OPHTHALMIC LENSES

PRENTICE

A TREATISE

ON

SIMPLE AND COMPOUND

OPHTHALMIC LENSES

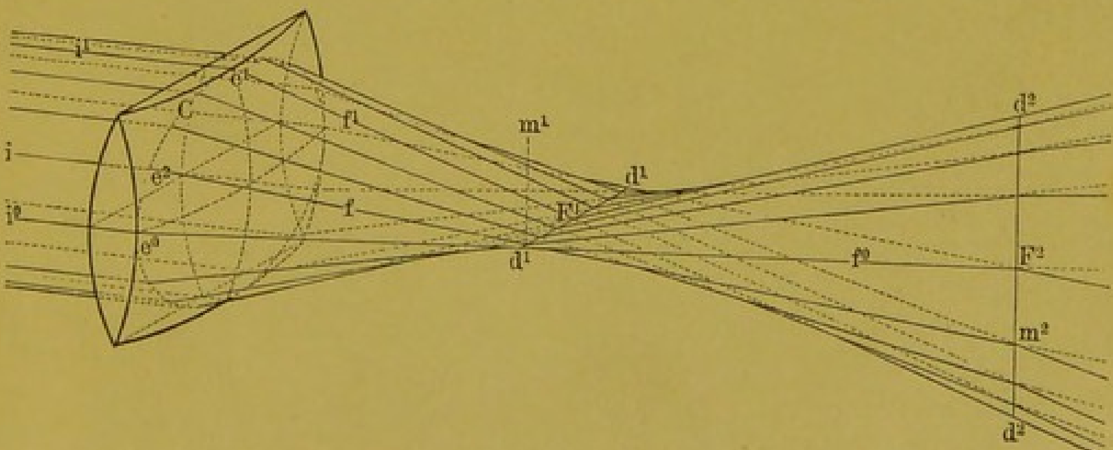
THEIR REFRACTION AND DIOPTRIC FORMULÆ

INCLUDING TABLES OF

CROSSED CYLINDERS AND THEIR SPHERO-CYLINDRICAL EQUIVALENTS

BY

CHAS. F. PRENTICE



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PREFACE.

In treating of Refraction by Simple and Compound Ophthalmic Lenses, I hope, more especially through graphical and analytical means, to guide the reader upon a path by which he may gain easy access to an understanding of the subject without recourse to mathematical dioptrics.

To better attain my purpose, I shall develop the principles involved in their primary or natural order of succession, and under the restriction of the supposition that the reader is a novice.

In the drawings, which are photo-engravings by the Moss Company from my own pen, I have intentionally exaggerated the curvatures to better contrast the characterizing differences in the lenses illustrated.

Should this limited treatise prove of interest, I trust the same will be weighed and accepted as an unpretentious effort to fill a vacancy strictly in the spirit of its issuance.

CHAS. F. PRENTICE.

New-York, Sept. 1, 1886.

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From advance proof of Harper's Weekly.

Judge Wallace, of the United States Circuit Court, has just rendered a decision upon an interesting and important point of the law of copyright. The case was that of Harper against Shoppell. The defendant had made an electrotype copy of an engraving that appeared in *Harper's Weekly*, the *Weekly* being copyrighted, and had sold his plate, which was afterward published in another periodical. Judge Wallace, after a full presentation of the facts, decides that, under the circumstances of the case, the defendant is liable in damages as a joint wrongdoer with the publisher of the periodical in which the reproduced engraving appeared.

The point involved was whether the copyright of a "book," as such, protects the engravings published in the same. This point has never come directly before the courts in this country, and Judge Wallace's decision establishes here the English rule, which, as stated by Judge Jessell, is that "a book must include every part of the book; it must include every print, design, or engraving which forms part of the book, as well as the letter-press therein, which forms another part of it."

The drawings, tables, and manuscript herein contained are therefore *individually* protected under this copyright.

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REFRACTION.

§ 1. The effect which transparent mediums produce on light projected obliquely to a surface in its passage from one medium to another of different density being termed Refraction, and as the proposed treatment of its manifestation by lenses is to be purely elementary, I may be pardoned for citing the primary law governing it, in which I shall confine myself strictly to parallel rays, in air, impinging upon and passing through transparent optical glass exclusively. In the accompanying diagram, Fig. 1, the medium glass is represented as being of very appreciable thickness, with parallel surfaces, intercepted by an oblique ray of light, i, in an isolated vertical section, a–b–c–d.

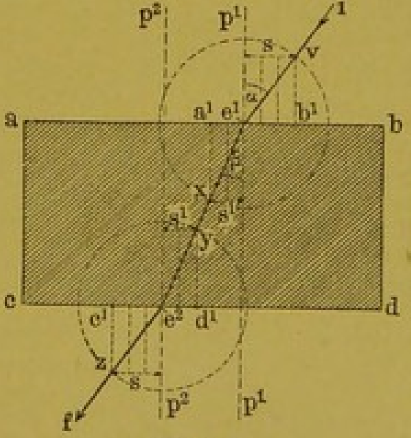

Fig. 1.

For convenience we shall term the ray prior to its contact with the glass, the incident ray, i; the ray during transit within the glass, the refracted ray, e^1 e^2;* and the refracted ray after exit, the final ray, f.

* The use of superior indices will not prove conflicting, as algebraic values are excluded.

§ 2. Refraction manifests itself by an acute bend in the direction of an oblique ray of light, i, at the point of entrance, e^1, in passing from one conducting medium to another, a–b–c–d, of different density. Hence, a ray passing from one into and through another medium is bent both at the point of entrance e^1 and of exit e^2.

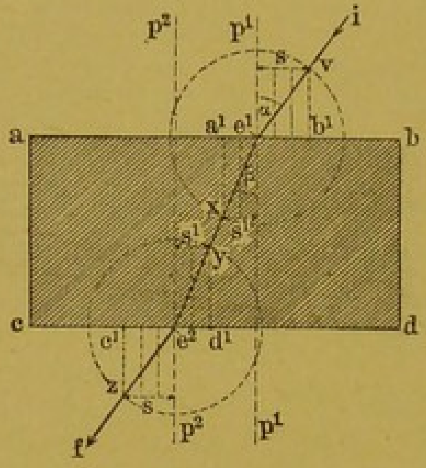

Fig. 1.

By virtue of the deflection or bend alluded to, the incident ray, i, must include a different angle, α, with the perpendicular p^1 from that, β, of the refracted ray e^1 e^2, and it is by the trigonometrical values s and s^1 of these angles, which have been found to bear a constant proportion to each other, that we are enabled to give expression to the amount of deflection sustained by a ray in passing from one medium to another.

§ 3. Experiment has shown that the proportion $\frac{s}{s^1}$ remains a constant value for any obliquity of a ray incident to the same medium, and yet, that it becomes a different value by substituting one medium for another.

It has therefore been considered expedient to establish the value of $\frac{s}{s^1}$ for all transparent mediums in the specific case of a ray passing from *air* into them, such values being known as the refractive indices of the substances.

To illustrate the graphical method by which we may arrive at the direction of the refracted ray, e^1 e^2, the index of refraction and the direction of the incident ray i being known, I shall select the index for crown glass = 1.5, by introducing the proportion $\frac{s}{s^1} = \frac{3}{2} = 1.5$ in the construction as follows:

After erecting the perpendicular p^1, take from a scale of equal parts the value for $s = 3$, and transfer it from e^1 beneath the ray i, upon the line e^1 b.

In the same manner transfer the value for $s^1 = 2$ from e^1 upon the line e^1 a, and in both points b^1 a^1, so established erect perpendiculars. The perpendicular at b^1 will intersect the ray, i, at a point, v, which limits the radius of a circle drawn from e^1 as a center; and by the circle's intersection with the perpendicular at a^1, the point, x, defining the direction of the ray, e^1 e^2, is fixed.

§ 4. As a ray of light is propagated backwards or forwards on the same path, the index of refraction from a denser medium into air is the inverse proportion from that of air into the medium, hence $\frac{s^1}{s}$ is the proportion by which the direction of the final ray, f, is to be determined when the direction of the ray, e^1 e^2, is known.

We therefore erect at e^2 the perpendicular p^2 and transfer the value of $s^1 = 2$ beneath the ray e^1 e^2 from e^2 upon the line e^2 d; likewise the value for $s = 3$ from e^2 upon the line e^2 c, and erect as before in the points d^1 and c^1 the perpendiculars.

The perpendicular at d^1 will intersect the ray, e^1 e^2, at a point, y, limiting the radius of a circle from the point e^2; the point, z, at the circle's intersection with the perpendicular in c^1 establishing the direction of the final ray, f.

As p^1 and p^2 are parallel, from the construction it follows that the ray f is parallel to i, and therefore of the same direction.

§ 5. Pursuant to the spirit of my intention to avoid mathematical formulæ, I shall seek to arrive at a conclusion respecting the deflection incurred by a ray in passing through a medium with oblique plane surfaces, confining myself as before to isolated vertical sections.

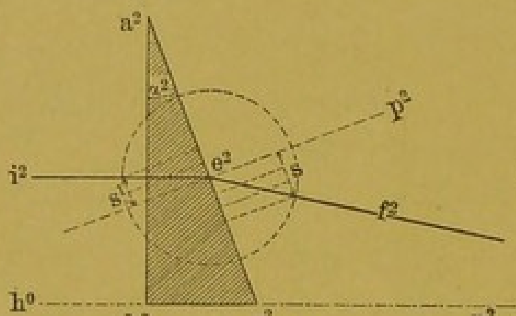

Fig. 2.

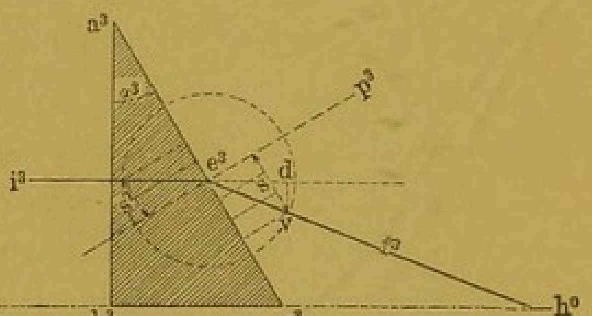

Fig. 3.

Specifically I shall select two right-angled prisms of varying angles, α^2 and α^3, with the rays i^2 and i^3 incident perpendicularly to the vertical sides $a^2\,b^2$ and $a^3\,b^3$, so as to avoid refraction on the incident sides, as shown in the vertical sections, Fig. 2 and Fig. 3, respectively. At e^2 and e^3 the rays i^2 and i^3 suffer refraction in the proportion $\frac{s^1}{s} = \frac{2}{3}$, according to § 4, and which, if carried out in construction as before indicated, determines the directions of f^2 and f^3, respectively, as shown.

In the future I shall have occasion to refer to the line dv, which is the perpendicular from v upon a line coincident with the ray i^3 when the latter is parallel to the base $b^3\,c^3$ of the prism, Fig. 3. Under such circumstances the displacement dv of the final ray f^3 is associated with a mathematical dependency upon the angle, α^3, of the prism, and the index of refraction $\frac{s}{s^1}$.

From the construction it follows that the final ray f^3 (Fig. 3) intersects the horizontal line h^0 at n^3, and f^2 (Fig. 2) at a more distant point, n^2, not shown. By a comparison of the prismatic section Fig. 2 with Fig 3, we observe that by a decrease of the angle from α^3 to α^2 the perpendicular p^2 has a greater tendency to parallelism with the horizontal line h^0 than p^3. Such parallelism being realized — when a^2 c^2 is parallel to a^2 b^2 or $\alpha^2 = 0^\circ$ — would result in the value s^1 vanishing in the incident ray i^2, and s in the final ray f^2, by virtue of the decrease of the angles of incidence and refraction in the proportion 2 to 3, thus establishing the coincidence of the incident and final rays, and placing the point n^2 of intersection at infinity respecting the horizontal line h^0.

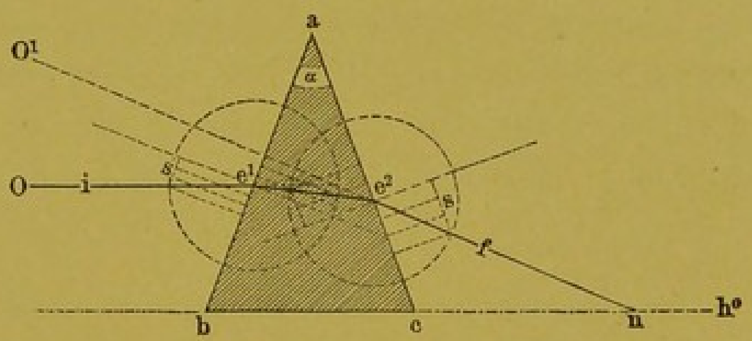

Fig. 4.

§ 6. In general we may therefore be permitted to assume that the greater the angle, α, of obliquity of the surfaces (Fig. 4) the greater will be the deflection of the final ray f, and the more proximate to the base b c of the prism will be its intersection n with the horizontal line h^0. Through practical experiment prismatic refraction manifests itself by an apparent change from the true position of an object, O, to that of its image O^1 when viewed by the observer's eye at n.

§ 7. In confining our observations to the relative directions of the incident and final rays, we may attain to a conception of the refraction for a medium included within plane surfaces by any of the following methods of impression:

1, a. The direction of a ray remains unchanged in passing through opposite parallel surfaces of a transparent medium, or

b. The incident ray i and the final ray f are parallel when the former is projected obliquely upon a transparent medium included within parallel surfaces.

2, a. The direction of a ray *is* changed in passing through opposite oblique surfaces, by a deflection of the final ray f toward the region of their greatest distance apart, or

b. The incident ray i and the final ray f are oblique when the former impinges upon a transparent medium included within oblique surfaces, or

c. The apex of the angle formed by an obliquity of the incident and final rays is always directed toward the apex of the angle of obliquity of the surfaces.

The law 2 finds its graphical demonstration in the following figures wherein I have introduced the medium glass as being intercepted by imaginary vertical and horizontal planes, V and H, coördinate at the point of exit e^2 for the final ray f.

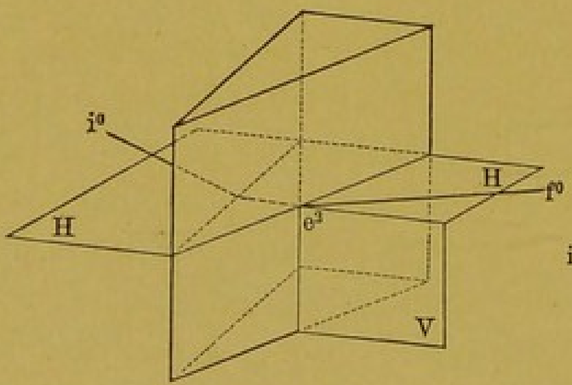

Fig. 5.

Prism, Base vertical; Refraction horizontal.

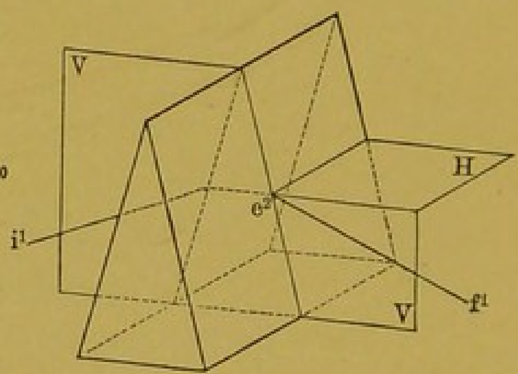

Fig. 6.

Prism, Base horizontal; Refraction vertical.

§ 8. The figures 5 and 6 are of particular interest to us, as they illustrate a very vital element in our future consideration of the refraction by cylindrical lenses, namely, that the refraction is confined to the plane whose intersection with the medium corresponds to the obliquity of the surfaces. Thus, for an obliquity of the surfaces in the horizontal plane H (Fig. 5), we find the refraction active in the horizontal plane (i^0 to f^0), and for an obliquity of the surfaces in the vertical plane V (Fig. 6), the refraction is active in the vertical plane (i^1 to f^1).

Here, in the sense that the final rays are confined to the plane of incidence, we may term the refraction passive in respect to its right-angled coördinate plane. Thus in Fig. 5 the refraction is passive with regard to the vertical plane, and in Fig. 6 with regard to the horizontal plane.

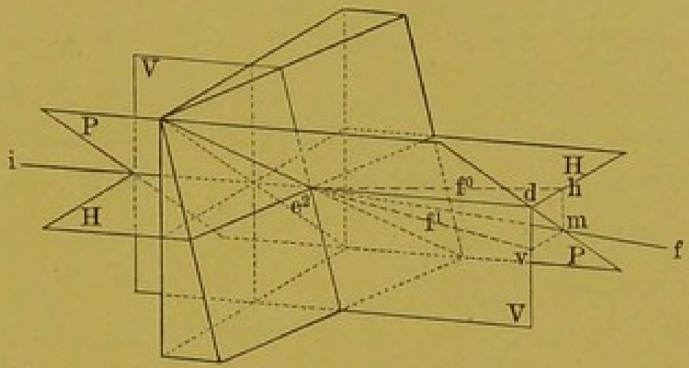

Fig. 7.

Prism, Base oblique; Refraction diametrically opposed.

§ 9. It is evident that the refraction is active in one and passive in the other plane for a medium of which the surfaces are oblique in but *one* plane, and to obtain the refraction active in both fixed planes an obliquity of the surfaces relative to each plane would be necessary. In such a medium (Fig. 7), if we consider the refraction merely with regard to the horizontal obliquity of the surfaces, the final ray would take the direction f^0–h, and, if independently for the vertical obliquity, the final ray would assume the direction f^1–v. Therefore, with due consideration of the obliquity in both planes, the refraction must include both properties of deflection and result in a final ray, f, which is directed to a point, m, defined by projection of the apportioned horizontal and vertical displacements, dh and dv. The figure of reference being merely a prism having its base set diagonally to the fixed right-angled coördinate system, the ray f is therefore also directed to the region of the greatest distance apart of the surfaces, through the point m, within a diagonally bisecting and oblique plane P.

§ 10. Directing our attention to the effect produced by substituting a segment of a circle for the line a^2 c^2 of the original prismatic section (Fig. 2), each succeeding point e^2, e^3, e^4 . . . . . (Fig. 8) may be considered as one of a prism varying in its angle α^2, α^3, α^4 . . . . . with that of its predecessor; and if the construction be carried out for each incident ray i^2, i^3, i^4 . . . . . the corresponding radial lines at the points, e^2, e^3, e^4 . . . . . in this case substituting the perpendicular p^2 heretofore mentioned, each final ray f^2, f^3, f^4 . . . . . will be found to intercept an arbitrarily selected base line, h^0, at the respective points n^2, n^3, n^4, to infinity.

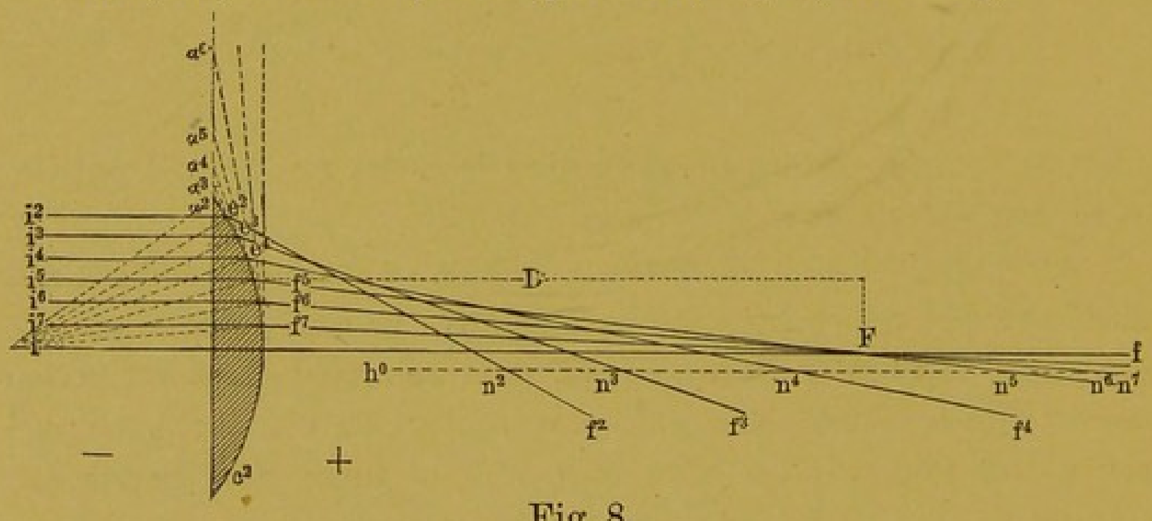

Fig. 8.

Plano-convex section.

In the so-called plano-convex section Fig. 8, the converging final rays f^5, f^6, f^7, corresponding to the more central incident parallel rays i^5, i^6, i^7,* establish points n^5, n^6, n^7 . . . . . to infinity, and possess the remarkable feature of intersecting each other at a common point F, termed the focal point, which is situated upon the central and direct ray i–f. According to § 4, rays emanating from the focal

* All future deductions refer exclusively to such rays.

point, F, will be emitted as parallel rays i^5, i^6, i^7 . . . . . . i. The points n^4, n^3, n^2, toward the refracting medium, correspond to the more eccentric incident rays, and, in the sense that these fail to assist in the harmony of a union of the final rays at the focal point, are to be considered a disturbing element, giving issue to what is termed "aberration." In the plano-concave section Fig. 9 the final rays f^5, f^6, f^7 . . . . . . are emitted as diverging rays, which may be considered as emanating from the so-called *virtual* focal point F, situated on that side of the section which corresponds to that of the incident rays.

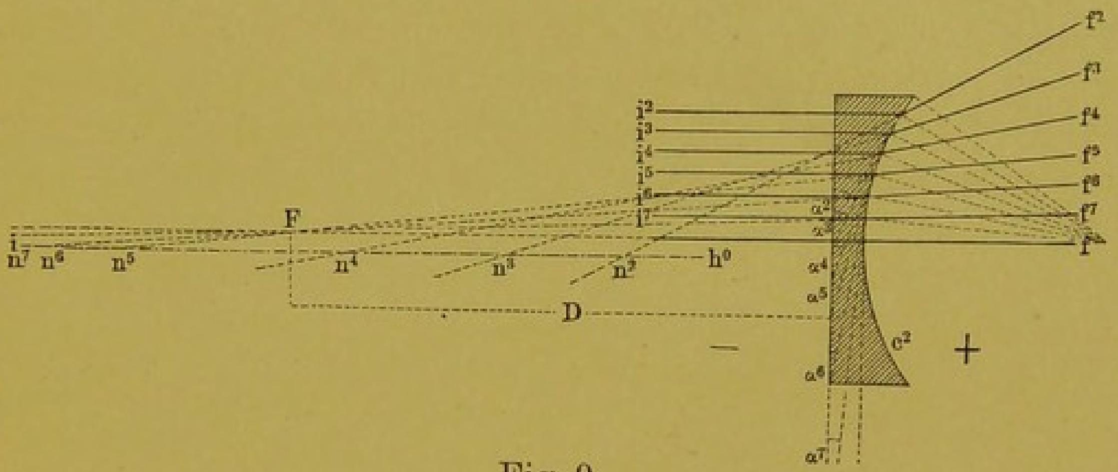

Fig. 9.

Plano-concave section.

§ 11. For either of the above sections it is also obvious that the more acute the curvature of the circle the greater proportionately will be the angles α^2, α^3, α^4, limiting the obliquity of the surfaces, and the more proximate to the medium will be the focal point F. Further, as the curvature of the circle is dependent upon the dimensions of the radius, the latter must prescribe the distance, D, of the focal point from the medium for which the index of refraction is known. This relationship involves mathematical formulæ for which I refer the reader to leading German authors* on the subject.

The greater the deflection of the final rays f^5, f^6, f^7, the shorter will be the distance D, or for an increase of the refraction we have a corresponding decrease of the focal distance.

* H. von Helmholtz, "Handbuch der Physiologischen Optik," § 9, 1886. Müller-Pouillet's "Lehrbuch der Physik," II. Band, 1876–1881. Dr. Adolph Wüllner, "Lehrbuch der Experimental Physik," Leipzig, 1883.

§ 12. If we seek to express the unit of refraction by the numeral 1, for a section of which the focal distance D is equal to one metre or 100 centimetres, sections of two, three, four times the refraction would find the expression of their focal distances in $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, the focal distance of the unit, or 50, $33\frac{1}{3}$, 25 centimetres, respectively.

The unit above mentioned has been termed one Dioptric,* and is adopted as the standard in measurements of ophthalmic refraction. Values beneath the unit are placed at 0.25D.,† 0.50D., and 0.75D., their respective focal distances being four metres or 400 centimetres, two metres or 200 centimetres, and one and one-third metres or $133\frac{1}{3}$ centimetres.

§ 13. Assuming the medium to divide the aerial space into negative and positive regions (Figs. 8, 9) as indicated by the sign − (minus) on the incident side of the medium, and the sign + (plus) behind the medium, we shall find the focal point on the positive side for all convex, and on the negative side for all concave sections.

In this sense the refraction for convex sections is considered positive, and for concave negative; so that for a numeral of 1 D., the refraction for Fig. 8 is expressed as being + 1 D., and for Fig. 9 — 1 D.

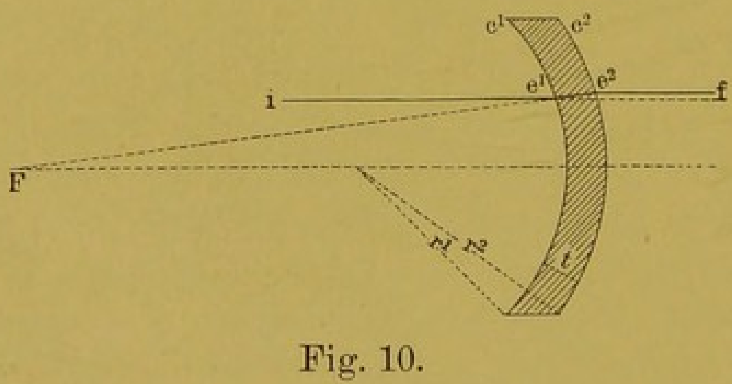

Fig. 10.

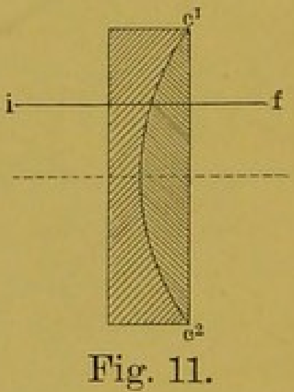

Fig. 11.

§ 14. By substituting, in Fig. 8, for the plane side a curvature c^1 concentric with c^2, the refractive effects of the sections, Fig. 8 and Fig. 9, are virtually united, as shown in Fig. 10. Owing to the concave curvature c^1, the incident ray i will assume the direction e^1–e^2, being coincident with the focal point F, which may also be practi-

* The full table of Dioptric numerals is given on page 41.

† D here being the abbreviation for Dioptric.

cally accepted as the focal point for the convex curvature c^2, provided the thickness, t, of the medium is created infinitely small in proportion to the radii r^1 and r^2.

Rays emanating from the focal point F for a convex curvature c^2 being emitted as parallel rays, § 10, it conditionally follows that the ray f will be parallel to the ray i. The neutralization is the more complete when the curvatures c^1 and c^2 are identical, and are brought in contact as shown in Fig. 11.

§ 15. Hence, in a pair of united convex and concave sections of identical curvature, it follows that the effect of the one is neutralized by the other respecting the existence of a focal point on either side of the medium.

§ 16. Prescribing the opposite curvatures to be *unequal*, the final rays will be accumulated at a focal point on that side of the medium which corresponds to the focal point for the more acute curvature.

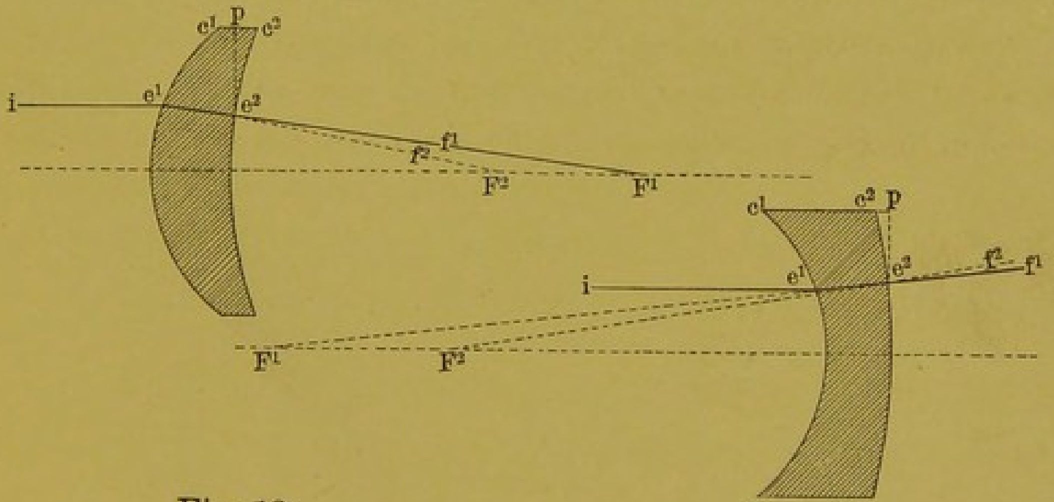

Fig. 12.

Periscopic convex section.

Fig. 13.

Periscopic concave section.

By reference to the periscopic convex and concave sections, Fig. 12 and Fig. 13, respectively, if we consider the refraction merely with respect to the front curvature c^1, disregarding the existence of a terminating back surface, the incident ray i will assume the direction of the ray e^1 f^1 toward the focal point F^1 then within the medium.

Considering a *plane* back surface e^2–p to exist, at the point e^2 the ray e^1 e^2 would suffer a second refraction and result in the ray e^2–f^2, directed to the focal point at F^2.

To eliminate this second or augmented refraction, it would be necessary for the ray e^1 e^2 to impinge upon the back surface c^2 perpendicularly at e^2.

A surface effecting this is obtained by giving it a curvature c^2 prescribed from the point F^1 as a center, in which specific event the ray e^1 e^2 traverses the radius of the circle or the perpendicular at e^2 for the surface c^2, thus fixing the point F^1 as the focal point for the respective periscopic convex and concave sections.

§ 17. Observation of the figures shows that the *lesser* curvature proportionately reduces the refraction of the more acute, the focal point F^1 for the periscopic sections being at a greater distance from the medium than the focal point F^2 for the plano-convex or concave. The more acute the curvature c^2, within the limits of parallelism with the curvature c^1, the more distant will be the focal point F^1 from the medium, so that the total refraction for the respective sections is equivalent to the difference of the apportioned numerals, and bears the sign corresponding to the more acute curvature c^1.

Supposing, in a periscopic convex section, 2.5D. to be the prescribed numeral of refraction for the convex, and 0.50D. for the concave side, the total refraction will be $2.5-0.50=2$D. convex, or $+2$D.

Similarly in a periscopic concave section, 2.5D. concave combined with 0.50D. convex equals $2.5-0.50=2$D. concave, or -2D.

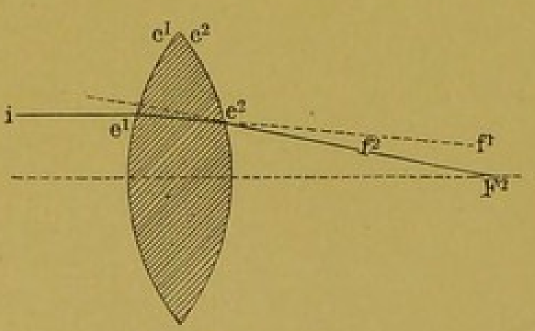

Fig. 14.

Double or Bi-convex section.

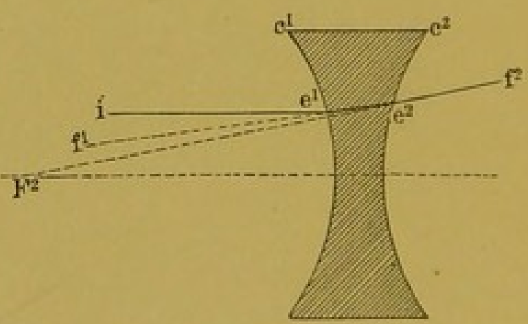

Fig. 15.

Double or Bi-concave section.

§ 18. In the bi-convex and bi-concave sections Fig. 14 and Fig. 15 it can be similarly shown that the curvature c^2 increases the refraction of c^1, so that the total refraction is expressed by the sum of the apportioned numerals and bears the sign associated with the nature of the respective sections.

Thus in either figures the numeral for c^1 being 1D., and for c^2 being 1.5D., the total refraction will be $1+1.5=2.5$D.

Convex, or +2.5D. for Fig. 14, and concave, or —2.5D. for Fig. 15.

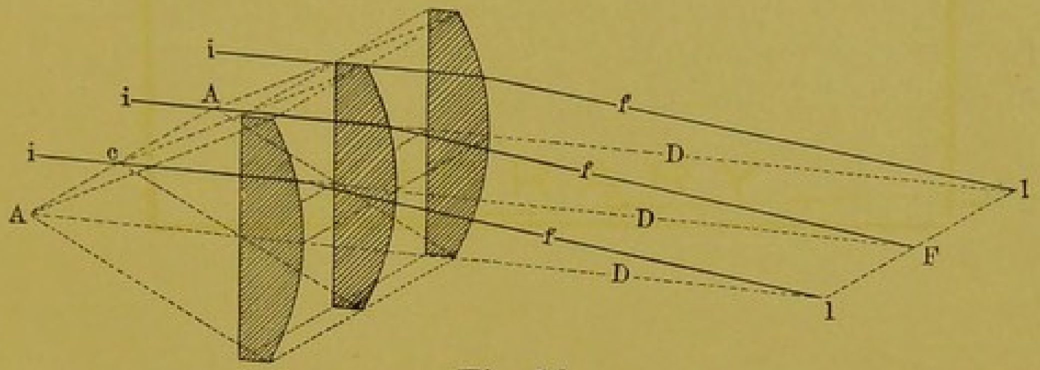

Fig. 16.

§ 19. For a medium (Fig. 16) composed of parallel vertical sections, each adjacent imaginary section has its corresponding focal point at the same distance (D) from the medium, so that the refraction for all central incident parallel rays becomes manifest by establishing a succession of these points, resulting in the so-called focal line l F l.

A similar succession of the radial centers (c) establishes a line (AcA), termed the axis of the so-created cylindrical medium or lense which is parallel to the focal line l F l and in the same plane.

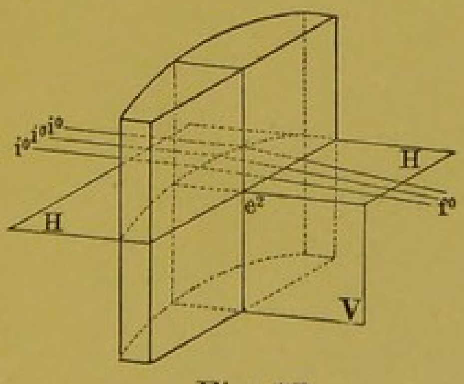

Fig. 17.

Axis vertical; Refraction horizontal.

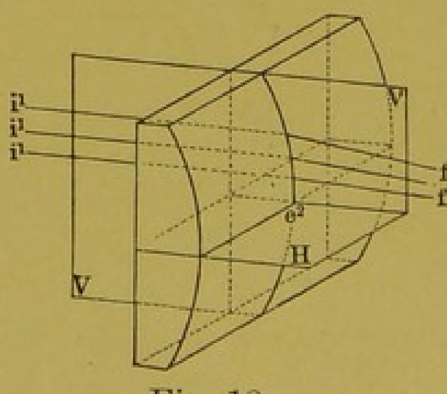

Fig. 18.

Axis horizontal; Refraction vertical.

Plano-convex Cylindrical Lenses.

§ 20. As in simple cylindrical lenses the surfaces have their defining obliquity in the plane which is perpendicular to the axis, we here also find the refraction active in this plane, and passive in the axial or right-angled coördinate plane (see figures 17–20), wherein, as before, i^0 and f^0 are associated with refraction in the horizontal, and i^1 and f^1 with refraction in the vertical plane.

In a practical experiment convex cylindrical refraction manifests itself by an apparent *increase*, and concave cylindrical refraction by an apparent *decrease* of the dimensions of an observed object in the plane which is at right angles to the axis. In the axial or diametrically opposed coördinate plane, the refraction being passive, corresponding dimensions remain unchanged.

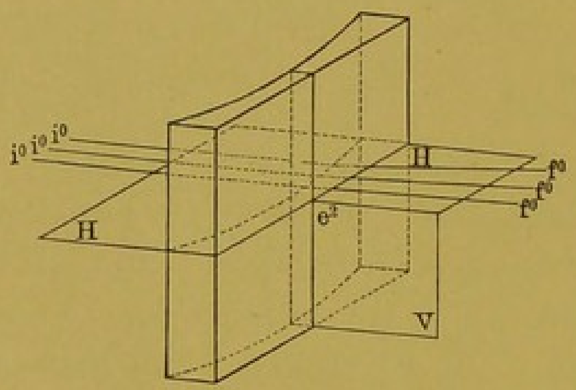

Fig. 19.

Axis vertical ; Refraction horizontal.

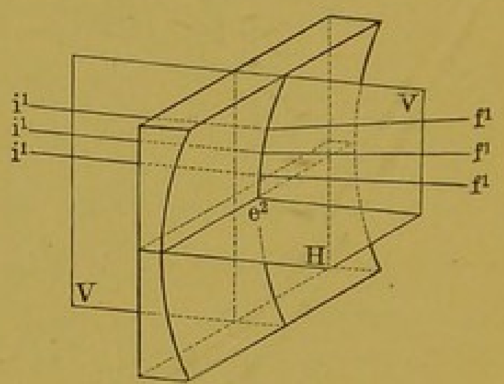

Fig. 20.

Axis horizontal ; Refraction vertical.

Plano-concave Cylindrical Lenses.

§ 21. To obtain the cylindrical refraction equally active in both planes, thereby reducing the focal line to a focal point, it would be necessary to combine identical cylinders, or, what is the same, create a single lens of which the opposite surfaces are right-angled coördinate cylindrical elements as shown in Fig. 21.

Under such circumstances, however, the focal line $l^1 F^1 l^1$ for the front surface c^1 is closer to the face of the lens than the focal line $l^2 F^2 l^2$ for the back surface c^2, which, in addition to the fact of it being difficult to insure the chief planes of refraction being at right-angles to each other in the construction of a bi-cylindrical lens, makes it obvious that these conditions, collectively, must increase the tendency to greater aberration.

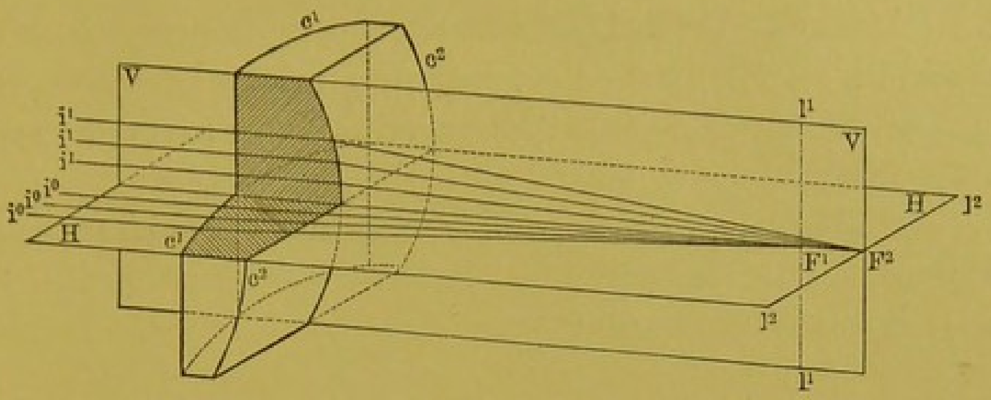

Fig. 21.

Double or Bi-cylindrical Lens.

§ 22. The greater the distance apart of the surfaces, c^1 and c^2, the greater will be the aberrative distance, F^1 to F^2. Yet, as the thickness of the lens may generally be accepted as a vanishing quantity in proportion to the focal distance, we may consider a common focal point to exist for both refracting surfaces.

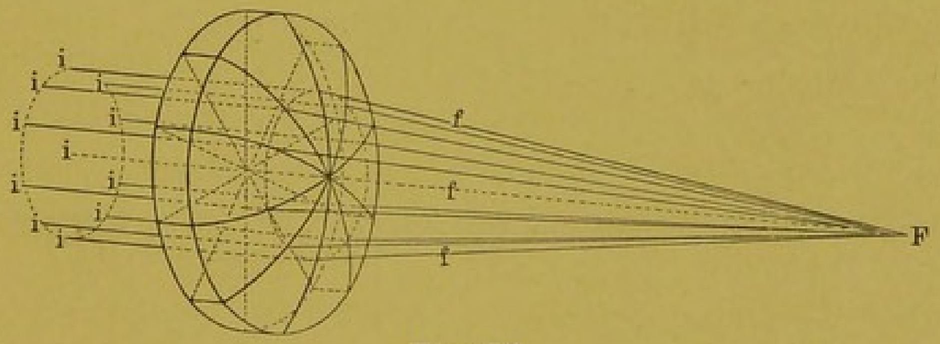

Fig. 22.

Plano-convex Spherical Lens.

§ 23. Practically, however, it would be better to create a surface which shall include within itself the activity of refraction for the vertical as well as horizontal planes. With this object in view, I shall select the isolated vertical section described in § 10, and cause it to be rotated upon the central incident and direct ray, i–f, as its so called optical axis, whereby a plano-convex spherical lens is obtained. (See Fig. 22.) Similar rotation of the sections Figs. 9, 12, 13, 14, and 15, inclusive, would result in the so-created spherical lenses being characterized and distinguished by the sections employed.

It is evident that the incident and final rays will retain their relative obliquity during the rotation, so that all incident parallel rays find their corresponding final rays in the resulting cone having its apex at F or the focal point.

To further illustrate, we may take advantage of §9 in its application in this instance to a medium of which one surface is *curved* and oblique in both right-angled coördinate planes.

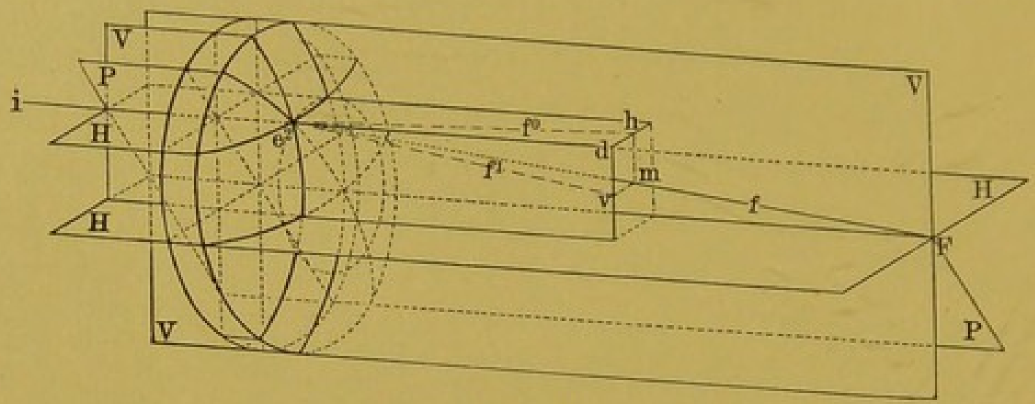

Fig. 23.

In the plano-convex spherical lens Fig. 23, if we consider the refraction at e^2 of the ray i merely with regard to the horizontal obliquity, the final ray would take the direction f^0–h, and, if independently for the vertical obliquity, the final ray would assume the direction f^1–v. Therefore, with due consideration of the obliquity in both planes or meridians, the refracted ray must include both properties of deflection, and result in a final ray f, which is directed to the focal point F through a point m, of the oblique plane P, defined by projection of the apportioned horizontal and vertical displacements dh and dv.

§ 24. Finally, we may therefore conclude that spherical refraction is equivalent to the refraction of right-angled crossed cylinders of identical curvature.

As in spherical lenses the refraction is active in two diametrically opposed coördinate planes or meridians, the observance of an object through such will create the impression of enlargement for a convex, and of reduction for a concave lens for both the lateral and longitudinal dimensions of the object.

COMPOUND LENSES.

I. CONVEX MERIDIANS.

§ 25. An asymmetrically-refracting or compound lens is one in which the principal diametrically-opposed sections include different degrees of refraction, in contradistinction to those hitherto mentioned, and in which uniform refraction took place either in one and not the other or equally in both meridians.

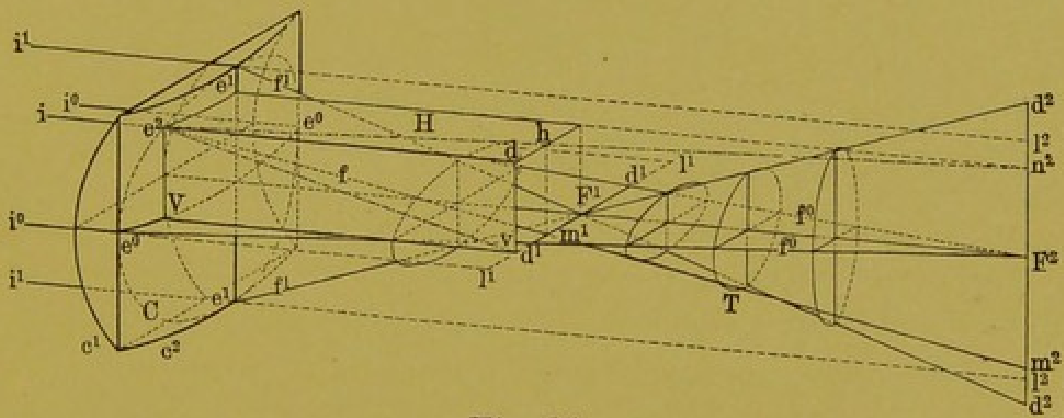

Fig. 24.

Convex Cylindro-cylindrical Lens (+ c^1 axis 180° ⊃ + c^2 axis 90°).

By reference to § 21, Fig. 21, it is evident that the aberrative distance F^1 to F^2 may also be definitely increased by prescribing different degrees of refraction for the active planes or sections of the opposite cylinders, in which event the focal point ascribed to the equally curved crossed cylindrical lens must be destroyed, and substituted by a pair of focal lines, which are separated by a distance equivalent to the difference of the respective focal distances of the diametrically-opposed *unequal* cylinders. Thus in the asymmetrically-refracting lens Fig. 24, represented as consisting of two crossed convex cylinders (c^1 and c^2) of unequal curvature, l^1 F^1 l^1 and l^2 F^2 l^2 will be the region of the respective focal lines, and their distance apart (F^1 to F^2) the aberrative distance, termed by Sturm the "focal interval."

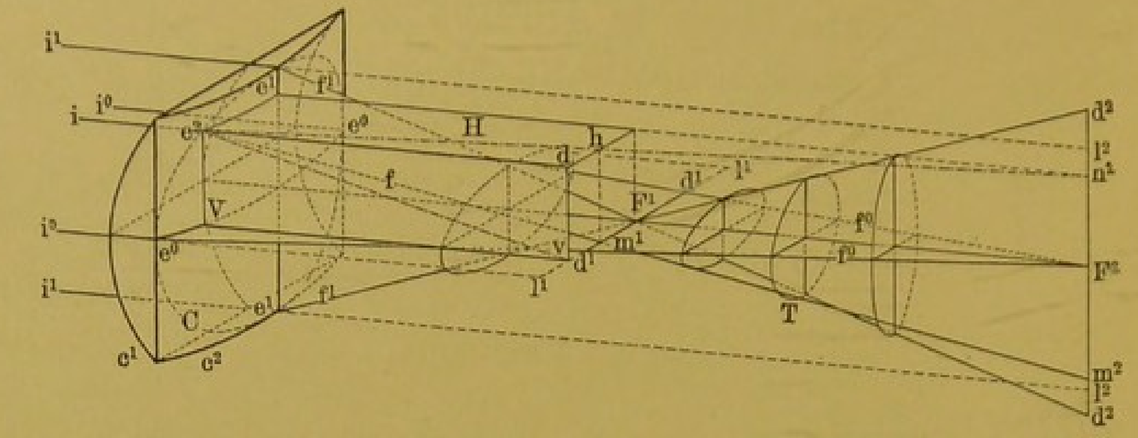

Fig. 24.

Convex Cylindro-cylindrical Lens (+ c^1 axis 180° $\supset$ + c^2 axis 90°).

As the cylinders are of equal length, the focal lines l^1 F^1 l^1 and l^2 F^2 l^2 would also be identical in this regard when the apportioned refractions of the cylinders are considered *independently* of each other.

§ 26. The combined refraction of the cylinders, however, definitely modifies this specific condition, and in the following manner:

The outermost incident rays i^0, in the central horizontal plane, which would have been directed to the points l^1 and l^1 for the cylinder c^1, will suffer horizontal displacement toward the point F^2, owing to the activity of the refraction in this plane for the cylinder c^2, and so establish points d^1 and d^1 of the focal line l^1 F^1 l^1 for the combined action of the cylinders c^1 and c^2 in the horizontal plane.

Similarly, the outermost incident rays i^1, in the central vertical plane, which would have been directed to the points l^2 and l^2 for the cylinder c^2, will suffer vertical refraction in this plane for the cylinder c^1 to F^1, crossing each other at this point and thereby intercepting the focal line l^2 F^2 l^2 at the points d^2 and d^2 for the combined action of the cylinders c^1 and c^2 in the vertical plane.

If we consider the refraction at the point e^2 of the circle C for the ray i merely with regard to the horizontal obliquity of the surfaces or the cylinder c^2, the final ray would take the direction e^2–h, intercepting the focal line of the cylinder c^2 at a correlative point n^2; and as all final rays for the cylinder c^1 above the central horizontal plane intercept the focal line d^1 F^1 d^1, it follows that by introduction of the cylinder c^1 the ray e^2 h must fall subject to the same influence for the combined action of the cylinders, thus depressing the ray e^2 h from the point h perpendicularly to m^1, and consequently also the point n^2 to m^2 within the focal line d^2 F^2 d^2.

By an analogous reasoning to § 9 we here also find the direction of the final ray f to be determined by projection of the apportioned horizontal and vertical displacements, dh and dv, which are solely dependent upon the refraction ascribed to the diametrically opposed active meridians of the cylinders c^1 and c^2.

Increased proximity of the point e^2 to e^0,* upon the circle C, will be associated with a further recession of m^1 from F^1, and with an approach of m^2 toward F^2 for these points of intersection of the final ray f with the respective focal lines at F^1 and F^2 within the correlative regions F^1 d^1 and d^2 F^2, the reverse being the case for an advancement of e^2 to e^1. (See illustration on title-page.)

§ 27. The total refraction for all incident parallel rays included within the area of the circle C will therefore also result in a limitation of the final rays to the region and magnitude of the focal lines d^1 F^1 d^1 and d^2 F^2 d^2.

Such final rays, if intercepted at intervals by a transverse perpendicular screen, in a practical experiment, would project themselves as elliptical areas of diffused light of proportionately varying size.

The longest and shortest diameters of the consecutive ellipses correspond to the meridians of least and greatest refraction, so that in the immediate vicinity of F^1, for instance, the ellipses have their longest diameters horizontally; whereas, in the vicinity of F^2, their longest diameters are vertical.

This naturally occasions a reversal of the ellipses respecting their diameters, at some point within the focal interval F^1–F^2, such point being determined where the vertical and horizontal displacements are alike, and the section consequently a circle, T.

The position of this circle relative to the limits of the "focal interval" might be termed the region of transition.

* In the illustrations for compound lenses, e^0 and e^1 have been selected to designate points of the horizontal and vertical planes respectively, the indices thereby harmonizing with i^0, f^0, and i^1, f^1, of correlative refraction.

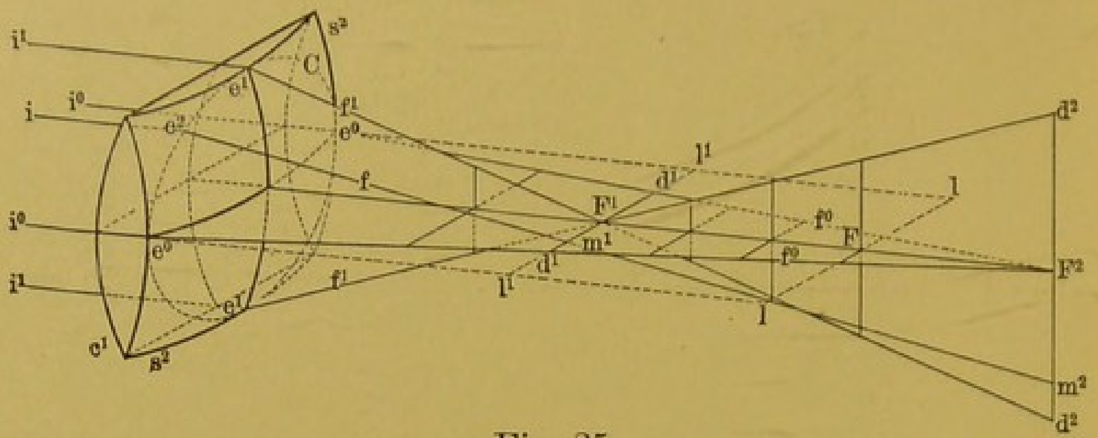

Fig. 25.

Convex Sphero-cylindrical Lens (+ $s^2 \subset + c^1$ axis 180°)—Double Form.

§ 28. Asymmetrical refraction in a lens is, however, preferably attained by combining a spherical with a cylindrical surface, the requisite conditions being fulfilled through the difference arising from the augmented or decreased refraction of the spherical surface by and in the active meridian of the cylinder.

To increase the refraction of a positive or negative spherical lens in one meridian, we may add to it the active meridian of a cylinder bearing the same sign; and to decrease it in the same meridian, we may combine it with the active meridian of a cylinder bearing the opposite sign.

(1) The combination of a positive spherical with a positive cylindrical surface would result in the section of *greatest* refraction being *double* convex; and,

(2) The combination of a positive spherical with a limited or less acute negative cylinder would result in the section of *least* refraction being *periscopic* convex.

Where the aforesaid combinations are spoken of, I shall take the liberty of ascribing to them the terms double and periscopic form, respectively.

§ 29. As the combination of crossed convex cylinders of unequal curvatures gave issue to a pair of focal lines, to the novice it may appear requisite that a focal point and a focal line should exist for a combination of a spherical with a cylindrical surface. In consequence, I shall endeavor to avert this possible though erroneous impression.

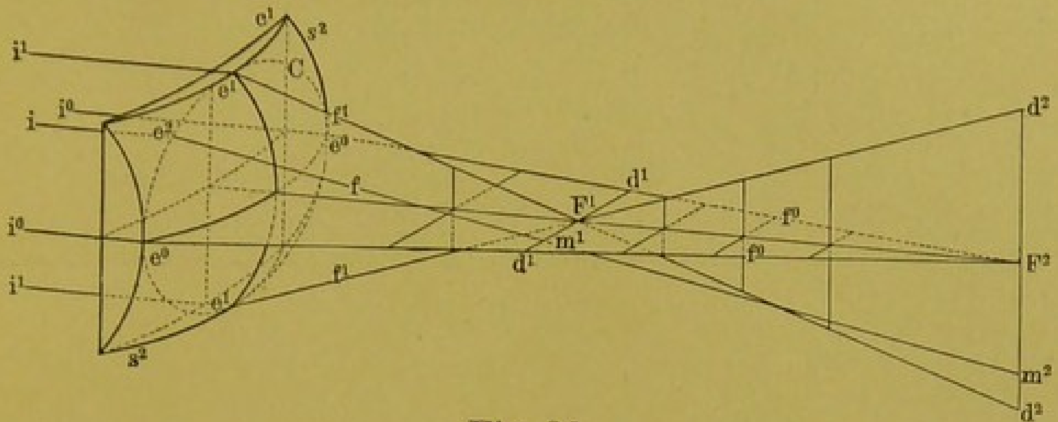

Fig. 26.

Convex Sphero-cylindrical Lens (+ s² ⊃ — c¹ axis 90°) — Periscopic Form.

In the convex sphero-cylindrical lens of double form Fig. 25, if we considered the refraction for each surface independently of the other, we should find a focal point at F^2 for the convex spherical surface, s^2, and a focal line, say, at 1 F 1, for the cylindrical surface c^1. Their combination giving issue to augmented refraction in the vertical plane, however, occasions a displacement of the focal line 1 F 1, to the region l^1 F^1 l^1.

The final rays from the outermost points, e^0, in the horizontal plane being directed to the focal point F^2, it is evident that the focal line l^1 F^1 l^1 must become subject to the limiting influence of the spherical refraction in this plane, thereby establishing the points d^1 and d^1, and restricting the magnitude of the focal line to d^1 F^1 d^1.

The final rays from the outermost points, e^1, in the vertical plane, which would have been directed to the focal point F^2, cross each other at F^1, the extremities being thereby displaced from F^2 to d^2 and d^2, thus resulting in the destruction of the focal point F^2, and prescribing a limitation of the rays to a *created* focal line d^2 F^2 d^2.

§ 30. The convex sphero-cylindrical lens of periscopic form Fig. 26 is constructed by combining a *limited* concave cylinder, c^1, with a convex spherical surface, s^2, the axis of the cylinder here being placed in the vertical instead of the horizontal plane for future purposes of reference.

In this case I have ascribed to the spherical surface s^2 a curvature corresponding to the focal point F^1, and to the cylindrical surface c^1, a curvature, which, acting in combination with its associated horizontal meridian of the spherical surface, causes the rays to accumulate at the focal line $d^2 F^2 d^2$. The reasons ascribed for the destruction of the focal point F^2, in the lens Fig. 25, are alike applicable to the creation of a focal line $d^1 F^1 d^1$ at F^1, in the present instance, as also to the restriction of the focal line to the magnitude $d^2 F^2 d^2$, for the cylinder c^1.

§ 31. The characteristic difference between the double and the periscopic form of asymmetrically-refracting lens exists merely in the fact that the focal lines and their respective elements of creation are interchanged. Thus in Fig. 25 the focal line $d^2 F^2 d^2$ corresponds to the initial effect of the spherical surface; whereas, in Fig. 26 the first focal line $d^1 F^1 d^1$ corresponds to the same.

§ 32. This difference, however, is not material, as it is evident that the magnitude and the distance of the focal lines from the lens are dependent upon the refraction ascribed to its two principal sections; and, since any two given points (d^1 and F^2, F^1 and d^2, m^1 and m^2) definitely fix the position of a line or ray in space, it is further obvious that the direction of all final rays will be identical for any lens* in which the right-angled coördinate meridians of greatest and least refraction are allotted the same.

§ 33. To demonstrate the analysis of formulæ for these equivalents I shall designate, for the respective figures, the refraction as being expressed by

Ia.	+ 3.5 cyl. axis 180°	⊂ +	1.5 cyl. axis 90°.	(Fig. 24.)
IIa.	+ 1.5 spherical	⊂ +	2 cyl. axis 180°.	(Fig. 25.)
IIIa.	+ 3.5 spherical	⊂ −	2 cyl. axis 90°.	(Fig. 26.)

It being necessary to become impressed with the meridians of greatest and least refraction, I have considered it expedient to picture these in their allotted planes of activity, V and H, as shown in their

* Wherein the rays are incident in the immediate vicinity of the optical axis, and the thickness of the lens is a vanishing quantity in proportion to the focal distances of the surfaces.

correlative sectional diagrams, Fig. 24a, Fig. 25a, Fig. 26a, and refer to them in the following, as occasion demands:

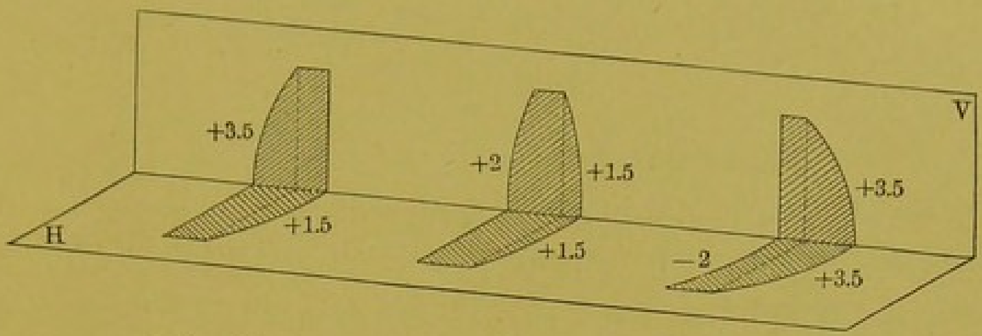

Fig. 24a. Fig. 25a. Fig. 26a.

Formula Ia. + 3.5 cyl. axis 180° ⊂⊃ + 1.5 cyl. axis 90°.

Refraction: Fig. 24a. } + 3.5 vertical ⊂⊃ + 1.5 horizontal = + 3.5V ⊂⊃ + 1.5H.

Formula IIa. + 1.5 spherical ⊂⊃ + 2 cyl. axis 180°.

Refraction: Fig. 25a. } + 2 + 1.5 vertical ⊂⊃ + 1.5 horizontal = + 3.5V ⊂⊃ + 1.5H.

Formula IIIa. + 3.5 spherical ⊂⊃ – 2 cyl. axis 90°.

Refraction: Fig. 26a. } + 3.5 vertical ⊂⊃ – 2 + 3.5 horizontal = + 3.5V ⊂⊃ +1.5H.

Pursuant to § 32 we find the lenses Ia, IIa, and IIIa to be asymmetrically-refracting equivalents.

§ 34. As the preference is generally given to the double form (Formula IIa), and, under certain circumstances, occasionally to the periscopic (Formula IIIa), I here only give the rules applicable for the conversion of the one into the other formula.

To convert the double into the periscopic form:

Rule 1. Place the *sum* of both numerals as the numeral for newly created spherical,* and combine with the same cylindrical numeral having its sign and axis reversed.

* The sign of the original spherical remaining unchanged.

To convert the periscopic into the double form:

Rule 2. Place the *difference* of both numerals as the numeral for newly created spherical,* and combine with the same cylindrical numeral having its sign and axis reversed.

§ 35. These lenses being designated for the correction of anomalies of ophthalmic refraction, in prescribing such it becomes necessary to indicate their position before the eye by allotting to the axis of the cylinder the noted degrees of rotation as indicated by the protractor or divisions of an oculist's trial frame. This, however, does not change the inherent properties of the lens, as the meridians of greatest and least refraction are ever and anon 90° apart between the limits of rotation, 0° and 180°.

Thus, in the instance of the formula:

$$+ 1.5 \text{ sph.} \subset + 0.50 \text{ cyl. axis } 130^\circ$$

the periscopic form would be expressed according to Rule 1, § 34, by

$$+ 2 \text{ sph.} \subset - 0.50 \text{ cyl. axis } 40^\circ.$$

Inversely, the former may be made the result of the latter by application of Rule 2, § 34.

A table giving the available combinations by crossed convex cylinders, from 0.25D. to 4D., is shown on page 43, and wherein, according to § 24, crossed convex cylinders of identical curvature are substituted by their spherical equivalents.

The diagonal column of spherical lenses divides the table into two sets of compound lenses which are duplicates in refraction, the one being a reversion of the other by a change in the axis of 90°.

Thus all lenses in the vertical columns *beneath* the spherical are correlative duplicates of the lenses in the horizontal columns to the *right* of the same spherical. ($A^1 = a^1$), ($A^2 = a^2$), ($A^3 = a^3$), ($B^1 = b^1$), ($B^2 = b^2$), ($B^3 = b^3$), etc.

* The sign of the original spherical remaining unchanged.

COMPOUND LENSES.

2. CONCAVE MERIDIANS.

§ 36. The preceding general principles are alike applicable to the similarly * planned concave compound lenses Figs. 27, 28, 29, in each of which the focal lines, and consequently also the focal interval and region of transition are virtual, and in the negative region before the lens.

All parallel rays incident upon and within the periphery of the circle C in either of the figures will therefore result in final rays behind the lens which appear to emanate from correlatively established virtual points d^1 and F^2, F^1 and d^2, m^1 and m^2, of and within the limits of the focal lines before the lens. For these lenses, respectively, I have ascribed the refraction as being expressed by

Ib. — 1.5 cyl. axis 180° ⊂ — 3.5 cyl. axis 90°. (Fig. 27.)

IIb. — 1.5 spherical ⊂ — 2 cyl. axis 90°. (Fig. 28.)

IIIb. — 3.5 spherical ⊂ + 2 cyl. axis 180°. (Fig. 29.)

and which, by a similar method of analysis to § 33 pursuant to § 32, will be found to be asymmetrically-refracting equivalents.

According to Rule 1, § 34, as an instance, the concave sphero-cylindrical lens

— 1.25 sph. ⊂ — 0.75 cyl. axis 160°

may be converted into the periscopic form

— 2 sph. ⊂ + 0.75 cyl. axis 70°,

and *vice versa*, according to Rule 2, § 34.

* The meridian of greatest refraction is here placed in the horizontal instead of the vertical plane.

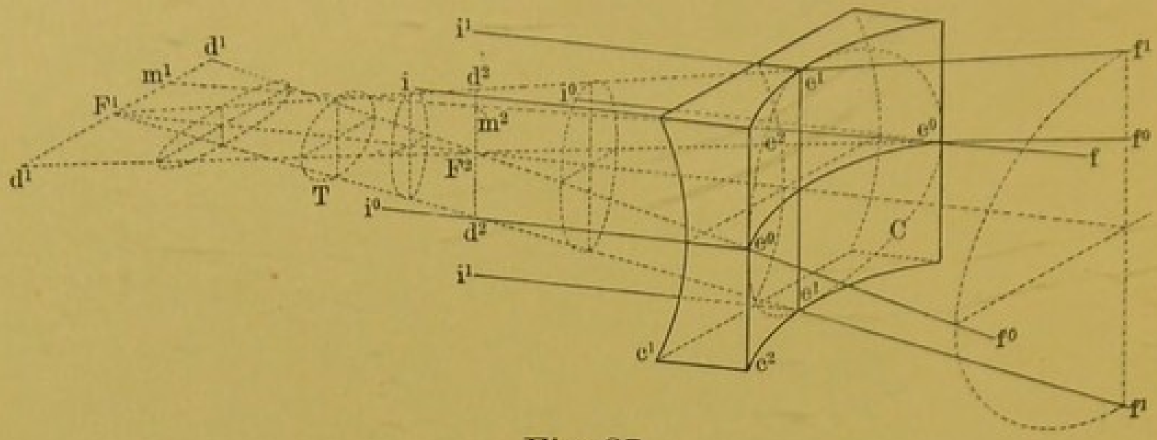

Fig. 27.

Concave Cylindro-cylindrical Lens ($-c^1$ axis 180° ⊂⊃ $-c^2$ axis 90°).

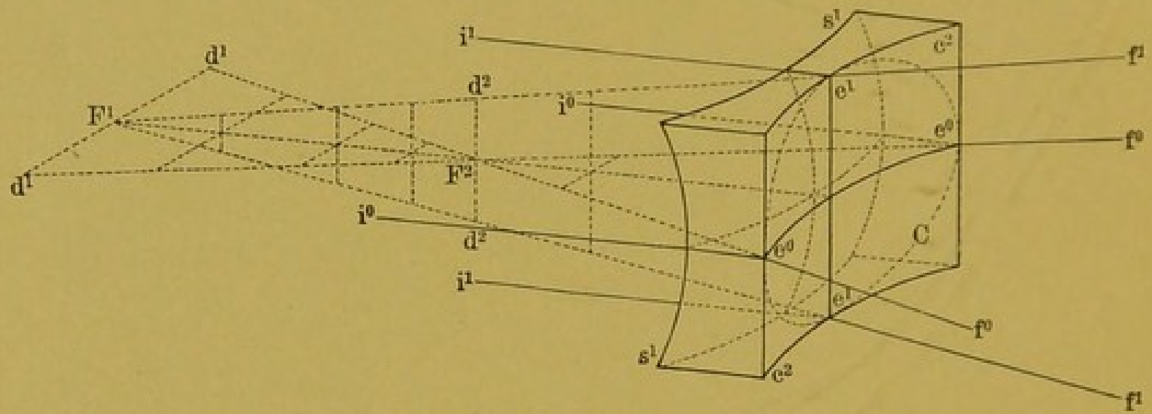

Fig. 28.

Concave Sphero-cylindrical Lens ($-s^1$ ⊂⊃ $-c^2$ axis 90°)—Double Form.

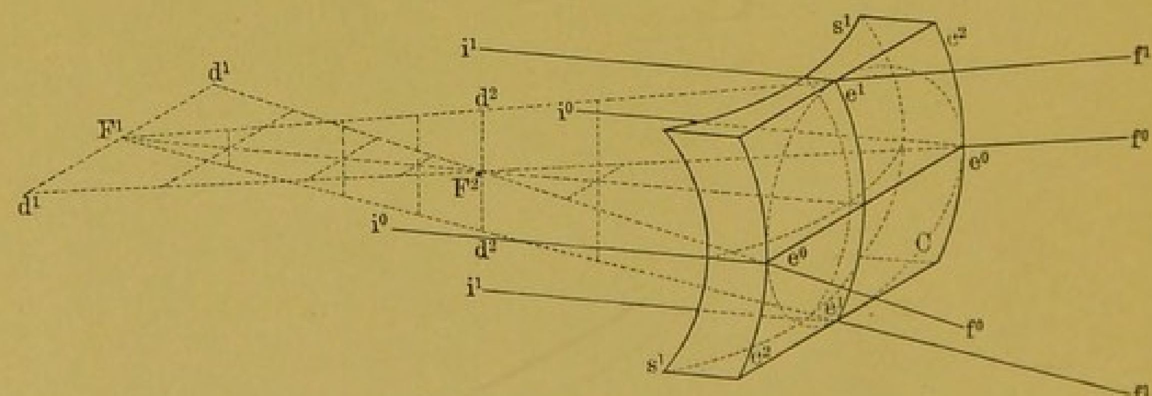

Fig. 29.

Concave Sphero-cylindrical Lens ($-s^1$ ⊂⊃ $+c^2$ axis 180°)—Periscopic Form.

In the above figures, i^0, e^0, and f^0 are associated with horizontal, and i^1, e^1, and f^1 with vertical refraction.

COMPOUND LENSES.

3. CONCAVO-CONVEX OR MIXED MERIDIANS.

§ 37. Hitherto we have prescribed different degrees of refraction, restricted to the same quality, convex or concave, for the chief right-

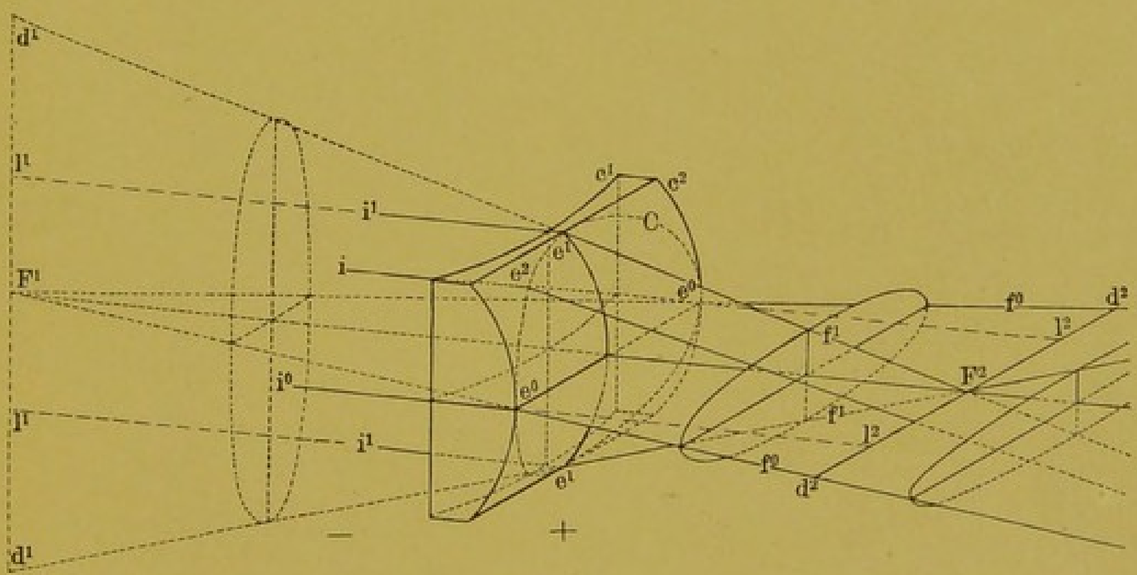

Fig. 30.

Concavo-convex Cylindro-cylindrical Lens (— c^1 axis 90° ⊂ + c^2 axis 180°).

angled sections. In contradistinction hereto, and as a final complication, we may combine in a lens different or even like degrees of refraction though of reverse quality ; namely, convex in one and concave in the other diametrically-opposed coördinate meridian. As an instance, I shall select the compound lens Fig. 30, represented as consisting of a plano-concave, c^1, and a plano-convex cylinder, c^2, so combined as to place their active meridians at right angles to each other.

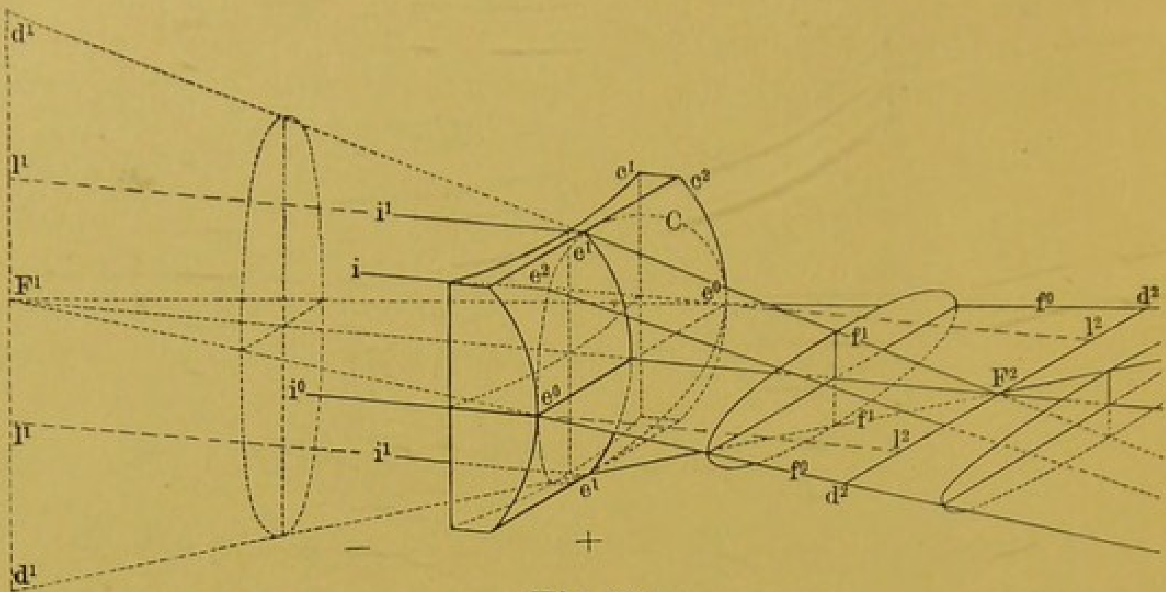

Fig. 30.

Concavo-convex Cylindro-cylindrical Lens (— c^1 axis 90° ⊃ + c^2 axis 180°).

Independently considered, each cylinder c^1 and c^2 would have its focal line $l^1\ F^1\ l^1$, and $l^2\ F^2\ l^2$, of original magnitude in the region of its sign − and + respectively, and consequently on opposite sides of the lens.

When associated, however, the final rays, which would have been restricted to the limits of the focal line $l^2\ F^2\ l^2$ for the cylinder c^2, will, by virtue of the dispersive effect of the cylinder c^1 in the horizontal plane, be confined to an augmented focal line $d^2\ F^2\ d^2$, within the limits d^2–d^2, for the outermost rays emanating from the point F^1 of the virtual focal line, $l^1\ F^1\ l^1$.

By a similar method of reasoning to § 26, all final rays within the limits of the circle C will be accorded associated vertical and horizontal refraction, culminating in their united intersection of a line $d^2\ F^2\ d^2$, of the horizontal plane in the positive region behind the lens. Interception of these rays by successive transverse vertical planes will manifest itself in a demonstration of similarly arranged ellipses respecting their greatest and least diameters, before and behind the focal line $d^2\ F^2\ d^2$. By projecting the final rays into the region of their apparent emanation from before the lens, we would attain to a similar increase of the virtual focal line $l^1\ F^1\ l^1$, to the magnitude $d^1\ F^1\ d^1$, and to a reversal of the so-defined ellipses respecting their greatest and least diameters, as shown by the dotted lines in the negative region (Fig. 30).

§ 38. Identical refraction is also preferably obtained in this instance by combinations of spherical and cylindrical surfaces.

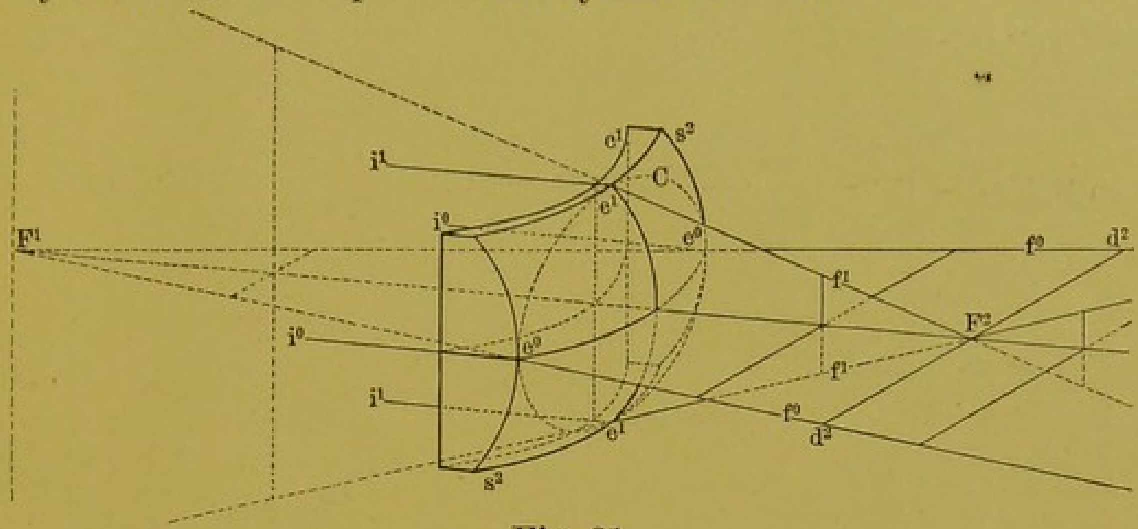

Fig. 31.

Concavo-convex Sphero-cylindrical Lens ($+ s^2 \supset - c^1$ axis 90°).

The combination of a convex spherical with a *more acute* concave cylindrical surface results in the periscopic section being concave, and the combination of a concave spherical with a *more acute* convex cylindrical surface results in the periscopic section being convex. The identity of the refraction for these combinations becomes apparent by reference to the concavo-convex sphero-cylindrical lenses Figs. 31 and 32, in which by a judicious selection of the respective spherical and cylindrical curvatures according to § 32, the demanded positive and negative elements of refraction for the principal meridians of the crossed cylindrical lens Fig. 30, are fulfilled.

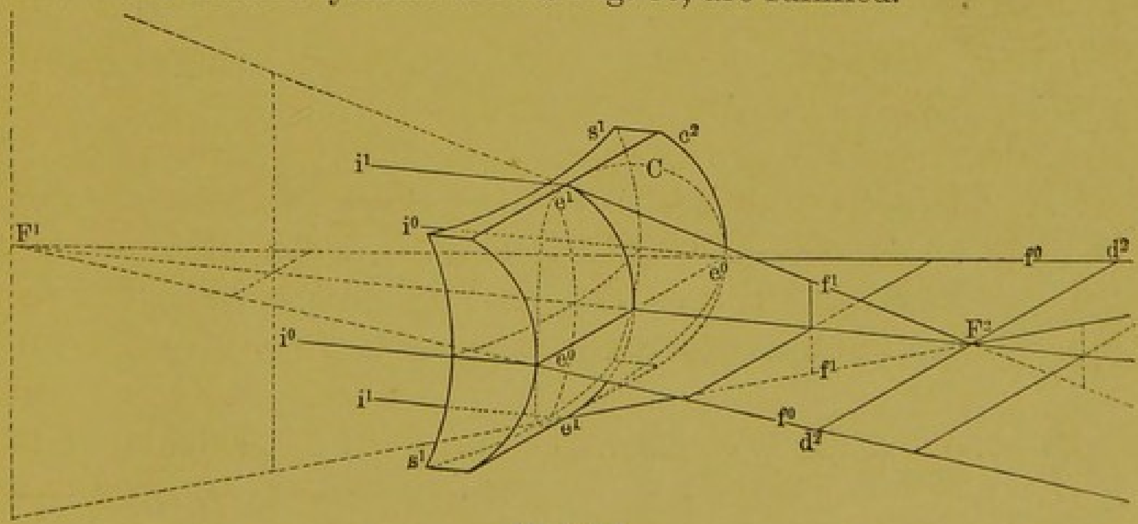

Fig. 32.

Concavo-convex Sphero-cylindrical Lens ($- s^1 \supset + c^2$ axis 180°).

To illustrate the equality of formulæ characterizing these equivalents, I refer to their correlative sectional diagrams Figs. 30c, 31c, 32c, in the order following:

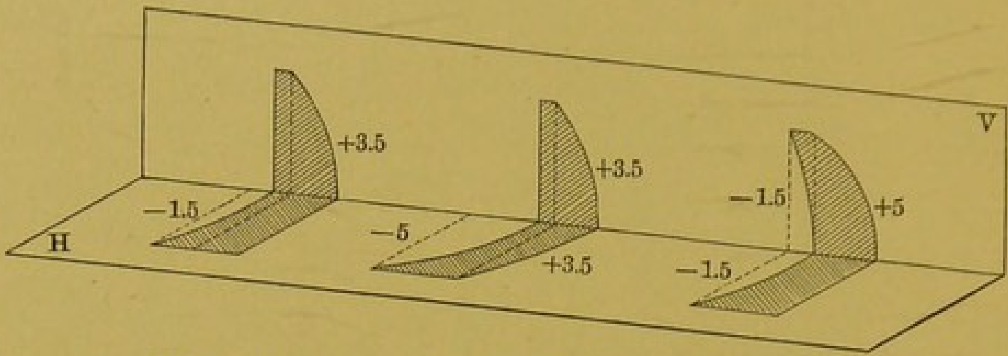

Fig. 30c. Fig. 31c. Fig. 32c.

Formula Ic. -1.5 cyl. axis $90^\circ \subset + 3.5$ cyl. axis 180°. (Fig. 30.)

Refraction: Fig. 30c. } -1.5 horizontal $\subset + 3.5$ vertical $= -1.5\text{H} \subset + 3.5\text{V}$.

Formula IIc. $+ 3.5$ spherical $\subset - 5$ cyl. axis 90°. (Fig. 31.)

Refraction: Fig. 31c. } $-5 + 3.5$ horizontal $\subset + 3.5$ vertical $= -1.5\text{H} \subset + 3.5\text{V}$.

Formula IIIc. $- 1.5$ spherical $\subset + 5$ cyl. axis 180°. (Fig. 32.)

Refraction: Fig. 32c. } -1.5 horizontal $\subset -1.5 + 5$ vertical $= -1.5\text{H} \subset + 3.5\text{V}$.

§ 39. These lenses being equivalents (see § 32), I here give the rule in exclusive demand, by reasons later given (§ 40), for converting the cylindro-cylindrical lens (Formula Ic), into the concavo-convex sphero-cylindrical lenses (Formulæ IIc. and IIIc).

Rule 3. Place the new numeral of *cylindrical* refraction equal to the sum of both numerals accompanied by both the sign and axis associated with either cylinder, and combine with the neglected cylindrical numeral bearing its correlative sign as spherical.

Comparison of the periscopic lenses Figs. 26 and 29 with the lenses Figs. 31 and 32, respectively, exhibits a striking similarity in construction, the characterizing difference being that the cylindrical curvatures exceed the spherical in the latter as against a reverse condition for the former.

§ 40. In a case of mixed astigmatism, demanding the foregoing correction, it becomes necessary to determine the chief meridians − 1.5 and + 3.5 independently of each other, thereby obtaining the combination expressed by Formula Ic, as by an endeavor to correct through introducing a *spherical* element in any proportion or wholly of either equivalent (Formula IIc or IIIc), an improvement in one meridian would always be attended by a proportionate derangement in the other, with a probability of the patient failing to appreciate the merits of its application.

It is only in consequence of this fact that the lenses of the Formulæ IIc and IIIc are rarely the direct issue of a diagnosis, whereas, in cases of regular compound astigmatism with congeneric meridians, the lenses IIa, IIIa and IIb, IIIb are most apt to be.

§ 41. Astigmatism has in the main been attributed to asymmetry of the cornea, though the crystalline lens is often found to be implicated; yet *specifically* in a case of mixed astigmatism *in which the crystalline lens does not assist*, it is improbable that the corneal surface can ever be of the form requisite to include reversed curvatures; so that in such an instance the Ametropia is rather more apt to be one in which an opposite quality of astigmatism is in excess of an existing Hypermetropia or Myopia, respectively. Accepting this to be the case, such an eye would fall heir to the features accredited to Hypermetropia or Myopia respecting the "nodal points" and "amplitude of accommodation"; wherefore, in prescribing either of the aforesaid sphero-cylindrical equivalents, a preference might be given to that form which would be commensurate with the inherent physical and physiological developments above alluded to.

ASYMMETRICAL SURFACES.

§ 42. The properties of asymmetrical refraction are also fulfilled in a lens by creating for it, opposite a plane side, a single surface including both curvatures allotted to the unequally refracting chief meridians.

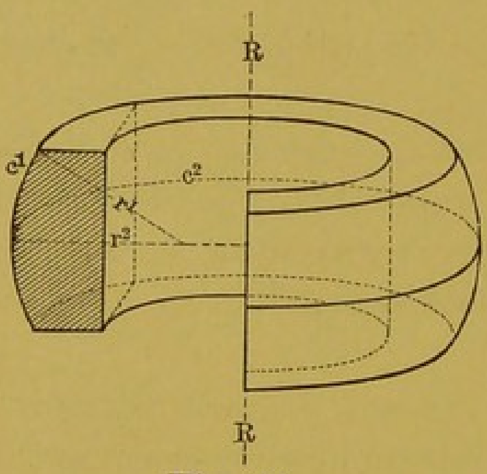

Fig. 33.

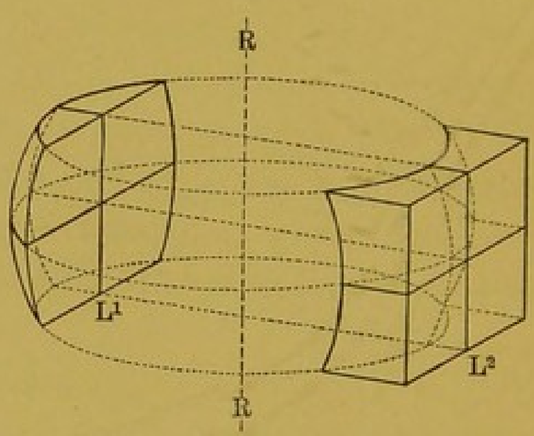

Fig. 34.

The character of such a surface is shown in Fig. 33, wherein I have selected a curvature c^1 prescribed by the requisite radius r^1 to effect a refraction of 3 Dioptrics, and by having caused it to be rotated upon a vertical axis R, a second curvature c^2 at right angles to c^1 is created having a radius r^2, so chosen as to correspond to a refraction of 2 Dioptrics.

In Fig. 34 two lenses are shown as being included within the surface so prescribed and an opposite plane side, the one being plano-convex L^1,—the other plano-concave L^2.

From the nature of the construction, it follows that these lenses are each possessed of 3D. refraction in the vertical and 2D. refraction in the horizontal meridian, so that the formulæ for the same might be expressed, respectively, by

(A_1) [+ 3D. Ref. 90° ⊂ + 2D. Ref. 180°] As. . . . . (L^1)

(B_1) [− 3D. Ref. 90° ⊂ − 2D. Ref. 180°] As. . . . . (L^2)

as a distinction to the correlative formulæ A_2 and B_2 for a pair of crossed cylinders of identical refraction

(A_2) + 3 cyl. axis 180° ⊂ + 2 cyl. axis 90°

(B_2) − 3 cyl. axis 180° ⊂ − 2 cyl. axis 90°

and their sphero-cylindrical equivalents, respectively—

(A_3) { + 2 sph. ⊂ + 1 cyl. axis 180° (Double Form)
+ 3 sph. ⊂ − 1 cyl. axis 90° (Periscopic Form)

(B_3) { − 2 sph. ⊂ − 1 cyl. axis 180° (Double Form)
− 3 sph. ⊂ + 1 cyl. axis 90° (Periscopic Form)

§ 43. The rotary body shown in Fig. 33 may also be considered to have been created by bending a simple cylindrical lens c^1 to the radius r^2.

In such an attempt, the lens of the Formula A_1 might be obtained by bending a 3D. cylindrical lens to a radius corresponding to a refraction of 2 Dioptrics, or a 2D. cylindrical lens to a radius corresponding to a refraction of 3 Dioptrics, in which event the latter would merely require to be turned 90° to correspond with the balance of formulæ. The inner or back surface would naturally also require to be restored to a plane as indicated by the parallelogram in dotted lines.

The suggested method being impracticable, the process of grinding must be resorted to, and, as this involves a far greater variety of tools than is commensurate with the practical advantages of the lenses so created, little has as yet been done in this direction except to demonstrate the feasibility of the operation.

§ 44. To exhibit the nature of the surface requisite to include the functions of mixed meridians, a curvature, c^2, of the horizontal refraction + 2D., is shown in Fig. 35 as being the result of a rotation of the concave curvature c^1 with the vertical refraction − 3D.

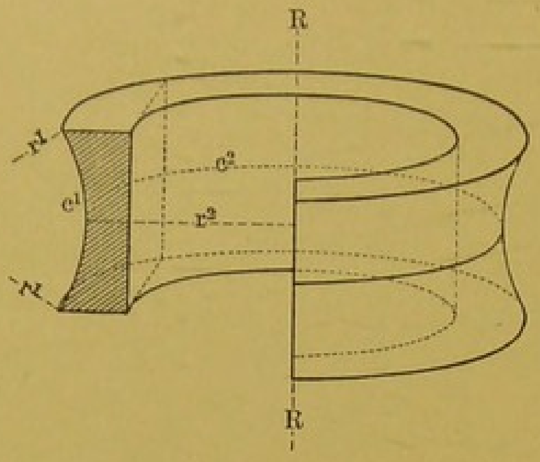

Fig. 35.

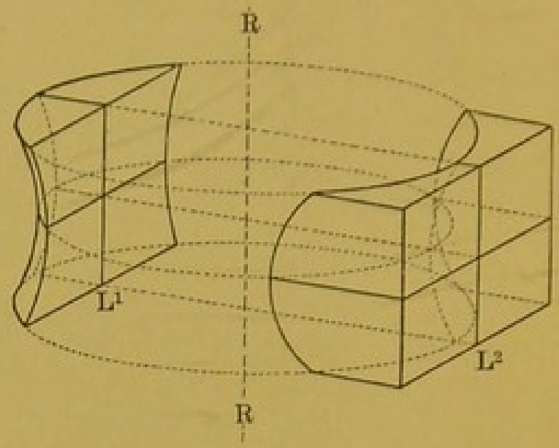

Fig. 36.

For the lenses shown in Fig. 36 as being provided with the above surface, it is obvious that each is possessed of a convex and a concave chief meridian or reversed curvatures, and the formulæ for the same is

(C_1) [$-$ 3D. Ref. 90° $\subset$ + 2D. Ref. 180°] As. . . . . (L^1)

(D_1) [+ 3D. Ref. 90° $\subset$ $-$ 2D. Ref. 180°] As. . . . . (L^2)

as a distinction to the Formulæ C_2 and D_2 for crossed cylinders of identical refraction

(C_2) $-$ 3 cyl. axis 180° $\subset$ + 2 cyl. axis 90°.

(D_2) + 3 cyl. axis 180° $\subset$ $-$ 2 cyl. axis 90°.

and their sphero-cylindrical equivalents, respectively—

(C_3) { + 2 sph. $\subset$ $-$ 5 cyl. axis 180°.
$-$ 3 sph. $\subset$ + 5 cyl. axis 90°.

(D_3) { $-$ 2 sph. $\subset$ + 5 cyl. axis 180°.
+ 3 sph. $\subset$ $-$ 5 cyl. axis 90°.

In concluding this treatise, I recommend the student to practice the transformation of optionally selected formulæ, by the application of the rules given, and in which he may further resort to the appended tables in verification of his work.

	NUMERALS OF REFRACTION.		
Focal Distances. Centimetres.	**Metric System.** Dioptrics. =	**Inch System.** Approximates.	**Focal Distances.** U. S. Standard Inches.
400.	**0.25**	**1:160**	$157\frac{1}{2}$
200.	**0.50**	**1:80**	$78\frac{3}{4}$
133.3	**0.75**	**1:53**	$52\frac{1}{2}$
100.	**1.**	**1:40**	$39\frac{3}{8}$
80.	**1.25**	**1:32**	$31\frac{1}{2}$
66.7	**1.50**	**1:26**	$26\frac{1}{4}$
57.1	**1.75**	**1:22**	$22\frac{1}{2}$
50.	**2.**	**1:20**	$19\frac{11}{16}$
44.4	**2.25**	**1:18**	$17\frac{1}{2}$
40.	**2.50**	**1:16**	$15\frac{3}{4}$
36.4	**2.75**	**1:14**	$14\frac{5}{16}$
33.3	**3.**	**1:13**	$13\frac{1}{8}$
30.8	**3.25**	**1:12**	$12\frac{1}{8}$
28.6	**3.50**	**1:11**	$11\frac{1}{4}$
25.	**4.**	**1:10**	$9\frac{7}{8}$
22.2	**4.50**	**1:9**	$8\frac{3}{4}$
20.	**5.**	**1:8**	$7\frac{7}{8}$
18.2	**5.50**	**1:7**	$7\frac{1}{8}$
16.7	**6.**	**$1:6\frac{1}{2}$**	$6\frac{9}{16}$
15.4	**6.50**	**1:6**	6
14.3	**7.**	**$1:5\frac{1}{2}$**	$5\frac{5}{8}$
12.5	**8.**	**1:5**	$4\frac{15}{16}$
11.1	**9.**	**$1:4\frac{1}{2}$**	$4\frac{3}{8}$
10.	**10.**	**1:4**	$3\frac{15}{16}$
9.1	**11.**	**$1:3\frac{1}{2}$**	$3\frac{9}{16}$
8.3	**12.**	**$1:3\frac{1}{4}$**	$3\frac{9}{32}$
7.7	**13.**	**1:3**	$3\frac{1}{32}$
7.1	**14.**	**$1:2\frac{3}{4}$**	$2\frac{13}{16}$
6.7	**15.**		$2\frac{5}{8}$
6.3	**16.**	**$1:2\frac{1}{2}$**	$2\frac{7}{16}$
5.5	**18.**	**$1:2\frac{1}{4}$**	$2\frac{3}{16}$
5.	**20.**	**1:2**	$1\frac{31}{32}$
2.5	**40.**	**1:1**	$\frac{63}{64}$

The above table has been arranged for comparison of the metric with the old system of numbering, in which 1 inch was adopted as the unit. A lens of 10, 20, or 40 inches focus is therefore represented as being $\frac{1}{10}$, $\frac{1}{20}$, or $\frac{1}{40}$ of the refraction of the old standard.

The focal distances have been calculated upon the basis: 1 metre=100 centimetres=39.37 U. S. standard inches, through dividing each of these equivalents by the Dioptric numerals. To render a harmony of the numerals of the two systems possible, it is found necessary to neglect slight fractional variations, as shown in the differences between the divisors in the 3d with the figures of the 4th column. 1 Dioptric being placed as equivalent to $\frac{1}{40}$, lenses of 2, 3, or 4 Dioptrics may be calculated as $\frac{2}{40}=\frac{1}{20}$, $\frac{3}{40}=\frac{1}{13}$, or $\frac{4}{40}=\frac{1}{10}$ respectively, without materially conflicting with the practical demands upon accuracy in a substitution of one system of numerals for the other.

I. TABLE OF CROSSED CYLINDERS AND THEIR SPHERO-CYLINDRICAL EQUIVALENTS.

CONVEX MERIDIANS.

DIOPTRICS.	+0.25 C. 180°	+0.50 C. 180°	+0.75 C. 180°	+1.00 C. 180°	+1.25 C. 180°	+1.50 C. 180°	+1.75 C. 180°	+2.00 C. 180°	+2.25 C. 180°	+2.50 C. 180°	+2.75 C. 180°	+3.00 C. 180°	+3.25 C. 180°	+3.50 C. 180°	+4.00 C. 180°
+0.25 C. 90°	(+0.25)	[a1] +0.25○+0.25⊖ +0.50○−0.25⦶	[a2] +0.25○+0.50⊖ +0.75○−0.50⦶	[a3] +0.25○+0.75⊖ +1.00○−0.75⦶	+0.25○+1.00⊖ +1.25○−1.00⦶	+0.25○+1.25⊖ +1.50○−1.25⦶	+0.25○+1.50⊖ +1.75○−1.50⦶	+0.25○+1.75⊖ +2.00○−1.75⦶	+0.25○+2.00⊖ +2.25○−2.00⦶	+0.25○+2.25⊖ +2.50○−2.25⦶	+0.25○+2.50⊖ +2.75○−2.50⦶	+0.25○+2.75⊖ +3.00○−2.75⦶	+0.25○+3.00⊖ +3.25○−3.00⦶	+0.25○+3.25⊖ +3.50○−3.25⦶	+0.25○+3.50⊖ +4.00○−3.50⦶
+0.50 C. 90°	[A1] +0.25○+0.25⦶ +0.50○−0.25⊖	(+0.50)	[b1] +0.50○+0.25⊖ +0.75○−0.25⦶	[b2] +0.50○+0.50⊖ +1.00○−0.50⦶	[b3] +0.50○+0.75⊖ +1.25○−0.75⦶	+0.50○+1.00⊖ +1.50○−1.00⦶	+0.50○+1.25⊖ +1.75○−1.25⦶	+0.50○+1.50⊖ +2.00○−1.50⦶	+0.50○+1.75⊖ +2.25○−1.75⦶	+0.50○+2.00⊖ +2.50○−2.00⦶	+0.50○+2.25⊖ +2.75○−2.25⦶	+0.50○+2.50⊖ +3.00○−2.50⦶	+0.50○+2.75⊖ +3.25○−2.75⦶	+0.50○+3.00⊖ +3.50○−3.00⦶	+0.50○+3.25⊖ +4.00○−3.25⦶
+0.75 C. 90°	[A2] +0.25○+0.50⦶ +0.75○−0.50⊖	[B1] +0.50○+0.25⦶ +0.75○−0.25⊖	(+0.75)	+0.75○+0.25⊖ +1.00○−0.25⦶	+0.75○+0.50⊖ +1.25○−0.50⦶	+0.75○+0.75⊖ +1.50○−0.75⦶	+0.75○+1.00⊖ +1.75○−1.00⦶	+0.75○+1.25⊖ +2.00○−1.25⦶	+0.75○+1.50⊖ +2.25○−1.50⦶	+0.75○+1.75⊖ +2.50○−1.75⦶	+0.75○+2.00⊖ +2.75○−2.00⦶	+0.75○+2.25⊖ +3.00○−2.25⦶	+0.75○+2.50⊖ +3.25○−2.50⦶	+0.75○+2.75⊖ +3.50○−2.75⦶	+0.75○+3.00⊖ +4.00○−3.00⦶
+1.00 C. 90°	[A3] +0.25○+0.75⦶ +1.00○−0.75⊖	[B2] +0.50○+0.50⦶ +1.00○−0.50⊖	+0.75○+0.25⦶ +1.00○−0.25⊖	(+1.00)	+1.00○+0.25⊖ +1.25○−0.25⦶	+1.00○+0.50⊖ +1.50○−0.50⦶	+1.00○+0.75⊖ +1.75○−0.75⦶	+1.00○+1.00⊖ +2.00○−1.00⦶	+1.00○+1.25⊖ +2.25○−1.25⦶	+1.00○+1.50⊖ +2.50○−1.50⦶	+1.00○+1.75⊖ +2.75○−1.75⦶	+1.00○+2.00⊖ +3.00○−2.00⦶	+1.00○+2.25⊖ +3.25○−2.25⦶	+1.00○+2.50⊖ +3.50○−2.50⦶	+1.00○+2.75⊖ +4.00○−2.75⦶
+1.25 C. 90°	+0.25○+1.00⦶ +1.25○−1.00⊖	[B3] +0.50○+0.75⦶ +1.25○−0.75⊖	+0.75○+0.50⦶ +1.25○−0.50⊖	+1.00○+0.25⦶ +1.25○−0.25⊖	(+1.25)	+1.25○+0.25⊖ +1.50○−0.25⦶	+1.25○+0.50⊖ +1.75○−0.50⦶	+1.25○+0.75⊖ +2.00○−0.75⦶	+1.25○+1.00⊖ +2.25○−1.00⦶	+1.25○+1.25⊖ +2.50○−1.25⦶	+1.25○+1.50⊖ +2.75○−1.50⦶	+1.25○+1.75⊖ +3.00○−1.75⦶	+1.25○+2.00⊖ +3.25○−2.00⦶	+1.25○+2.25⊖ +3.50○−2.25⦶	+1.25○+2.50⊖ +4.00○−2.50⦶
+1.50 C. 90°	+0.25○+1.25⦶ +1.50○−1.25⊖	+0.50○+1.00⦶ +1.50○−1.00⊖	+0.75○+0.75⦶ +1.50○−0.75⊖	+1.00○+0.50⦶ +1.50○−0.50⊖	+1.25○+0.25⦶ +1.50○−0.25⊖	(+1.50)	+1.50○+0.25⊖ +1.75○−0.25⦶	+1.50○+0.50⊖ +2.00○−0.50⦶	+1.50○+0.75⊖ +2.25○−0.75⦶	+1.50○+1.00⊖ +2.50○−1.00⦶	+1.50○+1.25⊖ +2.75○−1.25⦶	+1.50○+1.50⊖ +3.00○−1.50⦶	+1.50○+1.75⊖ +3.25○−1.75⦶	+1.50○+2.00⊖ +3.50○−2.00⦶	+1.50○+2.25⊖ +4.00○−2.25⦶
+1.75 C. 90°	+0.25○+1.50⦶ +1.75○−1.50⊖	+0.50○+1.25⦶ +1.75○−1.25⊖	+0.75○+1.00⦶ +1.75○−1.00⊖	+1.00○+0.75⦶ +1.75○−0.75⊖	+1.25○+0.50⦶ +1.75○−0.50⊖	+1.50○+0.25⦶ +1.75○−0.25⊖	(+1.75)	+1.75○+0.25⊖ +2.00○−0.25⦶	+1.75○+0.50⊖ +2.25○−0.50⦶	+1.75○+0.75⊖ +2.50○−0.75⦶	+1.75○+1.00⊖ +2.75○−1.00⦶	+1.75○+1.25⊖ +3.00○−1.25⦶	+1.75○+1.50⊖ +3.25○−1.50⦶	+1.75○+1.75⊖ +3.50○−1.75⦶	+1.75○+2.00⊖ +4.00○−2.00⦶
+2.00 C. 90°	+0.25○+1.75⦶ +2.00○−1.75⊖	+0.50○+1.50⦶ +2.00○−1.50⊖	+0.75○+1.25⦶ +2.00○−1.25⊖	+1.00○+1.00⦶ +2.00○−1.00⊖	+1.25○+0.75⦶ +2.00○−0.75⊖	+1.50○+0.50⦶ +2.00○−0.50⊖	+1.75○+0.25⦶ +2.00○−0.25⊖	(+2.00)	+2.00○+0.25⊖ +2.25○−0.25⦶	+2.00○+0.50⊖ +2.50○−0.50⦶	+2.00○+0.75⊖ +2.75○−0.75⦶	+2.00○+1.00⊖ +3.00○−1.00⦶	+2.00○+1.25⊖ +3.25○−1.25⦶	+2.00○+1.50⊖ +3.50○−1.50⦶	+2.00○+1.75⊖ +4.00○−1.75⦶
+2.25 C. 90°	+0.25○+2.00⦶ +2.25○−2.00⊖	+0.50○+1.75⦶ +2.25○−1.75⊖	+0.75○+1.50⦶ +2.25○−1.50⊖	+1.00○+1.25⦶ +2.25○−1.25⊖	+1.25○+1.00⦶ +2.25○−1.00⊖	+1.50○+0.75⦶ +2.25○−0.75⊖	+1.75○+0.50⦶ +2.25○−0.50⊖	+2.00○+0.25⦶ +2.25○−0.25⊖	(+2.25)	+2.25○+0.25⊖ +2.50○−0.25⦶	+2.25○+0.50⊖ +2.75○−0.50⦶	+2.25○+0.75⊖ +3.00○−0.75⦶	+2.25○+1.00⊖ +3.25○−1.00⦶	+2.25○+1.25⊖ +3.50○−1.25⦶	+2.25○+1.50⊖ +4.00○−1.50⦶
+2.50 C. 90°	+0.25○+2.25⦶ +2.50○−2.25⊖	+0.50○+2.00⦶ +2.50○−2.00⊖	+0.75○+1.75⦶ +2.50○−1.75⊖	+1.00○+1.50⦶ +2.50○−1.50⊖	+1.25○+1.25⦶ +2.50○−1.25⊖	+1.50○+1.00⦶ +2.50○−1.00⊖	+1.75○+0.75⦶ +2.50○−0.75⊖	+2.00○+0.50⦶ +2.50○−0.50⊖	+2.25○+0.25⦶ +2.50○−0.25⊖	(+2.50)	+2.50○+0.25⊖ +2.75○−0.25⦶	+2.50○+0.50⊖ +3.00○−0.50⦶	+2.50○+0.75⊖ +3.25○−0.75⦶	+2.50○+1.00⊖ +3.50○−1.00⦶	+2.50○+1.25⊖ +4.00○−1.25⦶
+2.75 C. 90°	+0.25○+2.50⦶ +2.75○−2.50⊖	+0.50○+2.25⦶ +2.75○−2.25⊖	+0.75○+2.00⦶ +2.75○−2.00⊖	+1.00○+1.75⦶ +2.75○−1.75⊖	+1.25○+1.50⦶ +2.75○−1.50⊖	+1.50○+1.25⦶ +2.75○−1.25⊖	+1.75○+1.00⦶ +2.75○−1.00⊖	+2.00○+0.75⦶ +2.75○−0.75⊖	+2.25○+0.50⦶ +2.75○−0.50⊖	+2.50○+0.25⦶ +2.75○−0.25⊖	(+2.75)	+2.75○+0.25⊖ +3.00○−0.25⦶	+2.75○+0.50⊖ +3.25○−0.50⦶	+2.75○+0.75⊖ +3.50○−0.75⦶	+2.75○+1.00⊖ +4.00○−1.00⦶
+3.00 C. 90°	+0.25○+2.75⦶ +3.00○−2.75⊖	+0.50○+2.50⦶ +3.00○−2.50⊖	+0.75○+2.25⦶ +3.00○−2.25⊖	+1.00○+2.00⦶ +3.00○−2.00⊖	+1.25○+1.75⦶ +3.00○−1.75⊖	+1.50○+1.50⦶ +3.00○−1.50⊖	+1.75○+1.25⦶ +3.00○−1.25⊖	+2.00○+1.00⦶ +3.00○−1.00⊖	+2.25○+0.75⦶ +3.00○−0.75⊖	+2.50○+0.50⦶ +3.00○−0.50⊖	+2.75○+0.25⦶ +3.00○−0.25⊖	(+3.00)	+3.00○+0.25⊖ +3.25○−0.25⦶	+3.00○+0.50⊖ +3.50○−0.50⦶	+3.00○+0.75⊖ +4.00○−0.75⦶
+3.25 C. 90°	+0.25○+3.00⦶ +3.25○−3.00⊖	+0.50○+2.75⦶ +3.25○−2.75⊖	+0.75○+2.50⦶ +3.25○−2.50⊖	+1.00○+2.25⦶ +3.25○−2.25⊖	+1.25○+2.00⦶ +3.25○−2.00⊖	+1.50○+1.75⦶ +3.25○−1.75⊖	+1.75○+1.50⦶ +3.25○−1.50⊖	+2.00○+1.25⦶ +3.25○−1.25⊖	+2.25○+1.00⦶ +3.25○−1.00⊖	+2.50○+0.75⦶ +3.25○−0.75⊖	+2.75○+0.50⦶ +3.25○−0.50⊖	+3.00○+0.25⦶ +3.25○−0.25⊖	(+3.25)	+3.25○+0.25⊖ +3.50○−0.25⦶	+3.25○+0.50⊖ +4.00○−0.50⦶
+3.50 C. 90°	+0.25○+3.25⦶ +3.50○−3.25⊖	+0.50○+3.00⦶ +3.50○−3.00⊖	+0.75○+2.75⦶ +3.50○−2.75⊖	+1.00○+2.50⦶ +3.50○−2.50⊖	+1.25○+2.25⦶ +3.50○−2.25⊖	+1.50○+2.00⦶ +3.50○−2.00⊖	+1.75○+1.75⦶ +3.50○−1.75⊖	+2.00○+1.50⦶ +3.50○−1.50⊖	+2.25○+1.25⦶ +3.50○−1.25⊖	+2.50○+1.00⦶ +3.50○−1.00⊖	+2.75○+0.75⦶ +3.50○−0.75⊖	+3.00○+0.50⦶ +3.50○−0.50⊖	+3.25○+0.25⦶ +3.50○−0.25⊖	(+3.50)	+3.50○+0.25⊖ +4.00○−0.25⦶
+4.00 C. 90°	+0.25○+3.50⦶ +4.00○−3.50⊖	+0.50○+3.25⦶ +4.00○−3.25⊖	+0.75○+3.00⦶ +4.00○−3.00⊖	+1.00○+2.75⦶ +4.00○−2.75⊖	+1.25○+2.50⦶ +4.00○−2.50⊖	+1.50○+2.25⦶ +4.00○−2.25⊖	+1.75○+2.00⦶ +4.00○−2.00⊖	+2.00○+1.75⦶ +4.00○−1.75⊖	+2.25○+1.50⦶ +4.00○−1.50⊖	+2.50○+1.25⦶ +4.00○−1.25⊖	+2.75○+1.00⦶ +4.00○−1.00⊖	+3.00○+0.75⦶ +4.00○−0.75⊖	+3.25○+0.50⦶ +4.00○−0.50⊖	+3.50○+0.25⦶ +4.00○−0.25⊖	(+4.00)

In the above formulæ the first numerals apply to spherical, and the second to cylindrical refraction, for which, in the appended signs, the upright and horizontal diameters (| and —) of the circles denote the axes 90° and 180°, respectively, or diametrically opposed axes.

With the exception of the diagonal column of spherical equivalents, each field contains both the double and periscopic form of convex sphero-cylindrical equivalent. For crossed concave cylinders it is merely necessary to reverse the signs + and — wherever they occur.

II. TABLE OF CROSSED CYLINDERS AND THEIR SPHERO-CYLINDRICAL EQUIVALENTS.

CONCAVO-CONVEX MERIDIANS.

DIOPTRICS.	+0.25C.180°	+0.50C.180°	+0.75C.180°	+1.00C.180°	+1.25C.180°	+1.50C.180°	+1.75C.180°	+2.00C.180°	+2.25C.180°	+2.50C.180°	+2.75C.180°	+3.00C.180°	+3.25C.180°	+3.50C.180°	+4.00C.180°
−0.25C.90°	+0.25⊂−0.50⦶ −0.25⊂+0.50⊖	+0.50⊂−0.75⦶ −0.25⊂+0.75⊖	+0.75⊂−1.00⦶ −0.25⊂+1.00⊖	+1.00⊂−1.25⦶ −0.25⊂+1.25⊖	+1.25⊂−1.50⦶ −0.25⊂+1.50⊖	+1.50⊂−1.75⦶ −0.25⊂+1.75⊖	+1.75⊂−2.00⦶ −0.25⊂+2.00⊖	+2.00⊂−2.25⦶ −0.25⊂+2.25⊖	+2.25⊂−2.50⦶ −0.25⊂+2.50⊖	+2.50⊂−2.75⦶ −0.25⊂+2.75⊖	+2.75⊂−3.00⦶ −0.25⊂+3.00⊖	+3.00⊂−3.25⦶ −0.25⊂+3.25⊖	+3.25⊂−3.50⦶ −0.25⊂+3.50⊖	+3.50⊂−3.75⦶ −0.25⊂+3.75⊖	+4.00⊂−4.25⦶ −0.25⊂+4.25⊖
−0.50C.90°	+0.25⊂−0.75⦶ −0.50⊂+0.75⊖	+0.50⊂−1.00⦶ −0.50⊂+1.00⊖	+0.75⊂−1.25⦶ −0.50⊂+1.25⊖	+1.00⊂−1.50⦶ −0.50⊂+1.50⊖	+1.25⊂−1.75⦶ −0.50⊂+1.75⊖	+1.50⊂−2.00⦶ −0.50⊂+2.00⊖	+1.75⊂−2.25⦶ −0.50⊂+2.25⊖	+2.00⊂−2.50⦶ −0.50⊂+2.50⊖	+2.25⊂−2.75⦶ −0.50⊂+2.75⊖	+2.50⊂−3.00⦶ −0.50⊂+3.00⊖	+2.75⊂−3.25⦶ −0.50⊂+3.25⊖	+3.00⊂−3.50⦶ −0.50⊂+3.50⊖	+3.25⊂−3.75⦶ −0.50⊂+3.75⊖	+3.50⊂−4.00⦶ −0.50⊂+4.00⊖	+4.00⊂−4.50⦶ −0.50⊂+4.50⊖
−0.75C.90°	+0.25⊂−1.00⦶ −0.75⊂+1.00⊖	+0.50⊂−1.25⦶ −0.75⊂+1.25⊖	+0.75⊂−1.50⦶ −0.75⊂+1.50⊖	+1.00⊂−1.75⦶ −0.75⊂+1.75⊖	+1.25⊂−2.00⦶ −0.75⊂+2.00⊖	+1.50⊂−2.25⦶ −0.75⊂+2.25⊖	+1.75⊂−2.50⦶ −0.75⊂+2.50⊖	+2.00⊂−2.75⦶ −0.75⊂+2.75⊖	+2.25⊂−3.00⦶ −0.75⊂+3.00⊖	+2.50⊂−3.25⦶ −0.75⊂+3.25⊖	+2.75⊂−3.50⦶ −0.75⊂+3.50⊖	+3.00⊂−3.75⦶ −0.75⊂+3.75⊖	+3.25⊂−4.00⦶ −0.75⊂+4.00⊖	+3.50⊂−4.25⦶ −0.75⊂+4.25⊖	+4.00⊂−4.75⦶ −0.75⊂+4.75⊖
−1.00C.90°	+0.25⊂−1.25⦶ −1.00⊂+1.25⊖	+0.50⊂−1.50⦶ −1.00⊂+1.50⊖	+0.75⊂−1.75⦶ −1.00⊂+1.75⊖	+1.00⊂−2.00⦶ −1.00⊂+2.00⊖	+1.25⊂−2.25⦶ −1.00⊂+2.25⊖	+1.50⊂−2.50⦶ −1.00⊂+2.50⊖	+1.75⊂−2.75⦶ −1.00⊂+2.75⊖	+2.00⊂−3.00⦶ −1.00⊂+3.00⊖	+2.25⊂−3.25⦶ −1.00⊂+3.25⊖	+2.50⊂−3.50⦶ −1.00⊂+3.50⊖	+2.75⊂−3.75⦶ −1.00⊂+3.75⊖	+3.00⊂−4.00⦶ −1.00⊂+4.00⊖	+3.25⊂−4.25⦶ −1.00⊂+4.25⊖	+3.50⊂−4.50⦶ −1.00⊂+4.50⊖	+4.00⊂−5.00⦶ −1.00⊂+5.00⊖
−1.25C.90°	+0.25⊂−1.50⦶ −1.25⊂+1.50⊖	+0.50⊂−1.75⦶ −1.25⊂+1.75⊖	+0.75⊂−2.00⦶ −1.25⊂+2.00⊖	+1.00⊂−2.25⦶ −1.25⊂+2.25⊖	+1.25⊂−2.50⦶ −1.25⊂+2.50⊖	+1.50⊂−2.75⦶ −1.25⊂+2.75⊖	+1.75⊂−3.00⦶ −1.25⊂+3.00⊖	+2.00⊂−3.25⦶ −1.25⊂+3.25⊖	+2.25⊂−3.50⦶ −1.25⊂+3.50⊖	+2.50⊂−3.75⦶ −1.25⊂+3.75⊖	+2.75⊂−4.00⦶ −1.25⊂+4.00⊖	+3.00⊂−4.25⦶ −1.25⊂+4.25⊖	+3.25⊂−4.50⦶ −1.25⊂+4.50⊖	+3.50⊂−4.75⦶ −1.25⊂+4.75⊖	+4.00⊂−5.25⦶ −1.25⊂+5.25⊖
−1.50C.90°	+0.25⊂−1.75⦶ −1.50⊂+1.75⊖	+0.50⊂−2.00⦶ −1.50⊂+2.00⊖	+0.75⊂−2.25⦶ −1.50⊂+2.25⊖	+1.00⊂−2.50⦶ −1.50⊂+2.50⊖	+1.25⊂−2.75⦶ −1.50⊂+2.75⊖	+1.50⊂−3.00⦶ −1.50⊂+3.00⊖	+1.75⊂−3.25⦶ −1.50⊂+3.25⊖	+2.00⊂−3.50⦶ −1.50⊂+3.50⊖	+2.25⊂−3.75⦶ −1.50⊂+3.75⊖	+2.50⊂−4.00⦶ −1.50⊂+4.00⊖	+2.75⊂−4.25⦶ −1.50⊂+4.25⊖	+3.00⊂−4.50⦶ −1.50⊂+4.50⊖	+3.25⊂−4.75⦶ −1.50⊂+4.75⊖	+3.50⊂−5.00⦶ −1.50⊂+5.00⊖	+4.00⊂−5.50⦶ −1.50⊂+5.50⊖
−1.75C.90°	+0.25⊂−2.00⦶ −1.75⊂+2.00⊖	+0.50⊂−2.25⦶ −1.75⊂+2.25⊖	+0.75⊂−2.50⦶ −1.75⊂+2.50⊖	+1.00⊂−2.75⦶ −1.75⊂+2.75⊖	+1.25⊂−3.00⦶ −1.75⊂+3.00⊖	+1.50⊂−3.25⦶ −1.75⊂+3.25⊖	+1.75⊂−3.50⦶ −1.75⊂+3.50⊖	+2.00⊂−3.75⦶ −1.75⊂+3.75⊖	+2.25⊂−4.00⦶ −1.75⊂+4.00⊖	+2.50⊂−4.25⦶ −1.75⊂+4.25⊖	+2.75⊂−4.50⦶ −1.75⊂+4.50⊖	+3.00⊂−4.75⦶ −1.75⊂+4.75⊖	+3.25⊂−5.00⦶ −1.75⊂+5.00⊖	+3.50⊂−5.25⦶ −1.75⊂+5.25⊖	+4.00⊂−5.75⦶ −1.75⊂+5.75⊖
−2.00C.90°	+0.25⊂−2.25⦶ −2.00⊂+2.25⊖	+0.50⊂−2.50⦶ −2.00⊂+2.50⊖	+0.75⊂−2.75⦶ −2.00⊂+2.75⊖	+1.00⊂−3.00⦶ −2.00⊂+3.00⊖	+1.25⊂−3.25⦶ −2.00⊂+3.25⊖	+1.50⊂−3.50⦶ −2.00⊂+3.50⊖	+1.75⊂−3.75⦶ −2.00⊂+3.75⊖	+2.00⊂−4.00⦶ −2.00⊂+4.00⊖	+2.25⊂−4.25⦶ −2.00⊂+4.25⊖	+2.50⊂−4.50⦶ −2.00⊂+4.50⊖	+2.75⊂−4.75⦶ −2.00⊂+4.75⊖	+3.00⊂−5.00⦶ −2.00⊂+5.00⊖	+3.25⊂−5.25⦶ −2.00⊂+5.25⊖	+3.50⊂−5.50⦶ −2.00⊂+5.50⊖	+4.00⊂−6.00⦶ −2.00⊂+6.00⊖
−2.25C.90°	+0.25⊂−2.50⦶ −2.25⊂+2.50⊖	+0.50⊂−2.75⦶ −2.25⊂+2.75⊖	+0.75⊂−3.00⦶ −2.25⊂+3.00⊖	+1.00⊂−3.25⦶ −2.25⊂+3.25⊖	+1.25⊂−3.50⦶ −2.25⊂+3.50⊖	+1.50⊂−3.75⦶ −2.25⊂+3.75⊖	+1.75⊂−4.00⦶ −2.25⊂+4.00⊖	+2.00⊂−4.25⦶ −2.25⊂+4.25⊖	+2.25⊂−4.50⦶ −2.25⊂+4.50⊖	+2.50⊂−4.75⦶ −2.25⊂+4.75⊖	+2.75⊂−5.00⦶ −2.25⊂+5.00⊖	+3.00⊂−5.25⦶ −2.25⊂+5.25⊖	+3.25⊂−5.50⦶ −2.25⊂+5.50⊖	+3.50⊂−5.75⦶ −2.25⊂+5.75⊖	+4.00⊂−6.25⦶ −2.25⊂+6.25⊖
−2.50C.90°	+0.25⊂−2.75⦶ −2.50⊂+2.75⊖	+0.50⊂−3.00⦶ −2.50⊂+3.00⊖	+0.75⊂−3.25⦶ −2.50⊂+3.25⊖	+1.00⊂−3.50⦶ −2.50⊂+3.50⊖	+1.25⊂−3.75⦶ −2.50⊂+3.75⊖	+1.50⊂−4.00⦶ −2.50⊂+4.00⊖	+1.75⊂−4.25⦶ −2.50⊂+4.25⊖	+2.00⊂−4.50⦶ −2.50⊂+4.50⊖	+2.25⊂−4.75⦶ −2.50⊂+4.75⊖	+2.50⊂−5.00⦶ −2.50⊂+5.00⊖	+2.75⊂−5.25⦶ −2.50⊂+5.25⊖	+3.00⊂−5.50⦶ −2.50⊂+5.50⊖	+3.25⊂−5.75⦶ −2.50⊂+5.75⊖	+3.50⊂−6.00⦶ −2.50⊂+6.00⊖	+4.00⊂−6.50⦶ −2.50⊂+6.50⊖
−2.75C.90°	+0.25⊂−3.00⦶ −2.75⊂+3.00⊖	+0.50⊂−3.25⦶ −2.75⊂+3.25⊖	+0.75⊂−3.50⦶ −2.75⊂+3.50⊖	+1.00⊂−3.75⦶ −2.75⊂+3.75⊖	+1.25⊂−4.00⦶ −2.75⊂+4.00⊖	+1.50⊂−4.25⦶ −2.75⊂+4.25⊖	+1.75⊂−4.50⦶ −2.75⊂+4.50⊖	+2.00⊂−4.75⦶ −2.75⊂+4.75⊖	+2.25⊂−5.00⦶ −2.75⊂+5.00⊖	+2.50⊂−5.25⦶ −2.75⊂+5.25⊖	+2.75⊂−5.50⦶ −2.75⊂+5.50⊖	+3.00⊂−5.75⦶ −2.75⊂+5.75⊖	+3.25⊂−6.00⦶ −2.75⊂+6.00⊖	+3.50⊂−6.25⦶ −2.75⊂+6.25⊖	+4.00⊂−6.75⦶ −2.75⊂+6.75⊖
−3.00C.90°	+0.25⊂−3.25⦶ −3.00⊂+3.25⊖	+0.50⊂−3.50⦶ −3.00⊂+3.50⊖	+0.75⊂−3.75⦶ −3.00⊂+3.75⊖	+1.00⊂−4.00⦶ −3.00⊂+4.00⊖	+1.25⊂−4.25⦶ −3.00⊂+4.25⊖	+1.50⊂−4.50⦶ −3.00⊂+4.50⊖	+1.75⊂−4.75⦶ −3.00⊂+4.75⊖	+2.00⊂−5.00⦶ −3.00⊂+5.00⊖	+2.25⊂−5.25⦶ −3.00⊂+5.25⊖	+2.50⊂−5.50⦶ −3.00⊂+5.50⊖	+2.75⊂−5.75⦶ −3.00⊂+5.75⊖	+3.00⊂−6.00⦶ −3.00⊂+6.00⊖	+3.25⊂−6.25⦶ −3.00⊂+6.25⊖	+3.50⊂−6.50⦶ −3.00⊂+6.50⊖	+4.00⊂−7.00⦶ −3.00⊂+7.00⊖
−3.25C.90°	+0.25⊂−3.50⦶ −3.25⊂+3.50⊖	+0.50⊂−3.75⦶ −3.25⊂+3.75⊖	+0.75⊂−4.00⦶ −3.25⊂+4.00⊖	+1.00⊂−4.25⦶ −3.25⊂+4.25⊖	+1.25⊂−4.50⦶ −3.25⊂+4.50⊖	+1.50⊂−4.75⦶ −3.25⊂+4.75⊖	+1.75⊂−5.00⦶ −3.25⊂+5.00⊖	+2.00⊂−5.25⦶ −3.25⊂+5.25⊖	+2.25⊂−5.50⦶ −3.25⊂+5.50⊖	+2.50⊂−5.75⦶ −2.25⊂+5.75⊖	+2.75⊂−6.00⦶ −3.25⊂+6.00⊖	+3.00⊂−6.25⦶ −3.25⊂+6.25⊖	+3.25⊂−6.50⦶ −3.25⊂+6.50⊖	+3.50⊂−6.75⦶ −3.25⊂+6.75⊖	+4.00⊂−7.25⦶ −3.25⊂+7.25⊖
−3.50C.90°	+0.25⊂−3.75⦶ −3.50⊂+3.75⊖	+0.50⊂−4.00⦶ −3.50⊂+4.00⊖	+0.75⊂−4.25⦶ −3.50⊂+4.25⊖	+1.00⊂−4.50⦶ −3.50⊂+4.50⊖	+1.25⊂−4.75⦶ −3.50⊂+4.75⊖	+1.50⊂−5.00⦶ −3.50⊂+5.00⊖	+1.75⊂−5.25⦶ −3.50⊂+5.25⊖	+2.00⊂−5.50⦶ −3.50⊂+5.50⊖	+2.25⊂−5.75⦶ −3.50⊂+5.75⊖	+2.50⊂−6.00⦶ −3.50⊂+6.00⊖	+2.75⊂−6.25⦶ −3.50⊂+6.25⊖	+3.00⊂−6.50⦶ −3.50⊂+6.50⊖	+3.25⊂−6.75⦶ −3.50⊂+6.75⊖	+3.50⊂−7.00⦶ −3.50⊂+7.00⊖	+4.00⊂−7.50⦶ −3.50⊂+7.50⊖
−4.00C.90°	+0.25⊂−4.25⦶ −4.00⊂+4.25⊖	+0.50⊂−4.50⦶ −4.00⊂+4.50⊖	+0.75⊂−4.75⦶ −4.00⊂+4.75⊖	+1.00⊂−5.00⦶ −4.00⊂+5.00⊖	+1.25⊂−5.25⦶ −4.00⊂+5.25⊖	+1.50⊂−5.50⦶ −4.00⊂+5.50⊖	+1.75⊂−5.75⦶ −4.00⊂+5.75⊖	+2.00⊂−6.00⦶ −4.00⊂+6.00⊖	+2.25⊂−6.25⦶ −4.00⊂+6.25⊖	+2.50⊂−6.50⦶ −4.00⊂+6.50⊖	+2.75⊂−6.75⦶ −4.00⊂+6.75⊖	+3.00⊂−7.00⦶ −4.00⊂+7.00⊖	+3.25⊂−7.25⦶ −4.00⊂+7.25⊖	+3.50⊂−7.50⦶ −4.00⊂+7.50⊖	+4.00⊂−8.00⦶ −4.00⊂+8.00⊖

In the above formulæ the first numerals apply to spherical, and the second to cylindrical refraction, for which, in the appended signs, the upright and horizontal diameters (| and —) of the circles denote the axes 90° and 180°, respectively, or diametrically opposed axes.

Practical equivalents are only to be obtained when the component numerals of formulæ coincide with those included in the adopted graduation of the dioptric scale, page 41.

For crossed cylinders of which the concave axes are at 180° and the convex at 90° it is merely necessary *to reverse the axes* throughout, and for any combination at right angles within the limits 0° and 180°, by substituting the new axis for that accompanying its correlative cylinder in the above table.