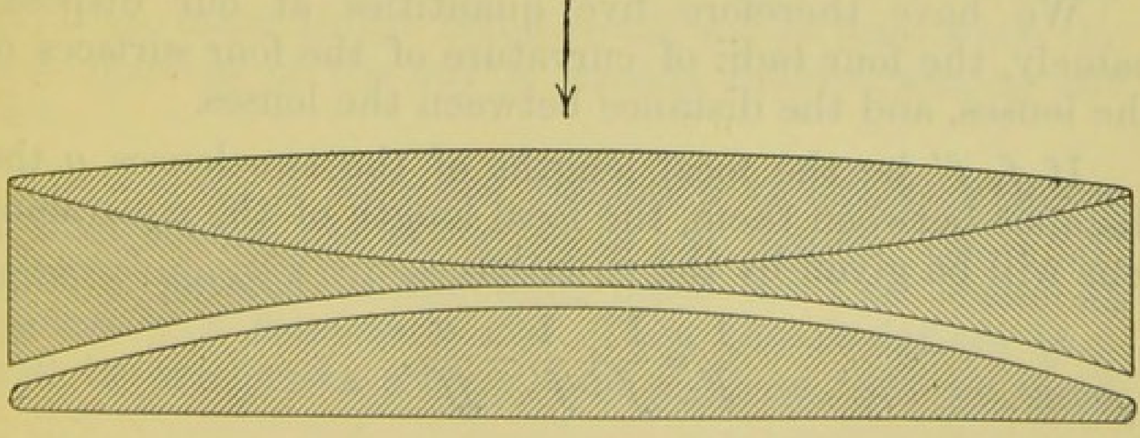
$$\frac{\varpi}{f}+\frac{\varpi'}{f'}=0.$$

These two equations determine f and f', so that no other condition can be satisfied which involves relations between the focal lengths.

The radii of curvature of the surfaces are chosen so as to eliminate as many as possible of the defects due to aberration.

In practice the calculations of the curves required to satisfy the conditions necessary to eliminate the effects of spherical aberration are usually neglected, and the final corrections are made by polishing. The curves are originally designed to give an approximate correction, and the rest is done by figuring the surface. It is thus the exception rather than the rule for the final curves to be truly spherical.

For a most interesting account of the methods of preparing and testing object-glasses, see a lecture by Sir Howard Grubb, F.R.S., printed in *Nature*, vol. XXXIV., 1886.



The diagram represents a new form of telescopic objective constructed by Messrs Cooke and Sons of York for a 12-in. refractor for Rio Janeiro, described in *Engineering*, 1894. It is made of some of the new varieties of glass produced by Messrs Schott and Gen of Jena. The first lens is convex and is made out of baryta light flint glass having a refractive index 1·5637 for the D ray and a dispersive power 1/50·6. The second lens is bi-concave and made of boro-silicate flint, whose refractive index is 1·54685 for the D ray and dispersive power 1/50·2. The third lens is made of slight silicate crown glass, of refractive index 1·5109, the dispersive power 1/60·4. A small space is left between the second and third lenses, with the object of making the correction for spherical aberration as complete as possible; and when the thickness of this space is properly proportioned, there is an entire absence of spherical aberration for all colours of the spectrum.

Eye-pieces.

149. In the Astronomical telescope instead of a single eye-glass it is usual to use a combination of two lenses separated by an interval. The introduction of a third lens between the object-glass and the eye-glass will increase the field of view of the instrument. For this reason it is usually called the *field-glass*.

The incidence of the pencils on the field-glass is not centrical, so that no advantage is gained by placing it close to the eye-glass. The two lenses of an eye-piece are therefore separated by an interval.

We have therefore five quantities at our disposal, namely, the four radii of curvature of the four surfaces of the lenses, and the distance between the lenses.

If f, f' be the focal lengths of the two lenses, a the distance between them, the focal length of the equivalent lens will be given by the equation

$$\frac{1}{F}=\frac{1}{f}+\frac{1}{f'}-\frac{a}{ff'}.$$

The focal length F of the combination will be a given quantity, so this is to be considered as one relation between the constants.

By far the most important defect of the image given by a single lens is that due to chromatic aberration. For a combination of two lenses separated by an interval, it is not possible to remove entirely the defects of this chromatic aberration. The defects of the image are two-fold, the coloured images are not in the same plane perpendicular to the axis of the telescope, and they are not of the same magnitude. Either of these defects can be removed but not both; and the first defect is of the less consequence and is therefore neglected. It is best to make the lenses of the same kind of glass, for then if the combination be achromatic as regards two colours, it will be perfectly achromatic, because there will be no irrationality of dispersion.

It has been shown in § 103 that the condition for this imperfect achromatism for two lenses of the same kind of glass is

$$a = \tfrac{1}{2}(f + f').$$

This is a second relation between the constants.

150. The errors of spherical aberration in eye-pieces are very complicated. Without entering into details connected with these defects, it will be understood that the errors will, in general, be reduced by diminishing the aberrations of extreme pencils, and that if the forms of the lenses be given, this effect will be produced by increasing their number and dividing the refraction. The resulting aberration, other things being equal, will be least when the whole bending of the ray is equally divided among the lenses.

The condition for equal refraction is easily obtained. We shall confine our attention to two lenses. Let a ray, originally parallel to the axis, meet the two lenses at distances y, y' from the axis. Then the deviations produced by the lenses are y/f, and y'/f', so that we must have $y/f = y'/f'$. But if θ be the inclination to the axis of the ray between the lenses

$$y' = y - a\theta, \text{ and } \theta = \frac{y}{f};$$

therefore $$y' = y\left(1 - \frac{a}{f}\right);$$

this gives $$f' = f\left(1 - \frac{a}{f}\right),$$

or finally, $$a = f - f'.$$

This condition, expressed in words, is that the interval between the lenses must be equal to the difference of their focal lengths. This is the principle on which Huyghens' eye-piece was constructed.

The preceding conditions only relate to the focal lengths and positions of the lenses, and are independent of their particular forms. The aberrations will depend largely on their forms; but the different defects previously mentioned in general require different and sometimes opposite forms for their correction. It is therefore necessary to sacrifice the perfection of the instrument in one respect to improve it in another which may be of more importance for the particular object for which it is intended. The theory of this part of the subject is however very troublesome, and it is but little attended to in practice. The lenses employed are almost invariably plano-convex or equi-convex lenses.

151. If we combine the condition of achromatism with the condition for equal refraction at the two lenses, we get the two equations

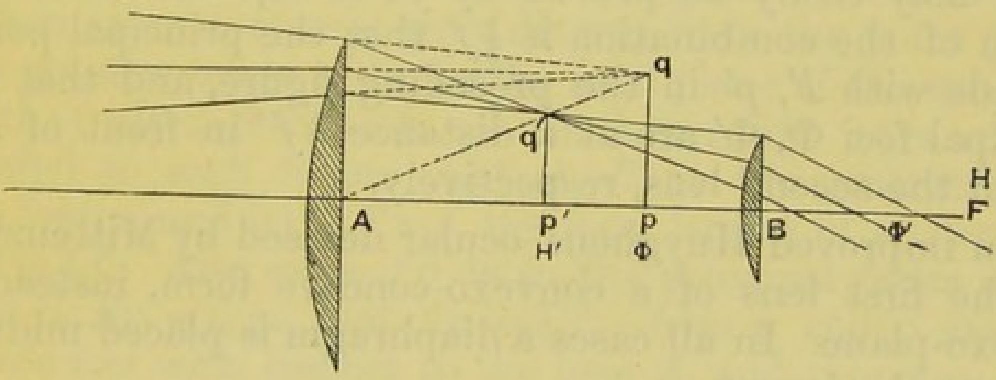
$$\left.\begin{aligned} a &= \tfrac{1}{2}(f + f') \\ a &= f - f' \end{aligned}\right\}.$$

From these equations we deduce

$$f = 3f', \qquad a = 2f'.$$

The eye-piece will therefore consist of two lenses, the field-glass having a focal length equal to three times that of the eye-glass, and the distance between them equal to twice the focal length of the eye-glass. This is the construction of Huyghens' eye-piece, invented by him to diminish the effects of aberration, by making the deviations of the rays at the two lenses equal. It was afterwards pointed out by Boscovich, that it also possessed the advantage of being achromatic.

The eye-piece is usually made with plano-convex lenses, the plane faces being next the eye. Rays proceeding from the object-glass would meet in q, qp being in the principal focal plane of the object-glass; the rays are caught by the field-glass before reaching q, and are brought to a focus at q', which is in the focus of the eye-glass, so that the rays will emerge parallel to each other.



Let A, B be the centres of the lenses, AF the focal length of the lens A; then since $AF = 3f'$, $AB = 2f'$, the point F is also the principal focus of the lens B. Since $q'p'$ is in the focal plane of the lens B, $Bp' = \frac{1}{2}AB$. Also, since p, p' are conjugate foci with respect to the lens A,

$$\frac{1}{Ap'} - \frac{1}{Ap} = \frac{1}{3f'}$$

and $Ap' = f'$; therefore $Ap = \frac{3}{2}f' = \frac{3}{4}AB$. Thus p is the middle point of AF.

Thus the field-glass must be placed between the object-glass and its principal focus, at a distance equal to half its own focal length from the latter.

This eye-piece cannot be used in telescopes where measurements by means of spider-lines or fine wires are to be made. For the principal focus of the object-glass is virtual. The wires could not be placed at the image qp, because there will be distortions in the image of the wires due to the eye-glass, while the image of the object will be distorted by excentrical refraction through both the field-glass and the eye-glass; so that the wires and the image will appear distorted in different degrees, and therefore the position of a point in the field would be estimated incorrectly by referring it to the wires. In all telescopes graduated by wires, for measurement, the field-glass must be beyond the principal focus of the object-glass; then the image and the wires if distorted at all are distorted equally, and therefore no error will result in the measurement.

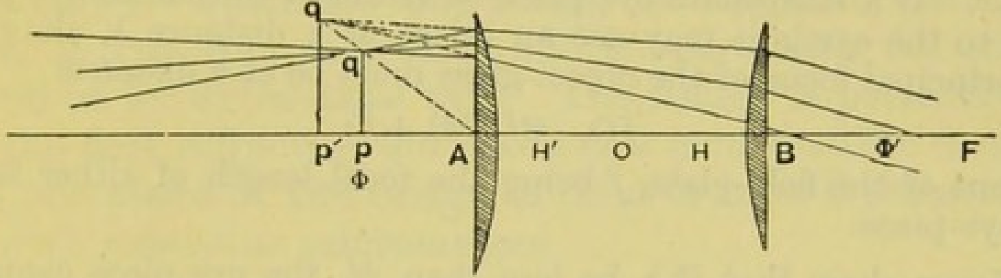
It may easily be proved by 71 *et seq.* that the focal length of the combination is $\frac{1}{2}f$, that the principal points coincide with F, p' in the preceding figure, and that the principal foci Φ, Φ' are at a distance $\frac{1}{2}f'$ in front of and behind the second lens, respectively.

An improved Huyghens' ocular devised by Mittenzwey has the first lens of a convexo-concave form, instead of convexo-plane. In all cases a diaphragm is placed midway between the lenses.

Ex. If Huyghens' eye-piece be arranged so that an eye placed close to the eye-lens may see an image at a distance λ, show that the eye-piece must be placed so that the principal focus of the object-glass may lie between the lenses of the eye-piece at a distance from the field-glass equal to $3f'(2f'+\lambda)/(f'+2\lambda)$.

152. In the common astronomical eye-piece, known as Ramsden's eye-piece, the two lenses are of equal focal length, and therefore the condition of achromatism requires that the distance between them should be equal to the focal length of either. But in this arrangement, the field-glass being exactly in the focal plane of the eye-glass, any dust which might happen to lie on it or any flaw in the glass would be magnified by the eye-glass and confuse the vision. The distance between the lenses is therefore made a little less than the focal length of either; and thus, though the eye-piece is not achromatic, the departure

from perfect achromatism will not be great. The lenses are usually plano-convex lenses with their curved surfaces turned towards each other, and the interval between them two-thirds of the focal length of either.



Rays proceeding from the object-glass converge to a focus at q in the principal focal plane of the object-glass, and after crossing at q meet the field-glass. Their direction is then altered, so that they diverge from the point q', and this point is made to lie in the focal plane of the eye-glass, so that after refraction at the latter, the rays emerge parallel to each other. Let A, B be the centres of the two lenses, and let $AF=f$, the focal length of either, then $AB=\frac{2}{3}f$. Also since $q'p'$ is in the principal focus of the lens B, $Bp'=f$, so that $Ap'=\frac{1}{3}f$. Also p and p' are conjugate foci with respect to the lens A, and therefore

$$\frac{1}{Ap}-\frac{1}{Ap'}=\frac{1}{f};$$

and $Ap'=\frac{1}{3}f$, therefore $Ap=\frac{1}{4}f$.

Thus the field-glass is placed beyond the focus of the object-glass at a distance from it equal to one-fourth of its own focal length.

The radii of the lenses are arranged so as to remedy as many of the defects of aberration as possible, and the indistinctness arising from this cause in this eye-piece is much less than in any of the other ordinary constructions.

The focal length of the combination is $\frac{3}{4}f$; its principal foci Φ, Φ' are at a distance of $\frac{1}{4}f$ from the lenses measured outwards, and the principal points H, H' are between the lenses, such that if O be the middle point of AB, $OH=OH'=\frac{1}{6}f$.

Achromatic symmetrical oculars are frequently made in which each lens of the combination consists of a double convex lens of crown glass, cemented to a meniscus of flint glass, the flint glass being outside in each case.

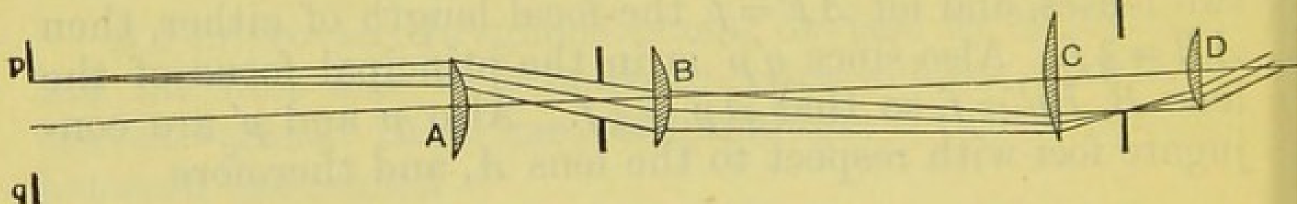
Ex. If a Ramsden's eye-piece be arranged so that an eye placed close to the eye-lens may see an image at a distance λ, show that the principal focus of the object-glass must be at a distance

$$f(\lambda - 2f)/(4\lambda + f)$$

in front of the field-glass, f being the focal length of either lens of the eye-piece.

Hence show that if λ be less than $2f$, the eye-piece cannot be made use of for astronomical observations.

153. There is another eye-piece in common use, known as the erecting eye-piece; it is used for terrestrial objects. A terrestrial telescope differs from an astronomical telescope only in having an erecting eye-piece, instead of an ordinary eye-piece.



One form of erecting eye-piece is shown in the figure. A and B are two convex lenses of equal focal length, placed at any distance from each other, pq is the image as formed by the object-glass. The third and fourth lenses, C and D, form an ordinary Huyghens' eye-piece. The images are formed between the first and second lenses, and between the third and fourth, and at the position of each image is placed a circular diaphragm. The distances between the four lenses are fixed; they are usually fitted into one tube, and adjustments for different distances are effected by pushing in or drawing out this eye-tube.

This eye-piece will not be achromatic, because the focal lengths of the two first lenses will not be the same for all colours; but occasionally each of the four lenses consists of a cemented achromatic doublet.

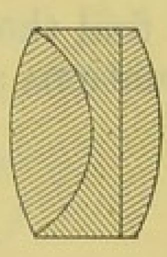
Besides these eye-pieces, there is another type frequently employed, consisting of three cemented lenses. Of these two may be mentioned. Steinheil's Aplanatic micrometer oculars consist of a bi-convex lens of crown glass, between two meniscus lenses of flint-glass, similar in form to Steinheil's aplanatic magnifiers, already described. The ocular is free from distortion and is achromatic for central as well as for extra-axial rays. The image is perfectly flat and the first refracting surface is at a considerable distance from the plane of the image, so that it is well adapted for use with cross-line micrometers.

Schröder's high power eye-piece consists of a double convex lens of soft-crown glass, a concavo-plane lens of dense flint and a plano-convex lens of soft-crown, cemented together. The values of the radii of the spherical surfaces are given as follows:—

$$\left.\begin{aligned} r_1 &= 80{\cdot}026 \\ r_2 &= 36{\cdot}536 \end{aligned}\right\} \text{soft-crown,}$$

$$\left.\begin{aligned} r_3 &= r_2 \\ r_4 &= \infty \end{aligned}\right\} \text{dense flint,}$$

$$\left.\begin{aligned} r_5 &= \infty \\ r_6 &= 80{\cdot}026 \end{aligned}\right\} \text{soft-crown.}$$



This lens can be used up to an aperture equal to half the focal length.

154. The position of a compound eye-piece when arranged for distinct vision, and the magnifying power of the instrument, may be found by considering the images formed as the rays pass through the instrument.

We shall suppose the object to be very distant, and that the instrument is arranged so that the rays of the emergent pencils are parallel to each other, and therefore the first image will be in the principal focus of the object-glass, and the last image will be in the principal focus of the eye-lens. Let x, x' be the distances of these images in front of, and behind, the field-lens, respectively, and let η, η' be the linear magnitudes of these images, and α, α'

the initial and final inclinations of the axis of the extreme pencil. Then, if f, f', f'' be the focal lengths of the three lenses, and a, a' the intervals between them,

$$\left.\begin{aligned} a &= x + f \\ a' &= x' + f'' \end{aligned}\right\},$$

and
$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{f'}.$$

The relation between the intervals a, a' is therefore

$$\frac{1}{a-f} + \frac{1}{a'-f''} = \frac{1}{f'}.$$

If we clear of fractions and add f'^2 to each side of this equation, it takes the form

$$(f+f'-a)(f'+f''-a') = f'^2.$$

To find the inclinations of the initial and final pencils, the equations are

$$\alpha = \frac{\eta}{f},$$

$$\alpha' = -\frac{\eta'}{f''},$$

and also
$$\frac{\eta}{x} = -\frac{\eta'}{x'};$$

and therefore the magnifying power is

$$\gamma = \frac{\alpha'}{\alpha} = -\frac{\eta'}{\eta}\frac{f}{f''},$$

or
$$\gamma = \frac{x'}{x}\frac{f}{f''}.$$

But
$$\frac{x'}{x} = \frac{x'}{f'} - 1 = \frac{a' - f' - f''}{f'},$$

and therefore

$$\gamma = -f\left\{\frac{1}{f'} + \frac{1}{f''} - \frac{a'}{f'f''}\right\}.$$

This formula might have been found directly by substituting for the two lenses of the eye-piece the equivalent lens, and then using the result already obtained for the magnifying power of the astronomical telescope consisting of two lenses.

In exactly the same way it may be shown that

$$\frac{1}{\gamma} = -f''\left\{\frac{1}{f} + \frac{1}{f'} - \frac{a}{ff'}\right\}.$$

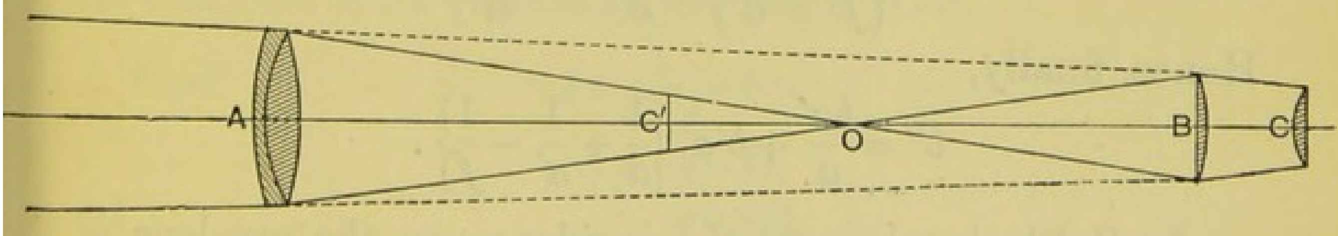
If we multiply these equations together we get the relation between the intervals already mentioned.

The object will be inverted, as in the ordinary astronomical telescope, unless a be greater than $f+f'$; that is, unless the distance between the first two lenses be greater than the sum of their focal lengths.

If the focal lengths of the lenses be given, and also the magnifying power, the intervals between the lenses are determined; for, from the preceding values of γ, we get

$$a = f + f' + \frac{ff'}{f''\gamma}, \quad a' = f' + f'' + \frac{f'f''\gamma}{f}.$$

155. The field of view is determined in the same way as in the ordinary astronomical telescope, supposing it to be governed by the first two lenses. The aperture of the third lens will then be chosen so as to allow of all the rays



to pass through. Thus suppose we consider the field of view which is seen by full pencils only. The bright field of view is determined by the equation

$$\Theta = \frac{fb' - b(a-f)}{fa},$$

where b, b' are the semi-diameters of the object-glass and field-lens, respectively, just as in the simple astronomical telescope.

The full cone of rays between the first two lenses will just fill both lenses, crossing at an intermediate point O. If after refraction at the field-lens the full cone of rays just fills the eye-lens, the cone of rays *before* refraction must just fill the image of the eye-lens formed by the field-lens. Let C' be the position of the image of the lens C, formed by the lens B.

Then, if $BC' = x$, we have

$$\frac{1}{x} + \frac{1}{a'} = \frac{1}{f'}.$$

Also, if η be the semi-diameter of the image, then by § 51

$$\frac{\eta}{b''} = -\frac{x}{a'}.$$

The condition that the cone between A and B shall just fill this image is

$$\frac{b' + \eta}{x} = \frac{b' + b}{a};$$

that is, taking the numerical value of η,

$$\frac{b'}{x} + \frac{b''}{a'} = \frac{b' + b}{a};$$

or

$$b' \left\{\frac{1}{f'} - \frac{1}{a'}\right\} + \frac{b''}{a'} = \frac{b' + b}{a}.$$

Hence, finally,

$$b'' = \frac{ba'}{a} + b'a' \left\{\frac{1}{a} + \frac{1}{a'} - \frac{1}{f}\right\}.$$

In Galileo's telescope the incidence on the eye-lens is centrical; the eye-lens used is therefore always a single concave lens, or achromatised combination of two or three lenses in contact.

156. *Ex.* Let O, A, B, C, D denote the lenses of an astronomical telescope with an erecting eye-piece, taken order beginning with the object-glass. The focal lengths are respectively 36, $1\frac{7}{8}$, $2\frac{1}{2}$, 2, $1\frac{1}{2}$ inches, and the distances OA, AB, BC, $37\frac{1}{2}$, $2\frac{1}{2}$, 3 inches

respectively. Find the position of the eye-lens when the instrument is adapted for an eye whose distance of distinct vision is 8 inches, and show that the magnifying power is then $40\frac{5}{7}$.

If u, v be the distances of the object and image, respectively, in front of and behind a convex lens of focal length f, we have

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f}.$$

As object consider the image formed in the principal focus of the object-glass; then using similar notation for all the subsequent lenses, with dashed letters, we find

$$u=\tfrac{3}{2},$$

and therefore $$\frac{1}{v}+\tfrac{2}{3}=\tfrac{8}{15},$$

which gives $$v=-\tfrac{15}{2}.$$

Hence $$u'=\tfrac{5}{2}-v=10.$$

Taking the next lens, we get

$$\frac{1}{v'}+\tfrac{1}{10}=\tfrac{2}{5};$$

therefore $$v'=\tfrac{10}{3},$$

and $$u''=3-v'=-\tfrac{1}{3}.$$

Similarly, considering the next lens in order

$$\frac{1}{v''}-3=\tfrac{1}{2};$$

and therefore $$v''=\tfrac{2}{7}.$$

If a denote the interval between the lenses C, D, we have

$$u'''=a-\tfrac{2}{7}.$$

The conditions of the question require that $v'''=-8$.

Thus $$-\tfrac{1}{8}+\frac{1}{a-\frac{2}{7}}=\tfrac{2}{3};$$

$$\therefore a-\tfrac{2}{7}=\tfrac{24}{19},$$

and $$a=\tfrac{206}{133}.$$

To find the magnifying power, let the magnitudes of the images in succession be x, x', x'', x''', x^{iv}. Then if α be the angle subtended by the object at the centre of the object-glass

$$\alpha=\frac{x}{36}.$$

Also the angle of vision is

$$\beta=-\frac{x^{\text{iv}}}{8}.$$

But by similar triangles

$$\frac{x'}{x} = -\frac{v}{u} = 5,$$

$$\frac{x''}{x'} = -\frac{v'}{u'} = -\tfrac{1}{3},$$

$$\frac{x'''}{x''} = -\frac{v''}{u''} = \tfrac{6}{7},$$

$$\frac{x^{\text{iv}}}{x'''} = -\frac{v'''}{u'''} = \tfrac{19}{3}.$$

Hence by multiplication

$$\frac{x^{\text{iv}}}{x} = -\tfrac{190}{21}.$$

The magnifying power is

$$\gamma = \frac{\beta}{a} = -\frac{9x^{\text{iv}}}{2x}$$

$$= \tfrac{285}{7},$$

that is $$\gamma = 40\tfrac{5}{7}.$$

Reflecting Telescopes.

157. If instead of a convex object-glass, a concave mirror be used to receive the rays proceeding from an object, an image of the object will be formed by the mirror, which, if the aperture be sufficiently large, may be viewed directly by means of an eye-piece placed in a suitable position, as in the case of the telescopes previously described. Such is the principle of Sir W. Herschel's telescope, which is the simplest of the reflecting telescopes.

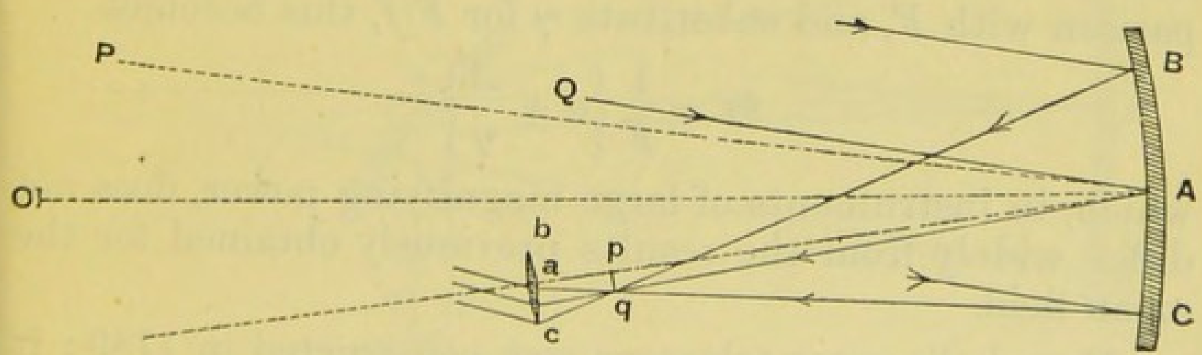
In order that the head of the observer may intercept as little light as possible, the axis of the mirror is slightly inclined to the axis of the tube in which it is fixed, and thus the image is thrown near the edge of the tube, where it is viewed through an eye-lens, or eye-piece, the observer having his back to the object and looking down into the tube. The obliquity of the incident pencil to the axis of the mirror will produce a slight distortion of the image, but the errors due to this cause are scarcely appreciable in the very large instruments to which this construction is alone applicable.

We shall suppose that the object is very distant, so that the image formed by the mirror will be in the principal focus of the mirror; and also that the instrument is to be adapted to the use of eyes with normal sight, so that the emergent rays must be parallel, and therefore the eye-lens must be placed in such a position that the first image may lie in its focal plane.

Now the angle which the object will subtend at the centre of the mirror, and therefore at the eye, will be equal to $-\eta/F$, where η is the linear magnitude of the image, and F the focal length of the mirror. And the angle under which the image will be seen by the eye will be η/f, f being the focal length of the eye-lens. The magnifying power is represented by the ratio of the latter to the former, and therefore

$$\gamma = -\frac{F}{f}.$$

This instrument therefore gives an inverted image. But since the observer has turned round and is looking at the image in front, the appearance of the object as seen is as if it were inverted top and bottom, but not from side to side.



The arrangement of the mirror and eye-lens is shown in the figure. BAC is the large spherical reflector, AO being its axis, and O its centre; AP is the axis of the tube and Aa the axis of the eye-lens, and these two lines are equally inclined to AO, the axis of the mirror. Bq, Aq, Cq, are three rays which are brought to a focus at q by the large reflector; the rays afterwards meet the eye-lens and finally emerge parallel to each other. The focus q and the corresponding point of the object lie in the same line through O, the centre of the reflector.

158. The field of view in this telescope is determined in exactly the same way as in the astronomical telescope. The distance between the lens and the centre is $F-f$, nearly; for $AO = 2F$, and $Ap = F$ and the inclination of Ap to AO is very small. If therefore a denote the semi-aperture of the eye-lens, and Θ half the field of view, as determined by the axes of extreme pencils, then

$$\Theta = \frac{a}{F-f}.$$

The focal length of the eye-piece in these instruments is very small in comparison with that of the mirror so that *the field of view is very nearly equal to the angle subtended by the eye-lens at the vertex of the large reflector.*

If we adopt the result previously obtained for the total field of view, and the bright field of view, we have

$$\Theta' = \frac{aF \pm Af}{F(F+f)},$$

where A denotes the semi-aperture of the object-mirror, and Θ' half the extreme field. If we neglect f in comparison with F, and substitute γ for F/f, this becomes

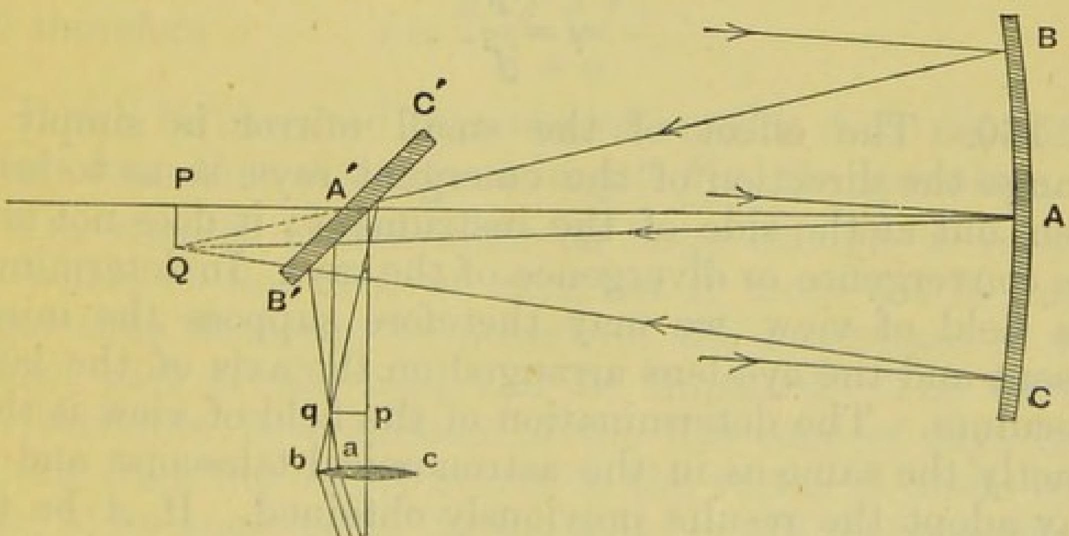
$$\Theta' = \frac{1}{F}\left\{a \pm \frac{A}{\gamma}\right\}$$

which, in instruments of large magnifying power, does not differ widely from the results previously obtained for the mean field.

Herschel's great telescope was constructed in 1789; it was 40 feet in length, and the great reflector was 50 inches in diameter. The quantity of light obtained by this instrument was so great as to enable its inventor to use eye-pieces of far shorter focal length than any previously used. Lord Rosse's telescope has a speculum of 53 feet focal length and 6 feet diameter.

Newton's Telescope.

159. The principle of the front view, as previously described, can only be used in instruments in which the aperture is very considerable, and to instruments of moderate aperture it is wholly inapplicable. In the telescope invented and constructed by Newton, the rays reflected by the object-reflector are received on a small plane mirror placed between the object-mirror and its principal focus. The plane of the mirror is inclined to the axis of the telescope at an angle of 45°, and the rays which tend to form an image in the principal focus of the object-reflector are reflected laterally and form an image near the side of the tube, equal and similar to the former, and similarly placed with regard to the plane mirror. This image, whose plane is parallel to the axis of the tube, is viewed through an eye-piece placed at the side of the instrument. Instead of a plane mirror, Newton used a rectangular isosceles prism of glass, through the sides of which the rays enter and emerge perpendicularly, being reflected totally at the hypotenuse. The reflexion at the hypotenuse being total, there is a much smaller loss of light in the reflexion than in the reflexion at a metal speculum.



The arrangement of the mirrors and the eye-lens is shown in the figure. BAC is the object-mirror, $B'A'C'$ the plane mirror, and bac the eye-lens. Rays BQ, AQ, CQ are reflected by the large mirror to a focus Q, where PQ is the principal focal plane of the reflector. But before they reach Q they are reflected by the small plane mirror and meet in q; after crossing at q they strike the eye-lens and emerge parallel to each other. The point Q and the corresponding point of the object lie on a line through the centre of the large reflector; also the image PQ and the second image qp are symmetrically placed with regard to the mirror $B'A'C'$, and qp is equal in magnitude to QP.

If F, f denote the focal lengths of the object-mirror and the eye-lens, and e, e' denote their distances from the centre of the plane mirror, then in the figure,

$$\left.\begin{aligned} A'P &= F - e \\ A'p &= e' - f \end{aligned}\right\},$$

since the first image is in the principal focal plane of the large mirror, and the last in that of the eye-lens. Hence, since $A'P = Ap$, we get

$$e + e' = F + f.$$

This is the condition of distinct vision with parallel rays.

The magnifying power may be found just as in the case of Herschel's telescope; the value of it is, as before,

$$\gamma = \frac{F}{f}.$$

160. The effect of the small mirror is simply to change the direction of the emergent rays, so as to bring them out at the side of the instrument; it does not alter the convergence or divergence of the rays. In determining the field of view, we may therefore suppose the mirror absent and the eye-lens arranged on the axis of the large speculum. The determination of the field of view is then exactly the same as in the astronomical telescope, and we may adopt the results previously obtained. If A be the semi-diameter of the large speculum, a that of the eye-lens, the bright field of view as seen by full pencils, will be

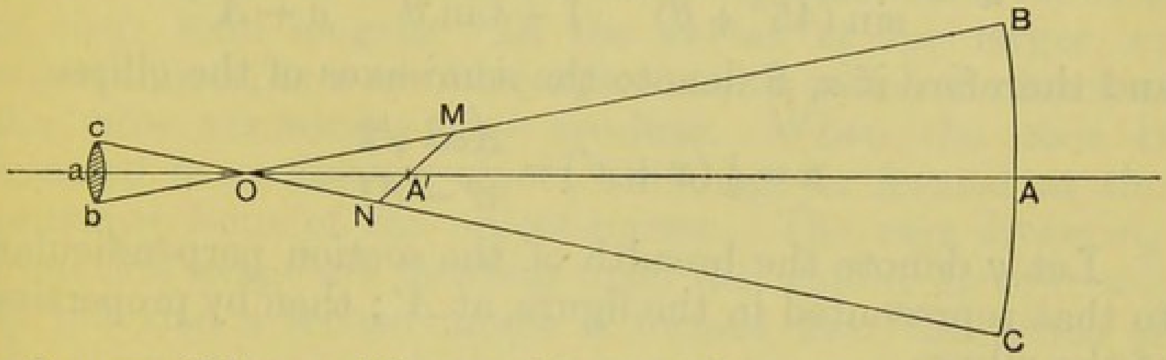
$$\Theta = \frac{1}{F} \cdot \frac{aF - Af}{F + f}.$$

If we neglect f in comparison with F in the denominator, we get

$$\Theta = \frac{1}{F}\left(a - \frac{A}{\gamma}\right),$$

as in Herschel's telescope.

161. The cone of rays within the instrument which contributes to fill the field of full pencils, will fill the large speculum and the eye-lens (supposed transferred to the axis of the instrument), crossing at a point O between the



mirror and lens. The position of O may easily be found, by similar triangles. Thus, if OA be denoted by c, we have

$$\frac{A}{c} = \frac{A + a}{F + f};$$

and therefore

$$c = \frac{A\,(F + f)}{A + a}.$$

It will be observed that the value of c does not differ widely from that of the focal length F of the speculum.

The smaller mirror must be large enough to receive the whole of this cone of rays, but it must not be made larger than necessary, or otherwise the brightness of the central part of the field will be impaired. The mirror will therefore be a section of the full cone of rays converging from the object-mirror to the vertex O, made by a plane at an angle of 45° to the axis; it will therefore be in the form of an ellipse.

Let the semi-vertical angle of the cone be θ, then

$$\tan\theta = \frac{A}{c}.$$

Suppose a section of the cone by a plane through the axis perpendicular to the plane mirror to be represented in the figure, MN being the section of the plane mirror; then MN will be the major axis of the ellipse. Denote the two portions of MN, as divided in the centre A', by x and x'. Then if $A'O$ be denoted by d,

$$x = \frac{d \sin\theta}{\sin(45^\circ - \theta)} = \frac{d\sqrt{2}\tan\theta}{1 - \tan\theta} = \frac{Ad\sqrt{2}}{c - A},$$

$$x' = \frac{d \sin\theta}{\sin(45^\circ + \theta)} = \frac{d\sqrt{2}\tan\theta}{1 + \tan\theta} = \frac{Ad\sqrt{2}}{c + A},$$

and therefore if α, β denote the semi-axes of the ellipse,

$$\alpha = \tfrac{1}{2}(x + x') = \frac{Acd\sqrt{2}}{c^2 - A^2}.$$

Let y denote the breadth of the section perpendicular to that represented in the figure, at A'; then by properties of the ellipse,

$$\frac{\beta^2}{\alpha^2} = \frac{y^2}{xx'}.$$

But y is the radius of the circular section of the cone through the point A', so that $y = Ad/c$; and therefore

$$\frac{\beta^2}{\alpha^2} = \frac{c^2 - A^2}{2c^2}.$$

If we give α its value, the corresponding value of β becomes

$$\beta = \frac{Ad}{\sqrt{c^2 - A^2}}.$$

The aperture of the object-mirror will be small compared with its focal length, and therefore A^2 may be neglected in comparison with c^2. The approximate values of α and β will therefore be

$$\alpha = \frac{Ad\sqrt{2}}{c},$$

$$\beta = \frac{Ad}{c},$$

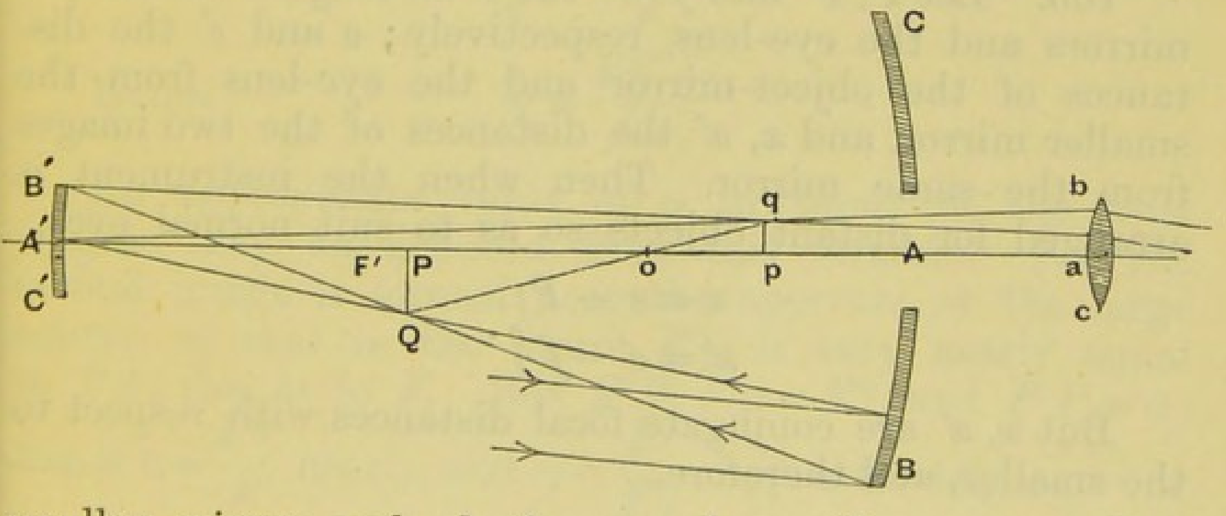
which are in the ratio of $\sqrt{2}$ to 1.

Gregory's Telescope.

162. The invention of the reflecting telescope is generally ascribed to James Gregory, who described the instrument now called by his name, in his *Optica promota*, published in the year 1663.

Gregory's telescope consists of two concave mirrors placed along the same axis with their concavities facing each other, and at an interval a little greater than the sum of their focal lengths. In the vertex of the larger, or object-mirror, is a circular aperture, to which is attached the tube containing the eye-lens. When the axis is directed to a distant object an image is formed at the principal focus of the object-mirror. The rays diverging from this image are incident upon the smaller mirror, and by reflexion a second image is formed near the vertex of the large mirror, and this image is viewed through the eye-lens, placed at a distance from it equal to its own focal length.

The arrangement of the mirrors and the lens is shown in the figure, in which AB is the object-mirror, $A'B'$ the



smaller mirror, and ab the eye-glass. Rays proceeding from a point of the object are reflected at the large mirror and are brought to a focus in Q, where PQ is in the principal focal plane; also Q and the corresponding point of the object lie on the same line through the centre of the large mirror. After passing Q, the rays diverge and are incident on the small mirror, and are brought to a focus at q; as before, the points Q, q lie on the same line through the centre of the small mirror. The eye-lens is placed in such a position that qp is in its principal focal plane, and therefore the rays of the pencil after passing through the eye-lens emerge parallel to each other.

In the original description of the instrument the large reflector was a paraboloid of revolution, and the smaller, a prolate spheroid whose foci are at P and p, the positions of the two images. With reflectors formed of these surfaces, there would be no aberration for rays in the centre of the field. It was for some time deemed hopeless to prepare mirrors having these forms, and the instrument was never constructed till after that of Newton.

Gregory's telescope is generally preferred to Newton's. Its superiority seems to arise from the fact that the two specula may be matched and their irregularities of form made to counteract each other; whereas in Newton's telescope there is nothing to compensate any defect in the form of the object-mirror, and experience shows that such mirrors can seldom be made truly spherical.

163. Let F, F' and f be the focal lengths of the two mirrors and the eye-lens, respectively; e and e' the distances of the object-mirror and the eye-lens from the smaller mirror, and x, x' the distances of the two images from the same mirror. Then when the instrument is arranged for distant objects so as to suit normal eyes,

$$\left.\begin{aligned} x &= e - F \\ x' &= e' - f \end{aligned}\right\}.$$

But x, x' are conjugate focal distances with respect to the smaller, and therefore,

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{F'}.$$

If we eliminate x, x', we get the equation

$$\frac{1}{e - F} + \frac{1}{e' - f} = \frac{1}{F'}.$$

This is the equation of condition for distinct vision in Gregory's telescope. The equation is similar to that previously obtained for a refracting telescope with three lenses.

The eye-glass is usually fixed in position and the adjustment to distinct vision effected by moving the smaller mirror by a fine screw.

164. Let a denote the distance between the principal foci of the two mirrors, so that

$$e = F + F' + a.$$

Let η, η' be the linear magnitudes of the first and second images. Then the angle subtended by the object to the eye is equal to $-\eta/F$, and the angle under which the last image is seen, is equal to η'/f. The magnifying power of the instrument is therefore

$$\gamma = -\frac{\eta'}{\eta} \cdot \frac{F}{f}.$$

But, by the theory of spherical mirrors (§ 37),

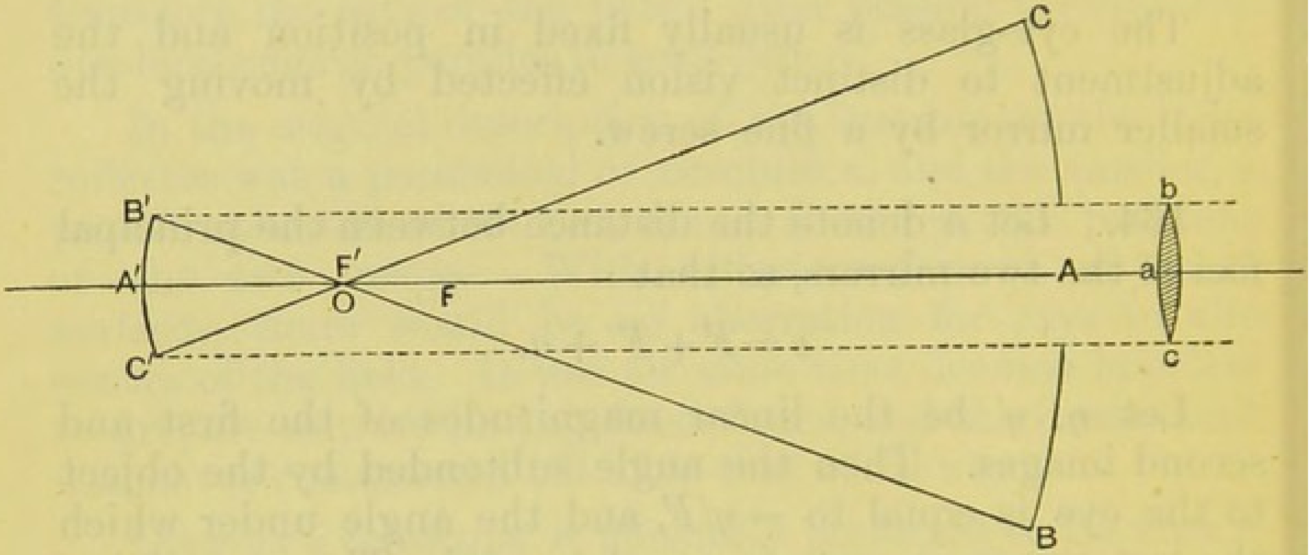
$$\frac{\eta}{\eta'} = -\frac{a}{F'}.$$

Hence
$$\gamma = \frac{FF'}{fa}.$$

An approximate value of γ in terms of the focal lengths may be deduced from this formula. For the second image is formed near the aperture of the large mirror, so that in the figure $F'p$ is very nearly equal to PA, that is to F. But $F'P \, . \, F'p = F'^2$, and $F'P = a$; hence $a = \frac{F'^2}{F}$ nearly, and $\gamma = \frac{F^2}{F'f}$, approximately.

165. We shall next consider the field of view, confining ourselves to the bright field, seen by full pencils. After reflexion at the large speculum the full cone of rays will fill the smaller speculum, crossing at the point O (see figure on next page), between the two specula. If the

focus F' coincide with the vertex O of this cone, the rays after reflexion at the small mirror will proceed parallel to the axis, and will fill an eye-lens of the



same diameter as the mirror. And this is the best arrangement; for the dimensions of the small mirror should not be too large, or otherwise it will intercept a considerable amount of the incidence pencil. On the other hand if the ratio of the aperture to the focal length be less than is indicated by the above conditions, the pencil after reflexion at the smaller mirror would diverge, and would need an eye-lens of greater magnitude than the small mirror to receive all the rays; but this is impossible, otherwise some of the incident light would fall directly on the lens. Hence if A, A' be the semi-diameters of the two mirrors, and e the distance between them, we must have

$$\frac{A'}{F'} = \frac{A + A'}{e}.$$

The field of view may now be regarded as limited by the large and small speculum. The method to be followed is the same as that employed in the case of the former telescopes. It will be found, however, that if the image of the smaller speculum be formed by reflexion in the larger, the image will have smaller linear dimensions than the large speculum, so that this image must be regarded as the entrance pupil.

For let $B'' A'' C''$ denote the image of $B' A' C'$ formed by the large speculum, A'' its semi-diameter.

Then $$\overline{FA'} = e - F,$$

and therefore $$\overline{FA''} = \frac{F^2}{e - F}.$$

The image will be inverted and its magnitude determined by the equation

$$\frac{A'}{A''} = -\frac{e - F}{F},$$

so that $$A'' = -A' \cdot \frac{F}{e - F}.$$

Now the two images PQ, pq in the figure of § 162 both lie on the same side of F', and therefore in the figure of this section F necessarily lies between F' and A.

Hence A'' is numerically less than $A' \cdot \overline{OA}/\overline{OA'}$, and therefore less than A.

The value of the bright field will therefore be determined by the equation

$$\Theta = \frac{A - A''}{\overline{AA''}}$$

$$= \frac{A - \dfrac{A'F}{e - F}}{F + \dfrac{F^2}{e - F}}$$

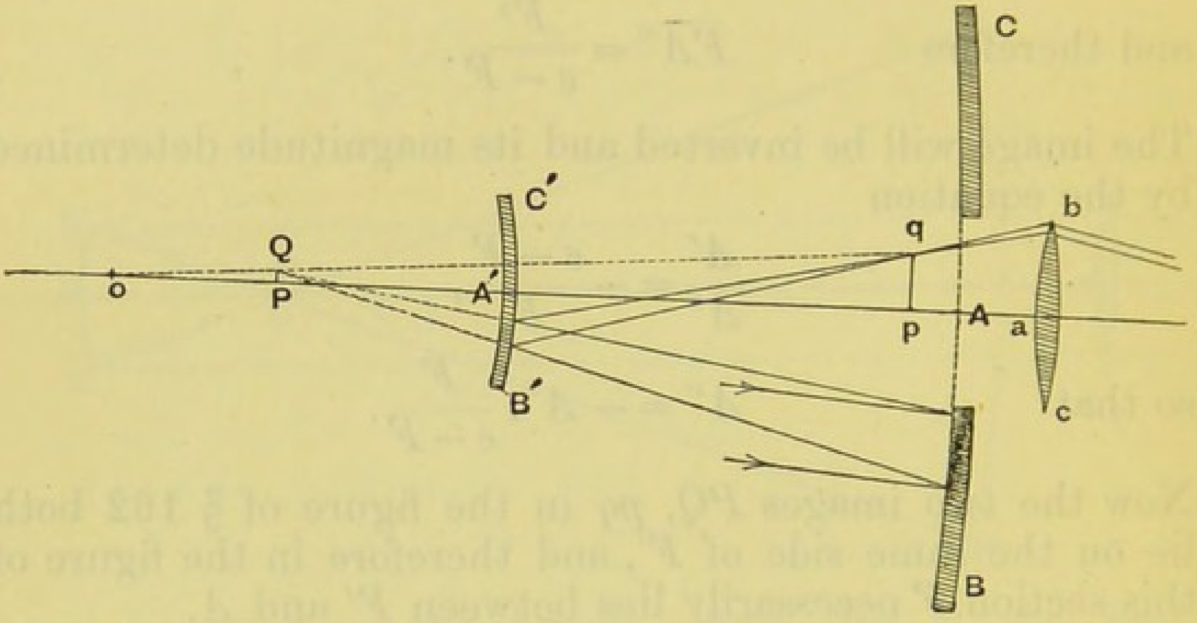
$$= \frac{A(e - F) - FA'}{eF}$$

$$= \frac{A}{F} - \frac{A + A'}{e},$$

or finally, $$\Theta = \frac{A}{F} - \frac{A'}{F'}.$$

In other words, the bright field of view is measured by the difference of the numerical measures of the apertures of the specula.

166. Another reflecting telescope was invented some years after Gregory's and Newton's telescopes by a Frenchman named Cassegrain, probably without any knowledge



of what had been done in England. Cassegrain's telescope only differs from Gregory's in having its small mirror convex instead of concave, and placed between the large mirror and its principal focus. The arrangement of the mirrors and images is shown in the figure.

The investigations to find the position of the mirrors and lenses so as to admit of distinct vision, and the magnifying power and the field of view in Gregory's telescope are all applicable to Cassegrain's; we have only to change the sign of F', the focal length of the small mirror, throughout.

The image will appear inverted, just as in the astronomical telescope.

The great 48-inch reflecting telescope constructed by Sir Howard Grubb for Melbourne University was of the Cassegrain type.

167. *Ex.* The focal lengths of the larger and smaller mirrors of a Gregorian telescope are 32 inches and 3 inches, and the distance between their principal foci $\frac{1}{4}$ inch; it is fitted with a Huyghenian eye-piece, the focal lengths of whose lenses are 3 and 1 inches. Prove that when the instrument is adjusted for normal vision the distance between the field-glass and the smaller mirror is $37\frac{1}{2}$ inches,

and that the magnifying power is 256. If the instrument be adjusted for vision at a distance of 12 inches, by altering the position of the smaller mirror, prove that the distance through which the latter must be moved is ·001 inch towards the large reflector.

Let e, e' be the distances of the object-mirror and the field-lens, respectively, from the small mirror, and x, x' the distances of the first two images from the same mirror. Then the second image must lie between the lenses of the eye-piece at a distance from the field-lens equal to $\frac{3}{4}$ the distance between the lenses, in order that the rays may emerge from the eye-lens parallel to each other. Hence

$$x' = e' + \tfrac{3}{2}.$$

Also the distance between the principal foci of the mirrors is $\frac{1}{4}$ inch, and therefore

$$x - F' = \tfrac{1}{4},$$

or $$x = 3\tfrac{1}{4}.$$

And since x, x' are conjugate focal distances,

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{F'};$$

that is, $$\frac{1}{x'} + \tfrac{4}{13} = \tfrac{1}{3},$$

and therefore $$x' = 39.$$

Hence, finally $$e' = 39 - \tfrac{3}{2} = 37\tfrac{1}{2}.$$

Let η, η', η'' be the linear magnitudes of the first three images. Then the angle subtended by the object to the eye is $-\eta/F$, and that subtended by the final image is $\eta''/1$. The magnifying power of the instrument is therefore

$$\gamma = -\frac{\eta'' F}{\eta}.$$

But we have $$\frac{\eta'}{\eta} = -\frac{x'}{x},$$

and $$\frac{\eta''}{\eta'} = \tfrac{2}{3}.$$

Hence $$\gamma = \tfrac{2}{3}\,\frac{x'}{x}\,.\,F$$

$$= \tfrac{2}{3} \times 39 \times \tfrac{4}{13} \times 32;$$

that is, $$\gamma = 256.$$

Next, let the instrument be arranged for an eye seeing at a distance of 12 inches, the adjustment being made by altering the position of the small mirror. Since the position of the eye-piece is unchanged, the value of $e' - e$ is unchanged, and therefore

$$e' - e = 2\tfrac{1}{4}.$$

Also $e = x + F = x + 32.$

Hence $e' = x + 34\frac{1}{4}.$

Let u, v represent the distances of the first and second images in front of and behind the field-lens, respectively, and u', v' similar quantities with reference to the eye-lens. Then we have

$$v' = -12;$$

and therefore $\frac{1}{u'} - \frac{1}{12} = 1,$

or $u' = \frac{12}{13}.$

Hence $v = 2 - u' = \frac{14}{13}.$

Also $\frac{1}{u} + \frac{13}{14} = \frac{1}{3},$

and therefore $u = -\frac{42}{25} = -1{\cdot}68.$

But $x' = e' - u$

$$= x + 34\tfrac{1}{4} + 1{\cdot}68,$$

or $x' = x + 35{\cdot}93.$

And since x, x' are conjugate focal distances, this gives

$$\frac{1}{x} + \frac{1}{x + 35{\cdot}93} = \frac{1}{3}.$$

This is a quadratic to find x, and if we solve it we find

$$x = 3{\cdot}249, \text{ very nearly.}$$

But the value of x in the previous arrangement of the instrument was 3·25.

Hence the small mirror must be moved through a distance ·001 inch nearer to the larger mirror.

The Compound Microscope.

168. In its simplest form the compound microscope, like the astronomical telescope, consists of two lenses, an object-glass or objective, as it is usually called, and an eye-glass or eye-piece. The objective has a very short focal length, and the object is placed at a distance from it slightly greater than the focal length; the objective then forms a real inverted image of the object, which is viewed through the eye-piece.

The objective is usually made up of a system of lenses, designed to diminish chromatic and spherical aberration. Very generally, there are three doublets, each consisting of a double-convex lens of crown-glass cemented to a plano-concave lens of flint, arranged to be achromatic for centrical pencils; these doublets are placed with their plane faces towards the incident light, the lens of shortest focal length being next the object, and their apertures increasing from the first outwards. In this way the apertures can be chosen that a pencil filling the first lens will just fill the other lenses in succession, so that diaphragms are unnecessary; this is a great advantage, because diaphragms will always introduce diffraction fringes which interfere with the definition in the outer parts of the field.

The magnifying power of the microscope has already been investigated in §§ 133, 134.

Ex. The lenses of a common astronomical telescope whose magnifying power is 16, and length from object-glass to eye-glass $8\frac{1}{2}$ inches, are arranged as a microscope to view an object placed $\frac{5}{8}$ inch from the object-glass; if the distance of vision be taken to be 8 inches, show that the magnifying power will be 8.

EXAMPLES.

1. The object-glass of an astronomical telescope has a focal length of 50 inches, and the focal length of each lens of the Ramsden's eye-piece is 2 inches; show that when adjusted for normal vision the distance between the field-glass and the object-glass is 50·5 inches, and that the magnifying power is 100/3.

Show that, in order to adjust the instrument for vision at a distance of 10 inches, the eye-piece must be pushed inwards through a distance $\frac{3}{14}$ of an inch, and that then the magnifying power is increased to 35.

2. The focal length of the object-glass of an astronomical telescope is 40 inches, and the focal lengths of four convex lenses forming an erecting eye-piece are, respectively, $\frac{3}{2}$, $\frac{1}{6}$, $\frac{3}{8}$, $\frac{3}{2}$ inches, reckoning backwards from the object-glass. The intervals between the first and second and between the second and third being one inch and half-an-inch, respectively, show that when the instrument is in adjustment for eyes which can see with parallel rays, the

distance of the eye-lens from the object-glass is $41\frac{1}{2}$ inches, and the magnifying power of the instrument $\frac{80}{3}$.

3. The object-glass of an astronomical telescope has an aperture of 1 foot and a magnifying power of 240. Show that if the aperture of the pupil be $\frac{1}{4}$ inch, the brightness of the image is to that of the object as 1 : 25.

4. A Galileo's telescope, the focal lengths of whose lenses are 6 inches and 1 inch has an objective aperture of $1\frac{1}{2}$ inches; find the field of view as determined by the axes of the extreme pencils, taking the aperture of the eye-pupil to be $\frac{3}{8}$ inch. Show that if the instrument be directed towards a graduated rod distant 160 yards, the length of rod visible through the instrument is 10 yards. If the instrument be arranged for vision at a distance of 10 inches, show that the length of rod visible is increased to about 10·68 yards.

5. A Galileo's telescope is adjusted so that a pencil from an object 289 feet distant emerges as a pencil of parallel rays; the focal length of the object-glass is one foot, and of the eye-glass one inch; show that if the axis be directed towards the sun, and a piece of paper be held 23 inches from the eye-glass, an image of the sun will be formed on the paper. The sun's apparent diameter being $\cot^{-1} 120$, show that the diameter of the image is 2·4 inches and that it is inverted.

6. If the focal length of the larger mirror of Newton's telescope be 20 feet, and its diameter 2 feet, find what portion of the incident light is necessarily stopped by the smaller mirror.

7. The focal lengths of the large and small mirrors of a Gregorian telescope are 18 and $\frac{18}{20}$ inches, respectively, and their distance from each other is 19 inches; if the focal length of the eye-glass be $\frac{1}{2}$ inch, show that the magnifying power is 684.

8. Show that, if in Gregory's telescope the focal length of the small mirror and of the eye-piece be each 2 inches, and the distance between the foci of the large mirror and of the eye-piece be 32 inches, and the telescope be adjusted so that rays from a distant point emerge in a state of parallelism, then the alteration needed for a person who can see best at a distance of 26 inches will be a motion of the small mirror of approximately ·0005 of an inch.

9. A Wollaston's doublet is formed of two convex lenses of focal lengths f, and $3f$, respectively, the distance between them being $2f$, and is in adjustment for viewing a small flat uncovered object; show that if a plate of glass whose thickness is $f/10$ be laid on the object, the instrument may be readjusted without altering the position of the lower lens, by increasing the distance between the lenses by $2(\mu-1)f/(6\mu-1)$.

10. In an astronomical telescope, in which the focal lengths of the object-glass and eye-glass are f, f' and their semi-diameters b, b',

respectively, show that for a person who can see distinctly at a distance a, the radius of the stop should be

$$\frac{a(fb'-f'b)+ff'b'}{ff'+a(f+f')}.$$

11. If the final image formed by the object-glass be at a distance from the eye-piece equal to the least distance of distinct vision, show that specks on the object-glass cannot be distinctly seen when the eye is close to the eye-piece, and find how far off the eye must be in order to see them distinctly.

12. Find the radius of the stop in an astronomical telescope for an observer who sees objects clearly at a distance λ; show that the stop may be greater than for a person seeing distinctly with parallel rays by $(b+b')f'^2/\lambda f$, if the square of f'/f be neglected, f and f', b, b' being the focal lengths and the semi-apertures of the two lenses, respectively.

13. In Huyghens' eye-piece the focal length of the field-lens is three times that of the eye-lens, and the distance between them is twice the focal length of the eye-lens; show that if the lenses be supposed thin, this combination, when focussed for normal vision, will also be in focus (except for aberration) if it be inverted, provided the eye-lens be brought back into the same position as before.

14. Show that if F be the focal length of the object-glass of an astronomical telescope fitted with a Ramsden's eye-piece whose equivalent focal length is f, and d the distance of distinct vision, the magnifying power of the telescope when viewing a very distant object is equal to the ratio $F(f+3d) : 3df$.

15. If F be the focal length of the object-glass of an astronomical telescope, which is fitted with a Ramsden's eye-piece whose field-glass is at a distance a from the object-glass, show that the magnifying power of the telescope is

$$\frac{F}{3(a-F)}.$$

16. Show that in an astronomical telescope fitted with a Ramsden's eye-piece, whatever the distance of distinct vision be, the eye must be placed in front of the eye-lens at a distance $\frac{1}{4}f$ from it, in order to catch all the rays that fall on the field-glass, and that then the magnifying power is equal to $\frac{4}{3}F/f$, F being the focal length of the object-glass and f that of either lens of the eye-piece.

Show that an observer whose distance of distinct vision is less than $\frac{9}{4}f$ cannot make use of the telescope for astronomical measurements.

17. Show that the radius of the stop in an astronomical telescope fitted with a Ramsden's eye-piece, which will intercept all but complete pencils, will be the smaller of two expressions $\frac{2}{3}r$ and

$$(4Fr-fR)/(4F+f),$$

where F and R are, respectively, the focal length and radius of the object-glass, and f and r similar quantities for either of the two equal lenses which compose the eye-piece.

18. Show that, if the focussing of the compound microscope were made by adjusting the eye-piece instead of the tube as a whole, the amount of adjustment would be increased in the ratio $(D/f-1)^2$ to 1, approximately, where D is the distance of the focal plane of the eye-piece from the objective, and f is the numerical focal length of the objective.

Show that if the outer surface of the objective, of radius of curvature r, be distant x from an object on which it is focussed in air, the magnification would be diminished

$$\{1-(\mu-1)x/r\}^{-1} \text{ times},$$

by dipping the face of the objective in liquid of index μ, in which the object has been immersed.

CHAPTER X.

Optical Instruments and Experiments.

169. If light be admitted into a darkened room through a very small aperture and be allowed to fall on a screen, an inverted picture of external objects will be formed upon the screen. A very narrow pencil of light proceeds from each point of the objects through the aperture and gives an image of the point on the screen. In consequence of the narrow limits of the aperture the image will of course be faint. If the opening be increased so as to admit more light, each pencil will be a cone of considerable breadth and will give a bright *patch* on the screen of the same shape as the hole, and therefore the image will become confused, and if the aperture be sufficiently enlarged the picture disappears altogether and becomes a bright patch. In order to produce a sharp image with a moderately large aperture a lens must be employed which must be arranged so as to give a real image at the distance of the screen. Objects outside having different distances from the lens will have images also at different distances. But if all the objects be at a distance large compared with the focal length, the images will all be very near to the principal focus of the lens.

This is the principle of the *camera obscura*. A box from which external light is excluded takes the place of the darkened room; instead of the screen is used a sensitive plate. An inverted image of external objects is thus printed on the plate and may be preserved in the form of a *photograph*.

The requisites for a good photographic lens are :—

(1) It must be achromatic, and indeed the focus for the chemical rays of the spectrum must be coincident with the focus for the brightest optical rays, so that when the lens is focussed so as to yield a sharp image to the eye, it will also yield a sharp image formed by the chemically active rays;

(2) It should admit the greatest possible quantity of light, and therefore should be made of transparent glass with as great an aperture as possible, and with not too many lenses. Lenses admitting a large quantity of light are called rapid lenses;

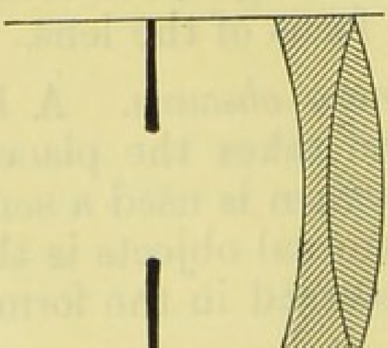
(3) Under certain circumstances it should bring to a focus without distortion objects lying within a wide angle (up to 110°);

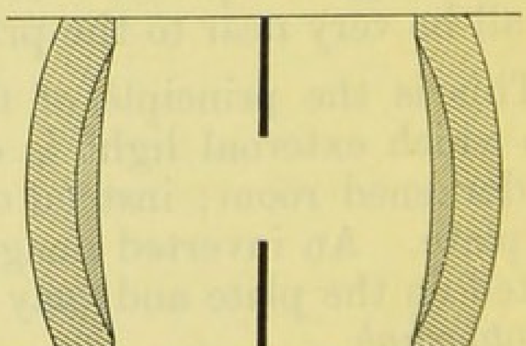
(4) Internal reflexions should be made harmless, as far as possible;

(5) For many purposes the lens should possess considerable *depth of focus*; that is, it should bring to a sharp focus objects lying at different distances from the lens.

170. The original form of photographic lens was a simple collective lens with a diaphragm in front. The radii of curvature and the position of the diaphragm were chosen so that the most important defects of the image might be removed. The usual form was a meniscus, with its concave surface turned towards the object, the diaphragm being in front of the lens. The lens could not be used with a greater aperture than about $f/30$.

The first great improvement was to use, instead of a single lens, a cemented doublet consisting of a double



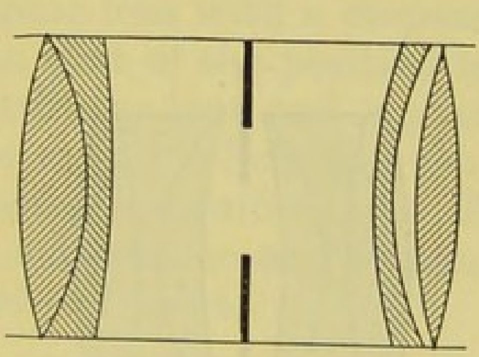


concave lens of flint glass joined to a double convex lens of crown; this combination can be made achromatic and free from spherical aberration near the axis. Such lenses are frequently used now for landscapes and can be made to work up to an aperture $f/15$ and a field of view of nearly 90°.

171. In 1866 A. Steinheil invented the symmetrical objective, consisting of two equal achromatised cemented doublets like those described above, placed in a symmetrical position with respect to the diaphragm. These lenses are very largely used now under the name of rapid rectilinear lenses.

Wide angled objectives of the same type are made by bringing the two doublets very near together.

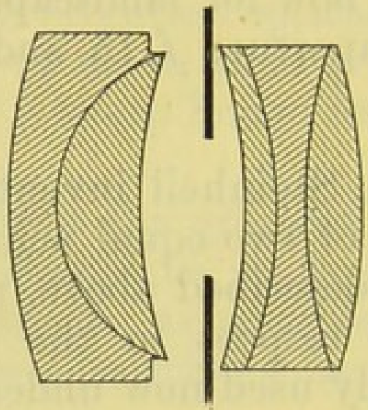
172. Of unsymmetrical objectives, one of the most famous is the Petzval portrait objective.



It consists of a cemented achromatic pair as the first element, with a back pair, separated by a small interval. The back pair may be moved nearer or further away from the front pair by a rack and screw-head. It is clear that for portrait lenses rapidity is a great advantage. At the same time very accurate definition is not necessary except in the centre of the picture; it is regarded as an advantage that there should be considerable diffusion or softness in the details of the background.

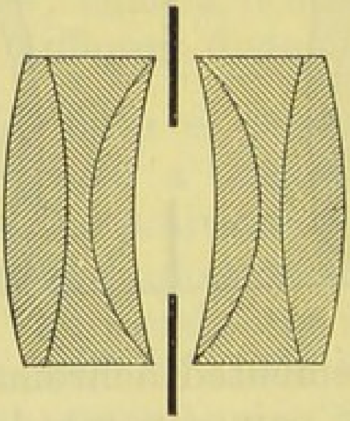
173. Two new lenses are here represented. The first is Dr Rudolf and Abbé's anastigmat, made by Zeiss of Jena. It consists of two elements, the first a cemented

doublet and the other a cemented triplet, each approximately achromatic in itself. The peculiarity of its construction is that in the first element the collective lens is



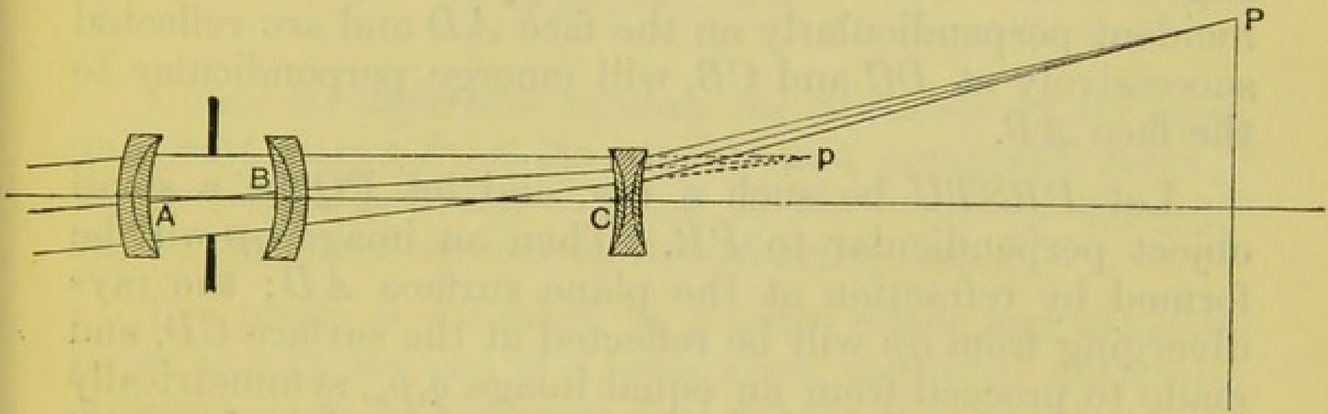
of smaller refractive index than the dispansive lens to which it is cemented, while in the back element the collective lenses are of greater refractive index than the dispansive lens.

The other new lens, known as the Goertz lens, consists of two symmetrical triplets. It is also claimed to be free from astigmatism.



174. With all lenses there is in addition to the image formed by refraction, several other fainter images formed partly by reflexion at one or other of the surfaces of the lenses. These subsidiary images are sometimes near the principal refractive images, and give rise to defects known as flare-spots. It is very important to cause these false images to fall at a considerable distance from the true image; then they will not seriously interfere with the true image or at most they will produce a general increase of illumination of the latter, which will only diminish the contrasts.

175. In order to photograph the details of a distant object with sufficient magnifying power to make the picture distinct, lenses of very long focal length are required. If these were made of the ordinary type, a camera with a very long extension and of considerable bulk would be necessary. To obviate this difficulty a so-called *telephotic* lens has recently been introduced.



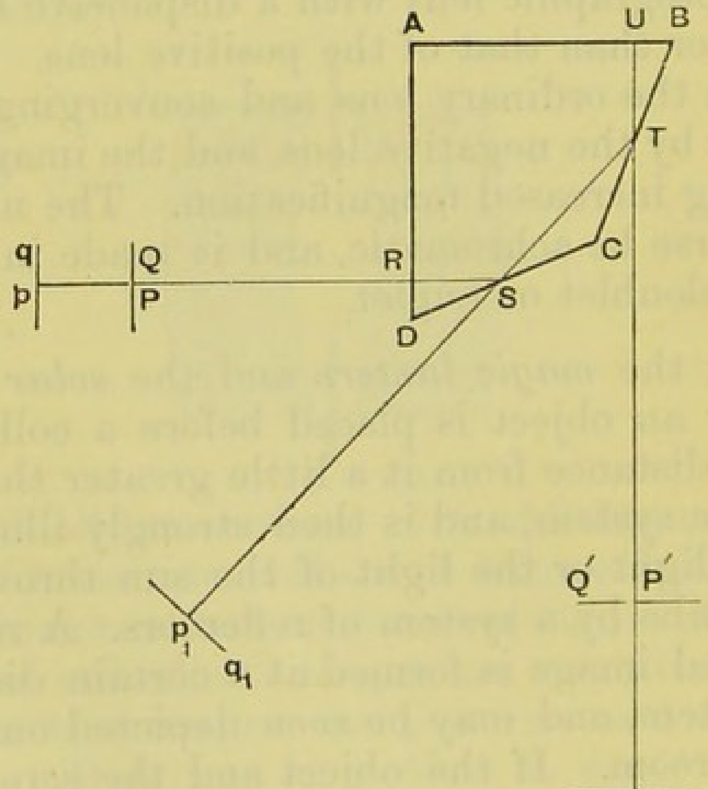
The necessary length of focus is obtained by uniting an ordinary photographic lens with a dispansive lens of focal length shorter than that of the positive lens. The pencils issuing from the ordinary lens and converging to a focus are received by the negative lens, and the image is thrown back, causing increased magnification. The negative lens must of course be achromatic, and is made in the form of a cemented doublet or triplet.

176. In the *magic lantern* and the *solar microscope*, a picture or an object is placed before a collective lens-system at a distance from it a little greater than the focal length of the system, and is then strongly illuminated by an artificial light or the light of the sun thrown into the axis of the tube by a system of reflectors. A real inverted and magnified image is formed at a certain distance from the lens-system, and may be seen depicted on a screen in a darkened room. If the object and the screen be fixed, the adjustment may be effected by moving the lens-system backwards or forwards in a sliding tube by means of a screw. The adjustment will always be possible, provided the distance of the screen from the object be greater than four times the focal length of the lens-system. [Cf. § 184.]

177. The *camera lucida*, invented by Wollaston, is an instrument of great use to the draughtsman, in preparing an accurate drawing of a building or a landscape.

Its essential feature is a quadrilateral prism of glass, represented in the adjoining figure. The angle A is a right angle, and the opposite angle is 135°, while the remaining angles B and D are equal; it follows that the angles B and D are each $67\frac{1}{2}$°. Rays of light which are incident perpendicularly on the face AD and are reflected successively at DC and CB, will emerge perpendicular to the face AB.

Let $PRSTU$ be such a ray, and let PQ be a small object perpendicular to PR. Then an image qp will be formed by refraction at the plane surface AD; the rays diverging from qp will be reflected at the surface CD, and made to proceed from an equal image q_1p_1, symmetrically placed on the other side of CD; the rays diverging from



q_1p_1 will be again reflected at the surface CB and another image $q'p'$ will be formed. Finally when the rays proceeding from $q'p'$ are refracted again into the air, they will proceed from an image $Q'P'$. Let PR the distance of the object from the first surface be denoted by x, and UP' the distance of the final image from the final surface AB by x', and let u, v, w, be the lengths of the three portions of the path within the prism. Then $pR = \mu x$. Also it is easy to see that $Up' = \mu x + u + v + w$, and therefore

$$x' = x + \frac{u + v + w}{\mu}.$$

Hence *the difference between the distances of the object and final image from the vertical and horizontal sides of the prism, respectively, is equal to the length of the path within the prism divided by its refractive index.*

The prism is mounted in a brass frame and attached by its axis to the end of a brass stem, the lower extremity of which may be clamped to a table; the length of the stem may be varied at pleasure by means of a sliding tube. The upper surface of the prism AB is furnished with an eye-stop of small aperture, which is adjusted so that the aperture is as nearly as possible bisected by the edge B; by this means only a small part of the surface AB is used, and the rest is covered. When the vertical face of the prism is turned towards the object, the observer looks downwards through the aperture and sees at the same time the image of the object through the uncovered portion of the prism, and the paper on which it is thrown through the remaining portion of the aperture. The image will be erect, because the rays from the upper part of the object proceed towards the upper part of the image.

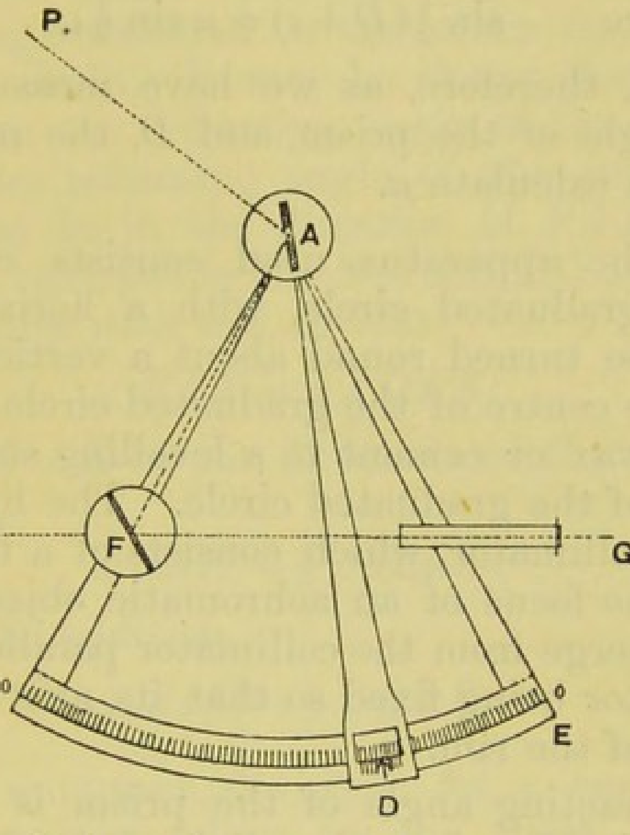
Since the dimensions of the prism are very small in comparison with the distance of the object, the distances of the object and image will be nearly equal. If the distance of the object from the prism be very different from the distance of the latter from the table, the image and the paper cannot be seen together distinctly. This may be remedied by a convex lens whose focal length is equal to the greatest distance of the prism from the table.

The lens is turned up horizontally under the prism, and the paper being in the principal focus, its image is thrown to an infinite distance and therefore made to coincide with the image of a remote object formed by the prism. The same correction may be made by placing a concave lens of the same focal length vertically in front of the vertical face of the prism. The rays proceeding from a distant object are made to diverge from an image whose distance is equal to the focal length; this image will therefore coincide with the paper after passing through the prism. The convex lens is to be used by normally sighted persons, the concave by short-sighted persons.

For near objects the adjustment of the distances is completed by varying the distance of the prism from the paper.

178. *Hadley's Sextant* is an instrument for measuring the angular distance between two distant points. It consists of a framework in the form of a sector of a circle, with a graduated arc, and two plane mirrors, whose planes are perpendicular to the plane of the sector. One of the mirrors A is moveable about an axis through the centre of the arc, and carries a pointer whose vernier slides along the graduated arc. The other mirror is fixed at F, and is parallel to the mirror A when the pointer of the latter is at E, the zero of the graduated scale; the lower part of this mirror only is silvered, so that rays of light may be transmitted directly through the upper part. The instrument is fitted with a small telescope G whose axis is directed towards the dividing line of the mirror F.

To measure the angular distance between any two points P, Q, the instrument is brought into the same plane with them and the telescope G is directed towards one of them, Q, which can be seen directly through the unsilvered part of the mirror F. The mirror A is then moved so that P, as seen through the telescope by a pencil reflected in succession at the mirrors A and F, appears to coincide with Q. In this arrangement the angular distance between the points P and Q is the deviation of the axis of the pencil by the two reflexions; and this is equal to twice the inclination of the mirrors.



The inclination of the mirrors may be read off the graduated scale. If the arc be graduated so that every half-degree may be read as a degree, the reading will give the angular distance between the two points without any further calculation.

Determination of Refractive Indices.

179. The general method of measuring the refractive index of a solid medium for any particular coloured ray of light, is to observe the minimum deviation of a ray of light of this colour, as it passes through a prism made out of the substance. It has been already seen that, when a ray of light passes through a prism with minimum deviation, its path is symmetrical with respect to the prism; so that with the usual notation

$$\phi = \psi, \ \phi' = \psi',$$

and therefore

$$\left. \begin{aligned} D + \iota &= 2\phi \\ \iota &= 2\phi' \end{aligned} \right\} .$$

If μ be the refractive index of the medium,

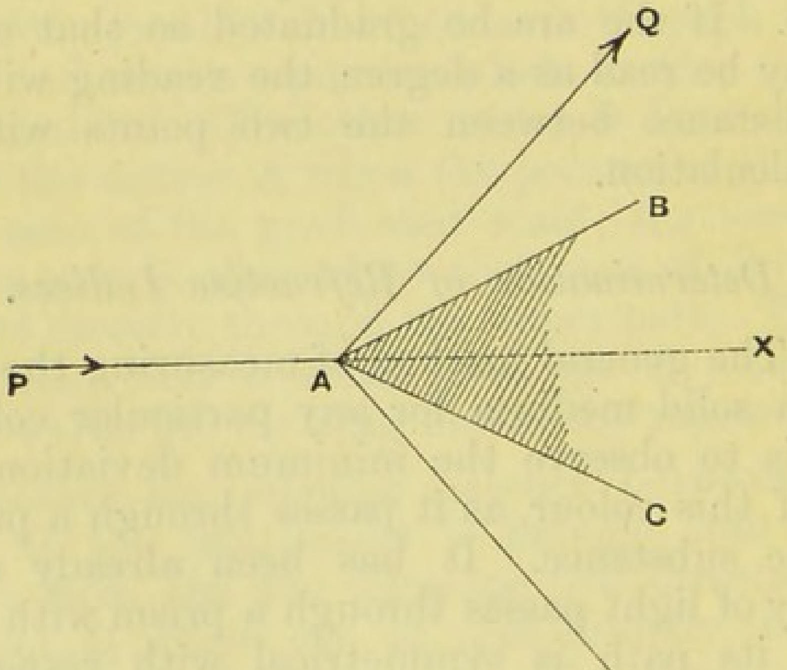
$$\sin\phi = \mu \sin\phi',$$

and therefore $\sin\frac{1}{2}(D+\iota) = \mu \sin\frac{1}{2}\iota.$

As soon, therefore, as we have measured ι, the refracting angle of the prism, and D, the minimum deviation, we can calculate μ.

180. The apparatus used consists essentially of a horizontal graduated circle, with a horizontal telescope which can be turned round about a vertical axis passing through the centre of the graduated circle. The prism is fixed with wax or cement to a levelling stand placed over the centre of the graduated circle. The light is supplied through a collimator, which consists of a fine vertical slit placed in the focus of an achromatic object-glass, so that the rays emerge from the collimator parallel to each other, the collimator being fixed so that its axis passes through the centre of the rim.

The refracting angle of the prism is first measured. The prism is placed so that light from the collimator is



reflected at both faces of the prism. The image of the slit as reflected at each face in succession is viewed by means of the telescope, the telescope being moved round till the image falls on the cross-wires of the telescope. The angle through which the telescope must be turned from seeing the image reflected in one face, in order to see the image reflected in the other face, is read off the graduated circle. It may be shown that this angle is equal to twice the refracting angle of the prism. For let BAC be the refracting angle of the prism, and let the incident ray be in the direction of PAX. Then, if AQ be the ray reflected in the face AB, AQ and AX must make equal angles with AB, so that

$$\angle BAX = \tfrac{1}{2} \angle QAX.$$

Similarly, if AR be the direction of the ray reflected in the face AC,

$$\angle CAX = \tfrac{1}{2} \angle RAX;$$

and therefore, by addition,

$$\angle BAC = \tfrac{1}{2} \angle QAR.$$

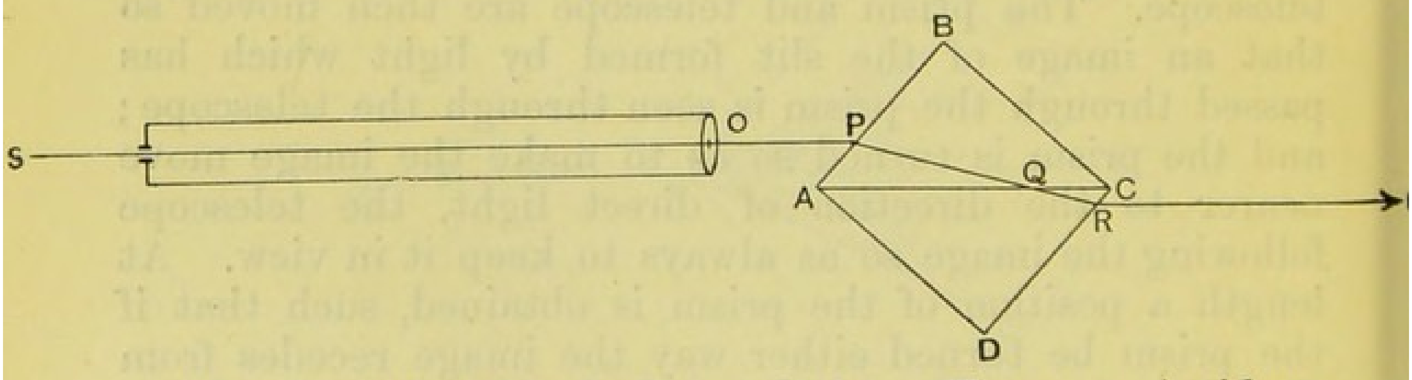
181. The minimum deviation for a ray of definite refrangibility, corresponding to any fixed line of the spectrum, is next measured. The slit is first viewed directly, the prism being turned so as not to obstruct all the light, and the telescope is moved until the line of the spectrum coincides with the cross-wires of the telescope. The prism and telescope are then moved so that an image of the slit formed by light which has passed through the prism is seen through the telescope; and the prism is turned so as to make the image move nearer to the direction of direct light, the telescope following the image so as always to keep it in view. At length a position of the prism is obtained, such that if the prism be turned either way the image recedes from the direction of the direct light; this position of the prism is therefore the position of minimum deviation. The telescope is moved until the line of the spectrum coincides again with the cross-wires of the telescope. The angle through which the telescope has been turned from the position of direct light is read off the graduated circle, and this angle is the minimum deviation required.

182. To measure the refractive index of a liquid, it is enclosed in a hollow prism of glass, made by cementing plates of glass together. The two sides of the plates however are never accurately parallel, and from the observed deviation it is necessary to subtract the small deviation caused by the empty prism.

The refractive indices of gases in given conditions as to temperature and pressure have been measured by a similar process. They must be enclosed in a tube, the ends of which are closed by two plates of glass placed very obliquely with reference to the axis of the tube.

The experiments of Biot and Arago on the refractive indices of gases showed that for gases the quantity $\mu^2 - 1$ is proportional to the density of the gas, a law which had been enunciated by Newton, who deduced it from his theory of emission.

183. A more modern instrument for measuring the refractive index of fluids is the Refractometer of Professor Abbé. This consists essentially of two equal rectangular prisms of equal angles made out of the same quality of highly refracting flint glass. The diagonal faces, which are placed in contact, leave space for a thin plane layer of the fluid whose refractive index is required. O is an objective in whose principal focus is



an illuminated slit, admitting light. The rays, incident on the prism, pass through it and emerge in a direction parallel to that of incidence, so long as the angle of incidence on the fluid layer is less than the critical angle. The compound prism is turned about so as to increase the angle of incidence on the fluid; as soon as the critical angle is reached, the light is totally reflected at the fluid and no longer emerges in the original direction, and to an eye near C, the light disappears. If we then measure the angle of incidence of the light on the first face of the prism, we have the means of expressing the refractive index of the fluid in terms of that of the prism.

Let θ be the angle of incidence, as measured, θ' the angle of emergence into the prism, α the angle CAB, γ the critical angle of the fluid, we have from the triangle PAQ;

$$\tfrac{1}{2}\pi + \gamma = \alpha + \tfrac{1}{2}\pi + \theta',$$

or

$$\gamma = \alpha + \theta'.$$

Also, if μ be the refractive index of the glass and μ' that of the fluid,

$$\sin\theta = \mu \sin\theta',$$

$$\sin\gamma = \frac{\mu'}{\mu}.$$

These equations are sufficient to determine μ'/μ.

The same apparatus may be used to measure the dispersive power of the fluid.

184. *To find the focal length of a thin convex lens.*

This is usually measured by adjusting the lens and an object, until the distance between the object and the image is a minimum; this distance is then four times the focal length. For, if u, v be the distances of the object and image in front of, and behind, the lens, respectively,

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f},$$

while the distance between the object and the image is given by the equation

$$u + v = x.$$

Combining these equations, we get

$$uv = xf,$$

and therefore

$$(u - v)^2 = x^2 - 4xf.$$

The quantity $(u - v)^2$ is always positive, and therefore the least value of x is equal to $4f$.

If the lens be concave, it is placed in contact with a convex lens, so that the whole combination may be collective; the focal length of the combination may be determined as before. If f, f' be the numerical focal lengths of the concave and convex lenses, respectively, F that of the combination,

$$\frac{1}{F} = \frac{1}{f'} - \frac{1}{f},$$

which determines f when f' is known.

Methods of Photometry.

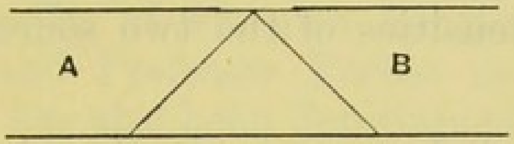
185. It has been shown in § 5 that when an element of a surface is illuminated by light proceeding from a source of intensity I, at a distance r, so that the axis of the pencil makes an angle θ with the normal to the element of surface, then the intensity of illumination is proportional to

$$\frac{I \cos \theta}{r^2}.$$

It is found that the eye is of itself unable to estimate the ratio of the intensities of two sources of light, but that it is an accurate judge of the equality of illumination of two illuminated surfaces when they are placed side by side. All methods of photometry depend therefore on the equalising of two illuminations.

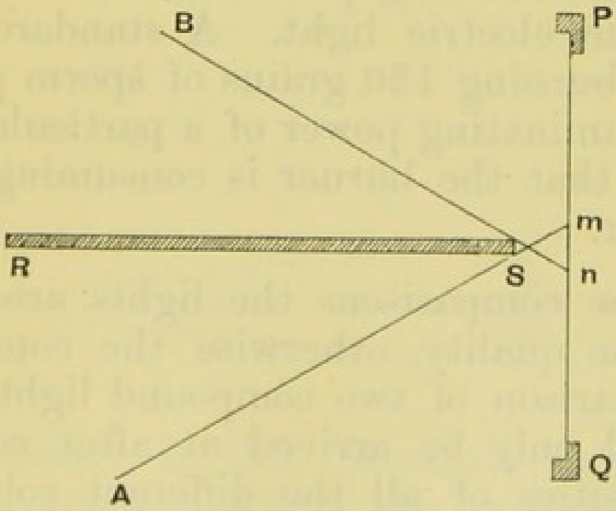
In order to compare the intensities of two sources of light, the two halves of a piece of thin porcelain are illuminated by the two sources, respectively, in such a way that either the light falls normally on the porcelain, or the lights from the two sources make equal angles with the plane of the porcelain. The distances of the lights are then adjusted so that the two halves of the porcelain are equally illuminated. Then the intensities of the sources are in the inverse proportion of the squares of their distances from the porcelain. This is the principle both of Ritchie's and of Foucault's photometers.

186. Ritchie's photometer consists of a rectangular box open at both ends. In the lid is a narrow strip of porcelain or oiled paper. The instrument is placed between the two sources to be compared, and the light is reflected up to the porcelain by two pieces of mirror (which must be cut from the same piece of glass) placed



at angles of 45° to the axis of the box. The box is then moved from one source towards the other until the two halves of the porcelain are equally illuminated, and the distances of the lights measured.

187. In Foucault's photometer the lights which are to be compared act separately on two different parts of the same vertical plate of thin transparent porcelain, PQ. RS is an opaque vertical screen which separates the two illuminations from one another. If this screen be so adjusted that the vertical planes ASm, BSn which limit the regions illuminated separately by the two sources A, B, intersect just in front of the lamina PQ, the dark band mn can be made as narrow as we please. The



distances of A and B are then adjusted so that the two portions of the lamina are equally illuminated.

188. In Rumford's photometer the intensities of the two shadows on a screen of a vertical rod due to the two lights are compared. The lights are arranged so that the shadows fall close together, and the shadow formed by one light is lighted by the light from the other source. The distances being so adjusted that the shadows are of equal intensity, the distances of the lights are measured, and thus the intensities of the two sources can be compared.

Bunsen invented a very simple photometer. If a spot of grease be made on a sheet of paper, then if the paper be equally illuminated on its two sides, the transparent spot cannot be seen except by close inspection. The sources of light are placed on opposite sides of the paper and their distances are so adjusted that the grease spot disappears; then the intensities of the sources are inversely as the squares of their distances from the paper. The adjustment should first be made from the side on which one source lies, then the screen should be turned round and the adjustment made from the side on which lies the other source, the same side of the paper being observed each time. The mean of these two positions will give a fairly accurate result.

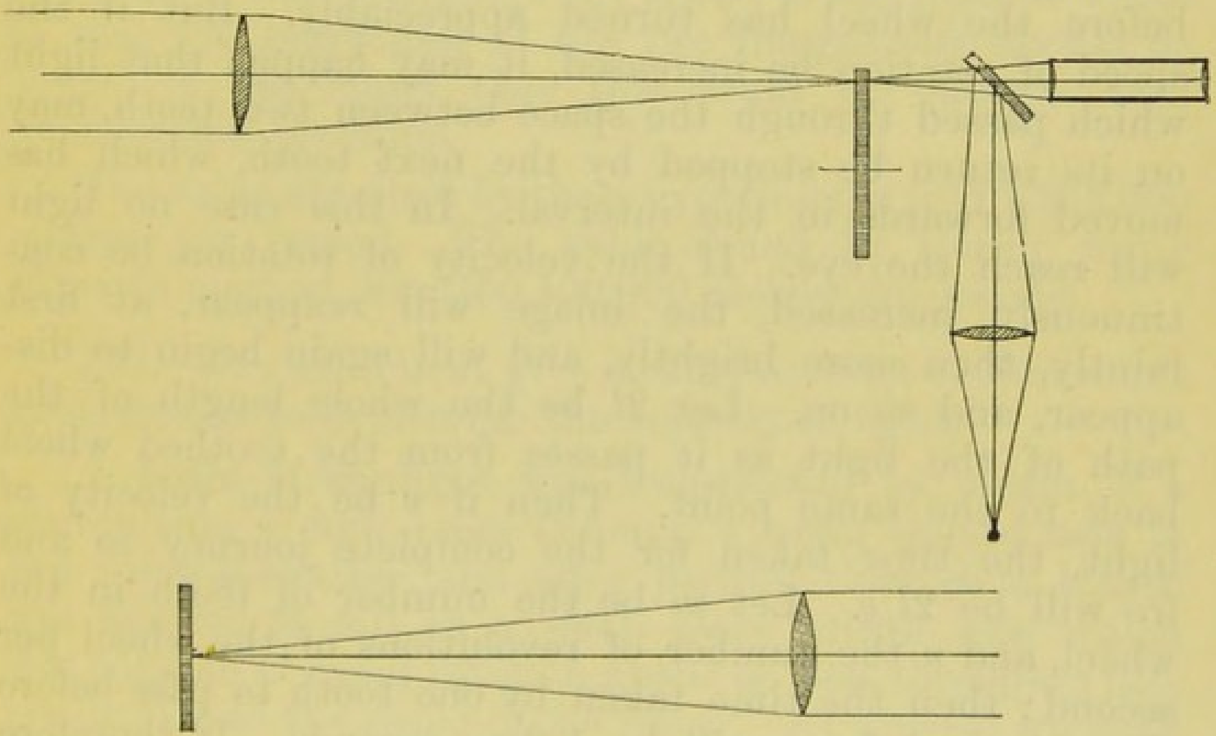
This is the photometer most usually employed to compare the illuminating power of different lights, such as gas and the electric light. A standard candle is a sperm candle burning 120 grains of sperm per hour. In stating the illuminating power of a particular gas-burner, it is supposed that the burner is consuming 5 cubic feet of gas per hour.

In all these comparisons the lights are supposed to be of the same quality, otherwise the comparison fails. A strict comparison of two compound lights of different qualities could only be arrived at after comparing the relative intensities of all the different coloured rays of the spectra given by the two lights, and tabulating the results.

Methods of determining the Velocity of Light.

189. There are two methods of determining the velocity of light by optical experiments, the one devised by Fizeau and the other by Foucault. Fizeau's experiments were repeated in 1876 by M. Cornu, and later a modification of Fizeau's method has been used by Dr Young and Professor Forbes in Scotland. The velocity of light has also been determined by A. A. Michelson, of the United States navy, who followed Foucault's method.

190. In Fizeau's experiments two astronomical telescopes several miles apart are arranged so that their axes are accurately parallel, the one telescope looking into the other. In one of the telescopes a mirror is placed at the focus of the object-glass, exactly perpendicular to the axis of the instrument. The observer stands at



the other telescope; in this instrument a plate of glass, inclined at an angle of 45° to the axis of the telescope, is placed between the eye-piece and the principal focus of the object-glass. Light is admitted through the side of the instrument and reflected down the tube by the plate-glass, the rays coming to a focus at the principal focus of the object-glass, so that they may emerge from the instrument in a direction parallel to its axis. These rays of light enter the object-glass of the distant telescope, are reflected back in the same direction by its mirror, and some of these rays after passing the object-glass will pass through the inclined plate of glass and enter the eye-piece and will be received by the eye in the usual manner. A wheel with a large number of fine teeth is rotated, so that the teeth pass in front of the focus of the object-glass. We shall suppose that the breadth of the teeth is equal to the interval between two consecutive teeth. When the wheel rotates comparatively slowly, but quickly enough for the intermittent light to make a continuous impression on the eye, the eye will see an image of the light; for the time taken to travel to the distant telescope and back again is so small that light which passes through the space between two teeth at starting will have time to return through the same space before the wheel has turned appreciably. But if the speed of rotation be increased, it may happen that light which passed through the space between two teeth, may on its return be stopped by the next tooth, which has moved forwards in the interval. In this case no light will reach the eye. If the velocity of rotation be continuously increased, the image will reappear, at first faintly, then more brightly, and will again begin to disappear, and so on. Let $2l$ be the whole length of the path of the light as it passes from the toothed wheel back to the same point. Then if v be the velocity of light, the time taken for the complete journey to and fro will be $2l/v$. Let m be the number of teeth in the wheel, and n the number of revolutions of the wheel per second; then the time taken by one tooth to pass before the principal focus will be $1/2mn$ seconds. If therefore the number of revolutions per second be such as to produce the first eclipse,

$$\frac{2l}{v} = \frac{1}{2mn},$$

or

$$v = 4mnl;$$

and if n be such as to cause the p^{th} eclipse, it may easily be seen that

$$v = \frac{4mnl}{2p-1}.$$

The distance l and the number of revolutions per second are observed, and then v is determined by these formulæ.

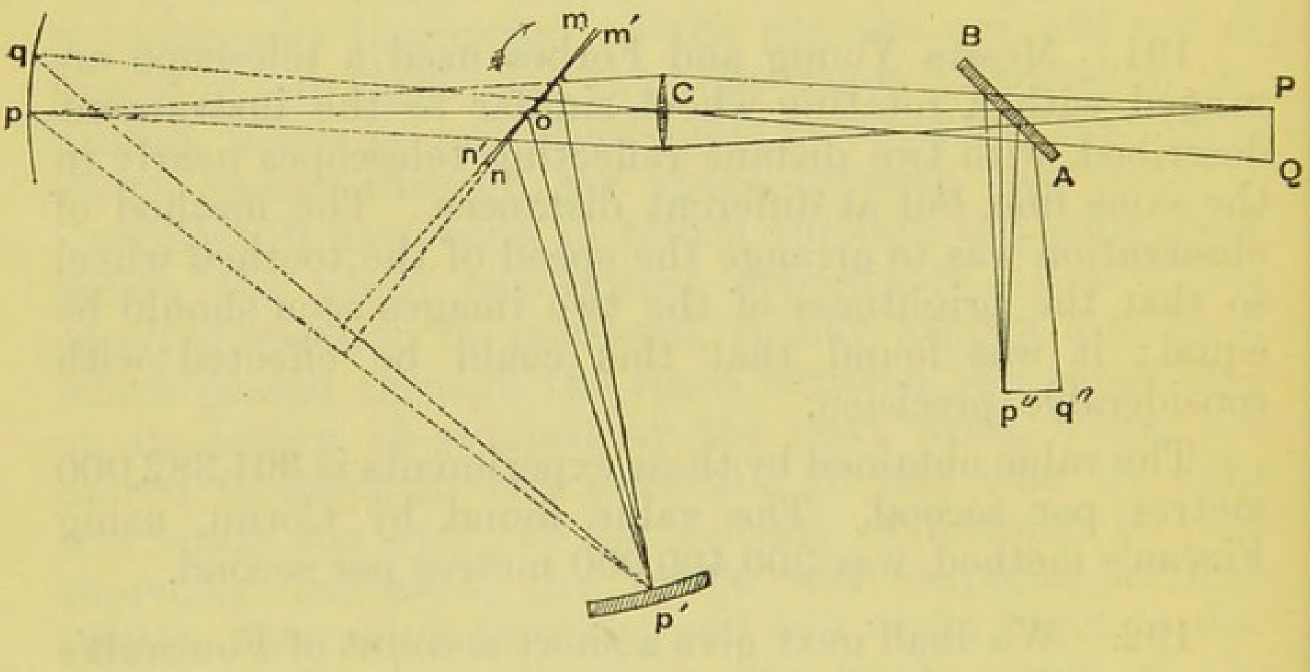
The imperfection of this method is that in actual experiments a total eclipse of the reflected rays is hardly ever reached; there is usually only a very great falling off in their intensity, and the exact moment which must be taken to represent the moment of eclipse cannot be determined with very great precision.

191. Messrs Young and Forbes used a telescope arranged with a rotating wheel, similar to the instrument described, with two distant reflecting telescopes nearly in the same line, but at different distances. The method of observation was to arrange the speed of the toothed wheel so that the brightness of the two images seen should be equal; it was found that this could be effected with considerable precision.

The value obtained by these experiments is 301,382,000 metres per second. The value found by Cornu, using Fizeau's method, was 300,400,000 metres per second.

192. We shall next give a short account of Foucault's experiments to determine the velocity of light.

A beam of sunlight was transmitted by means of a mirror into a dark room through a small square hole in the window-shutter, and after passing through a lens C was allowed to fall on a small plane mirror mon which was capable of rapid rotation about an axis through o perpendicular to the plane of the paper. At present we shall confine ourselves to the consideration of the path of a small pencil of the incident light which diverges from a point P of the aperture. This pencil, after passing through the lens, is made to converge to the point p; but before the rays reach p they are intercepted by the plane mirror *mon* and are reflected to the point p', where $op = op'$. At p' is placed a portion of a spherical mirror whose centre is o and radius op, which reflects the pencil back again in the same direction, and if the small plane mirror be at rest the pencil will retrace its original course back to P. Between P and the lens C is placed a sheet of plate glass, inclined at an angle of 45° to the axis PC; and part of the returning pencil is reflected at this piece of glass and is brought to a focus at p'', where it is viewed through a telescope. When the mirror *mon* is made to revolve slowly, the light will be returned only when the mirror *mon* is in a position to send light to the small mirror at p', and therefore the image p'' will be intermittent; but if the



velocity of rotation be increased up to about 30 revolutions per second, the impression produced in an observer's eye is continuous. So long as the mirror revolves with moderate velocity, the time taken by the light to travel from o to p' and back again is so short that the returning pencil reaches the mirror *mon* before it has appreciably changed its position; but if the velocity of rotation be greatly increased, until the mirror makes several hundred rotations per second, the mirror will have turned through a small angle during the time occupied by the reflected light in passing from o to p' and back again. The pencil returning from p' will be reflected by the mirror in its new position, and after reflexion will appear to diverge from a point q, where $oq = op'$, and after passing through the lens will be made to converge to a point Q on the line qC; the image by reflexion in the plate glass will therefore be at q'' instead of p'', where $p''q'' = PQ$.

Across the aperture through which the light was admitted was stretched a fine wire, whose position is represented by P, and the displacement of the image of this wire $p''q''$ can be measured by the aid of the observing telescope. Let the value of this displacement be δ.

193. Let n be the number of revolutions of the mirror per second; this can be determined by means of a siren. Also let $CP = a$, $Co = b$, and let $op' = r$. Then if v be the velocity of light, the time occupied by the light in passing from o to p' and back again will be

$$t = \frac{2r}{v}.$$

During this time the revolving mirror will have rotated through an angle $2\pi nt$ or $4\pi nr/v$.

The points p, q, p' lie on the circle whose centre is o; also the lines pp', qp' are respectively perpendicular to the two positions of the mirror, and therefore the angle $pp'q$ is equal to the angle between the two positions of the mirror, or to $4\pi nr/v$. It therefore follows that the arc pq subtends at the centre of the circle an angle $8\pi nr/v$; and therefore

$$pq = \frac{8\pi nr^2}{v}.$$

Also, by similar triangles, $PQ : pq = a : (b + r)$,

and therefore $$PQ = \frac{8\pi nr^2 a}{v\,(b + r)}.$$

This length PQ, being equal to $p''q''$, has been determined by observation to be δ, and therefore we get

$$v = \frac{8\pi nr^2 a}{\delta\,(b + r)},$$

an equation which expresses v in terms of quantities which can be measured.

Foucault found the velocity of light by this method to be 298,000,000 metres per second. The value obtained by Michelson by a slight modification of the same method was 299,940,000 metres per second.

The method employed by Foucault may be applied to the determination of the velocity of light in other transparent media, such as water. For this purpose a tube filled with the water, with its ends closed by plate glass, is placed between the revolving mirror and the small spherical mirror, so that part of the double journey is performed through water instead of air. It is found that light travels slower in water than in air.

CHAPTER XI.

THE RAINBOW.

194. THE first satisfactory explanation of the rainbow was given by Antonius de Dominis, archbishop of Spalatro, in a work *De Radiis Visus et Lucis*, published in 1611. He shows that the inner bow is formed by two refractions and one intermediate reflexion of the sun's light in drops of rain; and the outer bow by two refractions and two intermediate reflexions. This explanation was adopted by Descartes and was confirmed by experiments made with glass globes filled with water, and arranged so as to exhibit the colours of the two bows. It remained for Newton to add to the theory an explanation of the colours. The complete theory involves considerations which belong to Physical Optics and was developed by Sir G. Airy; we must confine ourselves to the approximate theory.

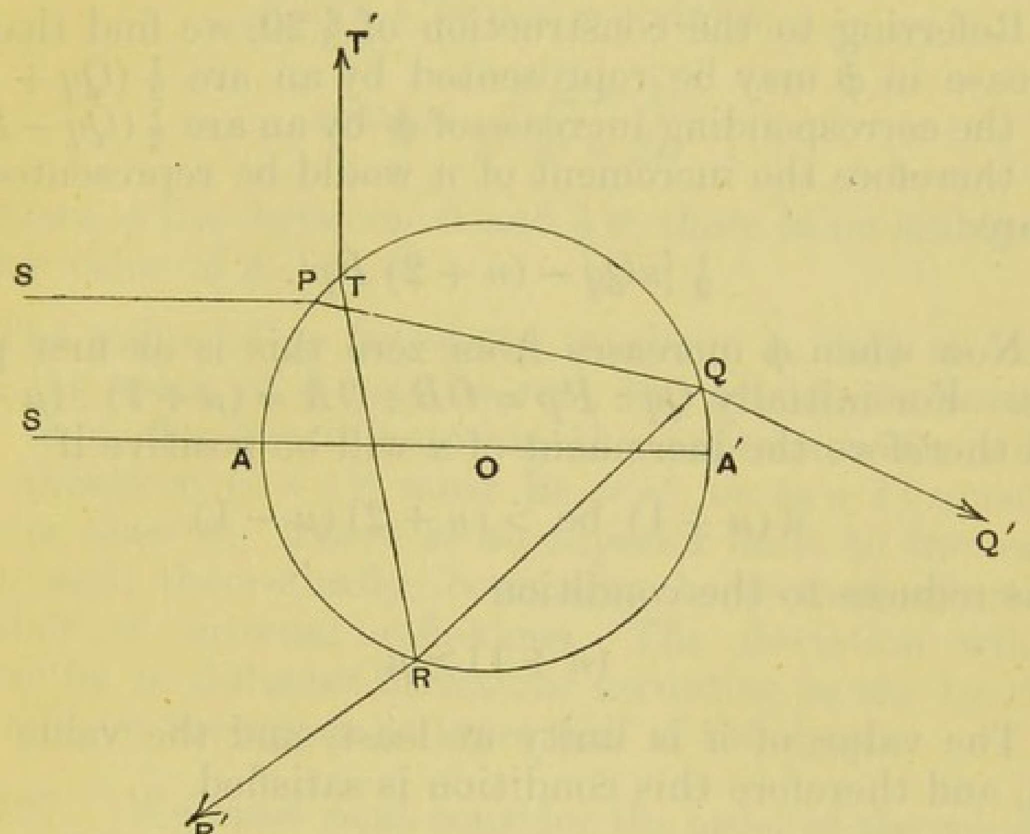
195. When the parallel rays of the sun strike a drop of water, part of the light will be scattered at the outer surface of the drop and serve to render the drop visible, and part will enter the drop by refraction; of those rays which enter the drop part will be refracted out of the drop at the incidence on the second surface of the drop, and part will be reflected back into the drop, and so on, for any number of incidences. Let us consider the rays which are incident in a plane of symmetry and which pass out of the drop by refraction after one internal reflexion; it is clear that they will not all emerge in the

same direction, for the deviation will depend on the angle of incidence. Moreover, if the angle of incidence increase uniformly the deviation will vary sometimes rapidly, sometimes more slowly; and the more slowly the deviation changes the less will be the divergence of the emergent rays. If therefore the emergent rays be received on a screen, the band will not be uniformly bright, but will be brightest in those parts where the divergence is least, that is, where the deviation changes most slowly. Now the changes of the deviation are slowest near a maximum or a minimum, and therefore at the spot where the deviation is a minimum the band will be much brighter than anywhere else. Within the direction of minimum deviation there will be no light transmitted.

If instead of a single drop, a shower of drops be illuminated by the rays of the sun, those drops whose positions are such that the rays emerge in the direction of the eye with minimum deviation will appear more brilliant than the others, and will be marked out against the cloud as specially bright. This phenomenon is the same in all planes which pass through the line joining the sun and the observer's eye, and therefore the assemblage of bright drops will form an arc of a circle whose centre is on this line, and whose angular radius as seen by the eye only depends on the refractive index of the light. The refractive index is not the same for all the rays of a solar beam, being greatest for the violet and least for the red rays, and therefore the position of the bright arc will not be the same for all the coloured rays of the solar beam. There will therefore be a series of coloured bands exhibiting the colours of the solar spectrum. This is the principle of the explanation of the rainbow.

196. Let SP be a ray of light incident on the drop of water at P, PQ the ray refracted into the drop; part of the light will pass out by refraction at Q along the line QQ', while another part will be reflected at Q along the line QR, where part will pass out by refraction and part

be reflected, and so on. Let ϕ be the angle of incidence at P, ϕ' the angle of refraction, so that $\sin\phi = \mu \sin\phi'$.



The deviation at P is therefore $\phi - \phi'$. When the ray is incident at Q, the angle of incidence is ϕ'; and therefore for the part which passes out at Q a second deviation equal to $\phi - \phi'$ in the same direction as before is produced. But for the part reflected at Q, the deviation is $\pi - 2\phi'$, and where the ray meets the surface again at R the angle of incidence is again ϕ'. If therefore the ray undergoes n internal reflexions and then passes out by refraction, the whole deviation will be

$$D = 2(\phi - \phi') + n(\pi - 2\phi').$$

The most efficacious rays, as we have seen, are those which make the deviation a maximum or minimum. To find the angle of incidence for these rays, we may use the method of Prof. P. G. Tait already mentioned in § 20. The value of the deviation may be written

$$D = n\pi - 2\{(n+1)\phi' - \phi\},$$

and therefore if

$$u = (n+1)\phi' - \phi,$$

we require to find the value of ϕ which will make u a maximum.

Referring to the construction of § 20, we find that an increase in ϕ may be represented by an arc $\frac{1}{2}(Qq + Pp)$, and the corresponding increase of ϕ' by an arc $\frac{1}{2}(Qq - Pp)$; and therefore the increment of u would be represented by an arc

$$\tfrac{1}{2}\{nQq - (n+2)\,Pp\}.$$

Now when ϕ increases from zero this is at first positive. For initially $Qq : Pp = OB : OA = (\mu+1) : (\mu-1)$, and therefore the increment of u will be positive if

$$n(\mu+1) \text{ be } > (n+2)(\mu-1).$$

This reduces to the condition

$$(n+1) > \mu.$$

The value of n is unity at least, and the value of μ is $\frac{4}{3}$, and therefore this condition is satisfied.

The increment of u will remain positive until a point is reached at which

$$nQq - (n+2)\,Pp = 0,$$

and after that will become negative. Thus the maximum value of u will be determined by the condition

$$nOQ = (n+2)\,OP.$$

This may be written

$$(n+1)(OQ - OP) = OQ + OP.$$

Thus if M be the middle point of PQ,

$$(n+1)\,PM = OM.$$

Now $\quad PM = CP\cos\phi, \quad OM = OC\cos\phi',$

and therefore

$$\mu\cos\phi' = (n+1)\cos\phi.$$

Besides this we have the equation of refraction,

$$\mu\sin\phi' = \sin\phi\,;$$

squaring and adding both members of this equation, we get

$$\mu^2 = (n+1)^2 \cos^2\phi + \sin^2\phi,$$

or

$$\cos\phi = \sqrt{\frac{\mu^2 - 1}{n^2 + 2n}}.$$

Since ϕ lies between 0 and $\frac{1}{2}\pi$, there is no ambiguity in this value of ϕ.

The value of μ for water is about $\frac{4}{3}$, and in order that the value of ϕ may be real, the numerator must be less than the denominator in the expression for $\cos\phi$; and therefore $(n+1)^2$ must be $> \mu^2$, or $(n+1)$ must be greater than $\frac{4}{3}$. There is no superior limit to the value of n, and, theoretically, bows may be formed after any number of internal reflexions. The deviation will of course be in different directions according as the incident ray falls on the upper or lower half of the drop.

197. We must next consider the order of the coloured rays by examining the changes in the direction of the most efficacious rays for different refractive indices.

This problem may be treated in the same way as before. If the refractive index be slightly increased, the point O recedes from the circle, since $OC = \mu CA$. If we consider two consecutive chords $O'PQ$, $O'pq$, instead of OPQ, Opq, the effect of the change from O to O' is to increase the arc Pp relatively to Qq; for if Qq be supposed to be unchanged, Pp will be increased. Thus the increment of u instead of being zero as before, will be negative, and the maximum value of u will be smaller than before. Hence the effect of an increase in μ is to increase the minimum deviation also; that is, *the minimum deviation is greatest for the violet rays and least for red rays.*

198. It has been shown that in order to produce a rainbow, at least one reflexion inside the drop is necessary. At each subsequent reflexion part of the light will be lost, and the corresponding rainbows will be fainter. The rainbow produced by one internal reflexion is called

the *primary rainbow.* The angle of incidence corresponding to the most efficacious rays is given by the formula,

$$\cos\phi = \sqrt{\frac{\mu^2 - 1}{3}},$$

and the deviation by the equation

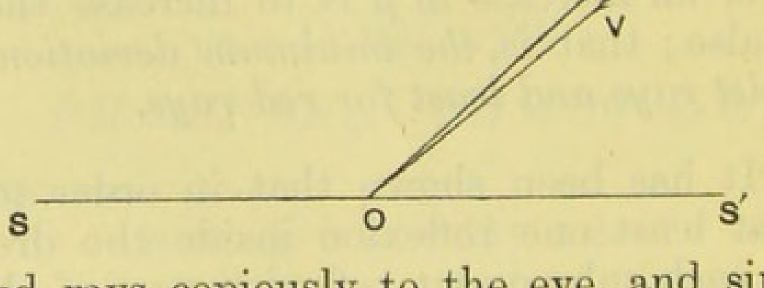
$$D = 2(\phi - \phi') + \pi - 2\phi'.$$

The refractive indices of water for red and violet rays, respectively, are $\frac{108}{81}$ and $\frac{109}{81}$. If these values be substituted for μ in the preceding formula, we find by the aid of trigonometrical tables the values of the deviations corresponding to these rays to be,

$$D_R = 137^\circ\, 58'\, 20'',$$
$$D_V = 139^\circ\, 43'\, 20''.$$

Let O be the eye of the spectator, and SOS' a line drawn in the direction of the sun's rays; then, if we make the angle $S'OR$ equal to the supplement of D_R, that is, equal to $42^\circ\, 1'\, 40''$, RO will be the direction in which the most efficacious red rays will enter the eye. Similarly, if an angle $S'OV$ be constructed equal to the supplement of D_V, that is, equal to $40^\circ\, 16'\, 40''$, VO will be the direction in which the most efficacious violet rays will enter the eye, and the intermediate coloured rays will enter in directions intermediate between RO and VO.

And, further, if the lines OR, OV revolve round the line OS' as an axis, it is clear that all the drops on the conical surface generated by the revolution of RO will



transmit red rays copiously to the eye, and similarly for the other colours. Thus to the eye there will appear a series of coloured arches with the violet rays innermost

The effect of the rays which strike the eye with greater deviation, will be to light up the cloud within the bow with faint light, while no light will reach the eye from drops lying outside the bow.

The separation of the colours is not perfect, but they overlap each other, so that some of the colours can scarcely be recognised. The reason of this, just as in Newton's experiment with the prism, is that the sun has an angular diameter of 33′, and as each point of the sun sends out rays we get a series of rainbows due to the different elements of the sun's surface all superimposed and confused together.

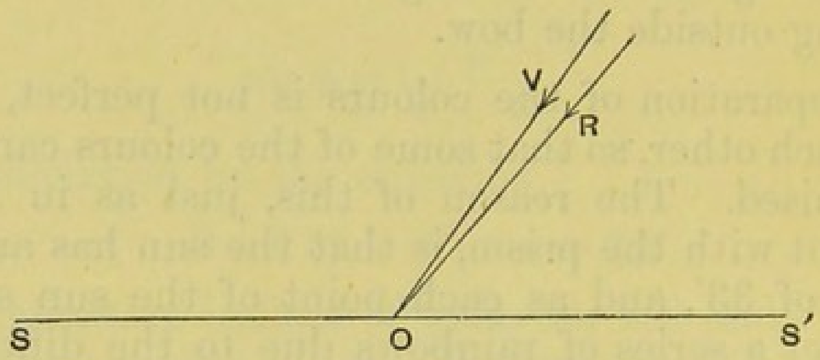
There is yet another set of rays which pass through the drop with minimum deviation, those which strike the drop on its lower side at the same angle of incidence as before. These are directed after refraction away from the earth, and are not seen by an observer on the earth; though they give bows which have sometimes been observed during balloon ascents, or on the summits of high mountains which lie above the clouds. When the sun is sufficiently near the horizon a complete circle may sometimes be seen in this manner.

199. When the rays undergo two internal reflexions they form a rainbow called the *secondary rainbow*. If we make $n = 2$, and substitute the same values of μ as before, we find

$$D_R = 230^\circ\ 58'\ 50'',$$
$$D_V = 234^\circ\ \ 9'\ 20''.$$

These deviations being greater than 180°, it is easy to see that the rays which reach the eye of an observer stationed on the earth are incident on the lower half of the drop.

Let SOS' be a line drawn through the observer's eye, in the direction of the sun's rays, and let angles $S'OR$, $S'OV$ be constructed, respectively equal to $D - 180^\circ$, $D_V - 180^\circ$, that is, to $50^\circ\ 58'\ 50''$, and $54^\circ\ 9'\ 20''$. Then RO, VO will be the directions of the most efficacious red and violet rays, respectively, and the phenomenon of the secondary rainbow may be deduced by revolving the



lines OR, OV about the line OS' as before. The order of the colours is inverted in this bow, the violet being outside and the red inside. The rays which reach the eye with greater deviation serve to light up the cloud outside the bow. The secondary bow will be less bright than the primary bow, for two reasons; first, the light has undergone two internal reflexions and has thereby been weakened, and secondly, there is a greater angular dispersion of the rays in this rainbow than in the primary bow.

200. These two rainbows are the only ones which are usually perceived, although the higher bows exist in theory. The third and fourth bows could never be seen except under special circumstances. For if we make $n = 3$, we find for red rays $D = 318° 24' = 360° - 41° 36'$. The direction of the rays will therefore pass behind the cloud, and to an observer stationed there it would be lost in the much brighter direct light from the sun.

If $n = 4$, $D = 360° + 44° 13'$. The case of four internal reflexions therefore differs little from the last; the efficacious rays will be incident on the upper half of the drop and will fall behind the cloud as before.

For the fifth arc, $D = 363° + 126°$, and the bow will have an angular radius of 54° and may be seen outside the secondary bow, especially in waterfalls where the drops are near the eye. The higher bows have never been seen except in laboratories under careful experimental conditions.

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